On the Class of Canonical Systems Corresponding to Matrix String Equations: General-Type and Explicit Fundamental Solutions and Weyl–Titchmarsh Theory

Alexander Sakhnovich

Received: January 18, 2021
Revised: May 20, 2021

Communicated by Heinz Siedentop

Abstract. Using linear similarity of a certain class of Volterra operators to the squared integration, we derive an important representation of the general-type fundamental solutions of the canonical systems corresponding to matrix string equations. Explicit fundamental solutions of such canonical systems are constructed (via the GBDT version of Darboux transformation) as well. Examples and applications to dynamical canonical systems are given. Explicit solutions of the dynamical canonical systems are constructed. Three appendices are dedicated to the Weyl–Titchmarsh theory for canonical systems, to the transformation of a subclass of canonical systems into matrix string equations (and of a smaller subclass of canonical systems into matrix Schrödinger equations), and to a linear similarity problem for Volterra operators.

2020 Mathematics Subject Classification: 34A05, 34B20, 34L40, 37J06, 46N20, 81Q05
Keywords and Phrases: Canonical system, matrix string equation, dynamical canonical system, fundamental solution, Volterra operator, Darboux matrix, explicit generalised eigenfunction

1 Introduction

Canonical (spectral canonical) systems have the form

$$w'(x,\lambda)=\mathrm{i}\lambda JH(x)w(x,\lambda), \quad J:=\begin{bmatrix}0 & I_p\\ I_p & 0\end{bmatrix} \quad \Big(w':=\frac{d}{dx}w\Big), \tag{1.1}$$

where i is the imaginary unit ($\mathrm{i}^2=-1$), λ is the so called spectral parameter, I_p is the $p\times p$ ($p\in\mathbb{N}$) identity matrix, $\mathbb{N}$ stands for the set of positive integer numbers, $H(x)$ is a $2p\times 2p$ matrix valued function (matrix function), and

$H(x) \geq 0$ (that is, the matrices $H(x)$ are self-adjoint and the eigenvalues of $H(x)$ are nonnegative). Canonical systems are important objects of analysis, being perhaps the most important class of the one-dimensional Hamiltonian systems and including (as subclasses) several classical equations. They have been actively studied in many already classical as well as in various recent works (see, e.g., [2, 9, 14, 16, 23, 24, 36–40, 45, 50, 55, 57] and numerous references therein). We will also consider (and construct explicit solutions) for a more general class

$$w'(x,\lambda) = \mathrm{i}\lambda j H(x) w(x,\lambda), \quad H(x) = H(x)^*, \quad j := \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \tag{1.2}$$

where $m_1, m_2 \in \mathbb{N}$. Here, we set

$$m_1 + m_2 =: m,$$

H is an $m \times m$ locally integrable matrix function, and $H(x)^*$ means the complex conjugate transpose of the matrix $H(x)$. System (1.2) will be called a *generalised canonical system* and the corresponding matrix function H will be called a *generalised Hamiltonian*.

In the case $m_1 = m_2 =: p$, it is easily checked that j and J are unitarily similar:

$$J = \Theta j \Theta^*, \quad \Theta := \frac{1}{\sqrt{2}} \begin{bmatrix} I_p & -I_p \\ I_p & I_p \end{bmatrix}, \tag{1.3}$$

that is (assuming $H(x) \geq 0$), system (1.2) is equivalent to (1.1) (see Appendix B for details). We call the system

$$w'(x,\lambda) = \mathrm{i}\lambda j H(x) w(x,\lambda), \quad H(x) \geq 0 \quad (m_1 = m_2 = p) \tag{1.4}$$

canonical (as well as the equivalent system (1.1)). The matrix function $H(x)$ is called the Hamiltonian of this system.

In most works on canonical systems, the less complicated 2×2 Hamiltonian case (i.e, the case $p = 1$) is dealt with although the cases with other values of p ($p > 1$) are equally important. Interesting recent works [12, 33, 58] on 2×2 canonical systems and string equations contain also some useful references. Here, we deal with the case of $2p \times 2p$ Hamiltonians ($p \geq 1$) although many results are new even for $p = 1$.

Well-known Dirac (or Dirac-type) systems are equivalent to a special subclass of canonical systems (see [16, 18, 45, 50, 55] and references therein). The Hamiltonians corresponding to Dirac systems (after we switch from the representation (1.1) to the representation (1.4)), have the form

$$H(x) = \gamma(x)^* \gamma(x), \quad \gamma(x) j \gamma(x)^* = -I_p, \tag{1.5}$$

where γ are $p \times 2p$ matrix functions. For instance, formulas (1.7), (1.11), (1.12), and (1.26) in [50] lead to the representation (1.5).

The Hamiltonians, which we consider in this paper, have the form

$$H(x) = \beta(x)^* \beta(x), \quad \beta(x) j \beta(x)^* = 0 \quad (p \geq 1), \tag{1.6}$$

where β are again $p \times 2p$ matrix functions. Thus, canonical systems (1.4) with Hamiltonians of the form (1.6) are dual in a certain way to the class of canonical systems corresponding to Dirac systems. Under some natural conditions, systems (1.4), (1.6) are also equivalent to the matrix string equations (see [55, Chapter 11] and Appendix B in our paper). Canonical systems (1.4) with some special Hamiltonians of the form (1.6) appear, for instance, as the linear systems auxiliary to *nonlinear second harmonic generation equations* [28, 29].

We note that first order symplectic systems $y'(x) = F(x)y(x)$ are actively studied (see [5, 10, 11, 32] and the references therein). In the reformulation for our case, simplecticity means the equality

$$F(x)^*j + jF(x) + \mu(x)F(x)^*jF(x) = 0.$$

Thus, generalised canonical systems (1.2) (where $m_1 = m_2$) are symplectic, with $\mu \equiv 0$. Canonical systems (1.4), (1.6), which are our main topic in this paper, remain symplectic for any choice of $\mu(x)$.

The normalization condition

$$\beta'(x)j\beta(x)^* = \mathrm{i}I_p \tag{1.7}$$

for Hamiltonians of the form (1.6) is essential for the construction of fundamental solutions and solving inverse problems. In Appendix B, we show that matrix Schrödinger equations may be transformed into canonical systems (1.4), (1.6), (1.7) satisfying certain additional condition. There is considerable interest in the generalised Schrödinger equations (e.g., in Schrödinger equations with distributional potentials as well as with matrix and operator potentials, see some references in [13, 20]). One can say that systems (1.4), (1.6), (1.7) present an important generalization of the matrix Schrödinger equations. Canonical systems with Hamiltonians satisfying (1.6), (1.7) were briefly considered in [54, 55]. However, local boundedness of β'' was required there instead of the local square-integrability of β'', which we require in the next section. In Section 2, we represent the fundamental solutions for this case as the transfer matrix function from [52, 54, 55]. For this purpose, we use the linear similarity of the operator $K = \mathrm{i}\beta(x)j\int_0^x \beta(t)^* \cdot dt$ to the operator (2.2) of squared integration as well as the form of the corresponding similarity transformation operator V (see Theorem C.1 and its proof in Appendix C).

The representation of the fundamental solutions in Section 2 is important in itself, and it presents also a crucial step in the study of the high energy asymptotics of Weyl–Titchmarsh functions [48] and in solving the inverse problem to recover canonical system from the spectral or Weyl–Titchmarsh function [49].

Some basic results and notions on the Weyl-Titchmarsh theory of the general-type canonical systems (1.4) are described in Appendix A. The results are conveniently reformulated in terms of system (1.4) instead of system (1.1), and, what is essentially more important, certain redundant conditions contained in [50, Appendix A] are removed.

In other sections of the paper, we study explicit solutions of systems (1.2) with generalised Hamiltonians $dj + \beta(x)^*\beta(x)$ as well as explicit solutions and corresponding Weyl–Titchmarsh (Weyl) functions of the canonical systems (1.4), (1.6). We note that explicit solutions of Dirac systems and the corresponding Weyl–Titchmarsh theory have been studied sufficiently well (see, e.g., [22, 41, 50]) but the situation with the systems (1.4), (1.6) is quite different.

Explicit solutions of canonical systems and their properties are of essential theoretical and applied interest. Various versions of Bäcklund-Darboux transformations and related dressing and commutation methods [7, 8, 17, 19, 25, 30, 34, 35, 59] are fruitful tools in the construction of explicit solutions of linear and integrable nonlinear equations. Bäcklund-Darboux transformations for canonical and dynamical canonical systems, respectively, were constructed in [42] and [44]. More precisely, GBDT (generalised Bäcklund-Darboux transformation) was constructed for these systems. It is important that GBDT (see, e.g., [22,30,41,46,50] and references therein) is characterized by the generalised matrix eigenvalues (not necessarily diagonal) and the corresponding generalised eigenfunctions. In Section 3, the generalised matrix eigenvalues and the generalised eigenfunctions are denoted by $\mathcal{A}$ and $\Lambda(x)$, respectively.

Although GBDT for canonical systems was obtained in [42], a crucial step of constructing the generalised eigenfunctions $\Lambda(x)$ (which is necessary for constructing explicitly Hamiltonians and fundamental solutions) is done in the present paper. More precisely, *the procedure works in the following way.* We start with some *initial systems* (1.2), where *initial Hamiltonians* $H(x)$ are comparatively simple, and construct explicitly the fundamental solutions and generalised eigenfunctions for these systems. (In particular, some considerations from [46] were helpful for this purpose.) Using generalised eigenfunctions, the *transformed generalised Hamiltonians* and so called *Darboux matrices* are constructed as well. Recall that Darboux matrix for generalised canonical systems is the matrix function $\Psi(x,\lambda)$ satisfying the equation

$$\Psi'(x,\lambda) = \mathrm{i}\lambda\big(j\widetilde{H}(x)\Psi(x,\lambda) - \Psi(x,\lambda)jH(x)\big),$$

where H is the initial generalised Hamiltonian and $\widetilde{H}$ the transformed one. In this way, we obtain fundamental solutions $\widetilde{W}$ for a wide class of the *transformed systems* (i.e., systems with the transformed generalised Hamiltonians $\widetilde{H}(x)$). Indeed, it is easy to see that $\widetilde{W}$ is expressed via the fundamental solution W of the initial system and the Darboux matrix, namely, $\widetilde{W}(x,\lambda) = \Psi(x,\lambda)W(x,\lambda)$. Some preliminaries on GBDT for the generalised canonical systems are given, and the transformed generalised Hamiltonians and Darboux matrices are constructed in Section 3. The generalised eigenfunctions are constructed explicitly in Section 4. Explicit formulas for fundamental solutions of the initial systems and for the Weyl functions of the transformed canonical systems on the semi-axis $[0,\infty)$ are established in Section 5. In Section 6, it is shown that the second equality in (1.6) (i.e., the equality $\beta(x)j\beta(x)^* = 0$) for the initial matrix function $\beta(x)$ yields the equalities $\widetilde{\beta}(x)j\widetilde{\beta}(x)^* = 0$ and $\widetilde{\beta}(x)'j\widetilde{\beta}(x)^* = \beta(x)'j\beta(x)^*$ for the transformed matrix function $\widetilde{\beta}(x)$. Some interesting examples are treated in Section 7.

There are important connections between spectral and dynamical characteristics as well as between spectral and dynamical systems (see, e.g., [3,6,27,43,44,56] and references therein). In particular, GBDT for the spectral canonical systems (1.4) is closely related to the GBDT for the dynamical canonical system

$$H(x)\frac{\partial}{\partial t}Y(x,t) = j\frac{\partial}{\partial x}Y(x,t) \quad (m_1 = m_2 = p), \quad H(x) \geq 0, \quad x \geq 0. \tag{1.8}$$

We note that the invertibility of $H(x)$ was assumed for the dynamical canonical system considered in [44], and system (1.8) slightly differs from the one in [44]. Dynamical canonical systems are of interest in mechanics and control theory (see, e.g., [26]). The GBDT formula for Y and some explicit examples of H and Y are also discussed in Section 7.

As usual, $\mathbb{R}$ stands for the real axis, $\mathbb{R}_+ = \{r : r \in \mathbb{R},\ r \geq 0\}$, $\mathbb{C}$ stands for the complex plane, the open upper half-plane is denoted by $\mathbb{C}_+$, and $\overline{a}$ means the complex conjugate of a. The notation $\Re(a)$ stands for the real part of a, and $\Im(a)$ denotes the imaginary part of a. The notation $\mathrm{diag}\{d_1, \ldots\}$ stands for the diagonal (or block diagonal) matrix with the entries (or blocks) $d_1, \ldots$ on the main diagonal. The space of square-integrable functions on $(0,\, b)$ $(0 < b \leq \infty)$ is denoted by $L_2(0, b)$ and the corresponding space of p-dimensional column vector functions is denoted by $L_2^p(0, b)$. By $L_2^{p\times q}(0, b)$ we denote the class of $p \times q$ matrix functions with the entries belonging to $L_2(0, b)$. The notation I stands for the identity operator. The norm $\|A\|$ of the $n \times n$ matrix A means the norm of A acting in the space ℓ_2^n of the sequences of length n. The class of bounded operators acting from Hilbert space $\mathcal{H}_1$ into Hilbert space $\mathcal{H}_2$ is denoted by $\mathbf{B}(\mathcal{H}_1, \mathcal{H}_2)$, and we set $\mathbf{B}(\mathcal{H}) := \mathbf{B}(\mathcal{H}, \mathcal{H})$.

