Hecke L-Functions and Fourier Coefficients of Covering Eisenstein Series

Fan Gao

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Abstract. We consider in this paper covering groups and Fourier coefficients of Eisenstein series for induced representations from certain distinguished theta representations. It is shown that one has global factorization of such Fourier coefficients, and the local unramified Whittaker function at the identity can be computed from the local scattering matrices. For a special family of covering groups of the general linear groups, we show that the Fourier coefficients of such Eisenstein series are reciprocals of Hecke L-functions, which recovers an earlier result by Suzuki for Kazhdan–Patterson covering groups. We also consider covers of the symplectic group and carry out a detailed analysis in the rank-two case.

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1 Introduction

For a linear algebraic group, Fourier coefficients of Eisenstein series for induced representations from a parabolic subgroup are important objects for the study of automorphic forms and L-functions. In particular, the Casselman–Shalika formula [Shi76, CS80] is a cornerstone for the subject. The formula expresses the value of the unique Whittaker function of a generic unramified representation in terms of L-functions for certain representations of the L-group on the dual side (see for example [Tam91, Proposition 1]). The uniqueness of such Whittaker function for local representations then further enables one to apply the Casselman–Shalika formula to obtain a uniform description of the partial global L-function that appears in the Fourier coefficients of Eisenstein series for linear algebraic groups. Such results, for example, were exploited in the work of Shahidi (see [Sha78, Sha81, Sha88, Sha90]) to develop and complete the theory of Langlands–Shahidi L-functions arising from Eisenstein series.

However, for finite degree central coverings of linear algebraic groups, the uniqueness of Whittaker functionals fails in general. This has the direct consequence that the analogous Casselman–Shalika formula takes a more sophisticated form. This failure was first systematically studied in [KP84, KP86] for certain covers of GL_r, which are naturally called Kazhdan–Patterson covers of GL_r. The investigation in [KP84] relies on the so-called scattering matrix arising from a map between two Whittaker spaces (i.e., the space of Whittaker functionals) induced from intertwining operators, while the method of [KP86] is via trace formula. With the same focus on such scattering matrices, the Casselman–Shalika formula in the covering setting was generalized in [Pat87, CO13, McN16, Suz97, GSS]. In a somewhat different direction, various forms of the Casselman–Shalika formula were also proved in connection with the theory of crystal basis, Demazure operators and representations of quantum groups, see [BBF11a, BN10, McN11, LLS14, LLL19, KL11]. Working with universal principal series, there are also other formulations of the Casselman–Shalika formula as in [BBF16, Pus, PP17, PP19], some including covers on the Kac-Moody groups.

As a result of such non-uniqueness, the computation of the Fourier coefficients of covering Eisenstein series becomes not as accessible as in the linear algebraic case. Indeed, if one considers the covering Borel Eisenstein series, then its Fourier coefficients are conjecturally just the Weyl group multiple Dirichlet series (WMDS), the theory of which has been developed and studied in much depth as in [BBC+06, BBF06, BBF08, BBFH07, BBFH12]. The theory of WMDS has proved to be important even for the theory of automorphic forms and L-functions for linear algebraic groups, especially concerning the analytic properties. See [BFH05, CFH06]. Such Weyl group multiple Dirichlet series (at least conjecturally) possess functional equations and meromorphic continuations; however, they are not Eulerian in general, a consequence of the fact that the covering torus is not abelian. Therefore, the difficulty of studying such Dirichlet series arises not only from the local representations, but also from the way (i.e., the twisted multiplicativity for WMDS) such local information manifests globally.

This multiplicity of Whittaker functionals lies in the heart of obstacles and difficulties of extending the theory of L-functions from linear algebraic groups to covering groups. This is especially the case for the Langlands–Shahidi method. However, in another direction of generalizing the classical doubling method of Piatetski-Shapiro and Rallis [GPSR87], there has been recent advance for studying L-functions for classical groups, including in the covering setting, see [CFGK19, CFK, Kap]. One crucial point is that in loc. cit., no assumption on the dimension of the Whittaker space is needed, and thus is especially applicable

to genuine representations of covering groups.
Nevertheless, it is natural to consider the subclass of representations for which such uniqueness holds. It is also reasonable to explore finer structure of the Whittaker space and consider some variants of the usual notion of Whittaker functionals. For this purpose, we briefly recall some notations in the local setting. Let $\overline{G}$ be an n-fold covering group over a non-archimedean local field F, which is a central extension of G by the group of n-th roots of unity in F, denoted by μ_n. Let $\overline{B} = \overline{T}U$ be its covering Borel subgroup. Let $Z(G)$ be the center of G, and $\overline{Z(G)} \subset \overline{G}$ the covering of $Z(G)$. Let ψ be a generic character of U. Let (π, V_π) be a genuine irreducible representation of $\overline{G}$, on which μ_n acts by a fixed embedding into $\mathbf{C}^\times$. We mention below some typical classes of groups and approaches considered in the literature.

- The twisted Jacquet module $J_{\pi,\psi}$, whose dual is the space $\mathrm{Wh}_\psi(\pi)$ of ψ-Whittaker functionals of π, is naturally a $U \times \overline{Z(G)}$-module. Here $\overline{Z(G)}$ is a Heisenberg type group, and its genuine irreducible representations are finite dimensional with the same dimension. If the uniqueness of ψ-Whittaker functionals for π holds, then $\overline{Z(G)}$ is necessarily abelian. The converse may not hold. However, assume that $\overline{Z(G)}$ is abelian. Let $\psi \times \mu$ be the representation of $U \times \overline{Z(G)}$, where μ is a fixed genuine character of $\overline{Z(G)}$. Consider π such that
$$\dim \mathrm{Hom}_{U \times \overline{Z(G)}}(J_{\pi,\psi}, \psi \times \mu) \leq 1. \tag{1}$$
If the equality holds, then π is said to possess a unique (ψ, μ)-Whittaker functional. For the two-fold Kazhdan–Patterson covering $\overline{\mathrm{GL}}_2$ with twisting parameter 0 (in the notation of [KP84]), it was shown by Gelbart, Howe, and Piatetski-Shapiro [GHPS79] that every irreducible genuine representation of this $\overline{\mathrm{GL}}_2$ has a unique (ψ, μ)-Whittaker functional up to a scalar. This fact was used in [GPS80] to study distinguished theta representations. Unfortunately, for covers of general linear groups, such (ψ, μ)-uniqueness does not hold for all genuine representations. In fact, it already fails for (appropriate) higher degree covers of GL_2.

- For a fixed n-fold covering $\overline{G}$, one can also study the subclass of representations where uniqueness of ψ-Whittaker functionals (or even (ψ, μ)-Whittaker functionals as above) holds. For instance, for coverings of the general linear groups, Kazhdan–Patterson initiated a representation theoretic analysis of the theta representations in [KP84] after the work of Kubota [Kub69]. In particular, the dimension of the space of ψ-Whittaker functionals for such a theta representation was determined in terms of the rank and degree of the covering group. We note that theta representations and their analogues, for example the Weil representations of the metaplectic double cover $\overline{\mathrm{Sp}}_{2r}^{(2)}$, have been extensively studied. In particular, as the Weil representation is the underlying key for the theta correspondence, it has been exploited to establish links between representations of $\overline{\mathrm{Sp}}_{2r}^{(2)}$ and the linear groups SO_{2m+1} or their inner forms. As the literature on this is vast, we simply refer the reader to [Gan14] and references therein for a quick review.

The theta representations we focus in this paper are just residues of the covering Borel Eisenstein series, as in [KP84]. Locally, the theta representation is the Langlands quotient of the "most reducible" standard module. Presumably, the family of theta representations should be the easiest one to study among all genuine representations of covering groups, as they should arise from liftings of characters of an appropriate linear group (see [Fli80] in the case of $\overline{\mathrm{GL}}_2$). Such theta representations not only play a special role for understanding genuine representations of covering groups, they are in fact very useful for understanding representations on linear algebraic groups.

For example, Bump and Hoffstein formulated several related conjectures [BH89] regarding the usage of theta representations in obtaining L-functions via the Rankin–Selberg method for genuine representations. One of their conjectures was extensively studied in the work of Suzuki [Suz91, Suz97]; in this direction, we also mention [FG15, Goe98, Gin18a]. Moreover, the work of Bump–Ginzburg [BG92] gives a Rankin–Selberg integral for the symmetric square L-functions of cuspidal representations π of GL_r, and their method relies crucially on properties of theta representations studied in [KP84]. The case of twisted symmetric square L-functions was treated by Takeda [Tak14]. See yet another recent work [FK19] of obtaining (quotient of) L-functions of π by using a Godement–Jacquet type integral involving theta representations. In another direction, starting with the work of Savin [Sav92] on representations distinguished by theta representations, the investigation was continued in [Kab01, Kab02]. Furthermore, in a series of works by Kaplan [Kap15, Kap16a, Kap16b, Kap17a, Kap17b], the theory was further studied and the author also found applications to the problems of computing certain periods. Notably, determining unipotent orbits of theta representations has also important applications, which is already clear in the work [BG92, Tak14, FK19] mentioned above, and is also one of the foci in the study by Friedberg and Ginzburg [FG18, FG17], Y.-Q. Cai [Cai19] and Leslie [Les19]. We also mention that the problem of the existence of cuspidal theta representations remains to be of interest and of challenge, see [PPS84, FG16] and references therein.

Compared to the above, our goal in this paper is of a different nature and is motivated from [BBL03]. It is to compute the Fourier coefficients of covering Eisenstein series induced from a genuine representation $\pi = \otimes_v \pi_v$ which possess unique nontrivial local and global Whittaker models. That is, locally we assume

$$\dim \mathrm{Wh}_{\psi_v}(\pi_v) = 1,$$

and globally up to scalar there is a unique nonzero ψ-Whittaker functional $\lambda_{\mathbb{A}}$ afforded by the Whittaker–Fourier coefficients. In this context, the ψ-Fourier coefficient of the Eisenstein series has a global factorization, and it is natural to ask what the Fourier-coefficient should be in terms of L-functions associated to the inducing data. Our paper concentrates on the case where the inducing representation is a global theta representation $\Theta(\overline{M}_{\mathbb{A}}, \chi)$ satisfying the above multiplicity-one property. We call such $\Theta(\overline{M}_{\mathbb{A}}, \chi)$ distinguished (see Definition 3.2). We briefly review some earlier work on this topic.

- In [BBL03], Banks, Bump and Lieman considered the case of degree n global Kazhdan–Patterson covering $\overline{\mathrm{GL}}_n$ and the covering parabolic subgroup $\overline{P} = \overline{M}N$ with

  $$M = \mathrm{GL}_{n-1} \times \mathrm{GL}_1.$$

  The representation of $\overline{M}$ is essentially a certain theta representation $\Theta(\chi)$ of $\overline{M}$, where χ is a genuine exceptional character. The ψ-Whittaker functionals for $\Theta(\chi)$ are not unique, and therefore for the induced representation $I(s, \Theta(\chi))$ of $\overline{\mathrm{GL}}_n$, uniqueness for ψ-Whittaker functionals also fails. However, in this case, $\overline{Z(\mathrm{GL}_n)}$ is abelian and it is shown in [BL94] that (ψ, μ)-Whittaker functional for $I(s, \Theta(\chi))$ is unique. This is the first key ingredient in [BBL03], which gives that the Fourier coefficients of Eisenstein series could be factorized into local Whittaker functions. The second key ingredient in [BBL03] is the local unramified computation, where the authors showed that the analogous Casselman–Shalika formula for $I(s, \Theta(\chi))$ involves a *quotient* of two Hecke L-functions for linear characters associated to χ.

- The main result of [BBL03], as remarked by the authors, is just a special case of the Bump–Hoffstein conjecture [BH89]. To recall the conjecture, we fix n and let $1 \leq r' \leq r \leq n-1$. Let Θ_r denote a theta representation of the n-fold Kazhdan–Patterson cover $\overline{\mathrm{GL}}_r$. Then Bump and Hoffstein conjectured that $L(s, \Theta_r \times \Theta^{\vee}_{n-r'})$ differs from $L(s, \Theta_{r'} \times \Theta^{\vee}_{n-r})$ by several Hecke L-functions in a precise way. Moreover, they conjectured that $L(s, \Theta_r \times \Theta^{\vee}_{n-r'})$ could be identified as the Fourier coefficient of an Eisenstein series of parabolic type (r, r') on the n-fold cover $\overline{\mathrm{GL}}_{r+r'}$. We refer the reader to the work of Bump and Hoffstein [BH87] for some early evidence and the proof by Suzuki [Suz97] for the Bump–Hoffstein conjecture in the function field case. The result in [BBL03] is just the Bump–Hoffstein conjecture in the case $r = 1, r' = n-1$. It should be noted that the proof in [BBL03] is different from that given in Suzuki [Suz97].

- It is important for our purpose to remark also that Suzuki [Suz97, §7.6] actually showed (as a special case of his general results) that if one considers degree $n-1$ covers of GL_n and the Eisenstein series built from theta representation of the degree $n-1$ cover of $M = \mathrm{GL}_{n-1} \times \mathrm{GL}_1$; then its Fourier coefficients involve just the *reciprocal* of a single Hecke L-function. It thus brings up contrast when we compare this with the degree n cover of GL_n treated in [BBL03].

- For the odd n-fold Savin coverings of GL_r (see [Sav]) such that $r = kn$, Kaplan [Kap] considered the Levi subgroup $M = \mathrm{GL}_n \times \mathrm{GL}_n \times ... \times \mathrm{GL}_n$ (k-copies). In this case, every theta representation of $\overline{M}$ is distinguished and the formula for the local unramified Whittaker function of $I(s, \Theta(\chi))$ involves several Hecke L-functions, see [Kap, Theorem 43]. For general Brylinski–Deligne covers of GL_r, Y.-Q. Cai [Cai20] obtained a formula for unramified Whittaker functions of GL_r for certain representations induced theta representations, even without assuming the uniqueness condition. His work generalizes that in [Suz97, §7] and [Kap, §2.3]. The methods in [Kap] and [Cai20] are the same, by adopting a crystal graph (and also Gelfand-Tsetlin pattern) description of the Whittaker function for covering groups, as developed in [BBF11b, BBC+12, McN11]. In fact, the methods used in [Kap, Cai20] was already employed by Ginzburg [Gin18b] to compute the value of a general unramified Whittaker function without assuming the uniqueness property, in order to verify a certain conjecture on the non-generic unramified representation of a covering group.

As alluded to above, in this paper we study the occurrence of Hecke L-functions as the Fourier coefficients of Eisenstein series induced from theta representations for general covering groups. In particular, we show that the setup is quite general, as expected. More precisely, with the assumption that the inducing theta representation $\Theta(\chi)$ is distinguished, the ψ-Fourier coefficient $E_\psi(1, f_s, \Theta(\chi))$ of the Eisenstein series can be factorized into a product of local Whittaker functions for the induced representations. To this end, we show in §4 that the local unramified computation is completely reduced to some combinatorial problems arising from the local scattering matrix $[\tau(w, \chi, \gamma, \gamma')]_{\gamma,\gamma' \in \overline{T}/\overline{A}}$. The main result for this part includes Proposition 4.3 and Theorem 4.5.

Theorem 1.1 (Theorem 4.5). *Let $\mathcal{W}^G_{v,1_{\overline{T}}}$ be the ψ-Whittaker functional of $I^{\overline{G}_v}_{\overline{B}_v}(s \cdot \omega_P, i({}^{w_M}\chi_v))$. If $\tau(w_M, \chi_v, 1_{\overline{T}}, 1_{\overline{T}}) \neq 0$, then as the unramified local component of $E_\psi(1, f_s, \Theta(\chi))$, we have*

$$\mathcal{W}^G_{f^0_{s,v}}(1) = \frac{\mathcal{W}^G_{v,1_{\overline{T}}}(1)}{\tau(w_M, \chi_v, 1_{\overline{T}}, 1_{\overline{T}})}.$$

Here $\mathcal{W}^G_{v,1_{\overline{T}}}(1)$ can be written in terms of $\tau(w, \chi, \gamma, \gamma')$ as well, see Proposition 3.5. To compute $\mathcal{W}^G_{f^0_{s,v}}(1)$, the key of Suzuki's work [Suz97] mentioned above is his overcoming the combinatorial difficulties with $[\tau(w, \chi, \gamma, \gamma')]_{\gamma,\gamma' \in \overline{T}/\overline{A}}$ for Kazhdan–Patterson coverings of GL_r by implementing some direct and sophisticated analysis of the matrix. On the other hand, the novelties in [Gin18b, Kap, Cai20] rely on an efficient application of the formula of unramified Whittaker functions in terms of the crystal graph descriptions.

As examples of computing $\mathcal{W}^G_{f^0_{s,v}}(1)$, we will first consider in §5 a special family of "nice" coverings of GL_r (see Definition 5.1) and show that the reciprocal of a Hecke L-function appears as the Fourier coefficients of Eisenstein series induced from parabolic subgroup of type $(r-1,1)$. Thus, our result is a generalization for the case of Kazhdan–Patterson coverings treated by Suzuki [Suz97]. However, our unramified computation follows [BBL03]. The main result in §5 is the following.

THEOREM 1.2 (Theorem 5.11). *Let $(n, \overline{\mathbf{GL}}_r)$ be a "nice" cover over F. Let ψ be a nontrivial character of $\mathbb{A}/F$. Let $\Theta(\chi)$ be the global theta representation of $\overline{M}_{\mathbb{A}}$ associated with an exceptional character χ for $\overline{M}_{\mathbb{A}}$. Let $S \subset |F|$ be a finite set of places such that: (i) S contains the archimedean places and $|n|_v = 1$ for all $v \in |F| - S$; (ii) χ_v and ψ_v are both unramified outside S. Let $E(g, f_s, \Theta(\chi))$ be the Eisenstein series on $\overline{\mathrm{GL}}_{r,\mathbb{A}}$ associated with $I(s, \Theta(\chi))$. Assume $\mu_{2n} \subset F^\times$. Then*

$$E_\psi(1, f_s, \Theta(\chi)) = L^S\left((r-1)(s+1), \chi_{\alpha_{r-1}}\right)^{-1} \cdot \prod_{v\in S} \mathcal{W}^{\mathrm{GL}_r}_{f_{s,v}}(1),$$

where $L^S(s, \chi_{\alpha_{r-1}}) = \prod_{v\notin S} L(s, (\chi_{\alpha_{r-1}})_v)$ is the partial Hecke L-function attached to $\chi_{\alpha_{r-1}}$.

There are several features of our treatment compared to that in [BBL03, Suz97, Kap, Cai20], which we would like to highlight below.

- Our formulation is for a nice class (see Definition 5.1) of covering groups of GL_r, which is captured by combinatorial constraints. In fact, we describe in §2 the covering groups in the Brylinski–Deligne framework and apply results in [Gao17]. Kazhdan–Patterson covers form a special family in the Brylinski–Deligne category. We hope that the usage of the Brylinski–Deligne language adds some transparency to the class of groups we focus on in this paper. For instance, the reader could readily specialize to the Kazhdan–Patterson covering groups and compare our result with [BBL03, Theorem 3.2] and [Suz97, §7.6]. See Example 5.12.

- For Kazhdan–Patterson coverings, the consideration of degree $(n-1)$ cover of GL_n (instead of n-fold covers considered in [BBL03]) is actually the crucial starting point. This family fits into the set-up described in §3–§4. More precisely, for degree $(n-1)$-cover of the Levi subgroup $\overline{\mathrm{GL}_{n-1} \times \mathrm{GL}_1}$, the theta representation $\Theta(\chi)$ is always ψ-distinguished (see Proposition 5.7). Therefore, we could refrain from considering the $Z(\overline{\mathrm{GL}}_n)$-structure of the ψ-Whittaker functionals for the induced representations $I(s, \Theta(\chi))$ on $\overline{\mathrm{GL}}_n$. In this regard, we do not need to prove the uniqueness of (ψ, μ)-Whittaker functionals for $I(s, \Theta(\chi))$ as in [BL94]. See Remark 5.6.

- Theorem 1.2 is parallel to [Kap, Theorem 43] but does not follow from it. On the other hand, the main result in [Cai20, Theorem 8.1] does recover Theorem 1.2 here. However, as mentioned above, the strategy for our proof is different, as we adopt the approach from [BBL03]. We also give an interpretation of the result on the dual side in §5.5. In particular, the Fourier coefficient could be interpreted in terms of certain adjoint L-function and zeta functions.

In the last section §6 of the paper, we consider covers of Sp_{2r} and induction from the Siegel parabolic subgroup. Under a certain condition on the degree of the covering group, we show that the theta representation of the Siegel Levi subgroup is distinguished. Thus the results in §3–§4 can be applied. However, we only carry out the detailed computation for $r = 2$, and the main result is Theorem 6.5, which shows that the reciprocal of a Hecke L-function appears in this case. In general, a difference between the cases when r is even or odd is expected. We mention some of these expected subtleties at the end of the paper.

We believe that for general covering groups, the Fourier coefficients of Eisenstein series induced from distinguished theta representations involve just Hecke L-functions. However, it seems to be still mysterious regarding the pattern of the occurrence of such L-functions. Indeed, the works of [BBL03, Suz97] and some covers of Sp_4 considered in this paper already suggest that at the moment one does not seem to have a uniformly simple description of the (even conjectural) pattern. The rank, degree and type of the covering group and the parabolic subgroups involved all play sensitive role and obey certain resonant relations here. We hope that the resonant relation between these data and the L-functions that could appear as Fourier coefficients of Eisenstein series will be predicted from a unified solution of the combinatorial problem involved in the future.

Lastly, we remark that at various places where we make the assumption that $\mu_{2n} \subset F^\times$ to avoid technical complications in our computations, we will explicate such assumption. However, the results are expected to hold under the (minimal and necessary) assumption $\mu_n \subset F^\times$ as well.

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2 Central extensions and covering groups

In this section, let F be a number field with the ring of adeles $\mathbb{A}$. Denote by $|F|$ the set of all places of F. Let F_v be the local field of F for a place $v \in |F|$. For a non-archimedean place v, denote by $O_v \subset F_v$ the ring of integers of F_v and $\varpi_v \in O_v$ a fixed uniformizer. To introduce covering groups, we follow the framework of Brylinski–Deligne [BD01], which is also based on the earlier work of Moore, Steinberg and Mastumoto etc. We refer the reader to [GGW18] for a historical review. Meanwhile, we also follow some notations and recall some results from [Wei18], [GG18] and [Gao17].

2.1 $\mathbf{K}_2$-EXTENSIONS

Let $\mathbf{G}$ be a split connected linear algebraic group over F with maximal split torus $\mathbf{T}$. Let

$$\{X,\ \Delta,\ \Phi;\ Y,\ \Delta^\vee,\ \Phi^\vee\}$$

be the based root datum of $\mathbf{G}$. Here X (resp. Y) is the character lattice (resp. cocharacter lattice) for $(\mathbf{G}, \mathbf{T})$. Choose a set $\Delta \subseteq \Phi$ of simple roots from the set of roots Φ, and $\Delta^\vee$ the corresponding simple coroots from $\Phi^\vee$. Write $Y^{\rm sc} \subseteq Y$ for the sublattice generated by $\Phi^\vee$. Let $\mathbf{B} = \mathbf{TU}$ be the Borel subgroup associated with Δ. Denote by $\mathbf{U}^- \subset \mathbf{G}$ the unipotent subgroup opposite $\mathbf{U}$.

Fix a Chevalley system of pinnings for $(\mathbf{G}, \mathbf{T})$. That is, we fix a set of compatible isomorphisms

$$\{e_\alpha : \mathbf{G}_{\rm a} \to \mathbf{U}_\alpha\}_{\alpha\in\Phi},$$

where $\mathbf{U}_\alpha \subseteq \mathbf{G}$ is the root subgroup associated with α. In particular, for each $\alpha \in \Phi$, there is a unique homomorphism $\varphi_\alpha : \mathbf{SL}_2 \to \mathbf{G}$ which restricts to $e_{\pm\alpha}$ on the upper and lower triangular subgroup of unipotent matrices of $\mathbf{SL}_2$.

