Reduced Whitehead Groups of Prime Exponent Algebras over p-Adic Curves

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Abstract. Let F be the function field of a curve over a p-adic field. Let D/F be a central division algebra of prime exponent ℓ which is different from p. Assume that F contains a primitive ℓ^{2th} root of unity. Then the abstract group $\mathrm{SK}_1(D) := \frac{\mathrm{SL}_1(D)}{[D^*,D^*]}$ is trivial.

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1 Introduction

The group $\mathrm{SL}_1(A)$ of reduced norm one elements of a finite dimensional central simple algebra A over a field K is one of the main and well-studied examples of simply connected almost simple algebraic groups of type A. The commutator subgroup $[A^*, A^*]$ is clearly contained in $\mathrm{SL}_1(A)$. Whether the reverse inclusion holds is however a far more subtle and difficult question to tackle. This problem was formulated by Tannaka and Artin independently in terms of $\mathrm{SK}_1(A)$ which is defined to be the abstract quotient group $\frac{\mathrm{SL}_1(A)}{[A^*,A^*]}$.

Question 1.1 (Tannaka-Artin, 1943). *Is* $\mathrm{SK}_1(A)$ *trivial?*

The Tannaka-Artin problem can be rephrased as a special case of the more general Kneser-Tits problem. For G, a semisimple simply connected isotropic K-group, let $G^+(K)$ denote the normal subgroup generated by the conjugates of the K-points of the unipotent radical of a proper K parabolic of G. One defines the reduced Whitehead group to be $W(G,K) := \frac{G(K)}{G^+(K)}$. The Kneser-Tits problem asks whether $W(G,K)$ is trivial.

The Tannaka-Artin problem was answered affirmatively for square-free index algebras over arbitrary fields ([W50]). It was also shown that $\mathrm{SK}_1(A)$ was trivial for *all* central simple algebras A defined over local or global fields ([NM43],[W50]) and it was widely believed that the Tannaka-Artin question had a positive answer in general. However Platonov's famous example ([P78]) of a biquarternion division algebra D over an iterated Laurent-series field $\mathbb{Q}_p((x))((y))$ with non-trivial $\mathrm{SK}_1(D)$ negatively settled the Tannaka-Artin problem and also gave rise to the first example of a non-rational simply connected almost simple algebraic K-group. Note that the cohomological dimension of the base field under consideration is 4. However, in the same paper by Platonov, it was also shown that the Tannaka-Artin problem has a positive answer for central simple algebras over fields of cohomological dimension ≤ 2.

In 1991, Suslin conjectured that if the index of the central simple algebra D/K is not square free, then $\mathrm{SK}_1(D)$ is *generically non-trivial*, i.e, there exists a field extension F/K such that $\mathrm{SK}_1(D \otimes_K F)$ is non-trivial ([Su91]). More formally, the Suslin invariant

$$\rho : \mathrm{SK}_1(D) \to \frac{\text{kernel}\left[\mathrm{H}^4_{et}\left(K, \mu_n^{\otimes 3}\right) \to \mathrm{H}^4_{et}\left(K(Y), \mu_n^{\otimes 3}\right)\right]}{[D] \bullet \mathrm{H}^2\left(K, \mu_n\right)},$$

where Y is the Severi-Brauer variety defined by D, a central division algebra of degree n, was conjectured to send the generic element to a non-trivial image. Suslin's conjecture was settled affirmatively by Merkurjev for algebras with indices divisible by 4 in ([M93], [M06]).

In the case when the index of D is 4, it is known that ρ is in fact an isomorphism (Rost, Chapter 17 [KMRT]; [M99]; [Su06]). Hence if $\operatorname{cd} K \leq 3$, then $\mathrm{SK}_1(D) = \{0\}$. This led Suslin to ask whether $\mathrm{SK}_1(D) = \{0\}$ for any central simple algebra D of index ℓ^2 where ℓ is a prime, over fields of cohomological dimension 3 ([Su06]).

In this paper, we settle this question affirmatively for exponent ℓ algebras over function fields of p-adic curves where ℓ is any odd prime not equal to p, assuming that our base field contains a primitive ℓ^{2th} root of unity (Theorem 13.8). The proof, whose strategy is outlined below, relies on the techniques of patching as developed by Harbater-Hartmann-Krashen (HHK) in ([HH10], [HHK09], [HHK14] & [HHK15]) and exploits the arithmetic of the base field to show triviality of the reduced Whitehead group.

Let $F = K(X)$ be the function field of a smooth projective geometrically integral curve X over a p-adic field K. Let D denote a central division algebra over F of exponent ℓ where ℓ is an odd prime different from p. Let $z \in \mathrm{SL}_1(D)$ lie in some maximal subfield M of D. We would like to show that z is a product of commutators. The results of Saltman and Wang ([S97], [S98], [W50]) along with standard Galois theory techniques help reduce to the case when D has index ℓ^2 and M contains a sub-cyclic degree ℓ extension Y/F. Let $\mathrm{N}_{M/Y}(z) = a$, which therefore has further norm one to F.

We now briefly explain our strategy (cf. Section 3.3) which essentially adapts

Platonov's argument ([P76]) to our situation. We split a into a product of suitable elements a_1 and a_2 in Y, where the case of each a_i is easier to handle. More precisely, we find elements $a_1, a_2 \in Y$ and degree ℓ sub-field extensions E_1/F, E_2/F in D which commute with Y such that a_j is a norm from YE_j of a product of commutators for each $j = 1, 2$. One can think of having *moved* the problem over to the fields E_js, which by construction are more "amenable" and where we can solve the problem. We then modify z by commutators so that the modified z (and hence also the original z) is a product of commutators (Proposition 3.6). The required E_js and a_js are constructed by HHK patching by prescribing compatible local data for an appropriate model $\mathcal{X}$ of X.

We now briefly mention what each section in the paper is about. The second section collects lemmata about the shape of units, norms of field extensions and reduced norms of algebras defined over some special complete fields encountered in the patching set-up. It also contains some class field theory lemmata which will be useful in approximating local data to get global objects. The third section sets forth patching notations, fixes a preliminary model $\mathcal{X}$ of X arranging some necessary divisors to be in good shape (i.e. normal crossing divisors with regular components) and gives the initial reductions which help simplify the problem. It also spells out the overall strategy adopted in the proof (mentioned above) in more precise detail.

The fourth and fifth sections classify into types, codimension one and closed points of $\mathcal{X}$ lying on the special fiber. Here, we also understand the configuration of the cyclic sub-extension Y/F and the shape of the norm one element $a \in Y$ at the fraction fields of the local rings at these points completed at their maximal ideals. The sixth section discusses further blowing up the model at closed points to eliminate certain types of closed points from the classification. It also constructs a partial dual graph and outputs a nine-colouring of it, which will help in ensuring compatibility of the local data at the *branches* in the patching problem. The seventh section gives patching data $(a_{1,P}, a_{2,P}, E_{1,P}, E_{2,P})$ at closed points P while the next two discuss their structure over the branches.

The tenth and eleven sections give patching data $(a_{1,\eta}, a_{2,\eta}, E_{1,\eta}, E_{2,\eta})$ at codimension one points η of $\mathcal{X}$ lying on the special fiber. We patch the data in the twelfth section by *spreading* $(a_{1,\eta}, a_{2,\eta}, E_{1,\eta}, E_{2,\eta})$ to work over open sets $U_\eta \ni \eta$ of the special fiber to get the required elements $a_1, a_2 \in Y$ and extensions $E_1, E_2/F$. The final section uses patching again to finally solve the problem over the E_js.

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2 Lemmata

2.1 Notations and terminology

Let ℓ be a prime and let J be a field which is not of characteristic ℓ containing ρ, a primitive ℓ^{th} root of unity. Then for $a, b \in J^*$, we let the symbol (a, b) denote the J-cyclic ℓ-algebra

$$(a, b) = J\langle i, j | i^\ell = a, j^\ell = b, ij = \rho ji\rangle.$$

If E/J is a cyclic extension of degree ℓ with $\mathrm{Gal}(E/J) = \langle\sigma\rangle$ and $b \in J^*$, we let the symbol (E, σ, b) (or (E, b) if the automorphism σ is clear from the context) denote the J-cyclic ℓ algebra

$$(E, \sigma, b) = \bigoplus_{i=0}^{\ell-1} u^i E,\ u^\ell = b,\ eu = u\sigma(e)\ \forall\, e \in E.$$

We also note that for central simple algebras (abbreviated as CSAs) D_1, D_2 over J, we use $D_1 = D_2$ to mean equality in $\mathrm{Br}(J)$, i.e $D_1 = D_2$ denotes that D_1/J and D_2/J are Brauer equivalent.
Let F be a complete discretely valued field with ring of integers R and residue field k. Let ℓ be a prime which is not equal to $\mathrm{char}(k)$ such that F contains a primitive ℓ^{th} root of unity. Let $\alpha \in \mathrm{Br}(F)$ be an element of order ℓ which is ramified at R. Recall the residue map $\partial_F : \mathrm{H}^2(F, \mu_\ell) \to \mathrm{H}^1(k, \mathbb{Z}/\ell Z)$. Let $\partial_F(\alpha) = \left(\overline{E}/k, \overline{\sigma}\right)$ where $\overline{E}/k$ is a cyclic extension of degree ℓ with Galois group generated by $\overline{\sigma}$.
Residual extension: There is a unique unramified cyclic extension E/F of degree ℓ with residue field $\overline{E}$. We call E the *lift of residue* of α at R or the *residual extension* of α at R.
Residual Brauer class: We define the residual class of α (depending on the choice of a parameter of R) as in ([S07]). Given a parameter π of R, let L denote the totally ramified extension $F\left(\sqrt[\ell]{\pi}\right)$ and S denote the ring of integers of L with residue field also k. Then $\alpha_L := \alpha \otimes_F L$ is unramified and hence is in $\mathrm{Br}(S)$.
Let $\beta \in \mathrm{Br}(k)$ denote the image of α_L. Then the *residual Brauer class* of α, denoted α_{rbc}, is defined to be the image of β in the unramified cohomology group $\mathrm{H}^2_{\text{nr}}(F, \mu_\ell)$ under the isomorphism $i_F : \mathrm{H}^2(k, \mu_\ell) \to \mathrm{H}^2_{\text{nr}}(F, \mu_\ell),\ \beta \rightsquigarrow \alpha_{\text{rbc}}$.

Lemma 2.1 ([S07], Proof of Proposition 0.6).

$$\alpha = \alpha_{\text{rbc}} + (E, \sigma, \pi) \text{ in } \mathrm{Br}(F).$$

2.2 Norms, reduced norms and index computations

Lemma 2.2 (cf. [PPS18], Lemma 2.7). *Let F be a field and ℓ, a prime not equal to the characteristic of F. Let Y/F be a cyclic extension of F or the split extension of degree ℓ and ψ, a generator of the Galois group of Y/F. Suppose that there exists an integer $m \geq 1$ such that F does not contain a primitive $\ell^{m\text{th}}$ root of unity. Let $\mu \in Y$ with $\mathrm{N}_{Y/F}(\mu) = 1$. Further assume that*

- *If Y/F is split, then $\mu = \left(g_i^{\ell}\right) \in \prod F$ for some $g_i \in F$.*
- *If Y/F is not split, then $\mu = g^{\ell^{2m}}$ for some $g \in Y$.*

Then there exists $h \in Y/F$ such that $\mu = h^{-\ell}\psi\left(h\right)^{\ell}$.

Lemma 2.3 (Totally ramified extensions (dim 1)). *Let R be a complete discretely valued ring with fraction field K and residue field k. Let ℓ be a prime which is not divisible by $\mathrm{char}(k)$ such that K contains a primitive $\ell^{2\text{th}}$ root of unity. Let L/K be a totally ramified extension of degree ℓ and let S be the integral closure of R in L. Then*

a. $L \simeq K\left(\sqrt[\ell]{\pi}\right)$ for some parameter π of K,

b. If $x \in R^*$ is a norm from L, then $x \in K^{*\ell}$,

c. Norm one elements in L are ℓ^{th} powers in S^*.

Proof. a. follows from ([PPS18], Lemma 2.4), while b. and c. are easy consequences of Hensel's lemma. □

Lemma 2.4. *Let A be a complete regular local ring of $\dim 2$ with fraction field F and finite residue field k. Let L/F be a cyclic extension of F of degree ℓ unramified on A, where ℓ is a prime not divisible by $\mathrm{char}(k)$. If $a \in A^*$, then it is a norm from L.*

Proof. Let σ be a generator of $\mathrm{Gal}(L/F)$. Since $a \in A^*$, the cyclic algebra (L, σ, a) is unramified and hence trivial in $\mathrm{Br}\,(F)$. □

Lemma 2.5 (Norm one elements of an unramified extension). *Let A be a complete regular local ring with fraction field F and finite residue field k. Let ℓ be a prime which is not divisible by $\mathrm{char}(k)$. Assume F contains a primitive ℓ^{th} root of unity. If Y is a degree ℓ field extension of F unramified on A, then norm one elements of Y/F which are integral over A are ℓ^{th} powers in Y.*

Proof. Let B denote the integral closure of A in Y and let k_1 be its residue field. Let $c \in Y$ be integral over A such that $\mathrm{N}_{Y/F}(c) = 1$. Hence $c \in B^*$, the minimal polynomial $g(t)$ of c in Y/F lies in $A[t]$ and is monic and irreducible. By the Henselian property of A, $\overline{g}(t)$ is irreducible (of the same degree) in $k[t]$ and is therefore the minimal polynomial of $\overline{c}$.
Since Y/F is unramified, $[Y : F] = [k_1 : k] = \ell$ and therefore $\mathrm{N}_{k_1/k}(\overline{c}) = (-1)^{\ell}\overline{g}(0)^{[k_1:k(\overline{c})]} = \overline{(-1)^{\ell}g(0)^{[Y:F(c)]}} = \overline{\mathrm{N}_{Y/F}(c)} = 1$. Now k_1/k is an extension of finite fields and hence the norm map $\mathrm{N} : k_1^* \to k^*$ is surjective. Since N is

also multiplicative, it induces a surjective map of groups $\tilde{\mathrm{N}} : \frac{k_1^*}{k_1^{*\ell}} \to \frac{k^*}{k^{*\ell}}$. Since ℓ is a prime not divisible by $\mathrm{char}(k)$ and F contains a primitive ℓ^{th} root of unity, ℓ divides $|k^*|$ and $|k_1^*|$. Thus both $\frac{k_1^*}{k_1^{*\ell}}$ and $\frac{k_1^*}{k_1^{*\ell}}$ are cyclic groups of order ℓ which shows that $\tilde{N}$ is injective as well.
Since $\mathrm{N}_{k_1/k}(\overline{c}) = 1$, this shows that $\overline{c} = \lambda^\ell$ for some $\lambda \in k_1$. Using the fact that B is Henselian as well, we see that c is also therefore an ℓ^{th} power in Y. $\square$

Lemma 2.6 (Norm one elements (dim 2)). *Let A be a complete regular local ring of $\dim 2$ with fraction field F and finite residue field k. Let ℓ be a prime not divisible by $\mathrm{char}(k)$. Assume F contains a primitive $\ell^{2\text{th}}$ root of unity. Let $Y = F\left(\sqrt[\ell]{u\pi^i\delta^j}\right)$ be a degree ℓ field extension of F where $u \in A^*$, (π, δ) form a system of parameters of A and $0 \le i, j \le \ell - 1$. Let $b \in Y$ be such that it is integral over A. If $\mathrm{N}_{Y/F}(b) = 1$, then $b \in Y^{*\ell}$.*

Proof. We split it into two cases depending on the ramification of Y/F. If Y/F is unramified and nonsplit, then by Lemma 2.5, b in an ℓ^{th} power.
If Y/F is ramified, then $Y = F\left(\sqrt[\ell]{u\pi^i\delta^j}\right)$ where $0 \le i, j \le \ell - 1$ with at least one of them non-zero. Let B denote the integral closure of A in Y. It is a complete local ring ([HS06], Theorem 4.3.4) with maximal ideal $\mathcal{M}_B$ and residue field k. Let $b \in B$ such that $\mathrm{N}_{Y/F}(b) = 1$. Let $a \in A^*$ be such that $\overline{a} = \overline{b}$. Thus $ba^{-1} \simeq 1 \mod \mathcal{M}_B$. Since B is complete and $\mathrm{char}(k) \neq \ell$, $b = a\lambda^{\ell^2}$ for some $\lambda \in B$. This implies that $\mathrm{N}_{Y/F}(b) = \left(a\,\mathrm{N}_{Y/F}(\lambda)^\ell\right)^\ell = 1$. Thus $a\,\mathrm{N}_{Y/F}(\lambda)^\ell = \rho$ where ρ is an ℓ^{th} root of unity. Hence a is equal to ρ up to ℓ^{th} powers in Y. Since F contains a primitive $\ell^{2\text{th}}$ root of unity, this shows a and hence b is an ℓ^{th} power in Y. $\square$

Lemma 2.7 (Reduced norms of an unramified algebra). *Let R be a complete discretely valued ring with fraction field K and residue field k of cohomological dimension ≤ 2. Let D_0 be an unramified central simple algebra over K of index ℓ where ℓ is a prime not divisible by $\mathrm{char}(k)$. Then every unit $u \in R^*$ is a reduced norm from D_0.*

Proof. By the results of Merkurjev and Suslin ([Se], Chapter II, Sec 4.5, Pg 88), the reduced norm of $\overline{D_0}$ is surjective. Thus the polynomial $\overline{\mathrm{Nrd}_{D_0}(\mathbf{x}) - u} = 0$ has a solution over k. By Hensel's Lemma, there exists a solution over K. $\square$

Lemma 2.8 (Splitting fields). *Let A be a complete regular local ring of $\dim 2$ with fraction field F and finite residue field k. Let ℓ be a prime not divisible by $\mathrm{char}(k)$ such that F contains a primitive ℓ^{th} root of unity. Let $D = (v, \pi)$ be an ℓ torsion algebra over F and $E = F\left(\sqrt[\ell]{u\pi^i\delta^j}\right)$ be a degree ℓ field extension of F where $u, v \in A^*$, (π, δ) form a system of parameters of A and $0 \le i, j \le \ell - 1$. Let $\widehat{A_{(\pi)}}$ be completion of $A_{(\pi)}$ at its maximal ideal and let its fraction field be denoted by F_B, which is a complete discretely valued field with parameter π and residue field k_B. If $D \otimes_F (E \otimes_F F_B)$ is split, then so is $D \otimes_F E$.*

Proof. If $i = j = 0$, then E is the unique unramified (on A) extension of F. Therefore $v \in E^{*\ell}$ and hence $D \otimes E = 0$.
If $i = 0, j \neq 0$, then $E \otimes F_B/F_B$ is an unramified extension, $\overline{E_B} := \overline{E \otimes F_B}$ is a totally ramified extension over k_B and the further residue field of $\overline{E_B}$ is k . Note that $D \otimes E \otimes F_B = 0$ implies that the residue $\overline{v} \in \overline{E_B}^{*\ell}$ and hence $\overline{\overline{v}} \in k^{*\ell}$. This implies $v \in A^{*\ell}$ and hence $D = 0$ to begin with.
If $i \neq 0$, without loss of generality we can assume $i = 1$ and $0 \leq j < \ell$. Thus $D \otimes E = \left(v, u^{-1}\delta^{-j}\right) = \left(\delta, v^j\right) \in \mathrm{Br}(E)$. Note that $E \otimes F_B/F_B$ is totally ramified and $\overline{E_B} = k_B$. Since $D \otimes E \otimes F_B = 0$, we see that $\left(\delta, v^j\right) = 0 \in \mathrm{Br}(E \otimes F_B)$ and hence $\left(\overline{\delta}, \overline{v}^j\right) \in \mathrm{Br}(k_B)$. This implies $\overline{\overline{v}}^j \in k^{*\ell}$ and hence $v \in A^{*\ell}$. Hence $D \otimes E = 0$. □

LEMMA 2.9 (Index formula, [JW90]). *Let R be a complete discretely valued ring with fraction field F. Let E be a cyclic unramified extension of F of degree m and let $\alpha = \alpha' + (E, \sigma, \pi)$ in $\mathrm{Br}(F)$ where π is a parameter of R, σ is a generator of E/F and α' is a central simple algebra of degree n unramified at R. Assume mn is invertible in R. Then* $\mathrm{index}\,(\alpha) = \mathrm{index}(\alpha' \otimes_F E)\,[E : F]$.

2.3 APPROXIMATING LOCAL DATA

For the rest of this section, ℓ will denote an odd prime, F, a global field with $\mathrm{char}(F) \neq \ell$ containing a primitive ℓ^{th} root of unity and D', a central simple algebra over F of index dividing ℓ. F_v will denote the completion of F at a place v of F and k_v, its residue field. $T = \{v_1, v_2, \ldots, v_r\}$ will be a finite set of places of F such that $\ell \neq \mathrm{char}(k_{v_i})$ for each $i \leq r$ and $D' \otimes F_v$ is split for every place $v \notin T$.

LEMMA 2.10 (An approximate cyclic extension). *Suppose that there exists $u' \in F^*$ and cyclic or split extensions E_{v_i}/F_{v_i} of degree ℓ for each $v_i \in T$ such that*

- *u' is a norm from E_{v_i}/F_{v_i} for each $v_i \in T$,*
- *$D' \otimes_F E_{v_i}$ is split for each $v_i \in T$.*

Then there exists a cyclic field extension E/F of degree ℓ such that

- *$E \otimes_F F_{v_i} \simeq E_i$ for each $v_i \in T$,*
- *u' is a norm from E/F,*
- *$D' \otimes_F E$ is split.*

Proof. Without loss of generality, assume that there exists a $v \in T$ such that E_v/F_v is a field extension. This can be done by expanding T to include a place v of F where $u' \in \mathcal{O}^*_{F_v}$ and choosing E_v to be the unique cyclic unramified field extension of degree ℓ over F_v.
Pick w_v to be so that the given $E_v \simeq \frac{F_v[t]}{(t^\ell - w_v)}$ for each $v \in T$. If $u' \in F^{*\ell}$, using weak approximation pick $w \in F$ so that up to ℓ^{th} powers, it matches $w_v \in F_v$ for

each $v \in T$. Then the field $E = F[t]/\left(t^\ell - w\right)$ satisfies the lemma. So we assume $u' \notin F^{*\ell}$ in the rest of the proof.
For each place $v \in T$, by hypothesis we know $(w_v, u') = 0 \in \mathrm{Br}(F_v)$. Hence pick $\theta_v \in \left(F\left(\sqrt[\ell]{u'}\right) \otimes F_v\right)^*$ so that $\mathrm{N}_{F(\sqrt[\ell]{u'})\otimes F_v/F_v}(\theta_v) = w_v$. By weak approximation, find $\theta \in F\left(\sqrt[\ell]{u'}\right)$ so that it matches θ_v up to ℓ^{th} powers. Set $w = \mathrm{N}_{F(\sqrt[\ell]{u'})/F}(\theta)$. Thus w matches with w_v up to ℓ^{th} powers and $(u', w) = 0 \in \mathrm{Br}(F)$. Set $E = F\left(\sqrt[\ell]{w}\right)$. This is a cyclic Galois extension of F which approximates the E_vs for each $v \in T$. By hypothesis, D' is split at places not in T and $E_v \otimes_F D'$ is split for every $v \in T$. Thus E splits D'. □

Lemma 2.11 (Another approximate cyclic extension). *Let $Y' = F\left(\sqrt[\ell]{u'}\right)$ be a cyclic field extension of degree ℓ where $u' \in F^* \setminus F^{*\ell}$. Let $a' \in Y'^* \setminus Y'^{*\ell}$ and L be the Galois closure of the compositum $Y'\left(\sqrt[\ell]{a'}\right)$ over F. Suppose that for each $v \in T$, there exist $w_v \in F_v^*$ and extensions $E_v := \frac{F_v[t]}{(t^\ell - w_v)}$ of F_v with the following properties:*

- *w_v is a norm from $L \otimes F_v/F_v$,*
- *$D' \otimes_F E_v$ is split,*
- *(w_v, a') is split over $Y' \otimes F_v$.*

Then there exists a cyclic field extension E/F of degree ℓ such that

- *$E \otimes_F F_v \simeq E_v$ for each $v \in T$,*
- *u' is a norm from E/F,*
- *$D' \otimes_F E$ is split,*
- *a' is a norm from $E \otimes_F Y'/Y'$.*

Proof. Without loss of generality, assume that there exists a $v \in T$ such that E_v/F_v is a field extension. This can be done by expanding T to include a place v of F with the following properties : 1) $u' \in \mathcal{O}^*_{F_v}$, 2) $a' \in \mathcal{O}^*_{Y'_x}$ for any place x of Y' lying over v, 3) $L \otimes F_v$ is a unramified extension of F_v (or a product of unramified extensions over F_v) and choosing $w_v \in O^*_{F_v} \setminus O^{*\ell}_{F_v}$ and E_v to be the unique cyclic unramified field extension of degree ℓ over F_v.
Let $z_v \in (L \otimes F_v)^*$ such that $\mathrm{N}_{L\otimes F_v/F_v}(z_v) = w_v$. By weak approximation, find $z \in L$ so that it matches up to ℓ^{th} powers with z_v for each $v \in T$. Set $\theta := \mathrm{N}_{L/Y'}(z)$ and set $w := \mathrm{N}_{Y'/F}(\theta) = \mathrm{N}_{L/F}(z)$. Thus w matches with the w_v up to ℓ^{th} powers. Clearly w is a norm from $Y' = F\left(\sqrt[\ell]{u'}\right)$ also and hence $(u', w) = 0 \in \mathrm{Br}(F)$. Set $E = F\left(\sqrt[\ell]{w}\right)$. Hence u' is a norm from E/F.
Note that E is an extension of F which approximates the given E_v for each $v \in T$. Since there exists some $v \in T$ such that E_v is a field, E/F is a nonsplit field extension,

which is clearly cyclic of degree ℓ. By hypothesis, D'/F is split at places not in T and $E_v \otimes_F D'$ is split for every $v \in T$. Thus E splits D'.
As $\theta = \mathrm{N}_{L/Y'}(z)$ and $Y' \subseteq Y'\left(\sqrt[\ell]{a'}\right) \subseteq L$, we have that $(a', \theta) = 0 \in \mathrm{Br}(Y')$. Given any $\psi \in \mathrm{Gal}(Y'/F)$, extend it to some $\tilde{\psi} \in \mathrm{Gal}(L/F)$. Then $\mathrm{N}_{L/Y'}\left(\tilde{\psi}(z)\right) = \psi(\theta)$. Hence $\psi(\theta)$ is a norm from L/Y' and so also from $Y'\left(\sqrt[\ell]{a'}\right)/Y'$. Therefore

$$(a', \psi(\theta)) = 0 \in \mathrm{Br}(Y') \; \forall \, \psi \in \mathrm{Gal}(Y'/F).$$

Finally, since $\mathrm{N}_{Y'/F}(\theta) = w$ and Y'/F is Galois, we have that $\prod_{\psi \in \mathrm{Gal}(Y/F)} \psi(\theta) = w$. Therefore

$$\prod_{\psi \in \mathrm{Gal}(Y', F)} (a', \psi(\theta)) = (a', w) = 0 \in \mathrm{Br}(Y').$$

□

Lemma 2.12 (Invariant algebras of global fields). *Let E/F be a cyclic extension of global fields of degree ℓ, where ℓ is a prime not divisible by any of the residual characteristics of F. Further assume that F contains a primitive ℓ^{th} root of unity. Let E_w denote the completion of E at any place w of E and $\mathcal{O}_{E_w}$, its valuation ring. Let $\mathrm{Gal}\,(E/F) = \langle \sigma \rangle$. Let $u \in F^*$, $b \in E^*$ be such that*

- *At every place w of E where E/F is ramified, u is an ℓ^{th} power in E_w^*,*
- *At every place w of E where E/F is unramified and inert, $u \in \mathcal{O}_{E_w}^*$ up to ℓ^{th} powers in E_w^*,*
- $(u, b) = (u, \sigma(b))$ in $\mathrm{H}^2(E, \mu_\ell)$.

Additionally, let T_0 be a finite set of places of F such that for each place $v \in T_0$, one is given $f_v \in F_v^*$ such that for any place w of E lying above v, $(u, b) = (u, f_v)$ in $\mathrm{H}^2(E_w, \mu_\ell)$.
Then there exists $f \in F^*$ such that

1. *$f = f_v \theta_v^\ell$ in F_v for some $\theta_v \in F_v$ for each $v \in T_0$,*
2. *$(u, b) = (u, f)$ in $\mathrm{H}^2(E, \mu_\ell)$.*

Proof. By Kummer theory, $E = F\left(\sqrt[\ell]{\psi}\right)$ for some ψ which generates $\frac{F^*}{F^{*\ell}}$. Note that if $u \in E^{*\ell}$, we can choose f by weak approximation such that f matches f_v up to ℓ^{th} powers. So for the remainder of the proof, we assume that $u \notin E^{*\ell}$. We also note that if $v \in T_0$ splits completely in E, then the hypothesis that $(u, b) = (u, \sigma(b))$ guarantees that the same f_v works for each place w above v.
Let T denote the union of T_0 and the finite set of places v of F which satisfy both the following conditions: 1) v is either unramified and inert or completely split in E, 2) There exists a place w of E lying above v at which either u or b is not a unit in $\mathcal{O}_{E_w}$.

For each place $v \in T$, we find $f_v \in F_v^*$ as follows:

Case 0 : For each $v \in T_0$, we choose the f_v given by the hypothesis.

Case I : Let $v \in T \setminus T_0$ be a place of F which is unramified and inert in E and let w be the place above v. Let $b = \tilde{b}_w \pi_v^s$ where $\tilde{b} \in \mathcal{O}_{E_w}^*$ and π_v is a parameter for F_v. Set $f_v = \pi_v^s$. Since by hypothesis, u is in $\mathcal{O}_{E_w}^*$ up to ℓ^{th} powers, we have that

$$(u, b) = \left(u, \tilde{b}_w\right) + (u, \pi_v^s) = (u, f_v) \in \mathrm{H}^2\left(E_w, \mu_\ell\right).$$

Case II : Let $v \in T \setminus T_0$ be a place of F which splits in E. Thus $E \otimes_F F_v = \prod_{i=1}^{\ell} F_v$ and let $b \in E = (b_1, b_2, \ldots, b_\ell) \in E \otimes_F F_v$. Thus $(u, b) = (u, \sigma(b))$ implies

$$(u, b_1) = (u, b_2) = \ldots = (u, b_\ell) \in \mathrm{H}^2\left(F_v, \mu_\ell\right) \qquad (*).$$

Set $f_v = b_1$. And thus (u, f_v) matches (u, b_i) over F_v for each i.
Since by hypothesis, $u \in E_w^{*\ell}$ for every w/v totally ramified, we have that for each place w lying over a place v not in T, (u, b) is split over E_w.
Note that since $u \notin E^{*\ell}$, it is not in $F^{*\ell}$ either and hence $F\left(\sqrt[\ell]{u}\right)$ is a cyclic Galois extension of degree ℓ over F. Then $L := F\left(\sqrt[\ell]{u}, \sqrt[\ell]{\psi}\right)$ is a Galois extension over F with Galois group $\frac{\mathbb{Z}}{\ell\mathbb{Z}} \times \frac{\mathbb{Z}}{\ell\mathbb{Z}}$.
By Chebotarev density, pick a place $\tilde{v}$ of F (there are infinitely many!) which is not in T such that 1) $\tilde{v}$ does not ramify in L, 2) $\sigma' \in \mathrm{Gal}(L/F) \leftrightarrow (1,1) \in \frac{\mathbb{Z}}{\ell\mathbb{Z}} \times \frac{\mathbb{Z}}{\ell\mathbb{Z}}$ is the *Frobenius automorphism* $\mathrm{Frob}_{\tilde{v}}$ of $L_x/F_{\tilde{v}}$ where x is any place lying above $\tilde{v}$. [Since L/F is abelian, the Frobenius automorphism does not depend on the choice of x]
Note that the residue field extension degree $[l_x : k_{\tilde{v}}] \leq \ell$. For if E_w is nonsplit unramified extension of $F_{\tilde{v}}$, then since $u \in \mathcal{O}_{F_{\tilde{v}}}^*$, we have $u \in E_w^{*\ell}$.
We have chosen σ' to be the non-trivial automorphism of L/F of order ℓ such that $\sigma'\left(\sqrt[\ell]{u}\right) = \rho \sqrt[\ell]{u}$ and $\sigma'\left(\sqrt[\ell]{\psi}\right) = \rho' \sqrt[\ell]{\psi}$, where ρ, ρ' are primitive ℓ^{th} roots of unity. This, by the choice of $\tilde{v}$ gives rise to the Frobenius automorphism of the residue field extensions $l_x/k_{\tilde{v}}$. Thus $l_x/k_{\tilde{v}}$ is a non-trivial extension, i.e., $\tilde{v}$ is not completely split in L. (The choice of Frobenius for a trivial extension is the identity map.)
Thus the residue field extension $l_x/k_{\tilde{v}}$ is degree ℓ with Galois group generated by $\overline{\sigma'}$. Note that since σ' fixes neither $\sqrt[\ell]{u}$ nor $\sqrt[\ell]{\psi}$, u and ψ are not ℓ^{th} powers in F_v. Thus E_w is unramified, nonsplit over $F_{\tilde{v}}$ and $u \notin F_{\tilde{v}}^{*\ell}$.

Finding *an* f'

Our first goal is to find an algebra $\alpha = (u, f') \in \mathrm{Br}(F)$ such that $\alpha \otimes_F E = (u, b) \in \mathrm{Br}(E)$. We find α by prescribing its shape α_v locally so that $\alpha_v \otimes_F E = (u, b) \in \mathrm{Br}\left(E_w\right)$ where w is any place lying over v.
For $v \in T$, choose $\alpha_v = (u, f_v)$. So $\alpha_v \otimes_F E = (u, b)$ in $\mathrm{Br}\left(E_w\right)$. For $v \notin T$ and $v \neq \tilde{v}$, choose $\alpha_v = 0 \in \mathrm{Br}\left(F_v\right)$. This matches with (u, b) over E_w since the latter is also split at these places. For $v = \tilde{v}$, let $\pi_{\tilde{v}}$ be a parameter of $F_{\tilde{v}}$. Choose $\alpha_{\tilde{v}} = (u, \pi_{\tilde{v}}^s)$ for an appropriate s so that $\sum_{v \in \Omega_F} \mathrm{inv}\left(\alpha_v\right) = 0 \in \frac{\mathbb{Q}}{\mathbb{Z}}$. This can be done since u is a unit at $\tilde{v}$ and $u \notin F_{\tilde{v}}^{*\ell}$.

Note that (u, b) is split over E_w for any $w|\tilde{v}$. Now since Cor is injective for local fields, $\alpha_{\tilde{v}} := (u, \pi_{\tilde{v}}^s)$ is split over E_w for $w|\tilde{v}$ because Cor : $\mathrm{H}^2(E_w, \mu_\ell) \to \mathrm{H}^2(F_{\tilde{v}}, \mu_\ell)$ sends $(u, \pi_{\tilde{v}}^s) \rightsquigarrow (u, \pi_{\tilde{v}}^{s\ell}) = 0$.

By the Albert-Hasse-Brauer-Noether theorem, there exists an $\alpha \in \mathrm{Br}(F)$ of order dividing ℓ such that $\alpha \otimes_F F_v = \alpha_v \in \mathrm{Br}(F_v)$. Also note that locally at each place, it is split by $F(\sqrt[\ell]{u})$, hence there exists an $f' \in F$ such that $\alpha = (u, f') \in \mathrm{Br}(F)$ since F contains a primitive ℓ^{th} root of unity.

Modifying f' so that it approximates f_v for each $v \in T$

By the choice of f', we have that $(u, f') = (u, f_v) \in \mathrm{Br}(F_v)$ for each $v \in T$. Hence for each $v \in T$, there exists $w_v \in F(\sqrt[\ell]{u}) \otimes F_v$ such that $\mathrm{N}_{F_v(\sqrt[\ell]{u})\otimes F_v/F_v}(w_v) = f'^{-1} f_v$.

By weak approximation, there exists a $w \in F(\sqrt[\ell]{u})$ such that for each $v \in T$, $w = w_v \gamma_v^\ell$ for $\gamma_v \in F(\sqrt[\ell]{u}) \otimes F_v$. This implies that $\mathrm{N}_{F(\sqrt[\ell]{u})/F}(w) \in f'^{-1} f_v F_v^{*\ell} \;\forall\, v \in T$.

Finally, set $f = f' \,\mathrm{N}(w)$. Therefore $(u, f') = (u, f) \in \mathrm{Br}(F)$ and $f \in f_v F_v^{*\ell} \;\forall\, v \in T$. □

3 Reductions and strategies

3.1 The set-up

Let K be a p-adic field with ring of integers $\mathcal{O}_K$ and residue field k. Let $F = K(X)$ be the function field of a smooth projective geometrically integral curve X over K. Let D denote a central division algebra over F of exponent ℓ where ℓ is a prime different from p. We want to prove triviality of $\mathrm{SK}_1(D)$. Since it is known that the index of D divides ℓ^2 ([S97], [S98]) and that $\mathrm{SK}_1(D)$ is trivial for square-free index algebras ([W50]), we assume from now on that the index of D is ℓ^2.

Note that in the case when $\ell = 2$, the works of Merkurjev and Rost ([M93], [M06], Rost, Chapter 17 [KMRT]; [M99]) lead to the more general result that $\mathrm{SK}_1(D) = \{0\}$ over cohomological dimension 3 fields. Thus, in this paper, we assume that $\ell \neq 2$. *We also make an additional assumption that F contains a primitive $\ell^{2^{\text{th}}}$ root of unity.*

Let $z \in \mathrm{SL}_1(D)$ and let M be a maximal subfield of D containing z. Thus $\mathrm{N}_{M/F}(z) = \mathrm{Nrd}_D(z) = 1$. We would like to show $z \in [D^*, D^*]$. Using ([P76], Lemma 2.2, Section 2.4) and ([A61], Chapter IV, Theorem 31), by a coprime to ℓ base change, we assume that M contains a cyclic degree ℓ sub-extension Y/F with $\mathrm{Gal}(Y/F) = \langle\psi\rangle$. Since F contains a primitive ℓ^{th} root of unity, by Kummer theory, we have $Y = F(\sqrt[\ell]{y})$ for some $y \in F^*$. Since $\mathrm{N}_{M/F}(z) = 1$, the element $a := \mathrm{N}_{M/Y}(z)$ is a norm one element of Y/F and by Hilbert 90, $a = b^{-1}\psi(b)$ for some $b \in Y$. We fix a choice of such a b.

3.1.1 A preliminary model

By resolution of singularities ([Lip75]), there exists a regular integral scheme[1] $\mathcal{X}$ with function field F equipped with a proper, flat and projective morphism $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$. Let X_0 denote its reduced special fiber. For each $x \in \mathcal{X}$, let the regular local ring at x on $\mathcal{X}$ be denoted by $A_x := \mathcal{O}_{\mathcal{X},x}$. Let the completion of A_x at its maximal ideal be denoted by $\widehat{A_x}$, the fraction field of $\widehat{A_x}$ by F_x and the residue field of $\widehat{A_x}$ by k_x. We also let D_x (resp. Y_x) denote $D \otimes_F F_x$ (resp. $Y \otimes_F F_x$). If $\eta \in X_0$ is a codimension one point of $\mathcal{X}$ and $P \in X_0$ is a closed point of $\mathcal{X}$ with P lying in the Zariski closure of η in $\mathcal{X}$, we let $F_{P,\eta}$ denote the *branch* field. More explicitly, if $(\pi_\eta) \in \widehat{A_P}$ denotes a prime defining η, then localization at this prime ideal yields a discrete valuation ring $\widehat{A_P}_{(\pi_\eta)}$. Completing this discrete valuation ring at its maximal ideal and further taking its field of fractions yields the branch field $F_{P,\eta}$. Thus F_P and F_η are both subfields of $F_{P,\eta}$. Let $k_{P,\eta}$ denote the residue field of $F_{P,\eta}$.

Since $\mathcal{X}$ is normal, for each codimension one point $x \in \mathcal{X}^{(1)}$, we choose an extension to a[2] discrete valuation $v_{(x)}$ on Y. Define $\operatorname{support}_{\mathcal{X}}(b) := \{x \in \mathcal{X}^{(1)} \mid \max_i \left(|v_{(x)}(\psi^i(b))|\right) > 0\}$ for the b which was fixed in Section 3.1 and let $\mathcal{J}_{\mathcal{X}} := \sum_{x \in \operatorname{support}_{\mathcal{X}}(b)} x$. Further set $\mathcal{H}_{\mathcal{X}}$ to be the divisor corresponding to the union of the reduced special fiber X_0, $\operatorname{div}_{\mathcal{X}}(y)$, $\mathcal{J}_{\mathcal{X}}$, the ramification locus of M and the ramification divisor of D in $\mathcal{X}$.

Notation: We say a divisor is in *good shape* if it is a union of regular curves in normal crossing.

Proposition 3.1. *There exists a regular proper model $\mathcal{X}$ of X over $\mathcal{O}_K$ such that $\mathcal{H}_{\mathcal{X}}$ is in good shape, i.e. is a union of regular curves in normal crossing in $\mathcal{X}$. Further, let $h : \mathcal{Y} \to \mathcal{X}$ denote the normal closure of the model $\mathcal{X}$ in Y. Let $x \in \mathcal{X}$ of codimension $1 \leq i \leq 2$ and let B_x denote the integral closure of $A_x = \mathcal{O}_{\mathcal{X},x}$ in Y. Then the following hold:*

a. *If Y_x is a field, then $h^{-1}(x) = \{y\}$ where $y \in \mathcal{Y}$ of codimension i and B_x is a local ring and isomorphic to $\mathcal{O}_{\mathcal{Y},y}$.*

b. *If $Y_x \simeq \prod F_x$, then $h^{-1}(x) = \{y_1, y_2, \ldots, y_\ell\}$, a set of ℓ points in $\mathcal{Y}$ of codimension i and B_x is semi-local with ℓ maximal ideals m_{y_i} for $1 \leq i \leq \ell$. Further, $(B_x)_{m_{y_i}} \simeq \mathcal{O}_{\mathcal{Y},y_i}$.*

Proof. Fix a preliminary regular proper model $\mathcal{X}'$ of X over $\mathcal{O}_K$. Construct $\mathcal{X}$ by blowing up $\mathcal{X}'$ at closed points of $\mathcal{X}'$ repeatedly ($p : \mathcal{X} \to \mathcal{X}'$) such that $\mathcal{H}'' := p^{-1}(\mathcal{H}_{\mathcal{X}'})$ is a union of regular curves in normal crossing. To prove that $\mathcal{H}_{\mathcal{X}}$ is in good shape, it suffices to show that $\mathcal{H}_{\mathcal{X}} \subseteq \mathcal{H}''$. By construction, the union of X_0, $\operatorname{div}_{\mathcal{X}}(y)$, $\operatorname{ram}_{\mathcal{X}}(M)$ and $\operatorname{ram}_{\mathcal{X}}([D])$ lies in $\mathcal{H}''$.

