# Torsors of Isotropic Reductive Groups over Laurent Polynomials

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Abstract. Let $k$ be a field of characteristic 0. Let $G$ be a reductive group over the ring of Laurent polynomials $R = k[x_1^{\pm 1}, ..., x_n^{\pm 1}]$. We prove that $G$ is isotropic over $R$ if and only if it is isotropic over the field of fractions $k(x_1, \ldots, x_n)$ of $R$, and if this is the case, then the natural map $H^1_{ét}(R, G) \to H^1_{ét}(k(x_1, \ldots, x_n), G)$ has trivial kernel and $G$ is loop reductive. In particular, we settle in positive the conjecture of V. Chernousov, P. Gille, and A. Pianzola that $H^1_{Zar}(R, G) = *$ for such groups $G$. We also deduce that if $G$ is a reductive group over $R$ of isotropic rank $\geq 2$, then the natural map of non-stable $K_1$-functors $K_1^G(R) \to K_1^G\big(k((x_1))...((x_n))\big)$ is injective, and an isomorphism if $G$ is moreover semisimple.

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## 1 Introduction

Let $k$ be a field of characteristic 0. Let $R = k[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ be the ring of Laurent polynomials over $k$. Let $G$ be a reductive group scheme over $R$ in the sense of [SGA3]. V. Chernousov, P. Gille, and A. Pianzola [ChGP17, Theorem 1.1] showed that classifying étale-locally trivial $G$-torsors over $R$ is equivalent to classifying Zariski-locally trivial torsors over $R$ for all twisted $R$-forms of $G$ that are loop reductive.

The notion of a loop reductive group (see § 2 for the definition) was introduced by P. Gille and A. Pianzola in [GP13] for the purpose of studying extended affine Lie algebras (EALAs), which are higher nullity generalizations of affine

Kac-Moody algebras [AABGP]. Any EALA can be reconstructed from its centerless core, which is a Lie torus in the sense of [Y, N]. The Realization theorem [ABFP, Theorem 3.3.1], together with [GP07, Theorem 5.13], implies that all Lie tori of nullity $n$ over an algebraically closed field $\overline{k}$ of characteristic 0, except for just one class called quantum tori, are Lie algebras of isotropic adjoint simple loop reductive groups over $\overline{k}[x_1^{\pm 1},...,x_n^{\pm 1}]$. Recall that a reductive group $G$ over $R$ is called isotropic if all semisimple quotients of $G$ contain $\mathbf{G}_{\mathrm{m},R}$. It is known that a reductive group $G$ over $R$ is loop reductive if and only if $G$ contains a maximal torus [GP13, Corollary 6.3].

With respect to the classification of $G$-torsors for loop reductive groups $G$, V. Chernousov, P. Gille, and A. Pianzola proposed the following conjecture.

Conjecture 1.1. [ChGP17, Conjecture 5.4] Let $k$ be a field of characteristic 0. Let $G$ be an isotropic loop reductive group over the ring of Laurent polynomials $R = k[x_1^{\pm 1},...,x_n^{\pm 1}]$. Then $H^1_{Zar}(R, G)$ is trivial.

If $G = \mathrm{GL}_n$, the conjecture amounts to the fact that all finitely generated projective modules over $k[x_1^{\pm 1},...,x_n^{\pm 1}]$ are free, and it was established (in arbitrary characteristic) by R. G. Swan [Swa78] relying on D. Quillen's proof of Serre's conjecture [Q76]. More generally, if $G$ is defined over $k$, the conjecture was proved by P. Gille and A. Pianzola [GP08] relying on a theorem of M. S. Raghunathan on the triviality of $k[x_1,\dots,x_n]$-torsors [Rag89]. Apart from that, the conjecture was previously known for several classes of groups if $k$ is algebraically closed and $n = 2$ (see Corollary 1.4 below); for some twisted forms of $\mathrm{GL}_n$ [Art95, ChGP17] and for orthogonal groups [Par83]. The isotropy condition in the statement is necessary, since there are anisotropic forms of $\mathrm{PGL}_n$ over $k[x_1^{\pm 1}, x_2^{\pm 2}]$ with non-trivial Zariski cohomology [GP07, Corollary 3.22].

We establish the above conjecture in full by proving the following more general statement.