Denote by W the Weyl group of $(\mathbf{G}, \mathbf{T})$, which we identify with the Weyl group of the coroot system. In particular, W is generated by simple reflections $\{\mathbb{w}_\alpha : \alpha^\vee \in \Delta^\vee\}$ for $Y \otimes \mathbf{Q}$. Let $l : W \to \mathbf{N}$ be the length function. Let $\mathbb{w}_G$ be the longest element in W.

Consider the algebro-geometric $\mathbf{K}_2$-extension $\overline{\mathbf{G}}$ of $\mathbf{G}$, which is categorically equivalent to the pairs $\{(D, \eta)\}$ (see [GG18, §2.6]). Here

$$\eta : Y^{\rm sc} \to F^\times$$

is a homomorphism. On the other hand,

$$D : Y \times Y \to \mathbf{Z}$$

is a (not necessarily symmetric) bilinear form on Y such that

$$Q(y) := D(y, y)$$

is a Weyl-invariant integer-valued quadratic form on Y. We call D a bisector following [Wei14, §2.1]. Let B_Q be the Weyl-invariant bilinear form associated to Q by

$$B_Q(y_1, y_2) = Q(y_1 + y_2) - Q(y_1) - Q(y_2).$$

Clearly, $D(y_1, y_2) + D(y_2, y_1) = B_Q(y_1, y_2)$. Any $\overline{\mathbf{G}}$ is, up to isomorphism, incarnated by (i.e. categorically associated to) a pair (D, η) for a bisector D and η.

The couple (D, η) plays the following role for the structure of $\overline{\mathbf{G}}$.

- First, the group $\overline{\mathbf{G}}$ splits canonically over any unipotent subgroup of $\mathbf{G}$. For $\alpha \in \Phi$ and $a \in \mathbf{G}_a$, denote by $\overline{e}_\alpha(a) \in \overline{\mathbf{G}}$ the canonical lifting of $e_\alpha(a) \in \mathbf{G}$. For $\alpha \in \Phi$ and $a \in \mathbf{G}_m$, define

$$w_\alpha(a) := e_\alpha(a) \cdot e_{-\alpha}(-a^{-1}) \cdot e_\alpha(a) \text{ and } \overline{w}_\alpha(a) := \overline{e}_\alpha(a) \cdot \overline{e}_{-\alpha}(-a^{-1}) \cdot \overline{e}_\alpha(a). \tag{2}$$

This gives natural representatives $w_\alpha := w_\alpha(1)$ in $\mathbf{G}$, and also $\overline{w}_\alpha := \overline{w}_\alpha(1)$ in $\overline{\mathbf{G}}$, of the Weyl element $\mathbb{w}_\alpha \in W$. Moreover, for any $h_\alpha(a) := \alpha^\vee(a) \in \mathbf{T}$, there is a natural lifting

$$\overline{h}_\alpha(a) := \overline{w}_\alpha(a) \cdot \overline{w}_\alpha(-1) \in \overline{\mathbf{T}}, \tag{3}$$

which depends only on the pinnings and the canonical unipotent splitting.

- Second, there is a section $\mathbf{s}$ of $\overline{\mathbf{T}}$ over $\mathbf{T}$ such that

$$\mathbf{s}(y_1(a)) \cdot \mathbf{s}(y_2(b)) = \{a, b\}^{D(y_1, y_2)} \cdot \mathbf{s}(y_1(a) \cdot y_2(b)) \tag{4}$$

for any $a, b \in \mathbf{G}_m$. Moreover, for $\alpha \in \Delta$ and the natural lifting $\overline{h}_\alpha(a)$ of $h_\alpha(a)$ above, one has

$$\overline{h}_\alpha(a) = \{\eta(\alpha^\vee), a\} \cdot \mathbf{s}(h_\alpha(a)) \in \overline{\mathbf{T}}. \tag{5}$$

- Third, let $w_\alpha \in \mathbf{G}$ be the above natural representative of $\mathbb{w}_\alpha \in W$. For any $\overline{y(a)} \in \overline{\mathbf{T}}$ with $y \in Y$ and $a \in \mathbf{G}_m$, one has

$$w_\alpha \cdot \overline{y(a)} \cdot w_\alpha^{-1} = \overline{y(a)} \cdot \overline{h}_\alpha(a^{-\langle y, \alpha \rangle}), \tag{6}$$

where $\langle -, - \rangle$ is the canonical paring between Y and X.

For every $\mathbb{w} = \mathbb{w}_r ... \mathbb{w}_2 \mathbb{w}_1$ in a minimal expansion, we choose representative $w \in G$ and $\overline{w} \in \overline{G}$ by

$$w := w_r ... w_2 w_1 \in \overline{G} \text{ and } \overline{w} = \overline{w}_r \cdot ... \overline{w}_2 \overline{w}_1 \in \overline{G}.$$

Here w_i and $\overline{w}_i$ are defined in (2) above. The representatives w and $\overline{w}$ are independent of the minimal expansion of $\mathbb{w}$. We write w_G for the representative of the longest $\mathbb{w}_G \in W$.

We remark that if the derived group of $\mathbf{G}$ is simply-connected, then the isomorphism class of $\overline{\mathbf{G}}$ is determined by the Weyl-invariant quadratic form Q. In particular, for such $\mathbf{G}$, any extension $\overline{\mathbf{G}}$ is incarnated by $(D, \eta = \mathbf{1})$ for some bisector D, up to isomorphism. In this paper, we assume that the composition

$$\eta_n : Y^{sc} \to F^\times \twoheadrightarrow F^\times/(F^\times)^n \tag{7}$$

of η with the obvious quotient is trivial.

Let $n \geq 1$. We assume that F contains the full group of n-th roots of unity, denoted by μ_n. An n-fold cover of $\mathbf{G}$, in the sense of [Wei18, Definition 1.2], is just the pair $(n, \overline{\mathbf{G}})$. The above relations among generators of $\overline{\mathbf{G}}$ will eventually give rise to some relations for the topological coverings.

2.2 Dual group and L-group

For a cover $(n, \overline{\mathbf{G}})$ associated to (D, η), with Q and B_Q arising from D, we define

$$Y_{Q,n} := \{y \in Y : B_Q(y, y') \in n\mathbf{Z} \text{ for all } y' \in Y\} \subset Y. \tag{8}$$

For every $\alpha^\vee \in \Phi^\vee$, define

$$n_\alpha := \frac{n}{\gcd(n, Q(\alpha^\vee))}$$

and

$$\alpha_{Q,n}^\vee := n_\alpha \alpha^\vee, \quad \alpha_{Q,n} := n_\alpha^{-1} \alpha.$$

Let $Y_{Q,n}^{sc} \subset Y_{Q,n}$ be the sublattice generated by $\Phi_{Q,n}^\vee := \{\alpha_{Q,n}^\vee : \alpha^\vee \in \Phi^\vee\}$. Denote $X_{Q,n} := \mathrm{Hom}_{\mathbf{Z}}(Y_{Q,n}, \mathbf{Z})$ and $\Phi_{Q,n} = \{\alpha_{Q,n} : \alpha \in \Phi\}$. We also write

$$\Delta_{Q,n}^\vee := \{\alpha_{Q,n}^\vee : \alpha^\vee \in \Delta^\vee\} \text{ and } \Delta_{Q,n} := \{\alpha_{Q,n} : \alpha \in \Delta\}.$$

Then

$$\left(Y_{Q,n},\ \Phi_{Q,n}^\vee,\ \Delta_{Q,n}^\vee;\ X_{Q,n},\ \Phi_{Q,n}^\vee, \Delta_{Q,n}\right)$$

forms a root datum. It gives a unique (up to unique isomorphism) pinned reductive group $\overline{\mathbf{G}}^\vee$ over $\mathbf{Z}$, called the dual group of $(n, \overline{\mathbf{G}})$. In particular, $Y_{Q,n}$ is the character lattice for $\overline{G}^\vee$ and $\Delta_{Q,n}^\vee$ the set of simple roots. Let $\overline{G}^\vee := \overline{\mathbf{G}}^\vee(\mathbf{C})$ be the associated complex dual group.
In [Wei14, Wei18], Weissman constructed the global L-group for $(n, \overline{\mathbf{G}})$ which is an extension

$$\overline{G}^\vee \hookrightarrow {}^L\overline{G}_{\mathbb{A}} \twoheadrightarrow W_F,$$

where W_F is the Weil group of F. There is also the local L-group

$$\overline{G}^\vee \hookrightarrow {}^L\overline{G}_v \twoheadrightarrow W_{F_v},$$

which is compatible with the global L-group. Moreover, the construction of L-group is functorial, and in particular it behaves well with respect to the restriction of $\overline{\mathbf{G}}$ to parabolic subgroups. For details on the construction and some properties regarding the L-group, we refer the reader to [Wei14, Wei18, GG18].

2.3 Weyl orbits

Let

$$\rho := \frac{1}{2} \sum_{\alpha^\vee > 0} \alpha^\vee$$

be the half sum of all positive coroots of $\mathbf{G}$. Denote by $\mathbb{w}(y)$ the natural Weyl group action on Y and $Y \otimes \mathbf{Q}$ generated by the reflections $\mathbb{w}_\alpha$. We consider the twisted Weyl-action

$$\mathbb{w}[y] := \mathbb{w}(y - \rho) + \rho.$$

Clearly Y is stable under this twisted action. Throughout the paper, we denote

$$y_\rho := y - \rho \in Y \otimes \mathbf{Q}$$

for any $y \in Y$, and thus $\mathbbm{w}[y] - y = \mathbbm{w}(y_\rho) - y_\rho$. From now on, by Weyl orbits in Y or $Y \otimes \mathbf{Q}$ we always refer to the ones with respect to the action $\mathbbm{w}[y]$. Write $\mathcal{O}^F$ for the set of free W-orbits in Y.

Let

$$\wp : Y \to Y/Y_{Q,n} \text{ and } \wp^{sc} : Y \to Y/Y_{Q,n}^{sc}$$

be the natural quotient maps. We call $\mathcal{O}_y \in \mathcal{O}^F$ a $Y_{Q,n}$-free orbit if $|\mathcal{O}_y| = |\wp(\mathcal{O}_y)|$, that is, if $\mathcal{O}_y$ and $\wp(\mathcal{O}_y)$ have the same size. Similarly, we call $\mathcal{O}_y \in \mathcal{O}^F$ a $Y_{Q,n}^{sc}$-free orbit if $|\mathcal{O}_y| = |\wp^{sc}(\mathcal{O}_y)|$. Denote

$$\mathcal{O}^F_{Q,n} := \left\{ \mathcal{O}_y \in \mathcal{O}^F : \mathcal{O}_y \text{ is } Y_{Q,n}\text{-free} \right\}$$

and

$$\mathcal{O}^F_{Q,n,sc} := \left\{ \mathcal{O}_y \in \mathcal{O}^F : \mathcal{O}_y \text{ is } Y_{Q,n}^{sc}\text{-free} \right\}.$$

Clearly, the inclusions $\mathcal{O}^F_{Q,n} \subset \mathcal{O}^F_{Q,n,sc} \subset \mathcal{O}^F$ hold.

2.4 Topological coverings

Write $G_\mathbb{A}$ for $\mathbf{G}(\mathbb{A})$ and G_v for $\mathbf{G}(F_v)$ for any place $v \in |F|$. We also denote $\mathbf{G}(F)$ by G_F. Recall that we assume $\mu_n \subset F^\times$.

The $\mathbf{K}_2$-extension $\overline{\mathbf{G}}$ gives rise to an n-fold global topological central covering

$$\mu_n \hookrightarrow \overline{G}_\mathbb{A} \stackrel{\phi}{\twoheadrightarrow} G_\mathbb{A},$$

which splits over G_F, and the splitting can be chosen in a canonical way (see [BD01, §10.4]). In fact it arises from (and thus is compatible with) the local covering

$$\mu_n \stackrel{i_v}{\hookrightarrow} \overline{G}_v \stackrel{\phi_v}{\twoheadrightarrow} G_v. \tag{9}$$

More precisely, let

$$(-,-)_{n,v} : F_v \times F_v \to \mu_n$$

be the local n-th Hilbert symbol. Then the local extension $\overline{G}_v$ arises from the central extension

$$\mathbf{K}_2(F_v) \hookrightarrow \overline{\mathbf{G}}(F_v) \stackrel{\phi_v}{\twoheadrightarrow} \mathbf{G}(F_v)$$

by push-out via the natural map $\mathbf{K}_2(F_v) \to \mu_n$ given by $\{a,b\} \mapsto (a,b)_{n,v}$. The extension (9) is a central extension of locally compact topological groups (with μ_n a finite and discrete group). The maps i_v and ϕ_v are continuous. We have topological isomorphisms $\mu_n \simeq i_v(\mu_n)$ and $\overline{G}_v/i_v(\mu_n) \simeq G_v$. Now $\overline{G}_\mathbb{A}$ is obtained from "gluing" the $\overline{G}_v$ together. For more details, see [BD01, §10].

For a subset $H \subset G_v$, denote $\overline{H} := \phi_v^{-1}(H)$. The relations for $\overline{\mathbf{G}}$ described in §2.1 give rise to the corresponding relations for $\overline{G}_v$. For example, inherited from (4), the group law on the covering torus $\overline{T}_v$ is given by

$$\mathbf{s}(y_1(a)) \cdot \mathbf{s}(y_2(b)) = (a,b)_n^{D(y_1,y_2)} \cdot \mathbf{s}(y_1(a) \cdot y_2(b)), \tag{10}$$

where $y_i \in Y$ and $a, b \in F_v^\times$. The commutator $[\overline{t}_1, \overline{t}_2] := \overline{t}_1 \overline{t}_2 \overline{t}_1^{-1} \overline{t}_2^{-1}$ on $\overline{T}_v$, which descends to a map $[-,-] : T_v \times T_v \to \mu_n$, is thus given by

$$[y_1(a), y_2(b)] = (a,b)_n^{B_Q(y_1,y_2)}.$$

For any group H, let $Z(H)$ be its center. We note that by [Wei09, Proposition 4.1] the center $Z(\overline{T}_v)$ of the covering torus $\overline{T}_v$ is equal to $\phi_v^{-1}(\mathrm{Im}(i_{Q,n}))$ where

$$i_{Q,n} : Y_{Q,n} \otimes F_v^\times \to T_v$$

is the isogeny induced from the embedding $Y_{Q,n} \subset Y$.

2.5 Local unramified representations

Let $v \in |F|$ be a non-archimedean place such that $|n|_v = 1$. Let $K_v \subset G_v$ be the hyperspecial maximal compact subgroup generated by $\mathbf{T}(O_v)$ and $e_\alpha(O_v)$ for all root α. With our assumption that η_n as in (7) is trivial, the group $\overline{G}_v$ splits over K_v (see [GG18, Theorem 4.2]) and we fix such a splitting s_{K_v}; call G_v an unramified group in this case. If no confusion arises, we will omit s_{K_v} and write $K_v \subset \overline{G}_v$ instead.
A genuine representation (π, V_π) of the n-fold cover $\overline{G}_v$ (or $\overline{G}_\mathbb{A}$) is such that μ_n acts on V_π by a fixed embedding $\mu_n \subset \mathbf{C}^\times$. For an unramified group $\overline{G}_v$, the representation (π, V_π) called unramified if $V_\pi^{K_v} \neq 0$. By the Satake isomorphism, we know that $\dim V_\pi^{K_v} = 1$ if π is unramified.
Since $e_\alpha(O_v)$ is a pro-p group and $|n|_v = 1$, we see that $\overline{G}_v$ splits canonically and uniquely over the unipotent subgroup $e_\alpha(O_v)$ (and not only over $e_\alpha(F_v)$), which is then also given by $e_\alpha(x) \mapsto \overline{e}_\alpha(x)$. Hence,

$$s_{K_v}(e_\alpha(u)) = \overline{e}_\alpha(u)$$

for every $u \in O_v$. Therefore,

$$\overline{h}_\alpha(u) = s_{K_v}(h_\alpha(u)) \in s_{K_v}(K_v) \subset \overline{G}_v \tag{11}$$

for every $u \in O_v^\times$ by the definition of $\overline{h}_\alpha(u)$ in (3). For $\overline{\mathrm{GL}}_r$, properties of such a splitting s_{K_v} are discussed more extensively in [KP84, Tak16].

3 Theta representations

In this section, we first introduce the global theta representations for $\overline{G}_{\mathbb{A}}$. For a discussion on general analytic properties of Eisenstein series, we refer the reader to [MW95]. For theta representations, which are residues of the Eisenstein series at the farthest hyperplanes, Kazhdan and Patterson carried out a detailed analysis for $\overline{\mathrm{GL}}_{r,\mathbb{A}}$ in [KP84, Section II]. Though the covering groups we consider here are general, the formulation in [KP84, §II] for $\overline{\mathrm{GL}}_r$ applies and we follow closely the exposition on theta representations there. Following this, we will compute for unramified local data the Whittaker function evaluated at dominant torus element for local theta representations, which generalizes the work of [KP84, Pat87].

3.1 Theta representations

By [Wei16, Theorem 4.15], $T_F \cdot Z(\overline{T}_{\mathbb{A}})$ is a maximal abelian subgroup of $\overline{T}_{\mathbb{A}}$, where $Z(\overline{T}_{\mathbb{A}}) = \otimes_v Z(\overline{T}_v)$. Let

$$\chi = \otimes_v \chi_v : Z(\overline{T}_{\mathbb{A}}) \to \mathbf{C}^\times$$

be a genuine character which is trivial on $T_F \cap Z(\overline{T}_{\mathbb{A}})$. Therefore, we could view χ as a genuine character of $T_F \cdot Z(\overline{T}_{\mathbb{A}})$ which is trivial on T_F; that is, χ is a genuine automorphic character.
For any $\alpha \in \Phi$, the map $F_v^\times \to \overline{T}_v$ given by $a_v \mapsto \overline{h}_\alpha(a_v^{n_\alpha})$ is a homomorphism. Therefore, we have a linear character

$$\chi_\alpha : \mathbb{A}^\times \to \mathbf{C}^\times$$

given by

$$\chi_\alpha((a_v)_v) := \chi\big((\overline{h}_\alpha(a_v^{n_\alpha}))_v\big).$$

For $x \in F$, since the canonical lifting of $e_\alpha(x) \in G_F \subset G_v$ into $\overline{G}_v$ agrees with the canonical lifting of G_F into $G_{\mathbb{A}}$ (cf. [MW95, Appendix I]), and that $\overline{h}_\alpha(x)$ is defined in terms of the unipotent elements, we see that $(\overline{h}_\alpha(x))_v \in \overline{G}_{\mathbb{A}}$ is the lifting of $h_\alpha(x) \in G_F$. Therefore, χ_α is an automorphic character; that is, it is trivial on $F^\times$.

Definition 3.1 ([KP84, page 113]). For any subset $\Delta' \subset \Delta$, a genuine character χ is called Δ'-exceptional (resp. Δ'-anti-exceptional) if $\chi_\alpha = |\cdot|_{\mathbb{A}}$ (resp. $\chi_\alpha = |\cdot|_{\mathbb{A}}^{-1}$) for every $\alpha \in \Delta'$, where $|\cdot| : \mathbb{A}^\times \to \mathbf{C}^\times$ is the idele norm of $\mathbb{A}^\times$. In the case $\Delta' = \Delta$, it is simply called exceptional or anti-exceptional respectively.

Let $\mathbf{P} = \mathbf{MN}$ be the parabolic subgroup associated to Δ'. If χ is Δ'-exceptional, we may call χ an exceptional character for $\overline{M}_{\mathbb{A}}$.
For a general χ, consider the induced representation

$$i(\chi) = \mathrm{Ind}_{T_F Z(\overline{T}_{\mathbb{A}})}^{\overline{T}_{\mathbb{A}}} \chi$$

of $\overline{T}_{\mathbb{A}}$. For each χ_v, let χ'_v be an extension of χ_v to an maximal abelian subgroup $A_v \subset \overline{T}_v$, where for almost all finite v we have $A_v = \mathbf{T}(O_v)Z(\overline{T}_v)$. Let $i(\chi_v) = \mathrm{Ind}_{A_v}^{\overline{T}_v}\chi'_v$ be the irreducible induced representation. The isomorphism class of $i(\chi_v)$ is independent of the choice of A_v and the extension χ'_v. We have $i(\chi) = \otimes_v i(\chi_v)$.

Now we define the induced representation of $\overline{G}_{\mathbb{A}}$ as follows. Recall that $\mathbf{B} = \mathbf{TU}$ is the Borel subgroup of $\mathbf{G}$. Let δ_B be the modular character of $\overline{B}_{\mathbb{A}}$. Then we have the induced representation $I(i(\chi)) = \mathrm{Ind}_{\overline{B}_{\mathbb{A}}}^{\overline{G}_{\mathbb{A}}}\big(i(\chi)\otimes\mathbf{1}\big)$, where $\mathbf{1}$ denotes the trivial representation of $U_{\mathbb{A}}$. One has $I(i(\chi)) = \otimes_v I(i(\chi_v))$, where the local space $I(i(\chi_v))$ consists of smooth functions $f_v : \overline{G}_v \to i(\chi_v)$ satisfying

$$f(\overline{b}_v \cdot \overline{g}_v) = \delta_{\overline{B}_v}^{1/2}(\overline{b}_v)\cdot i(\chi_v)(\overline{b}_v) f(\overline{g}_v)$$

for all $\overline{b}_v \in \overline{B}_v$ and $\overline{g}_v \in \overline{G}_v$. Here $\delta_{\overline{B}_v}$ is the pull-back of the modular character δ_{B_v} on B_v via the quotient $\overline{B}_v \twoheadrightarrow B_v$. By induction in stages, we also identify $I(i(\chi))$ as the representation $I(\chi)$ induced from $\chi\otimes\mathbf{1}$ on $(T_F Z(\overline{T}_{\mathbb{A}}))\ltimes U_{\mathbb{A}}$. We may use $I(\chi)$ for $I(i(\chi))$ interchangeably.

For the general notions of admissible and automorphic representations of $\overline{G}_{\mathbb{A}}$, which are derived from proper modification as in the linear algebraic case, we refer the reader to [Wei18, §8]. In particular, $I(\chi)$ defined above is an admissible representation.

Let $K = \prod_v K_v \subset G_{\mathbb{A}}$ be a maximal compact subgroup, where K_v is the hyperspecial maximal compact subgroup of G_v in §2.5 for almost all finite $v \in |F|$. The representation space of $I(\chi)$ could be identified with the space of right K-finite functions (see [MW95, II.1] and also [KP84, page 108-109])

$$f : U_{\mathbb{A}}T_F\backslash\overline{G}_{\mathbb{A}} = (T_F\backslash\overline{T}_{\mathbb{A}})\cdot K \longrightarrow \mathbf{C} \tag{12}$$

such that for each $\overline{k}\in\overline{K}$, the function $\overline{t}\mapsto f(\overline{t}\cdot\overline{k})$ belongs to $i(\chi)$. With this identification, for $f\in I(\chi)$ we define the Eisenstein series on $\overline{G}_{\mathbb{A}}$ by

$$E(g,\chi,f) = \sum_{\gamma\in B_F\backslash G_F} f(\gamma\cdot g) \text{ for } g\in\overline{G}_{\mathbb{A}},$$

where G_F is identified as a subgroup of $\overline{G}_{\mathbb{A}}$ via the canonical splitting.

For any $\mathbb{w} = \mathbb{w}_{\alpha_1}...\mathbb{w}_{\alpha_l}\in W$ in a minimal decomposition, let $w = w_{\alpha_1}...w_{\alpha_l}\in G_F$ be the representative which could be viewed as in $\overline{G}_{\mathbb{A}}$. For almost all v, we have $w = \overline{w}$, where $\overline{w}\in\overline{G}_v$ is the element given by (2). The two representations ${}^{w}i(\chi)$ and $i({}^{w}\chi)$ are isomorphic albeit not canonically. Let

$$r_{w,\chi} = \otimes_v r_{w,\chi_v} : {}^{w}i(\chi)\to i({}^{w}\chi)$$

be an isomorphism where for almost all finite $v\in|F|$, the isomorphism $r_{w,\chi_v} : {}^{w}i(\chi_v)\to i({}^{w}\chi_v)$ is the canonical one given by $r_{w,\chi_v}(f_v)(\overline{t}) = f_v(w^{-1}\overline{t}w)$, see [GSS, §3.6]. We also denote by $r_{w,\chi} : I({}^{w}i(\chi))\to I(i({}^{w}\chi))$ the induced isomorphism.