2 General-type fundamental solutions

In this section, we study canonical system (1.4) satisfying conditions (1.6) and (1.7):

$$H(x) = \beta(x)^*\beta(x), \quad \beta(x)j\beta(x)^* = 0, \quad \beta'(x)j\beta(x)^* = \mathrm{i}I_p. \tag{2.1}$$

Let us consider the system (1.4), (2.1) on some finite interval $[0, \mathbf{T}]$ $(\mathbf{T} > 0)$. The linear similarity of the operators $K \in \mathbf{B}\big(L_2^p(0, \mathbf{T})\big)$ and $A \in \mathbf{B}\big(L_2^p(0, \mathbf{T})\big)$, where

$$Kf = \mathrm{i}\beta(x)j\int_0^x \beta(t)^*f(t)dt, \quad Af = \int_0^x (t-x)f(t)dt, \tag{2.2}$$

is essential for us. Here, the operator A is introduced as the squared integration multiplied by -1. (Recall that in the case of Dirac systems the analog of A is the integration multiplied by i.) It is easy to see that

$$K - K^* = \mathrm{i}\beta(x)j\int_0^{\mathbf{T}} \beta(t)^* \cdot dt. \tag{2.3}$$

If $\beta''(x) \in L_2^{p\times 2p}(0, \mathbf{T})$, we have (according to Theorem C.1) $K = VAV^{-1}$, which we substitute into (2.3). Multiplying both parts of the derived equality by V^{-1} from the left and by $(V^*)^{-1}$ from the right, we obtain the operator identity

$$AS - SA^* = \mathrm{i}\Pi j \Pi^*, \tag{2.4}$$

where

$$S = V^{-1}(V^*)^{-1} > 0, \quad \Pi h = \Pi(x)h, \quad \Pi(x) := \big(V^{-1}\beta\big)(x), \tag{2.5}$$

$$\Pi \in \mathbf{B}\big(\mathbb{C}^{2p},\, L_2^p(0, \mathbf{T})\big), \quad \Pi(x) \in L_2^{p\times 2p}(0, \mathbf{T}), \quad h \in \mathbb{C}^{2p}. \tag{2.6}$$

Note that Π above is the operator of multiplication by the matrix function $\Pi(x)$ and the operator V^{-1} is applied to β (in the expression $V^{-1}\beta$) columnwise. The transfer matrix function corresponding to the so called S-node (i.e., to the triple $\{A, S, \Pi\}$ satisfying (2.4)) has the form

$$w_A(\lambda) = w_A(\mathbf{T}, \lambda) = I_{2p} - \mathrm{i}j\Pi^*S^{-1}(A - \lambda I)^{-1}\Pi, \tag{2.7}$$

and was first introduced and studied in [52]. We introduce the projectors $P_\ell \in \mathbf{B}\big(L_2^p(0,\mathbf{T}),\, L_2^p(0,\ell)\big)$:

$$\big(P_\ell f\big)(x) = f(x) \quad (0 < x < \ell, \quad \ell \le \mathbf{T}). \tag{2.8}$$

Now, we set

$$S_\ell = P_\ell S P_\ell^*, \quad V_\ell = P_\ell V P_\ell^*, \quad A_\ell = P_\ell A P_\ell^*, \quad \Pi_\ell = P_\ell \Pi, \tag{2.9}$$

$$w_A(\ell,\lambda) = I_{2p} - \mathrm{i} j \Pi_\ell^* S_\ell^{-1}(A_\ell - \lambda I)^{-1}\Pi_\ell. \tag{2.10}$$

Since V is a triangular operator, V^{-1} is triangular as wel, and we have $P_\ell V^{-1} = P_\ell V^{-1} P_\ell^* P_\ell$. Hence, taking into account (2.5) and (2.9) we derive

$$P_\ell V^{-1} P_\ell^* V_\ell = P_\ell V^{-1} P_\ell^* P_\ell V P_\ell^* = P_\ell V^{-1} V P_\ell^* = I, \tag{2.11}$$

$$S_\ell = P_\ell V^{-1}(V^*)^{-1} P_\ell^* = P_\ell V^{-1} P_\ell^* P_\ell (V^*)^{-1} P_\ell^*. \tag{2.12}$$

It follows that

$$V_\ell^{-1} = P_\ell V^{-1} P_\ell^*, \quad S_\ell = V_\ell^{-1}(V_\ell^*)^{-1}. \tag{2.13}$$

We have also $P_\ell A = P_\ell A P_\ell^* P_\ell$. Thus, multiplying both parts of (2.4) by P_ℓ from the left and by P_ℓ^* from the right (and using (2.9), (2.13), and the last equality in (2.5)) we obtain

$$A_\ell S_\ell - S_\ell A_\ell^* = \mathrm{i}\Pi_\ell j \Pi_\ell^*, \quad \Pi_\ell(x) = \big(V_\ell^{-1}\beta\big)(x) \quad (0 < x < \ell). \tag{2.14}$$

Clearly $w_A(\ell,\lambda)$ coincides with $w_A(\mathbf{T},\lambda)$ when $\ell = \mathbf{T}$.

Remark 2.1. *Relations* (2.10), (2.13) *and* (2.14) *show that S_ℓ and $w_A(\ell,\lambda)$ may be defined via V_ℓ (and $\beta(x)$ given on $[0,\ell]$) precisely in the same way as $w_A(\mathbf{T},\lambda)$ is constructed via V (and $\beta(x)$ given on $[0,\mathbf{T}]$). Moreover, according to Remark C.2, V_ℓ may be constructed in the same way as V, and so $w_A(\ell,\lambda)$ does not depend on the choice of $\beta(x)$ for $\ell < x < \mathbf{T}$ and the choice of $\mathbf{T} \ge \ell$. In particular, $w_A(\ell,\lambda)$ is uniquely defined on the semi-axis $0 < \ell < \infty$ for $\beta(x)$ considered on the semi-axis $0 \le x < \infty$.*

The fundamental solution of the canonical system (1.4), where Hamiltonian has the form (2.1) may be expressed via the transfer functions $w_A(\ell,\lambda)$ using continuous factorization theorem [55, p. 40] (see also [50, Theorem 1.20] as a more convenient for our purposes presentation).

Theorem 2.2. *Let the Hamiltonian of the canonical system* (1.4) *have the form* (2.1), *where $\beta(x)$ is a $p \times 2p$ two times differentiable matrix function. Assume that $\beta''(x) \in L_2^{p\times 2p}(0,\mathbf{T})$, if the canonical system is considered on the finite interval $[0,\beta]$, and that the entries of $\beta''(x)$ are locally square integrable, if the canonical system is considered on $[0,\infty)$.*
Then, the fundamental solution $W(x,\lambda)$ of the canonical system (normalised by $W(0,\lambda) = I_{2p}$) admits representation

$$W(\ell,\lambda) = w_A\Big(\ell, \frac{1}{\lambda}\Big). \tag{2.15}$$

Proof. First, we fix some $0 < \mathbf{T} < \infty$ and consider $\beta(x)$ on $[0, \mathbf{T}]$. It is easy to see that the projectors P_ℓ and the triple $\{A, S, \Pi\}$ satisfy conditions of [50, Theorem 1.20]. Hence, according [50, Theorem 1.20] the matrix function $w_A\left(\ell, \frac{1}{\lambda}\right)$ is the normalised fundamental solution of the canonical system (1.4) with Hamiltonian

$$H(\ell) = \frac{d}{d\ell}\int_0^\ell \Pi_\ell(x)^* S_\ell^{-1}\Pi_\ell(x)dx, \tag{2.16}$$

where S_ℓ^{-1} is applied to $\Pi_\ell(x)$ columnwise. Using the second equalities in (2.13) and (2.14), we rewrite (2.16) in the form

$$H(\ell) = \frac{d}{d\ell}\int_0^\ell \beta(x)^*\beta(x)dx = \beta(\ell)^*\beta(\ell), \tag{2.17}$$

and the statement of the theorem is proved on $[0, \mathbf{T}]$. Taking into account Remark 2.1, we see that the statement of the theorem is valid on $[0, \infty)$ as well. □

REMARK 2.3. *The operators $\mathcal{S}_\ell$ satisfying* (2.14) *are so called structured operators. The study of the structured operators in inverse problems takes roots in the seminal note [31] by M.G. Krein and was developed by L.A. Sakhnovich in [53–55].*

3 GBDT: Darboux matrices for generalised canonical systems

Let us consider systems (1.2) on finite or semi-infinite intervals. Without loss of generality, we choose either the intervals $\mathcal{I}_\mathbf{T} = [0, \mathbf{T}]$ $(\mathbf{T} < \infty)$ or the semi-axis $\mathbb{R}_+ = [0, \infty)$. We fix also an initial generalised Hamiltonian $H(x) = H(x)^*$. Given an initial $m \times m$ generalised Hamiltonian $H(x)$, each GBDT is (as usual) determined by some $n \times n$ matrices $\mathcal{A}$ and $\mathcal{S}(0) = \mathcal{S}(0)^*$ $(n \in \mathbb{N})$ and by an $n \times m$ matrix $\Lambda(0)$ which satisfy the matrix identity

$$\mathcal{A}\mathcal{S}(0) - \mathcal{S}(0)\mathcal{A}^* = \mathrm{i}\Lambda(0)j\Lambda(0)^*. \tag{3.1}$$

Taking into account the initial values $\Lambda(0)$ and $\mathcal{S}(0)$ (and using the matrix $\mathcal{A}$ and the matrix function $H(x)$) we introduce matrix functions $\Lambda(x)$ and $\mathcal{S}(x) = \mathcal{S}(x)^*$ via the equations:

$$\Lambda'(x) = -\mathrm{i}\mathcal{A}\Lambda(x)jH(x), \quad \mathcal{S}'(x) = \Lambda(x)jH(x)j\Lambda(x)^*. \tag{3.2}$$

It is easy to see that (3.1) and (3.2) yield [42] the identity

$$\mathcal{A}\mathcal{S}(x) - \mathcal{S}(x)\mathcal{A}^* \equiv \mathrm{i}\Lambda(x)j\Lambda(x)^*. \tag{3.3}$$

REMARK 3.1. *We note that, similar to the case of the general-type fundamental solutions in Section 2, we use also operator identities and transfer matrix function in Lev Sakhnovich form in our GBDT constructions. However, instead of the infinite-dimensional operators in Section 2, identities* (3.1) *and* (3.3) *are written for matrices. Here, we use calligraphic letter $\mathcal{A}$ and $\mathcal{S}$ instead A and S in Section 2* (*and the notation Λ instead of* Π) *for the elements of the S-node* (*of the triple* $\{\mathcal{A}, \mathcal{S}, \Lambda\}$). *The so called Darboux matrix from Darboux transformations is represented in GBDT* (*for each* x) *as the transfer matrix function. More precisely, we will show that in the points of invertibility of* $\mathcal{S}(x)$ (*for the case of the generalised canonical system*) *the Darboux matrix is expressed via*

$$w_{\mathcal{A}}(x,\lambda) = I_m - \mathrm{i}j\Lambda(x)^*\mathcal{S}(x)^{-1}(\mathcal{A}-\lambda I_n)^{-1}\Lambda(x) \tag{3.4}$$

(*see* (3.11)). *The dependence of* $\mathcal{S}, \Lambda$ *and* $w_{\mathcal{A}}$ *on* x *is of basic importance and greatly differs from the dependence of* S_ℓ, Π_ℓ *and* $w_A(\ell,\lambda)$ *on* ℓ *in Section 2.*

According to [42], $w_{\mathcal{A}}(x,\lambda)$ satisfies the equation

$$w_{\mathcal{A}}'(x,\lambda) = \big(\mathrm{i}\lambda jH(x) - \widetilde{q}_0(x)\big)w_{\mathcal{A}}(x,\lambda) - \mathrm{i}\lambda w_{\mathcal{A}}(x,\lambda)jH(x); \tag{3.5}$$

$$\widetilde{q}_0(x) := j\Lambda(x)^*\mathcal{S}(x)^{-1}\Lambda(x)jH(x) - jH(x)j\Lambda(x)^*\mathcal{S}(x)^{-1}\Lambda(x). \tag{3.6}$$

Note that (3.5) follows directly from (3.2)–(3.4). Moreover, (3.3) yields (see [42] or [50, (1.88)]):

$$w_{\mathcal{A}}(x,\overline{\mu})^*jw_{\mathcal{A}}(x,\lambda) = j + \mathrm{i}(\mu-\lambda) \times \Lambda(x)^*(\mathcal{A}^*-\mu I_n)^{-1}\mathcal{S}(x)^{-1}(\mathcal{A}-\lambda I_n)^{-1}\Lambda(x). \tag{3.7}$$

Relation (3.8) shows that, under conditions $\det((\mathcal{A}-\lambda I_n) \neq 0$ and $\det(\mathcal{A}^* - \lambda I_n) \neq 0$, $w_{\mathcal{A}}$ is invertible and

$$w_{\mathcal{A}}(x,\lambda)^{-1} = jw_{\mathcal{A}}(x,\overline{\lambda})^*j. \tag{3.8}$$

Further we assume that

$$\det \mathcal{A} \neq 0, \tag{3.9}$$

and so $w_{\mathcal{A}}(x,0)$ is well defined (in the points of invertibility of $\mathcal{S}(x)$). We note that (3.5) yields

$$w_{\mathcal{A}}'(x,0) = -\widetilde{q}_0(x)w_{\mathcal{A}}(x,0), \tag{3.10}$$

and we set

$$v(x,\lambda) := w_{\mathcal{A}}(x,0)^{-1}w_{\mathcal{A}}(x,\lambda). \tag{3.11}$$

Formulas (3.5), (3.10) and (3.11) imply that

$$v'(x,\lambda) = \mathrm{i}\lambda j\widetilde{H}(x)v(x,\lambda) - \mathrm{i}\lambda v(x,\lambda)jH(x), \tag{3.12}$$

$$j\widetilde{H}(x) = w_{\mathcal{A}}(x,0)^{-1}jH(x)w_{\mathcal{A}}(x,0). \tag{3.13}$$

Thus, one can see that $j\widetilde{H}$ is linear similar to jH. Moreover, in view of (3.8) we can rewrite (3.13) in the form

$$\widetilde{H}(x) = w_{\mathcal{A}}(x,0)^*H(x)w_{\mathcal{A}}(x,0). \tag{3.14}$$

According to (3.14), the equality $\widetilde{H}(x) = \widetilde{H}(x)^*$ is valid. Hence, $\widetilde{H}(x)$ is the *transformed* generalised Hamiltonian of the *transformed* generalised canonical system

$$\widetilde{w}'(x,\lambda) = \mathrm{i}\lambda j\widetilde{H}(x)w(x,\lambda), \quad \widetilde{H}(x) = \widetilde{H}(x)^* \quad (x \geq 0). \tag{3.15}$$

Clearly, $\widetilde{H} \geq 0$ if $H \geq 0$, and $\widetilde{H} > 0$ if $H > 0$. Therefore, in the case of an initial canonical system, the transformed system is also canonical. By virtue of (3.12), a fundamental solution $\widetilde{W}$ of the transformed system is given by the formula

$$\widetilde{W}(x,\lambda) = v(x,\lambda)W(x,\lambda), \tag{3.16}$$

where W is a fundamental solution of the initial system.

