

NON-COMMENSURABLE HYPERBOLIC MANIFOLDS
WITH THE SAME TRACE RING

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ABSTRACT. We prove that there are infinitely many pairwise non-commensurable hyperbolic n -manifolds that have the same ambient group and trace ring, for any $n \geq 3$. The manifolds can be chosen compact, if $n \geq 4$.

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INTRODUCTION

In the study of hyperbolic manifolds of dimension $n \geq 3$, the nicest family is (arguably) the family of arithmetic manifolds. At the core of their definition are two objects of algebraic nature: a field K and an algebraic group \mathbf{G} . In simple terms, an arithmetic lattice is then essentially just $\mathbf{G}(\mathcal{O}_K)$, and an arithmetic manifold the quotient of hyperbolic space by such a lattice.

In his 1971 article [13], Vinberg introduced similar objects for arbitrary Zariski-dense subgroups (in particular, lattices) in semisimple Lie groups. He defined the (adjoint) trace field (and ring) and the ambient group of such subgroups, and proved that they are invariant under commensurability. These are the main algebraic invariants used to study hyperbolic n -manifolds of arbitrary dimension; for $n \geq 4$ see for instance [4], and for $n = 3$ (and $n = 2$) essentially equivalent invariants are thoroughly studied in [7]. It is also an important tool when one is interested in proving nonarithmeticity and non-commensurability (see [9, 10]).

In the arithmetic case, these invariants completely determine the commensurability class [11, Prop. 2.5]. In fact, even more is true: if a manifold M_1 has the same ambient group and trace ring as an arithmetic manifold M_2 , then M_1 is

also arithmetic and commensurable with M_2 . It is hence tempting to hope that it still holds in the general case, i.e., that the pair (ambient group, trace ring) determines the commensurability class of an arbitrary hyperbolic manifold.

Alas, we prove that this does not hold in general: Theorems 2.1 and 3.1 below establish the existence of infinitely many pairwise non-commensurable hyperbolic manifolds with the same ambient group and trace ring, in the non-compact case for $n \geq 3$ and in the compact case for $n \geq 4$, respectively. Their proof is based on the (now classical) construction of Agol-Belolipetsky-Thomson [1, 2] of manifolds with short systole, and on the analysis of their trace ring carried out by the author in [9]. Observe that the case $n = 2$ was already known (see [7, Ex. 4.9.3]).

The article is structured as follows: in Section 1 we introduce the necessary background, Section 2 is devoted to the non-compact case and Section 3 to the compact case.

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1 BACKGROUND

1.1

In this paper all manifolds are assumed to be hyperbolic, complete without boundary and of finite volume. Equivalently, a manifold M is a quotient $M = \Gamma \backslash \mathbf{H}^n$ where \mathbf{H}^n denotes hyperbolic n -space and Γ is a torsion-free lattice in the Lie group $G = \text{Isom}(\mathbf{H}^n)$. We will use the so-called “ f -hyperboloid” models for hyperbolic space, defined as follows: For a real quadratic form f of signature $(n, 1)$, let

$$\mathbf{H}_f = \{x \in \mathbb{R}^{n+1} \mid f(x) = -1\} / \{\pm 1\},$$

with the Riemannian structure induced by setting $T_x \mathbf{H}_f = x^\perp$, the f -orthogonal complement of x . In this model, the isometry group $\text{Isom}(\mathbf{H}_f)$ is identified with $\mathbf{PO}_f(\mathbb{R})$, the real points of the algebraic group \mathbf{PO}_f . Observe that \mathbf{PO}_f is defined over the subfield $K \subset \mathbb{R}$ whenever f is.