Let $T_{w,\chi} = \bigotimes_v T_{w,\chi_v} : I(\chi) \to I({}^w\chi)$ be the intertwining operator given by

$$T_{w,\chi}(f)(g) = \int_{U_{\mathbb{A}}^w} r_{w,\chi}(f(w^{-1}ug))du,$$

where $U_{\mathbb{A}}^w = U_{\mathbb{A}} \cap wU_{\mathbb{A}}^- w^{-1}$. That is, $T_{w,\chi} = r_{w,\chi} \circ T(w, i(\chi))$, where $T(w, i(\chi)) : I(i(\chi)) \to I({}^w i(\chi))$ is the usual intertwining operator. Locally, the operator $T_{w,\chi_v} = r_{w,\chi_v} \circ T(w, i(\chi_v))$ is defined by analytic continuation of the integral

$$T_{w,\chi_v}(f_v)(g_v) = \int_{U_v^w} r_{w,\chi_v}(f_v(w^{-1}u_v g_v))du_v, \tag{13}$$

where $U_v^w = U_v \cap wU_v^- w^{-1}$.

For $\mathbb{w} \in W$, denote $\Phi_{\mathbb{w}} := \{\alpha \in \Phi : \alpha > 0 \text{ and } \mathbb{w}(\alpha) < 0\}$. For a non-archimedean v such that $|n|_v = 1$ and χ_v is an unramified character for an unramified group $\overline{G}_v$, denote the Gindikin-Karpelevich coefficient (see [Cas80, McN16, Gao18a]) by

$$c_{\mathsf{gk}}(w,\chi) := \prod_{\alpha \in \Phi_{\mathbb{w}}} c_{\mathsf{gk}}(w_\alpha, \chi_v), \text{ where } c_{\mathsf{gk}}(w_\alpha, \chi_v) = \frac{1 - q^{-1}\chi_v(\overline{h}_\alpha(\varpi_v^{n_\alpha}))}{1 - \chi_v(\overline{h}_\alpha(\varpi_v^{n_\alpha})}. \tag{14}$$

The intertwining operator $T_{w,\chi} : I(\chi_v) \to I({}^w\chi_v)$ gives

$$T_{w,\chi_v}(f_0) = c_{\mathsf{gk}}(w, \chi_v) \cdot f_0',$$

where $f_0 \in I(\chi_v)$ and $f_0' \in I({}^w\chi_v)$ are the normalized unramified vectors.

The set of genuine automorphic characters χ on $Z(\overline{T}_{\mathbb{A}})$ affords an analytic structure. The Eisenstein series $E(g,\chi,f)$ can be meromorphically continued as an operator on χ, and satisfies the following functional equation $E(g,\chi,f) = E(g, {}^w\chi, T(w,\chi)(f))$ for $\mathbb{w} \in W$. Moreover, the Eisenstein series have their "greatest" singularity for exceptional characters. Let χ be an exceptional character, define

$$\theta(g,f,\chi) = \lim_{\chi' \to \chi} \prod_{\substack{\alpha \in \Phi \\ \alpha > 0}} \frac{L(|\cdot|_{\mathbb{A}} \cdot \chi'_\alpha) \cdot \varepsilon(0, \chi'_\alpha)}{L(\chi'_\alpha)} \cdot E(g,\chi,f),$$

where $L(\chi')$ (resp. $\varepsilon(0,\chi')$) is the Hecke L-function (resp. ε-factor) associated with a Hecke character χ'. Let $\Theta(\overline{G}_{\mathbb{A}}, \chi)$ be the automorphic representation generated by $f \mapsto \theta(g,f,\chi)$. One has

$$\Theta(\overline{G}_{\mathbb{A}}, \chi) = \otimes_v \Theta(\overline{G}_v, \chi_v),$$

where the local representation $\Theta(\overline{G}_v, \chi_v)$ is realized as the unique Langlands quotient of $I(\chi_v)$, which is also the image of the local intertwining intertwining operator $T(w_G, \chi_v) : I(\chi_v) \to I({}^{w_G}\chi_v)$.

3.2 Whittaker models

Fix a nontrivial character $\psi : \mathbb{A}/F \to \mathbf{C}^\times$. By abuse of notation, denote by ψ the character on $U_\mathbb{A}$ such that its restriction to every $U_{\alpha,\mathbb{A}}$ for $\alpha \in \Delta$ is given by $\psi \circ e_\alpha^{-1}$. For an irreducible genuine automorphic representation (Π, V_Π) of $\overline{G}_\mathbb{A}$, consider the global ψ-Whittaker functional on V_Π by

$$\lambda_\mathbb{A}(f) = \int_{U_F\backslash U_\mathbb{A}} f(w_G^{-1}u)\psi(u)^{-1}du, \tag{15}$$

where $f \in V_\Pi$ is any function in the space.
On the other hand, for $v \in |F|$ and an irreducible genuine representation (π, V_π) of $\overline{G}_v$, denote by $\mathrm{Wh}_{\psi_v}(\pi)$ the space of continuous ψ_v-Whittaker functionals of V_π, i.e. the set of all continuous functionals

$$\lambda_v : V_\pi \to \mathbf{C}$$

such that $\lambda_v(\pi(u)f) = \psi_v(u)\cdot f$ for all $f \in V_\pi$ and $u \in U_v$. A remark is necessary on the topology on V_π and thus the continuity of such λ_v, see [Shal74, §3]. If v is non-archimedean, then V_π is endowed with the trivial locally convex topology for which every semi-norm is continuous. If v is an archimedean place, then by a genuine representation (π, V_π) we mean a genuine $(\mathfrak{g}_\mathbf{C}, \overline{K}_v)$-module with commuting action by the complexified Lie algebra $\mathfrak{g}_\mathbf{C}$ of G_v and a maximal compact subgroup $\overline{K}_v \subset \overline{G}_v$ (as the preimage of a maximal compact subgroup K_v of G_v), i.e.,

$$(\mathrm{Ad}(\overline{k})X)(v) = (\overline{k}Xk^{-1})(v)$$

for every $X \in \mathfrak{g}_\mathbf{C}, \overline{k} \in \overline{K}_v$ and $v \in V_\pi$. For archimedean v, we assume V_π is pre-unitary with respect to a norm $\|\cdot\|$, which then gives rise to a family of semi-norms $\{\|v\|_X = \|X(v)\| : X \in \mathfrak{g}_\mathbf{C}\}$; this gives a weak topology on V_π. Thus, we have specified the topology on V_π and also the expected continuity of λ_v for every $v \in |F|$.
If π is an unramified representation of $\overline{G}_v$ with a chosen unramified vector f_0, then one has the unramified Whittaker function

$$\mathcal{W}_{\lambda_v}(g) = \lambda_v(\pi(g)f_0), g \in \overline{G}_v \tag{16}$$

associated to λ_v.

Definition 3.2. A theta representation $\Theta(\overline{G}_\mathbb{A}, \chi)$ is called ψ-distinguished if the following two conditions hold:

- $\lambda_\mathbb{A}$ on $\Theta(\overline{G}_\mathbb{A}, \chi)$ is nonzero, and
- $\dim \mathrm{Wh}_{\psi_v}(\Theta(\overline{G}_v, \chi_v)) = 1$ for every $v \in |F|$.

As in the linear algebraic case, we have

Proposition 3.3. *Let $\Theta(\overline{G}_{\mathbb{A}}, \chi)$ be a ψ-distinguished theta representation. Then $\lambda_{\mathbb{A}}(\theta) = \prod_v \lambda_v(\theta_v)$ for $\theta = \otimes_v \theta_v \in \otimes_v \Theta(\overline{G}_v, \chi_v)$, where for almost all v, $\lambda_v \in Wh_{\psi_v}(\Theta(\overline{G}_v, \chi_v))$ is the unique normalized Whittaker functional such that $\lambda_v(\theta_v^0) = 1$ for the unramified vector $\theta_v^0 \in \Theta(\overline{G}_v, \chi_v)$.*

Remark 3.4. For a fixed ψ, an irreducible automorphic representation (Π, V_Π) of $\overline{G}_{\mathbb{A}}$ is called globally ψ-generic if $\lambda_{\mathbb{A}}(f) \neq 0$ for some $f \in V_\Pi$. Moreover, it is called locally ψ-generic if $\dim \mathrm{Wh}_{\psi_v}(\Theta(\overline{G}_v, \chi_v)) > 0$ for every $v \in |F|$. If $\overline{G}_{\mathbb{A}} = G_{\mathbb{A}}$, i.e., $n = 1$, then it is expected that global and local ψ-genericity are equivalent (see [Sha11]). For covering groups, Gelbart and Soudry [GS87] showed that locally ψ-generic (but not globally ψ-generic) cuspidal genuine representation of the double cover $\overline{\mathrm{SL}}_{2,\mathbb{A}}$ exists. However, it is expected that for theta representations of covering groups, local and global ψ-genericity agree. In particular, it should be sufficient to assume the second condition in Definition 3.2.

3.3 Unramified Whittaker function

We assume in this subsection that v is a non-archimedean place such that $\overline{G}_v$ is an unramified group. We first summarize some results from [Gao17] regarding values of Whittaker functions for principal series and local theta representations. For simplicity of notation, we omit the subscript v and write $F, \overline{T}, \chi, \psi$ for $F_v, \overline{T}_v, \chi_v, \psi_v$ etc. Thus, we assume that χ is unramified and ψ has conductor O.

Recall that χ is a genuine unramified character of $Z(\overline{T})$ and $\overline{A} \subset \overline{T}$ a maximal abelian subgroup. By abuse of notation, denote by χ an extension to $\overline{A}$. Let $\mathsf{Ftn}(i(\chi))$ be the vector space of functions $\mathbf{c}$ on $\overline{T}$ satisfying

$$\mathbf{c}(\overline{t} \cdot \overline{z}) = \mathbf{c}(\overline{t}) \cdot \chi(\overline{z}), \quad \overline{t} \in \overline{T} \text{ and } \overline{z} \in \overline{A}.$$

The support of $\mathbf{c} \in \mathsf{Ftn}(i(\chi))$ is a disjoint union of cosets in $\overline{T}/\overline{A}$.

Let $\{\gamma_i\} \subset \overline{T}$ be a chosen set of representatives of $\overline{T}/\overline{A}$, and consider $\mathbf{c}_{\gamma_i} \in \mathsf{Ftn}(i(\chi))$ which has support $\gamma_i \cdot \overline{A}$ and $\mathbf{c}_{\gamma_i}(\gamma_i) = 1$. It gives rise to a linear functional $l_{\gamma_i} \in i(\chi)^\vee$ such that $l_{\gamma_i}(\phi_{\gamma_j}) = \delta_{ij}$, where $\phi_{\gamma_j} \in i(\chi)$ is the unique element such that $\mathrm{supp}(\phi_{\gamma_j}) = \overline{A} \cdot \gamma_j^{-1}$ and $\phi_{\gamma_j}(\gamma_j^{-1}) = 1$. Then there is a natural isomorphism of vector spaces $\mathsf{Ftn}(i(\chi)) \simeq i(\chi)^\vee$ given by

$$\mathbf{c} \mapsto l_{\mathbf{c}} := \sum_{\gamma_i \in \overline{T}/\overline{A}} \mathbf{c}(\gamma_i) \cdot l_{\gamma_i}.$$

It can be checked easily that this isomorphism does not depend on the choice of representatives for $\overline{T}/\overline{A}$.

Furthermore, there is an isomorphism between $i(\chi)^\vee$ and the space $\mathrm{Wh}_\psi(I(\chi))$ of ψ-Whittaker functionals on $I(\chi)$ given by

$$l \mapsto \lambda_l$$

with

$$\lambda_l : I(\chi) \to \mathbf{C}, \quad f \mapsto l\left(\int_U f(\overline{w}_G^{-1} u)\psi(u)^{-1} du\right),$$

where $f \in I(\chi)$ is now viewed as an $i(\chi)$-valued function on $\overline{G}$. For any $\mathbf{c} \in \mathsf{Ftn}(i(\chi))$, write $\lambda_{\mathbf{c}} \in \mathrm{Wh}_\psi(I(\chi))$ for the ψ-Whittaker functional of $I(\chi)$ associated to $l_{\mathbf{c}}$. Therefore, $\mathbf{c} \mapsto \lambda_{\mathbf{c}}$ gives an isomorphism between $\mathsf{Ftn}(i(\chi))$ and $\mathrm{Wh}_\psi(I(\chi))$. For any $\gamma \in \overline{T}$, we will write

$$\lambda_\gamma := \lambda_{\mathbf{c}_\gamma}.$$

To avoid confusion, we may write λ^χ instead of λ for any $\lambda \in \mathrm{Wh}_\psi(I(\chi))$ to emphasize the underlying representation $I(\chi)$ involved.
The operator $T_{w,\chi} : I(\chi) \to I({}^w\chi)$ induces a homomorphism of vector spaces

$$T^*_{w,\chi} : \mathrm{Wh}_\psi(I({}^w\chi)) \to \mathrm{Wh}_\psi(I(\chi))$$

given by

$$\left\langle \lambda_{\mathbf{c}}^{{}^w\chi}, - \right\rangle \mapsto \left\langle \lambda_{\mathbf{c}}^{{}^w\chi}, T_{w,\chi}(-) \right\rangle$$

for any $\mathbf{c} \in \mathsf{Ftn}(i({}^w\chi))$. Let $\left\{\lambda_\gamma^{{}^w\chi}\right\}_{\gamma \in \overline{T}/\overline{A}}$ be a basis for $\mathrm{Wh}_\psi(I({}^w\chi))$, and $\left\{\lambda_{\gamma'}^{\chi}\right\}_{\gamma \in \overline{T}/\overline{A}}$ a basis for $\mathrm{Wh}_\psi(I(\chi))$. The map $T^*_{w,\chi}$ is then determined by the square matrix $[\tau(w,\chi,\gamma,\gamma')]_{\gamma,\gamma' \in \overline{T}/\overline{A}}$ such that

$$T^*_{w,\chi}(\lambda_\gamma^{{}^w\chi}) = \sum_{\gamma' \in \overline{T}/\overline{A}} \tau(w,\chi,\gamma,\gamma') \cdot \lambda_{\gamma'}^{\chi}.$$

We call the matrix $[\tau(w,\chi,\gamma,\gamma')]$ a scattering matrix (see [GSS, §3.6]). It satisfies some immediate properties:

- For $\mathbb{w} \in W$ and $\overline{z}, \overline{z}' \in \overline{A}$, the identity

$$\tau(w,\chi,\gamma \cdot \overline{z}, \gamma' \cdot \overline{z}') = ({}^w\chi)^{-1}(\overline{z}) \cdot \tau(w,\chi,\gamma,\gamma') \cdot \chi(\overline{z}') \tag{17}$$

  holds.

- For $\mathbb{w}_1, \mathbb{w}_2 \in W$ such that $l(\mathbb{w}_2\mathbb{w}_1) = l(\mathbb{w}_2) + l(\mathbb{w}_1)$, one has

$$\tau(w_2w_1,\chi,\gamma,\gamma') = \sum_{\gamma'' \in \overline{T}/\overline{A}} \tau(w_2, {}^{w_1}\chi, \gamma, \gamma'') \cdot \tau(w_1, \chi, \gamma'', \gamma'), \tag{18}$$

  which is referred to as the cocycle relation.

In view of the cocycle relation in (18), the understanding of $\tau(\chi, w, \gamma, \gamma')$ in principle is reduced to the case where $w = w_\alpha$ for some $\alpha \in \Delta$. For this purpose, we proceed to introduce first the Gauss sum.

Let du be the self-dual Haar measure of F such that $du(O) = 1$; thus, $du(O^\times) = 1 - q^{-1}$. The Gauss sum is defined by

$$G_\psi(a,b) = \int_{O^\times} (u,\varpi)_n^a \cdot \psi(\varpi^b u) du \text{ for } a, b \in \mathbf{Z}.$$

In particular, we are interest in

$$\mathbf{g}_\psi(k) := G_\psi(k, -1),$$

where $k \in \mathbf{Z}$ is any integer. We write henceforth

$$\xi := (-1, \varpi)_n \in \mathbf{C}^\times.$$

It is known that

$$\mathbf{g}_\psi(k) = \begin{cases} \xi^k \cdot \overline{\mathbf{g}_\psi(-k)} & \text{for any } k \in \mathbf{Z}, \\ -q^{-1} & \text{if } n|k, \\ \mathbf{g}_\psi(k) & \text{with } |\mathbf{g}_\psi(k)| = q^{-1/2} \text{ if } n \nmid k. \end{cases} \tag{19}$$

Here $\overline{z}$ denotes the complex conjugation of a complex number z. For any $y \in Y$, we write

$$\mathbf{s}_y := \mathbf{s}(y(\varpi)) \in \overline{T}.$$

If $\mu_{2n} \subset F^\times$ and thus $\xi = 1$, then the map $\mathbf{s} : Y \to \overline{T}$ defined above is a homomorphism by (10).

Suppose that $\gamma = \mathbf{s}_{y_1}$ and $\gamma' = \mathbf{s}_y$. Then it is shown in [KP84, McN16] (with some refinement from [Gao17]) that $\tau(\chi, w_\alpha, \gamma, \gamma')$ is determined as follows:

- We can write $\tau(\chi, w_\alpha, \gamma, \gamma') = \tau^1(\chi, w_\alpha, \gamma, \gamma') + \tau^2(w_\alpha, \chi, \gamma, \gamma')$ such that
$$\tau^i(w_\alpha, \chi, \gamma \cdot \overline{z}, \gamma' \cdot \overline{z}') = ({}^{w_\alpha}\chi)^{-1}(\overline{z}) \cdot \tau^i(w_\alpha, \chi, \gamma, \gamma') \cdot \chi(\overline{z}') \text{ for } \overline{z}, \overline{z}' \in \overline{A}.$$
- One has $\tau^1(w_\alpha, \chi, \gamma, \gamma') = 0$ unless $y_1 \equiv y \mod Y_{Q,n}$. Moreover, $\tau^2(\mathbb{w}_\alpha, \chi, \gamma, \gamma') = 0$ unless $y_1 \equiv \mathbb{w}_\alpha[y] \mod Y_{Q,n}$.
- If $y_1 = y$, then
$$\tau^1(w_\alpha, \chi, \gamma, \gamma') = (1 - q^{-1}) \frac{\chi(\overline{h}_\alpha(\varpi^{n_\alpha}))^{k_{y,\alpha}}}{1 - \chi(\overline{h}_\alpha(\varpi^{n_\alpha}))}, \text{ where } k_{y,\alpha} = \left\lceil \frac{\langle y, \alpha \rangle}{n_\alpha} \right\rceil.$$
If $y_1 = \mathbb{w}_\alpha[y]$, then
$$\tau^2(w_\alpha, \chi, \gamma, \gamma') = \xi^{\langle y_\rho, \alpha \rangle \cdot D(y, \alpha^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle\, Q(\alpha^\vee)).$$

The above is a generalization for the $\overline{\mathrm{GL}}_r$ case considered in [KP84].

Let $f^0 \in I(\chi)$ be the normalized unramified vector such that $f^0(1) \in i(\chi)$ is the unramified vector taking value 1 at $1_{\overline{T}}$. For any $\lambda \in \mathrm{Wh}_\psi(I(\chi))$, consider the associated unramified Whittaker function $\mathcal{W}_\lambda$ in (16). We also denote

$$\mathcal{W}_{\mathbf{c}} := \mathcal{W}_{\lambda_{\mathbf{c}}} \text{ and } \mathcal{W}_\gamma := \mathcal{W}_{\lambda_\gamma},$$

for any $\mathbf{c} \in \mathsf{Ftn}(i(\chi))$ and $\gamma \in \overline{T}$.

An element $\overline{t} \in \overline{T}$ is called dominant if $\overline{t} \cdot (U \cap K) \cdot \overline{t}^{-1} \subset K$.

PROPOSITION 3.5 ([Pat87, CO13]). *Let $I(\chi)$ be an unramified principal series of $\overline{G}$ and $\gamma \in \overline{T}$. Let $\mathcal{W}_\gamma$ be the Whittaker function associated to f^0. Then, $\mathcal{W}_\gamma(\overline{t}) = 0$ unless $\overline{t} \in \overline{T}$ is dominant. Moreover, for dominant $\overline{t}$, one has*

$$\mathcal{W}_\gamma(\overline{t}) = \delta_B^{1/2}(\overline{t}) \cdot \sum_{\mathrm{w} \in W} c_{\mathsf{gk}}(w_G w^{-1}, \chi) \cdot \tau(w, {}^{w^{-1}}\chi, \gamma, w_G \cdot \overline{t} \cdot w_G^{-1}),$$

where δ_B is the modular character of B.

Proof. For Kazhdan–Patterson covers of GL_r, the formula is given in [Pat87, CO13]. For general Brylinski–Deligne covers of GL_r, the details of the proof are given in [Gao18b, Proposition 3.3 and 3.4]. However, as noted there, the argument actually applies to covers of general reductive group, since the ingredients used in the proof are [McN16, Lemma 6.1, Theorem 8.1] and [Gao17, Corollary 3.5], which all hold for Brylinski–Deligne coverings of general reductive groups. □

Note that the above Proposition holds for any χ (not necessarily exceptional). Now if χ is exceptional, then every $\lambda_{\mathbf{c}} \in \mathrm{Wh}_\psi(I(\chi))$ that factors through the map

$$T_{w_G,\chi} : I(\chi) \to I({}^{w_G}\chi),$$

gives a ψ-Whittaker functional of $\Theta(\overline{G}, \chi)$. Moreover, it is shown in [Gao17] that we always have the bounds

$$\left|\wp(\mathcal{O}^{\digamma}_{Q,n})\right| \le \dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \chi)) \le \left|\wp(\mathcal{O}^{\digamma}_{Q,n,sc})\right|. \tag{20}$$

The proof of the first inequality in (20) is constructive. More precisely, it is shown in [Gao17] that any orbit $\mathcal{O}_y \in \mathcal{O}^{\digamma}_{Q,n}$ gives rise to an element $\mathbf{c}_{\mathcal{O}_y} \in \mathsf{Ftn}(i(\chi))$, whose associated nontrivial ψ-Whittaker functional $\lambda_{\mathcal{O}_y}$ of $I(\chi)$ factors through $T_{w_G,\chi}$. That is, $\lambda_{\mathcal{O}_y}$ can be viewed as a ψ-Whittaker functional on $\Theta(\overline{G}, \chi)$. In this case, let $\mathcal{W}_{\mathcal{O}_y} := \mathcal{W}_{\mathbf{c}_{\mathcal{O}_y}}$ be the Whittaker function associated to the unramified vector $\theta^0 := T_{w_G,\chi}(f^0)$ in $\Theta(\overline{G}, \chi)$.
As investigated in [Gao17], the case when the two bounds in (20) do not agree is quite subtle. However, we will consider in §5 and §6 of this paper only the case where $\mathcal{O}^{\digamma}_{Q,n} = \mathcal{O}^{\digamma}_{Q,n,sc}$. Therefore, we only recall below the construction of $\mathbf{c}_{\mathcal{O}_y} \in \mathsf{Ftn}(i(\chi))$ and some results for $\mathcal{W}_{\mathcal{O}_y}$ when $\mathcal{O}_y \in \mathcal{O}^{\digamma}_{Q,n}$, which are essentially already proved in [Gao18b].
The element $\mathbf{c}_{\mathcal{O}_y}$ for $\mathcal{O}_y \in \mathcal{O}^{\digamma}_{Q,n}$ is given in several steps as follows.