Remark 3.2. *If $H(x) \geq 0$ and $\mathcal{S}(0) > 0$, the second equation in (3.2) implies that $\mathcal{S}(x) > 0$ for $x \geq 0$. In particular, $\mathcal{S}(x)$ is invertible.*

Remark 3.3. *In view of (3.12) (or (3.16)) the matrix function $v(x,\lambda)$ is the so called Darboux matrix of the generalised canonical system.*
According to (3.7), (3.8) and (3.11), the representation of $v(x,\lambda)$ in terms of $\Lambda(x)$ and $\mathcal{S}(x)$ may be simplified. Namely, we have

$$\begin{aligned} v(x,\lambda) &= jw_{\mathcal{A}}(x,0)^* jw_{\mathcal{A}}(x,\lambda) \\ &= I_m - \mathrm{i}\lambda j\Lambda(x)^*(\mathcal{A}^*)^{-1}\mathcal{S}(x)^{-1}(\mathcal{A}-\lambda I_n)^{-1}\Lambda(x). \end{aligned} \tag{3.17}$$

4 Explicit solutions of the transformed generalised canonical systems

Consider the case, where the initial generalised Hamiltonian $H(x)$ has the form

$$H(x) = dj + \beta(x)^*\beta(x), \quad \beta(x) := \begin{bmatrix} \mathrm{e}^{\mathrm{i}cx}I_{m_1} & \mathrm{e}^{-\mathrm{i}cx}\alpha \end{bmatrix}. \tag{4.1}$$

Here, $\beta(x)$ is an $m_1 \times m$ matrix function, α is an $m_1 \times m_2$ matrix function and

$$c,\, d \in \mathbb{R}; \quad \alpha\alpha^* = I_{m_1} \quad (m_2 \geq m_1). \tag{4.2}$$

In view of (4.1) and (4.2), we have

$$\beta(x)j\beta(x)^* \equiv 0. \tag{4.3}$$

Recall that the matrix function $\Lambda(x)$ is determined by $\Lambda(0)$ and by the system

$$\Lambda'(x) = -\mathrm{i}\mathcal{A}\Lambda(x)jH(x). \tag{4.4}$$

We construct *generalised eigenfunction* $\Lambda(x)$ in the case (4.1) explicitly.

Proposition 4.1. *Let (4.1) and (4.2) hold. Then, the matrix function*

$$\Lambda(x) = \begin{bmatrix} \Phi_1(x) & \Phi_2(x) \end{bmatrix},$$

such that

$$\Phi_1(x) = \exp\{\mathrm{i}x(cI_n - d\mathcal{A})\}\big(\mathrm{e}^{\mathrm{i}xQ}f_1 + \mathrm{e}^{-\mathrm{i}xQ}f_2\big), \tag{4.5}$$

$$\begin{aligned} \Phi_2(x) = &\exp\{-\mathrm{i}x(cI_n + d\mathcal{A})\}\big(\mathrm{e}^{\mathrm{i}xQ}(\mathcal{A}+cI_n+Q)\mathcal{A}^{-1}f_1 \\ &+ \mathrm{e}^{-\mathrm{i}xQ}(\mathcal{A}+cI_n-Q)\mathcal{A}^{-1}f_2\big)\alpha, \end{aligned} \tag{4.6}$$

where f_k are $n \times m_1$ matrices, Q is an $n \times n$ matrix and

$$\mathcal{A}Q = Q\mathcal{A}, \quad Q^2 = c(2\mathcal{A}+cI_n), \tag{4.7}$$

satisfies (4.4).

Proof. Using (4.1)–(4.6), we derive

$$\Lambda(x)j\beta(x)^* = -\mathrm{e}^{-\mathrm{i}dx\mathcal{A}}\big(\mathrm{e}^{\mathrm{i}xQ}(cI_n+Q)\mathcal{A}^{-1}f_1 + \mathrm{e}^{-\mathrm{i}xQ}(cI_n-Q)\mathcal{A}^{-1}f_2\big). \quad (4.8)$$

It follows from (4.5) that

$$\frac{d}{dx}\Phi_1(x) = -\,\mathrm{i}d\mathcal{A}\Phi_1(x) + \mathrm{i}\exp\{\mathrm{i}x(cI_n - d\mathcal{A})\} \times \big(\mathrm{e}^{\mathrm{i}xQ}(cI_n+Q)f_1 + \mathrm{e}^{-\mathrm{i}xQ}(cI_n-Q)f_2\big). \quad (4.9)$$

According to (4.6), we have also

$$\frac{d}{dx}\Phi_2(x) = -\,\mathrm{i}d\mathcal{A}\Phi_2(x) + \mathrm{i}\exp\{-\mathrm{i}x(cI_n + d\mathcal{A})\} \times \big(\mathrm{e}^{\mathrm{i}xQ}(Q-cI_n)(\mathcal{A}+cI_n+Q)\mathcal{A}^{-1}f_1 + \mathrm{e}^{-\mathrm{i}xQ}(Q+cI_n)(Q-\mathcal{A}-cI_n)\mathcal{A}^{-1}f_2\big). \quad (4.10)$$

Since $\Lambda(x) = \begin{bmatrix}\Phi_1(x) & \Phi_2(x)\end{bmatrix}$ and $H(x)$ has the form (4.1) (where (4.2) holds) relations (4.7)–(4.10) imply (4.4). □

It is easy to see that one can set $Q = 0$ (in the Proposition 4.1) in the case $c = 0$. A more interesting case, where $c = 0$ and (4.7) holds, is generated by the matrices $\mathcal{A}$ and Q of the form

$$\mathcal{A} = \xi I_{2r} + \begin{bmatrix}0 & \mathcal{A}_{12}\\ 0 & 0\end{bmatrix} \quad (\xi \in \mathbb{C}), \quad Q = \begin{bmatrix}0 & Q_{12}\\ 0 & 0\end{bmatrix} \quad (4.11)$$

(where $\mathcal{A}$ and Q are $2r \times 2r$ matrices, $\mathcal{A}_{12}$ and Q_{12} are some $r \times r$ matrices) or by the block diagonal matrices with the blocks of the same form as the matrices on the right-hand sides of the equalities in (4.11).

The next immediate corollary of [46, Proposition B.1] and its proof (see also [47]) deals with the case $c \neq 0$.

Corollary 4.2. *Let $c \neq 0$, let $\det(2\mathcal{A} + cI_n) \neq 0$, and let $\mathcal{E}$ be the similarity transformation matrix and $\mathcal{J}$ Jordan normal form in the representation*

$$c(2\mathcal{A} + cI_n) = \mathcal{E}\mathcal{J}\mathcal{E}^{-1}. \quad (4.12)$$

Then, Q satisfying (4.7) may be constructed explicitly and has the form

$$Q = \mathcal{E}\mathcal{D}\mathcal{E}^{-1}, \quad (4.13)$$

where $\mathcal{D}$ is a block diagonal matrix with the blocks of the same orders as the corresponding Jordan blocks of $\mathcal{J}$. Moreover, the blocks of $\mathcal{D}$ are upper triangular Toeplitz matrices (or scalars if the corresponding blocks of $\mathcal{J}$ are scalars). If z is the eigenvalue of some block of $\mathcal{J}$, then the entries on the main diagonal of the corresponding block of $\mathcal{D}$ are equal to $\sqrt{z}$ (and one can fix any of the two possible values of $\sqrt{z}$ for this main diagonal).

Given generalised eigenfunction $\Lambda(x)$, one can construct (explicitly) the fundamental solution $\widetilde{W}(x,\lambda)$ of the transformed generalised canonical system using relations (3.4), (3.11), (3.16) and the second equality in (3.2). We note that an explicit expression for $W(x,\lambda)$, which we need for this purpose, is constructed similar to the way it is done in Proposition 5.1.

5 The case of the spectral canonical systems

1. It follows from (3.8), (3.14), and (4.3) that the transformed generalised Hamiltonians constructed in Section 4 have the form

$$\widetilde{H}(x) = dw_{\mathcal{A}}(x,0)^* j w_{\mathcal{A}}(x,0) + \widetilde{\beta}(x)^*\widetilde{\beta}(x) = dj + \widetilde{\beta}(x)^*\widetilde{\beta}(x), \tag{5.1}$$

$$\widetilde{\beta}(x) := \beta(x) w_{\mathcal{A}}(x,0), \quad \widetilde{\beta}(x) j \widetilde{\beta}(x)^* = \beta(x) j \beta(x)^* = 0. \tag{5.2}$$

Here, $\widetilde{\beta}$ is the corresponding transformation of β. Setting

$$d = 0, \quad m_1 = m_2 =: p, \tag{5.3}$$

we obtain a class of canonical systems

$$\widetilde{w}'(x,\lambda) = \mathrm{i}\lambda j \widetilde{H}(x)\widetilde{w}(x,\lambda), \quad \widetilde{H}(x) = \widetilde{\beta}(x)^*\widetilde{\beta}(x) \geq 0, \quad \widetilde{\beta}(x) j \widetilde{\beta}(x)^* = 0. \tag{5.4}$$

Further in the text *we normalise the fundamental solutions W and $\widetilde{W}$ of the systems* (1.2) *and* (3.15)*, respectively, setting*

$$W(0,\lambda) = \widetilde{W}(0,\lambda) = I_{2p}. \tag{5.5}$$

We write down the Hamiltonian $H(x)$ given by (4.1), (4.2), and (5.3) in the form

$$H(x) = \mathrm{e}^{-\mathrm{i}cxj}\mathcal{K}\mathrm{e}^{\mathrm{i}cxj}, \quad \mathcal{K} := \begin{bmatrix} I_p & \alpha \\ \alpha^* & I_p \end{bmatrix} \quad (\alpha\alpha^* = I_p). \tag{5.6}$$

Proposition 5.1. *The fundamental solution of the canonical system* (1.4)*, where $\Im(\lambda) \neq 0$, the Hamiltonian H is given by* (5.6) *and $c \neq 0$, has the form*

$$W(x,\lambda) = \mathrm{e}^{-\mathrm{i}cxj}E(\lambda)\begin{bmatrix} \mathrm{e}^{\mathrm{i}z_1(\lambda)x}I_p & 0 \\ 0 & \mathrm{e}^{\mathrm{i}z_2(\lambda)x}I_p \end{bmatrix}E(\lambda)^{-1}, \quad E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \tag{5.7}$$

$$E_i := \begin{bmatrix} -\alpha \\ \frac{1}{\lambda}(\lambda + c - z_i)I_p \end{bmatrix}, \quad z_i^2 = c(2\lambda + c) \quad (i = 1,2), \quad \Im(z_1) > 0. \tag{5.8}$$

Proof. It is easy to see that E is invertible (one may consider, for instance, the linear span of the rows of E, which coincides with $\mathbb{C}^{2p}$). Moreover, using the equality

$$\lambda - \frac{1}{\lambda}(\lambda + c)(\lambda + c - z_i) = \frac{z_i}{\lambda}(\lambda + c - z_i),$$

we have

$$EZE^{-1}E = EZ = (\lambda j\mathcal{K} + cj)E \quad \text{for} \quad Z = \mathrm{diag}\{z_1 I_p,\, z_2 I_p\}. \tag{5.9}$$

It follows that

$$EZE^{-1} = \lambda j \mathcal{K} + cj. \tag{5.10}$$

Relations (5.7) and (5.10) yield

$$W(x,\lambda) = \mathrm{e}^{-\mathrm{i}cxj}\mathrm{e}^{\mathrm{i}x(\lambda j\mathcal{K} + cj)}, \tag{5.11}$$

and for $W(x,\lambda)$ of the form (5.11) we immediately obtain

$$W'(x,\lambda)=\mathrm{i}\lambda j\mathrm{e}^{-\mathrm{i}cxj}\mathcal{K}\mathrm{e}^{\mathrm{i}cxj}W(x,\lambda), \quad W(0,\lambda)=I_{2p}. \tag{5.12}$$

Taking into account (5.6) and (5.12), we see that W given by (5.7), (5.8) is, indeed, the normalised fundamental solution of the canonical system described in the proposition. □

2. Further in this section, we assume that

$$\mathcal{S}(0)>0, \quad c\neq 0, \tag{5.13}$$

so that the statements of Remark 3.2 and Proposition 5.1 may be used. After normalization (5.5) formula (3.16) takes the form

$$\widetilde{W}(x,\lambda)=v(x,\lambda)W(x,\lambda)v(0,\lambda)^{-1}. \tag{5.14}$$

For system (5.1)–(5.3), in view of (3.11), (5.7) and (5.14) we obtain

$$\widetilde{\beta}(x)\widetilde{W}(x,\lambda)v(0,\lambda)E_1(\lambda)=\mathrm{e}^{\mathrm{i}z_1(\lambda)x}\beta(x)w_{\mathcal{A}}(x,\lambda)\mathrm{e}^{-\mathrm{i}cxj}E_1(\lambda). \tag{5.15}$$

Taking into account (5.15) (and some definitions and considerations on Weyl–Titchmarsh theory in Appendix A), we derive the following theorem.

Theorem 5.2. *Canonical system* (*considered on on* $[0,\infty)$) *with the Hamiltonian of the form* (5.1)*–*(5.3)*, where* (5.13) *holds, has a unique Weyl function* (*Weyl's limit point case*)*. This Weyl function is given explicitly by the formula*

$$\varphi(\lambda)=\begin{bmatrix}0 & I_p\end{bmatrix}v(0,\lambda)E_1(\lambda)\big(\begin{bmatrix}I_p & 0\end{bmatrix}v(0,\lambda)E_1(\lambda)\big)^{-1}, \tag{5.16}$$

where E_1 *has the form* (5.8).

Proof. First, we note that formulas (3.7)–(3.11) (and [50, Corollary E.3]) yield

$$v(0,\lambda)^*jv(0,\lambda)\geq j, \quad v(0,\lambda)jv(0,\lambda)^*\geq j \quad (\lambda\in\mathbb{C}_+). \tag{5.17}$$

It easily follows from (5.8) and (5.17) that

$$E_1\Big(-\frac{c}{2}+\varepsilon\mathrm{i}\Big)^*jE_1\Big(-\frac{c}{2}+\varepsilon\mathrm{i}\Big)>0, \quad \begin{bmatrix}I_p & 0\end{bmatrix}v(0,\lambda)jv(0,\lambda)^*\begin{bmatrix}I_p\\ 0\end{bmatrix}>0,$$

where the first inequality holds (at least) for small $\varepsilon>0$ and the second inequality holds for all $\lambda\in\mathbb{C}_+$ (excluding the part of spectrum of $\mathcal{A}$ situated in $\mathbb{C}_+$). Hence (see, e.g., [50, Proposition 1.43]),

$$\det\big(\begin{bmatrix}I_p & 0\end{bmatrix}v(0,\lambda)E_1(\lambda)\big)\not\equiv 0,$$

and so

$$\det\big(\begin{bmatrix}I_p & 0\end{bmatrix}v(0,\lambda)E_1(\lambda)\big)\neq 0 \quad \text{for} \quad \lambda\in\mathbb{C}_+, \tag{5.18}$$

excluding, possibly, some isolated points. In other words, $\varphi(\lambda)$ in (5.16) is well defined.