Theorem 1.2. *Let $k$ be a field of characteristic 0, and let $G$ be a reductive group over $R = k[x_1^{\pm 1},\dots,x_n^{\pm 1}]$. Set $K = k(x_1,\dots,x_n)$ and $F = k((x_1))\dots((x_n))$.*

1. *The following conditions on $G$ are equivalent:*
   - *(a) $G$ is isotropic;*
   - *(b) the algebraic $K$-group $G_K$ is isotropic;*
   - *(c) the algebraic $F$-group $G_F$ is isotropic.*
2. *If $G$ satisfies the equivalent conditions of (1), then $G$ is loop reductive and for any regular ring $A$ containing $k$, the natural map*
$$H^1_{\acute{e}t}\big(k[x_1^{\pm 1},\dots,x_n^{\pm 1}]\otimes_k A, G\big) \to H^1_{\acute{e}t}\big(k(x_1,\dots,x_n)\otimes_k A, G\big)$$
   *has trivial kernel.*

Corollary 1.3. *Let $k$ be a field of characteristic 0, and let $G$ be an isotropic reductive group over $R = k[x_1^{\pm 1},\dots,x_n^{\pm 1}]$. Then $H^1_{Zar}\big(R, G\big) = 1$.*

The proof of Theorem 1.2 relies on the "diagonal argument" trick for loop reductive groups [St16], on the established cases of the Serre–Grothendieck conjecture [PStV15, FP15], and on the classification results [ChGP17, Theorem 1.2]. The equivalence (1) was previously known for loop reductive groups [GP13, Corollary 7.4].
Using the above results and a known case of Serre's conjecture II [CTGP04], we also establish another conjecture of P. Gille and A. Pianzola.

Corollary 1.4. *[GP07, Conjecture 6.1] Let $k$ be an algebraically closed field of characteristic $0$. Let $G$ be a semisimple reductive group over $R = k[x_1^{\pm 1}, x_2^{\pm 1}]$ having no semisimple normal subgroups of type $A_n$, $n \geq 1$. Let $1 \to \mu \to G^{sc} \to G \to 1$ be the simply connected cover of $G$. Then the boundary map $H^1_{ét}(R, G) \to H^2_{ét}(R, \mu)$ is bijective. In particular, if $G$ is simply connected, then $H^1_{ét}(R, G)$ is trivial.*

In [GP07] P. Gille and A. Pianzola established this conjecture for groups of types $G_2$, $F_4$ and $E_8$. The case of groups of type $B_n$, $n \geq 2$, and of some groups of type $D_n$, follows from [Par83]. The groups of types $C_n$, $n \geq 6$, and $D_n$, $n \geq 8$, were covered in [SZ12]. Our proof covers all cases except for $E_8$, where we refer to [GP07].
As another corollary, we remove the assumption of loop reductivity in a previous result of the author concerning non-stable $K_1$-functors of isotropic reductive groups. For any commutative ring $R$, once a reductive group $G$ over $R$ is isotropic, then it contains a pair of opposite strictly proper parabolic $R$-subgroups $P$ and $P^-$ [SGA3, Exp. XXVI]. Under this assumption one can consider the following "large" subgroup of $G(R)$ generated by unipotent elements, $E_P(R) = \langle U_P(R), U_{P^-}(R)\rangle$ where $U_P$ and $U_{P^-}$ are the unipotent radicals of $P$ and $P^-$. The set of (left) cosets

$$G(R)/E_P(R) = K_1^{G,P}(R)$$

is called the non-stable $K_1$-functor associated to $G$ (and $P$), or the Whitehead group of $G$.
We say that $G$ has isotropic rank $\geq 2$, if all semisimple quotients of $G$ contain $(\mathbf{G}_{\mathrm{m},R})^2$. If this is the case, then $K_1^{G,P}(R)$ is independent of the choice of $P^{\pm}$ [PSt1] and we denote it by $K_1^G(R)$. The following result is a combination of [St16, Theorem 1.2] and Theorem 1.2. Its surjectivity part follows from [ChGP14, Theorem 14.3].