- For any $x \in \mathbf{R}$, let $\lceil x \rceil$ be the minimal integer such that $\lceil x \rceil \ge x$. For any $y \in Y$ and $\alpha^\vee \in \Delta^\vee$, write

$$\mathbf{t}(\mathrm{w}_\alpha, y) := \xi^{\langle y_\rho, \alpha\rangle \cdot D(y, \alpha^\vee)} \cdot q^{k_{y,\alpha}-1} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha\rangle\, Q(\alpha^\vee))^{-1},$$

  where

$$k_{y,\alpha} = \left\lceil \frac{\langle y, \alpha\rangle}{n_\alpha} \right\rceil.$$

For $\mathbbm{w} = \mathbbm{w}_k...\mathbbm{w}_2\mathbbm{w}_1 \in W$ in a minimum decomposition, define

$$\mathbf{t}(\mathbbm{w}, y) := \prod_{i=1}^{k} \mathbf{t}(\mathbbm{w}_i, \mathbbm{w}_{i-1}...\mathbbm{w}_1[y]),$$

which is well-defined and independent of the minimum expansion of $\mathbbm{w}$ (see [Gao17, Proposition 3.10]).

- Assign $\mathbf{c}_{\mathcal{O}_y}(\mathbf{s}_y) = 1$. For any $\mathbbm{w} \in W$, define

$$\mathbf{c}_{\mathcal{O}_y}(\mathbf{s}_{\mathbbm{w}[y]}) := \mathbf{t}(\mathbbm{w}, y) \cdot \mathbf{c}_{\mathcal{O}_y}(\mathbf{s}_y) = \mathbf{t}(\mathbbm{w}, y).$$

If $l(\mathbbm{w}_\alpha \mathbbm{w}) = 1 + l(\mathbbm{w})$ for some $\alpha \in \Delta$, then one sees immediately that

$$\mathbf{c}_{\mathcal{O}_y}(\mathbf{s}_{\mathbbm{w}_\alpha \mathbbm{w}[y]}) = \mathbf{t}(\mathbbm{w}_\alpha, \mathbbm{w}[y]) \cdot \mathbf{c}_{\mathcal{O}_y}(\mathbf{s}_{\mathbbm{w}[y]}).$$

- Extend $\mathbf{c}_{\mathcal{O}_y}$ to a function on $\overline{T}$ by

$$\mathbf{c}_{\mathcal{O}_y}(\mathbf{s}_{\mathbbm{w}[y]} \cdot \overline{z}) = \mathbf{c}_{\mathcal{O}_y}(\mathbf{s}_{\mathbbm{w}[y]}) \cdot \chi_v(\overline{z}), \ \overline{z} \in \overline{A}.$$

and

$$\mathbf{c}_{\mathcal{O}_y}(\overline{t}) = 0 \text{ if } \overline{t} \notin \bigcup_{\mathbbm{w} \in W} \mathbf{s}_{\mathbbm{w}[y]} \cdot \overline{A}. \tag{21}$$

Then $\mathbf{c}_{\mathcal{O}_y}$ is a well-defined element in $\mathsf{Ftn}(i(\chi))$. For the values of the Whittaker function $\mathcal{W}_{\mathcal{O}_y}$, we have the following.

Proposition 3.6. *Let $\mathcal{O}_y \in \mathcal{O}^F_{Q,n}$ and the Whittaker function $\mathcal{W}_{\mathcal{O}_y}$ as above. If $\overline{t} \in \overline{T}$ is not dominant, then $\mathcal{W}_{\mathcal{O}_y}(\overline{t}) = 0$. For dominant $\overline{t} \in \overline{T}$, one has*

$$\mathcal{W}_{\mathcal{O}_y}(\overline{t}) = c_{\mathsf{gk}}(w_G, \chi) \cdot \delta_B^{1/2}(\overline{t}) \cdot \mathbf{c}_{\mathcal{O}_y}(w_G \cdot \overline{t} \cdot w_G^{-1}).$$

In particular, for dominant $\mathbf{s}_z$ with $z \in Y$, if $\mu_{2n} \subset F^\times$, then

$$\mathcal{W}_{\mathcal{O}_y}(\mathbf{s}_z) = c_{\mathsf{gk}}(w_G, \chi) \cdot \delta_B^{1/2}(\mathbf{s}_z) \cdot \mathbf{c}_{\mathcal{O}_y}\big(\mathbf{s}_{\mathbbm{w}_G(z)}\big).$$

Proof. Again, for covers of the general linear groups, a detailed proof for the first equality is given in [Gao18b, Proposition 3.4]. However, the same as mentioned in the proof for Proposition 3.5, the argument in [Gao18b] actually applies to Brylinski–Deligne covers of general reductive group.
For the second equality, it follows from (6) and our assumption $\mu_{2n} \subset F^\times$ that $w \cdot \mathbf{s}_z \cdot w^{-1} = \mathbf{s}_{\mathbbm{w}(z)}$ for any $\mathbbm{w} \in W$; see [Gao18b, Lemma 2.1] for details. This completes the proof. □

4 Eisenstein series induced from theta representations

In this section, we consider a theta representation of the Levi subgroup of a maximal parabolic subgroup and Eisenstein series induced from it. If the theta representation is ψ-distinguished, then the Fourier coefficient of the Eisenstein series has a global factorization and thus the computation is completely reduced to the local case. For unramified places, we will show that the local computation can be reduced to some quantities involving the scattering matrix $[\tau(w, \chi, \gamma, \gamma')]$.

4.1 Maximal parabolic subgroup

We continue to use $(X, \Phi, \Delta; Y, \Phi^\vee, \Delta^\vee)$ to denote the root datum of $\mathbf{G}$. Consider a simple root $\beta \in \Delta$. Let $\mathbf{P} = \mathbf{MN}$ be the maximal parabolic subgroup of $\mathbf{G}$ associated with $\Delta \backslash \{\beta\}$. We write

$$(X, \Phi_M, \Delta_M;\ Y, \Phi_M^\vee, \Delta_M^\vee)$$

for the root datum of $\mathbf{M}$. Since $\mathbf{T} \subset \mathbf{M}$, the character and cocharacter lattices X and Y respectively are unchanged. However, we have $\Delta_M = \Delta \backslash \{\beta\}$ and $\Delta_M^\vee = \Delta^\vee \backslash \{\beta^\vee\}$. Denote by Y_M^{sc} the coroot lattice of $\mathbf{M}$, which is then the sublattice of Y^{sc} spanned by $\Delta_M^\vee$. Let $\mathbf{B}_M = \mathbf{TU}_M$ be the Borel subgroup of $\mathbf{M}$ corresponding to Δ_M.

Denote by $W_M \subset W$ the Weyl group of $(\mathbf{M}, \mathbf{T})$, where we reserve W for the Weyl group of $\mathbf{G}$. In general, to avoid confusion, we will use subscript to differentiate some structural data associated to $\mathbf{M}$ and $\mathbf{G}$. For example, $\mathbb{w}_G$ (resp. $\mathbb{w}_M$) denotes the longest element in W (resp. W_M).

Let $\mathbf{G}$ be a $\mathbf{K}_2$-extension associated to (D, η). Then by restriction, we obtain $\overline{\mathbf{P}} = \overline{\mathbf{M}}\mathbf{N}$. Thus, $\overline{\mathbf{M}}$ is associated to the pair $(D, \eta|_{Y_M^{sc}})$, where the quadratic form $Q(x) = D(x, x)$ carries only the W_M-invariance by applying the "forgetful" functor from the W-invariance. From $\overline{G}_{\mathbb{A}}$, we obtain by restriction the covering groups $\overline{P}_{\mathbb{A}}, \overline{M}_{\mathbb{A}}$ and their local analogues $\overline{P}_v$ and $\overline{M}_v$, which also arise from $\overline{\mathbf{P}}$ and $\overline{\mathbf{M}}$.

4.2 Fourier coefficients of Eisenstein series

Let $2\rho_P$ be the sum of positive roots in $\mathbf{N}$, define

$$\omega_P = \langle \rho_P, \beta^\vee \rangle^{-1} \cdot \rho_P.$$

Then $\omega_P \in X \otimes \mathbf{Q}$ is the fundamental weight associated with β. It is known that there exists a unique $\mathbb{w}_l \in W$ such that

$$\mathbb{w}_l(\Delta_M) \subseteq \Delta \text{ and } \mathbb{w}_l(\beta) \in \Phi^-.$$

Let $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$ be the maximal parabolic subgroup of $\mathbf{G}$ associated with $\mathbb{w}_l(\Delta_M)$. Then, $\mathbb{w}_l(\omega_P) = -\omega_{P'}$.

Consider the character group $X^*(\mathbf{M})$ of $\mathbf{M}$, and also the real and complex vector space

$$X^*(\mathbf{M})_{\mathbf{R}} = X^*(\mathbf{M}) \otimes_{\mathbf{Z}} \mathbf{R}, \quad X^*(\mathbf{M})_{\mathbf{C}} = X^*(\mathbf{M}) \otimes_{\mathbf{Z}} \mathbf{C}.$$

Any $\nu_o \in X^*(\mathbf{M})$ could be viewed as a character on $\mathbf{M}(\mathbb{A})$ valued in $\mathbb{A}^\times$. Further composition with the valuation of $\mathbb{A}^\times$ gives us a character of $\mathbf{M}(\mathbb{A})$ valued in $\mathbf{C}^\times$. Similarly, for any $\nu = \nu_o \otimes s \in X^*(\mathbf{M})_{\mathbf{C}}$, we denote by $\boldsymbol{\delta}^\nu$ the following character of $\mathbf{M}(\mathbb{A})$:

$$\boldsymbol{\delta}^\nu : \mathbf{M}(\mathbb{A}) \to \mathbf{C}, \quad m \mapsto |\nu_o(m)|_{\mathbb{A}}^s.$$

The relation between $\boldsymbol{\delta}$ and the modular character δ_P is $\boldsymbol{\delta}^{\rho_N \otimes 1} = \delta_P^{1/2}$. In the case of maximal parabolic subgroup, $X^*(\mathbf{M}/Z(\mathbf{G})) \otimes \mathbf{C}$ is of dimension one over $\mathbf{C}$ with $\omega_P \otimes 1$ or $\rho_P \otimes 1$ as a basis vector. Henceforth, we will write

$$\boldsymbol{\delta}^s := \boldsymbol{\delta}^{\omega_P \otimes s}, \quad s \in \mathbf{C}.$$

We have $\boldsymbol{\delta}^s = \prod_v \boldsymbol{\delta}_v^s$. For example for $\mathbf{SL}_2$ with positive root β, $\rho_P = \beta/2$ and $\omega_P = \rho_P$; then $\boldsymbol{\delta}^s = \delta_P^{s/2}$, with δ_P the modular character of the Borel subgroup P.

Let (π, V_π) be a genuine irreducible automorphic representation of $\overline{M}_{\mathbb{A}}$. We take $\boldsymbol{\delta}^s$ to be a character of the covering $\overline{M}_{\mathbb{A}}$ by the inflation via the surjection $\overline{M}_{\mathbb{A}} \twoheadrightarrow M_{\mathbb{A}}$. Now we consider the induced representation

$$I(s,\pi) := \mathrm{Ind}_{\overline{P}_{\mathbb{A}}}^{\overline{G}_{\mathbb{A}}} (\boldsymbol{\delta}^s \pi) \otimes \mathbf{1}.$$

We have the tensor product decomposition $I(s,\pi) = \bigotimes_v I(s,\pi_v)$, where $I(s,\pi_v)$ is unramified for almost all v. Similar to (12), an element $f_0 \in I(0,\pi)$ is identified with a right K-finite function

$$f : N_{\mathbb{A}} M_F \backslash \overline{G}_{\mathbb{A}} = (M_F \backslash \overline{M}_{\mathbb{A}}) \cdot \overline{K} \longrightarrow \mathbf{C} \tag{22}$$

such that for every $k \in \overline{K}$ the function $\overline{m} \mapsto f_0(\overline{m} \cdot \overline{k})$ lies in V_π. For every $s \in \mathbf{C}$, we define

$$f_s(\overline{m} \cdot \overline{k}) := \boldsymbol{\delta}^s(\overline{m}) \cdot f_0(\overline{m} \cdot \overline{k}) \in I(s,\pi),$$

and call such f_s a flat section. Consider the Eisenstein series

$$E(g, f_s, \pi) = \sum_{\gamma \in P_F \backslash G_F} f_s(\gamma g),$$

whose ψ-Fourier coefficient is given by

$$E_\psi(g, f_s, \pi) = \int_{U_F \backslash U_{\mathbb{A}}} E(ug, f_s, \pi) \psi(u)^{-1} du.$$

Suppose $f_v = \otimes_v f_{s,v} \in \bigotimes_v I(s, \pi_v)$. Locally, let λ_v be a ψ_v-Whittaker functional of π_v (which may be zero). Define λ_v^G on $I(s, \pi_v)$ by

$$\lambda_v^G(f_{s,v}) := \int_{N'_v} \lambda_v\big(f_{s,v}(\overline{w}_l^{-1}u)\big)\psi_v(u)^{-1}du.$$

Note that w_l and ψ are compatible, i.e., ψ and ${}^{w_l}\psi$ agree on $U_{M,\mathbb{A}}$. Thus λ_v^G is a well-defined ψ_v-Whittaker functional on $I(s, \pi_v)$. Associated to λ_v^G is the Whittaker function

$$\mathcal{W}^G_{f_{s,v}}(g_v) := \lambda_v^G\big(I(s, \pi_v))(g_v)f_{s,v})\big)$$

for any $g_v \in \overline{G}_v$.
Now we specialize to the case $\pi = \Theta(\overline{M}_{\mathbb{A}}, \chi)$. More precisely, let $\chi : Z(\overline{T}_{\mathbb{A}}) \to \mathbf{C}^\times$ be a genuine automorphic character. Assume also that χ is Δ_M-exceptional, i.e., χ is an exceptional character for the covering group $\overline{M}_{\mathbb{A}}$. One has the theta representation $\Theta(\overline{M}_{\mathbb{A}}, \chi)$ discussed in §3. Since we never consider theta representation of $\overline{G}_{\mathbb{A}}$ in this paper, we may write $\Theta(\chi) := \Theta(\overline{M}_{\mathbb{A}}, \chi)$ whenever no confusion arises.

Proposition 4.1. *Let $\Theta(\overline{M}_{\mathbb{A}}, \chi)$ be a ψ-distinguished theta representation. Then, for $f_s = \otimes_v f_{s,v} \in I(s, \Theta(\overline{M}_{\mathbb{A}}, \chi))$, we have*

$$E_\psi(g, f_s, \Theta(\overline{M}_{\mathbb{A}}, \chi)) = \prod_v \mathcal{W}^G_{f_{s,v}}(g_v),$$

where $g = \prod_v g_v \in \overline{G}_{\mathbb{A}}$.

Proof. This follows from the factorization of the Whittaker functional $\lambda_{\mathbb{A}} = \prod_v \lambda_v$ in Proposition 3.3 and the standard unfolding process for $E_\psi(g, f_s, \Theta(\overline{M}_{\mathbb{A}}, \chi))$. The argument, which follows the same line as in the linear algebraic case, can be found in [Sha10, Theorem 7.1.2 and Proposition 7.1.3]. □

Remark 4.2. The key for Proposition 4.1 is that the inducing theta representation is distinguished. Therefore, one could consider general (not necessarily maximal) parabolic subgroups and distinguished theta representations on the Levi subgroups. The analogous global factorization for the Fourier coefficients of Eisenstein series still holds.

4.3 Reduction of the unramified computation

In this subsection, assume $\Theta(\overline{M}_{\mathbb{A}}, \chi)$ is ψ-distinguished. Recall that for almost all v, we have $\lambda_v(\theta_v^0) = 1$ for the unramified vector $\theta_v^0 \in \Theta(\overline{M}_v, \chi_v)$. Consider v such that $\overline{G}_v$ is an unramified group and χ_v is an unramified character. Let $f^0_{s,v} \in I(s, \Theta(\overline{M}_v, \chi_v))$ be the normalized unramified vector such that

$$f^0_{s,v}(1) = \theta^0_v.$$

We would like to compute the local unramified component $\mathcal{W}^G_{f^0_{s,v}}(1)$ of $E_\psi(1, f_s, \Theta(\chi))$:

$$\mathcal{W}^G_{f^0_{s,v}}(1) = \int_{N'_v} \lambda_v\big(f^0_{s,v}(\overline{w}_l^{-1}u)\big)\psi_v(u)^{-1}du. \tag{23}$$

Recall that we use $\mathcal{W}_{\lambda_v}$ to denote the Whittaker function on $\overline{M}_v$ associated to θ^0_v and the ψ_v-Whittaker functional $\lambda_v : \Theta(\overline{M}_v, \chi_v) \to \mathbf{C}$. That is,

$$\mathcal{W}_{\lambda_v}(\overline{m}) = \lambda_v\big(\Theta(\overline{M}_v, \chi_v)(\overline{m})\theta^0_v\big)$$

such that $\mathcal{W}_{\lambda_v}(1) = 1$. There are two approaches of computing the integration in (23) we will use in this paper. The first approach relies on an explicit decomposition of $w_l^{-1}u$ as $\overline{m}\hat{u} \cdot k$ where $\overline{m} \in \overline{M}_v, \hat{u} \in N_v$ and $k \in K_v$. Then for each u, $\lambda_v(f^0_{s,v}(w_l^{-1}u))$ is essentially the Whittaker value $\mathcal{W}_{\lambda_v}(\overline{m})$. This approach is adapted in [BBL03] for Kazhdan–Patterson covers of $\overline{\mathrm{GL}}_r$, though the covers and theta representations considered there do not exactly fit in our context, as the key ingredient used in [BBL03] is the uniqueness of (ψ_v, μ_v)-Whittaker functionals for $I(s, \Theta(\chi_v))$.

The second approach of computing $\mathcal{W}^G_{f^0_{s,v}}$ is from adapting the idea in the linear algebraic case with proper modification. We will concentrate on this approach in the remaining part of this section. Let $\chi_v : Z(\overline{T}_v) \to \mathbf{C}^\times$ be an exceptional character for $\overline{M}_v$, i.e., it is Δ_M-exceptional. Write $I({}^{w_M}\chi_v) := \mathrm{Ind}_{\overline{B}_{M,v}}^{\overline{M}_v}\, {}^{w_M}i(\chi_v)$.

From the embedding

$$\Theta(\overline{M}_v, \chi_v) \hookrightarrow I({}^{w_M}\chi_v),$$

we have the surjective homomorphism of vector spaces

$$\mathsf{R} : \mathrm{Wh}_{\psi_v}\left(I({}^{w_M}\chi_v)\right) \twoheadrightarrow \mathrm{Wh}_{\psi_v}\left(\Theta(\overline{M}_v, \chi_v)\right),$$

where the image is of dimension one by the distinguished-ness of $\Theta(\overline{M}_{\mathbb{A}}, \chi)$. Let $\lambda'_v \in \mathrm{Wh}_{\psi_v}\left(I({}^{w_M}\chi_v)\right)$. We see that $\lambda'_v \in \mathsf{R}^{-1}(\lambda_v)$ if and only if $\lambda'_v(\theta^0_v) = 1$. One has the embeddings

$$I(s, \Theta(\overline{M}_v, \chi_v)) \hookrightarrow I\left(s, I({}^{w_M}\chi_v)\right) \hookrightarrow I_{\overline{B}_v}^{\overline{G}_v}(s \cdot \omega_P, i({}^{w_M}\chi_v)), \tag{24}$$

where $s \cdot \omega_P : \overline{T}_v \to \mathbf{C}^\times$ is the natural character obtained from the restriction $s \cdot \omega_P \in \mathrm{Hom}(\overline{M}_v, \mathbf{C}^\times)$. The first two induced representations in (24) are from $\overline{P}_v$ to $\overline{G}_v$. Let $\lambda'_v \in \mathsf{R}^{-1}(\lambda_v)$. Then it follows from the first embedding of (24) that

$$\lambda_v\left(f^0_{s,v}(\overline{w}_l^{-1}u)\right) = \lambda'_v\left(f^0_{s,v}(\overline{w}_l^{-1}u)\right),$$

where on the right hand side we view $f^0_{s,v}$ as the normalized unramified vector in $I\left(s, I({}^{w_M}\chi_v)\right)$. Note that $f^0_{s,v}(1) = \theta^0_v$ is the unramified vector in $I({}^{w_M}\chi_v)$.

Let $\mathbf{c} \in \mathsf{Ftn}(i({}^{w_M}\chi_v))$ be the element such that we have the correspondence

$$\mathbf{c} \leftrightarrow l_{\mathbf{c}} \leftrightarrow \lambda_v',$$

which is described in §3.3. Suppose $w_l^{-1}u = \overline{m}\hat{u}k$ with $\overline{m} \in \overline{M}_v, \hat{u} \in N_v$ and $k \in K$. Temporarily, denote $\sigma := I({}^{w_M}\chi_v)$. It then follows that

$$\begin{aligned}\lambda_v'\left(f_{s,v}^0(\overline{w}_l^{-1}u)\right) =&\lambda_v'\left(f_{s,v}^0(\overline{m}\cdot\hat{u}\cdot k)\right)\\ =&\lambda_v'\left((\boldsymbol{\delta}_v^s\cdot\delta_P^{1/2})(\overline{m})\cdot\sigma(\overline{m})\cdot f_{s,v}^0(1)\right)\\ =&(\boldsymbol{\delta}_v^s\cdot\delta_P^{1/2})(\overline{m})\cdot\lambda_v'\left(\sigma(\overline{m})\cdot\theta_v^0\right)\\ =&(\boldsymbol{\delta}_v^s\cdot\delta_P^{1/2})(\overline{m})\cdot l_{\mathbf{c}}\left(\int_{U_{M,v}}\left(\sigma(\overline{m})\theta_v^0\right)(\overline{w}_M^{-1}x)\psi_v^{-1}(x)dx\right)\end{aligned} \tag{25}$$

Therefore,

$$\begin{aligned}\mathcal{W}^G_{f^0_{s,v}}(1) =& \int_{N_v'}\lambda_v\left(f_{s,v}^0(\overline{w}_l^{-1}u)\right)\psi_v(u)^{-1}du\\ =& \int_{N_v'} l_{\mathbf{c}}\left((\boldsymbol{\delta}_v^s\cdot\delta_P^{1/2})(\overline{m})\cdot\int_{U_{M,v}}\theta_v^0(\overline{w}_M^{-1}x\cdot\overline{m})\psi^{-1}(x)dx\right)\psi_v(u)^{-1}du\\ =&l_{\mathbf{c}}\left(\int_{N_v'}(\boldsymbol{\delta}_v^s\cdot\delta_P^{1/2})(\overline{m})\cdot\int_{U_{M,v}}\theta_v^0(\overline{w}_M^{-1}x\cdot\overline{m})\psi^{-1}(x)dx\psi_v(u)^{-1}du\right)\\ =&l_{\mathbf{c}}\left(\int_{N_v'}\int_{U_{M,v}} f_{s,v}^0(\overline{w}_M^{-1}x\cdot\overline{m})(1)\cdot\psi^{-1}(x)\psi_v(u)^{-1}dxdu\right)\\ =&l_{\mathbf{c}}\left(\int_{N_v'}\int_{U_{M,v}} f_{s,v}^0(\overline{w}_M^{-1}x\cdot\overline{w}_l^{-1}u)(1)\cdot\psi^{-1}(x)\psi_v(u)^{-1}dxdu\right)\end{aligned} \tag{26}$$

Note that the function $f_{s,v}' : g \mapsto f_{s,v}^0(g)(1)$ is the unramified vector in $I_{\overline{B}_v}^{\overline{G}_v}(s\cdot\omega_P, i({}^{w_M}\chi_v))$, which is the image of $f_{s,v}^0$ in the second embedding of (24) above. By a change of variable, we get that

$$\mathcal{W}^G_{f^0_{s,v}}(1) = l_{\mathbf{c}}\left(\int_{U_v} f_{s,v}'(\overline{w}_G^{-1}u)\psi_v(u)^{-1}du\right),$$

where $\mathbb{w}_G$ is the longest Weyl element in W_G. By abuse of notation, we still use $f_{s,v}^0$ to denote the $i({}^{w_M}\chi_v)$-valued unramified vector in $I_{\overline{B}_v}^{\overline{G}_v}(s\cdot\omega_P, i({}^{w_M}\chi_v))$. In summary, we have shown the following.