Now, we will show that for such λ that $\Im\big(z_1(\lambda)\big)$ is sufficiently large (excluding, possibly, isolated points) the relation

$$\widetilde{\beta}(x)\widetilde{W}(x,\lambda)\begin{bmatrix} I_p \\ \varphi(\lambda)\end{bmatrix} \in L_2^{p\times p}(0,\infty) \tag{5.19}$$

is valid. Indeed, the matrix functions $\beta(x)$ and $\mathrm{e}^{-\mathrm{i}cxj}$ on the right-hand side of (5.15) are bounded. In view of (3.2), we have

$$\int_0^r \mathcal{S}(x)^{-1}\Lambda(x)j\beta(x)^*\beta(x)j\Lambda(x)^*\mathcal{S}(x)^{-1} = \mathcal{S}(0)^{-1} - \mathcal{S}(r)^{-1} \le \mathcal{S}(0)^{-1}.$$

Therefore, we obtain

$$\beta(x)j\Lambda(x)^*\mathcal{S}(x)^{-1} \in L_2^{p\times n}(0,\infty). \tag{5.20}$$

Finally, Proposition 4.1 and Corollary 4.2 show that the matrix function $\mathrm{e}^{\mathrm{i}z_1(\lambda)x}\Lambda(x)$ is bounded for sufficiently large values of $\Im\big(z_1(\lambda)\big)$. Taking into account the definition (3.4) of $w_{\mathcal{A}}$ and considerations above, we see that the right-hand side of (5.15) belongs $L_2^{p\times p}(0,\infty)$ (for sufficiently large values of $\Im\big(z_1(\lambda)\big)$). Thus, the left-hand side of (5.15) belongs $L_2^{p\times p}(0,\infty)$ as well, and so (5.19) holds for $\varphi(\lambda)$ given by (5.16).
Assume that for some $\lambda=\lambda_0\in\mathbb{C}_+$ we have (5.19) and have also

$$\widetilde{\beta}(x)\widetilde{W}(x,\lambda_0)\begin{bmatrix} I_p \\ \widehat{\varphi}(\lambda_0)\end{bmatrix} \in L_2^{p\times p}(0,\infty), \quad \varphi(\lambda_0)\neq\widehat{\varphi}(\lambda_0). \tag{5.21}$$

We will show (by contradiction) that this is impossible for sufficiently large values of $\Im\big(z_1(\lambda_0)\big)$. Indeed, since (5.19) implies

$$\widetilde{\beta}(x)\widetilde{W}(x,\lambda_0)v(0,\lambda_0)E_1(\lambda_0) \in L_2^{p\times p}(0,\infty),$$

additional relations (5.21) yield the existence of $f\in\mathbb{C}^p$ such that

$$\widetilde{\beta}(x)\widetilde{W}(x,\lambda_0)v(0,\lambda_0)E_2(\lambda_0)f \in L_2^{p\times 1}(0,\infty) \quad (f\neq 0). \tag{5.22}$$

On the other hand, taking into account that $z_2(\lambda)=-z_1(\lambda)$ we similar to (5.15) derive

$$\widetilde{\beta}(x)\widetilde{W}(x,\lambda)v(0,\lambda)E_2(\lambda) = \mathrm{e}^{-\mathrm{i}z_1(\lambda)x}\beta(x)w_{\mathcal{A}}(x,\lambda)\mathrm{e}^{-\mathrm{i}cxj}E_2(\lambda). \tag{5.23}$$

Next, we should consider $g(x,\lambda)=\beta(x)w_{\mathcal{A}}(x,\lambda)\mathrm{e}^{-\mathrm{i}cxj}$ in a more detailed way, and we note that according to (3.2), (3.4), (4.1), (4.2), (4.5), and (4.6) the entries g_{ik} of g admit representation

$$g_{ik}(x,\lambda) = \sum_{s=1}^{N_1} P_s(\lambda)x^{\ell_s}\mathrm{e}^{h_s x}\Big/\left(P(\lambda)\sum_{s=1}^{N_2} x^{n_s}\mathrm{e}^{\zeta_s x}\right). \tag{5.24}$$

where P and P_s are polynomials, and N_1, P_s, ℓ_s and h_s depend on i, k. Moreover, similar to (5.18) one can show that (excluding isolated points λ) we have

$$\det\big(g(x,\lambda)\widehat{E}(\lambda)\big)\neq 0, \quad \widehat{E}(\lambda) := \begin{bmatrix} -\alpha \\ \big((\lambda+c)/\lambda\big)I_p\end{bmatrix},$$

where $\widehat{E}$ is the "rational part" of E_2. It follows that

$$g(x,\lambda)E_2(\lambda)f \neq 0. \tag{5.25}$$

Taking into account (5.23)–(5.25), we see that (5.22) (and so (5.21)) does not hold for sufficiently large values of $\Im\big(z_1(\lambda_0)\big)$.
Since (5.21) does not hold for sufficiently large values of $\Im\big(z_1(\lambda_0)\big)$ (excluding, may be, isolated points), there is an open domain in $\mathbb{C}_+$, where $\varphi(\lambda)$ (given by (5.16)) is uniquely defined via (5.19). Thus, each Weyl function of our system coincides with $\varphi(\lambda)$ in this domain (see the Definition A.2 of the Weyl functions). Recall that Weyl functions are holomorphic in $\mathbb{C}_+$. Hence, the Weyl function of our system is unique (and its existence follows from Proposition A.1). We see that the Weyl function exists, is unique and coincides with $\varphi(\lambda)$ in some domain. Therefore, $\varphi(\lambda)$ given by (5.16) is the Weyl function and admits holomorphic continuation in all $\mathbb{C}_+$. □

Remark 5.3. *In Proposition 5.1 and Theorem 5.2, we assume that $c \neq 0$. The constructions are much simpler when $c = 0$. In particular, we recall that $\beta j \beta^* \equiv 0$ (see (4.3)). Moreover, $\mathcal{K}$ given in (5.6) equals $\beta^*\beta$ as $c = 0$. Hence, $\mathcal{K} j \mathcal{K} = 0$. Therefore, using (5.6) we have:*

$$H(x) \equiv \mathcal{K}, \quad W(x,\lambda) = \mathrm{e}^{\mathrm{i}\lambda x j \mathcal{K}} = I_{2p} + \mathrm{i}\lambda x j \mathcal{K}$$

for the case $c = 0$.

Remark 5.4. *The so called limit point/limit circle problem is of essential interest in spectral theory. In Theorem 5.2, we deal with the limit point case of the unique Weyl function. The Weyl function is also unique in the case of Dirac system on the semi-axis and of the corresponding canonical system even assuming that m_1 does no necessarily equals m_2 in (1.2) (see, e.g., [50, Section 2.2.1] and references therein). For the canonical system of the form*

$$u'(x) = z\widehat{J}H(x)u(x), \quad H(x) \geq 0, \quad \widehat{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad x \in [0,\infty),$$

where H is a 2×2 matrix function with real-valued entries, we have limit point case if and only if H is not integrable near infinity or, equivalently, on $[0,\infty)$ [9] (see also [1] and [37, Theorem 3.5]). For the canonical system (1.4), (1.6), (1.7), the situation is more complicated. We plan to study it later in greater detail.

6 Matrix string equation

Consider again the case of the initial canonical systems

$$w'(x,\lambda) = \mathrm{i}\lambda j H(x) w(x,\lambda),$$

where $H(x) = \beta(x)^*\beta(x)$ and $\beta(x)$ are $p \times 2p$ matrix functions. According to (3.14), the transformed Hamiltonians (of the GBDT-transformed canonical systems (3.15)) have the form

$$\widetilde{H}(x) = \widetilde{\beta}(x)^*\widetilde{\beta}(x), \quad \widetilde{\beta}(x) = \beta(x) w_{\mathcal{A}}(x,0). \tag{6.1}$$

When the matrix functions $\beta(x)$ have the form presented in (4.1) (and (4.2), (5.3) hold), our assertions below show (in view of Appendix B) that the considered transformed canonical systems correspond to a special subclass of string equations. Thus, our explicit formulas may be transferred for the case of string equations as explained in Remark B.3.

Proposition 6.1. *Let $\beta(x)$ satisfy the equality*

$$\beta(x)j\beta(x)^*=0. \tag{6.2}$$

Then, $\widetilde{\beta}(x)$ satisfies the relations

$$\widetilde{\beta}(x)j\widetilde{\beta}(x)^*=0, \quad \widetilde{\beta}(x)'j\widetilde{\beta}(x)^*=\beta(x)'j\beta(x)^*. \tag{6.3}$$

Proof. Recall (see, e.g., (3.8)) that

$$w_{\mathcal{A}}(x,0)jw_{\mathcal{A}}(x,0)^*=j. \tag{6.4}$$

The first equality in (6.3) easily follows from (6.4) (and was already stated in (5.2)). Formulas (3.10), (6.4) and the second equality in (6.1) imply that

$$\begin{aligned}\widetilde{\beta}(x)'j\widetilde{\beta}(x)^* &= \beta(x)'w_{\mathcal{A}}(x,0)jw_{\mathcal{A}}(x,0)^*\beta(x)^*+\beta(x)w_{\mathcal{A}}'(x,0)jw_{\mathcal{A}}(x,0)^*\beta(x)^* \\ &= \beta(x)'j\beta(x)^*-\beta(x)\widetilde{q}_0(x)j\beta(x)^*.\end{aligned} \tag{6.5}$$

The definition (3.6) of $\widetilde{q}_0$ and the equality (6.2) yield

$$\beta(x)\widetilde{q}_0(x)j\beta(x)^*=0. \tag{6.6}$$

The second equality in (6.3) is immediate from (6.5) and (6.6). □

Corollary 6.2. *Let $\beta(x)$ be given by (4.1), where $c=\frac{1}{2}$ and $\alpha\alpha^*=I_p$. Then, (6.2) holds and*

$$\widetilde{\beta}(x)'j\widetilde{\beta}(x)^*=\mathrm{i}I_p. \tag{6.7}$$

7 Examples and applications

1. Let us consider explicit examples of the Hamiltonians $\widetilde{H}(x)=\widetilde{\beta}(x)^*\widetilde{\beta}(x)$, corresponding Darboux matrices $v(x,\lambda)$, fundamental solutions $\widetilde{W}(x,\lambda)$, and Weyl functions $\varphi(\lambda)$.

Example 7.1. *In our first example, we assume that*

$$p=n=1, \quad \mathcal{A}=a\neq\overline{a} \quad (a\in\mathbb{C}), \quad c\neq 0, \quad d=0, \tag{7.1}$$

where the condition $a\neq\overline{a}$ provides an easy recovery of $\mathcal{S}(x)$ from (3.3).

Recall that according to the second equalities in (4.1), (4.2), and (4.7), we have

$$\beta(x):=\begin{bmatrix}\mathrm{e}^{\mathrm{i}cx} & \mathrm{e}^{-\mathrm{i}cx}\alpha\end{bmatrix}, \quad |\alpha|=1, \quad Q=\sqrt{2ac+c^2}. \tag{7.2}$$

In order to define the sign of the square root above, we assume that $\Im(Q)>0$. By virtue of (4.5), (4.6)) and (7.1), we obtain

$$\begin{aligned}a\Lambda(x)=&\big[a\big(f_1\mathrm{e}^{\mathrm{i}xQ}+f_2\mathrm{e}^{-\mathrm{i}xQ}\big) \quad \alpha\big((a+c+Q)f_1\mathrm{e}^{\mathrm{i}xQ}+(a+c-Q)f_2\mathrm{e}^{-\mathrm{i}xQ}\big)\big] \\ &\times\mathrm{e}^{\mathrm{i}cxj},\end{aligned} \tag{7.3}$$

where f_1 and f_2 are scalars and $a\Lambda$ is written down more conveniently than Λ. It follows from (3.3) and (7.3) that

$$\mathcal{S}(x) =\frac{\mathrm{i}}{a-\overline{a}}\Big(\big|f_1\mathrm{e}^{\mathrm{i}xQ}+f_2\mathrm{e}^{-\mathrm{i}xQ}\big|^2 - \frac{1}{|a|^2}\big|(a+c+Q)f_1\mathrm{e}^{\mathrm{i}xQ}+(a+c-Q)f_2\mathrm{e}^{-\mathrm{i}xQ}\big|^2\Big), \tag{7.4}$$

and the requirement $\mathcal{S}(0) > 0$ takes the form

$$\mathrm{i}(\overline{a}-a)\big(|a(f_1+f_2)|^2 - \big|(a+c+Q)f_1+(a+c-Q)f_2\big|^2\big) > 0. \tag{7.5}$$

Relations (3.4), (5.2) and (7.1)–(7.3) yield

$$\widetilde{\beta}(x) =\beta(x) - \frac{\mathrm{i}}{a|a|^2\mathcal{S}(x)}\Big(\overline{a}\big(\overline{f_1}\mathrm{e}^{-\mathrm{i}x\overline{Q}}+\overline{f_2}\mathrm{e}^{\mathrm{i}x\overline{Q}}\big) - \overline{\alpha}\big(\overline{(a+c+Q)f_1}\mathrm{e}^{-\mathrm{i}x\overline{Q}}+\overline{(a+c-Q)f_2}\mathrm{e}^{\mathrm{i}x\overline{Q}}\big)\Big)\big(a\Lambda(x)\big). \tag{7.6}$$

According to (3.17) and (7.1), the corresponding Darboux matrix is given by the formula

$$v(x,\lambda) = I_2 - \frac{\mathrm{i}\lambda}{\overline{a}|a|^2(a-\lambda)\mathcal{S}(x)}j\big(a\Lambda(x)\big)^*\big(a\Lambda(x)\big). \tag{7.7}$$

Formulas (5.7), (5.14) and (7.3), (7.7) give explicitly fundamental solutions of the canonical systems with $\widetilde{\beta}$ of the form (7.6). In view of (5.16) and (7.7), the Weyl functions φ of such canonical systems on $[0,\infty)$ have the form:

$$\varphi(\lambda) =\psi_1(\lambda)/\psi_2(\lambda), \tag{7.8}$$

$$\psi_1(\lambda) =\overline{a}|a|^2\mathcal{S}(0)(a-\lambda)\big(\lambda+c-z_1(\lambda)\big) + \mathrm{i}\overline{\alpha}\big((\overline{a}+c+\overline{Q})\overline{f_1}+(\overline{a}+c-\overline{Q})\overline{f_2}\big)\lambda h(\lambda), \tag{7.9}$$

$$\psi_2(\lambda) =\alpha\overline{a}|a|^2\mathcal{S}(0)(\lambda-a)\lambda - \mathrm{i}\overline{a}(\overline{f_1}+\overline{f_2})\lambda h(\lambda), \tag{7.10}$$

where $z_1(\lambda) = \sqrt{c(2\lambda+c)}$ $(\Im(z_1) > 0)$,

$$h(\lambda) := a\Lambda(0)E_1(\lambda) =\alpha\big((a+c+Q)f_1+(a+c-Q)f_2\big)\big(\lambda+c-z_1(\lambda)\big) - \alpha a(f_1+f_2)\lambda.$$

Example 7.2. *Now, assume that*

$$p=1,\quad n=2,\quad c=0,\quad d=0,\quad \mathcal{A}=\begin{bmatrix}\xi & a\\ 0 & \xi\end{bmatrix}\quad (\xi\in\mathbb{R},\ \xi\neq 0), \tag{7.11}$$

$$Q=\begin{bmatrix}0 & q\\ 0 & 0\end{bmatrix},\quad f_1=\begin{bmatrix}f\\ 0\end{bmatrix}\quad f_2=\begin{bmatrix}0\\ g\end{bmatrix};\quad q,f,g\in\mathbb{C},\quad f\neq 0,\quad g\neq 0.$$