Corollary 1.5. *Let $k$ be a field of characteristic $0$, and let $G$ be a reductive group of isotropic rank $\geq 2$ over $R = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the natural map*

$$K_1^G(R) \to K_1^G\big(k((x_1))...((x_n))\big)$$

*is injective. If $G$ is moreover semisimple, the map is an isomorphism.*

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## 2 Preliminaries on loop reductive groups

Let $X$ be a connected locally noetherian scheme. Let $\Omega$ be an algebraically closed field, and let $a : \operatorname{Spec}(\Omega) \to X$ be a geometric point of $X$.
Let $\mathrm{F\acute{E}t}_X$ be the category of finite étale covers of $X$, and let $F_a$ be the covariant functor from $\mathrm{F\acute{E}t}_X$ to the category of finite sets defined as follows. Let $Y$ be an object of $\mathrm{F\acute{E}t}_X$ with the structure morphism $f : Y \to X$. Then $F_a(Y)$ is the set of all $\Omega$-points of $Y$ above $a$, that is, the set of all morphisms $y : \operatorname{Spec}(\Omega) \to Y$ for which the diagram

$$\begin{array}{ccc} & & Y \\ & \overset{y}{\nearrow} & \downarrow_f \\ \operatorname{Spec}(\Omega) & \underset{a}{\to} & X \end{array}$$

commutes. The group of automorphisms of the functor $F_a$ is called the *(étale) fundamental group of $X$ at $a$*, and is denoted by $\pi_1(X, a)$. If $X$ is noetherian, then there is an $X$-scheme $X$ that represents $F_a$, that is, $F_a(Y) = \operatorname{Hom}_X(X, Y)$ for any object $Y$ of $\mathrm{F\acute{E}t}_X$ [GP13, Ch. 2, 2.1].
Assume, moreover, that $X$ is a geometrically connected noetherian $k$-scheme, where $k$ is a field, and $\Omega = \overline{k}$ is an algebraic closure of $k$. Let $a_0 : \operatorname{Spec}(\overline{k}) \to \operatorname{Spec}(k)$ be the morphism obtained by composing $a$ with the structure morphism $X \to \operatorname{Spec}(k)$. Set $\overline{X} = X \times_k \overline{k}$, and let $\overline{a} : \operatorname{Spec}(\overline{k}) \to \overline{X}$ be the induced geometric point of $\overline{X}$. Then there is a canonical short exact sequence of group homomorphisms

$$1 \to \pi_1(\overline{X}, \overline{a}) \to \pi_1(X, a) \to \pi_1(\operatorname{Spec}(k), a_0) \to 1, \tag{2.1}$$

and $\pi_1(\operatorname{Spec}(k), a_0) \cong \operatorname{Gal}(k^s/k)$, the Galois group of the separable closure $k^s$ of $k$ in $\overline{k}$ [SGA1, Exp. IX, Théorème 6.1].
Furthermore, let $G$ be a group scheme locally of finite presentation over $k$. The right action of $\pi_1(X, a)$ on $X$ induces an action of $\pi_1(X, a)$ on $G(X^{sc})$. One can show that this action is continuous with respect to the discrete topology on $G(X^{sc})$ [GP13, Proposition 2.3], and thus one may consider the non-abelian cohomology set $H^1\big(\pi_1(X, a), G(X^{sc})\big)$ in the sense of Serre [Se]. The group $\pi_1(X, a)$ acts on $G(k^s)$ via the homomorphism $\pi_1(X, a) \to \pi_1(\operatorname{Spec}(k), a_0)$ of (2.1), and we denote by $H^1\big(\pi_1(X, a), G(k^s)\big)$ the usual non-abelian Galois cohomology.
Let $H^1_{fppf}(X, G)$ be the faithfully flat Čech style cohomology of $X$ with values in $G$, i.e. the set of isomorphism classes of $G$-torsors over $X$ that are locally trivial with respect to the fppf topology.

Definition 2.1. [GP13, Definition 3.1] An fppf-locally trivial $G$-torsor $E$ over $X$ is called a *loop torsor*, if the isomorphism class of $E$ in $H^1_{fppf}(X, G)$ is in the image of the natural composite map

$$H^1\big(\pi_1(X, a), G(k^s)\big) \to H^1\big(\pi_1(X, a), G(X^{sc})\big) \to H^1_{fppf}\big(X, G\big).$$

Cocycles in the corresponding cocycle classes in $H^1_{fppf}\big(X, G\big)$ are called *loop cocycles*.