Proposition 4.3. *Let $\mathbf{c} \in \mathsf{Ftn}(i({}^{w_M}\chi_v))$ be such that the associated ψ_M-Whittaker functional $\lambda_{v,\mathbf{c}}'$ of $I({}^{w_M}\chi_v)$ is normalized, i.e. $\lambda_{v,\mathbf{c}}'(\theta_v^0) = 1$. Let $\lambda_{v,\mathbf{c}}^G$ be ψ-Whittaker functional of $I_{\overline{B}_v}^{\overline{G}_v}(s\cdot\omega_P, i({}^{w_M}\chi_v))$ associated to $\mathbf{c}$. Then*

$$\mathcal{W}^G_{f^0_{s,v}}(1) = \mathcal{W}^G_{v,\mathbf{c}}(1),$$

where $\mathcal{W}^G_{v,\mathbf{c}}$ *is the Whittaker function on* $\overline{G}_v$ *associated to* $f^0_{s,v}$ *and* $\lambda^G_{v,\mathbf{c}}$*. In particular, for any* $\gamma \in \overline{T}_v$ *such that* $\mathcal{W}^M_{v,\gamma}(1) \neq 0$*, where* $\mathcal{W}^M_{v,\gamma}$ *is the Whittaker function associated to* θ^0_v *and* $\lambda_{v,\gamma} : I({}^{w_M}\chi_v) \to \mathbf{C}$*, one has*

$$\mathcal{W}^G_{f^0_{s,v}}(1) = \frac{\mathcal{W}^G_{v,\gamma}(1)}{\mathcal{W}^M_{v,\gamma}(1)}.$$

We note that the values of both $\mathcal{W}^G_{v,\gamma}(1)$ and $\mathcal{W}^M_{v,\gamma}(1)$ are given by Proposition 3.5. A special case is to consider $\gamma = 1_{\overline{T}} = 1_{\overline{G}}$.

Lemma 4.4. *We have* $\lambda'_{v,1_{\overline{T}}}(\theta^0_v) = \tau(w_M, \chi_v, 1_{\overline{T}}, 1_{\overline{T}})$*; or equivalently,*

$$\mathcal{W}^M_{v,1_{\overline{T}}}(1) = \tau(w_M, \chi_v, 1_{\overline{T}}, 1_{\overline{T}}).$$

Proof. For the proof we temporarily denote $\chi'_v := {}^{w_M}\chi_v$, which is an anti-exceptional for $\overline{M}_v$. It follows from Proposition 3.5 that

$$\mathcal{W}^M_{v,1_{\overline{T}}}(1) = \sum_{w \in W_M} c_{\mathsf{gk}}(w_M w^{-1}, \chi'_v) \cdot \tau(w, {}^{w^{-1}}\chi'_v, 1_{\overline{T}}, 1_{\overline{T}}).$$

For any nontrivial $\mathbb{w} \in W_M$, the set $\Phi_{M,\mathbb{w}}$ contains an element $\alpha \in \Delta_M$. Fix such an α. It then follows from (14) and the fact $\chi'(\overline{h}_\alpha(\varpi^{n_\alpha})) = q$ that $c_{\mathsf{gk}}(w_\alpha, \chi') = 0$. Therefore $c_{\mathsf{gk}}(w, \chi') = 0$ as well. Thus,

$$\mathcal{W}_{v,1_{\overline{T}}}(1) = \tau(w_M, \chi_v, 1_{\overline{T}}, 1_{\overline{T}}).$$

This completes the proof. □

We remark that it is not clear if $\tau(w_M, \chi_v, 1_{\overline{T}}, 1_{\overline{T}}) \neq 0$ in general. Moreover, even if it is nonzero, it is not always equal to 1. However, by assuming its nonvanishing, we obtain the following more explicit form for $\mathcal{W}^G_{f^0_{s,v}}(1)$ from Proposition 4.3

Theorem 4.5. *Keep the above notations. Let* $\mathcal{W}^G_{v,1_{\overline{T}}}$ *be the* ψ*-Whittaker functional of* $I^{\overline{G}_v}_{\overline{B}_v}(s \cdot \omega_P, i({}^{w_M}\chi_v))$*. If* $\tau(w_M, \chi_v, 1_{\overline{T}}, 1_{\overline{T}}) \neq 0$*, then as the unramified local component of* $E_\psi(1, f_s, \Theta(\chi))$*, we have*

$$\mathcal{W}^G_{f^0_{s,v}}(1) = \frac{\mathcal{W}^G_{v,1_{\overline{T}}}(1)}{\tau(w_M, \chi_v, 1_{\overline{T}}, 1_{\overline{T}})}.$$

In view of Proposition 3.5 and Theorem 4.5, we see that to obtain an explicit form for $\mathcal{W}^G_{f^0_{s,v}}(1)$ amounts to some computation involving the scattering matrix. However, there are difficulties with high-rank groups. For covers of the general linear groups, such difficulties were overcome in the work by Suzuki [Suz97]. In §6, we will elaborate on Theorem 4.5 by considering the rank-two symplectic groups.

5 Covering groups of GL_r

In this section, we treat the case of certain "nice" covers $(n, \overline{\mathbf{GL}}_r)$. Our main result is Theorem 5.11. When specialised to topological Kazhdan–Patterson covers, these nice covers are just degree $n-1$ covers of $\mathrm{GL}_{n,\mathbb{A}}$, in which case the result was first proved in [Suz97, §7.6].

5.1 Coverings of $\mathbf{GL}_r$

Recall the set-up and notations in §2.1. Let

$$\{X,\ \Delta,\ \Phi;\ Y,\ \Delta^\vee,\ \Phi^\vee\}$$

be the root datum of $\mathbf{GL}_r, r \geq 2$ with a maximal split torus $\mathbf{T}$. To facilitate computation, we fix a basis

$$\{e_1, e_2, ..., e_r\}$$

for the cocharacter lattice Y of $\mathbf{T}$, and a basis $\{e_1^*, e_2^*, ..., e_r^*\}$ for the character lattice X of $\mathbf{T}$ such that for the natural pairing

$$\langle -, - \rangle : Y \times X \to \mathbf{Z},$$

one has $\langle e_i, e_j^* \rangle = \delta_{ij}$. Denote $\alpha_i^\vee := e_i - e_{i+1}$ and $\alpha_i := e_i^* - e_{i+1}^*$. We choose simple coroots

$$\Delta^\vee = \{\alpha_i^\vee : 1 \leq i \leq r-1\}$$

and corresponding simple roots $\Delta = \{\alpha_i : 1 \leq i \leq r-1\}$. Let $\mathbf{B} = \mathbf{TU}$ be the Borel subgroup of $\mathbf{GL}_r$ associated with Δ.

Any coroot is of the form $\alpha_{i,j}^\vee := e_i - e_j$ for $i \neq j$. In particular, with this notation, $\alpha_i^\vee = \alpha_{i,i+1}^\vee$. The positive coroots are

$$\Phi_+^\vee = \left\{\alpha_{i,j}^\vee : i < j\right\}.$$

Let $\{e_\alpha : \mathbf{G}_a \to \mathbf{U}_\alpha\}_{\alpha\in\Phi}$ be a Chevalley-Steinberg system of pinnings for $\mathbf{GL}_r$. For any $a \in \mathbf{G}_a$, we write $e_{i,j}(a)$ with $i \neq j$ for the unipotent element associated with $\alpha_{i,j}^\vee$. Similarly, for any $a \in \mathbf{G}_m$, we write $h_{i,j}(a)$ or $h_{\alpha_{i,j}}(a)$ for the element $\alpha_{i,j}^\vee(a)$.

Since the derived group of $\mathbf{GL}_r$ is simply-connected, the isomorphism classes of $\mathbf{K}_2$-extensions of $\mathbf{GL}_r$ over F are determined by Weyl-invariant integer-valued quadratic forms on Y. Let Q be such a quadratic form and B_Q the associated bilinear form. The quadratic form Q is determined from B_Q by $Q(x) = B_Q(x,x)/2$. For $\mathbf{GL}_r$, any Weyl-invariant integer-valued bilinear form B_Q is determined by two integers $\mathbf{p}, \mathbf{q} \in \mathbf{Z}$ such that

$$B_Q(e_i, e_i) = 2\mathbf{p} \text{ and } B_Q(e_i, e_j) = \mathbf{q} \text{ if } i \neq j. \tag{27}$$

For any coroot $\alpha^\vee \in \Phi^\vee$, one has $Q(\alpha^\vee) = 2\mathbf{p} - \mathbf{q}$. Let $\overline{\mathbf{GL}}_r$ be the $\mathbf{K}_2$-extension of $\mathbf{GL}_r$ with the underlying $\mathbf{p}$ and $\mathbf{q}$ understood.

In fact, we may choose (without loss of generality on the isomorphism class) a bisector D of the symmetric bilinear form B_Q as follows:

$$D(e_i, e_j) = \begin{cases} 0 & \text{if } i < j, \\ Q(e_i) = \mathbf{p} & \text{if } i = j, \\ B_Q(e_i, e_j) = \mathbf{q} & \text{if } i > j. \end{cases} \tag{28}$$

Also take $\eta = \mathbf{1}$. The group structure of $\overline{\mathbf{GL}}_r$ is described as in (2) to (6). We remark that for any $\mathbf{p}, \mathbf{q}$ such that $2\mathbf{p} - \mathbf{q} = -1$, we have the Kazhdan–Patterson extensions of $\mathbf{GL}_r$, whose topological coverings are studied in [KP84]. The parameter $\mathbf{p}$ is just the twisting parameter c in the notation of [KP84].
Note that the numbers $Q(\alpha^\vee)$ and n_α are both independent of the choice of $\alpha^\vee$, since $\mathbf{GL}_r$ is simply-laced.

Definition 5.1. An n-fold cover $(n, \overline{\mathbf{GL}}_r)$ is called nice if the following two conditions hold:

- n divides $2\mathbf{p}$, and
- $n_\alpha = r - 1$.

For example, the Kazhdan–Patterson extension $(n, \overline{\mathbf{GL}}_{n+1})$ with $\mathbf{p} = 0, \mathbf{q} = -1$ is a nice cover. As another example, $(n = 1, \overline{\mathbf{GL}}_2)$ is nice for any $\mathbf{p}$ and $\mathbf{q}$.

5.2 Weyl orbits

Let $\mathbf{P} = \mathbf{MN} \subset \mathbf{GL}_r$ be the maximal parabolic subgroup associated with the subset $\Delta \backslash \{\alpha_{r-1}^\vee\}$. Here, $\mathbf{M} \simeq \mathbf{GL}_{r-1} \times \mathbf{GL}_1$. The root datum of $\mathbf{M}$ is

$$(X, \Phi_M, \Delta_M;\ Y, \Phi_M^\vee, \Delta_M^\vee),$$

where $\Delta_M = \Delta_{\mathbf{GL}_{r-1}}$ and $\Delta_M^\vee = \Delta_{\mathbf{GL}_{r-1}}^\vee$ and similarly for Φ_M and $\Phi_M^\vee$. Also, $W_M = W_{\mathbf{GL}_{r-1}}$ is the Weyl group of $\mathbf{M}$ with $\mathbb{w}_M \in W_M$ the longest element. The unique $\mathbb{w}_l \in \mathrm{W}$ such that $\mathbb{w}_l(\Delta_M) \subseteq \Delta$ and $\mathbb{w}_l(\alpha) < 0$ for every $\alpha \in \mathbf{N}$ in this case is

$$\mathbb{w}_l = \mathbb{w}_{\alpha_1} \cdot \mathbb{w}_{\alpha_2} \cdot \ldots \cdot \mathbb{w}_{\alpha_{r-1}}.$$

Let $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$ be the parabolic associated with $\mathbb{w}_l(\Delta_M)$.
Temporarily, let $\mathbf{G}$ be either $\mathbf{GL}_r$, $\mathbf{M}$ or $\mathbf{GL}_{r-1} \subset \mathbf{M}$. Let $Y_\mathbf{G}$ be the cocharacter lattice of $\mathbf{G}$. Then from the data $(\mathbf{G}, Q, n)$, we have the lattice $Y_{Q,n,\mathbf{G}} \subset Y$ given by (8), that is,

$$Y_{Q,n,\mathbf{G}} := \{y \in Y_\mathbf{G} : B_Q(y, z) \in n\mathbf{Z} \text{ for all } z \in Y_\mathbf{G}\}.$$

Since $\mathbf{GL}_r$ and $\mathbf{M}$ have the same torus, we simply denote by $Y_{Q,n}$ for $Y_{Q,n,\mathbf{GL}_r} = Y_{Q,n,\mathbf{M}}$. An easy computation with (8) and (27) gives that

$$Y_{Q,n} = \left\{ \sum_{i=1}^r k_i e_i \in Y :\ Q(\alpha^\vee) \cdot k_j + \mathbf{q} \cdot \left(\sum_{i=1}^r k_i \right) \in n\mathbf{Z} \text{ for all } j \right\}. \tag{29}$$

Recall that $Y_{Q,n,\mathbf{M}}^{sc} \subseteq Y_{Q,n}$ is the sublattice generated by $\{\alpha_{Q,n}^\vee : \alpha \in \Phi_M\}$. We apply the discussion of Weyl orbits in §2.3 to the group $\mathbf{M}$. In particular, we consider in this section only W_M-orbits for the twisted action. If

$$\rho = \frac{1}{2} \sum_{\substack{\alpha^\vee \in \Phi_M^\vee \\ \alpha^\vee > 0}} \alpha^\vee$$

is the half sum of the positive coroots in $\Phi_M^\vee$; then $\mathcal{O}^F$ denote the set of all free W_M-orbits in Y with respect to the action $\mathbb{w}[y] := \mathbb{w}(y-\rho)+\rho$. Moreover, $\mathcal{O}_{Q,n}^F \subset \mathcal{O}^F$ denotes the set of $Y_{Q,n}$-free orbits and and $\mathcal{O}_{Q,n,sc}^F \subset \mathcal{O}^F$ the $Y_{Q,n,\mathbf{M}}^{sc}$-free orbits.
If $(n, \overline{\mathbf{GL}}_r)$ is a nice cover, then it follows from (29) that

$$Y_{Q,n} = \left\{ \sum_{i=1}^r k_i e_i : k_1 \equiv k_2 \equiv ... \equiv k_r \mod n_\alpha \right\}, \tag{30}$$

where $n_\alpha = r-1$ for a nice cover. Clearly, $n_\alpha \cdot Y_{\mathbf{GL}_{r-1}}$ is a sublattice of $Y_{Q,n}$. We see that

$$Y_{Q,n,\mathbf{GL}_{r-1}} = n_\alpha \cdot Y_{\mathbf{GL}_{r-1}}.$$

On the other hand, define

$$Y_0 = \left\{ x \cdot (\sum_{i=1}^r e_i) + k \cdot n_\alpha e_r : \ x \text{ and } k \in \mathbf{Z} \right\}.$$

It then follows easily:

Lemma 5.2. *Let $(n, \overline{\mathbf{GL}}_r)$ be a nice cover. Then $Y = Y_{\mathbf{GL}_{r-1}} + Y_0$. Moreover,*

$$Y_{Q,n} = Y_{Q,n,\mathbf{GL}_{r-1}} + Y_0,$$

where the intersection of $Y_{Q,n,\mathbf{GL}_{r-1}}$ and Y_0 is the one-dimensional lattice spanned by $n_\alpha \cdot \sum_{i=1}^{r-1} e_i$.

The following lemma plays a pivotal role in this section.

Lemma 5.3. *Let $(n, \overline{\mathbf{GL}}_r)$ be a nice cover. Then $\mathcal{O}_{Q,n}^F = \mathcal{O}_{Q,n,sc}^F$ and $\wp(\mathcal{O}_y) = \wp(\mathcal{O}_0)$ for every $\mathcal{O}_y \in \mathcal{O}_{Q,n}^F$. In particular, $\left|\wp(\mathcal{O}_{Q,n}^F)\right| = 1$.*

Proof. The argument is analogous to [Gao18b, Proposition 3.5], where covers of $\mathbf{GL}_r$ are treated (instead of the semisimple $\mathbf{M}$ here). For completeness, we give the details.
First, to show $\mathcal{O}_{Q,n}^F = \mathcal{O}_{Q,n,sc}^F$, it suffices to show $\mathcal{O}_{Q,n}^F \supset \mathcal{O}_{Q,n,sc}^F$. Let $Y_{\mathbf{M}}^{sc} = Y_{\mathbf{GL}_{r-1}}^{sc}$ be the coroot lattice of $\mathbf{M}$. We have $Y_{Q,n,\mathbf{M}}^{sc} = n_\alpha \cdot Y_{\mathbf{M}}^{sc}$. In view of Lemma 5.2, it is easy to see that

$$Y_{Q,n} \cap Y_{\mathbf{M}}^{sc} = Y_{Q,n,\mathbf{M}}^{sc}.$$

Now let $\mathcal{O}_y \in \mathcal{O}^F$ be a $Y^{sc}_{Q,n,\mathbf{M}}$-free orbit. Suppose that it is not $Y_{Q,n}$-free, then there exists a nontrivial $\mathbbm{w} \in W_M$ such that $\mathbbm{w}[y] - y \in Y_{Q,n}$. However, since $\mathbbm{w}[y] - y$ lies in $Y^{sc}_{\mathbf{M}}$, we see that $\mathbbm{w}[y] - y$ lies in $Y^{sc}_{Q,n,\mathbf{M}}$. That is, $\mathcal{O}_y$ is not $Y^{sc}_{Q,n,\mathbf{M}}$-free, and this is a contradiction. Therefore, $\mathcal{O}^F_{Q,n} = \mathcal{O}^F_{Q,n,sc}$.
Lastly, a straightforward combinatorial computation as in [Gao17, §3], which relies crucially on the fact that $n_\alpha = r-1$ for a nice cover, shows that $\wp(\mathcal{O}^F_{Q,n}) = \{\wp(\mathcal{O}_0)\}$ in this case. This completes the proof. ☐

We also record the following result which is important for the unramified computation in §5.4.

Lemma 5.4. *Let $(n, \overline{\mathbf{GL}}_r)$ be a nice cover. Let k be a natural number such that $1 \le k \le r-2$. Let z_k be the element $z_k = \sum_{i=r-k}^{r-1} e_i + (-ke_r) \in Y$. Then there exists a nontrivial element $\mathbbm{w} \in W_M$ such that $\mathbbm{w}[z_k] - z_k \in Y_{Q,n}$. Therefore, $\mathcal{O}_{z_k}$ is not $Y_{Q,n}$-free in this case.*

Proof. Let $y_k = \sum_{i=r-k}^{r-1} e_i$ be the element in $Y_{\mathbf{GL}_{r-1}}$. It suffices to show that $\mathbbm{w}[y_k] - y_k$ lies in $Y^{sc}_{Q,n,\mathbf{M}}$ for some nontrivial $\mathbbm{w} \in W_M$. For this, we note that

$$\rho = -\sum_{i=0}^{r-1} (i-1) \cdot e_i + \frac{r-2}{2} \cdot \Big(\sum_{i=1}^{r-1} e_i\Big).$$

Thus,

$$y_k - \rho = \sum_{i=1}^{r-1-k} (i-1)e_i + \sum_{i=r-k}^{r-1} i \cdot e_i.$$

It is clear that for any $\mathbbm{w} \in W_M$ we have $\mathbbm{w}[y] - y = \mathbbm{w}(y-\rho) - (y-\rho)$. Let $\mathbbm{w} \in W_M$ be corresponding to the permutation of x_1 and x_{r-1} in $\sum_{i=1}^{r-1} x_i e_i \in Y_{\mathbf{GL}_{r-1}}$. Then it follows that

$$\mathbbm{w}(y_k - \rho) - (y_k - \rho) = (r-1)(e_{r-1} - e_1) \in Y^{sc}_{Q,n,\mathbf{M}} \subset Y_{Q,n},$$

since $n_\alpha = r-1$ for nice covers. Therefore z_k is not $Y_{Q,n}$-free. This completes the proof. ☐

Let $n \ge 1$ be a natural number with $\mu_{2n} \subset F^\times$. We have the global n-fold cover $\overline{\mathrm{GL}}_{r,\mathbb{A}}$ and the local $\overline{\mathrm{GL}}_{r,v}$. The covering $\overline{\mathrm{GL}}_{r,\mathbb{A}}$ restricts to give the covering $\overline{M}_{r,\mathbb{A}}$ of $M_{r,\mathbb{A}}$. There are local coverings $\overline{M}_{r,v}$ of $M_{r,v} = \mathrm{GL}_{r-1,v} \times \mathrm{GL}_{1,v}$ from the restriction of $\overline{\mathrm{GL}}_{r,v}$. In general,

$$\overline{M}_{r,\mathbb{A}} \ncong \overline{\mathrm{GL}}_{r-1,\mathbb{A}} \times_{\mu_n} \overline{\mathrm{GL}}_{1,\mathbb{A}}.$$

That is, the coverings $\overline{\mathrm{GL}}_{r-1,\mathbb{A}}$ and $\overline{\mathrm{GL}}_{1,\mathbb{A}}$ do not commute in general. However, we do not rely on this property or its contrary, since we will carry out the analysis directly on $\overline{M}_{r,\mathbb{A}}$ and its local analogue. Let $\overline{T}_{\mathbb{A}}$ and $\overline{T}_v$ be the global and local covering torus.

Proposition 5.5. *Let* $(n, \overline{\mathbf{GL}}_r)$ *be a nice cover. Then,* $\overline{M}_{\mathbb{A}} = \overline{\mathrm{GL}}_{r-1,\mathbb{A}} \cdot Z(\overline{M}_{\mathbb{A}})$*, which is equivalent to the equality* $\overline{T}_{\mathbb{A}} = \overline{T}_{r-1,\mathbb{A}} \cdot Z(\overline{M}_{\mathbb{A}})$*, where* $\overline{T}_{r-1,\mathbb{A}} \subset \overline{\mathrm{GL}}_{r-1,\mathbb{A}}$ *is the* n*-fold cover of the torus* $T_{r-1,\mathbb{A}} \subset \mathrm{GL}_{r-1,\mathbb{A}}$*. Moreover,*

$$Z(\overline{T}_{\mathbb{A}}) = Z(\overline{T}_{r-1,\mathbb{A}}) \cdot Z(\overline{M}_{\mathbb{A}}).$$

The same holds for the local covering groups as well.

Proof. In fact, we will show that locally the two equalities $\overline{T}_v = \overline{T}_{r-1,v} \cdot Z(\overline{M}_v)$ and $Z(\overline{T}_v) = Z(\overline{T}_{r-1,v}) \cdot Z(\overline{M}_v)$ hold. We have $Z(\overline{M}_v) = Z(\overline{T}_v) \cap \overline{Z(M_v)}$. As noted in §2.4, $Z(\overline{T}_v) = \phi_v^{-1}(\mathrm{Im}(i_{Q,n}))$ where $i_{Q,n} : Y_{Q,n} \otimes F_v^\times \to T_v$ is the isogeny induced from $Y_{Q,n} \hookrightarrow Y$. By using the explicit form of $Y_{Q,n}$ in (30), we see that $Z(\overline{M}_v)$ is equal to the preimage in $\overline{T}_v$ of

$$i_{Q,n}(Y_0 \otimes F^\times) \subset T_v.$$

Since $Y = Y_{\mathrm{GL}_{r-1}} + Y_0$ by Lemma 5.2, we have $\overline{T}_v = \overline{T}_{r-1,v} \cdot Z(\overline{M}_v)$ and therefore $\overline{M}_v = \overline{\mathrm{GL}}_{r-1,v} \cdot Z(\overline{M}_v)$.
On the other hand, we have $Z(\overline{T}_{r-1,v}) = \phi_v^{-1}(\mathrm{Im}(i_{Q,n,\mathbf{GL}_{r-1}}))$, where

$$i_{Q,n,\mathbf{GL}_{r-1}} : Y_{Q,n,\mathbf{GL}_{r-1}} \otimes F^\times \to T_{r-1,v}$$

is induced from $Y_{Q,n} \hookrightarrow Y_{\mathbf{GL}_{r-1}}$. It then follows from Lemma 5.2 that $Z(\overline{T}_v) = Z(\overline{T}_{r-1,v}) \cdot Z(\overline{M}_v)$. This completes the proof. □

The above proposition shows that representations of $\overline{M}_{\mathbb{A}}$ and $\overline{\mathrm{GL}}_{r-1,\mathbb{A}}$ differ only by a central character of $Z(\overline{M}_{\mathbb{A}})$. Such difference is insignificant for our purpose in this paper.