In this case, we have

$$\beta=\begin{bmatrix}1 & \alpha\end{bmatrix},\quad \mathrm{e}^{\pm\mathrm{i}xQ}=I_2\pm\mathrm{i}xQ, \tag{7.12}$$

$$(\mathcal{A}-\lambda I_2)^{-1} = (\xi-\lambda)^{-1}I_2 - (\xi-\lambda)^{-2}\begin{bmatrix}0 & a\\ 0 & 0\end{bmatrix}. \tag{7.13}$$

Hence, formulas (4.5), (4.6) and simple calculations yield

$$\Lambda(x) = \begin{bmatrix} f - \mathrm{i}qgx & -qg\big(\mathrm{i}x + \xi^{-1}\big) \\ g & g \end{bmatrix}. \tag{7.14}$$

In view of (7.14), the required matrix identity (3.1) may be written in the form

$$\begin{bmatrix} a\mathcal{S}_{21}(0) - \overline{a}\mathcal{S}_{12}(0) & a\mathcal{S}_{22}(0) \\ -\overline{a}\mathcal{S}_{22}(0) & 0 \end{bmatrix} = \mathrm{i}\begin{bmatrix} |f|^2 - |qg\xi^{-1}|^2 & \overline{g}\big(f + gq\xi^{-1}\big) \\ g\big(\overline{f} + \overline{gq}\xi^{-1}\big) & 0 \end{bmatrix}, \tag{7.15}$$

where $\mathcal{S}_{ik}$ are the entries of $\mathcal{S}$. Hence, we cannot choose an arbitrary entry a in $\mathcal{A}$ but demand $f+gq\xi^{-1} \neq 0$ and choose a and $\mathcal{S}_{22}(0)$ satisfying the following conditions (which is always possible):

$$a\mathcal{S}_{22}(0) = \overline{g}\big(f + gq\xi^{-1}\big), \quad a \neq 0, \quad \mathcal{S}_{22}(0) > 0. \tag{7.16}$$

Next, we choose $\mathcal{S}_{12}(0)$ (and so $\mathcal{S}_{21}(0) = \overline{\mathcal{S}_{12}(0)}$) such that (7.15) holds, and we choose such $\mathcal{S}_{11}(0) > 0$ that $\mathcal{S}(0) > 0$.
Since $\xi \in \mathbb{R}$, we cannot use (3.3) in order to recover $\mathcal{S}(x)$ from $\Lambda(x)$ and construct $\mathcal{S}(x)$ in a different way. It follows from (4.1), (7.11) and (7.14) that

$$\Lambda(x)j\beta^* = \begin{bmatrix} C_1x + C_2 \\ C_3 \end{bmatrix}, \quad C_1 = \mathrm{i}(\overline{\alpha} - 1)qg, \quad C_2 = f + \overline{\alpha}qg\xi^{-1}, \tag{7.17}$$

$$C_3 = g(1 - \overline{\alpha}). \tag{7.18}$$

Therefore, the second equality in (3.2) yields

$$\begin{aligned} \mathcal{S}(x) &= \mathcal{S}(0) + \int_0^x \Lambda(t)j\beta^*\big(\Lambda(t)j\beta^*\big)^*dt \qquad (7.19) \\ &= \mathcal{S}(0) + \begin{bmatrix} \frac{1}{3}|C_1|^2x^3 + \Re\big(C_1\overline{C_2}\big)x^2 + |C_2|^2x & \frac{1}{2}C_1\overline{C_3}x^2 + C_2\overline{C_3}x \\ \frac{1}{2}\overline{C_1}C_3x^2 + \overline{C_2}C_3x & |C_3|^2x \end{bmatrix}. \end{aligned}$$

Using (7.17), we rewrite the equality (5.2) for $\widetilde{\beta}$ (transformed β) in the form

$$\widetilde{\beta}(x) = \begin{bmatrix} 1 & \alpha \end{bmatrix} - \mathrm{i}\begin{bmatrix} \overline{C_1}x + \overline{C_2} & \overline{C_3} \end{bmatrix}\mathcal{S}(x)^{-1}\mathcal{A}^{-1}\Lambda(x), \tag{7.20}$$

where $\mathcal{S}(x)$, $\mathcal{A}$ and $\Lambda(x)$ are given in (7.19), (7.11) and (7.14), respectively. Finally, the Darboux matrix $v(x,\lambda)$ is expressed via $\Lambda(x)$ and $\mathcal{S}(x)$ in (3.17), and the expression for the corresponding fundamental solution $\widetilde{W}$ follows from (5.14) and Remark 5.3.

2. Relations (3.2) and (3.3) imply an important equality (see [44, (2.13)]):

$$\big(\Lambda^*\mathcal{S}^{-1}\big)' = \mathrm{i}Hj\Lambda^*\mathcal{S}^{-1}\mathcal{A} + \widetilde{q}_0^*\Lambda^*\mathcal{S}^{-1}. \tag{7.21}$$

We assume that $\mathcal{S}(0) > 0$ and $H(x) \geq 0$, that is, $\mathcal{S}(x) > 0$ for $x \geq 0$, and so $\mathcal{S}(x)^{-1}$ is well defined (see Remark 3.2). In view of (3.8), (3.10) and (7.21), for $\widetilde{H}$ of the form (3.14) and Y given by

$$Y(x,t) = jw_{\mathcal{A}}(x,0)^*\Lambda(x)^*\mathcal{S}(x)^{-1}\mathrm{e}^{\mathrm{i}t\mathcal{A}}, \tag{7.22}$$

we have

$$\widetilde{H}(x)\frac{\partial}{\partial t}Y(x,t) = j\frac{\partial}{\partial x}Y(x,t) \quad (m_1 = m_2 = p), \quad x \geq 0. \tag{7.23}$$

In other words, the $2p \times n$ matrix function Y (or, equivalently, the columns of Y) satisfies the dynamical canonical system (7.23).
Taking into account (3.3) and (3.4), we rewrite $w_{\mathcal{A}}(x,0)^*\Lambda(x)^*\mathcal{S}(x)^{-1}$ in a simpler form (in terms of $\Lambda(x)$ and $\mathcal{S}(x)$):

$$w_{\mathcal{A}}(x,0)^*\Lambda(x)^*\mathcal{S}(x)^{-1} = \Lambda(x)^*\big(\mathcal{A}^*\big)^{-1}\mathcal{S}(x)^{-1}\mathcal{A}^{-1}.$$

Hence,

$$Y(x,t) = j\Lambda(x)^*\big(\mathcal{A}^*\big)^{-1}\mathcal{S}(x)^{-1}\mathrm{e}^{\mathrm{i}t\mathcal{A}}\mathcal{A}^{-1}. \tag{7.24}$$

Proposition 7.3. *Let the initial Hamiltonian $H(x) \geq 0$ be given, and let the relations* (3.1)*, $\mathcal{S}(0) > 0$, and $\det \mathcal{A} \neq 0$ hold. Then, Y of the form* (7.24) *satisfies dynamical canonical system* (7.23)*, where the transformed Hamiltonian $\widetilde{H}$ is given by* (3.14)*.*

In this way, explicit expressions for Λ and $\mathcal{S}$ in Examples 7.1 and 7.2 give us explicit expessions for $Y(x,t)$. Moreover, it is immediate from (7.11) that $\mathrm{e}^{\mathrm{i}t\mathcal{A}}$ in (7.24) takes under assumptions of Example 7.2 a simple form

$$\mathrm{e}^{\mathrm{i}t\mathcal{A}} = \mathrm{e}^{\mathrm{i}t\xi}\left(I_2 + \mathrm{i}ta\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right). \tag{7.25}$$

A Canonical systems: Weyl–Titchmarsh theory

Consider generalised canonical system (1.2). It is immediate that the fundamental solution W of (1.2) satisfies the equality

$$\frac{d}{dx}\big(W(x,\overline{\mu})^*jW(x,\lambda)\big) = \mathrm{i}(\lambda-\mu)W(x,\overline{\mu})^*H(x)W(x,\lambda). \tag{A.1}$$

In view of (A.1) (for the case $\mu = \overline{\lambda}$) and of the normalization $W(0,\lambda) = I_m$ always assumed in this appendix, we have

$$\int_0^r W(x,\lambda)^*H(x)W(x,\lambda)dx = \frac{\mathrm{i}}{\lambda-\overline{\lambda}}\big(j - W(r,\lambda)^*jW(r,\lambda)\big), \tag{A.2}$$

for $\lambda \notin \mathbb{R}$ and $r \geq 0$. Moreover, (A.1) for the case $\mu = \lambda$ implies that

$$W(r,\overline{\lambda})^*jW(r,\lambda) \equiv j \equiv W(r,\lambda)jW(r,\overline{\lambda})^*. \tag{A.3}$$

Further in the appendix, we will deal with the general-type (i.e., not necessarily related to explicit solutions) canonical system (1.4) on $[0,\infty)$. Since $H \geq 0$, formula (A.2) yields

$$W(r_2,\lambda)^*jW(r_2,\lambda) \leq W(r_1,\lambda)^*jW(r_1,\lambda) \leq j, \tag{A.4}$$

$$j \leq W(r_1,\overline{\lambda})^*jW(r_1,\overline{\lambda}) \leq W(r_2,\overline{\lambda})^*jW(r_2,\overline{\lambda}) \quad (r_1 \leq r_2, \quad \lambda \in \mathbb{C}_+). \tag{A.5}$$

Next, introduce the families $\mathcal{N}(r)$ of linear-fractional (Möbius) transformations

$$\begin{aligned}\phi(r,\lambda) =& \big(\mathcal{W}_{21}(r,\lambda)\mathcal{P}_1(\lambda) + \mathcal{W}_{22}(r,\lambda)\mathcal{P}_2(\lambda)\big)\\ &\times \big(\mathcal{W}_{11}(r,\lambda)\mathcal{P}_1(\lambda) + \mathcal{W}_{12}(r,\lambda)\mathcal{P}_2(\lambda)\big)^{-1},\end{aligned} \tag{A.6}$$

where $\mathcal{W}_{ik}$ and $\mathcal{P}_k$ are $p \times p$ matrix functions,

$$\mathcal{W}(r,\lambda) = \{\mathcal{W}_{ik}(r,\lambda)\}_{i,k=1}^2 := jW(r,\overline{\lambda})^* j, \tag{A.7}$$

and $\mathcal{P}_1(\lambda)$, $\mathcal{P}_2(\lambda)$ are pairs of meromorphic in $\mathbb{C}_+$ matrix functions (so called nonsingular pairs with property-j) such that

$$\mathcal{P}_1(\lambda)^*\mathcal{P}_1(\lambda) + \mathcal{P}_2(\lambda)^*\mathcal{P}_2(\lambda) > 0, \quad \begin{bmatrix}\mathcal{P}_1(\lambda)^* & \mathcal{P}_2(\lambda)^*\end{bmatrix} j \begin{bmatrix}\mathcal{P}_1(\lambda) \\ \mathcal{P}_2(\lambda)\end{bmatrix} \geq 0, \tag{A.8}$$

where the first inequality holds in one point (at least) of $\mathbb{C}_+$, and the second inequality holds in all the points of analyticity of $\mathcal{P}_k$ $(k = 1, 2)$.
It follows from (A.8) by contradiction that

$$\det\big(\mathcal{W}_{11}(r,\lambda)\mathcal{P}_1(\lambda) + \mathcal{W}_{12}(r,\lambda)\mathcal{P}_2(\lambda)\big) \not\equiv 0. \tag{A.9}$$

Indeed, formulas (A.3), (A.4) and (A.7) imply that $\mathcal{W}(r,\lambda)^* j \mathcal{W}(r,\lambda) \geq j$, which yields

$$\begin{bmatrix}\mathcal{P}_1(\lambda)^* & \mathcal{P}_2(\lambda)^*\end{bmatrix} \mathcal{W}(r,\lambda)^* j \mathcal{W}(r,\lambda) \begin{bmatrix}\mathcal{P}_1(\lambda) \\ \mathcal{P}_2(\lambda)\end{bmatrix} \geq 0 \tag{A.10}$$

in the points of analyticity of $\mathcal{P}_k(\lambda)$ in $\mathbb{C}_+$. On the other hand, if we have

$$\det\big(\mathcal{W}_{11}(r,\lambda)\mathcal{P}_1(\lambda) + \mathcal{W}_{12}(r,\lambda)\mathcal{P}_2(\lambda)\big) = 0, \tag{A.11}$$

then (for some $g \neq 0$)

$$\big(\mathcal{W}_{11}(r,\lambda)\mathcal{P}_1(\lambda) + \mathcal{W}_{12}(r,\lambda)\mathcal{P}_2(\lambda)\big) g = 0 \quad (g \in \mathbb{C}^p), \tag{A.12}$$

and so we obtain (for such λ in $\mathbb{C}_+$ that (A.8) and (A.11) hold):

$$g^* \begin{bmatrix}\mathcal{P}_1(\lambda)^* & \mathcal{P}_2(\lambda)^*\end{bmatrix} \mathcal{W}(r,\lambda)^* j \mathcal{W}(r,\lambda) \begin{bmatrix}\mathcal{P}_1(\lambda) \\ \mathcal{P}_2(\lambda)\end{bmatrix} g < 0. \tag{A.13}$$

Clearly, (A.13) contradicts (A.10).
Let us rewrite (A.6) in the form

$$\begin{bmatrix} I_p \\ \phi(\lambda)\end{bmatrix} = jW(r,\overline{\lambda})^* j \begin{bmatrix}\mathcal{P}_1(\lambda) \\ \mathcal{P}_2(\lambda)\end{bmatrix} \big(\mathcal{W}_{11}(r,\lambda)\mathcal{P}_1(\lambda) + \mathcal{W}_{12}(r,\lambda)\mathcal{P}_2(\lambda)\big)^{-1}.$$

Now, setting

$$\mathfrak{A}(r,\lambda) := W(r,\lambda)^* j W(r,\lambda), \tag{A.14}$$

and using (A.3), we see that formulas (A.6) and (A.8) (i.e., the relation $\phi \in \mathcal{N}(r)$) yield

$$\begin{bmatrix} I_p & \phi(\lambda)^*\end{bmatrix} \mathfrak{A}(r,\lambda) \begin{bmatrix} I_p \\ \phi(\lambda)\end{bmatrix} \geq 0. \tag{A.15}$$