[1] We would like to note in advance that we will finally work over a new model obtained from $\mathcal{X}$ by repeatedly blowing up closed points.

[2] In case the prime corresponding to x splits in Y, then x defines ℓ valuations $v_{x_1}, v_{x_2}, \ldots, v_{x_\ell}$ on Y. However $v_{x_i}(b) = v_{x_1}(\psi^{-i+1}(b))$. Set $v_{(x)} := v_{x_1}$.

Now let $\beta \in \mathcal{J}_{\mathcal{X}}$. If β is the generic point of the strict transform of a curve in $\mathcal{X}'$, then $p(\beta) \in \mathcal{J}_{\mathcal{X}'}$ and hence $\beta \in \mathcal{H}''$. On the other hand, if β lies on an exceptional curve of $p : \mathcal{X} \to \mathcal{X}'$, then clearly $\beta \in \mathcal{H}''$. Hence $\mathcal{H}_{\mathcal{X}}$ is in good shape.
We give the proof for the case when $x = P$, a closed point in $\mathcal{X}$. The proof for the case when x has codimension one is similar. Let $U \subset \mathcal{X}$ be an open affine neighbourhood containing P with coordinate ring A. Thus $h^{-1}(U)$ is affine with coordinate ring, say B, which is the integral closure of A in Y. Thus it follows that the integral closure of the local ring A_P in Y is B localized at the multiplicatively closed set $A \setminus P$ which we denoted by B_P. Since B_P is integral over A_P, the maximal ideals of B_P contract to the unique maximal ideal of A_P and hence correspond to the points in $h^{-1}(P)$. Since $\mathrm{Gal}(Y/F) \simeq \mathbb{Z}/\ell\mathbb{Z}$ acts transitively on $h^{-1}(P)$, it is clear that $h^{-1}(P)$ is either a singleton or a set of size ℓ.
Now it only remains to compare the shape of $Y_P := Y \otimes_F F_P$ and the size of $h^{-1}(P)$. By (Lemma 07N9, stacks-project), $B_P \otimes_{A_P} \widehat{A_P} \simeq_{Q_i \in h^{-1}(P)} \prod \widehat{\mathcal{O}_{\mathcal{Y},Q_i}}$ which is a (local) domain iff $|h^{-1}(P)| = 1$.
We have the following injective[3] A_P-morphism: $B_P \otimes_{A_P} \widehat{A_P} \hookrightarrow Y \otimes_{A_P} \widehat{A_P} \hookrightarrow Y \otimes_{A_P} F_P \simeq Y \otimes_F F_P := Y_P$. Thus if Y_P is a field, $B_P \otimes_{A_P} \widehat{A_P}$ has to be a domain and hence $|h^{-1}(P)| = 1$. Conversely, if $h^{-1}(P) = 1$, then $B_P \otimes_{A_P} \widehat{A_P}$ has to be a local domain. The above injection shows that $Y_P \simeq Y \otimes_{A_P} F_P$ lies in the fraction field of $B_P \otimes_{A_P} \widehat{A_P}$. Hence Y_P is a domain and hence a field. □

We continue to work this model $\mathcal{X}$ till the end of Section 5.

LEMMA 3.2. *Let P be a closed point in $\mathcal{X}$ lying on the Zariski closure of a codimension one point $\eta \in \mathcal{X}$. If $Y_\eta \simeq \prod F_\eta$, then $Y_P \simeq \prod F_P$.*

Proof. Let (π_P, δ_P) be a system of parameters of A_P such that π_P cuts out the curve $\overline{\eta}$ at P. Recall that $Y = F(\sqrt[\ell]{y})$ and that $\mathrm{div}(y)$ is arranged to be in good shape in $\mathcal{X}$. Since Y_η is split, so is $Y \otimes F_{P,\eta}$. Thus we can assume that up to ℓ^{th} powers, $y = v_P \delta_P^j$ for some unit $v_P \in \widehat{A_P}^*$ and $0 \leq j < \ell$ with $\overline{y} \in k_{P,\eta}^{*\ell}$. Recall that $k_{P,\eta}$ is a complete discretely valued field with $\overline{\delta_P}$ as a parameter. Thus $j = 0$ and since $v_P \in \widehat{A_P}^*$, $\overline{v_P} \in k_P^{*\ell}$. Hence $v_P \in \widehat{A_P}^{*\ell}$. This immediately implies that Y_P is split. □

3.1.2 FIXING PARAMETERS

Let $S_0 = \{P_1, P_2, \ldots, P_m\}$ denote the finite set of closed points of intersection of distinct irreducible curves in $\mathcal{H}_{\mathcal{X}}$. Expand S_0 if necessary so that it includes at least one closed point from each irreducible curve in $\mathcal{H}_{\mathcal{X}}$. We call the elements in the set S_0 to be *intersection* points.
Let N_0' denote the set of all codimension one points of $\mathcal{X}$ which lie in $\mathcal{H}_{\mathcal{X}}$ and let N_0 denote the subset $N_0' \cap X_0$. Using ([S98], Lemma), for each $\eta \in N_0'$, choose a

[3] As $\widehat{A_P}/A_P$ and Y/A_P are flat and $M \otimes_R N \simeq M \otimes_S N$ for S-modules M, N where S is a localisation of R.

function $\pi_\eta \in F$ such that $\operatorname{div}_{\mathcal{X}}(\pi_\eta) = \overline{\eta} + E_\eta$ where E_η avoids $N_0' \cup S_0$. Thus π_η is a parameter of F_η for each such η.
Further if $P \in S_0$ lies on two distinct irreducible curves C and C' in $\mathcal{H}_{\mathcal{X}}$ with generic points η and η' respectively. Then $(\pi_\eta, \pi_{\eta'})$ form a system of parameters of A_P. If $P \in S_0$ lies on exactly on one irreducible curve C of $\mathcal{H}_{\mathcal{X}}$ with generic point η, then π_η can be extended to a system of parameters $(\pi_\eta, \pi_{\eta'})$ of A_P for some prime $\pi_{\eta'}$ defining a curve C' with generic point η' cutting C transversally.
We choose this system of parameters for each $P \in S_0$. Let $\pi_P := \pi_\eta$ and $\delta_P := \pi_{\eta'}$. Since $\mathcal{H}_{\mathcal{X}}$ is in good shape and at P, the division algebra is ramified at most along C and C', using ([S97], Proposition 1.2) we see that there exist $\alpha' \in \operatorname{Br}(A_P)$, $u_P, v_P \in A_P^*$ and integer $0 < m < \ell$ such that $[D] \in \operatorname{Br}(F)$ is either equal to α' or $\alpha' + (u_P, \pi_P)$ or $\alpha' + (v_P, \delta_P)$ or $\alpha' + (u_P, \pi_P) + (v_P, \delta_P)$ or $\alpha' + (u_P \pi_P^m, v_P \delta_P)$.

3.2 The shape of a

The following propositions specify the shape of $a = \mathrm{N}_{M/Y}(z)$ (which is an element of norm one in Y/F) over the model $\mathcal{X}$.

Proposition 3.3. *Let $x \in X_0$ be such that Y_x is a field extension of F_x. Let $\widehat{B_x}$ be the integral closure of $\widehat{A_x}$ in Y_x. Then $a \in \widehat{B_x}^*$.*

Proof. Let us first look at the case when $x \in X_0$ is a codimension one point of $\mathcal{X}$. Thus F_x is a complete discretely valued field and therefore so is Y_x. Let π_{Y_x} be a parameter of Y_x and π_{F_x} be a parameter of F_x. Thus $a = u_x \pi_{Y_x}^j$ for some $u_x \in \widehat{B_x}^*$ and $j \in \mathbb{Z}$. Let e be the ramification degree of Y_x/F_x. Then there exists $v_x \in \widehat{A_x}^*$ such that $1 = \mathrm{N}_{Y_x/F_x}(a) = \mathrm{N}_{Y_x/F_x}\left(u_x \pi_{Y_x}^j\right) = v_x \pi_{F_x}^{\frac{j\ell}{e}}$. This implies $\frac{j\ell}{e} = 0$ which shows that $j = 0$ and that $a \in \widehat{B_x}^*$.
Now let $x = P \in X_0$ be a closed point of $\mathcal{X}$ and let B_P denote the integral closure of A_P in Y. By Proposition 3.1, B_P is local and isomorphic to $\mathcal{O}_{\mathcal{Y},Q}$ where $h : \mathcal{Y} \to \mathcal{X}$ denotes the normal closure of $\mathcal{X}$ in Y and $h^{-1}(P) = \{Q\}$.
If $P \notin \mathcal{H}_{\mathcal{X}}$, then $a = b^{-1}\psi(b) \in {(B_P)_I}^*$ for any height one prime ideal I of B_P. Since B_P is normal, we have $\cap_I (B_P)_I = B_P$. Therefore $a \in B_P$ and further since a is not contained in any height one prime ideal, $a \in B_P^*$.
Let $P \in \mathcal{H}_{\mathcal{X}}$ and (π_P, δ_P) be a system of parameters of A_P such that they cut out the irreducible curves in $\mathcal{H}_{\mathcal{X}}$ on which P lies. Thus $\operatorname{div}_{\operatorname{Spec} B_P}(a)$ is supported at most along primes of B_P lying over (π_P) and (δ_P). By Lemma 3.2 and Proposition 3.1, there exists exactly one prime lying over (π_P) and one over (δ_P). Since B_P is normal and $\mathrm{N}_{Y/F}(a) = 1$, we see that $a \in B_P^*$.
The canonical A_P-morphism $i : B_P \to Y \otimes_F F_P = Y_P$ sending $b' \rightsquigarrow b' \otimes 1$ is an injection. Since B_P is integral over A_P, we see that $i(B_P)$ is integral over $\widehat{A_P}$ and hence $i(B_P) \subseteq \widehat{B_P}$. Hence $a \in B_P^*$ implies $a \in \widehat{B_P}^*$ also. ☐

Proposition 3.4. *Let $P \in S_0$ such that $Y_P \simeq \prod_{i=1}^{\ell} F_P$. Let (π_P, δ_P) be the system of regular parameters at A_P fixed as in Section 3.1.2 and let $a = (a_i')_i$ where*

$a'_i \in F_P$. Then there exist $z_{i,P} \in \widehat{A_P}^*$ and $m_i, n_i \in \mathbb{Z}$ such that $a'_i = z_{i,P}\pi_P^{m_i}\delta_P^{n_i}$. Further $\sum m_i = \sum n_i = 0$ and $\prod z_{i,P} = 1$.

Proof. By Proposition 3.1, if $h : \mathcal{Y} \to \mathcal{X}$ denotes the normal closure of $\mathcal{X}$ in Y, then $h^{-1}(P) = \{Q_1, \dots, Q_\ell\}$. Further if B_P denotes the integral closure of A_P in Y, B_P is semi-local with maximal ideals $\{m_{Q_1}, \dots, m_{Q_\ell}\}$ with $(B_P)_{m_{Q_i}} \simeq \mathcal{O}_{\mathcal{Y},Q_i}$.
Let (π_P, δ_P) be a system of parameters of A_P such that they cut out the irreducible curves in $\mathcal{H}_\mathcal{X}$ on which P lies. As in the proof of Proposition 3.3, $\mathrm{div}_{\mathrm{Spec}\, B_P}(a)$ is supported at most along primes lying above (π_P) and (δ_P).
Since $Y = F(\sqrt[\ell]{y})$ where $\mathrm{div}(y)$ is arranged to be in good shape in $\mathcal{X}$ and Y_P is split, $\mathcal{O}_{\mathcal{Y},Q_i}$ is a regular local ring. Further $\widehat{\mathcal{O}_{\mathcal{Y},Q_i}} \simeq \widehat{A_P}$. Let $(\pi'_{Q_i}, \delta'_{Q_i})$ be a system of regular parameters where π'_{Q_i} (resp. δ'_{Q_i}) lies over π_P (resp. δ_P). Using the identification[4] $Y \subseteq F_P$ (via Q_1 say), we identify $Y \otimes F_P$ with $\prod F_P$.
Note that $\pi'_{Q_i} \in Y \otimes F_P$ gets identified with $\left(\pi'_{Q_i}, \pi'_{Q_{i+1}}, \dots, \pi'_{Q_{i-1}}\right) \in \prod F_P$ where each π'_{Q_j} is supported at most along (π_P) in $\widehat{A_P}$. Similarly δ'_{Q_i} gets identified with $\left(\delta'_{Q_i}, \delta'_{Q_{i+1}}, \dots, \delta'_{Q_{i-1}}\right)$ with each δ'_{Q_j} being supported at most along (δ_P) in $\widehat{A_P}$. Since a has norm 1, the proposition about the shape of a follows. □

Proposition 3.5. *Let $P \in X_0 \setminus S_0$ be a closed point of $\mathcal{X}$ such that it lies on exactly one irreducible curve (say C) of $\mathcal{H}_\mathcal{X}$. Further assume $Y_P \simeq \prod_{i=1}^{\ell} F_P$. Let (π_P, δ_P) be a system of regular parameters at A_P such that π_P defines C at P. Let $a = (a'_i)_i$ where $a'_i \in F_P$. Then there exist $z_{i,P} \in \widehat{A_P}^*$ and $m_i \in \mathbb{Z}$ such that $a'_i = z_{i,P}\pi_P^{m_i}$. Further $\sum m_i = 0$ and $\prod z_{i,P} = 1$.*

Proof. The proof is similar to that of the previous proposition except that a is now supported at most at primes lying above π_P. □

3.3 Strategy

Recall that we have $z \in M \cap \mathrm{SL}_1(D)$ with $\mathrm{N}_{M/Y}(z) = a$ and $\mathrm{N}_{Y/F}(a) = 1$ where M is a maximal subfield containing a cyclic subfield Y of degree ℓ. The goal is to show $z \in [D^*, D^*]$. We would like to split a into a product of *suitable* elements a_1 and a_2 lying in *nicer* subfields E_1 and E_2 respectively. More precisely, we would like to find elements $a_1, a_2 \in Y$ and field extensions E_1, E_2 such that for each $i = 1, 2$, the following hold:

1. $a_1 a_2 = a$.
2. E_i/F is a subfield of D of degree ℓ.
3. $E_i \subseteq C_D(Y)$ and $D \otimes E_i \otimes Y$ is split.
4. There exists $\theta_i \in YE_i \subseteq D$ such that $\mathrm{N}_{YE_i/Y}(\theta_i) = a_i$.

[4]Note that if $Y \subseteq F_P$ via a different Q_i, then the new identification of $Y \otimes F_P \simeq_{Q_i} \prod F_P$ differs from the old one $Y \otimes F_P \simeq_{Q_1} \prod F_P$ by an automorphism $\prod F_P \simeq \prod F_P$ permuting the components.

5. $\theta_i \in [D^*, D^*]$.

Note that properties (3), (4) and (5) force that $a_i \in \mathrm{Nrd}_Y(C_D(Y))$ and that $\mathrm{N}_{Y/F}(a_i) = 1$. The construction of such subfields E_i/F is useful in modifying z by commutators so that it is a product of commutators, as shown by the proposition below.

Proposition 3.6. *Let D, M, Y, z, a be as before. If there exist elements $a_1, a_2 \in Y$ and subfields E_1/F and E_2/F with properties (1) - (5) above, then z is a product of commutators.*

Proof. Let $D' := C_D(Y)$ which is a central divison algebra of index ℓ over Y. Since E_i commutes with Y in D , $\theta_i \in D'$. Since $z \in D'$, we have that $z\theta_2^{-1}\theta_1^{-1} \in D'$. Note that YE_i and M are maximal subfields of D'/Y. Thus

$$\mathrm{Nrd}_{D'}\left(z\theta_2^{-1}\theta_1^{-1}\right) = \mathrm{N}_{M/Y}(z)\,\mathrm{N}_{YE_2/Y}\left(\theta_2^{-1}\right)\mathrm{N}_{YE_1/Y}\left(\theta_1^{-1}\right) = aa_2^{-1}a_1^{-1} = 1.$$

Since D' is a central division algebra with square-free index, every reduced norm one element is a product of commutators ([W50]). Thus $z\theta_2^{-1}\theta_1^{-1} \subseteq [D'^*, D'^*] \subseteq [D^*, D^*]$. Since each $\theta_i \in [D^*, D^*]$ by hypothesis, $z \in [D^*, D^*]$ also. □

The rest of the paper is devoted to constructing E_i and a_i satisfying properties (1)-(5) listed above. This is done by applying the techniques of patching developed by Harbater-Hartmann-Krashen.

4 At codimension one points

Recall that N_0' denotes the set of all codimension one points of $\mathcal{X}$ which lie in $\mathcal{H}_{\mathcal{X}}$. For each $\eta \in N_0'$, let π_η be the parameter of F_η fixed as in Section 3.1.2.

Classification of points of N_0'

We say that $\eta \in N_0'$ is of

- Type 0 if the index of D_η is 1. Thus $\eta \notin \mathrm{ram}_{\mathcal{X}}(D)$.
- Type 1 if the index of D_η is ℓ. We further classify these points into subtypes.
  - Type 1a: if $\eta \notin \mathrm{ram}_{\mathcal{X}}(D)$. Thus D_η/F_η is an unramified index ℓ CSA.
  - Type 1b: if $\eta \in \mathrm{ram}_{\mathcal{X}}(D)$. Thus $D_\eta = D_0 + (u_\eta, \pi_\eta)$ where D_0/F_η is an unramified CSA and u_η is a unit in $\widehat{A_\eta}$. By Lemma 2.9, D_0 is split by the degree ℓ extension $F_\eta(\sqrt[\ell]{u_\eta})$ and hence $D_\eta = \mathrm{M}_\ell(u_\eta, v_\eta\pi_\eta)$ where u_η, v_η are units in $\widehat{A_\eta}$
- Type 2 if the index of D_η is ℓ^2. Thus $\eta \in \mathrm{ram}_{\mathcal{X}}(D)$ and $D_\eta = D_0 + (u_\eta, \pi_\eta)$ where $u_\eta \in \widehat{A_\eta}^*$ is not an ℓ^{th} power and D_0/F_η is an unramified CSA such that $D_0 \otimes F_\eta(\sqrt[\ell]{u_\eta})$ has index ℓ (Lemma 2.9).

Shapes of Y and a

For $\eta \in N_0'$, let $\widehat{B_\eta}$ denote the integral closure of $\widehat{A_\eta}$ in Y_η whenever the latter is a field extension of F_η. If $Y_\eta \simeq \prod F_\eta$, we let $a = \left(a'_{i,\eta}\right)_i$ where $a'_{i,\eta} \in F_\eta$. Since $\mathrm{N}_{Y/F}(a) = 1$, we have $\prod a'_{i,\eta} = 1 \in F_\eta$. We now classify Y_η into four types as follows:

- RAM : Y_η is of Type RAM if Y_η/F_η is a ramified extension.
- RES : Let η be of Type 1b or 2 (i.e $\eta \in \mathrm{ram}_{\mathcal{X}}(D)$). Then Y_η is of Type RES if it is the lift of residues as defined in Section 2.1. In particular, it is an unramified nonsplit extension of F_η.
- SPLIT : Y_η is of Type SPLIT if $Y_\eta \simeq \prod_{i=1}^{\ell} F_\eta$.
- NONRES : Y_η is of Type NONRES if it is none of the above types. That is, it is an unramified nonsplit extension of F_η and if $\eta \in \mathrm{ram}_{\mathcal{X}}(D)$, it is NOT the lift of residues.

Remark 4.1. *Thus if η is of Type 2, then Y_η cannot be of Type SPLIT.*

Lemma 4.2 (Along η of Type 1a). *Let $\eta \in N_0'$ be of Type 1a. Further assume that $Y_\eta \simeq \prod F_\eta$. Let $a = \left(a'_{i,\eta}\right) \in \prod F_\eta$ where each $a'_{i,\eta} \in F_\eta$. Then $a'_{i,\eta} = z_{i,\eta}\pi_\eta^{\ell m_i'} \in F_\eta$ where $z_{i,\eta} \in \widehat{A_\eta}^*$ and $m_i' \in \mathbb{Z}$.*

Proof. Let $a'_{i,\eta} = z_{i,\eta}\pi_\eta^{m_i}$ for $z_{i,\eta} \in \widehat{A_\eta}^*$ and $m_i \in \mathbb{Z}$. Since a is a reduced norm from $D \otimes Y$, we have $\left(z_{i,\eta}\pi_\eta^{m_i}\right)[D_\eta] = 0 \in \mathrm{H}^3\left(F_\eta, \mu_\ell\right)$ for each i. Since D_η is unramified and has index ℓ, by Lemma 2.7 $\left(z_{i,\eta}\right)[D_\eta] = 0$. Thus, taking residues along π_η shows that $m_i \cong 0 \mod \ell$. □

For ease of reference, we summarize possible shapes of Y and a at points of N_0' in the following table (cf. Lemmata 2.3, 4.2, Proposition 3.3) where we use the notations that $w_\eta', z_\eta \in \widehat{B_\eta}^*$, $u_\eta, z_{i,\eta} \in \widehat{A_\eta}^*$ and $u_\eta \notin \widehat{A_\eta}^{\ell}$, $m_i, m_i' \in \mathbb{Z}$ and D_0/F_η is an unramified CSA. Further $\mathrm{N}_{Y_\eta/F_\eta}\left(z_\eta\right) = 1$, $\mathrm{N}_{Y_\eta/F_\eta}\left(w_\eta'\right)^\ell = 1$, $\sum_{i=1}^{\ell} m_i = \sum_{i=1}^{\ell} m_i' = 0$ and $\prod_{i=1}^{\ell} z_{i,\eta} = 1$.

η	D_η	More information	Y_η	$a \in Y_\eta$
0	D_0	$\mathrm{index}(D_0) = 1$	RAM	${w'}_\eta^\ell$
0	D_0	$\mathrm{index}(D_0) = 1$	SPLIT	$\left(a'_{i,\eta} = z_{i,\eta}\pi_\eta^{m_i}\right)_i$
0	D_0	$\mathrm{index}(D_0) = 1$	NONRES	z_η
$1a$	D_0	$\mathrm{index}(D_0) = \ell$	RAM	${w'}_\eta^\ell$
$1a$	D_0	$\mathrm{index}(D_0) = \ell$	SPLIT	$\left(a'_{i,\eta} = z_{i,\eta}\pi_\eta^{\ell m'_i}\right)_i$
$1a$	D_0	$\mathrm{index}(D_0) = \ell$	NONRES	z_η
$1b$	$D_0 + (u_\eta, \pi_\eta)$	$\mathrm{index}(D_0 \otimes F_\eta(\sqrt[\ell]{u_\eta})) = 1$	RAM	${w'}_\eta^\ell$
$1b$	$D_0 + (u_\eta, \pi_\eta)$	$\mathrm{index}(D_0 \otimes F_\eta(\sqrt[\ell]{u_\eta})) = 1$	RES	z_η
$1b$	$D_0 + (u_\eta, \pi_\eta)$	$\mathrm{index}(D_0 \otimes F_\eta(\sqrt[\ell]{u_\eta})) = 1$	SPLIT	$\left(a'_{i,\eta} = z_{i,\eta}\pi_\eta^{m_i}\right)_i$
$1b$	$D_0 + (u_\eta, \pi_\eta)$	$\mathrm{index}(D_0 \otimes F_\eta(\sqrt[\ell]{u_\eta})) = 1$	NONRES	z_η
2	$D_0 + (u_\eta, \pi_\eta)$	$\mathrm{index}(D_0 \otimes F_\eta(\sqrt[\ell]{u_\eta})) = \ell$	RAM	${w'}_\eta^\ell$
2	$D_0 + (u_\eta, \pi_\eta)$	$\mathrm{index}(D_0 \otimes F_\eta(\sqrt[\ell]{u_\eta})) = \ell$	RES	z_η
2	$D_0 + (u_\eta, \pi_\eta)$	$\mathrm{index}(D_0 \otimes F_\eta(\sqrt[\ell]{u_\eta})) = \ell$	NONRES	z_η

Table 1: Shape of D, Y and a at $\eta \in N'_0$

Fixing residual Brauer classes for points in N'_0 along which D is ramified

For each $\eta \in N'_0$ of Type 1b or 2, we define $\beta_{rbc,\eta} \in \mathrm{Br}(F_\eta)$ as follows:
If Y_η is RAM, (so $Y_\eta = F_\eta\left(\sqrt[\ell]{w_\eta \pi_\eta}\right)$ for some $w_\eta \in \widehat{A_\eta}^*$), then there exists an unramified algebra $D_{0\eta}$ such that $D_\eta = D_{0\eta} + (u_\eta, w_\eta\pi_\eta) \in \mathrm{Br}\,(F_\eta)$. Set $\beta_{rbc,\eta} = [D_{0\eta}] \in \mathrm{Br}\,(F_\eta)$, i.e. set it to be the residual Brauer class with respect to parameter $w_\eta\pi_\eta$. In all other cases, set $\beta_{rbc,\eta}$ to be the residual Brauer class of D with respect to parameter π_η (cf. Section 2.1). Note that $\beta_{rbc,\eta}$ has index at most ℓ.

5 At closed points

Recall that S_0 denotes the finite set of closed points lying on $\mathcal{H}_\mathcal{X}$ chosen as in Section 3.1.2. We refer to points P in S_0 as *marked points* occasionally. In this section, we classify points in S_0 following ([S07]) in essence, study the configuration of Y at these points and also investigate the shape of a at some types of closed points P when $Y_P \simeq \prod F_P$.
Let $P \in S_0$ be the intersection of two distinct irreducible curves C and C' of $\mathcal{H}_\mathcal{X}$ with generic points η and η' in N'_0 respectively. Let π_P and δ_P be primes defining C and C' at P be as fixed in Section 3.1.2.

5.1 Classification of marked points

We use the following notations: u_P, v_P will denote units in A_P, D_{00}, the Brauer class of an algebra of $\mathrm{Br}\,(F)$ unramified at A_P, i.e. $D_{00} \in \mathrm{Br}\,(A_P)$. Superscripts s and ns on D_P are used to denote that the algebra D_P is split and non-split respectively. We sometimes refer to the irreducible curve with generic point $\eta \in N'_0$ as $\overline{\eta}$. We begin with a lemma (similar to Lemma 3.2) relating the shapes of D_η and D_P.

LEMMA 5.1. *If* $D_\eta = 0 \in \mathrm{Br}(F_\eta)$, *then* $D_P = 0 \in \mathrm{Br}(F_P)$.

Proof. Since D is unramified at η, $D_P = (v_P, \delta_P)$. Further as D_η is split, so is $D \otimes F_{P,\eta}$. This implies $(v_P, \delta_P) = 0 \in \mathrm{Br}\,(F_{P,\eta})$. That is $\left(\overline{v_P}, \overline{\delta_P}\right) = 0 \in \mathrm{Br}\,(k_{P,\eta})$. Recall that $k_{P,\eta}$ is a complete discretely valued field with $\overline{\delta_P}$ as a parameter. Since $v_P \in \widehat{A_P}^*$, $\overline{v_P} \in k_P^{*\ell}$ and hence $v_P \in \widehat{A_P}^{*\ell}$. This immediately implies that $D_P = 0$ in $\mathrm{Br}\,(F_P)$. ☐

REMARK 5.2. *Lemma 3.2 implies that if* $\eta, \eta' \in N_0'$ *are such that* Y_η *is of Type RAM and* $Y_{\eta'}$ *is of Type SPLIT, then* $\overline{\eta}$ *and* $\overline{\eta'}$ *cannot intersect.*

We now list[5] the types[6] of closed points in S_0 possible.
TYPE A: P is of Type A if both C and C' do not lie in the ramification locus of D. Further D is unramified at P and because the residue field is finite, D_P is split. Type A points are further subdivided as follows:

- TYPE A_{00}^s: Both η and η' are of Type 0. Thus D_η and $D_{\eta'}$ are split.
- TYPE A_{10}^s: Exactly one of η, η' is of Type 0. Thus the other, say η, is of Type 1a. So $D_{\eta'}$ is split whereas D_η is an unramified index ℓ CSA.
- *Type A_{11}^s: Both η and η' are of Type 1a.

TYPE B: P is of Type B if exactly one of C and C' lies in the ramification locus of D (say C). Thus η is of Type 1b or 2 and η', of Type 0 or 1a. Further $D = D_{00} + (u_P, \pi_P)$ in $\mathrm{Br}(F)$ and because the residue field is finite, $D_P = (u_P, \pi_P)$ in $\mathrm{Br}\,(F_P)$. Type B points are further subdivided as follows:

- TYPE B_{10}^s: η is of Type 1b and η' is of Type 0. Note that by Lemma 5.1, D_P is split.
- *Type B_{11}^s: η is of Type 1b, η' is of Type 1a and D_P is split.
- TYPE B_{11}^{ns}: η is of Type 1b, η' is of Type 1a and D_P is non-split.
- TYPE B_{20}^s: η is of Type 2 and η' is of Type 0. Note that by Lemma 5.1, D_P is split.
- *Type B_{21}^s: η is of Type 2, η' is of Type 1a and D_P is split.
- TYPE B_{21}^{ns}: η is of Type 2, η' is of Type 1a and D_P is non-split.

TYPE C: P is of Type C if both C and C' lie in the ramification locus of D. Thus η and η' are of Type 1b or 2. Further $D = D_{00} + (u_P, \pi_P) + (v_P, \delta_P)$ or $D_{00} + (u_P \pi_P^m, v_P \delta_P)$ in $\mathrm{Br}\,(F)$ for an integer m coprime to ℓ in $\mathrm{Br}(F)$.

[5] The order of the subscripts in the types of points do not matter. So for instance we will use both C_{12}^{Cold} and C_{21}^{Cold} to mean the same type of point.

[6] It will be shown in Proposition 6.2 following the classification that the starred ones can be eliminated by blowing up our model repeatedly.

Points P where $D = D_{00} + (u_P\pi_P^m, v_P\delta_P)$ were labelled *cold* points in ([S07]). Thus $D_P = (u_P\pi_P^m, v_P\delta_P)$ at a cold point P and the ramification data at C, $\partial_C([D])$ is given by $\left((v_P\delta_P)^{-m}\right)^{\frac{1}{\ell}}$ at P. Points P where $D = D_{00} + (u_P, \pi_P) + (v_P, \delta_P)$ were further subdivided depending on the shape of the finite subgroups $x = \langle\overline{u_P}\rangle$ and $y = \langle\overline{v_P}\rangle$ in $k_P^*/k_P^{*\ell}$ into *chilly* points (when $x = y \neq \{1\}$), *cool* points (when $x = y = \{1\}$), and *hot* points (when $x \neq y$). Since $k_P^*/k_P^{*\ell}$ is a cyclic group of order ℓ, the subgroups x, y have to be either trivial or all of $k_P^*/k_P^{*\ell}$.

If P is a chilly point, without loss of generality assume $\overline{u_P} = \overline{v_P}^j$ for some j coprime to ℓ. Thus $D_P = \left(v_P, \pi_P^j\delta_P\right) \in \mathrm{Br}(F_P)$ and the ramification data at C, $\partial_C([D])$ is given by $\left(v_P^j\right)^{\frac{1}{\ell}}$ at P. If P is a cool point, $D_P = \{0\} \in \mathrm{Br}(F_P)$. If P is a hot point, assume without loss of generality that $y = \{1\}$. Thus $D_P = (u_P, \pi_P) \in Br(F_P)$ and the ramification data at C, $\partial_C([D])$ is given by $(u_P)^{\frac{1}{\ell}}$ at P. We also recall that in this case, $D \otimes F_{\eta'}$ has index ℓ^2 ([S07], Proposition 0.5, Theorem 2.5) and hence η' is of Type 2. We continue to follow Saltman's convention while refining the classification as follows:

- Type C_{11}^{Cold} : η is of Type 1b, η' is of Type 1b and P is cold.
- Type C_{11}^{Chilly} : η is of Type 1b, η' is of Type 1b and P is chilly.
- *Type C_{11}^{Cool} : η is of Type 1b, η' is of Type 1b and P is cool.
- Type C_{12}^{Cold} : η is of Type 1b, η' is of Type 2 and P is cold.
- *Type C_{12}^{Chilly} : η is of Type 1b, η' is of Type 2 and P is chilly.
- *Type C_{12}^{Cool} : η is of Type 1b, η' is of Type 2 and P is cool.
- Type C_{12}^{Hot} : η is of Type 1b, η' is of Type 2 and P is hot.
- *Type C_{22}^{-} : η is of Type 2 and η' is of Type 2.

5.2 Shape of a when Y_P is split

We investigate the shape of a at some types of closed points $P \in S_0$ when $Y_P \simeq \prod F_P$. By Proposition 3.4, $a = \left(a'_{i,P}\right)_i \in \prod F_P$ where $a'_{i,P} = z_{i,P}\pi_P^{m_i}\delta_P^{n_i}$, $z_{i,P} \in \widehat{A_P}^*$ and $m_i, n_i \in \mathbb{Z}$ with $\sum m_i = \sum n_i = 0$.

Proposition 5.3. *Let $P \in S_0$ such that $Y_P \simeq \prod F_P$ and let $a = \left(a'_{i,P}\right)_i \in \prod F_P$ where $a'_{i,P} = z_{i,P}\pi_P^{m_i}\delta_P^{n_i}$ as above.*

1. *If P is a cold point with $D_P = (u_P\pi_P^m, v_P\delta_P)$ where $0 < m < \ell$, then $a'_{i,P} = (u_P\pi_P^m)^{sm_i}(v_P\delta_P)^{n_i}(w'_{i,P})^\ell\left(\pi_P^{-rm_i}\right)^\ell$ for some $w'_{i,P} \in \widehat{A_P}^*$ and $s, r \in \mathbb{Z}$ such that $sm = r\ell + 1$.*

2. *If P is a chilly point with $D_P = \left(v_P, \pi_P^j \delta_P\right)$ where $0 < j < \ell$, then $m_i = r_i\ell + jn_i$ where $r_i \in \mathbb{Z}$. Thus $\sum r_i = 0$ and $a'_{i,P} = z_{i,P}\left(\pi_P^j\delta_P\right)^{n_i}(\pi_P^{r_i})^\ell$.*

3. *If P is a hot point*[7] *or a Type B_{11}^{ns} point*[8] *with $D_P = (u_P, \pi_P)$, then $n_i = n'_i\ell$ where $n'_i \in \mathbb{Z}$. Thus $\sum n'_i = 0$ and $a'_{i,P} = z_{i,P}\pi_P^{m_i}\left(\delta_P^{n'_i}\right)^\ell$.*

4. *If P be a Type B_{21}^{ns} point*[9] *with $D_P = (u_P, \pi_P)$, then $m_i = 0$ and $n_i = n'_i\ell$ where $n'_i \in \mathbb{Z}$. Thus $\sum n'_i = 0$ and $a'_{i,P} = z_{i,P}\left(\delta_P^{n'_i}\right)^\ell$.*

Proof. Since a is a reduced norm from $D \otimes Y$, for each i, $\left(a'_{i,P}\right)[D] = 0 \in \mathrm{H}^3(F_P, \mu_\ell)$.
At a cold point:

$$\begin{aligned}
&\left(z_{i,P}\pi_P^{m_i}\delta_P^{n_i}\right)\left(u_P\pi_P^m, v_P\delta_P\right) = 0\\
\implies &\left(z_{i,P}\right)\left(\pi_P^m, \delta_P\right) + \left(\pi_P^{m_i}\right)\left(u_P, \delta_P\right) + \left(\delta_P^{n_i}\right)\left(\pi_P^m, v_P\right) = 0\\
\implies &\left(z_{i,P}^m\right)\left(\pi_P, \delta_P\right) + \left(u_P^{-m_i}\right)\left(\pi_P, \delta_P\right) + \left(v_P^{-mn_i}\right)\left(\pi_P, \delta_P\right) = 0\\
\implies &\left(z_{i,P}^m u_P^{-m_i} v_P^{-mn_i}\right)\left(\pi_P, \delta_P\right) = 0
\end{aligned}$$

Taking residues along π_P and then along $\overline{\delta_P}$, we see that $z_{i,P}^m = u_P^{m_i}v_P^{mn_i}{w''}_{i,P}^{\ell}$ for some $w''_{i,P} \in \widehat{A_P}^*$. Since $0 < m < \ell$, let $0 < s < \ell$ such that $sm = r\ell + 1$ for some $r \in \mathbb{Z}$. Taking s^{th} powers, we have $z_{i,P}^{r\ell+1} = u_P^{sm_i}v_P^{n_ir\ell+n_i}{w''}_{i,P}^{s\ell}$. Hence for some $w'_{i,P} \in \widehat{A_P}^*$,

$$\begin{aligned}
a'_{i,P} &= z_{i,P}\pi_P^{m_i}\delta_P^{n_i} = \left(u_P^s\pi_P\right)^{m_i}\left(v_P\delta_P\right)^{n_i}\left(v_P^{n_ir}{w''}_{i,P}^{s}z_{i,P}^{-r}\right)^\ell\\
&= \left(u_P\pi_P^m\right)^{sm_i}\left(\pi_P^{-rm_i\ell}\right)\left(v_P\delta_P\right)^{n_i}{w'}_{i,P}^{\ell}
\end{aligned}$$

At a chilly point:

$$\begin{aligned}
&\left(z_{i,P}\pi_P^{m_i}\delta_P^{n_i}\right)\left(v_P, \pi_P^j\delta_P\right) = 0\\
\implies &\left(\pi_P\right)\left(v_P^{m_i}, \delta_P\right) + \left(\delta_P\right)\left(v_P^{jn_i}, \pi_P\right) = 0\\
\implies &\left(v_P^{-m_i}\right)\left(\pi_P, \delta_P\right) + \left(v_P^{jn_i}\right)\left(\pi_P, \delta_P\right) = 0\\
\implies &\left(v_P^{jn_i-m_i}\right)\left(\pi_P, \delta_P\right) = 0.
\end{aligned}$$

[7] Here η is of Type 1b whereas η' is of Type 2 ([S07], Proposition 0.5, Theorem 2.5).
[8] Here η is of Type 1b whereas η' is of Type 1a.
[9] Here η is of Type 2 whereas η' is of Type 1a.

Taking residues along π_P and then along $\overline{\delta_P}$, we see that $m_i \cong jn_i \mod \ell$. Since $\sum m_i = \sum n_i = 0$ and $0 < j < \ell$, $\sum r_i = 0$.
At a hot / B_{11}^{ns} point : $(z_{i,P}\pi_P^{m_i}\delta_P^{n_i})(u_P, \pi_P) = 0$. Hence $(\delta_P^{n_i})(u_P, \pi_P) = 0$ and therefore $(u_P^{n_i})(\pi_P, \delta_P) = 0$. Taking residues along π_P and then along $\overline{\delta_P}$, we see that $n_i = n_i'\ell$ for some $n_i' \in \mathbb{Z}$. Since $\sum n_i = 0$, $\sum n_i' = 0$ also.
At a B_{21}^{ns} point : Since Y_P is split, Y_η is not of Type RAM. By Remark 4.1 and Proposition 3.3, a is a unit along η. Since a is arranged to be in good shape, we have $m_i = 0$. The same proof as in the previous case shows $n_i = n_i'\ell$ and $\sum n_i' = 0$. □

5.3 CONFIGURATION OF Y AT MARKED POINTS IN S_0

We record the configuration of Y at some types of the marked points in S_0. This is possible since the $\mathrm{div}_{\mathcal{X}}(y)$ is arranged to be in good shape where $Y = F(\sqrt[\ell]{y})$. We spell out the proof in the case when P is a C_{11}^{Cold} point. The other proofs follow in a similar fashion by using Lemma 3.2 and the fact that the shape of Y_P can be determined from that of Y_η and $Y_{\eta'}$ by going to the branch fields $Y \otimes F_{P,eta}$ and $Y \otimes F_{P,\eta'}$ (c.f proof of Lemma 3.2) along with Remarks 4.1 and 5.2.
In this subsection, we use the following notations in the tables: $0 < r < \ell$ and $w, u_P, v_P \in \widehat{A_P}^*$. L_P refers to the unique cyclic degree ℓ field extension of F_P unramified at $\widehat{A_P}$.