From now on, let $k$ be a field of characteristic 0, and let $X$ be the $k$-scheme

$$X = \mathrm{Spec}\big(k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]\big).$$

We fix once and for all an algebraic closure $\bar{k}$ of $k$ and a compatible set of primitive $m$-th roots of unity $=\xi_m \in \bar{k}$, $m \geq 1$.
P. Gille and A. Pianzola [GP08, Corollary 2.13], [GP13, Ch. 2, 2.3] computed the étale fundamental group of $X$ at the natural geometric point $e : \mathrm{Spec}\,\bar{k} \to X$ induced by the evaluation $x_1 = x_2 = \dots = x_n = 1$. Namely,

$$\pi_1(X, e) = \hat{\mathbb{Z}}(1)^n \rtimes \mathrm{Gal}(\bar{k}/k), \tag{2.2}$$

where $\hat{\mathbb{Z}}(1)$ denotes the profinite group $\varprojlim_m \mu_m(\bar{k})$ equipped with the natural action of $\mathrm{Gal}(\bar{k}/k)$.
For any reductive group scheme $G$ over $X$, we denote by $G_0$ the split, or Chevalley—Demazure reductive group in the sense of [SGA3] of the same type as $G$. The group $G$ is an étale-locally trivial twisted form of $G_0$ [SGA3, Exp. XXII, Corollaire 2.3], corresponding to a cocycle class $\xi$ in the étale cohomology set $H^1_{\acute{e}t}(X, \mathrm{Aut}(G_0)) \subseteq H^1_{fppf}(X, \mathrm{Aut}(G_0))$.

Definition 2.2. [GP13, Definition 3.4] The group scheme $G$ is called *loop reductive*, if it corresponds to a loop cocycle class, i.e. if $\xi$ is in the image of the natural map

$$H^1\big(\pi_1(X, e), \mathrm{Aut}(G_0)(\bar{k})\big) \to H^1_{\acute{e}t}\big(X, \mathrm{Aut}(G_0)\big).$$

The nature of the cocycles used to define loop reductive groups and the description (2.2) of the fundamental group of Laurent polynomials are used in the proof of the following result that is key to the proof of our Theorem 1.2.

Lemma 2.3 ("diagonal argument"). *[St16, Lemma 4.1] Let $k$ be a field of characteristic 0. Let $G$ be a loop reductive group scheme over $R = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. For any integer $d > 0$, denote by $f_{z,d}$ (respectively, $f_{w,d}$) the composition of $k$-homomorphisms*

$$R \to k[z_1^{\pm 1}, \dots, z_n^{\pm 1}, w_1^{\pm 1}, \dots, w_n^{\pm 1}] \\ \to k[z_1^{\pm 1}, \dots, z_n^{\pm 1}, (z_1 w_1^{-1})^{\pm \frac{1}{d}}, \dots, (z_n w_n^{-1})^{\pm \frac{1}{d}}]$$

*sending $x_i$ to $z_i$ (respectively, to $w_i$) for every $1 \leq i \leq n$. Then there is $d > 0$ such that*

$$f_{z,d}^*(G) \cong f_{w,d}^*(G)$$

*as group schemes over $k[z_1^{\pm 1}, \dots, z_n^{\pm 1}, (z_1 w_1^{-1})^{\pm \frac{1}{d}}, \dots, (z_n w_n^{-1})^{\pm \frac{1}{d}}]$.*

It turns out that loop reductive groups also admit the following internal characterisation.

Theorem. *[GP13, Corollary 6.3] A reductive group scheme $G$ over $X$ is loop reductive if and only if $G$ has a maximal torus.*

The definition of a maximal torus is as follows.

Definition 2.4. [SGA3, Exp. XII Définition 1.3] Let $S$ be a scheme, let $G$ be a group scheme of finite type over $S$, and let $T$ be a subgroup scheme of $G$. Then $T$ is a *maximal torus of* $G$, if $T$ is an $S$-torus and $T_{\overline{k(s)}}$ is a maximal torus of $G_{\overline{k(s)}}$ for all $s \in S$, where $\overline{k(s)}$ denotes an algebraic closure of $k(s)$.