1.2

Following Vinberg [13], we define the (*adjoint*) *trace field* of a Zariski-dense subgroup $\Gamma \subset G$ as the field $K = \mathbb{Q}(\text{tr Ad}(\gamma) \mid \gamma \in \Gamma)$, where Ad denotes the adjoint representation. If Γ is a lattice, K is a number field [14, Chap. 1, §6]. The (*adjoint*) *trace ring* of Γ is the ring A defined as the integral closure of $\mathbb{Z}[\text{tr Ad}(\gamma) \mid \gamma \in \Gamma]$. When K is a number field (in particular, when Γ is a lattice), we simply have

$$A = \mathcal{O}_K[\text{tr Ad}(\gamma) \mid \gamma \in \Gamma],$$

where \mathcal{O}_K denotes the ring of integers of K . Both the trace field and the trace ring are invariant under commensurability [13, Th. 3] (recall that two subgroups of a group are *commensurable* if they share a finite-index subgroup, up to conjugation).

Finally the *ambient group* of Γ is any algebraic group \mathbf{G} defined over K such that $\mathbf{G}(\mathbb{R}) \cong G = \text{Isom}(\mathbf{H}^n)$ and $\Gamma \subset \mathbf{G}(K)$ via this isomorphism. Such a group always exists (take the Zariski-closure of the image of Γ under the representation given by [13, Th. 1]). Furthermore, by Zariski-density of Γ it is unique up to K -isomorphism, and up to commensurability we actually have $\Gamma \subset \mathbf{G}(A)$.

1.3

With this machinery, it is easy to define arithmetic lattices. A lattice $\Gamma \subset G = \text{Isom}(\mathbf{H}^n)$ is *arithmetic* if:

1. its trace field K is a totally real number field,
2. its trace ring is \mathcal{O}_K ,
3. its ambient group \mathbf{G} is *admissible*, meaning that $\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R})$ is isomorphic (as a Lie group) to $G \times K$ with K compact.

Recall that here $\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) = \prod_{\sigma} \sigma \mathbf{G}(\mathbb{R})$, where the product is over all embeddings $\sigma: K \hookrightarrow \mathbb{R}$, and $\sigma \mathbf{G}$ denotes the σ -conjugate group of \mathbf{G} .

In this article, the only arithmetic lattices we will consider are those of the *simplest type*, meaning that their ambient group \mathbf{G} is isomorphic to \mathbf{PO}_f for some quadratic form f defined over the trace field K . For such a lattice Γ , the *arithmetic manifold* $M = \Gamma \backslash \mathbf{H}_f$ will be non-compact if and only if f is isotropic (see [3, Th. 11.6]). For $n \geq 4$ this forces that (and, for the simplest type, is actually equivalent to) $K = \mathbb{Q}$.

1.4

Let f be a signature $(n, 1)$ quadratic form. A *hyperplane* of \mathbf{H}_f is a codimension one totally geodesic subspace. Equivalently, it is the image in \mathbf{H}_f of the f -orthogonal complement $v^{\perp f}$ of a vector v with $f(v) > 0$. Two hyperplanes $R_1 = v_1^{\perp f}$ and $R_2 = v_2^{\perp f}$ are either *incident* if they meet in \mathbf{H}_f , *asymptotically parallel* if they meet at infinity, or *ultraparallel* otherwise. These correspond to the cases when the square of the f -scalar product $\langle v_1, v_2 \rangle_f^2$ is smaller than, equal to, or greater than $f(v_1)f(v_2)$ respectively, see [12, Th. 3.2.7]. In the ultraparallel case, there is a unique shortest geodesic between R_1 and R_2 and orthogonal to both. Its length is the (*hyperbolic*) *distance* $d(R_1, R_2)$ between R_1 and R_2 , and we have (see [12, Th. 3.2.8])

$$\cosh d(R_1, R_2) = \frac{|\langle v_1, v_2 \rangle_f|}{(f(v_1)f(v_2))^{1/2}}.$$

The image of a hyperplane $R \subset \mathbf{H}_f$ in a manifold $M = \Gamma \backslash \mathbf{H}_f$ is a *hypersurface* if the composite $R \hookrightarrow \mathbf{H}_f \twoheadrightarrow \Gamma \backslash \mathbf{H}_f$ is a covering onto its image.