Remark 5.6. The covering groups which appear in [BBL03] could be placed in the Brylinski–Deligne framework as follows. Consider the Kazhdan–Patterson $\mathbf{K}_2$-extension $\overline{\mathbf{GL}}_n$ with $\mathbf{p} = 0$ and $\mathbf{q} = 1$. Let $\flat : \mathbf{GL}_{n-1} \to \mathbf{GL}_n$ be the embedding given by $\flat(g) = (g, \det(g)^{-1}) \in \mathbf{SL}_n \subset \mathbf{GL}_n$. Denote by $\overline{\mathbf{GL}}^\flat_{n-1}$ the pull-back of $\overline{\mathbf{GL}}_n$ via $\flat$. The extension $\overline{\mathbf{GL}}^\flat_{n-1}$ also belongs to the Kazhdan–Patterson family, but is the one associated to $\mathbf{p} = -1, \mathbf{q} = -1$. We consider $\mathbf{M} = \mathbf{GL}_{n-1} \times \mathbf{GL}_1$. Since $\flat(\mathbf{GL}_{n-1}) \subset \mathbf{M}$, we have $\overline{\mathbf{GL}}^\flat_{n-1} \subset \overline{\mathbf{M}}$.
Now consider the arising topological n-fold covering groups. We have

$$\overline{\mathrm{GL}}^\flat_{n-1,\mathbb{A}} \cdot Z(\overline{M}_{\mathbb{A}}) \subset \overline{M}_{\mathbb{A}},$$

which however is not an equality in general. In fact, locally for $v \in |F|$, the quotient $\overline{M}_v / \big(\overline{\mathrm{GL}}^\flat_{n-1,v} \cdot Z(\overline{M}_v)\big)$ is equal to $F_v/(F_v)^n$. This explains the multi-dimension of ψ-Whittaker functionals for the representation π_s (in the notation of [BBL03, p. 171]) parabolically induced from theta representations. However, as a rectification, it is shown in [BL94] that the (ψ, μ)-Whittaker functional for π_s, where μ is any genuine character of the abelian group $\overline{Z(\mathrm{GL}_{n,v})}$, is unique.

We also remark that the reason for considering $\overline{\mathbf{GL}}^{\flat}_{n-1}$ in [BBL03] instead of $\overline{\mathbf{GL}}_{n-1}$, which is obtained from the restriction of $\overline{\mathbf{GL}}_n$, is that theta representations on the n-fold cover $\overline{\mathrm{GL}}^{\flat}_{n-1,\mathbb{A}}$ have a unique ψ-Whittaker functional. However, this is not true for the n-fold cover $\overline{\mathrm{GL}}_{n-1,\mathbb{A}}$ in general.

5.3 Theta representations and Eisenstein series

In view of Proposition 5.5, the automorphic representation for $\overline{M}_{\mathbb{A}}$ is essentially the same for $\overline{\mathrm{GL}}_{r-1,\mathbb{A}}$. From the general set-up in §3 and §4, recall that

$$\chi = \otimes_v \chi_v : Z(\overline{T}_{\mathbb{A}}) \to \mathbf{C}^\times$$

is a genuine character which is trivial on $T_F \cap Z(\overline{T}_{\mathbb{A}})$, which could be viewed as a genuine character on $T_F \cdot Z(\overline{T}_{\mathbb{A}})$ trivial on T_F. Then we have the Eisenstein series $E(g, \chi, f)$ on $\overline{M}_{\mathbb{A}}$. Moreover, assume that χ is an exceptional character for $\overline{M}_{\mathbb{A}}$, i.e. $\chi_\alpha = |\cdot|_{\mathbb{A}}$ for all $\alpha \in \Delta_M^\vee$. We obtain the theta representation $\Theta(\overline{M}_{\mathbb{A}}, \chi) = \otimes_v \Theta(\overline{M}_v, \chi_v)$ as the residue of the Eisenstein series $E(g, \chi, f)$.

Proposition 5.7. *Let $\overline{\mathbf{M}}$ be arising from a nice cover $(n, \overline{\mathbf{GL}}_r)$, and keep notations as above. Then $\Theta(\overline{M}_{\mathbb{A}}, \chi)$ is ψ-distinguished.*

Proof. Let v be a place such that $|n|_v = 1$. Then by [Gao17, Theorem 3.14] coupled with Lemma 5.3, we have $\dim \mathrm{Wh}_{\psi_v}(\Theta(\overline{M}_v, \chi_v)) = \left|\wp(\mathcal{O}^F_{Q,n})\right| = 1$. For this, we note that the computation in [Gao17] is based on assuming that χ_v is an exceptional and unramified character of $Z(\overline{T}_v)$. However, examining the argument shows that the "unramified" assumption on the whole center $Z(\overline{T}_v)$ is not necessary. Instead, what is used is just that χ_v is exceptional and that there exists an extension χ'_v to the maximal abelian subgroup $Z(\overline{T}_v) \cdot \mathbf{T}(O_v)$ such that χ'_v is trivial on $\overline{h}_\alpha(O_v^\times) \subset s_{K_v}(K_v)$ for $\alpha \in \Delta$. However, as shown in [KP84, page 77], such an extension χ'_v is always possible for $|n|_v = 1$. Moreover, since $\Theta(\overline{M}_v, \chi_v) \subset I({}^{w_M}\chi_v)$ is a subrepresentation of the principal series induced from the character χ_v, therefore the dimension of $\mathrm{Wh}_{\psi_v}(\Theta(\overline{M}_v, \chi_v))$ is independent of the nontrivial character ψ_v, and is computed in [Gao17] under the harmless assumption that ψ_v has conductor O_v.
By Proposition 5.5, we have $\chi = \chi^o \cdot \omega$, where χ^o (respectively ω) is an automorphic character of $Z(\overline{T}_{r-1,\mathbb{A}})$ (respectively $Z(\overline{M}_{\mathbb{A}})$) such that χ^o and ω agrees on the domain of intersection. Note that χ is an exceptional character for $\overline{M}_{\mathbb{A}}$ if and only if χ^o is an exceptional character for $\overline{\mathrm{GL}}_{r-1,\mathbb{A}}$. Therefore, $\Theta(\overline{M}_{\mathbb{A}}, \chi) = \Theta(\overline{\mathrm{GL}}_{r-1,\mathbb{A}}, \chi^o) \otimes \omega$, where the tensor is the standard one by Proposition 5.5. Locally, $\chi_v = \chi_v^o \otimes \omega_v$ and $\Theta(\overline{M}_v, \chi_v) = \Theta(\overline{\mathrm{GL}}_{r-1,v}, \chi_v^o) \otimes \omega_v$. Thus we have:

(C) $\dim \mathrm{Wh}_{\psi_v} \Theta(\overline{\mathrm{GL}}_{r-1,v}, \chi_v^o) = 1$ for all v such that $|n|_v = 1$.

Though we are considering more general covering groups $\overline{\mathrm{GL}}_{r-1,\mathbb{A}}$ here, examining the proof for [KP84, Theorem II 2.5] shows that the global argument applies in our context. Indeed, if $\mathbf{p} = 0$ and $\mathbf{q} = 1$, then $\overline{\mathrm{GL}}_{r-1,\mathbb{A}}$ is the $(r-1)$-fold Kazhdan–Patterson covering group. Note that the proof of loc. cit. only relies on (C) above and it gives that $\dim \mathrm{Wh}_{\psi_v} \Theta(\overline{\mathrm{GL}}_{r-1,v}, \chi_v^o) = 1$ for all $v \in |F|$. This in turn shows that $\dim \mathrm{Wh}_{\psi_v} \Theta(\overline{M}_v, \chi_v) = 1$ for all $v \in |F|$. The nonvanishing of $\lambda_{\mathbb{A}}$ follows from [KP84, Theorem II 2.2 and 2.5]. This completes the proof. □

REMARK 5.8. If we assume $\mu_{2n} \subset F^\times$, then $\overline{M}_v$ splits canonically over M_v for archimedean place v and thus $\overline{M}_v \simeq \mu_n \times M_v$; in this case, one has $\dim \mathrm{Wh}_{\psi_v}(\Theta(\overline{M}_v, \chi_v)) \le 1$. It can be shown that if $\overline{M}_v$ arise from a nice cover, then $\mathrm{Ind}_{\overline{M}_v N_v}^{\overline{\mathrm{GL}}_r}(\chi)$ is irreducible for archimedean v, and thus $\dim \mathrm{Wh}_{\psi_v}(\Theta(\overline{M}_v, \chi_v)) = 1$. We refer the reader to [KP84, Theorem I.6.4, I.6.5] for discussions in the case of Kazhdan–Patterson covers.

In the rest of this section, we will assume that $(n, \overline{\mathbf{GL}}_r)$ is a nice cover. Also, χ is an exceptional character for $\overline{M}_{\mathbb{A}}$.

Let $\delta_P : \overline{P}_{\mathbb{A}} \to \mathbf{C}^\times$ be the modular character of $\overline{P}_{\mathbb{A}}$. More explicitly, consider $\overline{m} \cdot u \in \overline{P}_{\mathbb{A}}$ with $\overline{m} \in \overline{M}_{\mathbb{A}}$ and $u \in N_{\mathbb{A}}$. Suppose that $m = (m_1, m_2) \in \mathrm{GL}_{r-1,\mathbb{A}} \times \mathrm{GL}_{1,\mathbb{A}}$; then explicitly,

$$\delta_P(\overline{m} \cdot u) = |\det(m_1)|_{\mathbb{A}} \cdot |m_2|_{\mathbb{A}}^{-(r-1)}.$$

We have $I(s, \Theta(\chi)) = \mathrm{Ind}_{\overline{P}_{\mathbb{A}}}^{\overline{G}_{\mathbb{A}}}(\delta_P^{s/r} \cdot \Theta(\chi))$ in this case, where the latter is the normalized induced representation. Taking $f_s \in I(s, \Theta(\chi))$ to be a flat section, consider the Eisenstein series $E(g, f_s, \Theta(\chi))$. Then, Proposition 5.7 coupled with Proposition 4.1 give that

$$E_\psi(1, f_s, \Theta(\chi)) = \prod_v \mathcal{W}_{f_{s,v}}^{\mathrm{GL}_r}(1)$$

where

$$\mathcal{W}_{f_{s,v}}^{\mathrm{GL}_r}(1) = \int_{N_v'} \lambda_v\big(f_{s,v}(\overline{w}_l^{-1} u)\big)\psi_v(u)^{-1} du.$$

Moreover, the results in §4 apply; in particular, Theorem 4.5 gives a description of $\mathcal{W}_{f_{s,v}}^{\mathrm{GL}_r}(1)$. However, to obtain a more explicit formula of $\mathcal{W}_{f_{s,v}}^{\mathrm{GL}_r}(1)$ in terms of L-functions, we carry out an alternate computation following the idea in [BBL03]. This will be the focus of the remaining part of this section.

5.4 LOCAL UNRAMIFIED COMPUTATIONS

We will carry out the computation with unramified data. Thus, we suppress the subscript v for all notations. In particular, F denotes a non-archimedean local field such that $|n| = 1$. We assume that χ is an unramified Δ_M-exceptional character and ψ has conductor O_F. We assume in this subsection $\mu_{2n} \subset F^\times$.

Let $f_s \in I(s, \Theta(\chi))$ be the unramified vector such that $f_s(1) = \theta^0$ is the normalized unramified vector in $\Theta(\chi)$. By Lemma 5.3 and the proof of Proposition 5.7, let $\lambda_{\mathcal{O}_0}$ be the unique Whittaker functional of $\Theta(\overline{M}, \chi)$ such that $\lambda_{\mathcal{O}_0}(\theta^0) = 1$. Recall that $\lambda_{\mathcal{O}_0}$ arises from $\mathbf{c}_{\mathcal{O}_0} \in \mathsf{Ftn}(i(\chi))$ and gives rise to the Whittaker function $\mathcal{W}_{\mathcal{O}_0}$ on $\overline{M}$ associated to θ^0. We would like to compute

$$\mathcal{W}_{f_s}^{\mathrm{GL}_r}(1) = \int_{N'} \lambda_{\mathcal{O}_0}\big(f_s(\overline{w}_l^{-1}u)\big)\psi(u)^{-1}du, \tag{31}$$

where $w_l = w_{\alpha_1}w_{\alpha_2}...w_{\alpha_{r-1}}$. We follow closely the paper [BBL03] to decompose explicitly $w_l^{-1}u = \overline{p} \cdot k$ for some $\overline{p} \in \overline{P}$ and $k \in K$.
Note that N' is abelian and any element u in N' can be written uniquely as

$$u_x = e_{1,r}(x_{r-1}) \cdot ... \cdot e_{1,3}(x_2) \cdot e_{1,2}(x_1)$$

for $x = (x_1, ..., x_{r-1}) \in F^{r-1}$. Now it is easy to see that for any i with $1 \le i \le r-2$, we have

$$e_{\alpha_{i+1}}(x_{i+1}) = (w_{\alpha_1}w_{\alpha_2}...w_{\alpha_i})^{-1} \cdot e_{1,i+2}(x_{i+1}) \cdot (w_{\alpha_1}w_{\alpha_2}...w_{\alpha_i}) \in U.$$

Since the splitting of unipotent subgroup U is $\overline{\mathrm{GL}}_r$-equivariant, it follows that

$$\overline{e}_{\alpha_{i+1}}(x_{i+1}) = (w_{\alpha_1}w_{\alpha_2}...w_{\alpha_i})^{-1} \cdot \overline{e}_{1,i+2}(x_{i+1}) \cdot (w_{\alpha_1}w_{\alpha_2}...w_{\alpha_i}) \in U.$$

Inductively, one obtains that

$$\overline{w_l}^{-1}\overline{u_x} = \prod_{i=r-1}^{1} \overline{w}_{\alpha_i}^{-1}\overline{e}_{\alpha_i}(x_i). \tag{32}$$

We also note that for any root α and $x \in F^\times$, we have

$$\overline{w}_{\alpha}^{-1}\overline{e}_{\alpha}(x) = \overline{h}_{\alpha}(x^{-1})\overline{e}_{\alpha}(-x) \cdot \overline{e}_{-\alpha}(-x^{-1}). \tag{33}$$

Moreover, for any $1 \le i \le r-1$ and $x \in F^\times$, one has

$$\prod_{j=i}^{r-1} \overline{h}_{\alpha_j}(x) = \prod_{j=i}^{r-1} \mathbf{s}\big(h_{\alpha_j}(x)\big) = \mathbf{s}\big(h_{i,r}(x)\big), \tag{34}$$

where the first equality follows from (5) and our assumption that η_n is trivial, the second equality follows from (4) and $(x,x)_n = 1$ (since we have assumed $\mu_{2n} \subset F^\times$). In particular, we see that

$$\mathbf{s}(h_{i,r}(x)) \in s_K(K)$$

for any $x \in O^\times$, since $\overline{h}_{a_j}(x) \in s_K(K)$, see (11).
The domain of integration of (31) is identified with F^{r-1}. Moreover, we could restrict to the domain to $(F - \{0\})^{r-1}$. For any $\mathfrak{f} = (f_1, ..., f_{r-1}) \in \mathbf{Z}^{r-1}$, define

$$R(\mathfrak{f}) = \left\{(x_1, x_2, ..., x_{r-1}) \in F^{r-1} : \mathrm{val}(x_i) = \mathfrak{f}_i \text{ for all } i\right\}.$$

Thus the integration in (31) is equal to

$$\sum_{\mathfrak{f}\in\mathbf{Z}^{r-1}}\int_{x\in R(\mathfrak{f})}\lambda_{\mathcal{O}_0}\big(f_s(\overline{w}_l^{-1}u_x)\big)\psi(u_x)^{-1}du_x. \tag{35}$$

For each $1\leq i\leq r-1$, define

$$\delta_i=\begin{cases}0, & \text{if } \mathfrak{f}_i\geq 0,\\ 1, & \text{if } \mathfrak{f}_i<0.\end{cases}$$

To simplify notation, we write

$$\mathrm{Int}(R(\mathfrak{f})):=\int_{x\in R(\mathfrak{f})}\lambda_{\mathcal{O}_0}\big(f_s(\overline{w}_l^{-1}u_x)\big)\psi(u_x)^{-1}du_x.$$

The relations (32)-(34) and (6) allow us to argue as in [BBL03, page 173-174]. We therefore obtain that $\mathrm{Int}(R(\mathfrak{f}))$ is equal to

$$\int_{x\in R(\mathfrak{f})}\lambda_{\mathcal{O}_0}\left(f_s\Big(\prod_{i=1}^{r-1}\mathbf{s}\big(h_{i,r}(x_i^{-\delta_i})\big)\Big)\right)\cdot\prod_{i=1}^{r-1}|x_i|_F^{\delta_i(r-i-1)}\cdot\psi(x_1)\cdot\prod_{j=2}^{r-1}\psi(\delta_{j-1}x_j)dx, \tag{36}$$

where we write dx for $\prod_i dx_i$. A change of variables $x_i\mapsto\varpi^{f_i}x_i$ for all $1\leq i\leq r-1$ gives that (36) is equal to

$$\prod_{i=1}^{r-1}q^{-\delta_i\mathfrak{f}_i(r-i-1)-\mathfrak{f}_i}$$
$$\cdot\int_{x_i\in O^\times}\lambda_{\mathcal{O}_0}\left(f_s\Big(\prod_{i=1}^{r-1}\mathbf{s}\big(h_{i,r}(\varpi^{-\delta_i\mathfrak{f}_i}x_i^{-\delta_i})\big)\Big)\right)\cdot\psi(\varpi^{\mathfrak{f}_1}x_1)\cdot\prod_{j=2}^{r-1}\psi(\varpi^{\delta_{j-1}\mathfrak{f}_j}x_j)dx. \tag{37}$$

Note that for any $1\leq i,j\leq r-1$, it follows from (4) that

$$\mathbf{s}\big(h_{i,r}(uv)\big)=\mathbf{s}\big(h_{i,r}(u)\big)\cdot\mathbf{s}\big(h_{i,r}(v)\big)\cdot(v,u)_n^{Q(\alpha_{i,r}^\vee)}$$

and

$$\mathbf{s}\big(h_{i,r}(u)\big)\cdot\mathbf{s}\big(h_{j,r}(v)\big)=\mathbf{s}\big(h_{j,r}(v)\big)\cdot\mathbf{s}\big(h_{i,r}(u)\big)\cdot(u,v)_n^{B(\alpha_{i,r}^\vee,\alpha_{j,r}^\vee)}.$$

Write $Q(\alpha^\vee)$ for the number which is independent of any coroot $\alpha^\vee$. By using the two equalities above, a simple computation gives that

$$\prod_{i=1}^{r-1}\mathbf{s}\big(h_{i,r}(\varpi^{-\delta_i\mathfrak{f}_i}x_i^{-\delta_i})\big)$$
$$=\left(\prod_{i=1}^{r-1}\mathbf{s}\big(h_{i,r}(\varpi^{-\delta_i\mathfrak{f}_i})\big)\right)\cdot\left(\prod_{i=1}^{r-1}\mathbf{s}\big(h_{i,r}(x_i^{-\delta_i})\big)\right)\cdot\prod_{i=1}^{r-1}\prod_{j=1}^{i}(\varpi^{\delta_i\mathfrak{f}_i},x_j^{\delta_j})_n^{-Q(\alpha^\vee)}$$

and therefore

$$\begin{aligned}&f_s\left(\prod_{i=1}^{r-1}\mathbf{s}\big(h_{i,r}(\varpi^{-\delta_i\mathfrak{f}_i}x_i^{-\delta_i})\big)\right)\\ =&f_s\left(\prod_{i=1}^{r-1}\mathbf{s}\big(h_{i,r}(\varpi^{-\delta_i\mathfrak{f}_i})\big)\right)\cdot\prod_{i=1}^{r-1}\prod_{j=1}^{i}(\varpi^{\delta_i\mathfrak{f}_i},x_j^{\delta_j})_n^{-Q(\alpha^\vee)}.\end{aligned}$$

Let $\delta\mathfrak{f}\in\mathbf{Z}^{r-1}\simeq Y_{\mathbf{GL}_{r-1}}$ be the element such that

$$(\delta\mathfrak{f})_i:=\delta_i\mathfrak{f}_i=\min(\mathfrak{f}_i,0)$$

for all $1\le i\le r-1$. Let $\delta\mathfrak{f}^*\in\mathbf{Z}^r\simeq Y$ be the element such that

$$(\delta\mathfrak{f}^*)_i=(\delta\mathfrak{f})_i \text{ for } 1\le i\le r-1 \text{ and } (\delta\mathfrak{f}^*)_r=-\sum_{i=1}^{r-1}(\delta\mathfrak{f})_i.$$

We have

$$\prod_{i=1}^{r-1}\mathbf{s}\big(h_{i,r}(\varpi^{-\delta_i\mathfrak{f}_i})\big)=\mathbf{s}(-\delta\mathfrak{f}^*(\varpi)).$$

As a summary, from (36) we see that

$$\begin{aligned}&\mathrm{Int}(R(\mathfrak{f}))\\ =&\Big(\prod_{i=1}^{r-1}q^{-\delta_i\mathfrak{f}_i(r-i-1)-\mathfrak{f}_i}\Big)\cdot\delta_P^{\frac{s}{r}+\frac{1}{2}}(\mathbf{s}_{-\delta\mathfrak{f}^*})\cdot\mathcal{W}_{\mathcal{O}_0}(\mathbf{s}_{-\delta\mathfrak{f}^*})\\ &\cdot\int_{x_i\in O^\times}\prod_{i=1}^{r-1}\prod_{j=1}^{i}(\varpi^{\delta_i\mathfrak{f}_i},x_j^{\delta_j})_n^{-Q(\alpha^\vee)}\cdot\psi(\varpi^{\mathfrak{f}_1}x_1)\cdot\prod_{j=2}^{r-1}\psi(\varpi^{\delta_{j-1}\mathfrak{f}_j}x_j)dx.\end{aligned}\tag{38}$$

Proposition 5.9. *The integration* $\mathrm{Int}(R(\mathfrak{f}))$ *is zero unless* $\delta\mathfrak{f}=0\in\mathbf{Z}^{r-1}$ *or* $\delta\mathfrak{f}$ *is such that* $(\delta\mathfrak{f})_i=-1$ *for all* $1\le i\le r-1$.

Proof. Note that $\mathcal{W}_{\mathcal{O}_0}(\mathbf{s}_{-\delta\mathfrak{f}^*})=0$ unless $-\delta\mathfrak{f}^*$ is Δ_M-dominant, that is, unless

$$(\delta\mathfrak{f})_1\le(\delta\mathfrak{f})_2\le...\le(\delta\mathfrak{f})_{r-1}.$$

This implies that $\mathcal{W}_\lambda(\mathbf{s}_{-\delta\mathfrak{f}^*})$ vanishes unless there exists some $0\le k\le r-1$ such that

$$\mathfrak{f}_1\le...\le\mathfrak{f}_k<0 \text{ and } \mathfrak{f}_{k+1}\ge...\ge\mathfrak{f}_{r-1}\ge 0.$$

On the other hand, an observation at the formula (38) shows that the integral is zero unless $\mathfrak{f}_1\ge-1$. Therefore, we may assume $1\le k\le r-2$ and that $\mathfrak{f}$ takes the form

$$\mathfrak{f}=(-1,-1,...,-1,\mathfrak{f}_{k+1},...,\mathfrak{f}_{r-1}),\tag{39}$$

where the first k-coordinates are -1, and $\mathfrak{f}_{k+1},...,\mathfrak{f}_{r-1}\ge 0$. Then it suffices to show that $\mathcal{W}_{\mathcal{O}_0}(\mathbf{s}_{-\delta\mathfrak{f}^*})=0$ for such $\mathfrak{f}$.