## 3 Some corollaries of the Serre–Grothendieck conjecture for isotropic groups

Throughout this section, $A$ denotes a regular ring containing an infinite field $k$, and $G$ denotes an isotropic reductive group over $A$.
The following statement was obtained in [St19] as a joint corollary of the corresponding statement for simply connected semisimple reductive groups [PStV15, Theorem 1.6], and of the result of I. Panin and R. Fedorov on the Serre–Grothendieck conjecture [FP15].
Theorem. *[St19, Theorem 4.2] Assume that $A$ is a semilocal domain, and let $K$ be its fraction field. Then for any $n \geq 1$ the natural map*

$$H^1_{\acute{e}t}(R[x_1, \dots, x_n], G) \to H^1_{\acute{e}t}(K[x_1, \dots, x_n], G)$$

*has trivial kernel.*

Lemma 3.1. *Assume that $A$ is local. Let $f(x) \in A[x]$ be a non-zero polynomial. Then $H^1_{\acute{e}t}(\mathbb{A}^1_A, G) \to H^1_{\acute{e}t}((\mathbb{A}^1_A)_f, G)$ has trivial kernel.*

*Proof.* Let $K$ be the fraction field of $A$. By [St19, Theorem 4.2] the map $H^1_{\acute{e}t}(A[x], G) \to H^1_{\acute{e}t}(K[x], G)$ has trivial kernel. By [CTO92, Proposition 2.2] the map $H^1_{\acute{e}t}(K[x], G) \to H^1_{\acute{e}t}(K(x), G)$ has trivial kernel. Hence the claim. ☐

The following lemma combines the previous one with a classical trick of Quillen [Q76].

Lemma 3.2. *Let $f(x) \in A[x]$ be a monic polynomial. Then $H^1_{\acute{e}t}(\mathbb{A}^1_A, G) \to H^1_{\acute{e}t}((\mathbb{A}^1_A)_f, G)$ has trivial kernel.*

*Proof.* Let $\xi \in H^1_{\acute{e}t}(\mathbb{A}^1_A, G)$ be in the kernel. Since $f$ is monic, for any maximal ideal $m$ of $A$ the image of $f$ in $A_m[x]$ is non-zero. Then by Lemma 3.1 the $G$-bundle $\xi|_{\mathbb{A}^1_{A_m}}$ is trivial. Since $A$ is regular, $G$ is $A$-linear by [Tho87, Corollary 3.2]. Then by [AHW18, Theorem 3.2.5] (see also [Mos08, Korollar 3.5.2]) the fact that for any maximal ideal $m$ of $A$ the $G$-bundle $\xi|_{\mathbb{A}^1_{A_m}}$ is trivial implies that $\xi$ is extended from $A$.
Set $y = x^{-1}$ and choose $g(y) \in A[y]$ so that $x^{\deg(f)} g(y) = f(x)$. Then $g(0) \in A^\times$ and $A[x]_{xf} = A[y]_{yg}$. We have $\mathbb{P}^1_A = \mathbb{A}^1_A \cup \mathrm{Spec}(A[y]_g)$, and $\mathbb{A}^1_A \cap \mathrm{Spec}(A[y]_g) =$

$(\mathbb{A}^1_A)_{xf}$. Hence we can extend $\xi$ to a bundle $\hat{\xi}$ on $\mathbb{P}^1_A$ by gluing it to a trivial bundle on $\operatorname{Spec}(A[y]_g)$. Let $\eta = \hat{\xi}|_{\operatorname{Spec}(A[y])}$. By assumption, $\eta$ is trivial on $\operatorname{Spec}(A[y]_g)$. Since $g(0) \in A^\times$, by the same argument as above $\eta$ is extended from $A$. However, $g(0)$ is invertible and $\eta$ is trivial at $y = 0$, hence $\eta$ is trivial. Hence $\xi$ is trivial at $x = y = 1$. Hence $\xi$ is trivial. □

Lemma 3.3. *Let $f(x) \in A[x]$ be a monic polynomial such that $f(0) \in A^\times$. Then $H^1_{ét}((\mathbb{A}^1_A)_x, G) \to H^1_{ét}((\mathbb{A}^1_A)_{xf}, G)$ has trivial kernel.*

*Proof.* Since $f(0) \in A^\times$, any $G$-bundle in the kernel can be extended to $\mathbb{A}^1_A$ by gluing it to a trivial $G$-bundle on $(\mathbb{A}^1_A)_f$. Then it is trivial by Lemma 3.2 applied to $xf$. □

Lemma 3.4. *For any $n \geq 0$ the natural map*

$$H^1_{ét}\big(A[t_1^{\pm 1}, \dots, t_n^{\pm 1}], G\big) \to H^1_{ét}\big(A \otimes_k k(t_1, \dots, t_n), G\big)$$