PROPOSITION 5.4 (At C_{11}^{Cold} points). *Let P be a C_{11}^{Cold} point and let $D_P = (u_P\pi_P^m, v_P\delta_P)$ for $0 < m < \ell$. Then the following table gives the possible configurations (including some symmetric situations) of Y at P.*

$Y_{\eta'}$	Y_η	Y_P
RAM	RAM	$F_P\left(\sqrt[\ell]{w\pi_P\delta_P^r}\right)$
RAM	RES	$F_P\left(\sqrt[\ell]{v_P\delta_P}\right)$
RAM	NONRES	$F_P\left(\sqrt[\ell]{w\delta_P}\right)$
RES	RAM	$F_P\left(\sqrt[\ell]{u_P\pi_P^m}\right)$
SPLIT	SPLIT	$\prod F_P$
SPLIT	NONRES	$\prod F_P$
NONRES	RAM	$F_P\left(\sqrt[\ell]{w\pi_P}\right)$
NONRES	SPLIT	$\prod F_P$
NONRES	NONRES	L_P or $\prod F_P$

Table 2: Shape of Y at C_{11}^{Cold} point P

Proof. If $Y_{\eta'}$ is RAM, by Remark 5.2, Y_η cannot be SPLIT. If $Y_{\eta'}$ is RES, then $Y_{P,\eta'} \simeq F_{P,\eta'}\left(\sqrt[\ell]{u_P\pi_P^m}\right)$, a field extension and $\overline{Y_{P,\eta'}}/k_{P,\eta'}$ is ramified. Hence Y_η is RAM. If $Y_{\eta'}$ is SPLIT, then $Y_P \simeq \prod F_P$ by Lemma 3.2. Hence Y_η cannot be RAM. It also cannot be RES by the same argument as above. Finally, if $Y_{\eta'}$ is NONRES, the same argument shows Y_η cannot be RES. Since Y/F is arranged to be in good shape in $\mathcal{X}$, the shape of Y_P can be determined from that of Y_η and $Y_{\eta'}$ in a similar manner as that in the proof of Lemma 3.2. □

PROPOSITION 5.5 (At C_{12}^{Cold} points). *Let P be a C_{12}^{Cold} point and let $D_P = (u_P\pi_P^m, v_P\delta_P)$ for $0 < m < \ell$. Assume without loss of generality that η' is of Type 2. Then the following table gives the possible configurations of Y at P.*

$Y_{\eta'}$	Y_η	Y_P
RAM	RAM	$F_P\left(\sqrt[\ell]{w\pi_P\delta_P^r}\right)$
RAM	RES	$F_P\left(\sqrt[\ell]{v_P\delta_P}\right)$
RAM	NONRES	$F_P\left(\sqrt[\ell]{w\delta_P}\right)$
RES	RAM	$F_P\left(\sqrt[\ell]{u_P\pi_P^m}\right)$
NONRES	RAM	$F_P\left(\sqrt[\ell]{w\pi_P}\right)$
NONRES	SPLIT	$\prod F_P$
NONRES	NONRES	L_P or $\prod F_P$

Table 3: Shape of Y at C_{12}^{Cold} point P

Proposition 5.6 (At C_{11}^{Chilly} points). *Let P be a C_{11}^{Chilly} point and let $D_P = \left(v_P, \pi_P^j\delta_P\right)$ where $0 < j < \ell$. Then the following table*[10] *gives the possible configurations of Y at P.*

$Y_{\eta'}$	Y_η	Y_P
RAM	RAM	$F_P\left(\sqrt[\ell]{w\pi_P\delta_P^r}\right)$
RAM	NONRES	$F_P\left(\sqrt[\ell]{w\delta_P}\right)$
RES	RES	L_P
RES	NONRES	L_P
SPLIT	SPLIT	$\prod F_P$
SPLIT	NONRES	$\prod F_P$
NONRES	RAM	$F_P\left(\sqrt[\ell]{w\pi_P}\right)$
NONRES	RES	L_P
NONRES	SPLIT	$\prod F_P$
NONRES	NONRES	L_P or $\prod F_P$

Table 4: Shape of Y at chilly point P

Proposition 5.7 (At C_{12}^{Hot} points). *Let P be a C_{12}^{Hot} point and let*[11] *$D_P = (u_P, \pi_P)$. If $Y_{\eta'}/F_{\eta'}$ is an unramified extension which is not RES, then it must be of Type NONRES. Further, Y_P is a non-split extension and hence $Y \otimes_F D \otimes_F F_P$ is split.*

Proof. By ([S97], [S98]), $[D] = [D_{00}] + (u_P, \pi_P) + (v_P, \delta_P) \in \mathrm{Br}(F)$ where D_{00} is unramified at A_P. By ([S07], Proposition 0.5, Theorem 2.5), $D \otimes_F F_{\eta'}$ is a division algebra and hence $Y_{\eta'}$ has to be a non-split field extension. Thus it is of Type NONRES.
Now $D = D_0 + (v_P, \delta_P) \in \mathrm{Br}(F_{\eta'})$ where $D_0 = [D_{00}] + (u_P, \pi_P)$ is unramified at η'. Since P is a hot point, D is ramified along both η and η'. Thus v_P is not an ℓ^{th} power in $F_{\eta'}$. Since $Y_{\eta'}$ is unramified and not RES, $\left[Y_{\eta'}\left(\sqrt[\ell]{v_P}\right) : Y_{\eta'}\right] = \ell$.
Thus by Lemma 2.9, ℓ equals index $(D \otimes Y_{\eta'})$. Thus,

$$\begin{aligned}\ell &= \operatorname{index}\left(D_0 \otimes Y_{\eta'}\left(\sqrt[\ell]{v_P}\right)\right)\left[Y_{\eta'}\left(\sqrt[\ell]{v_P}\right) : Y_{\eta'}\right] \\ &= \ell\left(\operatorname{index}\left(D_0 \otimes Y_{\eta'}\left(\sqrt[\ell]{v_P}\right)\right)\right).\end{aligned}$$

Thus $Y_{\eta'}\left(\sqrt[\ell]{v_P}\right)$ splits D_0 over $F_{\eta'}$ and hence also over the branch field $F_{P,\eta'}$. Note that $[D_0] = (u_P, \pi_P) \neq 0 \in \mathrm{Br}(F_{P,\eta'})$.

[10] It includes some symmetric situations.
[11] Thus η is of Type 1b whereas η' is of Type 2 ([S07], Proposition 0.5, Theorem 2.5).

Suppose that Y_P is split. Since P is a hot point, $\overline{v_P}$ is an ℓ^{th} power in k_P and hence $Y_P\left(\sqrt[\ell]{v_P}\right) \simeq \prod F_P$. Thus along the branch field, $Y_{P,\eta'}\left(\sqrt[\ell]{v_P}\right) \simeq \prod F_{P,\eta'}$ which cannot split the non-trivial algebra D_0. Thus we conclude that Y_P is a non-split field extension.

Since we have assumed that Y is in good shape and that $Y_{\eta'}$ is unramified at η', there exists $j \in \{0,1\}$ such that

$$Y_P = F_P\left(\sqrt[\ell]{w_P\pi_P^j}\right), w_P \in \widehat{A_P}^*.$$

If $j=0$, Y_P is the unique non-split unramified extension at P and has to be isomorphic to $F_P\left(\sqrt[\ell]{u_P}\right)$, which splits D_P. If $j=1$, then let $\lambda^\ell = w_P\pi_P$ for $\lambda \in Y_P$. Thus $D_P \otimes Y_P = (u_P, \pi_P) = \left(u_P, w_P^{-1}\right) \in \mathrm{Br}\,(Y_P)$ and hence split. □

Proposition 5.8 (At B^s_{10} points). *Let P be a B^s_{10} point. Assume without loss of generality that η is of Type 1b and η' is of Type 0. Then the following table gives the possible configurations of Y at P.*

$Y_{\eta'}$	Y_η	Y_P
RAM	RAM	$F_P\left(\sqrt[\ell]{w\pi_P^r\delta_P}\right)$
RAM	NONRES	$F_P\left(\sqrt[\ell]{w\delta_P}\right)$
SPLIT	RES	$\prod F_P$
SPLIT	SPLIT	$\prod F_P$
SPLIT	NONRES	$\prod F_P$
NONRES	RAM	$F_P\left(\sqrt[\ell]{w\pi_P}\right)$
NONRES	RES	$\prod F_P$
NONRES	SPLIT	$\prod F_P$
NONRES	NONRES	L_P or $\prod F_P$

Table 5: Shape of Y at B^s_{10} point P

Proposition 5.9 (At B^{ns}_{11} points). *Let P be a B^s_{11} point. Assume without loss of generality that η is of Type 1b and η' is of Type 1a. Then the following table gives the possible configurations of Y at P.*

$Y_{\eta'}$	Y_η	Y_P
RAM	RAM	$F_P\left(\sqrt[\ell]{w\pi_P^r\delta_P}\right)$
RAM	NONRES	$F_P\left(\sqrt[\ell]{w\delta_P}\right)$
SPLIT	SPLIT	$\prod F_P$
SPLIT	NONRES	$\prod F_P$
NONRES	RAM	$F_P\left(\sqrt[\ell]{w\pi_P}\right)$
NONRES	RES	L_P
NONRES	SPLIT	$\prod F_P$
NONRES	NONRES	L_P or $\prod F_P$

Table 6: Shape of Y at B^{ns}_{11} point P

6 Blowups

We repeatedly exploit the trick of blowing up[12] our model at closed points to make the model more amenable for patching. In this section, assume $P \in C \cap C'$ where C, C' are distinct irreducible curves in $\mathcal{H}_{\mathcal{X}}$ with generic points η and η' respectively. Let π_P and δ_P be primes defining C and C' at P as before. After blowing up the model at P once, let Σ denote the exceptional curve with generic point ϵ and let $\tilde{C}$ and $\tilde{C}'$ denote the strict transforms of C and C' respectively. Let the two new intersection points be Q_1 (where ϵ intersects $\tilde{C}$) and Q_2 (where ϵ intersects $\tilde{C}'$).

Lemma 6.1 (Blowing up a cold point). *Let P be a cold point and let $D_P = (u_P\pi_P^m, v_P\delta_P)$ where $0 < m < \ell$. Let $\phi : \mathcal{X}_1 \to \mathcal{X}$ denote the blowup at point P. Then the exceptional curve Σ obtained is of Type 1b and both $Q_1 = C \cap \Sigma$ and $Q_2 = C' \cap \Sigma$ are cold points.*

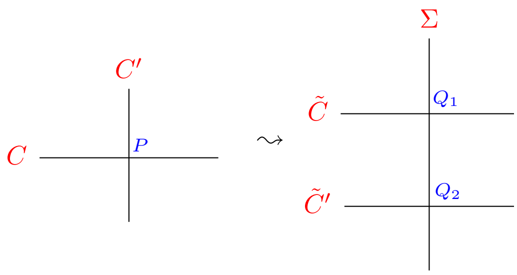


Figure 1: Blowup of a point

Proof. Look at the local blow-up $\mathcal{Z} := \operatorname{Proj}\left(\frac{\widehat{A_P}[x,y]}{(x\pi_P - y\delta_P)}\right) \to \operatorname{Spec}(\widehat{A_P})$ at the maximal ideal of $\widehat{A_P}$. Setting $t = y/x$, we have $\pi_P = t\delta_P$. Thus $\mathcal{Z}$ is the union of open affines $\operatorname{Spec}\frac{\widehat{A_P}[t]}{(\pi_P - t\delta_P)}$ and $\operatorname{Spec}\frac{\widehat{A_P}[\frac{1}{t}]}{(\frac{\pi_P}{t} - \delta_P)}$ glued appropriately.

By (Lemma 085S, stacks-project), we have $\widehat{\mathcal{O}_{\mathcal{X}_1,\epsilon}} =: \widehat{A_\epsilon} = \widehat{\left(\frac{\widehat{A_P}[t]}{(\pi_P - t\delta_P)}\right)_{(\pi_P,\delta_P)}}$. Thus in F_ϵ, the fraction field of $\widehat{A_\epsilon}$, both π_P and δ_P are parameters. Since $D_\epsilon = (u_P\pi_P^m, v_P\delta_P)$, it has index at most ℓ. The residue of D_ϵ is equal to $\overline{\frac{u_P\pi_P^m}{v_P^m\delta_P^m}} = \overline{u_Pv_P^{-m}t^m}$ which is non-trivial in the residue field $k_\epsilon = k_P(t)$. Therefore ϵ is of Type 1b.

[12]Note that with each blow up, the set S_0 for the new model is enlarged to include the intersection points of the exceptional curve and the closure of the strict transforms of $\mathcal{H}_{\mathcal{X}}$ and the set N_0' is expanded to include the generic point of the exceptional curve.

Note that $\widehat{\mathcal{O}_{\mathcal{X}_1,Q_1}} =: \widehat{A_{Q_1}} = \widehat{\left(\frac{\widehat{A_P}[t]}{(\pi_P - t\delta_P)}\right)_{(t,\delta_P)}}$ where t defines $\tilde{C}$ and δ_P defines Σ at Q_1 (cf. [S07], Pg 832, paragraph 1). Thus over F_{Q_1}, the fraction field of $\widehat{A_{Q_1}}$,

$$\begin{aligned} D_{Q_1} &= (u_P\delta_P^m t^m, v_P\delta_P) = (u_P t^m, v_P\delta_P) + (\delta_P^m, v_P\delta_P) \\ &= \left(u_P v_P^{-m} t^m, v_P\delta_P\right) \in \mathrm{Br}(F_{Q_1}). \end{aligned}$$

Similarly $\widehat{\mathcal{O}_{\mathcal{X}_1,Q_2}} =: \widehat{A_{Q_2}} = \widehat{\left(\frac{\widehat{A_P}[1/t]}{(\pi_P/t - \delta_P)}\right)_{(1/t,\pi_P)}}$ where $1/t$ defines $\tilde{C}'$ and π_P defines Σ at Q_2. Thus over F_{Q_2}, the fraction field of $\widehat{A_{Q_2}}$ and for s with $ms \cong 1 \mod \ell$,

$$\begin{aligned} D_{Q_2} &= \left(u_P\pi_P^m, v_P\frac{\pi_P}{t}\right) = \left(u_P\pi_P^m, v_P\frac{1}{t}\right) + (u_P\pi_P^m, \pi_P) \\ &= \left(u_P\pi_P^m, u_P^{-s}v_P\frac{1}{t}\right) \in \mathrm{Br}(F_{Q_2}). \end{aligned}$$

Hence both Q_1 and Q_2 are cold points. □

We now eliminate certain types of closed points listed in the classification in Section 5.1.

Proposition 6.2. *There exists a regular proper model such that S_0 does not contain points of Type $A^s_{11}, B^s_{11}, B^s_{21}, C^{Cool}_{11}, C^{Cool}_{12}, C^{Chilly}_{12}$ and C^-_{22}.*

Proof. Let P denote an intersection point of one of types listed in the proposition and let Σ and ϵ denote the exceptional curve and its generic point obtained after blowing up P once. The following subtypes can be avoided by blowing up the model once at P.

Type A^s_{11}: Since D_P is split, $D \otimes F_\epsilon$ is split too and hence ϵ is of Type 0. Thus the two new intersection points are obtained by Type 1a curves ($\tilde{C}$ or $\tilde{C}'$) intersecting a curve of Type 0 (Σ). $[A^s_{11} \mapsto A^s_{10} + A^s_{10}]$.

Type B^s_{i1} $(i = 1, 2)$: Since D_P is split, $D \otimes F_\epsilon$ is split too and hence ϵ is of Type 0. Thus the two new intersection points are obtained by Type 1b/2 or 1a curves ($\tilde{C}$ or $\tilde{C}'$) intersecting a curve of Type 0 (Σ). $[B^s_{i1} \mapsto B^s_{i0} + A^s_{i0}]$.

Type C^{Cool}_{1i} $(i = 1, 2)$: $\left[C^{Cool}_{11} \mapsto B^s_{10} + B^s_{i0}\right]$ (cf. [S07], Theorem 2.6).

Type C^{Chilly}_{12}: This subtype can be avoided by blowing up the model $\mathcal{X}$ consecutively. Since $D_P = (v_P, \pi_P^j\delta_P)$, after one blowup at P, $D \otimes F_\epsilon$ has index at most ℓ and hence ϵ is of Type 0 or 1. Thus the two new intersection points are Q_1 (Type 11/01 : where Σ intersects $\tilde{C}$) and Q_2 (Type 12/02 : where Σ intersects $\tilde{C}'$).

Let us investigate the case when ϵ is of Type 1b. Then as in the proof of Lemma 6.1, $D_{Q_2} = \left(v_P, \pi_P^{j+1}\frac{1}{t}\right)$ where $1/t$ defines $\tilde{C}'$ and π_P defines Σ at Q_2 with $\delta_P = \frac{\pi_P}{t}$. Hence Q_2 is again of Type C^{Chilly}_{12}. However the ramification along the Type 1b curve ($C \rightsquigarrow \Sigma$) has changed as evinced by the increase $j \rightsquigarrow j+1$. We can keep blowing up the intersection points of the strict transforms of C' and the exceptional curve repeatedly till the new exceptional curve is of Type 0 or 1a and thus eliminate intersection points of the shape C_{12}^{Chilly}.

TYPE C_{22}^{-}: This subtype can again be avoided by blowing up the model $\mathcal{X}$ an appropriate number of times at P. Since D_P has index at most ℓ, after one blowup, ϵ is of Type 0, 1a or 1b. Thus the two new intersection points are obtained by Type 2 curves ($\tilde{C}$ or $\tilde{C}'$) intersecting a curve of Type 0,1a or 1b (Σ). In case B_{21}^{s} or C_{12}^{Chilly} points are generated, further blow up as in the previous steps to eliminate them. □

6.1 LIMITING NEIGHBOURS

We introduce the terminology that the closed points P and Q in S_0 are *Type x neighbours* if they both lie on the closure (denoted $\overline{\eta}$) of some $\eta \in N_0'$ of Type x where $x \in \{0, 1a, 1b, 2\}$. Let $P, C, C', \eta, \eta', \pi_P, \delta_P, \tilde{C}, \tilde{C}', \Sigma, \epsilon$ be as before. We first begin with the following proposition that records the configuration of Y when $\mathcal{X}$ is blown up at a hot point P once.

PROPOSITION 6.3. *Let P be a hot point of $\mathcal{X}$ and let $\phi : \mathcal{X}_{new} \to \mathcal{X}$ be the blowup at P. Without loss of generality, let $D_P = (u_P, \pi_P)$. Then Q_2 is a hot point in $\mathcal{X}_{new}$ while Q_1 is a chilly point. Further the following table records possible configurations of Y_η, $Y_{\eta'}$ and Y_ϵ. In particular if $Y_{\eta'}$ is not of Type RAM, then Y_ϵ is not of Type NONRES.*

Proof. Since $D_P = (u_P, \pi_P)$ where $u_P \in \widehat{A_P}^*$, η is of Type 1b while η' is of Type 2. Thus $D_{Q_1} = (u_P, \delta_P t)$ where t defines $\tilde{C}$ and δ_P defines Σ at Q_1 where $\pi_P = t\delta_P$. Similarly $D_{Q_2} = (u_P, \pi_P)$ where $1/t$ defines $\tilde{C}'$ and π_P defines Σ at Q_2. Thus we have replaced P with hot point Q_2 and chilly point Q_1 in $\mathcal{X}_{new}$.

Let $\epsilon \in \mathcal{X}_{new}$ denote the generic point of the exceptional curve Σ and by abuse of notation, a parameter of F_ϵ. Since $D_{\eta'}$ is division ([S07], Theorem 2.5, Proposition 0.5), $Y_{\eta'}$ cannot be SPLIT. If $Y_{\eta'}$ is of Type RAM, then Y_η cannot be SPLIT or RES. If $Y_{\eta'}$ is of Type RES, then $Y_P \simeq \prod F_P$ and hence Y_η can only be SPLIT or NONRES. Finally observe that if $Y_{\eta'}$ is NONRES, then Y_P is non-split by Proposition 5.7 and hence Y_η cannot be SPLIT by Lemma 3.2.

Thus we have the following table (in which we use the notations $v \in \widehat{A_P}^*$, $w \in \widehat{A_\epsilon}^*$, $0 < r < \ell$ and $F_P\left(\sqrt[\ell]{u_P}\right)$ to be the unique degree ℓ unramified field extension of F_P).

$Y_{\eta'}$	Y_η	Y_P	Y_ϵ	Type of Y_ϵ
RAM	RAM	$F_P\left(\sqrt[\ell]{v\pi_P^r\delta_P}\right)$	$F_\epsilon\left(\sqrt[\ell]{w\epsilon^{r+1}}\right)$	RAM/NONRES
RAM	NONRES	$F_P\left(\sqrt[\ell]{v\delta_P}\right)$	$F_\epsilon\left(\sqrt[\ell]{w\epsilon}\right)$	RAM
RES	SPLIT	$\prod F_P$	$\prod F_\epsilon$	SPLIT
RES	NONRES	$\prod F_P$	$\prod F_\epsilon$	SPLIT
NONRES	RAM	$F_P\left(\sqrt[\ell]{v\pi_P}\right)$	$F_\epsilon\left(\sqrt[\ell]{w\epsilon}\right)$	RAM
NONRES	RES	$F_P\left(\sqrt[\ell]{u_P}\right)$	$F_\epsilon\left(\sqrt[\ell]{u_P}\right)$	RES
NONRES	NONRES	$F_P\left(\sqrt[\ell]{u_P}\right)$	$F_\epsilon\left(\sqrt[\ell]{u_P}\right)$	RES

Table 7: Table giving shape of Y at hot point Q_2

□

In the following proposition, we blow up further so as to arrange for a model $\mathcal{X}$ such that its marked points do not have any 'difficult' neighbours. This will be helpful when constructing $E_{1,\eta}$ and $E_{2,\eta}$ along codimension one points η lying in the special fiber X_0.

Proposition 6.4. *There exists a sequence of blowups $\phi : \mathcal{X}_{new} \to \mathcal{X}$ such that for any $\eta \in (N_0')_{\mathcal{X}_{new}}$, the following hold:*

1. *If η is of Type 0 containing a A^s_{10}, B^s_{10} or B^s_{20} marked point P, then there is at most only one other marked point $Q \in \overline{\eta}$ and it is of Type A^s_{00}.*
2. *If η is of Type 1b containing a C^{Chilly}_{11} marked point P, then there is at most only one other marked point $Q \in \overline{\eta}$ and it is of Type B^{ns}_{11} or C^{Chilly}_{11}.*
3. *If η is of Type 1b containing a C^{Hot}_{12} marked point P, then there is at most only one other marked point $Q \in \overline{\eta}$ and it is of Type B^{ns}_{11}.*

Proof. Let $P, C, C', \eta, \eta', \pi_P, \delta_P, \tilde{C}, \tilde{C}', \Sigma, \epsilon$ be as before. Recall that $Q_1 = \tilde{C} \cap \Sigma$ while $Q_2 = \tilde{C}' \cap \Sigma$ are the two new marked points obtained after blowing up at P. We investigate each case separately.
1. Let η be of Type 0 with a marked point P as above. Since D_P is split, ϵ is of Type 0 and has exactly two marked points Q_1 and Q_2 lying on it. For $\{e, f\} = \{1, 2\}$, we see that Q_e replaces P and has at most one Type 0 neighbour Q_f which is necessarily of Type A^s_{00}.
2. Let η be of Type 1b with a chilly point P. This case is reminiscent of the breaking of chilly loops in ([S07], Corollary 2.9).
If $D_P = (u_P, \pi_P^m \delta_P^n)$ for some unit u_P and $0 < m, n < \ell$, we say the algebra is of the shape $[m, n]_{C,C'}$. Let $x^{-1} = \frac{1}{x} \in (\mathbb{Z}/\ell\mathbb{Z})^*$. Then $[m, n]_{C,C'} = [1, nm^{-1}]_{C,C'} = [mn^{-1}, 1]_{C,C'}$ as $(u_P, \pi_P^m \delta_P^n) = \left(u_P, (\pi_P^m \delta_P^n)^{m^{-1}m}\right) = m\left(u_P, (\pi_P^m \delta_P^n)^{m^{-1}}\right) = \left(u_P^m, \pi_P \delta_P^{nm^{-1}}\right)$.
Since P is a C^{Chilly}_{11} point, D_P is of the shape $[1, j]_{C,C'}$ for some $0 < j < \ell$. After a single blow up, as in the proof of Proposition 6.2, $D_{Q_1} = [j, j+1]_{\tilde{C},\Sigma}$ and $D_{Q_2} = [1, j+1]_{\tilde{C}',\Sigma}$. Hence either $j + 1 \cong 0 \mod \ell$ and ϵ is a Type 1a curve [13] or $j + 1 < \ell$, ϵ is a Type 1b curve and both Q_1 and Q_2 are C^{Chilly}_{11} points again. If $j + 1 < \ell$, blow up the point Q_2 again. Repeating this process, we get a model $\mathcal{X}_1$ where the closure of the strict transform of C' intersects an exceptional curve of Type 1a. Carry out the same procedure on Q_1, the other intersection point till the closure of the strict transform of C also intersects an exceptional curve of Type 1a.
3. Let η be of Type 1b with a hot point P. By Proposition 6.3, blowing up the model at P yields a hot point Q_2 which has only Q_1, a chilly point, as a Type 1b neighbour. Now following the proof of the previous case and blowing up the chilly point Q_1 repeatedly, we see that Q_2 will only have a B^{ns}_{11} point as a Type 1b neighbour at most. ∎

[13] Note that when $j + 1 \cong 0 \mod \ell$, D_{Q_2} is still not a split algebra and hence ϵ cannot be a Type 0 curve by Lemma 5.1.

6.2 THE FINAL MODEL $\mathcal{X}$

Recall that $\mathcal{X}$ is arranged such that the divisor $\mathcal{H}_{\mathcal{X}}$ is in good shape. We note that this property is preserved under blowups (cf. proof of Proposition 3.1). Thus using Propositions 6.2, 6.4, from now on we can and do assume that our model $\mathcal{X}$ has no marked points of Type A_{11}^s, B_{11}^s, B_{21}^s, C_{11}^{Cool}, C_{12}^{Cool}, C_{12}^{Chilly} and C_{22}^-. Further we also assume that any C_{11}^{Chilly} point has only Type 1b neighbours which are either again C_{11}^{Chilly} or B_{11}^{ns}, any C_{12}^{Hot} point can be a Type 1b neighbour at most of one other point which should be of Type B_{11}^{ns} and any A_{10}^s, B_{10}^s or B_{20}^s point has at most one Type 0 neighbour which will necessarily be of Type A_{00}^s. Note also that in constructing such a model (cf. the proof of Proposition 6.4, hot point case), we would have blown up the original hot points exactly once and hence would have arranged for the shape of Y at any hot point in the final model to be as given by Proposition 6.3. We finally fix parameters π_η for each $\eta \in N_0'$ as in Section 3.1.2, which further determine a system of parameters for each $P \in S_0$.

6.3 GRAPHS

6.3.1 LABELLING CURVES WITH {CH, C, H, Z} LABELS

Let $\gamma \in N_0'$ be of Type 1b with $Y_\gamma \simeq \prod F_\gamma$. Using Proposition 6.4, we label it as follows:

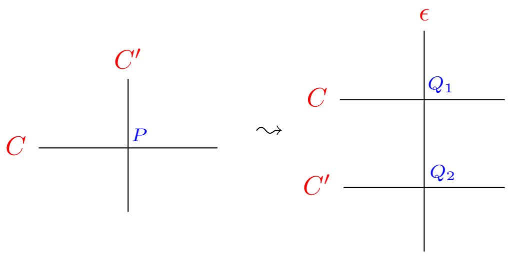
- γ is a CH-CURVE if $\overline{\gamma} \cap S_0$ contains a chilly point. Note that $\overline{\gamma} \cap S_0$ will consist of marked points of Types B_{11}^{ns} and C_{11}^{Chilly} only.
- γ is a C-CURVE if $\overline{\gamma} \cap S_0$ contains a cold point. Note that $\overline{\gamma} \cap S_0$ will consist of marked points of Types B_{10}^s, B_{11}^{ns}, C_{11}^{Cold} and C_{12}^{Cold} only.
- γ is a H-CURVE if $\overline{\gamma} \cap S_0$ contains a hot point. Note that $\overline{\gamma} \cap S_0$ will consist of marked points of Types B_{11}^{ns} and C_{12}^{Hot} only.
- γ is a Z-CURVE if it is not a Ch, C or H-curve. Note that $\overline{\gamma} \cap S_0$ will consist of marked points of Types B_{10}^s or B_{11}^{ns} only.

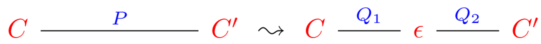
Thus the sets of Ch, C, H and Z-curves are mutually disjoint. Note also that when you blow up a cold point P on a C-curve η, then the exceptional curve obtained is again a C-curve and the two new marked points obtained are again cold points (Lemmata 3.2 and 6.1).

6.3.2 A PARTIAL DUAL GRAPH

In subsequent sections, we will prescribe patching data $E_{1,\eta}$ and $E_{2,\eta}$ for $\eta \in N_0 := N_0' \cap X_0$. Ensuring compatibility at branches can be, in part, turned into a colouring problem for *a partial dual graph* built as follows:
Construct an undirected graph Δ with vertex set V_Δ consisting of $\eta \in N_0'$ of Type 1b or 2. The edge set J_Δ consists of cold points in S_0. So if $\overline{\eta}, \overline{\eta'} \in V_\Delta$ intersect at a cold point P in our model, then they are joined by an edge labelled P. Note that

therefore multiple edges between distinct vertices are allowed, while self loops are not. Blowing up a cold point P has the effect of adding a vertex in middle of the edge P in Δ.





6.3.3 PRIMARY COLOURING OF Δ

We now present a combinatorial colouring proposition, reserving for later the explanation of the precise relevance of this to the patching problem. The following guarantees that after finitely many blowups of cold points, there exists a 'suitable' colouring of the vertices of Δ with the colours red (R), green (G), and blue (B). More precisely:

PROPOSITION 6.5. *There exists a sequence of blowups of cold points on C-curves, $\phi : \mathcal{X}_{new} \to \mathcal{X}$, such that the vertices of the new partial graph Δ_{new} can be coloured with colours blue (B), green (G) and red (R) such that*

1. *$\eta \in V_{\Delta_{new}}$ is coloured green if and only if η is not a C-curve*[14]*,*
2. *Any non-green vertex with an edge to a green vertex is coloured red,*
3. *Any non-green vertex with an edge to a red vertex is blue,*
4. *Any non-green vertex with an edge to a blue vertex is red.*

Proof. Without loss of generality, assume Δ is connected (otherwise repeat the same proof for each connected component). Let $W \subseteq V_\Delta$ denote the set of C-curves. Colour every $\eta \in V_\Delta \setminus W$ with green. Thus Δ is partially coloured. For $v \in V_\Delta$ which is uncoloured and $X \in \{R, G, B\}$, define the function $d(v, X)$ for any partial colouring of Δ as follows:

- Set $d(v, X) = 1$ if there is an edge between v and a vertex coloured X.
- Set $d(v, X) = 0$ if there is no edge between v and any vertex coloured X.

[14]Note that in particular if η is of Type 2, it is necessarily coloured green.

The following algorithm colours vertices in W with R and B in a compatible fashion.

Step 1: Colour with red (R), all uncoloured vertices $v \in W$ such that $d(v, G) = 1$. If no such vertices exist, colour an arbitrary uncoloured vertex with red (R).
Step 2: The previous step might lead to a situation where two red vertices are connected by an edge. For every such edge $P : \eta - \eta'$, blow up the cold point P (which note, is on two C-curves). As we have already observed, the exceptional curve obtained is again a C-curve and the new marked points are cold points. In the new partial dual graph, this introduces a new vertex (corresponding to the exceptional curve) breaking the edge P into two edges Q_1 and Q_2. Colour this new vertex with blue (B). If all vertices are coloured, terminate.
Step 3: Colour with blue(B), all uncoloured vertices v such that $d(v, R) = 1$. If no such vertices exist, colour an arbitrary uncoloured vertex with blue (B).
Step 4: The previous step might lead to a situation where two blue vertices are connected by an edge. For every such edge $P : \eta - \eta'$, blow up the cold point P (which note, is on two C-curves). As before, in the new partial dual graph, this introduces a new vertex (corresponding to the exceptional curve) breaking the edge P into two edges Q_1 and Q_2. Colour this new vertex with red (R). If all vertices are coloured, terminate.
Step 5: Colour with red (R), all uncoloured vertices $v \in W$ such that $d(v, B) = 1$. If no such vertices exist, colour an arbitrary uncoloured vertex with red (R).
Step 6: Go to Step 2.

Note that in Steps 1, 3 and 5 we colour at least one uncoloured vertex each time. In Steps 2 and 4, though we introduce new vertices, they always correspond to C-curves and we colour them with R or B in the same step. Since $|V_\triangle| < \infty$, the algorithm terminates after finitely many steps. Each partial colouring obtained satisfies Properties 1-4. Hence when the algorithm terminates, we will end up with a compatible colouring of $V_\triangle$. □

6.3.4 AN EXTENDED RAINBOW COLOURING OF Δ

We refine the colouring of Δ by colouring OVER η which are Ch, H or Z-curves as follows: Let $\eta \in V_\triangle$ be a Ch, H or Z-curve and let $a = \left(a'_{i,\eta}\right)_i \in \prod F_\eta$.

- If each $a'_{i,\eta}$ is a unit (up to ℓ^{th} powers) in $\widehat{A_\eta}$, then colour η violet (V) if it is a Ch-curve, indigo (I) if it is a H-curve, and black (Bl) if it is a Z-curve.

- If at least one $a'_{i,\eta}$ is not a unit (up to ℓ^{th} powers) in $\widehat{A_\eta}$, then colour η yellow (Ye) if it is a Ch-curve, orange (O) if it is a H-curve, and white (W) if it is a Z-curve.

Thus we get a nine-colouring of $V_\triangle$ with colours violet (V), indigo (I), blue (B), green (G), yellow (Ye), orange (O), red (R), black (Bl) and white(W).

Type 1b with split Y curves	Ch	H	Z	C
All $a_{i,\eta}$ units	Violet (V)	Indigo (I)	Black (Bl)	Red (R) or Blue (B)
Some $a_{i,\eta}$ not a unit	Yellow (Ye)	Orange (O)	White (W)	Red (R) or Blue (B)

Table 8: Extended rainbow colouring of V_Δ

7 Patching data at marked points in S_0

Let $P \in S_0$ be the intersection of two distinct irreducible curves C_1 and C_2 of $\mathcal{H}_\mathcal{X}$. Let η_1 and η_2 denote the generic points of C_1 and C_2 respectively. Let π_P and δ_P be primes defining C_1 and C_2 at P fixed as in Section 6.2. As before, if $Y_x \simeq \prod F_x$, we let $a = \left(a'_{i,x}\right)_i$, where $a'_{i,x} \in F_x$. We will now prescribe $E_{j,P}$ for $j = 1, 2$ at $P \in S_0$ in accordance with the following heuristic:

- If $\eta \in N'_0$ is of Type 0 or 1a, then both $E_{1,P}$ and $E_{2,P}$ should be unramified along η,
- If $\eta \in N'_0$ is coloured G, V, I or Bl, then both $E_{1,P}$ and $E_{2,P}$ should be unramified along η,
- If $\eta \in N'_0$ is coloured R, O, Ye or W, then $E_{1,P}$ should be ramified along η while $E_{2,P}$ should be unramified along η,
- If $\eta \in N'_0$ is coloured B, then $E_{1,P}$ should be unramified along η while $E_{2,P}$ should be ramified along η.

7.1 Points not of Type A^s_{00}

Proposition 7.1. *Let $P \in S_0$ be such that it is not of Type A^s_{00}. Then for each $j = 1, 2$, there exist cyclic degree ℓ extensions $E_{j,P}/F_P$ and elements $a_{j,P} \in Y_P$ such that*

1. $a_{1,P}a_{2,P} = a$.
2. *$D \otimes E_{j,P}$ has index at most ℓ.*
3. *$D \otimes Y \otimes E_{j,P}$ is split.*
4. *$a_{j,P}$ is a norm from $E_{j,P} \otimes Y_P/Y_P$.*
5. $\mathrm{N}_{Y_P/F_P}(a_{j,P}) = 1$.
6. *Each $E_{j,P}$ is either a split extension or $D \otimes E_{j,P}$ is split.*

Proof. We investigate each type of point separately. In every case, we will determine $E_{1,P}, E_{2,P}$ and $a_{1,P}$ and set $a_{2,P} = aa^{-1}_{1,P}$, thus ensuring that Property 1 holds. Since $\mathrm{N}(a) = 1$, Property 5 will also be satisfied provided $\mathrm{N}(a_{1,P}) = 1$. By ([S97], Proposition 1.2), Property 2 holds for any closed point.

We adopt the following notations in the proof: $u_P, v_P, w_P \in \widehat{A_P}^*$, $0 < r, s, m, j < \ell$. If Y_P is split, by Proposition 3.4, $a = \left(a'_{i,P}\right)$ where $a'_{i,P} = z_{i,P}\pi_P^{m_i}\delta_P^{n_i}$ where $m_i, n_i \in \mathbb{Z}$ and $z_{i,P} \in \widehat{A_P}^*$. Also since $\mathrm{N}(a) = 1$, we have $\prod z_{i,P} = 1$ and $\sum m_i = \sum n_i = 0$. L_P denotes the unique non-split degree ℓ extension of F_P unramified at $\widehat{A_P}$ and H_P, the extension $F_P\left(\sqrt[\ell]{u_P\pi_P^m + v_P\delta_P}\right)$.

TYPE A_{10}^s: Without loss of generality, assume η_1 is of Type 1a and η_2 is of Type 0. Note that by Lemma 5.1, D_P is split. The following choices for $E_{j,P}$ and $a_{j,P}$ satisfy Properties 1-6.

Row	η_1	η_2	$E_{1,P}$	$E_{2,P}$	$a_{1,P}$	$a_{2,P} = aa_{1,P}^{-1}$
0.1	1a	0	$\prod F_P$	$\prod F_P$	a	1

Table 9: Patching data at points of Type A_{10}^s

TYPE B_{10}^s: Without loss of generality, assume η_1 is of Type 1b and η_2 is of Type 0. By Proposition 6.4 and Section 6.3.1, η_1 cannot be a Ch or H-curve and hence isn't coloured V, I, Ye or O. The following table gives the choice for $E_{j,P}$ and $a_{j,P}$.

Row	η_1	$E_{1,P}$	$E_{2,P}$	$a_{1,P}$	$a_{2,P} = aa_{1,P}^{-1}$
1.1	R, W	$F_P\left(\sqrt[\ell]{\pi_P}\right)$	$\prod F_P$	$\left(\pi_P^{m_i}\right)_i$	$\left(z_{i,P}\delta_P^{n_i}\right)_i$
1.2	G, Bl	$\prod F_P$	$\prod F_P$	a	1
1.3	B	$\prod F_P$	$F_P\left(\sqrt[\ell]{\pi_P}\right)$	$\left(z_{i,P}\delta_P^{n_i}\right)_i$	$\left(\pi_P^{m_i}\right)_i$

Table 10: Patching data at points of Type B_{10}^s

Since D_P is itself split, Properties 3 and 6 hold while Property 4 holds by construction. Since $\sum m_i = \sum n_i = 0$ and $\prod z_{i,P} = 1$, we have $\mathrm{N}\left(a_{1,P}\right) = 1$. Hence Property 5 holds.

TYPE B_{11}^{ns}: Without loss of generality, assume η_1 is of Type 1b and η_2 is of Type 1a. Thus $D_P = (u_P, \pi_P) \in \mathrm{Br}\left(F_P\right)$. By Proposition 3.3, a is a unit in the integral closure of $\widehat{A_P}$ in Y_P if the latter is not split. By Proposition 5.3, if Y_P is split, $a'_{i,P} = z_{i,P}\pi_P^{m_i}\delta_P{}^{\ell n'_i}$ with $\sum n'_i = 0$. The following table[15] gives the choice for $E_{j,P}$ and $a_{j,P}$.

[15]Row 2.2* is a special case when η_1 is Type 1b and green with Y_{η_1} of Type RAM and η_2 of Type 1a with Y_{η_2} of Type NONRES. In this situation, we choose $E_{1,P} = E_{2,P} = \prod F_P$ while $a_{1,P} = a$ and $a_{2,P} = 1$.

Row	η_1	η_2	$E_{1,P}$	$E_{2,P}$	$a_{1,P}$	$a_{2,P} = aa_{1,P}^{-1}$
2.1	W, R, O, Ye		$F_P\left(\sqrt[\ell]{\pi_P}\right)$	L_P	$\left(\pi_P^{m_i}\right)_i$	$\left(z_{i,P}\delta_P{}^{\ell n'_i}\right)_i$
2.2*	G(RAM)	(NONRES)	$\prod F_P$	$\prod F_P$	a	1
2.2	Bl, I, G, V		L_P	L_P	a	1
2.3	B		L_P	$F_P\left(\sqrt[\ell]{\pi_P}\right)$	$\left(z_{i,P}\delta_P{}^{\ell n'_i}\right)_i$	$\left(\pi_P^{m_i}\right)_i$

Table 11: Patching data at points of Type B_{11}^{ns}

Since $D_P = (u_P, \pi_P)$ and u_P becomes an ℓ^{th} power in L_P, $E_{j,P}$ splits D in each case except Row 2.2*. In this row however, $Y_P = F_P\left(\sqrt[\ell]{w_P\pi_P}\right)$ and hence $D \otimes Y_P$ is split. Thus Properties 3 and 6 hold. By Lemmata 2.4 and 2.6, Property 4 holds. Since $\sum m_i = \sum n'_i = 0$ and $\prod z_{i,P} = 1$, we have $\mathrm{N}\left(a_{1,P}\right) = 1$. Hence Property 5 holds.

Type B_{20}^s: Without loss of generality, assume η_1 is Type 2 and η_2 is Type 0. Thus η_1 is coloured G. The following choices satisfy Properties 1-6.

Row	η_1	$E_{1,P}$	$E_{2,P}$	$a_{1,P}$	$a_{2,P} = aa_{1,P}^{-1}$
3.1	G	$\prod F_P$	$\prod F_P$	a	1

Table 12: Patching data at points of Type B_{20}^s

Type B_{21}^{ns}: Without loss of generality, assume η_1 is Type 2 and η_2 is Type 1a. Thus $D_P = (u_P, \pi_P) \in \mathrm{Br}\left(F_P\right)$. By Proposition 3.3, a is a unit in the integral closure of $\widehat{A_P}$ in Y_P if the latter is not split. By Proposition 5.3, if Y_P is split, $a'_{i,P} = z_{i,P}\delta_P{}^{\ell n'_i}$ with $\sum n'_i = 0$. The following table[16] gives the choice for $E_{j,P}$ and $a_{j,P}$.