Moreover, according to (A.4), (A.14) and (A.15), $\phi(\lambda)$ is holomorphic and contractive in $\mathbb{C}_+$. On the other hand, if meromorphic ϕ satisfies (A.15), we set

$$\begin{bmatrix}\mathcal{P}_1(\lambda) \\ \mathcal{P}_2(\lambda)\end{bmatrix} = W(r,\lambda) \begin{bmatrix} I_p \\ \phi(\lambda)\end{bmatrix}, \tag{A.16}$$

and see that the relations (A.6) are (A.8) are valid. Thus,

$$\phi(\lambda) \in \mathcal{N}(r) \tag{A.17}$$

is equivalent to (A.15). Therefore, according to (A.4), $\mathcal{N}(r_2)$ is embedded in $\mathcal{N}(r_1)$:

$$\mathcal{N}(r_2) \subseteq \mathcal{N}(r_1) \quad (r_1 < r_2). \tag{A.18}$$

By virtue of Montel's theorem, there is a sequence $\{\phi_k(\lambda)\}$ such that

$$\phi_k \in \mathcal{N}(r_k), \quad r_k \to \infty \quad (\text{for } k \to \infty), \tag{A.19}$$

and $\phi_k(\lambda)$ tend uniformly (on any compact in $\mathbb{C}_+$) to some matrix function $\varphi(\lambda)$. Thus, $\varphi(\lambda)$ is holomorphic and satisfies (A.15) for any $r > 0$. In other words,

$$\varphi(\lambda) \in \bigcap_{r>0} \mathcal{N}(r). \tag{A.20}$$

Let us write down $\mathcal{N}(r)$ in the Weyl matrix disk form. Taking into account (A.4) and (A.14), we obtain

$$-\mathfrak{A}_{22}(r_2,\lambda) \geq -\mathfrak{A}_{22}(r_1,\lambda) \geq I_p \quad (r_2 > r_1); \tag{A.21}$$

$$\mathfrak{A}(r_2,\lambda)^{-1} \geq \mathfrak{A}(r_1,\lambda)^{-1} \geq j, \tag{A.22}$$

$$\big(\mathfrak{A}(r,\lambda)^{-1}\big)_{11} = \big(\mathfrak{A}_{11}(r,\lambda) - \mathfrak{A}_{12}(r,\lambda)\mathfrak{A}_{22}(r,\lambda)^{-1}\mathfrak{A}_{21}(r,\lambda)\big)^{-1} \geq I_p, \tag{A.23}$$

where $\mathfrak{A}_{ik}(r,\lambda)$ and $\big(\mathfrak{A}(r,\lambda)^{-1}\big)_{ik}$, respectively, are $p \times p$ blocks of $\mathfrak{A}(r,\lambda)$ and $\mathfrak{A}(r,\lambda)^{-1}$. The invertibility of $\mathfrak{A}_{11} - \mathfrak{A}_{12}\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}$ in (A.23) follows from the invertibility of $\mathfrak{A}$ and $\mathfrak{A}_{22}$ (for this and for the equality in (A.23) see, e.g., [55, p. 21]). In particular, we derive from (A.21) and (A.23) that the following positive definite square roots are uniquely defined:

$$\rho_L(r,\lambda) = \big(-\mathfrak{A}_{22}(r_2,\lambda)^{-1}\big)^{1/2}, \tag{A.24}$$

$$\rho_R(r,\lambda) = \big(\mathfrak{A}_{11}(r,\lambda) - \mathfrak{A}_{12}(r,\lambda)\mathfrak{A}_{22}(r,\lambda)^{-1}\mathfrak{A}_{21}(r,\lambda)\big)^{1/2}. \tag{A.25}$$

Here, ρ_L and ρ_R are the so called left and right semi-radii of the Weyl disk. The inequality (A.15) may be rewritten in the form of the Weyl disk parametrization of the values $\phi(r,\lambda)$ (similar, for instance, to the parametrization [15, (2.19)] for Dirac systems):

$$\phi(r,\lambda) = \rho_L(r,\lambda)\omega(r,\lambda)\rho_R(r,\lambda) - \mathfrak{A}_{22}(r,\lambda)^{-1}\mathfrak{A}_{21}(r,\lambda) \quad (\omega^*\omega \leq I_p), \tag{A.26}$$

where $\omega(r,\lambda)$ are $p \times p$ matrices and $\phi \in \mathcal{N}_r$. Recall that the matrix inequality $B_2 \geq B_1 \geq 0$ yields $B_2^{1/2} \geq B_1^{1/2}$ (see, e.g., [4]). Hence, in view of (A.21)–(A.25) the left and right semi-radii are non-increasing.

By $L^2(H)$ we denote the space of vector functions on $\mathbb{R}_+$ with the scalar product

$$(f_1, f_2)_H = \int_0^\infty f_2(x)^* H(x) f_1(x) dx.$$

Proposition A.1. *Let $H(x)$ $(x \geq 0)$ be the Hamiltonian of a canonical system. Then, there is $\varphi(\lambda)$, which satisfies* (A.20)*, for this system. If* (A.20) *holds, the columns of $W(x,\lambda)\begin{bmatrix} I_p \\ \varphi(\lambda)\end{bmatrix}$ belong $L^2(H)$, that is,*

$$\int_0^\infty \begin{bmatrix} I_p & \varphi(\lambda)^* \end{bmatrix} W(x,\lambda)^* H(x) W(x,\lambda) \begin{bmatrix} I_p \\ \varphi(\lambda)\end{bmatrix} dx < \infty \quad (\lambda \in \mathbb{C}_+). \tag{A.27}$$

Proof. We already proved that $\bigcap_{r>0}\mathcal{N}(r)$ is non-empty. Moreover, in view of (A.15), for any φ satisfying (A.20) and any $r > 0$ we have

$$\begin{bmatrix} I_p & \varphi(\lambda)^* \end{bmatrix} \mathfrak{A}(r,\lambda) \begin{bmatrix} I_p \\ \varphi(\lambda)\end{bmatrix} \geq 0. \tag{A.28}$$

Taking into account (A.2) and (A.28), we derive

$$\begin{aligned} &\int_0^r \begin{bmatrix} I_p & \varphi(\lambda)^* \end{bmatrix} W(x,\lambda)^* H(x) W(x,\lambda) \begin{bmatrix} I_p \\ \varphi(\lambda)\end{bmatrix} dx \\ &\leq \frac{\mathrm{i}}{\lambda - \overline{\lambda}} \begin{bmatrix} I_p & \varphi(\lambda)^* \end{bmatrix} j \begin{bmatrix} I_p \\ \varphi(\lambda)\end{bmatrix} \leq \frac{\mathrm{i}}{\lambda - \overline{\lambda}} I_p, \end{aligned} \tag{A.29}$$

and (A.27) follows. □

Definition A.2. *Holomorphic (in $\mathbb{C}_+$) $p \times p$ matrix functions $\varphi(\lambda)$, such that the inequality* (A.27) *holds, are called Weyl–Titchmarsh (Weyl) functions of the canonical system* (1.4) *on $[0, \infty)$.*

Proposition A.1 implies that Weyl function always exists.

B Canonical systems and matrix string and Schrödinger equations: interconnections

1. In view of (1.3), canonical systems (1.4) with Hamiltonians $H(x)$ of the form (1.6) may be transformed into systems (1.1) with Hamiltonians $\mathcal{H}$:

$$\Upsilon'(x,\lambda) = \mathrm{i}\lambda J \mathcal{H}(x)\Upsilon(x,\lambda), \quad \mathcal{H} = \vartheta(x)^*\vartheta(x), \quad \vartheta(x)J\vartheta(x)^* = 0, \tag{B.1}$$

using the transformation

$$\Upsilon(x,\lambda) = \Theta w(x,\lambda), \quad \mathcal{H}(x) = \Theta H(x)\Theta^*, \quad \vartheta(x) = \beta(x)\Theta^*. \tag{B.2}$$

Clearly, the inverse transformation works as well, that is, systems (1.4), (1.6) and systems (B.1) are equivalent.

It will be convenient to repeat here the transformation (from [53, Ch. 4] or [55, Section 11.1]) of the system (B.1) into the matrix string equation. We partition $p \times 2p$ matrix function $\vartheta(x)$ into $p \times p$ blocks $\vartheta(x) = \begin{bmatrix} \vartheta_1(x) & \vartheta_2(x) \end{bmatrix}$. We assume that $\det(\vartheta_1(x)) \neq 0$, and we require also that $\vartheta_1(x)^{-1}\vartheta_2(x)$ is absolutely continuous and its derivative is invertible. We set

$$\mathcal{Y}(x,\lambda) = \vartheta(x)\Upsilon(x,\lambda), \quad \mathcal{Z}(x,\lambda) = \vartheta_1(x)^{-1}\vartheta(x)\Upsilon(x,\lambda), \tag{B.3}$$

$$\varkappa(x) := \Big(\mathrm{i}\big(\vartheta_1(x)^{-1}\vartheta_2(x)\big)'\Big)^{-1} = \varkappa(x)^*. \tag{B.4}$$

The self-adjointness of $\varkappa(x)$ above follows from the last equality in (B.1). According to (B.1) and (B.3), we have

$$\Upsilon'(x,\lambda) = \mathrm{i}\lambda J\vartheta(x)^*\mathcal{Y}(x,\lambda), \tag{B.5}$$

$$\mathcal{Z}'(x,\lambda) = \big(\vartheta_1(x)^{-1}\vartheta(x)\big)'\Upsilon(x,\lambda) = \Big[0 \quad \big(\vartheta_1(x)^{-1}\vartheta_2(x)\big)'\Big]\,\Upsilon(x,\lambda). \tag{B.6}$$

Finally, taking into account (B.4)–(B.6), we see that $\mathcal{Z}(x,\lambda)$ satisfies the matrix string equation

$$\frac{d}{dx}\Big(\varkappa(x)\frac{d}{dx}\mathcal{Z}(x,\lambda)\Big) = \lambda\vartheta_1(x)^*\mathcal{Y}(x,\lambda) = \lambda\omega(x)\mathcal{Z}(x,\lambda), \tag{B.7}$$

$$\omega(x) := \vartheta_1(x)^*\vartheta_1(x) > 0. \tag{B.8}$$

2. Now, consider the matrix Schrödinger equation

$$-\mathcal{Z}''(x,\lambda) + u(x)\mathcal{Z}(x,\lambda) = \lambda\mathcal{Z}(x,\lambda) \quad \big(u(x) = u(x)^*\big), \tag{B.9}$$

where u is a $p\times p$ matrix function. The transformation of (B.9) into the canonical system of the form (B.1), such that

$$\vartheta''(x) = u(x)\vartheta(x), \tag{B.10}$$

and $\vartheta(x)$ is normalised at $x=0$ by

$$B(0) = \Theta_1 := \frac{1}{\sqrt{2}}\begin{bmatrix}\mathrm{i}I_p & I_p\\ \mathrm{i}I_p & -I_p\end{bmatrix}, \tag{B.11}$$

where

$$B(x) := \begin{bmatrix}\vartheta(x)\\ \vartheta'(x)\end{bmatrix}, \tag{B.12}$$

is described in [55, Section 11.2]. The interconnections between the spectral theories of systems (B.1), (B.10) and equations (B.9) are also studied there. It is easily checked (see also [37] for the case $p=1$) that the above-mentioned transformation in [55, Section 11.2] works in the opposite direction as well. Namely, *starting from the canonical system* (B.1)*,* (B.10)*,* (B.11) *one comes to the Schrödinger equation* (B.9). Indeed, according to [55, (2.10)], we have

$$B(0)^*J_1B(0) = \Theta_1^*J_1\Theta_1 = J, \quad J_1 := \mathrm{i}\begin{bmatrix}0 & -I_p\\ I_p & 0\end{bmatrix}, \tag{B.13}$$

where J is given in (1.1). Moreover, the equalities (B.10) and (B.12) yield

$$B'(x) = \begin{bmatrix}0 & I_p\\ u(x) & 0\end{bmatrix}B(x). \tag{B.14}$$

The relations (B.13) and (B.14) imply that

$$B(x)^*J_1B(x) = B(0)^*J_1B(0) = J, \tag{B.15}$$

and so

$$B(x)JB(x)^* = J_1. \tag{B.16}$$

REMARK B.1. *Formula* (B.16) *shows that the equalities*

$$\vartheta(x)J\vartheta(x)^* = 0, \quad \vartheta'(x)J\vartheta(x)^* = \mathrm{i}I_p \tag{B.17}$$

follow from (B.10)–(B.12).

Finally, setting

$$\mathcal{Z}(x,\lambda) = \vartheta(x)\Upsilon(x,\lambda) \tag{B.18}$$

and taking into account (B.1), (B.10) and (B.17), we derive

$$\mathcal{Z}'' = u\mathcal{Z} - 2\lambda\mathcal{Z} + \lambda\mathcal{Z},$$

that is, $\mathcal{Z}(x,\lambda)$ satisfies matrix Schrödinger equation (B.9). Formula (B.18) describes the connection between the solutions of the canonical system (B.1), (B.10), (B.11) and of the corresponding Schrödinger equation (B.9).

3. Since matrix Schrödinger equations may be transformed (see [55]) into canonical systems satisfying (B.10)–(B.12) (and by virtue of Remark B.1), they are also equivalent to a subclass of canonical systems (1.4) with Hamiltonians of the form (2.1).

REMARK B.2. *It is easy to see that in the case of our explicit formulas* (4.1), (4.5), (4.6) *the matrix function* $\vartheta(x) = \beta(x)\Theta^*$ *satisfies the equality* (B.10), *where* $u = -c^2 I_p$. *However,* $\widetilde{\vartheta}(x) = \widetilde{\beta}(x)\Theta^*$ *does not satisfy* (B.10) (*excluding, possibly, some special cases*).

REMARK B.3. *Formulas* (B.3), (B.4), (B.7), *and* (B.8) *show that explicit expressions for the Hamiltonians and fundamental solutions constructed in this paper generate explicit expressions for the matrix string equations and their solutions as well.*

C ON LINEAR SIMILARITY TO SQUARED INTEGRATION

We will consider similarity transformations of linear integral operators K in $L_2^p(0,\mathbf{T})$ $(0<\mathbf{T}<\infty)$:

$$K = \mathrm{i}\beta(x)j\int_0^x \beta(t)^* \cdot dt, \quad \beta(x)j\beta(x)^* \equiv 0, \quad \beta'(x)j\beta(x)^* \equiv \mathrm{i}I_p, \tag{C.1}$$

where $\beta(x)$ is a $p\times 2p$ matrix function and

$$j = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}. \tag{C.2}$$

Recall that the operator A is introduced in (2.2). The class of operators $K = \int_0^x K(x,t)\cdot dt$, which are linear similar to A above, was studied (for the case of the scalar kernel function $K(x,t)$) in the essential for our considerations paper [51]. Here, we study an important special subclass (C.1) of such operators under reduced smoothness conditions on $K(x,t)$. We include the matrix case (i.e., the case $p>1$) and present a complete proof of the similarity result.