*has trivial kernel.*

*Proof.* We prove the claim by induction on $n$; the case $n = 0$ is trivial. Set $l = k(t_1, \dots, t_{n-1})$. By the inductive hypothesis, the map

$$H^1_{ét}\big(A[t_1^{\pm 1}, \dots, t_n^{\pm 1}], G\big) \to H^1_{ét}\big(A[t_n^{\pm 1}] \otimes_k l, G\big) = H^1_{ét}\big(A \otimes_k l[t_n^{\pm 1}], G\big)$$

has trivial kernel, so it remains to prove the triviality of the kernel for the map

$$H^1_{ét}\big(A \otimes_k l[t_n^{\pm 1}], G\big) \to H^1_{ét}\big(A \otimes_k l(t_n), G\big).$$

We have $l(t_n) = \varinjlim_g l[t_n]_{t_n g}$, where $g \in l[t_n]$ runs over all monic polynomials with $g(0) \in l^\times$. Since $H^1_{ét}(-, G)$ commutes with filtered direct limits, it remains to show that every map

$$H^1_{ét}(A \otimes_k l[t_n^{\pm 1}], G) \to H^1_{ét}(A \otimes_k l[t_n]_{t_n g}, G) \tag{3.1}$$

has trivial kernel. This is the claim of Lemma 3.3. □

Lemma 3.5. *Let $F$ be an isotropic reductive group over $A[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. Fix a set of integers $d_i > 0$, $1 \leq i \leq n$, and consider the $A[t_1, \dots, t_n]$-algebra homomorphism*

$$\psi : A[z_1^{\pm 1}, \dots, z_n^{\pm 1}, t_1, \dots, t_n] \xrightarrow{z_i \mapsto w_i t_i^{d_i}} A \otimes_k k(\mathbf{w})[t_1^{\pm 1}, \dots, t_n^{\pm 1}],$$

*where $\mathbf{w}$ stands for $w_1, \dots, w_n$. Then the induced map*

$$H^1_{ét}\big(A[z_1^{\pm 1}, \dots, z_n^{\pm 1}, t_1, \dots, t_n], F\big) \to H^1_{ét}\big(A \otimes_k k(\mathbf{w})[t_1^{\pm 1}, \dots, t_n^{\pm 1}], \psi^*(F)\big)$$

*has trivial kernel.*

*Proof.* We prove the claim by induction on $n \geq 0$. The case $n = 0$ is trivial. To simplify the notation, set

$$B = A[z_2^{\pm 1}, \dots, z_n^{\pm 1}, t_2, \dots, t_n]$$

and $z = z_1$, $t = t_1$, $w = w_1$. Let $\phi : B[z^{\pm 1}, t] \to B \otimes_k k(w)[t^{\pm 1}]$ be the $B[t]$-algebra homomorphism sending $z$ to $wt^d$. To prove the induction step for $n \geq 1$, it is enough to show that the induced map of étale cohomology

$$h : H^1_{ét}\big(B[z^{\pm 1}, t], F\big) \xrightarrow{z \mapsto wt^d} H^1_{ét}\big(B \otimes_k k(w)[t^{\pm 1}], \phi^*(F)\big)$$

has trivial kernel, where $F$ is defined over $B[z^{\pm 1}]$. Indeed, after that we can apply the induction assumption with $k$ substituted by $k(w_1)$ and $A$ substituted by $A \otimes_k k(w_1)[t_1^{\pm 1}]$.
We have

$$B \otimes_k k(w)[t^{\pm 1}] = \varinjlim_g B \otimes_k k[w^{\pm 1}]_g[t^{\pm 1}] = \varinjlim_g B \otimes_k k[w^{\pm 1}, t^{\pm 1}]_g,$$

where $g = g(w)$ runs over all monic polynomials in $k[w]$ with $g(0) \neq 0$. Let $N = \deg(g) \geq 1$. Since $\phi(z) = wt^d$, we have $g(w) = g(\phi(z)t^{-d}) = t^{-Nd} f(t)$, where $f(t)$ is a polynomial in $t$ with coefficients in $k[\phi(z)^{\pm 1}]$ such that its leading coefficient is in $k \setminus 0$, and $f(0) = \phi(z)^N$. Then

$$B \otimes_k k[w^{\pm 1}, t^{\pm 1}]_g = B \otimes_k k[\phi(z)^{\pm 1}, t]_{tf}.$$