Row	η_1	η_2	$E_{1,P}$	$E_{2,P}$	$a_{1,P}$	$a_{2,P} = aa_{1,P}^{-1}$
4.1*	G(RAM)	(NONRES)	$\prod F_P$	$\prod F_P$	a	1
4.1	G		L_P	L_P	a	1

Table 13: Patching data at points of Type B_{21}^{ns}

Since $D_P = (u_P, \pi_P)$ and u_P becomes an ℓ^{th} power in L_P, $E_{j,P}$ splits D in Row 4.1. In Row 4.1*, $Y_P = F_P\left(\sqrt[\ell]{w_P\pi_P}\right)$ and hence $D \otimes Y_P$ is split. Thus Properties 3 and 6 hold. By Lemmata 2.4 and 2.6, Property 4 holds. Since $\mathrm{N}(a) = 1$, so does Property 5.

Type C_{11}^{Chilly} : We assume that $D = D_{00} + (u_P, \pi_P) + (v_P, \delta_P) \in \mathrm{Br}(F)$ where D_{00} is unramified at A_P and $D_P = \left(v_P, \pi_P^j\delta_P\right)$ where $0 < j < \ell$. By Proposition

[16]Row 4.1* is a special case when η_1 is Type 2 with Y_{η_1} of Type RAM and η_2 of Type 1a with Y_{η_2} of Type NONRES. In this situation, we choose $E_{1,P} = E_{2,P} = \prod F_P$ while $a_{1,P} = a$ and $a_{2,P} = 1$.

5.3, if Y_P is split, then $a'_{i,P} = z_{i,P}\left(\pi_P^j\delta_P\right)^{n_i}\left(\pi_P^{r_i\ell}\right)$ where $m_i = jn_i + r_i\ell$ and hence $\sum r_i = 0$. In particular, a is a unit (up to ℓ^{th} powers) in $\widehat{A_{\eta_1}}^*$ if and only if it is is a unit (up to ℓ^{th} powers) in $\widehat{A_{\eta_2}}^*$.

Let $j = 1, 2$. Recall that if η_j is not a Ch-curve, then it is coloured G. The above discussion implies that if η_1 and η_2 are both Ch-curves, then they are both coloured Ye or both coloured V. Similarly if η_1 is coloured G and η_2 is a Ch-curve, then η_2 is coloured V. Likewise if η_2 is coloured G and η_1 is a Ch-curve, then η_1 is coloured V. Invoking Proposition 5.6, Table 13 below prescribes $E_{j,P}$ depending on the configuration of a, Y, η_1, η_2.

Since $D_P = \left(v_P, \pi_P^j\delta_P\right)$ and v_P becomes an ℓ^{th} power in L_P, $E_{j,P}$ splits D in each case. Thus Properties 3 and 6 hold. By Lemma 2.6, Property 4 holds for Rows 5.1-5.4, 5.8-5.9, 5.11 (when Y_P is a field). By Lemma 2.4, Property 4 holds in the remaining cases (except for $(E_{1,P}, a_{1,P})$ in Row 5.6, where it is clear by observation). Finally since $\mathrm{N}(a) = 1$ and for Row 5.6, $\sum n_i = \sum r_i = 0$ and $\prod z_{i,P} = 1$, $\mathrm{N}(a_{1,P}) = 1$ for all rows. Hence Property 5 holds.

Row	η_1	η_2	Y_{η_1}	Y_{η_2}	Y_P	$E_{1,P}$	$E_{2,P}$	$a_{1,P}$	$a_{2,P}$
5.1	G	G	RAM	RAM	$F_P\left(\sqrt[\ell]{w_P\pi_P^r\delta_P}\right)$	L_P	L_P	a	1
5.2	G	G	NONRES	RAM	$F_P\left(\sqrt[\ell]{w_P\delta_P}\right)$	L_P	L_P	a	1
5.3	G	G	RES	RES	L_P	L_P	L_P	a	1
5.4	G	G	NONRES	RES	L_P	L_P	L_P	a	1
5.5	V	V	SPLIT	SPLIT	$\prod F_P$	L_P	L_P	a	1
5.6	Ye	Ye	SPLIT	SPLIT	$\prod F_P$	$F_P\left(\sqrt[\ell]{\pi_P^j\delta_P}\right)$	L_P	$\left(\left(\pi_P^j\delta_P\right)^{n_i}\left(\pi_P^{r_i\ell}\right)\right)_i$	$(z_{i,P})_i$
5.7	G	V	NONRES	SPLIT	$\prod F_P$	L_P	L_P	a	1
5.8	G	G	RAM	NONRES	$F_P\left(\sqrt[\ell]{w_P\pi_P}\right)$	L_P	L_P	a	1
5.9	G	G	RES	NONRES	L_P	L_P	L_P	a	1
5.10	V	G	SPLIT	NONRES	$\prod F_P$	L_P	L_P	a	1
5.11	G	G	NONRES	NONRES	$L_P/\prod F_P$	L_P	L_P	a	1

Table 14: Patching data at points of Type C_{11}^{Chilly}

COLD POINTS: We assume that $D = D_{00} + (u_P\pi_P^m, v_P\delta_P) \in \mathrm{Br}(F)$ where D_{00} is unramified at A_P and $D_P = (u_P\pi_P^m, v_P\delta_P)$ where $0 < m < \ell$. By Proposition 5.3, if Y_P is split, then $a'_{i,P} = (u_P\pi_P^m)^{sm_i}(v_P\delta_P)^{n_i}\left(w'_{i,P}\pi_P^{-rm_i}\right)^{\ell}$ where $sm = r\ell + 1$ and $w'_{i,P} \in \widehat{A_P}^*$ with ${w'_{i,P}}^{\ell}u_P^{sm_i}v_P^{n_i} = z_{i,P}$. Set $x_{i,P} = \left(w'_{i,P}\pi_P^{-rm_i}\right)$. Since $\sum m_i = \sum n_i = 0$ and $\prod a'_{i,P} = 1$, clearly $\prod x_{i,P}^{\ell} = 1$.

TYPE C_{11}^{Cold}: Let $j = 1, 2$. If Y_{η_j} is of Type SPLIT, then since P is a cold point lying on it, η_j must be a C-curve. Thus it is coloured R or B. If η_1 is coloured G and η_2 is a C-curve, then by Proposition 6.5, η_2 will be coloured R. Similarly, if η_2 is coloured G and η_1 is a C-curve, then η_1 will be coloured R. Finally if both η_1 and η_2 are C-curves, then both of them cannot be of the same colour. Invoking Proposition 5.4, we prescribe the choices for $E_{j,P}$ and $a_{j,P}$ in the following table:

Row	η_1	η_2	Y_{η_1}	Y_{η_2}	Y_P	$E_{1,P}$	$E_{2,P}$	$a_{1,P}$	$a_{2,P}$
6.1	G	G	RAM	RAM	$F_P\left(\sqrt[\ell]{w_P\pi_P^r\delta_P}\right)$	H_P	H_P	a	1
6.2	G	G	RES	RAM	$F_P\left(\sqrt[\ell]{v_P\delta_P}\right)$	H_P	H_P	a	1
6.3	G	G	NONRES	RAM	$F_P\left(\sqrt[\ell]{w_P\delta_P}\right)$	H_P	H_P	a	1
6.4	G	G	RAM	RES	$F_P\left(\sqrt[\ell]{u_P\pi_P^m}\right)$	H_P	H_P	a	1
6.5	R	B	SPLIT	SPLIT	$\prod F_P$	$F_P\left(\sqrt[\ell]{u_P\pi_P^m}\right)$	$F_P\left(\sqrt[\ell]{v_P\delta_P}\right)$	$\left((u_P\pi_P^m)^{sm_i}(x_{i,P})^\ell\right)_i$	$((v_P\delta_P)^{n_i})_i$
6.6	B	R	SPLIT	SPLIT	$\prod F_P$	$F_P\left(\sqrt[\ell]{v_P\delta_P}\right)$	$F_P\left(\sqrt[\ell]{u_P\pi_P^m}\right)$	$((v_P\delta_P)^{n_i})_i$	$\left((u_P\pi_P^m)^{sm_i}(x_{i,P})^\ell\right)_i$
6.7	G	R	NONRES	SPLIT	$\prod F_P$	$F_P\left(\sqrt[\ell]{v_P\delta_P}\right)$	H_P	a	1
6.8	G	G	RAM	NONRES	$F_P\left(\sqrt[\ell]{w_P\pi_P}\right)$	H_P	H_P	a	1
6.9	R	G	SPLIT	NONRES	$\prod F_P$	$F_P\left(\sqrt[\ell]{u_P\pi_P^m}\right)$	H_P	a	1
6.10	G	G	NONRES	NONRES	$L_P / \prod F_P$	H_P	H_P	a	1

Table 15: Patching data at points of Type C_{11}^{Cold}

Since $D_P = (u_P\pi_P^m, v_P\delta_P)$, clearly $F_P\left(\sqrt[\ell]{u_P\pi_P^m}\right)$ and $F_P\left(\sqrt[\ell]{v_P\delta_P}\right)$ split it. Since the symbol algebra $(x, y) = (x + y, -yx^{-1})$, so does H_P. Thus Properties 3 and 6 hold. By Lemma 2.6, Property 4 holds for Rows 6.1-6.4, 6.8 (and for Row 6.10, if $Y_P = L_P$). By construction it also holds for Rows 6.5-6.6. In Row 6.7, a is a unit along η_1. Thus $a'_{i,P} = (v_P \delta_P)^{n_i} w^{\ell}_{i,P}$ which is a norm from $F_P\left(\sqrt[\ell]{v_P \delta_P}\right)$. A similar argument works for Row 6.9. In Row 6.10, if $Y_P = \prod F_P$, then since a is a unit along both η_1 and η_2, we have that $a'_{i,P} = w^{\ell}_{i,P}$. So each $a'_{i,P}$ is a norm from $E_{1,P}$. Thus Property 4 holds for C^{Cold}_{11} points. Finally since $\mathrm{N}(a) = 1$ and for Rows 6.6-6.7, $\sum m_i = \sum n_i = 0$ and $\prod x^{\ell}_{i,P} = 1$, we have $\mathrm{N}(a_{1,P}) = 1$ for all rows. Hence Property 5 holds.

Type C^{Cold}_{12} : Without loss of generality, assume η_2 is of Type 2. Hence it is coloured G. If Y_{η_1} is of Type SPLIT, then since P is a cold point lying on it, η_1 must be a C-curve. Thus it is coloured R or B. Since η_2 is coloured G, then by Proposition 6.5, η_1 will be coloured R in this case. Invoking Proposition 5.5, we prescribe the choices for $E_{j,P}$ and $a_{j,P}$ in the following table. The proof that Properties 1-6 hold is exactly similar to the Type C^{Cold}_{11} point case.

Row	η_1	η_2	Y_{η_1}	Y_{η_2}	Y_P	$E_{1,P}$	$E_{2,P}$	$a_{1,P}$	$a_{2,P} = aa^{-1}_{1,P}$
7.1	G	G	RAM	RAM	$F_P\left(\sqrt[\ell]{w_P \pi_P \delta^r_P}\right)$	H_P	H_P	a	1
7.2	G	G	RES	RAM	$F_P\left(\sqrt[\ell]{v_P \delta_P}\right)$	H_P	H_P	a	1
7.3	G	G	NONRES	RAM	$F_P\left(\sqrt[\ell]{w_P \delta_P}\right)$	H_P	H_P	a	1
7.4	G	G	RAM	RES	$F_P\left(\sqrt[\ell]{u_P \pi^m_P}\right)$	H_P	H_P	a	1
7.5	G	G	RAM	NONRES	$F_P\left(\sqrt[\ell]{w_P \pi_P}\right)$	H_P	H_P	a	1
7.6	R	G	SPLIT	NONRES	$\prod F_P$	$F_P\left(\sqrt[\ell]{u_P \pi^m_P}\right)$	H_P	a	1
7.7	G	G	NONRES	NONRES	$L_P / \prod F_P$	H_P	H_P	a	1

Table 16: Patching data at points of Type C^{Cold}_{12}

Type C^{Hot}_{12} : Without loss of generality, assume η_2 is of Type 2 and coloured G. If Y_{η_1} is of Type SPLIT, then since P is a hot point lying on it, η_1 must be a H-curve and coloured I or O. By Proposition 5.3, if Y_P is split, then $a'_{i,P} = z_{i,P} \pi^{m_i}_P \delta^{\ell n'_i}_P$ where $\sum n'_i = 0$. Invoking the table in Proposition 6.3, we prescribe the choices for $E_{j,P}$ and $a_{j,P}$ in the following table:

Row	η_1	η_2	Y_{η_1}	Y_{η_2}	Y_P	$E_{1,P}$	$E_{2,P}$	$a_{1,P}$	$a_{2,P} = aa^{-1}_{1,P}$
8.1	G	G	RAM	RAM	$F_P\left(\sqrt[\ell]{w_P \pi^r_P \delta_P}\right)$	L_P	L_P	a	1
8.2	G	G	NONRES	RAM	$F_P\left(\sqrt[\ell]{w_P \delta_P}\right)$	L_P	L_P	a	1
8.3	I	G	SPLIT	RES	$\prod F_P$	L_P	L_P	a	1
8.4	O	G	SPLIT	RES	$\prod F_P$	$F_P\left(\sqrt[\ell]{\pi_P}\right)$	L_P	$\left(\pi^{m_i}_P\right)_i$	$\left(z_{i,P} \delta^{\ell n'_i}_P\right)_i$
8.5	G	G	RAM	NONRES	$F_P\left(\sqrt[\ell]{w_P \pi_P}\right)$	$\prod F_P$	$\prod F_P$	a	1
8.6	G	G	RES	NONRES	L_P	$\prod F_P$	$\prod F_P$	a	1

Table 17: Patching data at points of Type C^{Hot}_{12}

Since $D_P = (u_P, \pi_P)$, clearly $F_P\left(\sqrt[\ell]{\pi_P}\right)$ and L_P splits it. Thus Properties 3 and 6 hold for Rows 8.1-8.4. For Rows 8.5-8.6, we observe that $D \otimes Y_P = 0$ and $E_{j,P}$ are split. Thus Properties 3 and 6 hold for all cases. By Lemma 2.6, Property 4 holds for Rows 8.1-8.2 and 8.5-8.6. In Row 8.3, the colours of η_1 and η_2 imply that a is a unit along η_2 and η_1 (up to ℓ^{th} powers). Hence each $a'_{i,P} = z_{i,P}\pi_P^{\ell m'_i}$ where $m_i = \ell m'_i$ is a unit in $\widehat{A_P}^*$ up to ℓ^{th} powers. Thus by Lemma 2.4, Property 4 holds here. For Row 8.4, clearly $\pi_P^{m_i}$ is a norm from $F_P\left(\sqrt[\ell]{\pi_P}\right)$. Appealing again to Lemma 2.4, we see that $z_{i,P}\delta_P^{\ell n'_i}$ are norms from L_P. Finally Property 5 holds because $\mathrm{N}(a) = 1$ and $\sum m_i = \sum n'_i = 0$. □

7.2 Points of Type A_{00}^s

Let P be of Type A_{00}^s. Thus η_1 and η_2 are both of Type 0. For $j = 1, 2$, let $\mathcal{C}_j := (\overline{\eta_j} \cap S_0) \setminus \{P\}$ denote the set of marked points on η_j apart from P. By Proposition 6.4, it is clear that if $Q_j \in \mathcal{C}_j$ is not of Type A_{00}^s, then $\mathcal{C}_j = \{Q_j\}$. In such a case, let γ_j denote the Type 1a/1b/2 curve such that $Q_j \in \overline{\eta_j} \cap \overline{\gamma_j} \cap S_0$. Note γ_j can only be coloured R, G, B, Bl, or W. We subdivide Type A_{00}^s points into three sub-types:
D1: $\mathcal{C}_j = \{Q_j\}$ where Q_j is not of Type A_{00}^s for $j = 1, 2$.
D2: $\mathcal{C}_j = \{Q_j\}$ where Q_j is not of Type A_{00}^s and $\mathcal{C}_{j'}$ is either empty or consists only of Type A_{00}^s points for $\{j, j'\} = \{1, 2\}$.
D3: $\mathcal{C}_j$ is either empty or consists only of Type A_{00}^s points for $j = 1, 2$.

Proposition 7.2. *Let $P \in S_0$ be such that it is of Type A_{00}^s. Set $E_{1,P} = E_{2,P} = \prod F_P$. Then there exist $a_{1,P}$, $a_{2,P} \in Y_P$ such that for $j = 1, 2$,*

1. $a_{1,P}a_{2,P} = a$.
2. $\mathrm{N}_{Y_P/F_P}(a_{j,P}) = 1$.
3. *$a_{j,P}$ is a norm from $E_{j,P} \otimes Y_P/Y_P$.*

Proof. Note that since we have chosen the split extension for each $E_{j,P}$, Property 3 holds for any choice of $a_{j,P}$. By Remark 5.2, note that if Y_{η_1} is of Type RAM, then Y_{η_2} cannot be of Type SPLIT and vice-versa. For the same reason, if γ_j is coloured red/blue/white/black, then Y_{η_j} cannot be of Type RAM. Finally if Y_{η_j} is of Type RAM, then by Proposition 3.3 and Lemmata 2.3 and 2.6, $a \in \mathcal{O}_{Y_{\eta_j}}^{*\ell}$ and $\mathcal{O}_{Y_P}^{*\ell}$. We prescribe $a_{j,P}$ as in the tables below[17] depending on the subtype and neighbours of P.
Subtype D3: Let P be of subtype D3. Thus $\mathcal{C}_j$ is empty or consists only of Type A_{00}^s points for $j = 1, 2$.

Row	$a_{1,P}$	$a_{2,P} = aa_{1,P}^{-1}$
11.1	a	1

Table 18: Patching data at points of subtype D3

[17] If Y_P is not split, set $m_i = n_i = 0$ and read the entry $\left(\pi_P^{m_i}\right)_i$ as 1 and $\left(z_{i,P}\delta_P^{n_i}\right)_i$ as a etc.

SUBTYPE D2: Let P be of subtype D2. Without loss of generality assume $\mathcal{C}_1 = \{Q_1\}$ where Q_1 is not of Type A^s_{00} and that $\mathcal{C}_2$ is empty or consists only of Type A^s_{00} points.

Row	Y_{η_1}	Y_{η_2}	Q_1	γ_1	Colour of γ_1	$a_{1,P}$	$a_{2,P} = aa_{1,P}^{-1}$
10.1	–	–	A^s_{10}	$1a$		a	1
10.2′	–	RAM	B^s_{10}	$1b$	R, W	a	1
10.2	–	Not RAM	B^s_{10}	$1b$	R, W	$(\delta_P^{n_i})_i$	$(z_{i,P}\pi_P^{m_i})_i$
10.3	–	–	B^s_{10}	$1b$	G, B, Bl	a	1
10.4	–	–	B^s_{20}	2	G	a	1

Table 19: Patching data at points of subtype D2

SUBTYPE D1: Let P be of subtype D1. For $j = 1, 2$, let $\mathcal{C}_j = \{Q_j\}$ where Q_j is not of Type A^s_{00}.

Row	Y_{η_1}	Y_{η_2}	Q_1	Q_2	γ_1	Colour of γ_1	γ_2	Colour of γ_2	$a_{1,P}$	$a_{2,P} = aa_{1,P}^{-1}$
9.1	–	–	A^s_{10}	A^s_{10}	$1a$		$1a$		a	1
9.2′	RAM	–	A^s_{10}	B^s_{10}	$1a$		$1b$	R, W	a	1
9.2	Not RAM	–	A^s_{10}	B^s_{10}	$1a$		$1b$	R, W	$(\pi_P^{m_i})_i$	$(z_{i,P}\delta_P^{n_i})_i$
9.3	–	–	A^s_{10}	B^s_{10}	$1a$		$1b$	G, B, Bl	a	1
9.4	–	–	A^s_{10}	B^s_{20}	$1a$		2	G	a	1
9.5′	–	RAM	B^s_{10}	A^s_{10}	$1b$	R, W	$1a$		a	1
9.5	–	Not RAM	B^s_{10}	A^s_{10}	$1b$	R, W	$1a$		$(\delta_P^{n_i})_i$	$(z_{i,P}\pi_P^{m_i})_i$
9.6	–	–	B^s_{10}	B^s_{10}	$1b$	R, W	$1b$	R, W	1	a
9.7′	–	RAM	B^s_{10}	B^s_{10}	$1b$	R, W	$1b$	G	a	1
9.7	–	Not RAM	B^s_{10}	B^s_{10}	$1b$	R, W	$1b$	G, B, Bl	$(\delta_P^{n_i})_i$	$(z_{i,P}\pi_P^{m_i})_i$
9.8′	–	RAM	B^s_{10}	B^s_{20}	$1b$	R, W	2	G	a	1
9.8	–	Not RAM	B^s_{10}	B^s_{20}	$1b$	R, W	2	G	$(\delta_P^{n_i})_i$	$(z_{i,P}\pi_P^{m_i})_i$
9.9	–	–	B^s_{10}	A^s_{10}	$1b$	G, B, Bl	$1a$		a	1
9.10′	RAM	–	B^s_{10}	B^s_{10}	$1b$	G	$1b$	R, W	a	1
9.10	Not RAM	–	B^s_{10}	B^s_{10}	$1b$	G, B, Bl	$1b$	R, W	$(\pi_P^{m_i})_i$	$(z_{i,P}\delta_P^{n_i})_i$
9.11	–	–	B^s_{10}	B^s_{10}	$1b$	G, B, Bl	$1b$	G, B, Bl	a	1
9.12	–	–	B^s_{10}	B^s_{20}	$1b$	G, B, Bl	2	G	a	1
9.13	–	–	B^s_{20}	A^s_{10}	2	G	$1a$		a	1
9.14′	RAM	–	B^s_{20}	B^s_{10}	2	G	$1b$	R, W	a	1
9.14	Not RAM	–	B^s_{20}	B^s_{10}	2	G	$1b$	R, W	$(\pi_P^{m_i})_i$	$(z_{i,P}\delta_P^{n_i})_i$
9.15	–	–	B^s_{20}	B^s_{10}	2	G	$1b$	G, B, Bl	a	1
9.16	–	–	B^s_{20}	B^s_{20}	2	G	2	G	a	1

Table 20: Patching data at points of subtype D1

□

8 Structure of $E_{j,P}$ and $a_{j,P}$ along branch fields

Let $P \in S_0$. Recall the choice of parameter, π_η, of F_η for each $\eta \in N_0'$ as in Section 6.2, which defines $\overline{\eta}$ at P if $P \in \overline{\eta}$. In this case, $\pi_P := \pi_\eta$ is part of the chosen system of parameters (π_P, δ_P) of A_P. In this section, we study the ramification and splitting properties of $E_{j,P}$ and the shape of $a_{j,P}$ for $j = 1, 2$ with respect to the colour and type of curves on which P lies. This will be useful when we construct extensions $E_{j,\eta}$ and elements $a_{j,\eta}$ for codimension one points $\eta \in N_0$.
We first begin by calculating how the lift of residues looks like along the residue fields $k_{P,\eta}$ of branch fields $F_{P,\eta}$.

Lemma 8.1. *Let $P \in S_0$ lie on the intersection of two distinct irreducible curves of $\mathcal{H}_{\mathcal{X}}$ with generic points η_1 and η_2. Let (π_P, δ_P) be the system of parameters at P chosen as in Section 6.2 such that π_P cuts out $\overline{\eta_1}$ and δ_P cuts out $\overline{\eta_2}$ at P. Let $\eta = \eta_1$ or η_2 and let H_η denote the lift of residues along η. Set $H_P' = \overline{H_\eta \otimes F_{P,\eta}}/k_{P,\eta}$ and $u' \in k_{P,\eta}^*/k_{P,\eta}^{*\ell}$ to be the residue of D over $F_{P,\eta}$. Then the following table gives the shape of $H_P'/k_{P,\eta}$ and $u' \in k_{P,\eta}$.*

Row	Location	P	$D_P \in \mathrm{Br}(F_P)$	(η, Type)	u'	$H_P'/k_{P,\eta}$	Description of H_P'
$a.$	Table 10	B_{10}^{s}	0	$(\eta_1, 1b)$	1	$\prod k_{P,\eta}$	Split
$b.$	Table 11	B_{11}^{ns}	(u_P, π_P)	$(\eta_1, 1b)$	$\overline{u_P}$	$k_{P,\eta}\left(\sqrt[\ell]{\overline{u_P}}\right)$	Unramified nonsplit
$c.$	Table 12	B_{20}^{s}	0	$(\eta_1, 2)$	1	$\prod k_{P,\eta}$	Split
$d.$	Table 13	B_{21}^{ns}	(u_P, π_P)	$(\eta_1, 2)$	$\overline{u_P}$	$k_{P,\eta}\left(\sqrt[\ell]{\overline{u_P}}\right)$	Unramified nonsplit
$e.$	Table 14	C_{11}^{Chilly}	$\left(v_P, \pi_P^j \delta_P\right)$	$(\eta_1, 1b)$	v_P^j	$k_{P,\eta}\left(\sqrt[\ell]{\overline{v_P}}\right)$	Unramified nonsplit
$f.$	Table 14	C_{11}^{Chilly}	$\left(v_P, \pi_P^j \delta_P\right)$	$(\eta_2, 1b)$	$\overline{v_P}$	$k_{P,\eta}\left(\sqrt[\ell]{\overline{v_P}}\right)$	Unramified nonsplit
$g.$	Table 15	C_{11}^{Cold}	$(u_P\pi_P^m, v_P\delta_P)$	$(\eta_1, 1b)$	$\overline{v_P^{-m}\delta_P^{-m}}$	$k_{P,\eta}\left(\sqrt[\ell]{\overline{v_P\delta_P}}\right)$	Ramified nonsplit
$h.$	Table 15	C_{11}^{Cold}	$(u_P\pi_P^m, v_P\delta_P)$	$(\eta_2, 1b)$	$\overline{u_P\pi_P^m}$	$k_{P,\eta}\left(\sqrt[\ell]{\overline{u_P\pi_P^m}}\right)$	Ramified nonsplit
$i.$	Table 16	C_{12}^{Cold}	$(u_P\pi_P^m, v_P\delta_P)$	$(\eta_1, 1b)$	$\overline{v_P^{-m}\delta_P^{-m}}$	$k_{P,\eta}\left(\sqrt[\ell]{\overline{v_P\delta_P}}\right)$	Ramified nonsplit
$j.$	Table 16	C_{12}^{Cold}	$(u_P\pi_P^m, v_P\delta_P)$	$(\eta_2, 2)$	$\overline{u_P\pi_P^m}$	$k_{P,\eta}\left(\sqrt[\ell]{\overline{u_P\pi_P^m}}\right)$	Ramified nonsplit
$l.$	Table 17	C_{12}^{Hot}	(u_P, π_P)	$(\eta_1, 1b)$	$\overline{u_P}$	$k_{P,\eta}\left(\sqrt[\ell]{\overline{u_P}}\right)$	Unramified nonsplit
$m.$	Table 17	C_{12}^{Hot}	(u_P, π_P)	$(\eta_2, 2)$	1	$\prod k_{P,\eta}$	Split

Table 21: The shape of the lift of residues

In the following, we let $\pi_\eta = \pi_P$ be the prime defining η at P and let δ_P denote the other prime completing the system of parameters at P. We also let L_P denote the the unique degree ℓ extension of F_P unramified at $\widehat{A_P}$.

Proposition 8.2 (Violet/Indigo/Black). *Let $\eta \in N_0'$ and $P \in \overline{\eta} \cap S_0$. Assume further that η is coloured violet, indigo or black. Let $j = 1$ or 2 and let $E_{j,P}$ be as prescribed in Proposition 7.1. Then*

1. *$E_{j,P} = L_P$ if η is coloured violet or indigo.*
2. *$a_{1,P} = a$ and $a_{2,P} = 1$.*

3. *$E_{j,P} \otimes F_{P,\eta}$ is an unramified extension of $F_{P,\eta}$ and matches with the lift of residues at η as etale algebras over $F_{P,\eta}$.*

4. *$E_{j,P} \otimes F_{P,\eta}$ splits D in $\mathrm{Br}\,(F_{P,\eta})$.*

Proof. An inspection of Row 1.2, 2.2, 5.5, 5.7, 5.10 and 8.3 of the tables in Proposition 7.1 immediately shows that Properties 1-4 hold (Lemma 8.1). ☐

PROPOSITION 8.3 (Blue). *Let $\eta \in N_0'$ and $P \in \overline{\eta} \cap S_0$. Assume further that η is coloured blue. Let $D_\eta \simeq M_\ell\,(u_\eta, w_\eta \pi_\eta)$ for units $w_\eta, u_\eta \in \widehat{A_\eta}^*$. Let $j = 1$ or 2 and let $E_{j,P}$ be as prescribed in Proposition 7.1. Then*

1. *$E_{1,P} \otimes F_{P,\eta}$ is an unramified extension of $F_{P,\eta}$ and matches with the lift of residues at η as etale algebras over $F_{P,\eta}$.*

2. *$E_{2,P} \otimes F_{P,\eta}$ is a ramified extension of $F_{P,\eta}$.*

3. *$a_{1,P}$ is a unit along η.*

4. *$E_{j,P} \otimes F_{P,\eta}$ splits D in $\mathrm{Br}\,(F_{P,\eta})$.*

5. *There exist $w_P, x_{i,P} \in \widehat{A_P}$ for $i \leq \ell$ which are units along η such that*

   (a) $E_{2,P} \simeq F_P[t]/(t^\ell - w_P\pi_P)$ and $a_{2,P} = \left((w_P\pi_P)^{m_{i,P}}\, x_{i,P}^\ell\right)_i$ for $m_{i,P} \in \mathbb{Z}$.

   (b) $\left(w_P w_\eta^{-1}, u_\eta\right) = 0 \in \mathrm{Br}\,(F_{P,\eta})$.

Proof. Since η is coloured blue, it has to be Type 1b C-curve. Thus P can either be a point of Type B_{10}^s, B_{11}^{ns} or C_{11}^{Cold}. (It cannot be a C_{12}^{Cold} point because then the other curve will be of Type 2 and hence green in colour. And therefore η would have to be red). We mention the relevant rows of the tables in Proposition 7.1 below (whence Properties 1-4 become clear) and give a proof of Property 5a & 5b in each case.
Row 1.3 of Table 10: Here $w_P = 1 = x_{i,P}$. Since P is a B_{10}^s point, by Lemma 8.1, $u_\eta \in F_{P,\eta}^{*\ell}$ and hence Property 5a & 5b is satisfied.
Row 2.3 of Table 11: Here $w_P = 1 = x_{i,P}$. Since P is a B_{11}^{ns} point, by Lemma 8.1, the lift of residues along η matches with L_P along $F_{P,\eta}$. Writing $D = D_{00} + (u_P, \pi_P) \in \mathrm{Br}(F)$ where D_00 is unramified at A_P, we have $(u_P, \pi_P) = (u_\eta, w_\eta\pi_\eta) \in \mathrm{Br}\,(F_{P,\eta})$. Since $\pi_P = \pi_\eta$, comparing residues we have that $u_P \cong u_\eta$ up to ℓ^{th} powers in $F_{P,\eta}$, and hence $(u_\eta, w_\eta) = 0$. As $w_P = 1$ here, Property 5b is proved.
Rows 6.5 and 6.6 of Table 15: We only investigate Row 6.6 (as the proof for Row 6.5 is similar in nature). Since P is a C_{11}^{Cold} point, by Lemma 8.1, the lift of residues along η_1 matches with $F_P\left(\sqrt[\ell]{v_P\delta_P}\right)$ along F_{P,η_1}.
Unravelling the expression for $a_{2,P}$ from Row 6.6, we see it is $\left((u_P^s\pi_P)^{m_i}\, {w'_{i,P}}^\ell\right)_i$ where $sm = r\ell + 1$ and $w'_{i,P} \in \widehat{A_P}^*$. Also $E_{2,P} = F_P\left(\sqrt[\ell]{u_P\pi_P^m}\right) = F_P\left(\sqrt[\ell]{u_P^s\pi_P}\right)$. Thus $w_P = u_P^s$ and $x_{i,P} = w'_{i,P}$. Writing $D = D_{00} + (u_P\pi_P^m, v_P\delta_P) \in \mathrm{Br}(F)$ for D_00 unramified at A_P, we have $(u_P\pi_P^m, v_P\delta_P) = (u_\eta, w_\eta\pi_\eta)$ in $\mathrm{Br}\,(F_{P,\eta})$.

As $\pi_\eta = \pi_P$, comparing residues as before, we have that $(v_P\delta_P)^{-m} \cong u_\eta$ up to ℓ^{th} powers in $F_{P,\eta}$.
Hence $(u_P\pi_P^m, v_P\delta_P) = \left(w_\eta^m\pi_\eta^m, v_P\delta_P\right)$. This implies $(u_P, v_P\delta_P) = \left(w_\eta^m, v_P\delta_P\right)$. Hence $\left(u_P w_\eta^{-m}, v_P\delta_P\right) = 0$ and so $\left(u_P^s w_\eta^{-1}, (v_P\delta_P)^m\right) = 0$. Thus $\left(w_P w_\eta^{-1}, u_\eta\right) = 0$. □

Proposition 8.4 (Green(1)). *Let $\eta \in N_0'$ be of Type 1b or 2 and let $P \in \overline{\eta} \cap S_0$. Assume further that Y_η is of Type NONRES. Let $j = 1$ or 2 and let $E_{j,P}$ be as prescribed in Proposition 7.1. Then*

1. *η is coloured green.*
2. *$a_{1,P} = a$ and $a_{2,P} = 1$.*
3. *$E_{j,\eta} \otimes F_{P,\eta}$ is an unramified extension of $F_{P,\eta}$ and matches with the lift of residues at η as etale algebras over $F_{P,\eta}$.*

Proof. Property 1 is obvious (Section 6.3.3). An inspection of Rows 1.2, 2.2, 3.1, 4.1, 5.2, 5.4, 5.7-5.11, 6.3, 6.7-6.10, 7.3, 7.5-7.7, 8.2 and 8.5-8.6 of the Tables in Proposition 7.1 shows that Properties 2 and 3 also hold (Lemma 8.1). □

Proposition 8.5 (Green(2)). *Let $\eta \in N_0'$ and $P \in \overline{\eta} \cap S_0$. Assume further that one of the following holds:*

1. *η is of Type 1b and coloured green and Y_η is not of Type NONRES,*
2. *η is of Type 2 (and hence coloured green) and Y_η is not of Type NONRES.*

Let $j = 1$ or 2 and let $E_{j,P}$ be as prescribed in Proposition 7.1. Then,

1. *$E_{j,P} \otimes F_{P,\eta}$ is an unramified (possibly split) extension of $F_{P,\eta}$.*
2. *$E_{j,P} \otimes \beta_{rbc,\eta} = 0$ where $\beta_{rbc,\eta}$ is as defined in Section 4.*
3. *If P is not a hot point, $a_{1,P} = a$ and $a_{2,P} = 1$.*
4. *If P is a hot point and Y_P is not split, then $a_{1,P} = a$ and $a_{2,P} = 1$.*
5. *If P is a hot point and Y_P is split, then $a_{2,P}$ is a unit in $\prod \widehat{A_P}$ up to ℓ^{th} powers.*

Proof. The hypothesis implies Y_η is of Type RES or RAM. We investigate each case separately. Note that Properties 1, 3-5 will be clear from inspection of the relevant rows in the tables in Proposition 7.1 (which we will mention subsequently).
RAM: Let Y_η be of Type RAM. Then $Y_\eta = F_\eta\left(\sqrt[\ell]{w_\eta\pi_\eta}\right)$ for some unit $w_\eta \in \widehat{A_\eta}^*$ and $\beta_{rbc,\eta} \in \operatorname{Br}(F_\eta)$ with $D\otimes F_\eta = \beta_{rbc,\eta} + (u_\eta, w_\eta\pi_\eta)$. If P is not a cold point write $D = D_{00} + (u_P, \pi_P) + (v_P, \delta_P)$ where D_{00} is unramified at A_P and $u_P, v_P \in A_P^*$. Thus $D \otimes F_{P,\eta} = (u_P, \pi_\eta) + (v_P, \delta_P)$. Hence in $\operatorname{Br}(F_{P,\eta})$, we have $(u_P, \pi_\eta) + (v_P, \delta_P) = \beta_{rbc,\eta} + (u_\eta, w_\eta\pi_\eta)$. This implies that

$$\begin{aligned}\beta_{rbc,\eta} &= (v_P, \delta_P) + (u_P, \pi_\eta) - (u_\eta, w_\eta\pi_\eta)\\ &= (v_P, \delta_P) + (w_\eta, u_\eta) + \left(u_P u_\eta^{-1}, \pi_\eta\right).\end{aligned}$$

Comparing residues, we get $\beta_{rbc,\eta} \otimes F_{P,\eta} = ((v_P, \delta_P) + (w_\eta, u_\eta)) \otimes F_{P,\eta}$, where $\overline{u_\eta}$ is the residue of $D \otimes F_\eta$. We investigate the relevant rows of the Tables in Proposition 7.1.

η of Type 1b:

Row 1.2 of Table 10: Here $(v_P, \delta_P) = 0 \in \mathrm{Br}(F)$ itself. Computing residues along η, we see that $\overline{u_\eta} \in k_{P,\eta}$ is an ℓ^{th} power. So $\beta_{rbc,\eta}$ is already split over $F_{P,\eta}$.

Row 2.2^* *of Table 11*: Here $(v_P, \delta_P) = 0 \in \mathrm{Br}(F)$ itself. Computing residues along η, we see that $\overline{u_\eta} \cong \overline{u_P} \in k_{P,\eta}$ up to ℓ^{th} powers. Also since Y_{η_2} is of Type NONRES, $\overline{w_\eta} \in \mathcal{O}^*_{k_{P,\eta}}$. So $\beta_{rbc,\eta}$ is already split over $F_{P,\eta}$.

Row 2.2 of Table 11: Here $(v_P, \delta_P) = 0 \in \mathrm{Br}(F)$ itself. Computing residues along η, we see that $\overline{u_\eta} \cong \overline{u_P} \in k_{P,\eta}$ up to ℓ^{th} powers. Thus the choice of $E_{j,P} = L_P$ splits $\beta_{rbc,\eta}$ over $F_{P,\eta}$.

Rows 5.1, 5.2 and 5.8 of Table 14: Since P is a chilly point, we have $\overline{u_P} = \overline{v_P}^j$. Here $E_{j,P} = L_P$. Note that (v_P, δ_P) is split by L_P as v_P becomes an ℓ^{th} power in L_P. Computing residues along η, we see $\overline{u_\eta} \cong \overline{v_P}^j \in k_{P,\eta}$ up to ℓ^{th} powers. Thus L_P splits (u_η, w_η) over $F_{P,\eta}$ also.

Row 8.1 of Table 17: Since P is a hot point and η is of Type $1b$, $(v_P, \delta_P) = 0 \in \mathrm{Br}(F_P)$ as $v_P \in \widehat{A_P}^{*\ell}$. Computing residues along η, we see $\overline{u_\eta} \cong \overline{u_P} \in k_{P,\eta}$ up to ℓ^{th} powers. Here $E_{j,P} = L_P$ which splits $(w_\eta, u_\eta) = \beta_{rbc,\eta}$ over $F_{P,\eta}$.

Row 8.5 of Table 17: Since P is a hot point and η is of Type 1b, $(v_P, \delta_P) = 0 \in \mathrm{Br}(F_P)$ as $v_P \in \widehat{A_P}^{*\ell}$. Computing residues along η, we see $\overline{u_\eta} \cong \overline{u_P} \in k_{P,\eta}$ up to ℓ^{th} powers. Since Y_{η_2} has to be of Type NONRES in this configuration, we see that $\overline{w_\eta}$ has to have valuation $\cong 0 \mod \ell$ in the complete discretely valued field $k_{P,\eta}$ with parameter $\overline{\delta_P}$.

Putting this together, we see that $\overline{(u_\eta, w_\eta)}$ is unramified over local field $k_{P,\eta}$, hence trivial. Therefore (u_η, w_η) is trivial over $F_{P,\eta}$. So $\beta_{rbc,\eta}$ is already split over $F_{P,\eta}$.

η of Type 2:

Row 3.1 of Table 12: Here $(v_P, \delta_P) = 0 \in \mathrm{Br}(F)$ itself. Computing residues along η, we see that $\overline{u_\eta} \in k_{P,\eta}$ is an ℓ^{th} power. So $\beta_{rbc,\eta}$ is already split over $F_{P,\eta}$.

Row 4.1^* *of Table 13*: Here $(v_P, \delta_P) = 0 \in \mathrm{Br}(F)$ itself. Computing residues along η, we see that $\overline{u_\eta} \cong \overline{u_P} \in k_{P,\eta}$ up to ℓ^{th} powers. Also since Y_{η_2} is of Type NONRES, $\overline{w_\eta} \in \mathcal{O}^*_{k_{P,\eta}}$. So $\beta_{rbc,\eta}$ is already split over $F_{P,\eta}$.

Row 4.1 of Table 13: Here $(v_P, \delta_P) = 0 \in \mathrm{Br}(F)$ itself. Computing residues along η, we see that $\overline{u_\eta} \cong \overline{u_P} \in k_{P,\eta}$ up to ℓ^{th} powers. Thus the choice of $E_{j,P} = L_P$ splits $\beta_{rbc,\eta}$ over $F_{P,\eta}$.

Rows 8.1-8.2 of Table 17: Since P is a hot point and η is of Type 2, computing residues along η, we see $\overline{u_\eta} \in k^{*\ell}_{P,\eta}$ and therefore $\beta_{rbc,\eta} = (v_P, \delta_P)$. Since $E_{j,P} = L_P$, it splits $\beta_{rbc,\eta}$ over $F_{P,\eta}$.