Theorem C.1. *Let the operator K be given by the first equality in (C.1), and let $\beta(x)$ satisfy the second and the third equalities in (C.1). Assume that $\beta(x)$ is two times differentiable and the entries of $\beta''(x)$ are square-integrable, that is, $\beta''(x) \in L_2^{p\times 2p}(0,\mathbf{T})$. Then, K is linear similar to A :*

$$K = VAV^{-1}, \quad V = u(x)\big(I + \int_0^x \mathcal{V}(x,t)\, \cdot\, dt\big), \tag{C.3}$$

where $u(x)$ is a two times differentiable $p\times p$ matrix function (such that $u^* = u^{-1}$ and $u'' \in L_2^{p\times p}(0,\mathbf{T})$), and

$$\sup \|\mathcal{V}(x,t)\| < \infty \quad (0 \le t \le x \le \mathbf{T}). \tag{C.4}$$

Proof. In the proof, we construct an operator V, which satisfies theorem's conditions. This V is closely related to *transformation operators in inverse spectral and scattering theories.*

Step 1. Together with K, we consider the operators:

$$\breve{K} := \mathrm{i}\beta''(x)j\int_0^x \beta(t)^* \,\cdot\, dt, \quad (I-\breve{K})^{-1} = I + \int_0^x \mathcal{R}(x,t)\,\cdot\, dt. \tag{C.5}$$

The operator $\breve{K}$ has a semi-separable kernel, and so (see, e.g., [21, Section IX.2]) the matrix function $\mathcal{R}$ in (C.5) has the form

$$\mathcal{R}(x,t) = \mathrm{i}\beta''(x)u_1(x)u_1(t)^{-1}j\beta(t)^* \quad (0 \le t \le x), \tag{C.6}$$

where the $2p\times 2p$ matrix function u_1 is the normalised fundamental solution of the system

$$u_1'(x) = \mathrm{i}j\beta(x)^*\beta''(x)u_1(x), \quad u_1(0) = I_{2p}. \tag{C.7}$$

Introduce the $p \times p$ matrix function $g(x)$ by the equalities

$$g(0) = I_p, \quad g'(0) = \frac{\mathrm{i}}{2}\beta'(0)j\beta'(0)^*, \tag{C.8}$$

$$g''(x) = \mathrm{i}(I-\breve{K})\Big(\beta''(x)u_1(x)\big(j\beta'(0)^* + \frac{\mathrm{i}}{2}j\beta(0)^*\beta'(0)j\beta'(0)^*\big)\Big), \tag{C.9}$$

where the operator $(I-\breve{K})$ on the right-hand side of (C.9) is applied columnwise to the $p\times p$ matrix function above. Further in the proof, we study the matrix function

$$y(x,z) := (I - z^2K)^{-1}g(x). \tag{C.10}$$

(Since K has a semi-separable kernel, one can write down a more explicit expression for y as well.) Differentiating two times both parts of the equality $(I-z^2K)y(x,z) = g(x)$ and taking into account (C.1), (C.5), we derive

$$\begin{aligned} y''(x,z) &= g''(x) - z^2y(x,z) + \mathrm{i}z^2\beta''(x)j\int_0^x \beta(t)^*y(t,z)dt \\ &= g''(x) - z^2(I-\breve{K})y(x,z). \end{aligned} \tag{C.11}$$

It easily follows also from (C.1), (C.8), and (C.10) that

$$y(0,z) = I_p, \quad y'(0,z) = \frac{\mathrm{i}}{2}\beta'(0)j\beta'(0)^*. \tag{C.12}$$

Applying (columnwise) $(I - \breve{K})^{-1}$ to both parts of (C.11), using (C.5) and (C.9), and integrating by parts, we obtain

$$\begin{aligned} y''(x,z) =& -\int_0^x \mathcal{R}(x,t)y''(t,z)dt \\ & + \mathrm{i}\beta''(x)u_1(x)\big(j\beta'(0)^* + \frac{\mathrm{i}}{2}j\beta(0)^*\beta'(0)j\beta'(0)^*\big) - z^2 y(x,z) \\ =& -\mathcal{R}(x,t)y'(t,z)\Big|_0^x + \Big(\frac{\partial \mathcal{R}}{\partial t}(x,t)y(t,z)\Big)\Big|_0^x \\ & + \mathrm{i}\beta''(x)u_1(x)\big(j\beta'(0)^* + \frac{\mathrm{i}}{2}j\beta(0)^*\beta'(0)j\beta'(0)^*\big) - z^2 y(x,z) \\ & - \int_0^x \Big(\frac{\partial^2}{\partial t^2}\mathcal{R}(x,t)\Big)y(t,z)dt. \end{aligned} \tag{C.13}$$

In view of the last equality in (C.1), we have $\big(\beta'(x)j\beta(x)^*\big)' = 0$, which yields

$$u_2(x) := \mathrm{i}\beta''(x)j\beta(x)^* = -\mathrm{i}\beta'(x)j\beta'(x)^*. \tag{C.14}$$

Here, u_2 is a $p \times p$ matrix function. Relations (C.6), (C.7), and (C.14) imply:

$$\mathcal{R}(x,x) = u_2(x); \quad \frac{\partial}{\partial t}\mathcal{R}(x,t)\Big|_{t=x} = -u_2(x)^2 + u_3(x), \tag{C.15}$$

$$u_2(x)' = u_3(x)^* - u_3(x), \quad u_3(x) := \mathrm{i}\beta''(x)j\beta'(x)^*; \tag{C.16}$$

$$\frac{\partial}{\partial t}\mathcal{R}(x,t)\Big|_{t=0} = \mathrm{i}\beta''(x)u_1(x)\big(j\beta'(0)^* - j\beta(0)^*u_2(0)\big); \tag{C.17}$$

$$\mathcal{R}(x,0) = \mathrm{i}\beta''(x)u_1(x)j\beta(0)^*. \tag{C.18}$$

Taking into account (C.12) and (C.14)–(C.17), we rewrite (C.13) in the form:

$$\begin{aligned} y''(x,z) + z^2 y(x,z) =& -u_2(x)y'(x,z) - \big(u_2(x)^2 - u_3(x)\big)y(x,z) \\ & - \int_0^x \Big(\frac{\partial^2}{\partial t^2}\mathcal{R}(x,t)\Big)y(t,z)dt, \end{aligned} \tag{C.19}$$

where

$$\frac{\partial^2}{\partial t^2}\mathcal{R}(x,t) = h_1(x)h_2(t), \quad h_1 \in L_2^{p\times 2p}(0,\mathbf{T}), \quad h_2 \in L_2^{2p\times p}(0,\mathbf{T}); \tag{C.20}$$

and

$$\begin{aligned} h_1(x) := \mathrm{i}\beta''(x)u_1(x), \quad h_2(t) :=& u_1(t)^{-1}\big(j\beta(t)^*u_2(t)^2 - j\beta(t)^*u_3(t)^* \\ & - j\beta'(t)^*u_2(t) + j\beta''(t)^*\big). \end{aligned} \tag{C.21}$$

We introduce the $p \times p$ matrix functions $y_1(x,z)$ and $u(x)$ by the equalities

$$y_1(x,z) = u(x)^{-1}y(x,z); \quad u'(x) = -\frac{1}{2}u_2(x)u(x), \quad u(0) = I_p. \tag{C.22}$$

Since $u_2^* = -u_2$ and $u(0) = I_p$, we obtain $u^* = u^{-1}$. Thus, using formulas (C.16), (C.20), and (C.22), we rewrite (C.19) in the form

$$y_1''(x,z) + z^2 y_1(x,z) = u_4(x)y_1(x,z) - u(x)^* h_1(x)\int_0^x h_2(t)u(t)y_1(t,z)dt, \tag{C.23}$$

$$u_4(x) := u(x)^*\Big(\frac{1}{2}\big(u_3(x)+u_3(x)^*\big) - \frac{3}{4}u_2(x)^2\Big)u(x). \tag{C.24}$$

In view of (C.12) and (C.22), the initial conditions for y_1 take the form

$$y_1(0,z) = I_p, \quad y_1'(0,z) = y'(0,z) - u'(0) = 0. \tag{C.25}$$

Step 2. Let us construct the solution of system (C.23) with the initial conditions (C.25) as a series

$$y_1(x,z) = \sum_{k=0}^{\infty}\psi_k(x,z), \quad \psi_0(x,z) = \cos(zx)I_p, \tag{C.26}$$

$$\begin{aligned}\psi_k(x,z) := \int_0^x \cos\big(z(x-t)\big)\int_0^t \Big(u_4(s)\psi_{k-1}(s,z) \\ - \int_0^s \mathcal{F}(s,\eta))\psi_{k-1}(\eta,z)d\eta\Big)dsdt \quad (k\geq 1),\end{aligned} \tag{C.27}$$

$$\mathcal{F}(x,t) := u(x)^* h_1(x)h_2(t)u(t). \tag{C.28}$$

Clearly, for $k \geq 1$ we have

$$\psi_k''(x,z) = -z^2\psi_k(x,z) + u_4(x,z)\psi_{k-1}(x,z) - \int_0^x \mathcal{F}(x,\eta)\psi_{k-1}(\eta,z)d\eta.$$

Thus, if the corresponding series converge, the matrix function y_1 given by (C.26)–(C.28) satisfies (C.23). Convergences follow from the representation

$$\psi_k(x,z) = \int_0^x \cos(z\zeta)\mathcal{V}_k(x,\zeta)d\zeta \quad (k \geq 1), \tag{C.29}$$

which is proved by induction. Indeed, setting $k=1$ in (C.27), taking into account that $\psi_0(s,z) = \cos(zs)I_p$ and

$$\cos\big(z(x-t)\big)\cos(zs) = \frac{1}{2}\Big(\cos\big(z(x-t-s)\big) + \cos\big(z(x-t+s)\big)\Big), \tag{C.30}$$

$$\cos\big(z(x-t)\big)\cos(z\eta) = \frac{1}{2}\Big(\cos\big(z(x-t-\eta)\big) + \cos\big(z(x-t+\eta)\big)\Big),$$

and changing variables and order of integration, we derive:

$$\psi_1(x,z) = \int_0^x \cos(z\zeta)\mathcal{V}_1(x,\zeta)d\zeta, \tag{C.31}$$

$$\begin{aligned}\mathcal{V}_1(x,\zeta) =& \frac{1}{2}\Bigg(\int_0^{(x+\zeta)/2} u_4(t)dt + \int_0^{(x-\zeta)/2} u_4(t)dt - \int_{(x+\zeta)/2}^x \breve{\mathcal{F}}(t,x-t+\zeta)dt \\ &- \int_{(x-\zeta)/2}^{x-\zeta}\breve{\mathcal{F}}(t,x-t-\zeta)dt - \int_{(x-\zeta)}^x \breve{\mathcal{F}}(t,\zeta+t-x)dt\Bigg),\end{aligned} \tag{C.32}$$

$$\breve{\mathcal{F}}(t,\eta) := \int_\eta^t \mathcal{F}(s,\eta)ds. \tag{C.33}$$

Assuming that (C.29) holds for $k-1$ and ψ_k is given by (C.27), we use a similar procedure (i.e., formulas of the (C.30) type, change of variables and change of order of integration) and obtain (C.29) for k, where

$$\begin{aligned}
2\mathcal{V}_k(x,\zeta) = & \int_{x-\zeta}^{x}\int_{\zeta+t-x}^{t} u_4(s)\mathcal{V}_{k-1}(s,\zeta+t-x)dsdt \\
& + \int_{(x+\zeta)/2}^{x}\int_{\zeta+x-t}^{t} u_4(s)\mathcal{V}_{k-1}(s,\zeta+x-t)dsdt \\
& + \int_{(x-\zeta)/2}^{x-\zeta}\int_{x-t-\zeta}^{t} u_4(s)\mathcal{V}_{k-1}(s,x-t-\zeta)dsdt \\
& - \int_{x-\zeta}^{x}\int_{\zeta+t-x}^{t}\int_{\zeta+t-x}^{s} \mathcal{F}(s,\eta)\mathcal{V}_{k-1}(\eta,\zeta+t-x)d\eta dsdt \\
& - \int_{(x+\zeta)/2}^{x}\int_{\zeta+x-t}^{t}\int_{\zeta+x-t}^{s} \mathcal{F}(s,\eta)\mathcal{V}_{k-1}(\eta,\zeta+x-t)d\eta dsdt \\
& - \int_{(x-\zeta)/2}^{x-\zeta}\int_{x-t-\zeta}^{t}\int_{x-t-\zeta}^{s} \mathcal{F}(s,\eta)\mathcal{V}_{k-1}(\eta,x-t-\zeta)d\eta dsdt.
\end{aligned} \tag{C.34}$$

Thus, the representation (C.29) is proved. Moreover, one can choose such $C(T)=C>0$ that

$$\int_0^{\mathbf{T}}\|h_k(t)\|dt \le C \quad (k=1,2), \qquad \int_0^{\mathbf{T}}\|u_4(t)\|dt \le C^2, \tag{C.35}$$

$$\sup_{0\le\zeta\le x\le\mathbf{T}}\|\mathcal{V}_1(x,\zeta)\| \le C. \tag{C.36}$$

We assume that the inequalities (C.35) and (C.36) hold. In particular, the inequality

$$\|\mathcal{V}_k(x,\zeta)\| \le \frac{(3C^2)^{k-1}}{(k-1)!}Cx^{k-1} \quad (k\ge 1) \tag{C.37}$$

is fulfilled for $k=1$. If (C.37) is valid for $\mathcal{V}_{k-1}$, relations (C.34)–(C.36) imply that (C.37) is valid for $\mathcal{V}_k$. Hence, (C.37) is proved. Therefore, the series $\sum_{k=1}^{\infty}\|\mathcal{V}_k(x,\zeta)\|$ is convergent. Thus, the series in (C.26) converges as well, and (in view of (C.26), (C.29), (C.37)) we have

$$y_1(x,z) = \cos(zx)I_p + \int_0^x \cos(z\zeta)\mathcal{V}(x,\zeta)d\zeta, \tag{C.38}$$

$$\mathcal{V}(x,\zeta) := \sum_{k=1}^{\infty}\mathcal{V}_k(x,\zeta), \qquad \sup_{0\le\zeta\le x\le\mathbf{T}}\|\mathcal{V}(x,\zeta)\| < \infty. \tag{C.39}$$

It is immediate from (C.38) and (C.39) that $y_1(0,z)=I_p$. In order to calculate the initial value $y_1'(0,z)$, we differentiate both sides of (C.27) and obtain

$$\begin{aligned}
\psi_k'(x,z) = & -z\int_0^x \sin\big(z(x-t)\big)\int_0^t \Big(u_4(s)\psi_{k-1}(s,z) \\
& - \int_0^s \mathcal{F}(s,\eta))\psi_{k-1}(\eta,z)d\eta\Big)dsdt \\
& + \int_0^x \Big(u_4(s)\psi_{k-1}(s,z) - \int_0^s \mathcal{F}(s,\eta))\psi_{k-1}(\eta,z)d\eta\Big)ds \quad (k\ge 1).
\end{aligned} \tag{C.40}$$

Now, the equality $y_1'(0,z)=0$ easily follows from (C.26) and (C.40). Summing up, we have shown (in this step of the proof) that y_1 of the form (C.38) satisfies (C.23) and (C.25).