For $\mathfrak{f}$ in (39), we have that $\delta_i = 1$ if $1 \le i \le k$ and $\delta_i = 0$ for $k+1 \le i \le r-1$. Therefore

$$-\delta\mathfrak{f}^* = (\underbrace{1, 1, ..., 1}_{k}, \underbrace{0, 0, ..., 0}_{r-1-k}, -k) \in \mathbf{Z}^r \simeq Y,$$

which gives

$$\mathbb{w}_0(-\delta\mathfrak{f}^*) = (\underbrace{0, 0, ..., 0}_{r-1-k}, \underbrace{1, 1, ..., 1}_{k}, -k) \in Y.$$

By Proposition 3.6 and the defining property of $\mathbf{c}_{\mathcal{O}_0}$ in (21), we deduce that $\mathcal{W}_{\mathcal{O}_0}(\mathbf{s}_{-\delta\mathfrak{f}^*}) = 0$ unless $\mathbb{w}_0(-\delta\mathfrak{f}^*)$ lies in $\mathcal{O}_0 + Y_{Q,n}$. By Lemma 5.4, we see that this is not possible if $1 \le k \le r-2$. This completes the proof. □

Thus, to compute (35), we only need to consider two cases in the above Proposition.

First, let

$$\mathscr{P} = \left\{ \mathfrak{f} \in \mathbf{Z}^{r-1} : \mathfrak{f}_i \ge 0 \text{ for all } i \right\} \subset \mathbf{Z}^{r-1}.$$

Then it follows from (38) that

$$\begin{aligned} &\sum_{\mathfrak{f} \in \mathscr{P}} \mathrm{Int}(R(\mathfrak{f})) \\ &= \sum_{\mathfrak{f} \in \mathscr{P}} q^{-\sum_i^{r-1} \mathfrak{f}_i} \cdot \int_{x_i \in O^\times} \psi(x_i) dx = (1-q^{-1})^{r-1} \cdot \sum_{\mathfrak{f} \in \mathscr{P}} q^{-\sum_i^{r-1} \mathfrak{f}_i} = 1. \end{aligned} \tag{40}$$

Now for the second case, we have $\mathfrak{f} = (-1, -1, ..., -1) \in \mathbf{Z}^{r-1}$ with all coordinates equal to 1. This gives that

$$-\delta\mathfrak{f} = (\underbrace{1, 1, ..., 1}_{r-1}, -(r-1)) \in \mathbf{Z}^r.$$

It then follows from (38) that in this case

$$
\begin{aligned}
&\mathrm{Int}(R(\mathfrak{f}))\\
=&\Big(\prod_{i=1}^{r-1} q^{r-i}\Big) \cdot \delta_P^{\frac{s}{r}+\frac{1}{2}}(\mathbf{s}_{-\delta\mathfrak{f}^*}) \cdot \mathcal{W}_{\mathcal{O}_0}(\mathbf{s}_{-\delta\mathfrak{f}^*})\\
&\qquad \cdot \int_{x_i\in O^\times} \prod_{i=1}^{r-1}\prod_{j=1}^{i} (\varpi^{-1}, x_j)^{-Q(\alpha^\vee)} \cdot \prod_{j=1}^{r-1} \psi(\varpi^{-1}x_j)dx\\
=&q^{-(r-1)s} \cdot \mathcal{W}_{\mathcal{O}_0}(\mathbf{s}_{-\delta\mathfrak{f}^*}) \int_{x_i\in O^\times} \prod_{j=1}^{r-1}\prod_{i=j}^{r-1} (\varpi, x_j)_n^{Q(\alpha^\vee)} \cdot \prod_{j=1}^{r-1} \psi(\varpi^{-1}x_j)dx\\
=&q^{-(r-1)s} \cdot \mathcal{W}_{\mathcal{O}_0}(\mathbf{s}_{-\delta\mathfrak{f}^*}) \cdot \prod_{j=1}^{r-1} \Big(\int_{x_i\in O^\times} (\varpi, x_j)_n^{(r-j)Q(\alpha^\vee)} \cdot \psi(\varpi^{-1}x_i)dx_j\Big)\\
=&q^{-(r-1)s} \cdot \mathcal{W}_{\mathcal{O}_0}(\mathbf{s}_{-\delta\mathfrak{f}^*}) \cdot \prod_{j=0}^{n_\alpha-1} \mathbf{g}_{\psi^{-1}}(j\cdot Q(\alpha^\vee))\\
=&q^{-(r-1)s} \cdot \delta_{B_M}^{1/2}(\mathbf{s}_{-\delta\mathfrak{f}^*}) \cdot \mathbf{c}_{\mathcal{O}_0}(\mathbf{s}_{\mathbb{w}_0(-\delta\mathfrak{f}^*)}) \cdot \prod_{j=0}^{n_\alpha-1} \mathbf{g}_{\psi^{-1}}(j\cdot Q(\alpha^\vee))\\
=&q^{-(r-1)s} \cdot \mathbf{c}_{\mathcal{O}_0}(\mathbf{s}_{-\delta\mathfrak{f}^*}) \cdot \prod_{j=0}^{n_\alpha-1} \mathbf{g}_{\psi^{-1}}(j\cdot Q(\alpha^\vee)),
\end{aligned}
$$

where the second last equality follows from Proposition 3.6. It is easy to see that

$$-\delta\mathfrak{f} = \mathbb{w}_{\alpha_{r-2}}...\mathbb{w}_{\alpha_2}\mathbb{w}_{\alpha_1}[0] + (r-1)\cdot\alpha_{r-1}^\vee.$$

We also note the equality $\mathbf{s}_{n_\alpha \alpha_{r-1}^\vee} = \overline{h}_{\alpha_{r-1}}(\varpi)^{n_\alpha} = \overline{h}_{\alpha_{r-1}}(\varpi^{n_\alpha})$. Therefore, it follows that

$$
\begin{aligned}
&\mathbf{c}_{\mathcal{O}_0}(\mathbf{s}_{-\delta\mathfrak{f}})\\
=&\mathbf{c}_{\mathcal{O}_0}\big(\mathbf{s}_{\mathbb{w}_{\alpha_{r-2}}...\mathbb{w}_{\alpha_2}\mathbb{w}_{\alpha_1}[0]}\big) \cdot \chi\big(\overline{h}_{\alpha_{r-1}}(\varpi^{n_\alpha})\big)\\
=&\Big(\prod_{i=1}^{r-2} \mathbf{t}(\mathbb{w}_{\alpha_i}, \mathbb{w}_{\alpha_{i-1}}...\mathbb{w}_{\alpha_1}[0])\Big) \cdot \cdot\chi\big(\overline{h}_{\alpha_{r-1}}(\varpi^{n_\alpha})\big)\\
=&\Big(\prod_{i=1}^{r-2} q^{\lceil \frac{-(i-1)}{n_\alpha}\rceil -1} \cdot \mathbf{g}_{\psi^{-1}}(-iQ(\alpha^\vee))^{-1}\Big) \cdot \chi\big(\overline{h}_{\alpha_{r-1}}(\varpi^{n_\alpha})\big)\\
=&q^{-(r-2)} \cdot \chi\big(\overline{h}_{\alpha_{r-1}}(\varpi^{n_\alpha})\big) \cdot \prod_{i=1}^{n_\alpha-1} \mathbf{g}_{\psi^{-1}}(-iQ(\alpha^\vee))^{-1}\\
=&q^{-(r-2)} \cdot \chi\big(\overline{h}_{\alpha_{r-1}}(\varpi^{n_\alpha})\big) \cdot \prod_{j=1}^{n_\alpha-1} \mathbf{g}_{\psi^{-1}}(jQ(\alpha^\vee))^{-1}.
\end{aligned}
$$

Hence, in the second case,

$$\begin{aligned}\operatorname{Int}(R(\mathfrak{f})) =&q^{-(r-1)s} \cdot \mathbf{c}_{\mathcal{O}_0}(\mathbf{s}_{-\delta\mathfrak{f}}) \cdot \prod_{j=0}^{n_\alpha - 1} \mathbf{g}_{\psi^{-1}}(j \cdot Q(\alpha^\vee)) \\ =&q^{-(r-1)s} \cdot q^{-(r-2)} \cdot \chi\Big(\overline{h}_{\alpha_{r-1}}(\varpi^{n_\alpha})\Big) \cdot \mathbf{g}_{\psi^{-1}}(0) \\ =& - q^{-(r-1)(s+1)} \cdot \chi\Big(\overline{h}_{\alpha_{r-1}}(\varpi^{n_\alpha})\Big).\end{aligned} \tag{41}$$

By combining (40) and (41), we get

Proposition 5.10. *Let $f_s \in I(s, \Theta(\chi))$ be the normalized unramified vector. Then*

$$\mathcal{W}_{f_s}^{\mathrm{GL}_r}(1) = L\Big((r-1)(s+1), \chi_{\alpha_{r-1}}\Big)^{-1}.$$

We note that the above result also follows from [Cai20, Theorem 8.1] with a proper interpretation of the result there. See [Cai20, Remark 8.2 (3)].

5.5 An interpretation

We would like to interpret the Hecke L-function $L(s, \chi_{\alpha_{r-1}})$ on the dual side. For a cover $(n, \overline{\mathbf{GL}}_r)$ and the cover $\overline{\mathbf{T}}$ obtained from restriction, one has the (local) compatible L-group extensions

$$\begin{array}{ccccc} \overline{G}^\vee & \hookrightarrow & {}^L\overline{G} & \twoheadrightarrow & W_F \\ \uparrow & & \uparrow & & \| \\ \overline{T}^\vee & \hookrightarrow & {}^L\overline{T} & \twoheadrightarrow & W_F. \end{array}$$

Let $\overline{B}^\vee = \overline{T}^\vee \overline{U}^\vee \subset \overline{G}^\vee$ be the Borel subgroup associated to the simple roots $\Delta_{Q,n}^\vee$ of $\overline{G}^\vee$. Let $\mathrm{Lie}(\overline{U}^\vee)$ be the Lie algebra of $\overline{U}^\vee$. Let

$$Ad : {}^L\overline{T} \to GL(\mathrm{Lie}(\overline{U}^\vee))$$

be the adjoint representation. The space $\mathrm{Lie}(\overline{U}^\vee)$ is spanned by eigenvectors for Ad denoted by $E_{\alpha_{Q,n}^\vee}, \alpha \in \Phi^+$. For any $\alpha \in \Phi^+$, the one-dimensional space $\mathbf{C} \cdot E_{\alpha_{Q,n}^\vee}$ is invariant under Ad and thus we have a character

$$Ad_\alpha : {}^L\overline{T} \to GL(\mathbf{C} \cdot E_{\alpha_{Q,n}^\vee}).$$

Now let $\chi : Z(\overline{T}) \to \mathbf{C}^\times$ be an unramified genuine character and $i(\chi)$ the irreducible representation of $\overline{T}$. By the local Langlands correspondence for covering torus (see [Wei18, §10] or [GG18, §8]), we have an associated splitting of ${}^L\overline{T}$ over W_F:

$$\rho_\chi : W_F \to {}^L\overline{T}.$$

This gives rise to an Artin representation $Ad_\alpha \circ \rho_\chi : W_F \to \mathbf{C}^\times$. Let Frob $\in W_F$ be the geometric Frobenius class in W_F. We define the local Artin L-function as

$$L(s, Ad_\alpha \circ \rho_\chi) = \left(1 - q^{-s} \cdot \rho_\chi \circ Ad_\alpha(\mathrm{Frob})\right)^{-1}.$$

It follows from [Gao18a, Theorem 7.8] that

$$\rho_\chi \circ Ad_\alpha(\mathrm{Frob}) = \chi(\overline{h}_\alpha(\varpi^{n_\alpha}))$$

and therefore

$$L(s, \chi_\alpha) = L(s, Ad_\alpha \circ \rho_\chi). \tag{42}$$

In particular, it applies to the case $\alpha = \alpha_{r-1}$. We note that if χ is an Δ_M-exceptional character, then for all $1 \le i \le r-2$,

$$L(s, \chi_{\alpha_i}) = L(s, Ad_{\alpha_i} \circ \rho_\chi) = \zeta(s+1). \tag{43}$$

In view of this, one may also give another interpretation of $L(s, \chi_{\alpha_{r-1}})$ as follows. Let $\overline{M}^\vee$ and ${}^L\overline{M}$ be the dual and L-group for the covering Levi subgroup $(n, \overline{\mathbf{M}})$ respectively. Let $\overline{N}^\vee \subset \overline{G}^\vee$ be the unipotent subgroup generated by $\{E_{\alpha^\vee_{i,Q,n}} : 1 \le i \le r-1\}$. Then $\overline{M}^\vee \overline{N}^\vee$ is a parabolic subgroup of $\overline{G}^\vee$, and the adjoint representation $Ad_M : {}^L\overline{M} \to GL(\mathrm{Lie}(\overline{N}^\vee))$ is irreducible. From the natural inclusion ${}^L\overline{T} \hookrightarrow {}^L\overline{M}$ (arising from the construction of L-groups), one has an unramified representation

$$Ad_M \circ \rho_\chi : W_F \to {}^L\overline{T} \hookrightarrow {}^L\overline{M} \to GL(\mathrm{Lie}(\overline{N}^\vee)).$$

Now suppose χ is Δ_M-exceptional. It then follows from (42) and (43) that

$$L(s, \chi_{\alpha_{r-1}}) = \frac{L(s, Ad_M \circ \rho_\chi)}{\zeta(s+1)^{r-2}}, \tag{44}$$

where $L(s, Ad_M \circ \rho_\chi)$ denotes the Artin L-function for $Ad_M \circ \rho_\chi$.
Since the above discussion is for unramified characters, it could be globalised to give global partial L-functions.

5.6 Main result

The main result in this section thus follows immediately from combining Proposition 4.1 and Proposition 5.10. We restore the global notations and give a summary as follows.

Theorem 5.11. *Let $(n, \overline{\mathbf{GL}}_r)$ be a nice cover over F. Let ψ be a nontrivial character of $\mathbb{A}/F$. Let $\Theta(\chi)$ be the global theta representation of $\overline{M}_\mathbb{A}$ associated with an exceptional character χ for $\overline{M}_\mathbb{A}$. Let $S \subset |F|$ be a finite set of places such that: (i) S contains the archimedean places and $|n|_v = 1$ for all $v \in |F| - S$; (ii) χ_v and ψ_v are both unramified outside S. Let $E(g, f_s, \Theta(\chi))$ be* *the Eisenstein series on* $\overline{\mathrm{GL}}_{r,\mathbb{A}}$ *associated with* $I(s,\Theta(\chi))$*. Assume* $\mu_{2n} \subset F^\times$*. Then*

$$E_\psi(1, f_s, \Theta(\chi)) = L^S\left((r-1)(s+1), \chi_{\alpha_{r-1}}\right)^{-1} \cdot \prod_{v\in S} \mathcal{W}^{\mathrm{GL}_r}_{f_{s,v}}(1),$$

where $L^S(s,\chi_{\alpha_{r-1}}) = \prod_{v\notin S} L(s,(\chi_{\alpha_{r-1}})_v)$ *is the partial Hecke L-function attached to* $\chi_{\alpha_{r-1}}$*.*

The above formula for $E_\psi(1, f_s, \Theta(\chi))$ is expected to be the same with just the assumption $\mu_n \subset F^\times$.

EXAMPLE 5.12. Let $\mathbf{p} = 0, \mathbf{q} = 1$. In this case, $n = n_\alpha$. Let $r = n+1$. This is the case of n-fold cover of $\overline{\mathrm{GL}}_{n+1,\mathbb{A}}$ with twisting parameter $c = 0$ (in the notation of [KP84]). Then Theorem 5.11 yields that

$$E_\psi(1, f_s, \Theta(\chi)) = L^S\left(n(s+1), \chi_{\alpha_n}\right)^{-1} \cdot \prod_{v\in S} \mathcal{W}^{\mathrm{GL}_r}_{f_{s,v}}(1).$$

In particular, for $n = 1$, $\Theta(\chi)$ is just the linear character χ of $T_\mathbb{A} \subset \mathrm{GL}_{2,\mathbb{A}}$. Then the formula for $E_\psi(1, f_s, \chi)$ is just the Casselman–Shalika formula for $\mathrm{GL}_{2,\mathbb{A}}$.

Besides the linear $\mathrm{GL}_{2,\mathbb{A}}$ case in the above example, there is another rank-one example arising from the Savin's class of extension of $\mathbf{GL}_r$ (see [Sav] and [Gao18b, §2.1]).

EXAMPLE 5.13. Let $r = 2, n = 2$ and $\mathbf{p} = 1, \mathbf{q} = 0$. The associated $(n = 2, \overline{\mathbf{GL}}_2)$ is a nice cover. Note that the covering torus $\overline{T}_\mathbb{A}$ is abelian in this case, which however does not split over $T_\mathbb{A}$, since $\overline{T}_v$ does not split over T_v for general places $v \in |F|$. Therefore, the covering group

$$\mu_2 \hookrightarrow \overline{\mathrm{GL}}_{2,\mathbb{A}} \twoheadrightarrow \mathrm{GL}_{2,\mathbb{A}}$$

is nontrivial. To give another description of $\overline{\mathrm{GL}}_{2,\mathbb{A}}$, we consider the extension $\overline{\mathbf{GL}}_1$ of $\mathbf{GL}_1$ determined by $Q(e) = \mathbf{p} = 1$. Let $\overline{\mathrm{GL}}_{1,\mathbb{A}}$ be the double cover, which is then abelian. Then $\overline{\mathrm{GL}}_{2,\mathbb{A}}$ is the pull-back from $\overline{\mathrm{GL}}_{1,\mathbb{A}}$ via the determinant map $\mathrm{GL}_{2,\mathbb{A}} \to \mathrm{GL}_{1,\mathbb{A}}$. In view of this, there is an automorphic genuine character of $\overline{\mathrm{GL}}_{2,\mathbb{A}}$. Therefore, representation theory of $\overline{\mathrm{GL}}_{2,\mathbb{A}}$ can be reduced to that of $\mathrm{GL}_{2,\mathbb{A}}$.
In any case, let χ be a genuine character of $\overline{T}_\mathbb{A}$, and χ_α the associated linear character of $\mathbb{A}/F^\times$. Here $\alpha = \alpha_1$ is the unique simple root with $Q(\alpha^\vee) = 1$ and $n_\alpha = 1$. Then Theorem 5.11 gives that

$$E_\psi(1, f_s, \chi) = L^S\left(s+1, \chi_\alpha\right)^{-1} \cdot \prod_{v\in S} \mathcal{W}^{\mathrm{GL}_2}_{f_{s,v}}(1).$$

Remark 5.14. The class of nice covers $(n, \overline{\mathbf{GL}}_r)$ is singled out in this section for the sole reason that we are considering parabolic subgroup with Levi subgroup $\mathbf{GL}_r \times \mathbf{GL}_1$. However, one may consider other maximal parabolic subgroup with Levi subgroup $\mathbf{M} = \mathbf{GL}_{r_1} \times \mathbf{GL}_{r_2}$ such that $r_1 + r_2 = n$. For instance, if $r = 2n$, we could consider the Kazhdan–Patterson cover $(n, \overline{\mathbf{GL}}_{2n})$ and the n-fold covering group $\overline{M}_{\mathbb{A}}$ with $\mathbf{M} = \mathbf{GL}_n \times \mathbf{GL}_n$. Then the analogue of Proposition 5.7 holds, i.e., theta representation of $\overline{M}_{\mathbb{A}}$ is distinguished in this case. This example thus fits into the set-up in §3 and §4, and results there apply. In fact, it follows from [Suz97, §7.6] that in this case, the Fourier coefficients will involve n Hecke L-functions associated to the exceptional character.
Moreover, as mentioned in Remark 4.2, one could consider general parabolic subgroups. For example, consider the Kazhdan–Patterson cover $(n, \overline{\mathbf{GL}}_{3n})$. Let $\mathbf{P} = \mathbf{MN}$ with $\mathbf{M} = \mathbf{GL}_n \times \mathbf{GL}_n \times \mathbf{GL}_n$. Any theta representation of the n-fold cover $\overline{M}_{\mathbb{A}}$ is also distinguished. However, the Fourier coefficients of Eisenstein series, which have global factorization as well, are presumably even more difficult to compute than the maximal parabolic case.

6 Covering groups of Sp_{2r}

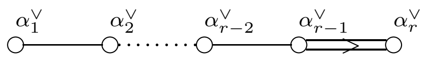
In this section, we will consider covers of $\mathbf{Sp}_{2r}$. For the computation of the Fourier coefficients, we will invoke Theorem 4.5. However, we will eventually concentrate on covers of $\mathbf{Sp}_4$ for combinatorial difficulties.

6.1 $\mathbf{K}_2$-extensions and covering groups

Let

$$(X, \Phi, \Delta; Y, \Phi^\vee, \Delta^\vee)$$

be the root datum of $\mathbf{Sp}_{2r}$. Consider the Dynkin diagram for the simple coroots for $\mathbf{Sp}_{2r}$:



Thus, $\alpha_r^\vee$ is the unique short simple coroot. Let $Y = Y^{\mathrm{sc}}$ be the cocharacter lattice of $\mathbf{Sp}_{2r}$ generated by $\Delta^\vee = \langle \alpha_1^\vee, \alpha_2^\vee, ..., \alpha_{r-1}^\vee, \alpha_r^\vee \rangle$. The isomorphism class of $\mathbf{K}_2$-extensions $\overline{\mathbf{Sp}}_{2r}$ is determined by the Weyl-invariant quadratic form Q on Y. Let Q be the unique Weyl-invariant quadratic form on Y such such $Q(\alpha_r^\vee) = -1$. Then the bilinear form B_Q is given by

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} -2 & \text{if } i = j = r; \\ -4 & \text{if } 1 \le i = j \le r-1; \\ 2 & \text{if } j = i+1; \\ 0 & \text{if } \alpha_i^\vee, \alpha_j^\vee \text{ are not adjacent.} \end{cases}$$

Let $\mathbf{P} = \mathbf{MN}$ be the Siegel parabolic subgroup associated to $\Delta_M := \Delta \backslash \{\alpha_r^\vee\}$. We have $\mathbf{M} \simeq \mathbf{GL}_r$. The restriction of $\overline{\mathbf{Sp}}_{2r}$ to $\mathbf{M}$ gives rise to the extension $\overline{\mathbf{GL}}_r$ associated with $\mathbf{p} = -1$ and $\mathbf{q} = 0$. This extension $\overline{\mathbf{GL}}_r$ does not belong to the Kazhdan–Patterson class (since $2\mathbf{p} - \mathbf{q} = -2$ in this case); however, it was already studied by Savin [Sav].

For a general cover $(n, \overline{\mathbf{Sp}}_{2r})$, define

$$n_0 = \begin{cases} n & \text{if } n \text{ is odd,} \\ n/2 & \text{otherwise.} \end{cases}$$

A simple computation for (8) gives that

$$Y_{Q,n} = n_0 \cdot Y^{sc} = \left\{ \sum_{i=1}^{r} k_i \alpha_i^\vee : n_0 | k_i \right\}.$$

Again, since $Y_{Q,n,\mathbf{Sp}_{2r}} = Y_{Q,n,\mathbf{M}}$, we have omitted the subscript in $Y_{Q,n}$. On the other hand,

$$Y_{Q,n,\mathbf{M}}^{sc} = n_0 \cdot Y_{\mathbf{M}}^{sc}.$$

Consider W_M-free orbits in Y. Let $\wp : Y \twoheadrightarrow Y/Y_{Q,n}$ and $\wp : Y \twoheadrightarrow Y/Y_{Q,n,\mathbf{M}}^{sc}$ be the quotient maps. Consider the $Y_{Q,n}$-free and $Y_{Q,n,\mathbf{M}}^{sc}$-free orbits defined exactly as in §2.3. Let $\mathcal{O}_{Q,n}^F$ (resp. $\mathcal{O}_{Q,n,sc}^F$) be the set of free orbits in Y which are also $Y_{Q,n}$-free (resp. $Y_{Q,n,\mathbf{M}}^{sc}$-free).