Now let P be a cold point and write $D = D_{00} + (u_P\pi_P^m, v_P\delta_P)$ where D_{00} is unramified at A_P and $u_P, v_P \in A_P^*$. Thus $D \otimes F_{P,\eta} = \left(u_P\pi_\eta^m, v_P\delta_P\right)$. Thus in $\mathrm{Br}\left(F_{P,\eta}\right)$, we have $\left(u_P\pi_\eta^m, v_P\delta_P\right) = \beta_{rbc,\eta} + (u_\eta, w_\eta\pi_\eta)$. This implies that

$$\begin{aligned}\beta_{rbc,\eta} &= \left(u_P\pi_\eta^m, v_P\delta_P\right) - (u_\eta, w_\eta\pi_\eta)\\ &= (u_P, v_P\delta_P) + (w_\eta, u_\eta) + (\pi_\eta, v_P^m\delta_P^m u_\eta)\,.\end{aligned}$$

Comparing residues, we get $u_\eta \cong v_P^{-m}\delta_P^{-m}$ up to ℓ^{th} powers and $\beta_{rbc,\eta}\otimes F_{P,\eta}$ equals $((u_P, v_P\delta_P) + (w_\eta, u_\eta)) \otimes F_{P,\eta}$ which is $\left(u_P w_\eta^{-m}, v_P\delta_P\right) \otimes F_{P,\eta}$ in $\operatorname{Br}(F_{P,\eta})$.
Rows 6.1-6.4 and 6.8 of Table 15 and Rows 7.1-7.5 of Table 16 are relevant here. In each case, $E_{j,P} = F_P\left(\sqrt[\ell]{u_P\pi_P^m + v_P\delta_P}\right)$. Thus $E_{j,P}\otimes F_{P,\eta} = F_{P,\eta}\left(\sqrt[\ell]{v_P\delta_P}\right)$ is unramified and clearly splits $\beta_{rbc,\eta}$ over $F_{P,\eta}$.
RES: Let Y_η be of Type RES. Then $Y_\eta = F_\eta\left(\sqrt[\ell]{u_\eta}\right)$ and $\beta_{rbc,\eta} \in \operatorname{Br}(F_\eta)$ with $D\otimes F_\eta = \beta_{rbc,\eta} + (u_\eta, \pi_\eta)$. If P is not a cold point, write $D = D_{00} + (u_P, \pi_P) + (v_P, \delta_P) \in Br(F)$ where D_{00} is unramified at A_P and $u_P, v_P \in A_P^*$. Thus $D\otimes F_{P,\eta} = (u_P, \pi_\eta) + (v_P, \delta_P)$. Hence in $\operatorname{Br}(F_{P,\eta})$, we have $(u_P, \pi_\eta) + (v_P, \delta_P) = \beta_{rbc,\eta} + (u_\eta, \pi_\eta)$ which implies $\beta_{rbc,\eta} - (v_P, \delta_P) = \left(u_\eta^{-1}u_P, \pi_\eta\right)$. Comparing residues, we see that $u_\eta \cong u_P$ up to ℓ^{th} powers and $\beta_{rbc,\eta}\otimes F_{P,\eta} = (v_P, \delta_P)\otimes F_{P,\eta}$.
If P is a cold point, $D = D_{00} + (u_P\pi_P^m, v_P\delta_P) \in \operatorname{Br}(F)$. Following a similar argument, we get $\beta_{rbc,\eta}$ equals $\left(u_P\pi_\eta^m, v_P\delta_P\right) - (u_\eta, \pi_\eta)$ which equals $(u_P, v_P\delta_P) + (\pi_\eta, v_P^m\delta_P^m u_\eta)$ in $\operatorname{Br}(F_{P,\eta})$. Comparing residues, we see that $\beta_{rbc,\eta}\otimes F_{P,\eta} = (u_P, v_P\delta_P)\otimes F_{P,\eta}$. We investigate the relevant rows of the Tables in Proposition 7.1.
η of Type 1b:
Row 1.2 of Table 10: Here $(v_P, \delta_P) = 0 \in \operatorname{Br}(F)$ itself.
Row 2.2 of Table 11: Here $(v_P, \delta_P) = 0 \in \operatorname{Br}(F)$ itself.
Rows 5.3-5.4 and 5.9 of Table 14: Since P is a chilly point, we have $\overline{u_P} = \overline{v_P}^j$. In any case (v_P, δ_P) is split by L_P as v_P becomes an ℓ^{th} power in L_P.
Rows 6.2 and 6.4 of Table 15 and Row 7.2 of Table 16: Since P is a cold point, we are interested in splitting $(u_P, v_P\delta_P)$. Here $E_{j,P}$ is $F_P\left(\sqrt[\ell]{u_P\pi_P^m + v_P\delta_P}\right)$ and hence $E_{j,P}\otimes F_{P,\eta} = F_{P,\eta}\left(\sqrt[\ell]{v_P\delta_P}\right)$ which clearly splits $\beta_{rbc,\eta}$.
Row 8.6 of Table 17: Since P is a hot point and η is of Type 1b, $\beta_{rbc,\eta} = (v_P, \delta_P)$ where $v_P \in \widehat{A_P}^{*\ell}$. Thus $\beta_{rbc,\eta}$ is already split over $F_{P,\eta}$.

η of Type 2:
Row 3.1 of Table 12: Here $(v_P, \delta_P) = 0 \in \operatorname{Br}(F)$ itself.
Row 4.1 of Table 13: Here $(v_P, \delta_P) = 0 \in \operatorname{Br}(F)$ itself.
Row 7.4 of Table 16: Since P is a cold point, we are interested in splitting $(u_P, v_P\delta_P)$. Here $E_{j,P}$ is $F_P\left(\sqrt[\ell]{u_P\pi_P^m + v_P\delta_P}\right)$ and hence $E_{j,P}\otimes F_{P,\eta} = F_{P,\eta}\left(\sqrt[\ell]{v_P\delta_P}\right)$ which clearly splits $\beta_{rbc,\eta}$.
Rows 8.3- 8.4 of Table 17: Since P is a hot point and η is of Type 2, $\beta_{rbc,\eta} = (v_P, \delta_P)$ where $v_P \in \widehat{A_P}^*$ but not an ℓ^{th} power. Here $E_{j,P}$ is either L_P or $F_P\left(\sqrt[\ell]{\delta_P}\right)$. In either case, it splits $\beta_{rbc,\eta}$ over $F_{P,\eta}$. □

Proposition 8.6 (Yellow/Orange/Red/White). *Let $\eta \in N_0'$ and $P \in \overline{\eta}\cap S_0$. Assume further that η is coloured yellow, orange, red or white. Let $D \simeq M_\ell(u_\eta, w_\eta\pi_\eta)$ for units $w_\eta, u_\eta \in \widehat{A_\eta}^*$. Let $j = 1$ or 2 and let $E_{j,P}$ be as prescribed in Proposition 7.1. Then,*

1. *$E_{1,P}\otimes F_{P,\eta}$ is a ramified extension of $F_{P,\eta}$.*
2. *$E_{2,P} = L_P$ if η is coloured yellow or orange.*
3. *$a_{2,P}$ is a unit along η.*

4. *$E_{2,P} \otimes F_{P,\eta}$ is an unramified extension of $F_{P,\eta}$ and matches with the lift of residues at η as etale algebras over $F_{P,\eta}$.*

5. *$E_{j,P} \otimes F_{P,\eta}$ splits D in* $\mathrm{Br}\,(F_{P,\eta})$.

6. *There exist $w_P, x_{i,P} \in \widehat{A_P}$ for $i \leq \ell$ which are units along η such that*

   (a) $E_{1,P} \simeq F_P[t]/(t^\ell - w_P\pi_P)$ and $a_{1,P} = \left((w_P\pi_P)^{m_{i,P}}\, x_{i,P}^\ell\right)_i$ for $m_{i,P} \in \mathbb{Z}$.

   (b) $\left(w_P w_\eta^{-1}, u_\eta\right) = 0 \in \mathrm{Br}\,(F_{P,\eta})$.

Proof. We will mention the relevant rows of the tables in Proposition 7.1 below, whence Properties 1-5 become clear (Lemma 8.1 for Property 4). We will give a proof of Property 6a & 6b in each case.
Row 1.1 of Table 10: η is coloured red/white. Here $w_P = 1 = x_{i,P}$. The proof is similar to that of Proposition 8.3 5(b) for Row 1.3.
Row 2.1 of Table 11: η is coloured yellow/orange/red/white. Here $w_P = 1 = x_{i,P}$. Since P is a B_{11}^{ns} point, $D_P = (u_P, \pi_P)$ is nonsplit. The proof is similar to that of Proposition 8.3 5(b) for Row 2.3.
Row 5.6 of Table 14: η is coloured yellow. Unravelling the expression for $a_{1,P}$ from Row 5.6, we see it is $(\pi_P^{m_i}\delta_P^{n_i})_i$ where $m_i = r_i\ell + jn_i$. Let $sj \cong 1 \mod \ell$. Thus $n_i = r_i'\ell + sm_i$ for some r_i'. Hence $a_{1,P} = \left((\pi_P\delta_P^s)^{m_i}\,\delta_P^{\ell r_i'}\right)_i$. As $E_{1,P} = F_P\left(\sqrt[\ell]{\pi_P^j\delta_P}\right) = F_P\left(\sqrt[\ell]{\pi_P\delta_P^s}\right)$, we have $w_P = \delta_P^s$ and $x_{i,P} = \delta_P^{r_i'}$ here. Writing $D = D_{00} + (u_P, \pi_P) + (v_P, \delta_P) \in \mathrm{Br}(F)$ for D_{00} unramified at A_P, we have $\left(v_P, \pi_P^j\delta_P\right) = (u_\eta, w_\eta\pi_\eta)$ in $\mathrm{Br}\,(F_{P,\eta})$. Since $\pi_P = \pi_\eta$, comparing residues we have $v_P^j \cong u_\eta$ up to ℓ^{th} powers, and hence $(u_\eta, w_\eta) = (v_P, \delta_P)$. Since $sj \cong 1 \mod \ell$, we have $v_P \cong u_\eta^s$ up to ℓ^{th} powers. Therefore $(u_\eta, w_\eta) = (u_\eta, \delta_P^s)$ which implies $\left(u_\eta, \delta_P^s w_\eta^{-1}\right) = 0$. Hence Properties 6a & 6b hold.
Rows 6.5, 6.6, 6.7 and 6.9 of Table 15 and Row 7.6 of Table 16: η is coloured red and $D_P = (u_P\pi_P^m, v_P\delta_P)$. The proof is similar to that of Proposition 8.3 5(b) of Rows 6.5-6.6.
Row 8.4 of Table 17: η is coloured orange. Here $w_P = 1 = x_{i,P}$. Writing $D = D_{00} + (u_P, \pi_P) + (v_P, \delta_P) \in \mathrm{Br}(F)$ for D_{00} unramified at A_P, we have $(u_P, \pi_P) = (u_\eta, w_\eta\pi_\eta)$ in $\mathrm{Br}\,(F_{P,\eta})$. Since $\pi_P = \pi_\eta$, comparing residues we have that $u_P \cong u_\eta$ up to ℓ^{th} powers, and hence $(u_\eta, w_\eta) = 0$. Hence Properties 6a & 6b hold. □

PROPOSITION 8.7 (0/1a). *Let $\eta \in N_0'$ be of Type 1a or 0 and $P \in \overline{\eta} \cap S_0$. Let $j = 1$ or 2 and let $E_{j,P}$ be as prescribed in Propositions 7.1 and 7.2. Then*

1. *$E_{j,P} \otimes F_{P,\eta}$ is an unramified (possibly split) extension of $F_{P,\eta}$.*

2. *If Y_η is of Type RAM, then $a_{1,P} = a$ and $a_{2,P} = 1$.*

Further if η is of Type 0, then there exist j, j' such that $\{j, j'\} = \{1, 2\}$ such that for every $P \in \overline{\eta} \cap S_0$, the element $a_{j',P}$ is a unit in $\mathcal{O}_{Y \otimes F_{P,\eta}}$ and $E_{j,P} \simeq \prod F_P$.

Proof. By an inspection of Tables 9, 10, 11, 12, 13 and the choice of $E_{j,P}$ in Proposition 7.2, it is clear that they are unramified along η. If Y_η is of Type RAM, by Remark 5.2 it cannot intersect $\eta' \in N_0'$ where $Y_{\eta'}$ is of Type SPLIT. Hence it cannot intersect $\eta' \in N_0'$ which is coloured V, I, B, Ye, O, R, W or Bl. Inspecting Tables 9, 10, 11, 12, 13, 18, 19 and 20 shows that in this case, $a_{1,P} = a$ and $a_{2,P} = 1$.

Now let us assume η is of Type 0 and let $\mathcal{P}_\eta' = \overline{\eta} \cap S_0$. Then one of the following holds (Proposition 6.4 and proof of Proposition 7.2): Case A) $\mathcal{P}_\eta' = \{Q\}$ where Q is not of Type A_{00}^s, Case B) $\mathcal{P}_\eta' = \{Q, Q'\}$ where Q is not of Type A_{00}^s and Q' is of Type A_{00}^s. Case C) $\mathcal{P}_\eta'$ consists of only Type A_{00}^s points.

Case A/B: Let $Q \in \overline{\eta} \cap \overline{\eta'}$ (and $Q' \in \overline{\eta} \cap \overline{\gamma}$ in Case B). Note that Q can be of Type A_{10}^s, B_{10}^s or B_{20}^s. Thus η' can be Type 1a or coloured red, green, blue, white or black (and γ is of Type 0 in Case B while Q' has to be of subtype D1 or D2 as defined in Section 7.2).

Set $j = 1$ and $j' = 2$ if η' is of Type 1a or coloured green, blue or black. Set $j = 2$ and $j' = 1$ if η' is coloured red or white. An inspection of Tables 9, 10 and 12 for Case A and Tables 19 and 20 for Case B[18] verifies that our choice of j and j' is compatible.

Case C: In this case each $P_i \in \mathcal{P}_\eta'$ is of Type A_{00}^s. Thus it has to be of subtype D2 or D3. Set $j = 1$ and $j' = 2$. An inspection of Tables 18 and 19 verifies the compatibility of this choice. □

9 Understanding $E_{j,P}$ in terms of norms from some extensions

In this section, we continue to assume that $\eta \in N_0'$ with $P \in S_0 \cap \overline{\eta}$ being the intersection of two distinct irreducible curves with generic points η_1 and η_2. Let $\pi_P, \delta_P, \pi_\eta$ be as before. We study $E_{j,P}$ (as prescribed in Propositions 7.1 and 7.2) vis-á-vis norms from some related extensions.

9.1 When η is 1b or 2 and Y_η is RAM

Let $\eta = \eta_1$ or η_2 be of Type 1b or 2 with Y_η of Type RAM. By Proposition 8.5, $E_{j,P}$ is unramified along η for $j = 1, 2$ and splits $\beta_{rbc,\eta}$ over $F_{P,\eta}$. Let $Y_\eta = F_\eta\left(\sqrt[\ell]{w_\eta \pi_\eta}\right)$ where $w_\eta \in \widehat{A_\eta}^*$. Thus $D = \beta_{rbc,\eta} + (u_\eta, w_\eta\pi_\eta)$ where $u' = \overline{u_\eta} \in k_\eta/k_\eta^{*\ell}$ is the residue of D_η. In this subsection, we show that u' is locally a norm from $\overline{E_{j,P}}$. This will be useful in the final part of this paper (Section 13) where we show that the constructed E_js are *good*.

Proposition 9.1. *Let η, u' and P be as above. Then there exist $w_{1,P}, w_{2,P} \in k_{P,\eta}$ such that for $j = 1$ or 2,*

1. $\overline{E_{j,P} \otimes F_{P,\eta}} = k_{P,\eta}[t]/(t^\ell - w_{j,P})$
2. $(w_{j,P}, u') = 0 \in \mathrm{Br}\,(k_{P,\eta})$.

[18] Appeal to Lemma 2.6 and Proposition 3.3 for Rows 9.5', 9.7', 9.8' and 10.2'.

Proof. Since η is Type 1b/2 and Y_η is RAM, η is coloured green. We investigate the relevant rows in the tables given in the proof of Proposition 7.1.

For the following situations, choose $w_{1,P} = w_{2,P} = 1$.

- B^s_{10} point, η of Type 1b (cf. Row 1.2 of Table 10 and Row a in Table 21)
- some B^{ns}_{11} points(*), η of Type 1b (cf. Row 2.2* of Table 11 and Row b in Table 21)
- B^s_{20} point, η of Type 2 (cf. Row 3.1 of Table 12 and Row c in Table 21)
- some B^{ns}_{21} points(*), η of Type 2 (cf. Row 4.1* of Table 13 and Row d in Table 21)
- C^{Hot}_{12} point, η of Type 1b (cf. Row 8.5 of Table 17 and Row l in Table 21).

For the following situations, choose $w_{1,P} = w_{2,P} = u'$.

- B^{ns}_{11} point, η of Type 1b : (cf. Row 2.2 of Table 11 and Row b in Table 21)
- B^{ns}_{21} point, η of Type 2 : (cf. Row 4.1 of Table 13 and Row d in Table 21)
- C^{Chilly}_{11} point, η of Type 1b : (cf. Rows 5.1, 5.2 and 5.8 of Table 14 and Rows e,f in Table 21)
- C^{Cold}_{11} point, η of Type 1b : (cf. Rows 6.1-6.4 and 6.8 of Table 15 and Rows g,h in Table 21)
- C^{Cold}_{12} point, η of Type 1b : (cf. Rows 7.1, 7.4 and 7.5 of Table 16 and Row i in Table 21)
- C^{Cold}_{12} point, η of Type 2 : (cf. Rows 7.1-7.3 of Table 16 and Row j in Table 21)
- C^{Hot}_{12} point, η of Type 1b : (cf. Row 8.1 of Table 17 and Row l in Table 21).

For C^{Hot}_{12} points, η of Type 2 (cf. Rows 8.1-8.2 of Table 17 and Row m in Table 21), choose $w_{1,P} = w_{2,P} \in \mathcal{O}^*_{k_{P,\eta}} \setminus \mathcal{O}^{*\ell}_{k_{P,\eta}}$. Since $u' = 1$ here, $(w_{j,P}, u') = 0$. □

9.2 WHEN η IS 1B OR 2 AND Y_η IS RES

Let $\eta = \eta_1$ or η_2 be of Type 1b or 2 with Y_η of Type RES. In this subsection, we will define certain extensions $\tilde{\mathrm{L}}_P$ and $\tilde{\mathrm{L}}'_P$ of $k_{P,\eta}$ and understand $E_{j,P}$ in terms of norms from these extensions. This will be helpful in constructing $E_{j,\eta}$ over F_η by approximating local data.

Since Y_η is of Type RES, we have $Y_\eta \simeq F_\eta\left(\sqrt[\ell]{u_\eta}\right)$ where $u' = \overline{u_\eta} \in k_\eta / k^{*\ell}_\eta$ is the residue of $D \otimes F_\eta$. Recall that $\mathrm{Gal}\,(Y_\eta/F_\eta) = \langle\psi\rangle$. Let Y' be $\overline{Y \otimes F_\eta}$ over k_η and by abuse of notation, let $\mathrm{Gal}\,(Y'/k_\eta) = \langle\psi\rangle$ also. Finally let Y'_P be $\overline{Y \otimes F_{P,\eta}}$ over $k_{P,\eta}$ with an induced action of ψ.

Note that if Y'_P is split, then $Y'_P \simeq \prod k_{P,\eta}$ where $x \in Y'$ is identified with the tuple $\big(x, \psi(x), \ldots, \psi^{\ell-1}(x)\big)$. Note that ψ acts on $\prod k_{P,\eta}$ by permutations. That is, for $x_i \in k_{P,\eta}$,

$$\psi(x_1, x_2, \ldots, x_\ell) = (x_2, x_3, \ldots, x_\ell, x_1).$$

Let $a_{j,P}$ and $E_{j,P}$ be as prescribed in Propositions 7.1 and 7.2. By Proposition 8.5, $E_{j,P}$ is unramified along η for $j = 1, 2$ and splits $\beta_{rbc,\eta}$ over $F_{P,\eta}$. Also $a_{j,P}$ are units along η (Proposition 3.3). Set $b_P := \overline{a_{1,P}}, b'_P := \overline{a_{2,P}}$ in Y'_P. If Y'_P is split, then set $b_P = (b_{i,P})_i \in \prod k_{P,\eta}$ and $b'_P = \left(b'_{i,P}\right)_i \in \prod k_{P,\eta}$.

When Y'_P is not split, set

$$\tilde{\mathrm{L}}_P = Y'_P\left(\sqrt[\ell]{b_P}, \sqrt[\ell]{\psi(b_P)}, \dots, \sqrt[\ell]{\psi^{\ell-1}(b_P)}\right)$$
$$\tilde{\mathrm{L}}'_P = Y'_P\left(\sqrt[\ell]{b'_P}, \sqrt[\ell]{\psi(b'_P)}, \dots, \sqrt[\ell]{\psi^{\ell-1}(b'_P)}\right).$$

Since Y'_P is a nonsplit extension of $k_{P,\eta}$ and $\mathrm{N}\,(b_P) = 1 = \mathrm{N}\,(b'_P)$, first of all b_P and b'_P are units in $\mathcal{O}_{Y',P}$. Also by Lemmata 2.3 and 2.5, we have $b_P, b'_P \in {Y'_P}^{*\ell}$. Hence $\psi^j\,(b_P)\,, \psi^j\,(b'_P)$ are all ℓ^{th} powers in Y'_P also. Therefore $\tilde{\mathrm{L}}_P = Y'_P = \tilde{\mathrm{L}}'_P$.
When Y'_P is split, set

$$\tilde{\mathrm{L}}_P = k_{P,\eta}\left(\sqrt[\ell]{b_{1,P}}, \sqrt[\ell]{b_{2,P}}, \dots, \sqrt[\ell]{b_{\ell,P}}\right)$$
$$\tilde{\mathrm{L}}'_P = k_{P,\eta}\left(\sqrt[\ell]{b'_{1,P}}, \sqrt[\ell]{b'_{2,P}}, \dots, \sqrt[\ell]{b'_{\ell,P}}\right)$$

Note that in either case $\tilde{\mathrm{L}}_P\,/k_{P,\eta}$ and $\tilde{\mathrm{L}}'_P\,/k_{P,\eta}$ are Galois extensions.

Proposition 9.2. *Let η, P, u', $\tilde{\mathrm{L}}_P$ and $\tilde{\mathrm{L}}'_P$ be as above. There exist $w_P, w'_P \in k_{P,\eta}$, $z_P \in \tilde{\mathrm{L}}_P$ and $z'_P \in \tilde{\mathrm{L}}'_P$ such that*

1. $\overline{E_{1,P} \otimes F_{P,\eta}} = k_{P,\eta}[t]/(t^\ell - w_P)$, $\mathrm{N}_{\tilde{\mathrm{L}}_P\,/k_{P,\eta}}(z_P) = w_P$ *and* $(w_P, u') = 0 \in \mathrm{Br}\,(k_{P,\eta})$.
2. $\overline{E_{2,P} \otimes F_{P,\eta}} = k_{P,\eta}[t]/(t^\ell - w'_P)$, $\mathrm{N}_{\tilde{\mathrm{L}}'_P\,/k_{P,\eta}}(z'_P) = w'_P$ *and* $(w'_P, u') = 0 \in \mathrm{Br}\,(k_{P,\eta})$.

Proof. Since η is of Type 1b or 2 and Y_η is RES, η is coloured green. We investigate the relevant rows in the tables given in the proof of Proposition 7.1.
η of Type 1b:
Choose $w_P = w'_P = z_P = z'_P = 1$ for B^s_{10} points (cf. Row 1.2 of Table 10 and Row a in Table 21) and C^{Hot}_{12} points (cf. Row 8.6 of Table 17 and Row l in Table 21).
For the following situations, $Y'_P = k_{P,\eta}$ is a nonsplit (unramified/ramified extension). Thus $\tilde{\mathrm{L}}_P = \tilde{\mathrm{L}}'_P = Y'_P = k_{P,\eta}\left(\sqrt[\ell]{u'}\right)$. Choose $w_P = w'_P = u'$ and $z_P = z'_P = \sqrt[\ell]{u'}$.
- B^{ns}_{11} point : (cf Row 2.2 of Table 11 and Row b in Table 21)
-C^{Chilly}_{11} point : (cf. Rows 5.3, 5.4 and 5.9 of Table 14 and Rows e,f in Table 21)
-C^{Cold}_{11} point : (cf. Rows 6.2 and 6.4 of Table 15 and Row g,h in Table 21)
-C^{Cold}_{12} point : (cf Row 7.2 of Table 16 and Row i in Table 21)
η of Type 2:
Choose $w_P = w'_P = z_P = z'_P = 1$ for B^s_{20} points (cf Row 3.1 of Table 12 and Row c in Table 21).
For the following situations, $Y'_P/k_{P,\eta}$ is a nonsplit (unramified/ramified extension). Thus $\tilde{\mathrm{L}}_P = \tilde{\mathrm{L}}'_P = Y'_P = k_{P,\eta}\left(\sqrt[\ell]{u'}\right)$. Choose $w_P = w'_P = u'$ and $z_P = z'_P = \sqrt[\ell]{u'}$.
- B^{ns}_{21} point : (cf. Row 4.1 of Table 13 and Row d in Table 21)
- C^{Cold}_{12} point : (cf Row 7.4 of Table 16 and Row j in Table 21).

We are left with the case of hot points. Rows 8.3-8.4 of Table 17 are relevant here (with $\eta = \eta_2$). Note that by Row m of Table 21, we know Y'_P is split. Hence $u' \in k^{*\ell}_{P,\eta}$ and therefore $(w_P, u') = (w'_P, u') = 0$ for whatever be the choice of w_P and w'_P. However we need to be more careful in making our choice to ensure the existence of z_P and z'_P.

Row 8.3 of Table 17: Observe η_1 is coloured indigo. This implies that the $b_{i,P}$ are all units in the local field $k_{P,\eta}$. Hence $\tilde{\mathrm{L}}_P$ is an unramified extension of $k_{P,\eta}$. By construction, $b'_{i,P} = 1$ and hence $\tilde{\mathrm{L}}'_P = k_{P,\eta}$. Thus all units of $k_{P,\eta}$ are norms from $\tilde{\mathrm{L}}_P$ and $\tilde{\mathrm{L}}'_P$. Choose $w_P = w'_P$ to be a unit in $k_{P,\eta}$ which is not an ℓ^{th} power and $z_P \in \tilde{\mathrm{L}}_P$ such that $\mathrm{N}_{L_P/k_{P,\eta}}(z_P) = w_P$. Also set $z'_P = w'_P$.

Row 8.4 of Table 17: By choice $b_{i,P} = \overline{\pi_P^{m_i}}$. Hence we see that $\tilde{\mathrm{L}}_P = k_{P,\eta}\left(\sqrt[\ell]{\overline{\pi_P}}\right)$. Set $w_P = \overline{\pi_P}$ which is clearly a norm from $\tilde{\mathrm{L}}_P$. Note that $b'_{i,P} = \overline{z_{i,P}}$. Since $z_{i,P}$ are units in $\widehat{A_P}^*$, we see that $\tilde{\mathrm{L}}'_P$ is an unramified extension of $k_{P,\eta}$. Thus all units of $k_{P,\eta}$ are norms from $\tilde{\mathrm{L}}'_P$. As before choose w'_P to be a unit in $\mathcal{O}_{k_{P,\eta}}$ which is not an ℓ^{th} power and $z'_P \in \tilde{\mathrm{L}}'_P$ such that $\mathrm{N}_{\tilde{\mathrm{L}}'_P/k_{P,\eta}}(z'_P) = w'_P$. □

9.3 When η is 1a

Let $\eta = \eta_1 \in N_0$ be of Type 1a. For convenience, we again summarize the choice of $E_{j,P}$ at points $P \in \overline{\eta} \cap S_0$ for $j = 1, 2$ while also tabulating the shape of Y in Table 22. Both extensions are unramified along η. By Proposition 3.3 and Lemma 4.2, a is a unit along η up to ℓ^{th} powers.

9.3.1 When Y_η is NONRES

Proposition 9.3. *Let $\eta = \eta_1$ be of Type 1a and let $P \in \overline{\eta} \cap S_0$. Further assume Y_η is of Type NONRES. Let $Y' = \overline{Y_\eta} = k_\eta\left(\sqrt[\ell]{u'}\right)$ and $Y'_P = Y' \otimes k_{P,\eta}$. If Y'_P is nonsplit, set $\tilde{\mathrm{L}}_P = \tilde{\mathrm{L}}'_P = Y'_P$. If $Y'_P = \prod k_{P,\eta}$, let $\overline{a_{1,P}} = (b_{i,P})_i$ and $\overline{a_{2,P}} = \left(b'_{i,P}\right)_i$ in $\prod k_{P,\eta}$ and set $\tilde{\mathrm{L}}_P = k_{P,\eta}\left(\sqrt[\ell]{b_{1,P}}, \sqrt[\ell]{b_{2,P}}, \ldots, \sqrt[\ell]{b_{\ell,P}}\right)$ and $\tilde{\mathrm{L}}'_P = k_{P,\eta}\left(\sqrt[\ell]{b'_{1,P}}, \sqrt[\ell]{b'_{2,P}}, \ldots, \sqrt[\ell]{b'_{\ell,P}}\right)$.*

Then there exist $w_P, w'_P \in k_{P,\eta}$, $z_P \in \tilde{\mathrm{L}}_P$ and $z'_P \in \tilde{\mathrm{L}}'_P$ such that

1. $\overline{E_{1,P} \otimes F_{P,\eta}} = k_{P,\eta}[t]/(t^\ell - w_P)$ *and* $\mathrm{N}_{\tilde{\mathrm{L}}_P/k_{P,\eta}}(z_P) = w_P$.
2. $\overline{E_{2,P} \otimes F_{P,\eta}} = k_{P,\eta}[t]/(t^\ell - w'_P)$ *and* $\mathrm{N}_{\tilde{\mathrm{L}}'_P/k_{P,\eta}}(z'_P) = w'_P$.

Proof. In Table 22, the Rows NR.1-NR.12 are relevant. We only give the choices of w_P and w'_P from which existence of z_P, z'_P become clear. For Rows NR.1-NR.4 and NR.10, choose $w_P = w'_P = 1$. For Row NR.7, $\tilde{L}_P = k_{P,\eta}\left(\sqrt[\ell]{\overline{\delta_P}}\right)$ is a ramified extension and $\tilde{L}'_P/k_{P,\eta}$ is an unramified extension. Choose $w_P = \overline{\delta_P}$ and w'_P to be any unit in $\mathcal{O}_{k_{P,\eta}}$ which is not an ℓ^{th} power. For Row NR.8, a similar proof as

Row	η_2−colour	η_2	P	Y_{η_1}	Y_{η_2}	Y_P	$E_{1,P}$	$E_{2,P}$	$a_{1,P}$	$a_{2,P}$
$R.1$		0	A^s_{10}	RAM	RAM	$F_P\left(\sqrt[\ell]{w_P\pi_P^r\delta_P}\right)$	$\prod F_P$	$\prod F_P$	a	1
$R.2$		0	A^s_{10}	RAM	NONRES	$F_P(\sqrt[\ell]{w_P\pi_P})$	$\prod F_P$	$\prod F_P$	a	1
$R.3$	G	1b	B^{ns}_{11}	RAM	RAM	$F_P\left(\sqrt[\ell]{w_P\pi_P^r\delta_P}\right)$	L_P	L_P	a	1
$R.4$	G	1b	B^{ns}_{11}	RAM	NONRES	$F_P(\sqrt[\ell]{w_P\pi_P})$	L_P	L_P	a	1
$R.5$	G	2	B^{ns}_{12}	RAM	RAM	$F_P\left(\sqrt[\ell]{w_P\pi_P^r\delta_P}\right)$	L_P	L_P	a	1
$R.6$	G	2	B^{ns}_{12}	RAM	NONRES	$F_P(\sqrt[\ell]{w_P\pi_P})$	L_P	L_P	a	1
$S.1$		0	A^s_{10}	SPLIT	SPLIT	$\prod F_P$	$\prod F_P$	$\prod F_P$	a	1
$S.2$		0	A^s_{10}	SPLIT	NONRES	$\prod F_P$	$\prod F_P$	$\prod F_P$	a	1
$S.3$	Bl,I,V	1b	B^{ns}_{11}	SPLIT	SPLIT	$\prod F_P$	L_P	L_P	a	1
$S.4$	W,R,O,Ye	1b	B^{ns}_{11}	SPLIT	SPLIT	$\prod F_P$	$F_P\left(\sqrt[\ell]{\delta_P}\right)$	L_P	$(\delta_P^{n_i})$	$\left(z_{i,P}\pi_P^{\ell m'_i}\right)$
$S.5$	B	1b	B^{ns}_{11}	SPLIT	SPLIT	$\prod F_P$	L_P	$F_P\left(\sqrt[\ell]{\delta_P}\right)$	$\left(z_{i,P}\pi_P^{\ell m'_i}\right)$	$(\delta_P^{n_i})$
$S.6$	G	1b	B^{ns}_{11}	SPLIT	NONRES	$\prod F_P$	L_P	L_P	a	1
$S.7$	G	2	B^{ns}_{12}	SPLIT	NONRES	$\prod F_P$	L_P	L_P	a	1
$NR.1$		0	A^s_{10}	NONRES	RAM	$F_P\left(\sqrt[\ell]{w_P\delta_P}\right)$	$\prod F_P$	$\prod F_P$	a	1
$NR.2$		0	A^s_{10}	NONRES	SPLIT	$\prod F_P$	$\prod F_P$	$\prod F_P$	a	1
$NR.3$		0	A^s_{10}	NONRES	NONRES	$F_P(\sqrt[\ell]{w_P})$	$\prod F_P$	$\prod F_P$	a	1
$NR.4$	G	1b	B^{ns}_{11}	NONRES	RAM	$F_P\left(\sqrt[\ell]{w_P\delta_P}\right)$	$\prod F_P$	$\prod F_P$	a	1
$NR.5$	G	1b	B^{ns}_{11}	NONRES	RES	L_P	L_P	L_P	a	1
$NR.6$	Bl,I,V	1b	B^{ns}_{11}	NONRES	SPLIT	$\prod F_P$	L_P	L_P	a	1
$NR.7$	W,R,O,Ye	1b	B^{ns}_{11}	NONRES	SPLIT	$\prod F_P$	$F_P\left(\sqrt[\ell]{\delta_P}\right)$	L_P	$(\delta_P^{n_i})$	$\left(z_{i,P}\pi_P^{\ell m'_i}\right)$
$NR.8$	B	1b	B^{ns}_{11}	NONRES	SPLIT	$\prod F_P$	L_P	$F_P\left(\sqrt[\ell]{\delta_P}\right)$	$\left(z_{i,P}\pi_P^{\ell m'_i}\right)$	$(\delta_P^{n_i})$
$NR.9$	G	1b	B^{ns}_{11}	NONRES	NONRES	$F_P(\sqrt[\ell]{w_P})$	L_P	L_P	a	1
$NR.10$	G	2	B^{ns}_{12}	NONRES	RAM	$F_P\left(\sqrt[\ell]{w_P\delta_P}\right)$	$\prod F_P$	$\prod F_P$	a	1
$NR.11$	G	2	B^{ns}_{12}	NONRES	RES	$\prod F_P$	L_P	L_P	a	1
$NR.12$	G	2	B^{ns}_{12}	NONRES	NONRES	$F_P(\sqrt[\ell]{w_P})$	L_P	L_P	a	1

Table 22: Patching data at closed points when η_1 is of Type 1a

in Row NR.7 works. For the remaining rows, $\tilde{L}_P$ and $\tilde{L}'_P$ are unramified extensions of $k_{P,\eta}$. So again choose $w_P = w'_P$ to be any unit in $\mathcal{O}_{k_{P,\eta}}$ which is not an ℓ^{th} power. $\square$

9.3.2 WHEN Y_η IS SPLIT

PROPOSITION 9.4. *Let $\eta = \eta_1$ be of Type 1a and let $P \in \overline{\eta} \cap S_0$. Further assume Y_η is of Type SPLIT. Let*[19] $\overline{a_{1,P}} = (b_{i,P})$ *(resp.* $\overline{a_{2,P}} = (b'_{i,P})$*). Set* $X_P = k_{P,\eta}\left(\sqrt[\ell]{b_{1,P}}, \ldots, \sqrt[\ell]{b_{\ell,P}}\right)$ *and* $X'_P = k_{P,\eta}\left(\sqrt[\ell]{b'_{1,P}}, \ldots, \sqrt[\ell]{b'_{\ell,P}}\right)$.
Then there exist $w_P, w'_P \in k_{P,\eta}$, $z_P \in X_P$ and $z'_P \in X'_P$ such that

1. $\overline{E_{1,P} \otimes F_{P,\eta}} = k_{P,\eta}[t]/(t^\ell - w_P)$ *and* $w_P = \mathrm{N}_{X_P/k_{P,\eta}}(z_P)$.
2. $\overline{E_{2,P} \otimes F_{P,\eta}} = k_{P,\eta}[t]/(t^\ell - w'_P)$ *and* $w'_P = \mathrm{N}_{X'_P/k_{P,\eta}}(z'_P)$.

Proof. In Table 22, the Rows S.1-S.7 are relevant. We only give the choices of w_P and w'_P from which the existence of z_P, z'_P become clear.
For Rows S.1-S.2, choose $w_P = w'_P = 1$. For Row S.4, $X_P = k_{P,\eta}\left(\sqrt[\ell]{\overline{\delta_P}}\right)$ is a ramified extension and $X'_P/k_{P,\eta}$ is an unramified extension. Choose $w_P = \overline{\delta_P}$ and w'_P to be any unit in $\mathcal{O}_{k_{P,\eta}}$ which is not an ℓ^{th} power. For Row S.5, a similar proof as in Row S.4 works. For Rows S.3, S.6 and S.7, X_P and X'_P are unramified extensions of $k_{P,\eta}$. So choose $w_P = w'_P$ to be any unit in $\mathcal{O}_{k_{P,\eta}}$ which is not an ℓ^{th} power. $\square$

10 PATCHING DATA AT COLOURED POINTS OF N_0

Let $\eta \in N_0$ be of Type 1b or 2 where N_0 denotes the subset $N'_0 \cap X_0$. Let π_η denote the parameter of F_η fixed in Section 6.2 and let $\beta_{rbc,\eta}$ be as defined in Section 4. For $j = 1, 2$ and any $P \in S_0$, let $E_{j,P}$ and $a_{j,P}$ be as prescribed in Propositions 7.1. We now prescribe the choices for $E_{j,\eta}$ and $a_{j,\eta}$.

PROPOSITION 10.1 (Violet/Indigo/Black). *Let $\eta \in N_0$ be coloured violet, indigo or black. Set $E_{1,\eta}$ and $E_{2,\eta}$ to be the lift of residues at $\widehat{A_\eta}$. Further, set $a_{1,\eta} = a$ and $a_{2,\eta} = 1$. Then for $j = 1, 2$, we have*

1. $a_{1,\eta}a_{2,\eta} = a$.
2. $D \otimes E_{j,\eta}$ *is split.*
3. $a_{j,\eta}$ *is a norm from* $E_{j,\eta} \otimes Y_\eta/Y_\eta$.
4. $\mathrm{N}_{Y_\eta/F_\eta}(a_{j,\eta}) = 1$.
5. $E_{j,\eta} \otimes F_{P,\eta} \simeq E_{j,P} \otimes F_{P,\eta}$ *for each point* $P \in S_0 \cap \overline{\eta}$.
6. $a_{j,P} = a_{j,\eta} \in Y \otimes F_{P,\eta}$ *for each point* $P \in S_0 \cap \overline{\eta}$.

[19] Modify $a_{j,P}$ by ℓth powers of the parameter π_η if needed to define $(b_{i,P})$ and $(b'_{i,P})$.

Proof. Recall that we have $D = \beta_{rbc,\eta} + (u_\eta, \pi_\eta)$ in $\operatorname{Br}(F_\eta)$ where $\beta_{rbc,\eta}$ is an unramified algebra with index at most ℓ. And by construction, $E_{j,\eta} \simeq F_\eta\left(\sqrt[\ell]{u_\eta}\right)$. Properties 1 & 4 are immediate, while Properties 5 & 6 follow from Proposition 8.2. Since D is ramified at η, we have $[E_{j,\eta} : F_\eta] = \ell$. Since η is of Type 1b, $D_\eta \simeq \mathrm{M}_\ell(u_\eta, w_\eta\pi_\eta)$ for some unit $w_\eta \in \widehat{A_\eta}^*$. Thus $D \otimes E_{j,\eta}$ is split which shows Property 2 holds.

Since η is coloured violet/indigo/black, then it has to be a Ch/H/Z curve respectively. In particular η is a Type 1b curve with Y_η of Type SPLIT and $a = \left(a'_{i,\eta}\right)_i$ where each $a'_{i,\eta}$ is a unit up to ℓ^{th} powers in $\widehat{A_\eta}$. Therefore the fact that $a \in \operatorname{Nrd}(D \otimes Y_\eta)$ translates to $\left(a'_{i,\eta}\right)(u_\eta, \pi_\eta) = 0$ in $\mathrm{H}^3(F_\eta, \mu_\ell)$ for all i (Lemma 2.7). This implies by taking residues that $\overline{\left(u_\eta, a'_{i,\eta}\right)} = 0$ in $\mathrm{H}^2(k_\eta, \mu_\ell)$ for all i. Hence each $a'_{i,\eta}$ is a norm from $E_{1,\eta}$. Since clearly 1 is a norm from $E_{2,\eta}$, Property 3 holds. □

Proposition 10.2 (Blue). *Let $\eta \in N_0$ be coloured blue. Set $E_{1,\eta}$ to be the lift of residues at $\widehat{A_\eta}$. Then there exists a ramified cyclic extension $E_{2,\eta}/F_\eta$ of degree ℓ and elements $a_{1,\eta} = (\tilde{a}_{1,i,\eta})_i$ and $a_{2,\eta} = (\tilde{a}_{2,i,\eta})_i \in \prod F_\eta$ such that for $j = 1, 2$, the following holds:*

1. *$a_{1,\eta}a_{2,\eta} = a \in Y_\eta$, i.e. $\tilde{a}_{1,i,\eta}\tilde{a}_{2,i,\eta} = a'_{i,\eta}$ for each i.*
2. *$D \otimes E_{j,\eta}$ is split.*
3. *$a_{j,\eta}$ is a norm from $E_{j,\eta} \otimes Y_\eta/Y_\eta$, i.e . $\tilde{a}_{j,i,\eta}$ is a norm from $E_{j,\eta}$ for each i.*
4. *$\mathrm{N}_{Y_\eta/F_\eta}(a_{j,\eta}) = 1$, i.e. $\prod_i \tilde{a}_{j,i,\eta} = 1$.*
5. *$E_{j,\eta} \otimes F_{P,\eta} \simeq E_{j,P} \otimes F_{P,\eta}$ for each point $P \in S_0 \cap \overline{\eta}$.*
6. *$\tilde{a}_{j,i,\eta}\mu_{j,i,P,\eta} = \tilde{a}_{j,i,P} \in F_{P,\eta}$ for all i at each point $P \in S_0 \cap \overline{\eta}$ for some $\mu_{j,i,P,\eta} \in F_{P,\eta}^{*\ell}$ such that $\prod_i \mu_{j,i,P,\eta} = 1$.*
7. *$E_{1,\eta}/F_\eta$ is unramified and cyclic of degree ℓ.*

Proof. Since η is coloured blue, it is a C curve of Type 1b with Y_η of Type SPLIT. Write $D \simeq \mathrm{M}_\ell(u_\eta, w_\eta\pi_\eta)$ where $w_\eta, u_\eta \in \widehat{A_\eta}^*$ and $a = \left(a'_{i,\eta}\right)_i$ where $a'_{i,\eta} = x'_i\pi_\eta^{m_i}$ with $x'_i \in \widehat{A_\eta}^*$. Because $\mathrm{N}(a) = 1$, we have $\sum m_i = 0$ and $\prod x'_i = 1$.