Step 3. Next, we show that the solution y_1 of (C.23), (C.25) is unique. Multiplying both parts of (C.23) by the operator A (given in (2.2)) and using (C.25), we derive

$$By_1 = I_p + z^2 Ay_1, \tag{C.41}$$

$$Bf = f - \int_0^x \mathcal{B}(x,t)f(t)dt := f + A(u_4 f) - A\int_0^x \mathcal{F}(x,t)f(t)dt. \tag{C.42}$$

From (2.2), (C.28), (C.41) and (C.42), after simple calculations we obtain

$$\mathcal{B}(x,t) = \begin{bmatrix} I_p & xI_p & u_7(x)\end{bmatrix}\begin{bmatrix} u_5(t)-tu_4(t) \\ u_4(t)+u_6(t) \\ h_2(t)u(t)\end{bmatrix}, \tag{C.43}$$

$$u_5(t) := -\int_0^t su(s)^* h_1(s)ds\, h_2(t)u(t), \quad u_6(t) := \int_0^t u(s)^* h_1(s)ds\, h_2(t)u(t),$$

$$u_7(x) := \int_0^x (s-x)u(s)^* h_1(s)ds. \tag{C.44}$$

Here, u_4 is given in (C.24) and the following transformation is used:

$$\begin{aligned}\int_0^x (t-x)\int_0^t \mathcal{F}(t,s)\cdot ds dt &= \int_0^x (s-x)\int_t^x \mathcal{F}(s,t)ds\cdot dt \\ &= \int_0^x\int_0^x (s-x)u(s)^* h_1(s)ds\, h_2(t)u(t)\cdot dt \\ &\quad - \int_0^x\int_0^t (s-x)u(s)^* h_1(s)ds\, h_2(t)u(t)\cdot dt.\end{aligned} \tag{C.45}$$

Since B is a triangular operator and the integral part of B has a semi-separable kernel, it easily follows (see, e.g., [21, Section IX.2]) that B is invertible and B^{-1} is a bounded operator. (In fact, the integral part of B is a Volterra operator from Hilbert-Schmidt class and $B^{-1}-I$ is again a triangular Volterra operator with a semi-separable kernel.) Thus, we rewrite (C.41) as

$$y_1 = (I - z^2 B^{-1}A)^{-1}B^{-1}I_p. \tag{C.46}$$

Now, it is easy to see that y_1 is unique. Recall that this unique solution admits representation (C.3), and so, taking into account (C.22), we obtain

$$y(x,z) = V\big(\cos(zx)I_p\big), \tag{C.47}$$

where V is given by the second equality in (C.3) and is applied to $\cos(zx)I_p$ columnwise. One easily checks that

$$(I - z^2 A)^{-1}I_p = \cos(zx)I_p. \tag{C.48}$$

In view of (C.10), (C.47), and (C.48), we have

$$(I - z^2 K)^{-1}g = V(I - z^2 A)^{-1}I_p. \tag{C.49}$$

Presenting the resolvents in both parts of (C.49) as series, we rewrite (C.49) in the form $K^n g = VA^n I_p$. In particular, setting $n = 0$, we derive $g = VI_p$. The substitution $g = VI_p$ into $K^n g = VA^n I_p$ yields

$$K^n VI_p = VA^n I_p \quad (n \geq 0). \tag{C.50}$$

It follows that

$$(KV)A^n I_p = K(VA^n I_p) = K^{n+1} VI_p = VA^{n+1} I_p = (VA)A^n I_p. \tag{C.51}$$

One can easily see (using, e.g., Weierstrass approximation theorem) that the closed linear span of the columns of the matrix functions $A^n I_p$ $(n \geq 0)$ coincides with $L_2^p(0, \mathbf{T})$. Therefore, (C.51) implies $KV = VA$, and (C.3) follows. The required properties of u and $\mathcal{V}$ have already been proved. □

Remark C.2. *It is important for the study of the canonical systems on the semi-axis* $[0, \infty)$ *that, according to* (C.26)–(C.28)*,* (C.29)*, and* (C.39)*, the matrix function* $\mathcal{V}(x, \zeta)$ *in the domain* $0 \leq \zeta \leq x \leq \ell$ *is uniquely determined by* $\beta(x)$ *on* $[0, \ell]$ (*and does not depend on the choice of* $\beta(x)$ *for* $\ell < x < \mathbf{T}$ *and the choice of* $\mathbf{T} \geq \ell$).

Acknowledgments

The author is grateful to the referee for useful remarks. This research was supported by the Austrian Science Fund (FWF) under Grant No. P29177.

References

[1] K. Acharya, *An alternate proof of the de Branges theorem on canonical systems*, ISRN Math. Anal. (2014), Art. 704607.

[2] D.Z. Arov and H. Dym, *Bitangential direct and inverse problems for systems of integral and differential equations,* Cambridge University Press, Cambridge, 2012.

[3] M.I. Belishev and V.S. Mikhailov, *Inverse problem for a one-dimensional dynamical Dirac system (BC-method),* Inverse Problems 30 (2014), Art. 125013.

[4] R. Bellman, *Some inequalities for the square root of a positive definite matrix,* Linear Algebra Appl. 1 (1968), 321–324.

[5] M. Bohner and O. Došlý, *Oscillation of symplectic dynamic systems,* ANZIAM J. 46 (2004), 17–32.

[6] J. Breuer, E. Ryckman, and B. Simon, *Equality of the spectral and dynamical definitions of reflection,* Comm. Math. Phys. 295 (2010), 531–550.

[7] J.L. Cieslinski, *Algebraic construction of the Darboux matrix revisited*, J. Phys. A 42 (2009), Art. 404003.

[8] A. Constantin and R. Ivanov, *Dressing method for the Degasperis-Procesi equation*, Stud. Appl. Math. 138 (2017), 205–226.

[9] L. de Branges, *Hilbert spaces of entire functions,* Prentice-Hall, Englewood Cliffs, N.J., 1968.

[10] O. Došlý, J. Elyseeva, and R. Šimon Hilscher, *Symplectic difference systems: oscillation and spectral theory,* Birkhäuser/Springer, Cham, 2019.

[11] O. Došlý and R. Šimon Hilscher, *Disconjugacy, transformations and quadratic functionals for symplectic dynamic systems on time scales,* J. Differ. Equations Appl. 7 (2001), no. 2, 265–295.

[12] J. Eckhardt and A. Kostenko, *The inverse spectral problem for indefinite strings*, Invent. Math. 204 (2016), 939–977.

[13] J. Eckhardt, F. Gesztesy, R. Nichols, A. Sakhnovich, and G. Teschl, *Inverse spectral problems for Schrödinger-type operators with distributional matrix-valued potentials,* Differential Integral Equations 28 (2015), 505–522.

[14] J. Eckhardt, A. Kostenko, and G. Teschl, *Spectral asymptotics for canonical systems,* J. Reine Angew. Math. 736 (2018), 285–315.

[15] B. Fritzsche, B. Kirstein, I.Ya. Roitberg, and A.L. Sakhnovich, *Weyl theory and explicit solutions of direct and inverse problems for Dirac system with a rectangular matrix potential*, Oper. Matrices 7 (2013), 183–196.

[16] B. Fritzsche, B. Kirstein, and A.L. Sakhnovich, *Weyl functions of generalized Dirac systems: integral representation, the inverse problem and discrete interpolation,* J. Anal. Math. 116 (2012), 17–51.

[17] F. Gesztesy, *A complete spectral characterization of the double commutation method*, J. Funct. Anal. 117 (1993), 401–446.

[18] F. Gesztesy and A. Sakhnovich, *The inverse approach to Dirac-type systems based on the A-function concept,* J. Funct. Anal. 279 (2020), Art. 108609.

[19] F. Gesztesy and G. Teschl, *On the double commutation method*, Proc. Amer. Math. Soc. 124 (1996), 1831–1840.

[20] F. Gesztesy, R. Weikard, and M. Zinchenko, *On spectral theory for Schrödinger operators with operator-valued potentials*, J. Differential Equations 255 (2013), 1784–1827.

[21] I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Classes of linear operators,* Vol. I, Birkhäuser, Basel, 1990.

[22] I. Gohberg, M.A. Kaashoek, and A.L. Sakhnovich, *Pseudocanonical systems with rational Weyl functions: explicit formulas and applications*, J. Differential Equations 146 (1998), 375–398.

[23] I.C. Gohberg and M.G. Krein, *Theory and applications of Volterra operators in Hilbert space,* Transl. Math. Monographs, 24, Amer. Math. Soc., Providence, RI, 1970.

[24] L. Golinskii and I. Mikhailova (edited by V.P. Potapov), *Hilbert spaces of entire functions as a J-theory subject*. In: Oper. Theory Adv. Appl. 95, pp. 205–251, Birkhäuser, Basel, 1997.

[25] C.H. Gu, H. Hu and Z. Zhou, *Darboux transformations in integrable systems. Theory and their applications to geometry,* Springer, Dordrecht, 2005.

[26] B. Jacob, K. Morris, and H. Zwart, *C_0-semigroups for hyperbolic partial differential equations on a one-dimensional spatial domain,* J. Evol. Equ. 15 (2015), 493–502.

[27] V. Jakšić, B. Landon, and A. Panati, *A note on reflectionless Jacobi matrices,* Comm. Math. Phys. 332 (2014), 827–838.

[28] D.J. Kaup, *Simple harmonic generation: an exact method of solution,* Stud. Appl. Math. 59 (1978), 25–35.

[29] D.J. Kaup and H. Steudel, *Recent results on second harmonic generation,* Contemporary Math. 326 (2003), 33–48.

[30] A. Kostenko, A. Sakhnovich, and G. Teschl, *Commutation methods for Schrödinger operators with strongly singular potentials,* Math. Nachr. 285 (2012), no. 4, 392–410.

[31] M.G. Krein, *Continuous analogues of propositions on polynomials orthogonal on the unit circle* (Russian), Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 637–640.

[32] W. Kratz and R. Šimon Hilscher, *Rayleigh principle for linear Hamiltonian systems without controllability,* ESAIM Control Optim. Calc. Var. 18 (2012), 501–519.

[33] H. Langer, *Transfer functions and local spectral uniqueness for Sturm-Liouville operators, canonical systems and strings*, Integral Equations Operator Theory 85 (2016), 1–23.

[34] V.A. Marchenko, *Nonlinear equations and operator algebras*, D. Reidel, Dordrecht, 1988.

[35] V.B. Matveev and M.A. Salle, *Darboux transformations and solitons*, Springer, Berlin, 1991.

[36] V. Mogilevskii, *Spectral and pseudospectral functions of Hamiltonian systems: development of the results by Arov–Dym and Sakhnovich,* Methods Funct. Anal. Topology 21 (2015), no. 4, 370–402.

[37] C. Remling, *Spectral theory of canonical systems*, De Gruyter, Berlin, 2018.

[38] R. Romanov, *Order problem for canonical systems and a conjecture of Valent,* Trans. Amer. Math. Soc. 369 (2017), 1061–1078.

[39] R. Romanov and H. Woracek, *Canonical systems with discrete spectrum,* J. Funct. Anal. 278 (2020), Art. 108318.

[40] J. Rovnyak and L.A. Sakhnovich, *Pseudospectral functions for canonical differential systems. II,* in Oper. Theory Adv. Appl. 218, Birkhäuser/Springer, Basel, 2012, pp. 583–612.

[41] A.L. Sakhnovich, *Dressing procedure for solutions of nonlinear equations and the method of operator identities,* Inverse Problems 10 (1994), 699–710.

[42] A.L. Sakhnovich, *Iterated Bäcklund–Darboux transform for canonical systems,* J. Funct. Anal. 144 (1997), 359–370.

[43] A.L. Sakhnovich, *Dynamical and spectral Dirac systems: response function and inverse problems,* J. Math. Phys. 56 (2015), Art. 112702.

[44] A.L. Sakhnovich, *Dynamical canonical systems and their explicit solutions,* Discrete Contin. Dyn. Syst. 37 (2017), 1679–1689.

[45] A.L. Sakhnovich, *New "Verblunsky-type" coefficients of block Toeplitz and Hankel matrices and of corresponding Dirac and canonical systems,* J. Approx. Theory 237 (2019), 186–209.

[46] A.L. Sakhnovich, *Einstein, σ-model and Ernst-type equations and non-isospectral GBDT version of Darboux transformation*, Preprint, 2020, arXiv:2003.13024.

[47] A.L. Sakhnovich, *On the classes of explicit solutions of Dirac, dynamical Dirac and Dirac-Weyl systems with non-vanishing at infinity potentials, their properties and applications*, J. Differential Equations 275 (2021), 250–269.

[48] A.L. Sakhnovich, *Generalised canonical systems related to matrix string equations: corresponding structured operators and high-energy asymptotics of the Weyl functions*, Preprint, 2020, arXiv:2103.03203.

[49] A.L. Sakhnovich, *On the solution of the inverse problem for a class of canonical systems corresponding to matrix string equations*, Preprint, 2021, arXiv:2104.07612.

[50] A.L. Sakhnovich, L.A. Sakhnovich and I.Ya. Roitberg, *Inverse problems and nonlinear evolution equations. Solutions, Darboux matrices and Weyl–Titchmarsh functions*, De Gruyter, Berlin, 2013.

[51] L.A. Sakhnovich, *The spectral analysis of Volterra operators and some inverse problems* (Russian), Dokl. Akad. Nauk SSSR (N.S.) 115 (1957), 666–669.

[52] L.A. Sakhnovich, *On the factorization of the transfer matrix function,* Sov. Math. Dokl. 17 (1976), 203–207.

[53] L.A. Sakhnovich, *Factorization problems and operator identities,* Russian Math. Surveys 41 (1986), 1–64.

[54] L.A. Sakhnovich, *The method of operator identities and problems in analysis,* St. Petersburg Math. J. 5 (1994), 1–69.

[55] L.A. Sakhnovich, *Spectral theory of canonical differential systems, method of operator identities,* Birkhäuser, Basel, 1999.

[56] L.A. Sakhnovich, *Dirac equation: the stationary and dynamical scattering problems,* in Oper. Theory Adv. Appl. 263, pp. 407–424, Birkhäuser/Springer, Cham, 2018.

[57] M. Suzuki, *An inverse problem for a class of canonical systems having Hamiltonians of determinant one*, J. Funct. Anal. 279 (2020), Art. 108699.

[58] H. Woracek, *Asymptotics of eigenvalues for a class of singular Krein strings*, Collect. Math. 66 (2015), 469–479.

[59] V.E. Zakharov and A.V. Mikhailov, *On the integrability of classical spinor models in two-dimensional space-time,* Comm. Math. Phys. 74 (1980), 21–40.

Alexander Sakhnovich
Faculty of Mathematics
University of Vienna
A-1090 Vienna
Austria
oleksandr.sakhnovych@univie.ac.at