Definition 6.1. An n-fold cover $(n, \overline{\mathbf{Sp}}_{2r})$ is called nice if the following conditions hold:

- if r is odd, then $n = r$ or $2r$;
- if r is even, then $n = 2r$.

Lemma 6.2. *For a general cover $(n, \overline{\mathbf{Sp}}_{2r})$, we have $Y_{Q,n} \cap Y_{\mathbf{M}}^{sc} = Y_{Q,n,\mathbf{M}}^{sc}$ and therefore $\mathcal{O}_{Q,n}^F = \mathcal{O}_{Q,n,sc}^F$. Moreover, if $(n, \overline{\mathbf{Sp}}_{2r})$ is a nice cover, then $\left|\wp(\mathcal{O}_{Q,n}^F)\right| = 1$.*

Proof. This follows from [Gao18b, Proposition 3.5] by noting that $Y_{Q,n} = n_0 \cdot Y_{\mathbf{M}}$ for any cover $(n, \overline{\mathbf{Sp}}_{2r})$, and moreover $n_0 = r$ if $(n, \overline{\mathbf{Sp}}_{2r})$ is nice. □

6.2 Fourier coefficient of Eisenstein series

Let $\overline{\mathrm{Sp}}_{2r,\mathbb{A}}$ and $\overline{M}_{\mathbb{A}}$ be the n-fold covering groups arising from $\overline{\mathbf{Sp}}_{2r}$ and $\overline{\mathbf{M}}$. In fact, such covers (of semisimple and simply-connected groups) were already studied earlier by Moore [Moo68] , Steinberg [Ste62], Matsumoto [Mat69] and others.

Let χ be a Δ_M-exceptional automorphic character of $Z(\overline{T}_{\mathbb{A}})$. That is,

$$\chi_{\alpha_i} = |\cdot|_{\mathbb{A}}$$

for all $1 \le i \le r-1$. Let $\Theta(\overline{M}_{\mathbb{A}}, \chi) = \otimes_v \Theta(\overline{M}_v, \chi_v)$ be the theta representation of $\overline{M}_{\mathbb{A}}$ associated to χ.

Proposition 6.3. *If $(n, \overline{\mathbf{Sp}}_{2r})$ is a nice cover, then the theta representation $\Theta(\overline{M}_{\mathbb{A}}, \chi)$ is ψ-distinguished.*

Proof. The proof is the same as Proposition 5.7. Note that the essential ingredient used is that for v such that $|n|_v = 1$, we have

$$\dim \mathrm{Wh}_{\psi_v}(\Theta(\overline{M}_v, \chi_v)) = 1,$$

which follows from [Gao17, Theorem 3.14] and Lemma 6.2. □

Let $(n, \overline{\mathbf{Sp}}_{2r})$ be a nice cover. Let $I(s, \Theta(\overline{M}_{\mathbb{A}}, \chi)) = \mathrm{Ind}_{\overline{P}_{\mathbb{A}}}^{\mathrm{Sp}_{4,\mathbb{A}}}\big(|\det(\cdot)|_{\mathbb{A}}^{s} \cdot \Theta(\overline{M}_{\mathbb{A}}, \chi) \otimes \mathbf{1}\big)$ be the induced representation and $E(s, f_s, \Theta(\chi))$ the Eisenstein series from $I(s, \Theta(\overline{M}_{\mathbb{A}}, \chi))$. Note that the parabolic subgroup $\mathbf{P}$ is self-associated, i.e. $\mathbf{P}' = \mathbf{P} = \mathbf{MN}$. It thus follows from Proposition 4.3 that

$$E_\psi(1, f_s, \Theta(\chi)) = \prod_v \mathcal{W}^G_{f_{s,v}}(1)$$

where

$$\mathcal{W}^G_{f_{s,v}}(1) = \int_{N_v} \lambda_v\big(f_{s,v}(\overline{w}_l^{-1} u)\big)\psi_v(u)^{-1} du.$$

We will compute the value $\mathcal{W}^G_{f_{s,v}}(1)$ for almost all v when $r = 2$.

6.3 Local unramified computation

In the rest of this section, we consider the nice cover $(n = 4, \overline{\mathbf{Sp}}_4)$. Denote

$$\alpha_3^\vee := 2\alpha_2^\vee + \alpha_1^\vee, \text{ and } \alpha_4^\vee := \alpha_1^\vee + \alpha_3^\vee.$$

The element $\mathbb{w}_l \in W$ such that $\mathbb{w}_l(\alpha_1^\vee) \in \Delta$ and $\mathbb{w}_l(\alpha_i^\vee) < 0$ for all $i = 2, 3, 4$ is

$$\mathbb{w}_l = \mathbb{w}_{\alpha_2}\mathbb{w}_{\alpha_1}\mathbb{w}_{\alpha_2}.$$

For simplicity, we suppress the subscript v for all notations. Again, F denotes a non-archimedean local field such that $|n| = 1$. Also, χ is an unramified exceptional character and ψ has conductor O_F.
Recall that χ is an unramified character of $Z(\overline{T})$ such that $\chi_{\alpha_1} = |\cdot|$. Note that in this case,

$$w_M = w_{\alpha_1}.$$

By Lemma 4.4, we have

$$\mathcal{W}^M_{1_{\overline{T}}}(1) = \tau(w_M, \chi, \mathbf{s}(0), \mathbf{s}(0)) = \frac{1 - q^{-1}}{1 - \chi\left(\overline{h}_{\alpha_1}(\varpi^{n_\alpha})\right)} = 1,$$

where the last equality follows from the fact that χ is Δ_M-exceptional. It follows from Theorem 4.5 that

$$\mathcal{W}^G_{f_s^0}(1) = \mathcal{W}^G_{1_{\overline{T}}}(1),$$

where $\mathcal{W}^G_{1_{\overline{T}}}$ is the Whittaker function on $\overline{\mathrm{Sp}}_4$ arising from the Whittaker functional $\lambda^G_{1_{\overline{T}}}$ on $I(s\omega_P, {}^{w_M}\chi)$.

6.4 Computation of $\mathcal{W}^G_{1_{\overline{T}}}(1)$

For convenience, we write $\chi' = {}^{w_M}\chi$, which is Δ_M-anti-exceptional. First we note that

$$I(s\omega_P, \chi') \simeq I(\chi'_s)$$

where χ'_s is the genuine character of $Z(\overline{T})$ given by

$$\chi'_s(\overline{t}) = \chi'(\overline{t}) \cdot |\det(t)|^s \text{ for } \overline{t} \in Z(\overline{T}).$$

In particular, χ'_s is Δ_M-anti-exceptional. By Proposition 3.5, for the principal series $I(\chi'_s)$ on $\overline{\mathrm{Sp}}_4$, one has

$$\mathcal{W}^G_{1_{\overline{T}}}(1) = \sum_{\mathbb{w} \in W_G} c_{\mathsf{gk}}(w_G w^{-1}, \chi'_s) \tau(w, {}^{w^{-1}}\chi'_s, 1_{\overline{T}}, 1_{\overline{T}}). \tag{45}$$

From now, we write

$$\mathbb{w}_i := \mathbb{w}_{\alpha_i}$$

for the Weyl element corresponding to the simple coroot $\alpha_i^\vee$ for $i = 1, 2$. Set

$$W^\flat := \{\mathbb{w}_1, \mathbb{w}_1\mathbb{w}_2, \mathbb{w}_1\mathbb{w}_2\mathbb{w}_1, \mathbb{w}_1\mathbb{w}_2\mathbb{w}_1\mathbb{w}_2\} \subset W_G.$$

The longest $\mathbb{w}_G \in W_G$ is just $\mathbb{w}_1\mathbb{w}_2\mathbb{w}_1\mathbb{w}_2$.

Lemma 6.4. *If $\mathbb{w} \in W_G - W^\flat$, then $c_{\mathsf{gk}}(w_G w^{-1}, \chi'_s) = 0$.*

Proof. Assume $\mathbb{w} \in W_G - W^\flat$, then $\alpha_1^\vee \in \Phi_{\mathbb{w}_G \mathbb{w}^{-1}}$. It then follows from (14) that $c_{\mathsf{gk}}(w_1, \chi'_s) = 0$ and therefore $c_{\mathsf{gk}}(w_G w^{-1}, \chi'_s) = 0$. □

By the above Lemma, it suffices to compute the terms in (45) for $\mathbb{w} \in W^\flat$. For this purpose, for any root $\alpha \in \Phi$ of $\mathbf{Sp}_4$, denote

$$\chi'_{s,\alpha} := \chi'_s\left(\overline{h}_\alpha(\varpi^{n_\alpha})\right).$$

There are relations as follows:

$$\chi'_{s,\alpha_4} = q \cdot \chi'_{s,\alpha_3}, \quad \chi'_{s,\alpha_3} = q \cdot \chi'_{s,\alpha_2}. \tag{46}$$

It also follows that $\chi'_{s,\alpha_4} = q^2 \cdot \chi'_{s,\alpha_2}$.
From now on, we assume $\mu_{2n} \subset F^\times$ to simplify the computation. Note that in this case $\xi = (-1, n) = 1$.

6.4.1 For $\mathbb{w} = \mathbb{w}_1$

In this case, one has

$$\begin{aligned}
&c_{\mathsf{gk}}(w_G w_1^{-1}, \chi'_s) \cdot \tau(w_1, {}^{w_1^{-1}}\chi'_s, 1_{\overline{T}}, 1_{\overline{T}}) \\
=&\frac{1-q^{-1}\chi'_{s,\alpha_2}}{1-\chi'_{s,\alpha_2}} \cdot \frac{1-q^{-1}\chi'_{s,\alpha_3}}{1-\chi'_{s,\alpha_3}} \cdot \frac{1-q^{-1}\chi'_{s,\alpha_4}}{1-\chi'_{s,\alpha_4}} \cdot \tau(w_1, {}^{w_1^{-1}}\chi'_s, 1_{\overline{T}}, 1_{\overline{T}}) \\
=&\frac{1-q^{-1}\chi'_{s,\alpha_2}}{1-\chi'_{s,\alpha_4}} \cdot \frac{1-q^{-1}}{1-(\chi'_{s,\alpha_1})^{-1}} \\
=&\frac{1-q^{-1}\chi'_{s,\alpha_2}}{1-\chi'_{s,\alpha_4}}.
\end{aligned}$$

6.4.2 For $\mathbb{w} = \mathbb{w}_1\mathbb{w}_2$

We first compute

$$\begin{aligned}
&\tau(w_1 w_2, {}^{(w_1w_2)^{-1}}\chi'_s, 1_{\overline{T}}, 1_{\overline{T}}) \\
=&\sum_{\gamma \in \overline{T}/\overline{A}} \tau(w_1, {}^{w_1^{-1}}\chi'_s, 1_{\overline{T}}, \gamma) \cdot \tau(w_2, {}^{(w_1w_2)^{-1}}\chi'_s, \gamma, 1_{\overline{T}}) \text{ by (18)} \\
=&\tau(w_1, {}^{w_1^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_0) \cdot \tau(w_2, {}^{(w_1w_2)^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_0) \\
&\quad + \tau(w_1, {}^{w_1^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_{\mathbb{w}_1[0]}) \cdot \tau(w_2, {}^{(w_1w_2)^{-1}}\chi'_s, \mathbf{s}_{\mathbb{w}_1[0]}, \mathbf{s}_0) \\
=&\tau(w_2, {}^{(w_1w_2)^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_0) \\
=&\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_4})^{-1}}.
\end{aligned}$$

It follows that in this case

$$\begin{aligned}
&c_{\mathsf{gk}}(w_G (w_1w_2)^{-1}, \chi'_s) \cdot \tau(w_1w_2, {}^{(w_1w_2)^{-1}}\chi'_s, 1_{\overline{T}}, 1_{\overline{T}}) \\
=&\frac{1-q^{-1}\chi'_{s,\alpha_2}}{1-\chi'_{s,\alpha_2}} \cdot \frac{1-q^{-1}\chi'_{s,\alpha_3}}{1-\chi'_{s,\alpha_3}} \cdot \frac{1-q^{-1}}{1-(\chi'_{s,\alpha_4})^{-1}} \\
=&\frac{1-q^{-1}\chi'_{s,\alpha_2}}{1-\chi'_{s,\alpha_3}} \cdot \frac{1-q^{-1}}{1-(\chi'_{s,\alpha_4})^{-1}} \text{ by (46).}
\end{aligned}$$

6.4.3 For $\mathbb{w} = \mathbb{w}_1\mathbb{w}_2\mathbb{w}_1$

In this case, we obtain

$$\begin{aligned}
&\tau(w_1w_2w_1, {}^{(w_1w_2w_1)^{-1}}\chi'_s, 1_{\overline{T}}, 1_{\overline{T}})\\
=&\sum_{\gamma\in\overline{T}/\overline{A}} \tau(w_1w_2, {}^{(w_1w_2)^{-1}}\chi'_s, 1_{\overline{T}}, \gamma)\cdot \tau(w_1, {}^{(w_1w_2w_1)^{-1}}\chi'_s, \gamma, 1_{\overline{T}})\\
=&\tau(w_1w_2, {}^{(w_1w_2)^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_0)\cdot \tau(w_1, {}^{(w_1w_2w_1)^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_0)\\
&\quad + \tau(w_1w_2, {}^{(w_1w_2)^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_{\mathbb{w}_1[0]})\cdot \tau(w_1, {}^{(w_1w_2w_1)^{-1}}\chi'_s, \mathbf{s}_{\mathbb{w}_1[0]}, \mathbf{s}_0)\\
=&\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_4})^{-1}}\cdot\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_3})^{-1}} + \mathbf{g}_{\psi^{-1}}(Q(\alpha_1^\vee))\cdot\tau(w_1w_2, {}^{(w_1w_2)^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_{\mathbb{w}_1[0]})\\
=&\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_4})^{-1}}\cdot\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_3})^{-1}} + q^{-1}\cdot\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_4})^{-1}}\\
=&\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_4})^{-1}}\cdot\frac{1-q^{-1}(\chi'_{s,\alpha_3})^{-1}}{1-(\chi'_{s,\alpha_3})^{-1}}\\
=&\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_3})^{-1}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&c_{\mathsf{gk}}(w_G(w_1w_2w_1)^{-1}, \chi'_s)\cdot\tau(w_1w_2w_1, {}^{(w_1w_2w_1)^{-1}}\chi'_s, 1_{\overline{T}}, 1_{\overline{T}})\\
=&\frac{1-q^{-1}\chi'_{s,\alpha_2}}{1-\chi'_{s,\alpha_2}}\cdot\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_3})^{-1}}.
\end{aligned}$$

6.4.4 For $\mathbb{w} = \mathbb{w}_G$

In this case, we have $c_{\mathsf{gk}}(w_G(w_G)^{-1}, \chi'_s) = 1$. Now,

$$\begin{aligned}
&\tau(w_G, {}^{w_G^{-1}}\chi'_s, 1_{\overline{T}}, 1_{\overline{T}})\\
=&\sum_{\gamma\in\overline{T}/\overline{A}} \tau(w_1w_2w_1, {}^{(w_1w_2w_1)^{-1}}\chi'_s, 1_{\overline{T}}, \gamma)\cdot\tau(w_2, {}^{w_G^{-1}}\chi'_s, \gamma, 1_{\overline{T}})\\
=&\tau(w_1w_2w_1, {}^{(w_1w_2w_1)^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_0)\cdot\tau(w_2, {}^{w_G^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_0)\\
&\quad + \tau(w_1w_2w_1, {}^{(w_1w_2w_1)^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_{\mathbb{w}_2[0]})\cdot\tau(w_2, {}^{w_G^{-1}}\chi'_s, \mathbf{s}_{\mathbb{w}_2[0]}, \mathbf{s}_0)\\
=&\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_3})^{-1}}\cdot\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_2})^{-1}}\\
&\quad + \mathbf{g}_{\psi^{-1}}(-Q(\alpha_2^\vee))\cdot\tau(w_1w_2w_1, {}^{(w_1w_2w_1)^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_{\mathbb{w}_2[0]}).
\end{aligned}$$

A simple computation as in §6.4.3 gives that

$$\tau(w_1w_2w_1, {}^{(w_1w_2w_1)^{-1}}\chi'_s, \mathbf{s}_0, \mathbf{s}_{\mathbb{w}_2[0]}) = \mathbf{g}_{\psi^{-1}}(Q(\alpha_2^\vee))\cdot\frac{1-q^{-1}\chi'_{s,\alpha_3}}{1-(\chi'_{s,\alpha_3})^{-1}}.$$

It follows that

$$\begin{aligned}&c_{\mathsf{gk}}(w_G(w_G)^{-1},\chi'_s)\cdot\tau(w_G,{}^{w_G^{-1}}\chi'_s,1_{\overline{T}},1_{\overline{T}})\\=&\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_3})^{-1}}\cdot\frac{1-q^{-1}}{1-(\chi'_{s,\alpha_2})^{-1}}+q^{-1}\cdot\frac{1-q^{-1}\chi'_{s,\alpha_3}}{1-(\chi'_{s,\alpha_3})^{-1}}.\end{aligned}$$

Combining the four terms above corresponding to the elements in $W^\flat$, and simplifying the formula, we obtain that

$$\mathcal{W}^G_{1_{\overline{T}}}(1)=1-q^{-1}\chi'_{s,\alpha_2}=1-q^{-4s-3}\cdot\chi(\overline{h}_{\alpha_2}(\varpi^{n_{\alpha_2}}))=L(4s+3,\chi_{\alpha_2})^{-1}.$$

One can give an interpretation of $L(s,\chi_{\alpha_2})$ on the dual side as in §5.5. However, we will omit the details.

6.5 Summary of result

We summarize the discussion above in the following theorem.

Theorem 6.5. *Let $\overline{\mathrm{Sp}}_{4,\mathbb{A}}$ be the 4-fold cover determined by $Q(\alpha_2^\vee)=-1$. Let ψ be a nontrivial character of $\mathbb{A}/F$. Let $\Theta(\chi)$ be the global theta representation of the covering Siegel Levi subgroup $\overline{M}_{\mathbb{A}}$ associated with an exceptional character χ. Let $S\subset|F|$ be a finite set of places such that: (i) S contains the archimedean places and $|n|_v=1$ for all $v\in|F|-S$; (ii) χ_v and ψ_v are both unramified outside S. Let $E(g,f_s,\Theta(\chi))$ be the Eisenstein series on $\overline{\mathrm{Sp}}_{4,\mathbb{A}}$ associated with $I(s,\Theta(\chi))$. Assume $\mu_{2n}\subset F^\times$. Then*

$$E_\psi(1,f_s,\Theta(\chi))=L^S\left(4s+3,\chi_{\alpha_2}\right)^{-1}\cdot\prod_{v\in S}\mathcal{W}^G_{f_{s,v}}(1),$$

where $L^S(s,\chi_{\alpha_2})=\prod_{v\notin S}L(s,(\chi_{\alpha_2})_v)$ is the partial Hecke L-function associated to χ_{α_2}.

Remark 6.6. For r an even number, we could consider the $2r$-fold cover of $\overline{\mathrm{Sp}}_{2r,\mathbb{A}}$ associated to $Q(\alpha^\vee)=-1$ for any short coroot $\alpha^\vee$. Let $\Theta(\overline{\mathrm{GL}}_{r,\mathbb{A}},\chi)$ be a theta representation of the Siegel Levi subgroup $\overline{\mathrm{GL}}_{r,\mathbb{A}}$. It is expected that Theorem 6.5 generalizes to this case. That is, $E_\psi(1,f_s,\Theta(\chi))$ can be expressed as the reciprocal of the L-function $L(s,\chi_{\alpha_r})$.

For r an odd number, we could consider both r-fold cover $\overline{\mathrm{Sp}}^{(r)}_{2r,\mathbb{A}}$ and $2r$-fold cover $\overline{\mathrm{Sp}}^{(2r)}_{2r,\mathbb{A}}$. For both coverings, the theta representation of $\overline{\mathrm{GL}}_{r,\mathbb{A}}$ is distinguished. However, it is expected that in the formula for $E_\psi(1,f_s,\Theta(\chi))$, the reciprocal of a single L-function appears in the $\overline{\mathrm{Sp}}^{(r)}_{2r,\mathbb{A}}$ case, while a quotient of two Hecke L-functions in the case of $\overline{\mathrm{Sp}}^{(2r)}_{2r,\mathbb{A}}$. Indeed, this dichotomy already appears in the case $r=1$. For the double cover $\overline{\mathrm{SL}}_{2,\mathbb{A}}$, one has (see [Szp09])

$$E_\psi(1,f_s,\chi)=\frac{L^S(s+\frac{1}{2},\chi^\flat)}{L^S(2s+1,\chi_\alpha)}\cdot\prod_{v\in S}\mathcal{W}^G_{f_{s,v}}(1).$$

Here the linear character $\chi_v^\flat$ for $v \notin S$ is given by

$$\chi_v^\flat(u) = \chi_v(\overline{h}_\alpha(u)) \cdot \gamma_\psi(u),$$

where $\gamma_\psi(u) \in \mu_4$ is the Weil-factor. Also $\chi_\alpha(u) = \chi_v(\overline{h}_\alpha(u^2))$.

REMARK 6.7. We could consider the cover $(n = 2, \overline{\mathbf{Sp}_4})$ and the non-Siegel parabolic subgroup $\mathbf{P} = \mathbf{MN}$ with $\mathbf{M} = \mathbf{GL}_1 \times \mathbf{SL}_2$. Note that $\overline{M}_\mathbb{A} \simeq \overline{\mathrm{GL}}_{1,\mathbb{A}} \times_{\mu_2} \overline{\mathrm{SL}}_{2,\mathbb{A}}$ in this case. Moreover, the covering torus $\overline{T}_\mathbb{A} \subset \overline{\mathrm{Sp}}_{4,\mathbb{A}}$ is abelian and $\overline{T}_\mathbb{A} = \overline{\mathrm{GL}}_{1,\mathbb{A}} \times_{\mu_2} \overline{T}_{\mathrm{SL}_2,\mathbb{A}}$. Therefore, a genuine character χ of $\overline{T}_\mathbb{A}$ is of the form

$$\chi = \xi \otimes \mu,$$

where ξ is a genuine character of $\overline{\mathrm{GL}}_{1,\mathbb{A}}$ and μ for $\overline{T}_{\mathrm{SL}_2,\mathbb{A}}$. The character χ is exceptional if and only if μ is exceptional.
For fixed ψ, there is a ψ-distinguished theta representation $\Theta(\overline{M}_\mathbb{A}, \chi) \simeq \xi \otimes \Theta(\overline{\mathrm{SL}}_{2,\mathbb{A}}, \mu)$. One could consider the Fourier coefficients $E_\psi(1, f_s, \Theta(\overline{M}_\mathbb{A}, \chi))$. Then it follows from [Szp09] that

$$E_\psi(1, f_s, \Theta(\chi)) = \frac{L^S(s + \frac{1}{2}, \xi^\flat)}{L^S(2s+1, \xi_\alpha) L_\psi^S(s+1, \xi \times \Theta(\mu))} \cdot \prod_{v \in S} \mathcal{W}_{f_{s,v}}^{\mathrm{Sp}_4}(1),$$

where $\Theta(\mu) := \Theta(\overline{\mathrm{SL}}_{2,\mathbb{A}}, \mu)$ is the theta representation for $\overline{\mathrm{SL}}_{2,\mathbb{A}}$ and $\xi^\flat$ is as in the preceding remark.

The above consideration shows that the "pattern" of L-functions that could appear in $E_\psi(1, f_s, \Theta(\chi))$ is a delicate issue. However, we hope that it could be predicted from a unified solution to the combinatorial problem arising from Theorem 4.5.

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Fan Gao
School of Mathematical Sciences
Yuquan Campus
Zhejiang University
Hangzhou
China 310027
gaofan@zju.edu.cn