Let $P \in \overline{\eta} \cap S_0$. By Proposition 8.3, $E_{2,P} = \frac{F_P[t]}{(t^\ell - w_P\pi_P)}$ and $a_{2,P} = (\tilde{a}_{2,i,P})$ where $\tilde{a}_{2,i,P} = (w_P\pi_P)^{m_{i,P}} x_{i,P}^\ell$ for some $w_P, x_{i,P} \in \widehat{A_P}$ which are units along η. Further $E_{1,P}$ matches the lift of residues along $F_{P,\eta}$ and $a_{1,P} = (\tilde{a}_{1,i,P})$ where $\tilde{a}_{1,i,P}$ are all units along η. Thus, since a is arranged to be in good shape, we have $m_i = m_{i,P}$.

Let $\tilde{X}_\eta = F_\eta\left(\sqrt[\ell]{u_\eta}\right)$ and let $X_\eta = k_\eta\left(\sqrt[\ell]{\overline{u_\eta}}\right)$. Our goal is to find a $\theta_\eta \in \widehat{A_\eta}^*$ which is a norm from $\tilde{X}_\eta$ so that $\overline{w_\eta\theta_\eta}$ is close to $\overline{w_P}$ in $k_{P,\eta}$ for each $P \in \overline{\eta} \cap S_0$.

If such a θ_η exists, then we set

$$\begin{aligned} E_{2,\eta} &= F_\eta\left(\sqrt[\ell]{w_\eta\theta_\eta\pi_\eta}\right), \\ E_{1,\eta} &= \tilde{X}_\eta = F_\eta\left(\sqrt[\ell]{u_\eta}\right), \\ \tilde{a}_{2,i,\eta} &= (w_\eta\theta_\eta\pi_\eta)^{m_i}\ \forall\, i \leq \ell-1, \\ \tilde{a}_{2,\ell,\eta} &= (\tilde{a}_{2,1,\eta}\dots\tilde{a}_{2,\ell-1,\eta})^{-1}, \\ a_{1,\eta} &= aa_{2,\eta}^{-1}. \end{aligned}$$

Thus clearly Properties 1, 4, 5, 6 and 7 hold. Since θ_η is assumed to be a norm from $\tilde{X}_\eta$, we have $(u_\eta, \theta_\eta) = 0 \in \mathrm{Br}(F_\eta)$. Hence $D = (u_\eta, w_\eta\theta_\eta\pi_\eta) \in \mathrm{Br}(F_\eta)$, which therefore implies Property 2 holds.

Let us check that Property 3 holds. Clearly $\tilde{a}_{2,i,\eta} = (w_\eta\theta_\eta\pi_\eta)^{m_i}$ is a norm from $E_{2,\eta}$ for each $i \leq \ell-1$. Since 1 is a norm always, so is $\tilde{a}_{2,\ell,\eta}$. It is left to show that $\tilde{a}_{1,i,\eta} = x_i'(w_\eta\theta_\eta)^{-m_i}$ is a norm from $\tilde{X}_\eta$ for each $i \leq \ell-1$ (which will automatically imply $\tilde{a}_{1,i,\eta}$ is a norm from $\tilde{X}_\eta$ also as $\mathrm{N}(a_{1,\eta}) = 1$).

Since each $a'_{i,\eta}$ is a reduced norm of D, we have $(u_\eta, w_\eta\theta_\eta\pi_\eta)\left(a'_{i,\eta}\right) = 0$. This implies $(u_\eta, w_\eta\theta_\eta\pi_\eta)\left((w_\eta\theta_\eta\pi_\eta)^{m_i}x_i'(w_\eta\theta_\eta)^{-m_i}\right) = 0$ and hence $(u_\eta, w_\eta\theta_\eta\pi_\eta)\left(x_i'(w_\eta\theta_\eta)^{-m_i}\right) = 0$. Taking residues, we see $\overline{\left(u_\eta, x_i'(w_\eta\theta_\eta)^{-m_i}\right)} = 0$ and thus each $\tilde{a}_{1,i,\eta}$ is a norm from $\tilde{X}_\eta$.

Now let us find θ_η. Recall that X_η is the residue of D_η. For each $P \in \overline{\eta} \cap S_0$, by Proposition 8.3 5(b), we know that $\overline{w_P w_\eta^{-1}}$ is a norm from $X_\eta \otimes k_{P,\eta}$. Thus for each $P \in \overline{\eta} \cap S_0$, let $z_{P,\eta} \in X_\eta \otimes k_{P,\eta}$ such that $\mathrm{N}(z_{P,\eta}) = \overline{w_P w_\eta^{-1}}$. By weak approximation, find $z \in X_\eta$ which is close to each $z_{P,\eta}$. Set $\theta = \mathrm{N}_{X_\eta/k_\eta}(z) \in k_\eta$ and let θ_η denote its lift in F_η. This θ_η satisfies the required properties. ☐

PROPOSITION 10.3 (Green(1)). *Let $\eta \in N_0$ be of Type 1b/2 with Y_η of Type NON-RES. Set $E_{1,\eta}$ and $E_{2,\eta}$ to be the lift of residues at $\widehat{A_\eta}$. Further, set $a_{1,\eta} = a$ and $a_{2,\eta} = 1$. Then for $j = 1, 2$, we have*

1. $a_{1,\eta}a_{2,\eta} = a$
2. $D \otimes E_{j,\eta}$ *has index at most* ℓ.
3. $D \otimes Y \otimes E_{j,\eta}$ *is split.*
4. $a_{j,\eta}$ *is a norm from* $E_{j,\eta} \otimes Y_\eta / Y_\eta$.
5. $\mathrm{N}_{Y_\eta/F_\eta}(a_{j,\eta}) = 1$.
6. $E_{j,\eta} \otimes F_{P,\eta} \simeq E_{j,P} \otimes F_{P,\eta}$ *for each point* $P \in S_0 \cap \overline{\eta}$.
7. $a_{j,P} = a_{j,\eta} \in Y \otimes F_{P,\eta}$ *for each point* $P \in S_0 \cap \overline{\eta}$.

Proof. By hypothesis, η is coloured green. Recall that we have $D = \beta_{rbc,\eta} + (u_\eta, \pi_\eta)$ in $\mathrm{Br}(F_\eta)$ where $\beta_{rbc,\eta}$ is unramified with index at most ℓ. By construction, $E_{j,\eta} \simeq F_\eta\left(\sqrt[\ell]{u_\eta}\right)$. Properties 1 & 5 are immediate while Properties 6 & 7 follow from Proposition 8.4. Note that $D \otimes E_{j,\eta} = \beta_{rbc,\eta} \otimes E_{j,\eta} \in \mathrm{Br}(F_\eta)$ and hence Property 2 holds.

Since D is ramified at η, we have $[E_{j,\eta} : F_\eta] = \ell$. As Y_η is NONRES, we have $E_{j,\eta} \otimes Y_\eta / Y_\eta$ is a field extension of degree ℓ. Then by Lemma 2.9, the index of $(D \otimes Y_\eta)$ equals index $\left(\beta_{rbc,\eta} \otimes_{F_\eta} Y_\eta \otimes_{F_\eta} E_{j,\eta}\right)[E_{j,\eta} \otimes Y_\eta : Y_\eta]$ which is index $\left(\beta_{rbc,\eta} \otimes_{F_\eta} Y_\eta \otimes_{F_\eta} E_{j,\eta}\right) \times \ell$. Since index $(D \otimes Y_\eta) \leq \ell$, we see that $E_{j,\eta} \otimes_{F_\eta} Y_\eta$ splits $\beta_{rbc,\eta}$ and hence also D, implying Property 3.

By hypothesis, $a \in \mathrm{Nrd}(D \otimes Y_\eta)$. That is, $(a)[\beta_{rbc,\eta}] + (a)(u_\eta, \pi_\eta) = 0$ in $\mathrm{H}^3(Y_\eta, \mu_\ell)$. By Proposition 3.3, a is a unit at η. By Lemma 2.7, a is a reduced norm of $\beta_{rbc,\eta} \otimes Y_\eta$ and therefore we have that $(a)(u_\eta, \pi_\eta) = 0$ in $\mathrm{H}^3(Y_\eta, \mu_\ell)$. Thus by taking residues, we see that $\overline{(u_\eta, a)} = 0$ in $\mathrm{H}^2\left(\overline{Y_\eta}, \mu_\ell\right)$ which would imply that a is a norm from $E_{j,\eta} \otimes_{F_\eta} Y_\eta / Y_\eta$. Hence Property 4 holds. □

Proposition 10.4 (Green(2)-RAM). *Let $\eta \in N_0$ be of Type 1b or 2 and let Y_η be of Type RAM. Set $a_{1,\eta} = a$ and $a_{2,\eta} = 1$. Then for $j = 1, 2$, there exist $E_{j,\eta}/F_\eta$, unramified cyclic extensions of degree ℓ such that*

1. $a_{1,\eta} a_{2,\eta} = a$
2. *$E_{j,\eta}$ splits the residual Brauer class $\beta_{rbc,\eta}$.*
3. *$D \otimes E_{j,\eta}$ has index at most ℓ and $D \otimes Y \otimes E_{j,\eta}$ is split.*
4. *$a_{j,\eta}$ is a norm from $E_{j,\eta} \otimes Y_\eta / Y_\eta$.*
5. $\mathrm{N}_{Y_\eta / F_\eta}(a_{j,\eta}) = 1$.
6. *$E_{j,\eta} \otimes F_{P,\eta} \simeq E_{j,P} \otimes F_{P,\eta}$ for each point $P \in S_0 \cap \overline{\eta}$.*
7. *$a_{j,\eta} = a_{j,P} \in Y \otimes F_{P,\eta}$ for each point $P \in S_0 \cap \overline{\eta}$.*
8. *The residue of D_η is a norm from $\overline{E_{j,\eta}}/k_\eta$.*

Proof. By hypothesis, η is coloured green. Properties 1 & 5 are immediate. By Lemma 2.3, $a = a_{1,\eta} \in Y_\eta^{*\ell}$. Since $a_{2,\eta} = 1$, Property 4 holds for whatever degree ℓ extensions $E_{j,\eta}$ we choose. We first construct $\overline{E_{j,\eta}}/k_\eta$ and then set $E_{j,\eta}/k_\eta$ to be the unramified lift of $\overline{E_{j,\eta}}/k_\eta$. We would like to apply Lemma 2.10 to construct $\overline{E_{j,\eta}}$.

Recall that we have $D = \beta_{rbc,\eta} + (u_\eta, w_\eta \pi_\eta)$ in $\mathrm{Br}(F_\eta)$ where $Y_\eta \simeq F_\eta(\sqrt[\ell]{w_\eta \pi_\eta})$ for $w_\eta \in \widehat{A_\eta}^*$. Let $D' = \overline{\beta_{rbc,\eta}}$, the residual Brauer class considered over the residue field k_η. Thus D' is a central simple algebra of exponent and index at most ℓ over global field k_η. Let $u' := \overline{u_\eta} \in k_\eta$. Let $\mathcal{P}'_\eta := \overline{\eta} \cap S_0$. Let $\mathcal{Q}'_\eta$ denote the set of closed points $Q \in \overline{\eta}$ not in $\mathcal{P}'_\eta$ such that $D' \otimes k_{Q,\eta} \neq 0$.

For $P \in \mathcal{P}'_\eta$, set $E'_{j,P} := \overline{E_{j,P} \otimes F_{P,\eta}}/k_{P,\eta}$ and let $w_{1,P}, w_{2,P} \in k^*_{P,\eta}$ be the ones obtained from Proposition 9.1. So $\overline{E_{j,P} \otimes F_{P,\eta}} = k_{P,\eta}[t]/(t^\ell - w_{j,P})$ and

$(u', w_{j,P}) = 0$. For $Q \in \mathcal{Q}'_\eta$, set $E'_{1,Q}$ and $E'_{2,Q}$ to be the unique unramified field extension of $k_{Q,\eta}$ and set $w_{1,Q} = w_{2,Q}$ be any unit in $\mathcal{O}_{k_{Q,\eta}}$ which is not an ℓ^{th} power. Let us verify that the hypotheses of Lemma 2.10 hold now. Let $P \in \mathcal{P}'_\eta \cup \mathcal{Q}'_\eta$. We first check that $(w_{j,P}, u') = 0 \in \operatorname{Br}(k_{P,\eta})$. For $P \in \mathcal{P}'_\eta$, this is assured by Proposition 9.1. For $P \in \mathcal{Q}'_\eta$, u' is a unit in $\mathcal{O}_{k_{P,\eta}}$. Since $w_{j,P}$ is a unit, by local class field theory, $(w_{j,P}, u') = 0$.
Next we verify that $D' \otimes E'_{j,P}$ is trivial. For $P \in \mathcal{P}'_\eta$, this is assured by Proposition 8.5. For $P \in \mathcal{Q}'_\eta$, since each $E'_{j,P}$ is a nonsplit unramified extension of degree ℓ, local class field theory guarantees that it will split any index ℓ algebra over $k_{P,\eta}$.
Clearly for each $Q \notin \left(\mathcal{P}'_\eta \cup \mathcal{Q}'_\eta\right)$, $D' \otimes k_{Q,\eta}$ is split.
Thus Lemma 2.10 can be used to construct $\overline{E_{1,\eta}}$ and $\overline{E_{2,\eta}}$ over k_η. Setting $E_{1,\eta}$ and $E_{2,\eta}$ to be their respective unramified lifts over F_η, it is immediate that Properties 2, 6, and 8 are satisfied. Property 7 is guaranteed by again using Proposition 8.5.
To complete the proof of Property 3, note that as $E_{j,\eta}$ splits $\beta_{rbc,\eta}$ and $Y_\eta = F_\eta\left(\sqrt[\ell]{w_\eta \pi_\eta}\right)$ and $D = \beta_{rbc,\eta} + (u_\eta, w_\eta \pi_\eta) \in \operatorname{Br}(F_\eta)$, it is immediate that $\operatorname{index}(D \otimes_F E_{i,\eta}) \leq \ell$ and that $D \otimes Y_\eta \otimes E_{j,\eta}$ is split. □

Proposition 10.5 (Green(2)-RES). *Let $\eta \in N_0$ be of Type 1b or 2 and let Y_η be of Type RES. Then for $j = 1, 2$, there exist $E_{j,\eta}/F_\eta$, unramified cyclic extensions of degree ℓ and elements $a_{1,\eta}, a_{2,\eta} \in \mathcal{O}_{Y_\eta}$ such that*

1. $a_{1,\eta} a_{2,\eta} = a$.
2. $E_{j,\eta}$ *splits* $\beta_{rbc,\eta}$.
3. $D \otimes E_{j,\eta}$ *has index at most* ℓ *and* $D \otimes Y \otimes E_{j,\eta}$ *is split.*
4. $a_{j,\eta}$ *is a norm from* $E_{j,\eta} \otimes Y_\eta / Y_\eta$.
5. $\mathrm{N}_{Y_\eta/F_\eta}(a_{j,\eta}) = 1$.
6. $E_{j,\eta} \otimes F_{P,\eta} \simeq E_{j,P} \otimes F_{P,\eta}$ *for each point* $P \in S_0 \cap \overline{\eta}$.
7. $a_{j,\eta} \mu_{j,P,\eta} = a_{j,P}$ *in* $Y \otimes F_{P,\eta}$ *for each point* $P \in S_0 \cap \overline{\eta}$ *for some* $\mu_{j,P,\eta} \in \mathcal{O}_{Y \otimes F_{P,\eta}}$ *such that* $\mu_{j,P,\eta} \cong 1 \bmod (\pi_\eta)$ *and* $\mathrm{N}(\mu_{j,P,\eta}) = 1$.
8. *The residue of* $D \otimes F_\eta$ *is a norm from* $\overline{E_{j,\eta}}/k_\eta$.

Proof. By hypothesis, η is coloured green. Since Y_η is of Type RES, by Lemma 3.3 we have that $a \in \mathcal{O}^*_{Y,\eta}$. Recall that we have $D = \beta_{rbc,\eta} + (u_\eta, \pi_\eta)$ in $\operatorname{Br}(F_\eta)$. Set $u' := \overline{u_\eta} \in k_\eta$, $Y' := \overline{Y_\eta} = k_\eta\left(\sqrt[\ell]{\overline{u'}}\right)$ and $a' = \overline{a} \in Y'$ and let $\operatorname{Gal}(Y'/k_\eta) = \langle \psi \rangle$. Let $\mathcal{P}'_\eta := \overline{\eta} \cap S_0$. By Proposition 8.5, $a_{j,P}$ is a unit along η. First let's construct $a'_1 \in Y'$ approximating $\overline{a_{1,P}} \in \overline{Y \otimes F_P \otimes F_{P,\eta}}$ for each $P \in \mathcal{P}'_\eta$. Since $\mathrm{N}(\overline{a_{1,P}}) = 1$, by Hilbert 90 there exists $c_P \in Y' \otimes k_{P,\eta}$ such that $c_P^{-1} \psi(c_P) = \overline{a_{1,P}}$. Using weak approximation, find $c \in Y'$ which is close to c_P for each $P \in \mathcal{P}'_\eta$. Set $a'_1 = c^{-1}\psi(c)$ and set $a'_2 = a' {a'_1}^{-1}$. Let $a_{1,\eta}$ denote a lift of a'_1 and let $a_{2,\eta} = a a_{1,\eta}^{-1}$.

Therefore Properties 1, 5 & 7 are immediate. We will first construct $\overline{E_{j,\eta}}/k_\eta$ and then set $E_{j,\eta}/k_\eta$ to be the unramified lift of $\overline{E_{j,\eta}}/k_\eta$. We appeal to Lemma 2.11 to construct $\overline{E_{j,\eta}}$.

Let $D' = \overline{\beta_{rbc,\eta}}$, the residual Brauer class considered over the residue field k_η. Thus D' is a central simple algebra of exponent and index at most ℓ over global field k_η. Let $\mathcal{Q}'_\eta$ denote the set of closed points $Q \in \overline{\eta}$ not in $\mathcal{P}'_\eta$ such that $D' \otimes k_{Q,\eta} \neq 0$.

Let $j = 1$ or 2. For $P \in \mathcal{P}'_\eta$, set $E'_{j,P} := \overline{E_{j,P} \otimes F_{P,\eta}}/k_{P,\eta}$. For $P \in \mathcal{Q}'_\eta$, set $E'_{j,P}$ to be the unique unramified field extension of $k_{P,\eta}$. Let $\tilde{\mathrm{L}}$ denote the Galois closure of $Y'(\sqrt[\ell]{a'_1})$ and let $\tilde{\mathrm{L}}'$ denote the Galois closure of $Y'(\sqrt[\ell]{a'_2})$.

Whenever $Y' \otimes k_{Q,\eta}$ is not split, since a'_1 and a'_2 have norm 1, they are also units and in fact ℓ^{th} powers in the complete discretely valued field $Y' \otimes k_{Q,\eta}$ for every $Q \in \overline{\eta}$ (Lemmata 2.3 and 2.5). Further, since Y_η is RES, we note that Y' is unramified except at points $P \in S_0 \cap \overline{\eta}$ of Type C_{11}^{Cold} or C_{12}^{Cold}. For each $P \in \mathcal{P}'_\eta$, recall the extensions $\tilde{\mathrm{L}}_P, \tilde{\mathrm{L}}'_P$ defined in Section 9.2. Note that the extension $\tilde{\mathrm{L}} \otimes_{k_\eta} k_{P,\eta} \simeq \prod_{i=1}^{g} \tilde{\mathrm{L}}_P$ and extension $\tilde{\mathrm{L}}' \otimes_{k_\eta} k_{P,\eta} \simeq \prod_{j=1}^{h} \tilde{\mathrm{L}}'_P$ for some $g, h \geq 1$.

Proposition 9.2 says that there exist $w_P, w'_P \in k_{P,\eta}$, $z_P \in \tilde{\mathrm{L}}_P$ and $z'_P \in \tilde{\mathrm{L}}'_P$ so that $E'_{1,P} = k_{P,\eta}[t]/\left(t^\ell - w_P\right)$, $\mathrm{N}_{\tilde{\mathrm{L}}_P/k_{P,\eta}}(z_P) = w_P$, $E'_{2,P} = k_{P,\eta}[t]/\left(t^\ell - w'_P\right)$ and $\mathrm{N}_{\tilde{\mathrm{L}}'_P/k_{P,\eta}}(z'_P) = w'_P$. For each $P \in \mathcal{Q}'_\eta$, let $w_P = w'_P$ be any unit in $\mathcal{O}_{k_{P,\eta}}$ which is not an ℓ^{th} power.

We now construct[20] the extensions $\overline{E_{j,\eta}}$ using Lemma 2.11 by verifying that the hypotheses of the same hold. Let $P \in \mathcal{P}'_\eta \cup \mathcal{Q}'_\eta$.

We need to find $\tilde{z}_P \in \tilde{\mathrm{L}} \otimes k_{P,\eta} = \prod \tilde{\mathrm{L}}_P$ (respectively $\tilde{z}'_P \in \tilde{\mathrm{L}}' \otimes k_{P,\eta} = \prod \tilde{\mathrm{L}}'_P$) such that its norm to $k_{P,\eta}$ is w_P (resp w'_P). For $P \in \mathcal{P}'_\eta$, set $\tilde{z_P} = (z_P, 1, 1, \ldots, 1)$ and $\tilde{z'_P} = (z'_P, 1, 1, \ldots, 1)$ and use Proposition 9.2 to conclude the proof in this case. For $P \in \mathcal{Q}'_\eta$, we claim that Y'_P is a nonsplit unramified extension of $k_{P,\eta}$. This is because of the following:

Write $D = D_{00} + (u_P, \pi_P) \in \mathrm{Br}(F)$ where $u_P \in \widehat{A_P}^*$ and π_P is a prime corresponding to the curve $\overline{\eta}$ ([S97]). We also have $D = \beta_{rbc,\eta} + (u_\eta, \pi_\eta) \in \mathrm{Br}\,(F_\eta)$. Note that $\pi_P = \pi_\eta w_\eta \in F_\eta$ where $w_\eta \in \widehat{A_\eta}^*$. Comparing these two expressions in $\mathrm{Br}\,(F_{P,\eta})$, we see $(u_P, w_\eta) + (u_P, \pi_\eta) = \beta_{rbc,\eta} + (u_\eta, \pi_\eta)$. Taking residues, we see that $\overline{u_P u_\eta}^{-1} = 1$ up to ℓ^{th} powers in $k_{P,\eta}$. And hence $\beta_{rbc,\eta} = (u_\eta, w_\eta) \in \mathrm{Br}\,(F_{P,\eta})$. Now we are looking at a place $P \notin \mathcal{P}'_\eta$ such that this algebra is not trivial. In particular, this implies $\overline{u_\eta}$ is not an ℓ^{th} power. Therefore Y'_P is not split.

As observed before, this further implies a'_1 and a'_2 are units (and in fact ℓ^{th} powers) in $\mathcal{O}_{Y'_P}$. Thus $\tilde{\mathrm{L}} \otimes k_{P,\eta} = \prod \tilde{\mathrm{L}}_P$ and $\tilde{\mathrm{L}}' \otimes k_{P,\eta} = \prod \tilde{\mathrm{L}}'_P$ where $\tilde{\mathrm{L}}_P = \tilde{\mathrm{L}}'_P = Y'_P$. Since Y'_P is unramified nonsplit extension of $k_{P,\eta}$, every unit of $k_{P,\eta}$ is a norm from it and hence from $\tilde{L}_P$ and $\tilde{L}'_P$, which finishes the proof of this case.

We need to verify that $D' \otimes E'_{j,P}$ is trivial for all $P \in \overline{\eta}$. For $P \in \mathcal{P}'_\eta$, use Proposi-

[20] If a'_j is an ℓ^{th} power in Y', Property 4 is automatically satisfied for $a_{j,\eta}$. A check of the relevant rows mentioned in Proposition 8.5 show u' is a norm from $\overline{E_{j,P}} \otimes k_{P,\eta}$. We can then use Lemma 2.10 to construct $\overline{E_{j,\eta}}$.

tion 8.5 to conclude the proof in this case. For $P \in \mathcal{Q}'_\eta$, since each $E'_{j,P}$ is a nonsplit unramified extension of degree ℓ, local class field theory guarantees that it will split any index ℓ algebra over $k_{P,\eta}$. Also clearly for each $Q \notin \left(\mathcal{P}'_\eta \cup \mathcal{Q}'_\eta\right)$, $D' \otimes k_{Q,\eta}$ is split already.

We need to verify that $(w_P, a'_1)_y = (w'_P, a'_2)_y = 0$ for every valuation $y \in \Omega_{Y'}$ lying over P. For $P \in \mathcal{P}'_\eta$, this is assured by Proposition 7.1 (4). For $P \in \mathcal{Q}'_\eta$, we have already noted that Y'_P is unramified and nonsplit over $k_{P,\eta}$. By Lemma 2.5, a'_1 and a'_2 are ℓ^{th} powers in $Y' \otimes k_{P,\eta}$. So $(w_P, a'_1) = 0 = (w'_P, a'_2)$.

Thus Lemma 2.11 can be used to construct $\overline{E_{j,\eta}}$ over k_η for $j = 1, 2$. Setting $E_{j,\eta}$ to be their respective unramified lifts over F_η, it is immediate that Properties 2, 4, 6 and 8 are satisfied. To complete the proof of Property 3, note that as $E_{j,\eta}$ splits $\beta_{rbc,\eta}$, $Y_\eta = F_\eta\left(\sqrt[\ell]{u_\eta}\right)$ and $D = \beta_{rbc,\eta} + (u_\eta, \pi_\eta) \in \mathrm{Br}\,(F_\eta)$, it is immediate that index $(D \otimes_F E_{j,\eta}) \leq \ell$ and that $D \otimes Y_\eta \otimes E_{j,\eta}$ is split. □

Proposition 10.6 (Yellow/Orange/Red/White). *Let $\eta \in N_0$ be coloured yellow, orange or white. Set $E_{2,\eta}$ to be the lift of residues at $\widehat{A_\eta}$. Then there exists a ramified cyclic extension $E_{1,\eta}/F_\eta$ of degree ℓ and elements $a_{1,\eta} = (\tilde{a}_{1,i,\eta})$ and $a_{2,\eta} = (\tilde{a}_{2,i,\eta}) \in \prod F_\eta$ such that for $j = 1, 2$, the following holds:*

1. *$a_{1,\eta}a_{2,\eta} = a \in Y_\eta$, i.e. $\tilde{a}_{1,i,\eta}\tilde{a}_{2,i,\eta} = a'_{i,\eta}$ for each i.*
2. *$E_{j,\eta}$ splits D.*
3. *$a_{j,\eta}$ is a norm from $E_{j,\eta} \otimes Y_\eta/Y_\eta$, i.e . $\tilde{a}_{j,i,\eta}$ is a norm from $E_{j,\eta}$ for each i.*
4. *$\mathrm{N}_{Y_\eta/F_\eta}(a_{j,\eta}) = 1$, i.e. $\prod_i \tilde{a}_{j,i,\eta} = 1$.*
5. *$E_{j,\eta} \otimes F_{P,\eta} \simeq E_{j,P} \otimes F_{P,\eta}$ for each point $P \in S_0 \cap \overline{\eta}$.*
6. *$\tilde{a}_{j,i,\eta}\mu_{j,i,P,\eta} = \tilde{a}_{j,i,P} \in F_{P,\eta}$ for all i at each point $P \in S_0 \cap \overline{\eta}$ for some $\mu_{j,i,P,\eta} \in F^{*\ell}_{P,\eta}$ such that $\prod_i \mu_{j,i,P,\eta} = 1$.*
7. *$E_{2,\eta}/F_\eta$ is unramified and cyclic of degree ℓ.*

Proof. By hypothesis, η is a Ch/H/C/Z curve of Type 1b with Y_η of Type SPLIT. The proof is similar to the proof of Proposition 10.2 (we appeal to Proposition 8.6 to ensure compatibility at branches). □

11 Patching data at uncoloured points of N_0

Let $\eta \in N_0$ be of Type 1a or 0 and let π_η be a parameter of F_η as before. Set $\mathcal{P}'_\eta := \overline{\eta} \cap S_0$. If η is of Type 0, set $\mathcal{Q}'_\eta := \emptyset$. If η is of Type 1a, set $D' = \overline{D \otimes F_\eta}$ over the residue field k_η. Thus D' is a central simple algebra over the global field k_η of exponent and index dividing ℓ. Let $\mathcal{Q}'_\eta$ denote the set of closed points $Q \in \overline{\eta}$ not in $\mathcal{P}'_\eta$ such that $D' \otimes k_{Q,\eta} \neq 0$. For $j = 1, 2$ and any $P \in S_0$, let $E_{j,P}$ and $a_{j,P}$ be as prescribed in Propositions 7.1 and 7.2. We now prescribe the choices for $E_{j,\eta}$ and $a_{j,\eta}$. Tables 9, 10, 11, 12, 13, 18, 19 and 20 are relevant in this section.

Proposition 11.1 (0/1a-RAM). *Let $\eta \in N_0$ be of Type 0 or 1a and let Y_η be of Type RAM. Set $a_{1,\eta} = a$ and $a_{2,\eta} = 1$. Then for $j = 1, 2$, there exist $E_{j,\eta}/F_\eta$, unramified cyclic extensions of degree ℓ such that*

1. $a_{1,\eta}a_{2,\eta} = a$ *in* Y_η.
2. $D \otimes E_{j,\eta}$ *is split. If η is of Type 0, then* $E_{j,\eta} \simeq \prod F_\eta$.
3. $a_{j,\eta}$ *is a norm from* $E_{j,\eta} \otimes Y_\eta/Y_\eta$.
4. $\mathrm{N}_{Y_\eta/F_\eta}(a_{j,\eta}) = 1$.
5. $E_{j,\eta} \otimes F_{P,\eta} \simeq E_{j,P} \otimes F_{P,\eta}$ *for each point* $P \in S_0 \cap \overline{\eta}$.
6. $a_{j,\eta} = a_{j,P} \in Y \otimes F_{P,\eta}$ *for each point* $P \in S_0 \cap \overline{\eta}$.

Proof. By Proposition 3.3 and Lemma 2.3, $a \in \mathcal{O}^{*\ell}_{Y_\eta}$. By Proposition 8.7, $a_{1,P} = a$ and $a_{2,P} = 1$ for each $P \in \mathcal{P}'_\eta$. Hence Properties 1, 3, 4 and 6 hold.
Let η be of Type 0. Note that by Remark 5.2, η cannot intersect $\eta' \in N'_0$ with $Y_{\eta'}$ of Type SPLIT. By inspection of the relevant tables, we see that $E_{1,P} = E_{2,P} = \prod F_P$ for any $P \in \mathcal{P}'_\eta$. Set $E_{j,\eta} = \prod F_\eta$. Hence Properties 2 & 5 hold in this case.
Let η be of Type 1a. Let $j = 1$ or 2. For $P \in \mathcal{P}'_\eta$, set $E'_{j,P} := \overline{E_{j,P} \otimes F_{P,\eta}}/k_{P,\eta}$. For $P \in \mathcal{Q}'_\eta$, set $E'_{j,P}$ to be the unique unramified field extension of $k_{P,\eta}$ of degree ℓ. $D' \otimes E'_{j,P}$ is trivial for all $P \in \mathcal{P}'_\eta \cup \mathcal{Q}'_\eta$ (cf. Proof of Proposition 7.1 for $P \in \mathcal{P}'_\eta$ and local class field theory for $P \in \mathcal{Q}'_\eta$). Also clearly for each $Q \in \overline{\eta} \setminus (\mathcal{P}'_\eta \cup \mathcal{Q}'_\eta)$, $D' \otimes k_{Q,\eta}$ is split already. Set $u' = 1$ and use Lemma 2.10 to construct $\overline{E_{j,\eta}}$. Set $E_{j,\eta}/F_\eta$ to be the unramified lift of $\overline{E_{j,\eta}}/k_\eta$ to see that Properties 2 & 5 hold. □

Proposition 11.2 (0/1a-SPLIT). *Let $\eta \in N_0$ be of Type 0 or 1a and let Y_η be of Type SPLIT. Then for $j = 1, 2$, there exist $E_{j,\eta}/F_\eta$, unramified cyclic extensions of degree ℓ and elements $a_{j,\eta} = (\tilde{a}_{j,i,\eta})_i \in \prod F_\eta$ such that*

1. $a_{1,\eta}a_{2,\eta} = a = \left(a'_{i,\eta}\right)_i$ *in Y_η, i.e* $\tilde{a}_{1,i,\eta}\tilde{a}_{2,i,\eta} = a'_{i,\eta}$ *for each i.*
2. $D \otimes E_{j,\eta}$ *is split. If η is of Type 0, then* $E_{j,\eta} \simeq \prod F_\eta$.
3. $a_{j,\eta}$ *is a norm from $E_{j,\eta} \otimes Y_\eta/Y_\eta$, i.e. $\tilde{a}_{j,i,\eta}$ is a norm from $E_{j,\eta}$ for each i.*
4. $\mathrm{N}_{Y_\eta/F_\eta}(a_{j,\eta}) = 1$, *i.e.* $\prod \tilde{a}_{j,i,\eta} = 1$.
5. $E_{j,\eta} \otimes F_{P,\eta} \simeq E_{j,P} \otimes F_{P,\eta}$ *for each point* $P \in S_0 \cap \overline{\eta}$.
6. $\tilde{a}_{j,i,\eta}\mu_{j,i,P,\eta} = \tilde{a}_{j,i,P} \in F_{P,\eta}$ *for all i at each point $P \in S_0 \cap \overline{\eta}$ for some $\mu_{j,i,P,\eta} \in F^{*\ell}_{P,\eta}$ such that* $\prod_i \mu_{j,i,P,\eta} = 1$.

Proof. Let $a = \left(a'_{i,\eta}\right) \in \prod F_\eta$ where $a'_{i,\eta} = x'_i\pi_\eta^{m_i}$ where $m_i \in \mathbb{Z}$ and $x'_i \in \widehat{A_\eta}^*$. Because $\mathrm{N}(a) = 1$, we have $\sum m_i = 0$ and $\prod x'_i = 1$.
Let η be of Type 0. Since D_η is already split, Property 2 is satisfied. Choose $\{j, j'\} = \{1, 2\}$ as in Proposition 8.7.

First let's construct $a_{j,\eta} \in \prod F_\eta$ approximating $a_{j,P}$ for each $P \in \mathcal{P}'_\eta$. By inspection of the relevant tables, we see that $a_{j,P} = (\tilde{a}_{j,i,P})$ is such that $\tilde{a}_{j,i,P} = x_{i,P}\pi_P^{m_i}$ where $x_{i,P} \in \mathcal{O}^*_{F_{P,\eta}}$. Since $\mathrm{N}(a_{j,P}) = 1$, we have $\prod x_{i,P} = 1$. For $1 \le i \le \ell - 1$ by weak approximation, find $c_i \in k^*_\eta$ which is close to $\overline{x_{i,P}}$ in $k^*_{P,\eta}$ and let $\tilde{c}_i$ be a lift of c_i in $\widehat{A_\eta}^*$. Let $c_\ell = \prod_{r=1}^{\ell-1}(c_r)^{-1}$ and $\tilde{c}_\ell = \prod_{r=1}^{\ell-1}(\tilde{c}_r)^{-1}$. Set $\tilde{a}_{j,i,\eta} = \tilde{c}_i\pi_\eta^{m_i}$ and $a_{j',\eta} = aa_{j,\eta}^{-1}$. Thus $\tilde{a}_{j',i,\eta} = x'_i\tilde{c}_i^{-1} \in \widehat{A_\eta}^*$. Let $c'_i = \overline{\tilde{a}_{j',i,\eta}} \in k_\eta$. Therefore Properties 1, 4 & 6 are immediate.

Set $E_{j,\eta} = \prod F_\eta$. Note that by Proposition 8.7, $E_{j,P} = \prod F_P$. Thus Properties 3 & 5 are satisfied for $a_{j,\eta}$ and $E_{j,\eta}$. For $P \in \mathcal{P}'_\eta$, set $E'_{j',P} := \overline{E_{j',P} \otimes F_{P,\eta}}/k_{P,\eta}$. Let $X' = k_\eta\left(\sqrt[\ell]{c'_1}, \ldots, \sqrt[\ell]{c'_\ell}\right)$. By inspection of the relevant tables, we find that one of the following hold for $P \in \mathcal{P}'_\eta$:

- $E'_{j',P} = \prod k_{P,\eta}$: In this case, set $w'_P = 1$.
- $E'_{j',P} = k_{P,\eta}\left(\sqrt[\ell]{\overline{\delta_P}}\right)$: In this case, also note that $c'_i = \overline{\delta_P^{n_i}}$ up to ℓ^{th} powers. Hence $X' \otimes k_{P,\eta} = k_{P,\eta}\left(\sqrt[\ell]{\overline{\delta_P}}\right)$. Set $w'_P = \overline{\delta_P}$.

Thus for $P \in \mathcal{P}'_\eta$ we have found $w'_P \in k_{P,\eta}$ so that $E'_{j',P} = k_{P,\eta}[t]/\left(t^\ell - w'_P\right)$ and $w'_P = \mathrm{N}_{X'\otimes k_{P,\eta}/k_{P,\eta}}(z'_P)$ for suitable elements z'_P. By weak approximation, we can find $z' \in X'$ close to z'_P. Let $w' = \mathrm{N}(z')$. Set $\overline{E_{j',\eta}} = k_\eta[t]/(t^\ell - w')$. Thus $(w', c'_i) = 0$ in $\mathrm{Br}(k_\eta)$ for all i. Let $E_{j',\eta}$ be the unramified lift of $\overline{E_{j',\eta}}$. This shows that Properties 3 & 5 hold for $a_{j',\eta}$ and $E_{j',\eta}$ as well.

Let η be of Type 1a. By Lemma 4.2 we have that $m_i = \ell m'_i$ and $a'_{i,\eta} = x'_i\pi_\eta^{\ell m'_i}$. For $Q \in \mathcal{Q}'_\eta$, set $a_{1,Q} = a$ and $a_{2,Q} = 1$. Since a is arranged to be in good shape (Proposition 3.5), we see that at these points $a_{1,Q} = \left(x_{i,Q}\pi_Q^{\ell m'_i}\right)$ where π_Q is some prime in a regular system of parameters defining η at Q and $x_{i,Q} \in \widehat{A_Q}^*$.

First let's construct $a_{1,\eta} \in \prod F_\eta$ approximating $a_{1,P}$ for each $P \in \mathcal{P}'_\eta \cup \mathcal{Q}'_\eta$. By the above discussion and inspection of the relevant tables, we see that in $\prod F_{P,\eta}$, $a_{1,P} = (x_{i,P})_i$ or $\left(x_{i,P}\pi_P^{\ell m'_i}\right)_i$ where $x_{i,P} \in \widehat{A_{P,\eta}}^*$. Since $\mathrm{N}(a_{1,P}) = 1$, we have $\prod x_{i,P} = 1$. For $1 \le i \le \ell - 1$, by weak approximation, find $c_i \in k_\eta$ which is close to $\overline{x_{i,P}}$ in $k_{P,\eta}$ and let $\tilde{c}_i$ be a lift of c_i in F_η. Let $c_\ell = \prod_{r=1}^{\ell-1}(c_r)^{-1}$, $\tilde{c}_\ell = \prod_{r=1}^{\ell-1}(\tilde{c}_r)^{-1}$ and $c'_r = \overline{x'_r\tilde{c}_r^{-1}}$ for $r \le \ell$. Let $a_{1,\eta} = (\tilde{c}_i)$ and let $a_{2,\eta} = aa_{1,\eta}^{-1}$. Thus Properties 1, 4 & 6 are immediate.

Let $j = 1$ or 2. For $P \in \mathcal{P}'_\eta$, set $E'_{j,P} := \overline{E_{j,P} \otimes F_{P,\eta}}/k_{P,\eta}$. For $P \in \mathcal{Q}'_\eta$, set $E'_{j,P}$ to be the unique unramified field extension of $k_{P,\eta}$. $D' \otimes E'_{j,P}$ is trivial for all $P \in \mathcal{P}'_\eta \cup \mathcal{Q}'_\eta$ (cf. Proof of Proposition 7.1 for $P \in \mathcal{P}'_\eta$ and local class field theory for $P \in \mathcal{Q}'_\eta$). Also clearly for each $Q \in \overline{\eta} \setminus \left(\mathcal{P}'_\eta \cup \mathcal{Q}'_\eta\right)$, $D' \otimes k_{Q,\eta}$ is split already.