Finally, if $M = \Gamma \backslash \mathbf{H}_f$ is a manifold, its *systole* is the length of the shortest geodesic in M . It actually equals the *minimal translation length* of Γ , defined as $\min\{d(x, \gamma x) \mid x \in \mathbf{H}_f, \gamma \in \Gamma \text{ hyperbolic}\}$. Note that if M is arithmetic of trace field K , there is a constant ϵ depending only on K and the dimension n such that the systole of M is at least ϵ (see Remark 5.7 and the remark after Conjecture 10.4 in [5]).

1.5

The manifolds we will be interested in (we call them *doubly-cut gluings*) arise from a specific gluing construction we now describe. They were introduced by Belolipetsky and Thomson [2], generalizing ideas of Agol [1]. We merely outline their construction here, and refer to [9] for a precise definition.

Let $\Gamma \subset \mathbf{PO}_f(\mathcal{O}_K)$ be an arithmetic lattice and let $R_1 = v_1^\perp$ and $R_2 = v_2^\perp$ be hyperplanes that are not incident (in \mathbf{H}_f). Assume that both vectors v_1, v_2 are actually in K^{n+1} ; we will say that R_1 and R_2 are *rational* or *K -rational*. Then a *doubly-cut gluing* constructed from this data is a manifold obtained using the following procedure:

1. Select a finite-index subgroup $\Gamma_1 \subset \Gamma$ such that the hyperplanes R_1 and R_2 project down to disjoint *hypersurfaces* in $M_1 = \Gamma_1 \backslash \mathbf{H}_f$.
2. Cut the manifold M_1 open at the two hypersurfaces and form its completion M_2 ; it is a manifold with boundary.
3. Form its *double* M_3 by identifying the boundary components of two mirrored copies of M_2 .

The hyperplanes R_1 and R_2 are the *cut hyperplanes*. See [9, Sect. 1.2] for a justification of the existence of the subgroup Γ_1 as well as other technical details.

If R_1 and R_2 are ultraparallel, the image in M_3 of the geodesic segment orthogonal to R_1 and R_2 , together with its “mirrored copy”, forms a closed geodesic in M_3 of length $2d(R_1, R_2)$. Thus with hyperplanes getting closer and closer we get doubly-cut gluings with shorter and shorter systole.

2 THE NON-COMPACT CASE

In this section we prove the following theorem.

THEOREM 2.1. *For every $n \geq 3$, there exist infinitely many non-compact pairwise non-commensurable hyperbolic n -manifolds having the same ambient group and trace ring.*

Proof. Fix $n \geq 3$. The manifolds we will construct are doubly-cut gluings with carefully chosen cut hyperplanes. Let $f = -x_0^2 + x_1^2 + \dots + x_n^2$ be the standard signature $(n, 1)$ quadratic form, and $R_0 \subset \mathbf{H}_f$ the hyperplane defined by $R_0 = v^{\perp f}$ with $v = (0, 1, 0, \dots, 0)$. For a rational hyperplane $R = w^{\perp f}$, ultraparallel to R_0 , we let M_w denote a doubly cut gluing constructed using the arithmetic manifold $\mathbf{PO}_f(\mathbb{Z})$ and the cut hyperplanes R_0 and R .

The goal is to find a sequence $(w_k)_{k \geq 1} \subset \mathbb{Z}^{n+1}$ such that the following holds:

1. The squared norm $f(w_k) = \langle w_k, w_k \rangle_f$ is positive, the hyperplanes R_0 and $R_k = w_k^{\perp f}$ are ultraparallel and

$$d(R_0, R_k) \longrightarrow 0 \quad \text{as } k \rightarrow \infty.$$

2. The trace rings A_k of the manifolds $M_{w_k} = \Gamma_{w_k} \backslash \mathbf{H}_f$ are all contained in a finitely generated subring A of \mathbb{Q} .