The group scheme $\phi^*(F)$ is defined over $B \otimes_k k[\phi(z)^{\pm 1}]$. Both terminal coefficients of $tf(t)$ are invertible in $k[\phi(z)^{\pm 1}]$, hence by Lemma 3.2 applied to the regular ring $B \otimes_k k[\phi(z)^{\pm 1}]$ the map

$$\begin{aligned} H^1_{ét}\big(B[z^{\pm 1}, t], F\big) \xrightarrow{z \mapsto wt^d} & H^1_{ét}\big(B \otimes_k k[w^{\pm 1}, t^{\pm 1}]_g, \phi^*(F)\big) \\ & = H^1_{ét}\big(B \otimes_k k[\phi(z)^{\pm 1}, t]_{tf}, \phi^*(F)\big) \end{aligned}$$

has trivial kernel.
Since $H^1_{ét}(-, F)$ commutes with filtered direct limits [Mar07], we conclude that $h$ has trivial kernel. □

## 4 Proof of the main results

Lemma 4.1. *Let $k$ be a field of characteristic $0$, and let $G$ be an isotropic loop reductive group over $R = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. For any regular ring $A$ containing $k$, the natural map*

$$H^1_{ét}\big(k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \otimes_k A, G\big) \to H^1_{ét}\big(k(x_1, \dots, x_n) \otimes_k A, G\big)$$

*has trivial kernel.*

*Proof.* We apply Lemma 2.3 to $G$. Set

$$t_i = (z_i w_i^{-1})^{1/d}, \quad 1 \leq i \leq n,$$

where $z_i, w_i$, and $d$ are as in that Lemma. Note that this is equivalent to

$$z_i = w_i t_i^d, \quad 1 \leq i \leq n.$$

We denote by $G_z$ the group scheme over $k[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ which is the pull-back of $G$ under the $k$-isomorphism

$$k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \xrightarrow{x_i \mapsto z_i} k[z_1^{\pm 1}, \dots, z_n^{\pm 1}].$$

The group scheme $G_w$ over $k[w_1^{\pm 1}, \dots, w_n^{\pm 1}]$ is defined analogously. By Lemma 2.3 $G_z$ and $G_w$ are isomorphic after pull-back to

$$k[z_1^{\pm 1}, \dots, z_n^{\pm 1}, t_1^{\pm 1}, \dots, t_n^{\pm 1}] = k[w_1^{\pm 1}, \dots, w_n^{\pm 1}, t_1^{\pm 1}, \dots, t_n^{\pm 1}].$$

Consider the following commutative diagram, where $\mathbf{x}$ stands for $x_1, \dots, x_n$, $\mathbf{z}$ stands for $z_1, \dots, z_n$, etc.

$$\begin{array}{ccc}
H^1_{ét}\Big(k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \otimes_k A, G\Big) & \xrightarrow{j_1} & H^1_{ét}\Big(k(\mathbf{x}) \otimes_k A, G\Big) \\
\downarrow {\scriptstyle f_1 :\, x_i \mapsto z_i} & & \downarrow {\scriptstyle f_2 :\, x_i \mapsto z_i} \\
H^1_{ét}\Big(k[z_1^{\pm 1}, \dots, z_n^{\pm 1}, t_1, \dots, t_n] \otimes_k A, G_z\Big) & & H^1_{ét}\Big(k(\mathbf{z}, \mathbf{t}) \otimes_k A, G_z\Big) \\
\downarrow {\scriptstyle h :\, z_i \mapsto w_i t_i^d} & & \\
H^1_{ét}\Big(k(\mathbf{w})[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes_k A, G_z\Big) & & \downarrow {\scriptstyle g_2 :\, z_i \mapsto w_i t_i^d} \cong \\
\downarrow {\scriptstyle g_1} \cong & & \\
H^1_{ét}\Big(k(\mathbf{w})[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes_k A, G_w\Big) & \xrightarrow{j_2} & H^1_{ét}\Big(k(\mathbf{w}, \mathbf{t}) \otimes_k A, G_w\Big)
\end{array}$$

The horizontal maps $j_1$ and $j_2$ are the natural ones, and all maps always take variables $t_i$ to $t_i$, $1 \leq i \leq n$, and $A$ to $A$. The bijections $g_1$ and $g_2$ exist by Lemma 2.3.