Let $X = k_\eta\left(\sqrt[\ell]{c_1}, \ldots, \sqrt[\ell]{c_\ell}\right)$ and let $X' = k_\eta\left(\sqrt[\ell]{c'_1}, \ldots, \sqrt[\ell]{c'_\ell}\right)$. Then $X \otimes_{k_\eta} k_{P,\eta} \simeq \prod_{i=1}^{g} X_P$ (resp. $X' \otimes_{k_\eta} k_{P,\eta} \simeq \prod_{j=1}^{h} X'_P$) where X_P and X'_P are as in Proposi-

tion 9.4 if $P \in \mathcal{Q}'_\eta$ and are unramified field extensions[21] of $k_{P,\eta}$ if $P \in \mathcal{Q}'_\eta$.
For each $P \in \mathcal{Q}'_\eta$, let $w_P = w'_P$ be any unit in $\mathcal{O}_{k_{P,\eta}}$ which is not an ℓ^{th} power, which therefore are norms from unramified extensions X_P and X'_P respectively. For each $P \in \mathcal{P}'_\eta$ choose $w_P, w'_P \in k_{P,\eta}$ as in Proposition 9.4. Thus for $P \in \mathcal{P}'_\eta \cup \mathcal{Q}'_\eta$, we have $E'_{1,P} = k_{P,\eta}[t]/\left(t^\ell - w_P\right)$, $E'_{2,P} = k_{P,\eta}[t]/\left(t^\ell - w'_P\right)$ with $w_P = \mathrm{N}_{X\otimes k_{P,\eta}/k_{P,\eta}}(z_P)$ and $w'_P = \mathrm{N}_{X'\otimes k_{P,\eta}/k_{P,\eta}}(z'_P)$ for suitable elements z_P and z'_P. By weak approximation, we can find $z \in X$ and $z' \in X'$ close to z_P and z'_P respectively. Let $w = \mathrm{N}(z)$ and $w' = \mathrm{N}(z')$. Set $E'_1 = k_\eta[t]/(t^\ell - w)$ and $E'_2 = k_\eta[t]/(t^\ell - w')$. Thus $(w, c_i) = 0$ and $(w', c'_i) = 0$ in $\mathrm{Br}(k_\eta)$ for all i. Let $E_{j,\eta}$ be unramified lifts of E'_j. These extensions approximate $E_{j,P}$ and Properties 2, 3 & 5 hold. □

Proposition 11.3 (0/1a-NONRES). *Let $\eta \in N_0$ be of Type 0 or 1a and let Y_η be of Type NONRES. Then for $j = 1, 2$, there exist $E_{j,\eta}/F_\eta$, unramified cyclic extensions of degree ℓ and elements $a_{j,\eta} \in \mathcal{O}_{Y,\eta}$ such that*

1. *$a_{1,\eta}a_{2,\eta} = a$ in Y_η.*
2. *D_η is split if η is of Type 0. Else $D \otimes E_{j,\eta}$ has index at most ℓ and $D \otimes Y \otimes E_{j,\eta}$ is split. Further if η is of Type 0, then $E_{j,\eta} \simeq \prod F_\eta$.*
3. *$a_{j,\eta}$ is a norm from $E_{j,\eta} \otimes Y_\eta/Y_\eta$.*
4. *$\mathrm{N}_{Y_\eta/F_\eta}(a_{j,\eta}) = 1$.*
5. *$E_{j,\eta} \otimes F_{P,\eta} \simeq E_{j,P} \otimes F_{P,\eta}$ for each point $P \in S_0 \cap \overline{\eta}$.*
6. *$a_{j,\eta}\mu_{j,P,\eta} = a_{j,P}$ in $Y \otimes F_{P,\eta}$ for each point $P \in S_0 \cap \overline{\eta}$ for some $\mu_{j,P,\eta} \in \mathcal{O}_{Y\otimes F_{P,\eta}}$ such that $\mu_{j,P,\eta} \cong 1 \bmod (\pi_\eta)$ and $\mathrm{N}(\mu_{j,P,\eta}) = 1$.*

Proof. By Proposition 3.3 we have that $a \in \mathcal{O}^*_{Y,\eta}$. Let $Y' = \overline{Y_\eta} = k_\eta\left(\sqrt[\ell]{u'}\right)$, $a' = \overline{a} \in Y'$ and $\mathrm{Gal}(Y'/k_\eta) = \langle\psi\rangle$. For $P \in \mathcal{Q}'_\eta$, set $a_{1,P} = a$ and $a_{2,P} = 1$. Further since a is in good shape and $P \notin S_0$, we see that $a_{j,P}$ are units along η and further, $\overline{a_{j,P}} \in \mathcal{O}^*_{Y'_P}$. Since Y is also arranged to be in good shape, Y'_P is an unramified (possibly split) extension of $k_{P,\eta}$. By inspecting the relevant tables, we see that $a_{j,P}$ are units along η for $P \in \mathcal{P}'_\eta$ also.
First let's construct $a'_1 \in Y'$ approximating $\overline{a_{1,P}} \in \overline{Y \otimes F_{P,\eta}}$ for each $P \in \mathcal{P}'_\eta \cup \mathcal{Q}'_\eta$. Since $\mathrm{N}(\overline{a_{1,P}}) = 1$, by Hilbert 90 there exists $c_P \in Y' \otimes k_{P,\eta}$ such that $c_P^{-1}\psi(c_P) = \overline{a_{1,P}}$. Using weak approximation, find $c \in Y'$ which is close to c_P for each P. Set $a'_1 = c^{-1}\psi(c)$ and set $a'_2 = a'{a'_1}^{-1}$. Let $a_{1,\eta}$ denote a lift of a'_1 and let $a_{2,\eta} = aa^{-1}_{1,\eta}$. Then Properties 1, 4 & 6 are immediate.
Let η be of Type 0. Property 2 is satisfied by the definition of Type 0. Choose $\{j, j'\} = \{1, 2\}$ as in Proposition 8.7. Set $E_{j,\eta} = \prod F_\eta$. Since by the same proposition, $E_{j,P} = \prod F_P$, Properties 3 & 5 are satisfied for $a_{j,\eta}$ and $E_{j,\eta}$.

[21] since $x_{i,P} \in \widehat{A_P}^*$ at these points.

For $P \in \mathcal{P}'_\eta$, set $E'_{j',P} := \overline{E_{j',P} \otimes F_{P,\eta}}/k_{P,\eta}$. Let $\tilde{\mathrm{L}}'$ denote the Galois closure of $Y'(\sqrt[\ell]{a'_{j'}})$. Letting $D' = 0 \in \mathrm{Br}(k_\eta)$, we would like to apply Lemma 2.11 to construct $E'_{j'} = \overline{E_{j',\eta}}$ first. Thus for each $P \in \mathcal{P}'_\eta$, we would like to first find $w'_P \in k_{P,\eta}$ and $z'_P \in \tilde{\mathrm{L}}' \otimes k_{P,\eta}$ so that $E'_{j',P} = k_{P,\eta}[t]/\left(t^\ell - w'_P\right)$ and $\mathrm{N}_{\tilde{\mathrm{L}}' \otimes k_{P,\eta}/k_{P,\eta}}(z'_P) = w'_P$.
By inspection of the relevant tables, we find that one of the following hold :

- $E'_{j',P} = \prod k_{P,\eta}$: In this case, set $w'_P = 1$.
- $E'_{j',P} = k_{P,\eta}\left(\sqrt[\ell]{\overline{\delta_P}}\right)$: In this case, also note that $a_{j',P} = \left(\overline{\delta_P^{n_i}}\right)_i$. Hence $\tilde{\mathrm{L}}' \otimes k_{P,\eta} = k_{P,\eta}\left(\sqrt[\ell]{\overline{\delta_P}}\right)$. So set $w'_P = \overline{\delta_P}$

Similarly it is an immediate check that $\left(w'_P, a_{j'}\right)_y = 0$ for every valuation $y \in \Omega_{Y'}$ lying over P. Thus Lemma 2.11 can be used to construct $E'_{j'}$. Setting $E_{j',\eta}$ to be its unramified lift over F_η, it is immediate that Properties 3 & 5 are satisfied.
LET η BE OF TYPE 1A. Let $j = 1$ or 2. We would like to apply Lemma 2.11 to first construct $E'_j = \overline{E_{j,\eta}}$. For $P \in \mathcal{P}'_\eta$, set $E'_{j,P} := \overline{E_{j,P} \otimes F_{P,\eta}}/k_{P,\eta}$. For $P \in \mathcal{Q}'_\eta$, set $E'_{j,P}$ to be the unique unramified field extension of $k_{P,\eta}$ of degree ℓ.
$D' \otimes Y' \otimes E'_{j,P}$ is trivial for all $P \in \mathcal{P}'_\eta \cup \mathcal{Q}'_\eta$ (cf. Proof of Proposition 7.1 for $P \in \mathcal{P}'_\eta$ and local class field theory for $P \in \mathcal{Q}'_\eta$). Also clearly for each $Q \in \overline{\eta} \setminus \left(\mathcal{P}'_\eta \cup \mathcal{Q}'_\eta\right)$, $D' \otimes k_{Q,\eta}$ is split already.
Let $\tilde{\mathrm{L}}$ denote the Galois closure of $Y'(\sqrt[\ell]{a'_1})$ and let $\tilde{\mathrm{L}}'$ denote the Galois closure of $Y'(\sqrt[\ell]{a'_2})$. Note that whenever $Y' \otimes k_{Q,\eta}$ is not split, since a'_1 and a'_2 have norm 1, they are also units (and in fact ℓ^{th} powers) in the complete discretely valued field $Y' \otimes k_{Q,\eta}$ for every $Q \in \overline{\eta}$ by Lemmata 2.3 and 2.5. Then as in the previous proof, $\tilde{\mathrm{L}} \otimes_{k_\eta} k_{P,\eta} \simeq \prod_{i=1}^{g} \tilde{\mathrm{L}}_P$ (resp. $\tilde{\mathrm{L}}' \otimes_{k_\eta} k_{P,\eta} \simeq \prod_{j=1}^{h} \tilde{\mathrm{L}}'_P$) where $\tilde{\mathrm{L}}_P$ and $\tilde{\mathrm{L}}'_P$ are as in Proposition 9.3 if $P \in \mathcal{P}'_\eta$ and are unramified field extensions[22] if $P \in \mathcal{Q}'_\eta$.
For each $P \in \mathcal{Q}'_\eta$, let $w_P = w'_P$ be any unit in $\mathcal{O}_{k_{P,\eta}}$ which is not an ℓ^{th} power, which therefore are norms from unramified extensions $\tilde{\mathrm{L}}_P$ and $\tilde{\mathrm{L}}'_P$ respectively. For each $P \in \mathcal{P}'_\eta$ choose $w_P, w'_P \in k_{P,\eta}$ as in Proposition 9.3. Thus for $P \in \mathcal{P}'_\eta \cup \mathcal{Q}'_\eta$, we have $E'_{1,P} = k_{P,\eta}[t]/\left(t^\ell - w_P\right)$, $E'_{2,P} = k_{P,\eta}[t]/\left(t^\ell - w'_P\right)$ and $\mathrm{N}_{\tilde{\mathrm{L}}_P/k_{P,\eta}}(z_P) = w_P$, $\mathrm{N}_{\tilde{\mathrm{L}}'_P/k_{P,\eta}}(z'_P) = w'_P$for suitable elements z_P and z'_P.
We would like to use a modified version of Lemma 2.11. Let $P \in \mathcal{P}'_\eta \cup \mathcal{Q}'_\eta$. By the above discussion, we can find $\tilde{z}_P \in \tilde{\mathrm{L}} \otimes k_{P,\eta} = \prod \tilde{\mathrm{L}}_P$ (resp $\tilde{z}'_P \in \tilde{\mathrm{L}}' \otimes k_{P,\eta} = \prod \tilde{\mathrm{L}}'_P$) such that its norm to $k_{P,\eta}$ is w_P (resp w'_P). We verify that $\left(w_P, a'_1\right)_y = \left(w'_P, a'_2\right)_y = 0$ for every valuation $y \in \Omega_{Y'}$ lying over P. For $P \in \mathcal{P}'_\eta$, this is by Proposition 7.1 (4). For $P \in \mathcal{Q}'_\eta$, by construction a'_1 is a unit in $\mathcal{O}^*_{Y'_P}$ while $a'_2 = (1)$. Since $w_P \in \mathcal{O}^*_{k_{P,\eta}}$ also, we are done in this case. Thus Lemma 2.11 can be used to construct E'_1 and E'_2 over k_η though E'_i will not split D'. Setting $E_{1,\eta}$ and $E_{2,\eta}$ to be their respective unramified lifts over F_η, we see that Properties 2, 3 & 6 are satisfied. □

[22]By the remark in the beginning of this proof, Y'_P is an unramified (possibly split) extension and a'_1 and a'_2 are units at these points.

12 Spreading and patching of E_j and a_j

Recall that $F = K(X)$ is the function field of a smooth projective geometrically integral curve X over a p-adic field K and $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$, a normal proper model of F as fixed in Section 6.2. We recall some further notation from ([HH10], Section 6) and ([HHK09], Section 3.3). Let $\eta \in N_0$ and let $U_\eta \subset X_0$ be a non-empty open subset containing η. Then A_{U_η} denotes the ring of functions regular on U_η. Fix a parameter t of K. Thus $t \in A_{U_\eta}$. Then $\widehat{A_{U_\eta}}$ denotes the completion of A_U at ideal (t) and F_U, the fraction field of $\widehat{A_{U_\eta}}$. Further $F \subseteq F_{U_\eta} \subseteq F_\eta$. Let π_η be the parameter of F_η fixed as in Section 6.2.

Proposition 12.1. *Let $j = 1$ or 2. For each $\eta \in N_0$, there exist a neighbourhood U_η of η in X_0 such that $U_\eta \subseteq \overline{\eta} \setminus S_0$, elements $a_{j,U_\eta} \in Y \otimes F_{U_\eta}$ and cyclic or split extensions $E_{j,U_\eta}/F_{U_\eta}$ of degree ℓ such that*

1. $a_{1,U_\eta} a_{2,U_\eta} = a$.
2. *$D \otimes E_{j,U_\eta}$ has index dividing ℓ.*
3. *$D \otimes Y \otimes E_{j,U_\eta}$ is split.*
4. *a_{j,U_η} is a norm from $Y \otimes E_{j,U_\eta}/Y \otimes F_{U,\eta}$.*
5. $\mathrm{N}_{Y \otimes F_{U_\eta}}\left(a_{j,U_\eta}\right) = 1$.
6. $E_{j,U_\eta} \otimes F_\eta \simeq E_{j,\eta}$.
7. *$E_{j,U_\eta} \simeq \prod F_{U_\eta}$ whenever $E_{j,\eta} \simeq \prod F_\eta$.*
8. *$E_{j,U_\eta} \simeq F_{U_\eta}[t]/\left(t^\ell - e_{j,U_\eta}\right)$ for some $e_{j,U_\eta} \in \widehat{A_{U_\eta}}$. Further if $E_{j,\eta}$ is unramified, then $e_{j,U_\eta} \in \widehat{A_{U_\eta}}^*$.*
9. *$a_{j,U_\eta} v_{j,\eta}^\ell = a_{j,\eta} \in Y \otimes F_\eta$ for some $v_{j,\eta} \in Y \otimes F_\eta$ of norm one.*
10. *$D \otimes F_{U_\eta}$ is split whenever $D \otimes F_\eta$ is split.*
11. *$D \otimes E_{j,U_\eta}$ is split whenever $D \otimes E_{j,\eta}$ is split.*

Proof. By the propositions in Section 10 and 11, we know $D \otimes E_{j,\eta} \otimes Y$ is split and $D \otimes E_{j,\eta}$ has index dividing ℓ. Further, we also know $E_{j,\eta} = F_\eta[t]/(t^\ell - e'_{j,\eta})$ where $e'_{j,\eta} = \pi_\eta^{\epsilon_j} e_{j,\eta}$ with $\epsilon_j \in \{0,1\}$ and $e_{j,\eta} \in \widehat{A_\eta}^*$. Finally we have norm one elements $a_{j,\eta} \in Y_\eta$ such that $a_{1,\eta} a_{2,\eta} = a$ and $\left(a_{j,\eta}, e'_{j,\eta}\right) = 0 \in \operatorname{Br}(Y_\eta)$.
For $d = 2$ or $\ell + 1$, by ([HHK15], Proposition 5.8) and ([KMRT], Proposition 1.17), there exists non-empty open set V'_η of η such that $D \otimes F_{V'_\eta}$ (resp. $D \otimes Y \otimes F_{V'_\eta}$) has index $< d$ whenever $D \otimes F_\eta$ (resp $D \otimes Y \otimes F_\eta$) has index $< d$. If $E_{j,\eta} \simeq \prod F_\eta$, set $V_{j,\eta} := V'_\eta$. Set $E_{V_{j,\eta}} = \prod F_{V_{j,\eta}}$ and $e_{j,V_{j,\eta}} = 1$. Then Properties 2, 3, 6, 7, 8 , 10 & 11 clearly hold.

If $E_{j,\eta}/F_\eta$ is a field extension, choose $e_j \in F^*$ such that $e_j^{-1}e_{j,\eta}$ is 1 mod (π_η) in $\widehat{A_\eta}$. Set $e'_j = \pi_\eta^{\epsilon_j}e_j \in F^*$ and $E'_j = F[t]/(t^\ell - e'_j)$. Since $e'_j = e'_{j,\eta}x^\ell$ for some $x \in \widehat{A_\eta}^*$, $E'_j \otimes F_\eta \simeq E_{j,\eta}$. Now again by ([HHK15], Proposition 5.8) and ([KMRT], Proposition 1.17), for $d = 2$ or $\ell + 1$, there exists non-empty open set $V_{j,\eta} \subseteq V'_\eta$ of η such that $D \otimes E'_j \otimes F_{V_{j,\eta}}$ (resp. $D \otimes Y \otimes E'_j \otimes F_{V_{j,\eta}}$) has index $< d$ whenever $D \otimes E'_j \otimes F_\eta$ (resp $D \otimes Y \otimes E'_j \otimes F_\eta$) has index $< d$. Setting $E_{j,V_{j,\eta}} := E'_j \otimes F_{V_{j,\eta}}$, it is clear that Properties $2, 3, 6, 7, 10$ and 11 hold. Shrink $V_{j,\eta}$ further to assume $e_j \in \widehat{A_{V_{j,\eta}}}^*$. Setting $e_{V_{j,\eta}} := e'_j$, it is clear that Property 8 holds. Shrink $V_{1,\eta}$ and $V_{2,\eta}$ to assume they are both equal and call them V_η. To address Properties $1, 4, 5$ and 9, we distinguish between the cases when Y_η/F_η is a field extension and when $Y_\eta \simeq \prod F_\eta$.

Suppose that Y_η/F_η is a field extension: Let F_η^h be the henselization of F at the discrete valuation η. Set $Y_\eta^h = Y \otimes_F F_\eta^h$ and identify it as a subfield of Y_η via the canonical morphism $Y_\eta^h \to Y_\eta$. Let $\tilde{\pi}_\eta \in Y_\eta^h$ be a parameter. Then $\tilde{\pi}_\eta$ is also a parameter in Y_η. Since $\mathrm{N}_{Y_\eta/F_\eta}(a_{1,\eta}) = 1$, by Hilbert 90, let $a_{1,\eta} = b_{1,\eta}^{-1}\psi(b_{1,\eta})$ for some $b_{1,\eta} \in Y_\eta^*$. Write $b_{1,\eta} = u_\eta\tilde{\pi}_\eta^r$ for some $u_\eta \in Y_\eta$ which is a unit at η. Since $u_\eta \in Y_\eta$ is a unit at η, by ([Ar69], Theorem 1.10), there exists $u_\eta^h \in Y_\eta^h$ such that $u_\eta^h \equiv u_\eta$ modulo the maximal ideal of valuation ring of Y_η. Let $b_{1,\eta}^h = u_\eta^h\tilde{\pi}_\eta^r \in Y_\eta^h$. Set $a_{1,\eta}^h = \left(b_{1,\eta}^h\right)^{-1}\psi\left(b_{1,\eta}^h\right)$. Thus $\mathrm{N}(a_{1,\eta}^h) = 1$ and $a_{1,\eta}^h a_{1,\eta}^{-1}$ is a norm one element in Y_η which is 1 modulo the maximal ideal of valuation ring of Y_η. Thus by Lemma 2.2 again, $a_{1,\eta}^h\left(v_{1,\eta}\right)^\ell = a_{1,\eta}$ for some $v_{1,\eta} \in Y_\eta$ of norm one. Thus $\left(a_{1,\eta}, e'_1\right) = \left(a_{1,\eta}^h, e'_1\right) = 0 \in \mathrm{Br}(Y_\eta)$ and we have $\left(a_{1,\eta}^h, e'_1\right) = 0 \in \mathrm{Br}(Y_\eta^h)$ (cf. proof of ([HHK14], Proposition 3.2.2)).

Since F_η^h is the filtered direct limit of the fields F_V, where V ranges over the non-empty open subset of η ([HHK14], Lemma 2.2.1), there exist a non-empty open subset $U_\eta \subseteq V_\eta$ of η and $a_{1,U_\eta} \in Y \otimes F_{U_\eta}$ such that $N_{Y\otimes F_{U_\eta}/F_{U_\eta}}(a_{1,U_\eta}) = 1$ and the image of a_{1,U_η} in Y_η^h is equal to $a_{1,\eta}^h$.

By shrinking U_η, we can assume that $\left(a_{1,U_\eta}, e'_1\right) = 0 \in \mathrm{Br}\left(Y_{U_\eta}\right)$ ([HHK14], Proposition 3.2.2). Hence Property 4 holds for a_{1,U_η}. Finally set $a_{2,U_\eta} = aa_{1,U_\eta}^{-1}$. Thus for $j = 1$ and 2, it is clear that Properties $1, 5$ and 9 are satisfied. Since $(a_{2,\eta}, e'_2) = \left(a_{2,U_\eta}, e'_2\right) = 0 \in \mathrm{Br}(Y_\eta)$, by using ([HHK14], Proposition 3.2.2) again and shrinking U_η, we can show that Property 4 holds for a_{2,U_η} also.

Suppose that Y_η is split: Then shrink V_η further such that $Y \otimes F_{V_\eta} \simeq \prod F_{V_\eta}$ also ([Ar69], Theorem 1.10 & [HHK14], Lemma 2.2.1). We have $a_{1,\eta} = (\tilde{a}_{1,i,\eta})_{i\leq\ell}$ where $\tilde{a}_{1,i,\eta} = c_{i,\eta}\pi_\eta^{m_i} \in F_\eta$ for $m_i \in \mathbb{Z}$ and $c_{i,\eta} \in \widehat{A_\eta}^*$. For $1 \leq i \leq \ell - 1$, choose $c_i \in F^*$ such that $c_i^{-1}c_{i,\eta}$ is 1 mod (π_η) in $\widehat{A_\eta}$.

Set $\tilde{a}_{1,i,V_\eta} = c_i\pi_\eta^{m_i}$ for $i \leq \ell - 1$ and set $\tilde{a}_{1,V_\eta,\ell} = \left(\prod_{r=1}^{\ell-1}\tilde{a}_{1,r,V_\eta},\right)^{-1}$. Finally set $a_{1,V_\eta} := \left(\tilde{a}_{1,i,V_\eta}\right)_i$ and $a_{2,V_\eta} = \left(\tilde{a}_{2,i,V_\eta}\right)_i = \left(a_{1,V_\eta}\right)^{-1}$ in $\prod F_{V_\eta}$. Thus Properties 1, 5 and 9 (using[23] Lemma 2.2) are satisfied. Let $j = 1$ or 2 and $i \leq \ell$. Since

[23]There exists an $m \geq 1$ such that F_η doesn't contain a primitive ℓ^mth root of unity. Look at any branch field $F_{P,\eta} \supset F_\eta$. Its residue field $k_{P,\eta}$ has further residue field k_P, a finite field with characteristic $\neq \ell$.

$\left(a_{j,\eta}, e'_j\right) = 0 \in \operatorname{Br}(Y_\eta)$, we have $\left(\tilde{a}_{j,i,V_\eta}, e'_j\right) = 0 \in \operatorname{Br}(F_\eta)$. By ([HHK14], Proposition 3.2.2) and shrinking further if necessary, there exists a neighbourhood $U_\eta \subseteq V_\eta$ of η such that $\left(\tilde{a}_{j,i,V_\eta}, e'_j\right) = 0 \in \operatorname{Br}(F_{U_\eta})$ which shows that Property 4 holds. ☐

Recall that $\mathcal{P}'_\eta$ denotes the finite set of marked closed points in $\overline{\eta} \cap S_0$. For each η in N_0, choose U_η as in Proposition 12.1 and let $\mathcal{R}'_\eta$ denote the finite set of closed points $(\overline{\eta} \setminus U_\eta) \setminus S_0$.

Proposition 12.2. *Let $j = 1$ or 2 and let $\eta \in N_0$. For each $P \in \mathcal{P}'_\eta \cup \mathcal{R}'_\eta$, there exist elements $a_{j,P} \in Y_P$ and cyclic or split extensions $E_{j,P}/F_P$ of degree ℓ such that*

1. $a_{1,P}a_{2,P} = a$.
2. $D \otimes E_{j,P}$ *has index at most* ℓ.
3. $D \otimes Y \otimes E_{j,P}$ *is split.*
4. $a_{j,P}$ *is a norm from* $Y \otimes E_{j,P}/Y \otimes F_P$.
5. $\mathrm{N}_{Y\otimes F_P}(a_{j,P}) = 1$.
6. $E_{j,U_\eta} \otimes F_{P,\eta} \simeq E_{j,P} \otimes F_{P,\eta}$.
7. $E_{j,P} \simeq \prod F_P$ *or* $D \otimes E_{j,P}$ *is split.*
8. *There exists* $\mu_{j,P,\eta} \in (Y \otimes F_{P,\eta})^*$ *such that* $a_{j,P} = a_{j,\eta}\mu_{j,P,\eta}$ *where* $\mathrm{N}(\mu_{j,P,\eta}) = 1$ *and*
   - $\mu_{j,P,\eta} = 1$ *if* Y_η *is of Type RAM.*
   - $\mu_{j,P,\eta} = (\mu_{j,i,P,\eta})_i$ *for* $i \leq \ell$ *where* $\mu_{j,i,P,\eta} \in F^{*\ell}_{P,\eta}$ *if* Y_η *is of Type SPLIT.*
   - $\mu_{j,P,\eta} \cong 1 \mod (\pi_\eta)$ *if* Y_η *is of Type RES/NONRES.*

Proof. If $P \in \mathcal{P}'_\eta$, the proof follows from Propositions 7.1, 7.2 and those in Section 10. Assume therefore that $P \in \mathcal{R}'_\eta$, i.e. it is a curve point. For $j = 1, 2$, let $E_{j,\eta} = F_\eta[t]/\left(t^\ell - e_{j,\eta}\right)$ where $e_{j,\eta} \in F_\eta$ with $v_\eta(e_{j,\eta}) = 0$ or 1. Let (π_P, δ_P) denote a system of regular parameters at A_P such that $D_P = (u_P, \pi_P)$ where $u_P \in \widehat{A_P}^*$ ([S97]). Let $\pi_\eta = \theta_P\pi_P$ in F_η where $\theta_P \in \widehat{A_\eta}^*$. Let $e_{j,\eta} = x_{j,\eta}\pi_P^{\epsilon_j} \in F_{P,\eta}$ where $x_j \in \widehat{A_{P,\eta}}^*$ and $\epsilon_j \in \{0,1\}$. Let $\overline{x_{j,\eta}} = y_j\overline{\delta_P}^{r_j}$ up to ℓ^{th} powers where $y_j \in \mathcal{O}^*_{k_{P,\eta}}$ and $0 \leq r_j < \ell$.
Let $\tilde{y_j} \in \widehat{A_P}^*$ be such that it matches with $\overline{y_j}$ in k_P. Set $e_{j,P} = \tilde{y_j}\delta_P^{r_j}\pi_P^{\epsilon_j}$ and $E_{j,P} = F_P[t]/\left(t^\ell - e_{j,P}\right)$. Using Proposition 12.1, Property 6 is satisfied. Also note that by ([S97]), Property 2 is satisfied. As Y is arranged to be in good shape, Y_P is unramified or $Y_P = F_P\left(\sqrt[\ell]{v_P\pi_P}\right)$ where $v_P \in \widehat{A_P}^*$. Thus if Y_P is not split, then $D \otimes Y_P$ is already split.
As D is ramified at most along η at P, this implies by Lemma 2.8 that if $D \otimes E_{j,P} \otimes F_{P,\eta}$ is split, then so is $D \otimes E_{j,P}$. Using Propositions in Section 10, it is clear that

$D \otimes Y \otimes E_{j,\eta}$ is split and hence so is $D \otimes Y \otimes E_{j,P} \otimes F_{P,\eta}$. If Y_P is split, therefore we see that $D \otimes E_{j,P} \otimes F_{P,\eta}$ is split. Hence Property 3 is satisfied.
Note that Property 7 holds in the following situations:

- η is of Type 0 or 1a. This is because $D = 0 \in \mathrm{Br}(F_P)$ already.
- $E_{j,P} = \prod F_P$.
- $D \otimes E_{j,P} \otimes F_{P,\eta}$ is split.
- Y_η is of Type SPLIT as the discussion above shows.
- $E_{j,P} = L_P$, the unique field extension of F_P of degree ℓ unramified at $\widehat{A_P}$. This is because u_P becomes an ℓ^{th} power in $E_{j,P}$.

Recall that we choose $E_{j,\eta}$ to be ramified along η in some cases only when Y_η is SPLIT, where Property 7 already holds. Thus to check that this Property holds in general, we have to investigate only the cases when $E_{j,P} = F_P[t]/(t^\ell - \tilde{y_j}\delta_P^{r_j})$ where $0 < r_j < \ell$.
We now discuss the proof of the rest of the Properties 1-8 depending on the type of Y_η.
Y_η *is RAM*: Thus η can be of Type 0, 1, 1b and coloured green or 2. Set $a_{1,P} = a$ and $a_{2,P} = 1$ to see Properties 1, 5 & 8 hold by construction (Propositions 10.4 and 11.1). By Lemma 2.6 and Proposition 3.3, these are ℓ^{th} powers in Y_P and hence Property 4 holds. To check that Property 7 holds, we can assume η is of Type 1b or 2. Proposition 10.4 also implies that each $E_{j,P} \otimes F_{P,\eta}$ is unramified. Thus the only case to check is when $E_{j,P} \simeq F_P[t]/(t^\ell - \tilde{y_j}\delta_P)$. However, the same proposition gives that $\overline{u_\eta}$ is a norm from $\overline{E_{j,\eta}}$ where $\overline{u_P} = \overline{u_\eta} \in k_{P,\eta}$ is the residue of D along the branch. Since we are in the case when $\overline{E_{j,\eta} \otimes k_{P,\eta}}$ is ramified, this implies that $\overline{u_P} \in k_P^{*\ell}$ and hence $u_P \in \widehat{A_P}^{*\ell}$. Therefore $D = 0 \in \mathrm{Br}(F_P)$ already.
Y_η *is SPLIT*: For $j = 1, 2$, write $a_{j,\eta} = (\tilde{a}_{j,i,\eta})_i \in \prod F_{P,\eta}$ where $\tilde{a}_{j,i,\eta} = x_{j,i,P}\pi_P^{m_{j,i}}$ where $x_{j,i,P} \in \widehat{A_{P,\eta}}^*$. Let $\overline{x_{j,i,P}} = x'_{j,i,P}\overline{\delta_P}^{s_{j,i}} \in k_{P,\eta}$ where $x'_{j,i,P} \in \mathcal{O}^*_{k_{P,\eta}}$. Let $\tilde{x'}_{j,i,P} \in \widehat{A_P}^*$ be a lift of $\overline{x'_{j,i,P}}$.
Set $a_{1,P} = (\tilde{a}_{1,i,P})_i$ where $\tilde{a}_{1,i,P} = \tilde{x'}_{1,i,P}\pi_P^{m_{1,i}}\delta_P^{s_{1,i}}$ for $1 \leq i \leq \ell - 1$. Set $\tilde{a}_{1,\ell,P} = (\tilde{a}_{1,1,P} \ldots \tilde{a}_{1,\ell-1,P})^{-1}$. And set $a_{2,P} = aa_{1,P}^{-1}$. Thus Properties 1, 5 & 8 hold. We have already checked that Property 7 holds in this case (Y_η being SPLIT).
Since $\tilde{a}_{j,i,\eta}$ is a norm from $E_{j,\eta}$ for each i, we have $(\tilde{a}_{j,i,\eta}, e_{j,P}) = (\tilde{a}_{j,i,P}, e_{j,P}) = 0 \in \mathrm{Br}\,(F_{P,\eta})$. By construction, $(\tilde{a}_{j,i,P}, e_{j,P})$ is ramified at most along π_P and δ_P in $\mathrm{Br}\,(F_P)$. Hence by ([PPS18], Corollary 5.5), we have $(\tilde{a}_{j,i,P}, e_{j,P}) = 0 \in \mathrm{Br}\,(F_P)$ also for each i. Therefore Property 4 holds.
Y_η *is RES/NONRES*: Since P is a curve point, Y is arranged to be in good shape and Y_η/F_η is unramified, we have $Y_P = F_P[t]/\left(t^\ell - v_P\right)$ for some $v_P \in \widehat{A_P}^*$. Hence Y_P is either split or L_P, the unique unramified extension of F_P of degree ℓ. Thus $Y \otimes F_{P,\eta}$ is unramified over $F_{P,\eta}$ as also $\overline{Y \otimes F_{P,\eta}}$ over $k_{P,\eta}$. Note that $a_{j,\eta} \in \mathcal{O}^*_{Y\otimes F_{P,\eta}}$ by construction and $E_{j,\eta}$ is unramified along η (cf. proofs of Propositions 10.3, 10.5 and 11.3).

Let $\overline{a_{j,\eta}} = x'_j \overline{\delta_P}^{s_j} \in \overline{Y \otimes F_{P,\eta}}$ where $x'_j \in \mathcal{O}^*_{\overline{Y \otimes F_{P,\eta}}}$. Set $a_{1,P} = \tilde{x'_1} \delta_P^{s_1} \in Y_P$ where $\tilde{x'_1} \in Y_P$ is a lift of $\overline{x'_1}$ and $a_2 = a a_{1,P}^{-1}$. Thus Properties 1, 5 & 8 hold. Since $a_{j,\eta}$ is a norm from $E_{j,\eta}$, we have $(a_{j,\eta}, e_{j,P}) = (a_{j,P}, e_{j,P}) = 0 \in \mathrm{Br}(Y \otimes F_{P,\eta})$. Note that Y_P is an unramified extension of F_P. By construction, $(a_{j,P}, e_{j,P})$ is ramified at most along π_P and δ_P in $\mathrm{Br}(Y_P)$. Hence by ([PPS18], Corollary 5.5), we have $(a_{j,P}, e_{j,P}) = 0 \in \mathrm{Br}(Y_P)$ also. Therefore Property 4 holds.
To check Property 7, we can assume η is Type 1b or 2. When Y_η is of Type NONRES, $E_{j,\eta}$ is the lift of residues. Thus, $E_{j,P} = L_P$ or $\prod F_P$ where we have checked that Property 7 holds. When Y_η is of Type RES, Proposition 10.5 guarantees that $\overline{u_\eta}$ is a norm from $\overline{E_{j,\eta}}$ where $\overline{u_P} = \overline{u_\eta} \in k_{P,\eta}$ is the residue of D along the branch. Arguing as in the case when Y_η is Type RAM, we are done. ☐

Proposition 12.3. *Let $j = 1$ or 2 and let $\eta \in N_0$. Let $\mathrm{Gal}(Y/F) = \langle \psi \rangle$. For each $P \in \mathcal{P}'_\eta \cup \mathcal{R}'_\eta$, there exist elements $h_{j,P,\eta} \in Y \otimes F_{P,\eta}$ such that*

$$a_{j,U_\eta} h_{j,P,\eta}^{-\ell} \psi\left(h_{j,P,\eta}\right)^\ell = a_{j,P} \in Y \otimes F_{P,\eta}.$$

Proof. Let $m \geq 1$ be[24] such that $F_{P,\eta}$ does not contain a primitive ℓ^m-th root of unity. By Propositions 12.1 and 12.2, there exist norm one elements $v_{j,\eta}$ and $\mu_{j,P,\eta}$ in $Y \otimes F_{P,\eta}$ such that $a_{j,U_\eta} v_{j,\eta}^\ell \mu_{j,P,\eta} = a_{j,P} \in Y \otimes F_{P,\eta}$.
Proposition 12.2 also gives us that $\mu_{j,P,\eta} \in (Y \otimes F_{P,\eta})^{*\ell^{2m}}$ if $Y \otimes F_{P,\eta}$ is a field extension and $\mu_{j,P,\eta} \in \prod F_{P,\eta}^{*\ell}$ if $Y \otimes F_{P,\eta}$ is split. Therefore by Lemma 2.2 and Hilbert 90, there exists $h_{j,P,\eta} \in Y \otimes F_{P,\eta}$ such that $a_{j,U_\eta} h_{j,P,\eta}^{-\ell} \psi\left(h_{j,P,\eta}\right)^\ell = a_{j,P}$. ☐

Remark 12.4. *Note that $\{\mathcal{P}'_\eta \cup \mathcal{R}'_\eta, U_\eta\}_{\eta \in N_0}$ forms a patching set $\mathcal{P}$ as in defined in ([HH10]).*

Proposition 12.5. *Let $j = 1$ or 2. Then there exist E_j/F, degree ℓ extensions of F which are subfields of D/F and elements $a_j \in Y$ such that*

- *$a_1 a_2 = a$ and $\mathrm{N}_{Y/F}(a_j) = 1$.*
- *$E_j \otimes_F F_{U_\eta} \simeq E_{j,U_\eta}$ and $E_j \otimes_F F_P \simeq E_{j,P}$ for the patching set-up $\mathcal{P}$.*
- *$D \otimes E_j \otimes Y$ is split and $E_j \subseteq C_D(Y)$.*
- *There exist $\theta_j \in E_j Y \subseteq D$ such that $\mathrm{N}_{E_j Y/Y}(\theta_j) = a_j$.*

Proof. Let $j = 1$ or 2. In this proof, by $x \in \mathcal{P}$ we mean $x \in \{U_\eta, P'_\eta \cup \mathcal{R}'_\eta\}$ of the patching set up $\mathcal{P}$ defined in Remark 12.4.
From Propositions 12.1 and 12.2, we see that by ([HH10], Theorem 7.1), there exists a degree ℓ etale algebra $\tilde{E_j}/F$ such that $\tilde{E_j} \otimes_F F_{U_\eta} \simeq E_{j,U_\eta}$ and $\tilde{E_j} \otimes_F F_P \simeq E_{j,P}$ for the patching set-up $\mathcal{P}$. Since at least one of the $E_{j,P}$ (or the E_{j,U_η}) is a nonsplit field extension, clearly $\tilde{E_j}/F$ is a field.

[24] As before, such an m exists because the residue field k_P of its residue field $k_{P,\eta}$ is a finite field (of characteristic not ℓ).

These propositions also guarantee that $\operatorname{index}(D \otimes_F E_{j,x}) \leq \ell$ for each $x \in \mathcal{P}$. Therefore by ([HHK09], Theorem 5.1), we have that $\operatorname{index}\left(D \otimes_F \tilde{E}_j\right) \leq \ell$ and hence there exists a subfield of D isomorphic to $\tilde{E}_j/F$ which we again call $\tilde{E}_j$.
We also have that $Y_x \otimes_F E_{j,x}$ splits D for each $x \in \mathcal{P}$. Thus $D \otimes_F Y \otimes_F \tilde{E}_j$ is split. And therefore $C_D(Y) \otimes_Y (Y \otimes_F \tilde{E}_j)$ is split. As D is a divison algebra of degree ℓ^2, we have that $C_D(Y)/Y$ is division of degree ℓ, and hence $Y \otimes_F \tilde{E}_j$ splits $C_D(Y)$. Thus it is a degree ℓ field extension of Y and therefore a degree ℓ^2 field extension of F.
Since $Y \otimes_F \tilde{E}_j$ is a splitting field of D, which is a division algebra of degree ℓ^2, there exists L'_j, a maximal subfield of D which is isomorphic to $Y \otimes_F \tilde{E}_j$. Let E''_j denote the subfield of L'_j which is isomorphic to $\{1\} \otimes_F \tilde{E}_j$ in L'_j and Y'_j, the isomorphic copy of $Y \otimes_F \{1\}$. Thus E''_j and Y'_j are commuting degree ℓ subfields of D.
By Skolem-Noether, $Y = b_j Y'_j b_j^{-1} \subseteq D$ for some unit $b_j \in D^*$. Set $L_j = b_j L'_j b_j^{-1} \subseteq D$ and $E_j = b_j E''_j b_j^{-1} \subseteq D$. Thus E_j and Y commute in D (they are subfields of the maximal subfield L_j).
We now construct $a_1 \in Y$ using the norm one elements $a_{1,x} \in Y \otimes F_x$ for $x \in \mathcal{P}$. By Proposition 12.3, for each branch in the patching set-up corresponding to a pair (U_η, P), we have $a_{1,P} = a_{1,U_\eta} h_{1,P,\eta}^{-\ell} \psi(h_{1,P,\eta})^\ell$ for some $h_{1,P,\eta} \in (Y \otimes F_{P,\eta})^*$ where $\operatorname{Gal}(Y/F) = \langle\psi\rangle$. By simultaneous factorization for curves for the rational group $\mathrm{R}_{Y/F}(\mathbb{G}_m)$ ([HHK09], Theorem 3.6), we can find $h_{1,x} \in (Y \otimes F_x)^*$ for each $x \in \mathcal{P}$ such that for every pair (U_η, P), we have $h_{1,P,\eta} = h_{1,U_\eta} h_{1,P}^{-1}$. Thus for every branch defined by (U_η, P),

$$\begin{aligned}
&a_{1,P} = a_{1,U_\eta} h_{1,P,\eta}^{-\ell} \psi(h_{1,P,\eta})^\ell \\
&\implies a_{1,P} = a_{1,U_\eta} h_{1,U_\eta}^{-\ell} h_{1,P}^{\ell} \psi\left(h_{1,U_\eta}\right)^\ell \psi(h_{1,P})^{-\ell} \\
&\implies a_{1,P} h_{1,P}^{-\ell} \psi(h_{1,P})^\ell = a_{1,U_\eta} h_{1,U_\eta}^{-\ell} \psi\left(h_{1,U_\eta}\right)^\ell .
\end{aligned}$$

Let $x \in \mathcal{P}$. Thus by ([HH10], Proposition 6.3 & Theorem 6.4), we have an element $a_1 \in Y$ such that $a_1 = a_{1,x} h_{1,x}^{-\ell} \psi(h_{1,x})^\ell \in Y \otimes F_x$ and $\mathrm{N}(a_1) = 1$. Set $a_2 = a a_1^{-1}$. Note that $a_j \cong a_{j,x}$ up to ℓ^{th} powers in $Y \otimes F_x$.
Now we only have to verify that a_j is a norm from $E_j Y$. Without loss of generality let $j = 1$ (the same proof works for $j = 2$). By Propositions 12.1 and 12.2, we see that $(a_{1,x}, E_{1,x})_{Y_x}$ is split for each x. This implies that $a_{1,x}$ and hence a_1 is a norm from $E_1 Y \otimes_Y Y \otimes_F F_x$ over $Y \otimes F_x$ as a_1 differs from each $a_{1,x}$ by an ℓ^{th} power.
There exists a field extension N/Y of degree coprime to ℓ such that $E_1 Y \otimes_Y N$ is a cyclic field extension of degree ℓ ([A61], Chapter IV, Theorem 31). Let $\mathcal{Y}$ (resp. $\mathcal{Z}$) denote the normal closure of $\mathcal{X}$ in Y (resp. N) with special fiber Y_0 (resp. Z_0). Let $\gamma : Z_0 \to Y_0$ and $\phi : Y_0 \to X_0$ be the induced morphisms. Then, as in the proof of ([PPS18], Proposition 7.5), we have induced patching systems $\mathcal{Y}'$ of Y_0 (resp. $\mathcal{Z}'$ of Z_0) consisting of open sets U_y (resp. U_z) and closed points P_y (resp. P_z) such that $F_U \subset Y_{U_y} \subset N_{U_z}$, $F_P \subset Y_{P_y} \subset N_{P_z}$ for $U, P \in \mathcal{P}$ with $\gamma(U_z) \subset U_y$, $\phi(U_y) \subset U$, $\gamma(P_z) = P_y$ and $\phi(P_y) = P$.