If we can find such a sequence, we can conclude using the following argument due to Agol [1] and generalized by Belolipetsky and Thomson [2]. First, observe that as the distance between R_0 and R_k tends to zero, the systole of the manifolds M_{w_k} also tends to zero (see the explanations of the previous section), and thus they are eventually all nonarithmetic.

Assume there are only finitely many commensurability classes among the M_{w_k} . Up to passing to a subsequence, we can assume that all M_{w_k} are commensurable and nonarithmetic. Hence the lattices Γ_{w_k} are all contained in the commensurator $\text{Comm}(\Gamma_{w_1})$ of, say, Γ_{w_1} . Since Γ_{w_1} is nonarithmetic, its commensurator is itself a lattice (see [8, Th. 1]), and thus has a positive minimal translation length. This contradicts the fact that the minimal translation length of Γ_{w_k} tends to zero.

Now up to finding a subsequence, we can assume that the M_{w_k} are all pairwise non-commensurable. Since there are only a finite number of subrings of A , we can find a further subsequence such that all elements M_{w_k} have the same trace ring, completing the proof.

We are left with finding such a sequence $(w_k)_{k \geq 1} \subset \mathbb{Z}^{n+1}$. First recall that, as explained above,

$$(\cosh d(R_0, R_k))^2 = \frac{\langle v, w_k \rangle_f^2}{f(v)f(w_k)} = \frac{w_{k,1}^2}{f(w_k)}.$$

Thus (a) is equivalent to

$$w_{k,1}^2 > f(w_k) > 0 \quad \text{and} \quad \frac{w_{k,1}^2}{f(w_k)} \longrightarrow 1. \tag{1}$$

(The first two inequalities mean that the hyperplanes are ultraparallel and that $w_k^{\perp f}$ indeed defines a hyperplane.)

Using Lemma 2.6 of [9], we get that the trace rings A_k of the M_{w_k} all satisfy:

$$A_k \subset \mathbb{Z} \left[\frac{1}{f(w_k)} \right] = \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right],$$

where p_1, \dots, p_r are the prime factors of $f(w_k)$, and the equality follows from elementary properties of subrings of \mathbb{Q} . Hence (b) can be achieved by finding a sequence w_k such that all integers $f(w_k)$ are T -smooth for some fixed constant T (an integer is T -smooth, or T -friable if all its prime factors are less than T). We are ready to construct our sequence $(w_k)_{k \geq 1}$. Let $T > 1$ be fixed and let $(r_k)_{k \geq 1}$ be any infinite sequence of positive T -smooth integers. For each r_k we can find integers $b_k > 0$ and $0 \leq c_k \leq 2b_k$ such that

$$r_k = b_k^2 - 2c_k - 1,$$

since all positive integers are contained in the union

$$\bigcup_{b \in \mathbb{Z}, b > 0} \{b^2 - 2c - 1 \mid c \in \mathbb{Z}, 0 \leq c \leq 2b\}.$$

Defining

$$w_k = (c_k + 1, b_k, c_k, 0, \dots, 0) \in \mathbb{Z}^{n+1},$$

we see that

$$f(w_k) = -(c_k + 1)^2 + b_k^2 + c_k^2 = b_k^2 - 2c_k - 1 = r_k$$

and hence all these square norms are T -smooth.

Moreover, it is obvious that for our choices of b_k, c_k , we have

$$b_k^2 > b_k^2 - 2c_k - 1 > 0,$$

and since

$$\frac{b_k^2}{b_k^2 - 2c_k - 1} = \frac{b_k^2}{b_k^2 + o(b_k^2)} \rightarrow 1,$$

the proof is complete. \square

REMARK 2.2. If we let $r_k = p^k$ for some fixed prime p , we get that all our manifolds have trace ring *exactly* $\mathbb{Z}[\frac{1}{p}]$.