In order to prove that $j_1$ has trivial kernel, it is enough to show that all maps $j_2, g_1, h, f_1$ have trivial kernels. The map $j_2$ has trivial kernel by Lemma 3.4. As explained above, $g_1$ is bijective. The map $h$ is has trivial kernel by Lemma 3.5. Finally, the map $f_1$ has trivial kernel, since it has a retraction. Therefore, the map $j_1$ has trivial kernel. □

*Proof of Theorem 1.2.* To prove the first statement of the theorem, it is enough to show that if $G_F$ is isotropic, then the same holds for $G$. Also, we can assume from the start that $G$ is an adjoint reductive group over $R$. Then $G$ is an inner twisted form of a uniquely determined quasi-split adjoint reductive $R$-group $G_{qs}$, given by a cocycle class $\xi \in H^1_{ét}(S, G_{qs})$ [SGA3, Exp. XXIV 3.12.1]. By definition, $G_{qs}$ contains a maximal $R$-torus, hence it is loop reductive.

By [ChGP17, Theorem 5.2] there is a cocycle $\eta \in H^1_{ét}(R, G_{qs})$ such that the corresponding twisted group $H = {}^{\eta}G_{qs}$ is also loop reductive, and $\xi \in H^1_{Zar}(R, H)$. Then $G_K \cong H_K$ and $G_F \cong H_F$. Since $H$ is loop reductive, by [GP13, Corollary 7.4] $H$ is isotropic if and only if $H_K$ is isotropic if and only if $H_F$ is isotropic. Thus, if $G_F$ is isotropic, then $H$ is isotropic over $R$. Then by Lemma 4.1 we have $H^1_{Zar}(R, H) = 1$. Then $G \cong H$. Consequently, $G$ is isotropic over $R$ and also $G$ is loop reductive.

To prove the second statement of the theorem, we note that the adjoint group $G^{ad} = G/\operatorname{Cent}(G)$ is loop reductive by the above argument. Then $G$ is also loop reductive, since the maximal tori of $G$ and $G^{ad}$ are in bijective correspondence by [SGA3, Exp. XII 4.7.c]. Then the rest of the second statement holds by Lemma 4.1. □

*Proof of Corollary 1.4.* It was proved in [GP07, Theorem 3.17] that boundary map

$$\delta_G : H^1_{ét}(R, G) \to H^2_{ét}(R, \mu)$$

induces a bijection between $H^1_{loop}(R, G)$ and $H^1_{ét}(R, \mu)$, where $H^1_{loop}(R, G) \subset H^1_{ét}(R, G)$ is the subset of loop torsors, i.e. such $G$-torsors that the correspongsing twisted form of $G$ is loop reductive. In particular, the boundary map is surjective, and it remains to prove that it is injective. Also, [GP07, Theorem 2.7] implies the conjecture for groups of pure type $E_8$. If groups of this type occur as normal subgroups in $G$, then they are necessarily direct factors, since they have trivial centers. Hence we can assume that $G$ does not have semisimple normal subgroups of types $E_8$ or $A_n$, $n \geq 1$.

Set $K = k(x_1, x_2)$. Since $K$ has cohomological dimension 2, and for central simple algebras over a finite extension of $K$ index coincides with exponent [dJ04], the group $G_K$ is subject to [CTGP04, Theorems 1.2 and 2.1]. The latter theorems imply that $G_K$ is isotropic over $K$, and that $H^1_{ét}(K, G^{sc}) = 1$. In particular, $H^1_{ét}(K, G) \to H^2_{ét}(K, \mu)$ is bijective.

By Theorem 1.2 the fact that $G_K$ is isotropic implies that $G$ is also isotropic and is loop reductive. Then $G^{sc}$ is loop reductive as well, and isotropic, since the maximal tori and parabolic subgroups of $G$ and $G^{sc}$ are in bijective correspondence. Then Theorem 1.2 applied to $G^{sc}$ implies that $H^1_{ét}(R, G^{sc}) \to H^1_{ét}(K, G^{sc})$ has trivial kernel. Hence $H^1_{ét}(R, G^{sc})$ is trivial and $\delta_G$ has trivial kernel. Since all fibers of $\delta_G$ are in bijective correspondence with kernels of $\delta_{G'}$ for suitable twisted forms $G'$ of $G$, we conclude that $\delta_G$ is injective. □

*Proof of Corollary 1.5.* Under the additional assumption that $G$ is loop reductive, the claim holds by [St16, Theorem 1.2]. This assumption is made redundant by Theorem 1.2. □

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