Then for $x = U$ or P, we have the following commutative diagram induced by norm maps

$$\begin{array}{ccccc} E_1Y \otimes_Y Y \otimes_F F_x & \longrightarrow & E_1Y \otimes_Y Y_{x_y} & \longrightarrow & ((E_1Y \otimes_Y N) \otimes_N N_{x_z}) \\ \downarrow & & \downarrow & & \downarrow \\ Y \otimes_F F_x & \longrightarrow & Y_{x_y} & \longrightarrow & N_{x_z} \end{array}$$

This shows that $(a_1, E_1Y \otimes_Y N)$ is trivial over each N_{x_z} for each $x_z \in \mathcal{Z}'$ and hence trivial over N ([HHK09], Theorem 5.1). Thus a_1 is a norm of the extension $E_1Y \otimes_Y N/N$ and hence $a_1^{[N:Y]}$ is a norm of E_1Y/Y. Since $[N : Y]$ is coprime to ℓ, this implies that there exists $\theta_1 \in E_1Y$ such that $\mathrm{N}_{E_1Y/Y}(\theta_1) = a_1$. □

13 Solving the problem over E_j

Recall that we started with $z \in \mathrm{SL}_1(D)$ living in a maximal subfield M of D which contains a cyclic degree ℓ subfield $F \subseteq Y \subseteq M$ with $\mathrm{N}_{M/Y}(z) := a$. Let a_j, E_j and θ_j be as in Proposition 12.5 for $j = 1, 2$. Note that $\mathrm{N}_{E_jY/F}(\theta_j) = 1$ and hence $\theta_j \in \mathrm{SL}_1(D)$. If we can prove that $\theta_j \in [D^*, D^*]$, then by Proposition 3.6, $z \in [D^*, D^*]$. Let $c_j := \mathrm{N}_{E_jY/E_j}(\theta_j)$. Since the proofs for the cases $j = 1$ and $j = 2$ are similar, without loss of generality, assume $j = 1$. We also drop the suffixes in the remainder of this paper, i.e. we set $E := E_1$, $\theta := \theta_1$, $c := c_1$ etc.

13.1 Strategy à la Platonov

To show $\theta \in [D^*, D^*]$, we adapt the basic strategy underlying the proof of the triviality of $\mathrm{SK}_1(D)$ over global fields ([P76], Theorem 5.4) as follows:
There exists a suitable[25] field extension N/F such that $[N : F]$ is coprime to ℓ with $E_N := E \otimes_F N$, a cyclic subfield of $D_N := D \otimes N$. By ([P76], Lemma 2.2, Section 2.4), it suffices to show that $\theta \in [D_N^*, D_N^*]$. Let $Y_N := Y \otimes_F N$ and $\mathrm{Gal}(E_N/N) = \langle\sigma\rangle$. Note that $\theta \in E_NY_N \subseteq C_{D_N}(E_N)$ and $\mathrm{N}_{E_NY_N/E_N}(\theta) = c$. Therefore the further norm, $\mathrm{N}_{E_N/N}(c) = 1$. Now, because E_N/N is a cyclic extension with Galois group $\langle\sigma\rangle$, by Hilbert 90, there exists a $b \in E_N$ such that $c = b^{-1}\sigma(b) \in E_N$. Note that $c = b^{-1}\sigma(b)$ is a reduced norm in E_N from $C_{D_N}(E_N)$.

Proposition 13.1. *For N, σ, b, c as above, if there exists $f \in N$ such that bf is a reduced norm in E_N from $(C_{D_N}(E_N))$, then $\theta \in [D_N^*, D_N^*]$.*

Proof. Set $b' = bf$. Note that $c = b^{-1}\sigma(b) = (bf)^{-1}\sigma(bf) = b'^{-1}\sigma(b')$. By Skolem Noether, extend $\sigma : E_N \to E_N \subseteq D_N$ to an automorphism of D_N given by

[25] We can and do choose the coprime extension N/F carefully as follows: Let N' be the Galois closure of E/F and take N to be the fixed field of an ℓ-Sylow group of $\mathrm{Gal}(N/F)$. Thus $[N : F]$ is coprime to ℓ and $E \otimes_F N = N'$ which is indeed cyclic over N of degree ℓ (cf. [A61], Chapter IV, Theorem 31).

$\tilde{\sigma} = \mathrm{Int}(v) : D_N \to D_N,\ d \rightsquigarrow vdv^{-1}$. Note that $\tilde{\sigma}$ restricts to an N-automorphism of $C_{D_N}(E_N)$ since $\tilde{\sigma}|_{E_N} = \sigma$. Set $D_1 = C_{D_N}(E_N)$.
By hypothesis, there exists $g \in D_1$ such that $\mathrm{Nrd}_{D_1/E_N}(g) = b'$. Thus,

$$\begin{aligned}\mathrm{Nrd}_{D_1/E_N}\left(g^{-1}vgv^{-1}\right) &= \mathrm{Nrd}_{D_1/E_N}\left(g^{-1}\right)\mathrm{Nrd}_{D_1/E_N}\left(vgv^{-1}\right)\\ &= b'^{-1}\,\mathrm{Nrd}_{D_1/E_N}\left(\tilde{\sigma}(g)\right)\\ &= b'^{-1}\tilde{\sigma}\left(\mathrm{Nrd}_{D_1/E_N}(g)\right)\\ &= b'^{-1}\sigma(b')\\ &= c\end{aligned}$$

Since $\mathrm{Nrd}_{D_1/E_N}(\theta) = c$, we have $\mathrm{Nrd}_{D_1/E_N}\left(\theta vg^{-1}v^{-1}g\right) = 1$. Since D_1 is a central simple algebra of square-free index ℓ, $\mathrm{SL}_1(D_1) = [D_1^*, D_1^*]$ ([W50]). Hence we have that $\theta vg^{-1}v^{-1}g \subseteq [D_1^*, D_1^*] \subseteq [D_N^*, D_N^*]$. □

We will find $f \in N$ satisfying the hypothesis of Proposition 13.1 by patching suitable elements $f_x \in (N \otimes_F F_x)^*$ for x in a refinement of the patching system $\mathcal{P}$ used to construct E (Remark 12.4).

13.1.1 The shapes of E_N and b

We investigate the shape of E after the coprime base change N. Let $x \in \mathcal{P}$. Since $E_x := E \otimes_F F_x$ is a cyclic extension by construction and $E_N = N'$, the Galois closure of E/F, we see that $E_N \otimes_F F_x \simeq \prod_{[N:F]} E_x$. Let $N \otimes_F F_x = \prod_{i=1}^{r_x} N_{i,x}$. Since $[E_x : F_x] = \ell$, this forces each $N_{i,x}$ to be isomorphic to F_x or E_x. Hence $E_N \otimes_F F_x$ as an $N \otimes_F F_x$ algebra is the product of an appropriate number of copies of the cyclic extensions E_x/F_x and the split extensions $\prod_\ell E_x/E_x$.
Let $b \otimes 1 \in E_N \otimes_F F_x$ correspond to the entry $\prod_q b_q \times \prod_i (b_{i,1,P}, b_{i,2,P}, \ldots, b_{i,\ell,P})$ in $\prod_q E_x/F_x \times \prod_i \left(\prod_\ell E_x/E_x\right)$. The σ action is componentwise and further in $\prod_\ell E_x/E_x$, it permutes the entries of each tuple $\left(b_{i,j,P}\right)_{j\le\ell}$ amongst themselves, i.e. $\sigma\left(\prod_i (b_{i,j,P})_{j\le\ell}\right) = \prod_i \left(b_{i,\sigma(j),P}\right)_{j\le\ell}$. The σ action on the E_x/F_x components can be similarly described if $E_x \simeq \prod_\ell F_x$ is itself split.
For $\eta \in N_0$ and closed point $P \in \overline{\eta}$, let $x = \eta$ or (P, η). Then we denote the integral closure of $\widehat{A_x}$ in $E \otimes F_x$ by $\widehat{B_x}$. Let its residue field be denoted k'_x. Similarly, let $\widehat{C_x}$ denote the integral closure of $\widehat{A_x}$ in $N \otimes F_x$ with residue field k''_x. Thus $\widehat{C_x} \simeq \prod \widehat{A_x} \times \prod \widehat{B_x}$.
We begin with the following broad modification of b: Let $\eta \in N_0$ be such that E_η/F_η is an unramified field extension. Thus $E_N \otimes_F F_\eta / N \otimes_F F_\eta$ is the unramified (possibly split or partially split) extension $\prod E_\eta/F_\eta \times \prod \left(\prod_\ell E_\eta/E_\eta\right)$. By weak approximation, modify b by a suitable element of N so that if $b = \prod_q b_q \times \prod_i (b_{i,1,\eta}, b_{i,2,\eta}, \ldots, b_{i,\ell,\eta})$ in $\prod_q E_\eta/F_\eta \times \prod_i \left(\prod_\ell E_\eta/E_\eta\right)$, then

1. Each b_q living in any component of shape E_η/F_η is a unit in $\widehat{B_\eta}$. This can be done by knocking off an appropriate power of π_η from F_η.

2. If $D \otimes E_\eta$ is an unramified algebra of index ℓ, then each entry $b_{i,j,\eta}$ in the tuple $(b_{i,j,\eta})_{j\leq\ell}$ living in any component of shape $\prod_\ell E_\eta/E_\eta$ is a unit in $\widehat{B_\eta}$. This can be done as follows:

   Let m_j denote the valuation of $b_{i,j,\eta}$ in E_η. It suffices to check that all m_js are equal because then we can again make $(b_{i,j,\eta})_{j\leq\ell} \in \prod \widehat{B_\eta}^*$ by knocking off $\pi_\eta^{m_j}$ from E_η. Since $c = b^{-1}\sigma(b)$ is a reduced norm from $D \otimes E$, this implies $b_{i,1,\eta}^{-1} b_{i,j,\eta}$ is a reduced norm from $D \otimes E_\eta$ for each $j \leq \ell$. Since every unit in $\widehat{B_\eta}$ is a reduced norm from $D \otimes E_\eta$ (Proposition 2.7), this forces all valuations m_j to equal each other.

3. If $D \otimes F_\eta$ is an unramified algebra of index ℓ and if $E_\eta \simeq \prod F_\eta$ is split, then each $b_q = (b_{q,j,\eta})_{j\leq\ell}$ living in any component of shape E_η/F_η is a unit in $\widehat{B_\eta}$, i.e. each $b_{q,j,\eta} \in \widehat{A_\eta}^*$. This can be achieved by a similar argument as in 2).

13.2 Preliminary patching data of f

Recall that for each $\eta \in N_0$, $\mathcal{P}'_\eta := \overline{\eta} \cap S_0$ and $\mathcal{R}'_\eta := (\overline{\eta} \setminus U_\eta) \setminus S_0$. Thus $S_0 = \cup_{\eta\in N_0} \mathcal{P}'_\eta$.

Proposition 13.2 (f at closed points). *Let $\eta \in N_0$ and $P \in \mathcal{P}'_\eta \cup \mathcal{R}'_\eta$. Then there exists $f_P \in (N \otimes_F F_P)^*$ such that bf_P is a reduced norm from $D_N \otimes E_P$. Further $f_P \in \widehat{C_{P,\eta}}^*$.*

Proof. If $D \otimes E_P$ is split, set $f_P = 1$. Note that bf_P (and indeed any other element in $N \otimes E_P$) is a reduced norm from $D_N \otimes E_P$. Clearly $1 \in \widehat{C_{P,\eta}}^*$.
Therefore assume $D \otimes E_P$ is not split and let $P \in \overline{\eta} \cap \overline{\eta'}$. We can in fact pinpoint precisely when this happens by a closer inspection of the proofs of Propositions 7.1 and 12.2 - at points in Rows 2.2* of Table 11, 4.1* of Table 13, 8.5-8.6 of Table 17 and at some innocuous curve points in $\mathcal{R}'_\eta$ where both Y_η and $Y_{\eta'}$ are not SPLIT. Note that in all these cases, E_η/F_η and $E_{\eta'}/F_{\eta'}$ are unramified field extensions by construction, $\{\eta, \eta'\} = \{\text{Type 1b}, \text{Type 1a}\}$ or $\{\text{Type 1b}, \text{Type 2}\}$ and $D_P \simeq (u_P, \pi_P)$ for some unit $u_P \in \widehat{A_P}^*$ and π_P defines one of η or η' at P.
By Proposition 12.2, $E_P \simeq \prod F_P$ and therefore $E_N \otimes_F F_P/N \otimes_F F_P \simeq \prod(\prod_\ell F_P/F_P)$. Let $b \otimes 1$ correspond to the entry $\prod_i (b_{i,1,P}, b_{i,2,P}, \ldots, b_{i,\ell,P})$. As discussed before, σ permutes the entries of each tuple $(b_{i,j,P})_{j\leq\ell}$ amongst themselves. Since $c = b^{-1}\sigma(b) \in \mathrm{Nrd}_{E_N}(C_{D_N}(E_N))$, we have that $(b_{i,1,P})[D_P] = (b_{i,2,P})[D_P] = \ldots = (b_{i,\ell,P})[D_P] \in \mathrm{H}^3(F_P, \mu_\ell)$. Let $b_{i,j,P}$ have valuation m_j in $F_{P,\eta}$.
We first look at the case when π_P defines η. Set $f_{i,P} := b_{i,1,P}^{-1}\pi_P^{m_1}$. Since π_P is a parameter of $F_{P,\eta}$ also, $f_{i,P}$ is a unit along η. Define $f_P = \prod_i (f_{i,P}) \in N \otimes F_P$. Thus $f_P \in \widehat{C_{P,\eta}}^*$. It now suffices to see that each $b_{i,j,P} f_{i,P}$ is a reduced norm from D_P. For $j = 1$, we have $(b_{i,1,P} f_{i,P})[D_P] = (\pi_P^{m_1})(u_P, \pi_P) = 0$. Since the cup-products $(b_{i,j,P})[D_P]$ equal each other for $j \leq \ell$, we have $(b_{i,j,P} f_{i,P})[D_P] = 0$ for each j.

Identifying $\mathrm{H}^1(F_P, \mathrm{SL}_1(D_P))$ with $F_P^*/\operatorname{Nrd}(D_P)^*$, recall Suslin's invariant

$$R : \mathrm{H}^1(F_P, \mathrm{SL}_1(D_P)) \to \mathrm{H}^3\left(F_P, \mu_\ell^{\otimes 2}\right),\ \lambda \rightsquigarrow (\lambda) \cup [D_P].$$

Since index of D_P is ℓ and in particular square-free, R is injective ([MS82], Theorem 12.2). Hence $b_{i,j,P} f_{i,P}$ is a reduced norm from D_P and hence we are done in this case.

Now let's look at the case when π_P defines η'. Thus η is either of Type 1a or P is a hot point and η is of Type 2 with Y_η of Type NONRES (Rows 8.5-8.6 of Table 17). In either case $D \otimes E_\eta$ is an unramified index ℓ algebra[26]. Thus by our initial modification, $b_{i,\eta} \in \widehat{B_\eta}^*$ already which shows that $b_{i,j,P} \in \widehat{A_{P,\eta}}^*$ for each $j \leq \ell$. Since π_P is a unit along η now, so is $f_{i,P}$. □

PROPOSITION 13.3 (f at codimension one points). *Let $\eta \in N_0$. Then there exists $f_\eta \in \widehat{C_\eta}^* \subset N \otimes F_\eta$ such that*

- $f_\eta = f_P \phi_{P,\eta}^\ell \in N \otimes F_{P,\eta}$ *for some* $\phi_{P,\eta} \in (N \otimes F_{P,\eta})^*$ *for each* $P \in \mathcal{P}'_\eta \cup \mathcal{R}'_\eta$.
- $b f_\eta$ *is a reduced norm from* $D_N \otimes E_\eta$.

Proof. Note that by Proposition 13.2, we see that $f_P \in \widehat{C_{P,\eta}}^*$ for each $P \in \mathcal{P}'_\eta \cup \mathcal{R}'_\eta$.
η IS OF TYPE 0: D_η is split and so is $D \otimes F_P$ for every $P \in \overline{\eta}$. Thus each $f_P = 1$ by choice for every marked point P on $\overline{\eta}$ (Proposition 13.2). Choose $f_\eta = 1$. Clearly $b f_\eta$ (and indeed any other element in $N \otimes E_\eta$) is a reduced norm from $D_N \otimes E_\eta$.
η IS OF TYPE 1A: By construction, E_η is unramified (Propositions 11.1, 11.2 and 11.3). By weak approximation, find $\overline{f_\eta} \in k''_\eta$ which is close to $\overline{f_P} \in k''_{P,\eta}$ for each marked P in $\overline{\eta}$. Let f_η be a lift of $\overline{f_\eta}$ in $\widehat{C_\eta}^*$.
If $D \otimes E_\eta$ is split, then clearly $b f_\eta$ (and indeed any other element in $N \otimes E_\eta$) is a reduced norm from $D_N \otimes E_\eta$. So assume $D \otimes E_\eta$ is not split. Since η is Type 1a, D_η is an unramified index ℓ algebra and hence so is $D \otimes E_\eta$. By our initial modification of b, this implies all components of b are units along η. Thus by Lemma 2.7, $b f_\eta$ is a reduced norm from $D_N \otimes E_\eta$.
η IS OF TYPE 1B/2: Let $u_\eta \in F_\eta$ such that $u' = \overline{u_\eta} \in k_\eta / k_\eta^{*\ell}$ is the residue of D_η. There are three possible shapes of E_η.
Shape A: E_η is a ramified/unramified field extension which splits D_η (Propositions 10.1, 10.2 and 10.6).
Shape B: E_η is the lift of residues which might or might not split D_η (Proposition 10.3). Though in particular, it is an unramified field extension of F_η.
Shape C: E_η / F_η is an unramified field extension which is not the lift of residues of F_η. Then u' is a norm from $\overline{E_\eta}$ and $E_\eta \otimes \beta_{\mathrm{rbc},\eta}$ is split. (Propositions 10.4 and 10.5).
For each shape, we prescribe $f_\eta \in N \otimes F_\eta$ as follows:
E_η *of Shape A/B*: As before find $\overline{f_\eta} \in k''_\eta$ which is close to $\overline{f_P} \in k''_{P,\eta}$ for each $P \in \mathcal{P}'_\eta \cup \mathcal{R}'_\eta$. Let f_η be a lift of $\overline{f_\eta}$ in $\widehat{C_\eta}^*$. If E_η is of Shape A , since $D \otimes E_\eta$ is split, every element in $N \otimes E_\eta$ is a reduced norm from $D_N \otimes E_\eta$.

[26] It is unramified if η is Type 1a and by Proposition 10.3 otherwise. It is non-split since $D \otimes E_P$ and hence $D \otimes E_{P,\eta}$ is non-split.

Let E_η be of Shape B. Note that $D = \beta_{rbc,\eta} + (u_\eta, \pi_\eta) \in \mathrm{Br}(F_\eta)$ where π_η is a parameter of F_η. Thus $D \otimes E_\eta$ is an unramified algebra. If it is split, our choice of f_η clearly works. So assume $D \otimes E_\eta$ has index ℓ. Then by our initial modification of b, each component of b is a unit along η. Thus by Lemma 2.7, bf_η is a reduced norm from $D_N \otimes E_\eta$.

E_η *of Shape C* : Note that $D = \beta_{rbc,\eta} + (u_\eta, \pi_\eta) \in \mathrm{Br}(F_\eta)$ where π_η is a parameter of F_η. Since E_η splits $\beta_{rbc,\eta}$, we have $D \otimes E_\eta = (u_\eta, \pi_\eta)$. Let $E_N \otimes F_\eta / N \otimes F_\eta \simeq \prod_q E_\eta/F_\eta \times \prod_i \left(\prod_\ell E_\eta/E_\eta\right)$ and let $b = \prod b_q \times \prod_i (b_{i,j,\eta})_{j \leq \ell}$. We will prescribe $f_\eta = \prod f_q \times \prod_i e_i \in N \otimes F_\eta \simeq \prod_q F_\eta \times \prod_i E_\eta$ by prescribing each of its components $f_q \in \widehat{A_\eta}^*$ and $e_i \in \widehat{B_\eta}^*$ individually.

Let us look at the case of $b_q \in E_\eta/F_\eta$. By our initial modification of b, we have $b_q \in \widehat{B_\eta}^*$ for each q and hence $\sigma(b_q) \in \widehat{B_\eta}^*$ also. Set $E' = \overline{E_\eta}$ and $b' = \overline{b_q} \in E'$. By abuse of notation, let $\mathrm{Gal}(E'/k_\eta) = \langle \sigma \rangle$ also. Since c is a reduced norm from $D_N \otimes E$, $\left(b_q^{-1}\sigma(b_q)\right)(u_\eta, \pi_\eta) = 0 \in \mathrm{H}^3(E_\eta, \mu_\ell)$. This gives $\left(b'^{-1}\sigma(b'), u'\right) = 0 \in \mathrm{H}^2(E', \mu_\ell)$. Thus $(b', u') = (\sigma(b'), u') \in \mathrm{H}^2(E', \mu_\ell)$.

We would like to apply Lemma 2.12 to find an $\overline{f_q} \in k_\eta$ and hence an $f_q \in F_\eta$ with the required properties. To do so, we proceed to verify that the rest of the hypotheses of the lemma are indeed satisfied by u', E'/k_η and b'.

By ([S97], [S98], Proposition 1.2), we see that the residue u' is up to ℓ^{th} powers, a unit at almost all places v of k_η except at those given by cold points (Type C-Cold) P on $\overline{\eta}$. Recall that by the choice of E_P at cold points (cf Tables 15 and 16), at such places $E'/k_{P,\eta}$ is given by adjoining the ℓ^{th} root of the residue u' and hence $u' \in {E'_{P,\eta}}^{\ell}$. In particular, this discussion shows that at every place v where E' is unramified and inert, $u' \in \mathcal{O}^*_{E'_v}$ up to ℓ^{th} powers in ${E'_v}^*$.

Let w be a place where E'/k_η is ramified. We have already seen that if w corresponds to a cold point P, then $u' \in {E'_w}^{*\ell}$. Therefore assume w corresponds to a non-cold point P. Hence $u' \in \mathcal{O}^*_{k_{P,\eta}}$. Since we know u' is a norm from E' and hence from E'_w, Lemma 2.3 implies that $u' \in {E'_w}^{*\ell}$.

Finally for $P \in \mathcal{P}'_\eta \cup \mathcal{R}'_\eta$, we have $(b_q f_{q,P})$ is a reduced norm from $D_N \otimes E_P$. This implies $(b_q f_{q,P})(u_\eta, \pi_\eta) = 0 \in \mathrm{H}^3(E_{P,\eta}, \mu_\ell)$. Taking residues, this implies $(u', b') = \left(u', \overline{f_{q,P}}^{-1}\right) \in E' \otimes k_{P,\eta}$.

Thus we can apply Lemma 2.12 to find $f_q \in F_\eta$ such that $f_q \equiv f_{q,P} \in F_{P,\eta}$ up to ℓ^{th} powers for marked points $P \in \mathcal{P}'_\eta \cup \mathcal{R}'_\eta$. Further, $(u_\eta, b_q f_q) = 0 \in \mathrm{Br}(E_\eta)$. This implies $(u_\eta, \pi_\eta)(b_q f_q) = 0$. Since $D \otimes E_\eta = (u_\eta, \pi_\eta)$, we have $b_q f_q \in \mathrm{Nrd}_{E_\eta}(D \otimes E_\eta)$ using injectivity of Suslin's invariant for index ℓ algebras again ([MS82], Theorem 12.2).

Now let us look at the case of $(b_{i,j,\eta})_{j \leq \ell} \in \left(\prod_\ell E_\eta\right)/E_\eta$. Since c is a reduced norm from $D_N \otimes E$, we have that $(b_{i,1,\eta})[D \otimes E_\eta] = (b_{i,2,\eta})[D \otimes E_\eta] = \ldots = (b_{i,\ell,\eta})[D \otimes E_\eta] \in \mathrm{H}^3(E_\eta, \mu_\ell)$. Let $b_{i,j,\eta}$ have valuation m_j in E_η.

Set $e'_i := b_{i,1,\eta}^{-1}\pi_\eta^{m_1}$. Since π_η is a parameter of F_η and hence of E_η also, $e'_i \in \widehat{B_\eta}^*$. Since $(b_{i,1,\eta}e'_i)[D \otimes E_\eta] = (\pi_\eta^{m_1})(u_\eta, \pi_\eta) = 0$. Since the cup-products $(b_{i,j,\eta})[D \otimes E_\eta]$ equal each other for $j \leq \ell$, we have $0 = (b_{i,j,\eta}e'_i)[D \otimes E_\eta]$ for each j. Thus by

injectivity of Suslin's invariant for index ℓ algebras, each $b_{i,j,\eta}e_i'$ is a reduced norm from $D \otimes E_\eta$.
However, e_i' might not approximate the choice at marked points on $\overline{\eta}$. So we find a suitable correcting factor $\theta \in \widehat{B_\eta}^*$ such that $\theta \in \mathrm{Nrd}(D \otimes E_\eta)$ and $e_i'\theta$ is close to the choice along marked points. Then $e_i = e_i'\theta$ is still in $\widehat{B_\eta}^*$ and each $b_{i,j,\eta}e_i$ is a reduced norm from $D \otimes E_\eta$.
Note that since $D \otimes E_{P,\eta}$ is still ramified, if $P \in \mathcal{P}_\eta' \cup \mathcal{R}_\eta'$, then $D \otimes E_P \simeq (u_P, \pi_P)$ where π_P defines η at P. Let $\pi_P = \pi_\eta \theta'$ for some $\theta' \in \widehat{A_\eta^*}$. Following the proof of Proposition 13.2, we see that we are in the case when η is Type 1b/2 and $E_P \simeq \prod F_P$. Thus $E_{P,\eta} \simeq \prod F_{P,\eta}$ and under this identification, $(b_{i,j,\eta})_{j\leq \ell} \in (\prod_\ell E_\eta / E_\eta)$ goes to $(\sigma^{j-1}(b_{i,1,\eta}), \sigma^{j-1}(b_{i,2,\eta}), \ldots, \sigma^{j-1}(b_{i,\ell,\eta}))_{j\leq\ell}$ in $\prod_\ell (\prod_\ell F_{P,\eta}/F_{P,\eta})$ over the branch. Our choice of $e_{i,P}$ along the branch corresponds to

$$\left(b_{i,1,\eta}^{-1}\pi_P^{m_1}, \sigma(b_{i,1,\eta})^{-1}\pi_P^{m_1}, \ldots, \sigma^{\ell-1}(b_{i,1,\eta})^{-1}\pi_P^{m_1}\right) \in \prod \widehat{A_{P,\eta}}^* \simeq \widehat{B_{P,\eta}}^*,$$

i.e. $e_{i,P} = b_{i,1,\eta}^{-1}\pi_P^{m_1} \in \widehat{B_{P,\eta}}^*$ and $e_{i,P}{e_i'}^{-1} = \theta'^{m_1} \in \widehat{B_{P,\eta}}^*$.
Since both π_η and π_P are reduced norms from $D \otimes E_{P,\eta}$, so is θ' and hence θ'^{m_1}. Therefore $(\theta'^{m_1}, u_\eta) = 0 \in \mathrm{H}^2(F_{P,\eta}, \mu_\ell)$ and $(\overline{\theta'}^{m_1}, \overline{u_\eta}) = 0 \in \mathrm{H}^2(k_{P,\eta}, \mu_\ell)$. Find $\theta_1 \in F_{P,\eta}\left(\sqrt[\ell]{u_\eta}\right)$ such that $\mathrm{N}(\theta_1) = \theta'^{m_1}$. Note that since $D \otimes E_P = (u_P, \pi_P)$, $F_{P,\eta}\left(\sqrt[\ell]{u_\eta}\right)$ is an unramified field extension of $F_{P,\eta}$. Choose $\tilde{\theta}_1 \in \mathcal{O}^*_{F_\eta\left(\sqrt[\ell]{u_\eta}\right)}$ such that its image is close to θ_1 and set $\theta = \mathrm{N}\left(\tilde{\theta}_1\right) \in F_\eta$. □

13.3 Spreading and patching of f

Proposition 13.4. *For each η in N_0, there exist a neighbourhood U_η' of η in X_0 with $U_\eta' \subseteq \overline{\eta} \setminus \left(\mathcal{P}_\eta' \cup \mathcal{R}_\eta'\right)$ and an $f_{U_\eta'} \in N \otimes F_{U_\eta'}$ such that*

1. *$U_\eta' \subseteq U_\eta$ where U_η are the neighbourhoods in the patching set up $\mathcal{P}$*
2. *$bf_{U_\eta'}$ is a reduced norm from $D_N \otimes E \otimes F_{U_\eta'}$.*
3. *$f_{U_\eta'} \cong f_\eta$ up to ℓ^{th} powers in $N \otimes F_\eta$.*

Proof. By Proposition 13.3, we see that $f_\eta \in \widehat{C_\eta}^*$ and that $bf_\eta \in \mathrm{Nrd}(D_N \otimes E_\eta)$. Thus $(bf_\eta)([D_N \otimes E_\eta]) = 0 \in \mathrm{H}^3(N \otimes E_\eta, \mu_\ell)$. Let $f' \in N^*$ such that $f_\eta^{-1}f'$ is 1 mod the maximal ideal of $\widehat{C_\eta}$. Note that $f_\eta = f'x^\ell \in (N \otimes F_\eta)^*$ for some $x \in \widehat{C_\eta}^*$ and hence $(bf')([D_N \otimes E_\eta]) = 0 \in \mathrm{H}^3(N \otimes E_\eta, \mu_\ell)$. By ([PPS18], proof of Lemma 7.2 & [HHK14], proof of Proposition 3.2.2) and shrinking further if necessary, there exists a neighbourhood $U_\eta' \subseteq U_\eta$ of η such that $(bf')([D_N \otimes E \otimes F_{U_\eta'}]) = 0 \in \mathrm{H}^3\left(E_N \otimes F_{U_\eta'}, \mu_\ell\right)$. Since $D_N \otimes E$ has index ℓ, by injectivity of Suslin's invariant ([MS82], Theorem 12.2), we have bf' is a reduced norm from $D_N \otimes E \otimes F_{U_\eta'}$. The element $f_{U_\eta'} := f'$ has the required properties. □

Remark 13.5. *Let T'_η denote the finite set of closed points $\overline{\eta} \setminus \left(U'_\eta \cup \mathcal{P}'_\eta \cup \mathcal{R}'_\eta\right)$. Thus $\{\mathcal{P}'_\eta \cup \mathcal{R}'_\eta \cup \mathcal{T}'_\eta, U'_\eta\}_{\eta \in N_0}$ form a patching set up $\mathcal{P}'$ as in defined in ([HH10]).*

Proposition 13.6. *Let $\eta \in N_0$ and let $P \in \left(\mathcal{P}'_\eta \cup \mathcal{R}'_\eta \cup T'_\eta\right)$. Then there exists $f_P \in N \otimes F_P$ such that*

1. *bf_P is a reduced norm from $D_N \otimes E_P$.*
2. *$f_{U'_\eta}\psi^{\ell}_{P,\eta} = f_P$ in $N \otimes F_{P,\eta}$ for some $\psi_{P,\eta} \in N \otimes F_{P,\eta}$.*

Proof. If $P \in \mathcal{P}'_\eta \cup \mathcal{R}'_\eta$, the proposition follows from Propositions 13.2, 13.3 and 13.4. Hence assume $P \in \mathcal{T}'_\eta$. In particular, this implies $P \in U_\eta$ where U_η is the neighbourhood of η in the patching system $\mathcal{P}$ defined in Remark 12.4. Hence $F_{U_\eta} \subseteq F_P$. Let (π_P, δ_P) be a system of regular parameters of A_P where π_P defines the curve $\overline{\eta}$ at P.

We choose f_P depending the shape of $D \otimes E_P$ as follows:

$D \otimes E \otimes F_P$ is split: Since $N \otimes F_P$ is dense in $N \otimes F_{P,\eta}$, pick an f_P here which approximates $f_\eta \in N \otimes F_\eta$ treated as an element over the branch, i.e. $f_\eta \in N \otimes F_{P,\eta}$. The proposition is clearly true for this choice of f_P.

$D \otimes E \otimes F_P$ is not split: Since P is a curve point, we have $D \neq 0 \in \mathrm{Br}(F_P)$ possibly only if η is of Type 1b or 2, in which case $D = (u_P, \pi_P) \in \mathrm{Br}\,(F_P)$ where $u_P \in \widehat{A_P}^*$ ([S97]). Let $u_\eta \in F_\eta$ be such that $u' = \overline{u_\eta} \in k^*_\eta / k^{*\ell}_\eta$ is the residue of D_η. Thus $\overline{u_P} \cong u' \in k_{P,\eta}$ up to ℓ^{th} powers.

Except when η is coloured green, $D \otimes E_\eta$ is split by construction (Propositions in 10). By Proposition 12.1, this implies $D \otimes E \otimes F_{U_\eta}$ is split and hence so is $D \otimes E \otimes F_P$. When η is coloured green, by Propositions 10.3, 10.4 and 10.5, E_η / F_η is unramified. By Proposition 12.1, this implies $E \otimes F_{U_\eta} \simeq \frac{F_{U_\eta}[t]}{(t^\ell - e)}$ for some unit $e \in \widehat{A_U}^*$. Therefore $E \otimes F_P = \prod F_P$ or L_P, the unique field extension of F_P of degree ℓ unramified at $\widehat{A_P}$. If $E \otimes F_P$ is a nonsplit field extension, then $D \otimes E \otimes F_P$ is split. Thus $E \otimes F_P \simeq \prod F_P$. Therefore $E_N \otimes F_P / N \otimes F_P \simeq \prod_i \left(\prod_\ell F_P\right) / F_P$. Let us look at the i-th component $\left(\prod_\ell F_P\right) / F_P$ in $E_N \otimes F_P / N \otimes F_P$. We will prescribe f_P by prescribing each of its components $f_i \in F_P$.

Let $b_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,\ell}) \in \prod F_P$. By Proposition 13.3, we have $f_{P,\eta} \in \widehat{A_{P,\eta}}^*$ such that for each j, we have $(b_{i,j} f_{P,\eta})\,(u_P, \pi_P) = 0 \in \mathrm{H}^3\,(F_{P,\eta}, \mu_\ell)$. Let $b_{i,1}$ have valuation m_1 in $F_{P,\eta}$ and let $b'_{i,1} := b_{i,1} f_{P,\eta} \pi_P^{-m_1} \in \widehat{A_{P,\eta}}^*$. Thus $\left(b'_{i,1}\right)(u_P, \pi_P) = 0$ also and taking residues, we get $\left(\overline{b'_{i,1}}, \overline{u_P}\right) = 0 \in \mathrm{H}^2\,(k_{P,\eta}, \mu_\ell)$. Since $\left(b'_{i,1}, u_P\right)$ is unramified over $F_{P,\eta}$, it is also split over $F_{P,\eta}$ and we see that there exists $\theta_1 \in F_{P,\eta}\left(\sqrt[\ell]{u_P}\right)$ such that $\mathrm{N}\,(\theta_1) = b'_{i,1}$.

Since we are in the case when $D \otimes E \otimes F_P$ is not split, $u_P \notin F^{*\ell}_P$. Therefore $F_{P,\eta}\left(\sqrt[\ell]{u_P}\right)$ is an unramified field extension of $F_{P,\eta}$ as also its residue field $k_{P,\eta}\left(\sqrt[\ell]{\overline{u_P}}\right) / k_{P,\eta}$. As $b'_{i,1} \in \widehat{A_{P,\eta}}^*$, clearly $\theta_1 \in \mathcal{O}^*_{F_{P,\eta}(\sqrt[\ell]{u_P})}$. Let $\overline{\theta_1} = \theta' \overline{\delta_P}^m$ where $\theta' \in \mathcal{O}_{k_{P,\eta}(\sqrt[\ell]{\overline{u_P}})}$ and $m \in \mathbb{Z}$. Find $\tilde{\theta}' \in \mathcal{O}^*_{F_P(\sqrt[\ell]{u_P})}$ such that its image matches that of θ'. Set $\tilde{\theta}_1 = \tilde{\theta}' \delta_P^m \in F_P\left(\sqrt[\ell]{u_P}\right)$. Thus $\tilde{\theta}_1 \cong \theta_1$ up to ℓ^{th} powers in $F_{P,\eta}\left(\sqrt[\ell]{u_P}\right)$. Set $f_{i,P} = b_{i,1}^{-1}\,\mathrm{N}\left(\tilde{\theta}_1\right) \pi_P^{m_1} \in F_P$.

Thus by construction $(b_{i,1}f_{i,P})(u_P,\pi_P) = 0 \in \mathrm{H}^3(F_P,\mu_\ell)$ and $f_{i,P} \cong f_{P,\eta}$ up to ℓ^{th} powers in $F_{P,\eta}$. Finally, since $b^{-1}\sigma(b)$ is a reduced norm from $D \otimes E_P$, we have that for every j, the cup-products $(b_{i,j})(u_P,\pi_P)$ are all equal in $\mathrm{H}^3(F_P,\mu_\ell)$. Thus, we also have that for each j, $(b_{i,j}f_{i,P})(u_P,\pi_P) = 0 \in \mathrm{H}^3(F_P,\mu_\ell)$. Again using injectivity of Suslin's invariant for index ℓ algebras ([MS82], Theorem 12.2), we can argue as before that this implies $b_{i,j}f_{i,P}$ is a reduced norm from $D \otimes F_P$ for each j and hence that $b_i f_{i,P}$ is a reduced norm from $D \otimes E \otimes F_P$. □

We are now in a position to find $f \in N$ satisfying the hypothesis of Proposition 13.1.

PROPOSITION 13.7. *There exists $f \in N$ such that $bf \in \mathrm{Nrd}_{E_N}(C_{D_N}(E_N))$.*

Proof. By Propositions 13.4 and 13.6, we have $f_x \in N \otimes F_x$ for $x \in \{U'_\eta, \mathcal{P}'_\eta \cup \mathcal{R}'_\eta \cup \mathcal{T}'_\eta\}_{\eta \in N_0}$ in the patching set-up $\mathcal{P}'$ defined in Remark 13.5 such that for $bf_x \in \mathrm{Nrd}\, D_N \otimes E_x$. Further for each branch in the patching set-up corresponding to a pair (U'_η, P), we have $f_P = f_{U'_\eta}\psi^\ell_{P,\eta}$ for some $\psi_{P,\eta} \in N \otimes F^*_{P,\eta}$.
By *simultaneous factorization for curves* for the rational group $R_{N/F}\mathbb{G}_m$ ([HHK09], Theorem 3.6), we can find $\psi_x \in (N \otimes F_x)^*$ for each $x \in \mathcal{P}'$ such that for every branch defined by (U'_η, P), we have $\psi_{P,\eta} = \psi_{U'_\eta}\psi_P^{-1}$. Thus we have $f_{U'_\eta}\psi^\ell_{U'_\eta} = f_P\psi^\ell_P$ for every branch (U'_η, P). Therefore there exists $f \in N$ such that $f = f_x\psi^\ell_x \in N \otimes F_x$ for each $x \in \{U'_\eta, P\}$ ([HH10], Proposition 6.3 & Theorem 6.4). Thus $bf \in \mathrm{Nrd}(D_N \otimes E_x)$ and therefore $(bf) \cup [D_N \otimes E_x] = 0 \in \mathrm{H}^3(N \otimes E_x, \mu_\ell)$ for each $x \in \mathcal{P}'$. This implies $(bf) \cup [D_N \otimes E] = 0 \in \mathrm{H}^3(E_N, \mu_\ell)$ ([PPS18], proofs of Proposition 7.1 & 7.4). Injectivity of Suslin's invariant for index ℓ algebras ([MS82]) shows that $bf \in \mathrm{Nrd}(D_N \otimes E)$ which proves the proposition. □

Thus we have our main theorem:

THEOREM 13.8. *Let F be the function field of a curve over a p-adic field. Let D/F be a central division algebra of prime exponent ℓ which is different from p. Assume that F contains a primitive ℓ^{2th} root of unity. Then $\mathrm{SK}_1(D)$ is trivial.*

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