3 THE COMPACT CASE

This method can be adapted to work over number fields, thus producing compact manifolds with same ambient group and trace rings:

THEOREM 3.1. *For every $n \geq 4$, there exist infinitely many compact pairwise non-commensurable hyperbolic n -manifolds having the same ambient group and trace ring.*

Proof. Let $K = \mathbb{Q}(\sqrt{5})$, and let f denote the following quadratic form in $n+1 \geq 4$ variables:

$$f = -\sqrt{5}x_0^2 + x_1^2 + \dots + x_n^2.$$

As in the proof of the previous theorem, we will construct a sequence of doubly-cut gluings $(M_{w_k})_{k \geq 1}$, this time using the f -hyperboloid model \mathbf{H}_f for hyperbolic space. The cut hyperplanes for M_{w_k} will be R_0 and R_k , where $R_0 = \{x_1 = 0\} = v^\perp \subset \mathbf{H}_f$, with $v = (0, 1, 0, \dots, 0)$ and $R_k = w_k^\perp$ is to be determined.

Let $w = (\alpha, \beta, \gamma_1, \gamma_2, \gamma_3, 0, \dots, 0) \in \mathcal{O}_K^{n+1}$, where $\mathcal{O}_K = \mathbb{Z} \left[\frac{1+\sqrt{5}}{2} \right]$ is the ring of integers of K . By Maass' Theorem (see [6, Th. 14.3.2, p. 193]), any totally positive element of \mathcal{O}_K can be realized as a sum of three squares in \mathcal{O}_K . Thus for any totally positive $\epsilon \in \mathcal{O}_K$, we can find values for $\gamma_1, \gamma_2, \gamma_3$ such that

$$f(w) = -\sqrt{5}\alpha^2 + \beta^2 + \epsilon$$

We will take $R_k = w_k^\perp$ for some w_k of the above form. As before, our goal is to find w_k such that the systole of M_{w_k} goes to zero while its trace ring remains fixed.

Let $\rho \in \mathbb{Z}[\sqrt{5}]$ be such that $\sigma(\rho)^2 > \rho > \sigma(\rho) > 1$, where σ is the non-trivial automorphism of K . (For instance, one can take $\rho = 6 + \sqrt{5}$). Write $\rho^k = u_k + \sqrt{5}v_k$ with $u_k, v_k \in \mathbb{Z}$, and define

$$\begin{aligned} \alpha_k &= [x] + \sqrt{5} & \text{and} & & \beta_k &= \lfloor \sqrt{5}y \rfloor + \sqrt{5}y \quad \text{where} \\ x &= \sqrt{\frac{\sigma(\rho^k)}{\sqrt{5}}}, & \text{and} & & y &= \left\lfloor \sqrt{\frac{u_k}{10}} \right\rfloor. \end{aligned}$$

A tedious but straightforward calculation shows that

$$-\sqrt{5}\alpha_k^2 + \beta_k^2 = \rho^k - \epsilon$$

where $\epsilon \in \mathbb{Z}[\sqrt{5}]$ is in $O(\beta_k)$ and totally positive for k large enough.

From the discussion above, it follows that there are $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{O}_K$ such that if $w_k = (\alpha_k, \beta_k, \gamma_1, \gamma_2, \gamma_3, 0, \dots, 0)$ we have

$$f(w_k) = \rho^k - \epsilon + \epsilon = \rho^k.$$

We are ready to conclude the proof. As previously, the conditions that R_k is a hyperplane not intersecting R_0 , and that the systole of M_{w_k} tends to zero can be fulfilled by choosing w_k satisfying equation (1) in the proof of Theorem 2.1. We claim that this is the case for our w_k , for k large enough.

To prove that $\frac{w_{k,1}^2}{f(w_k)} \rightarrow 1$, observe first that

$$\frac{\sqrt{5}v_k}{u_k} = \frac{\rho^k - \sigma(\rho^k)}{\rho^k + \sigma(\rho^k)} = \frac{1 - \left(\frac{\sigma(\rho)}{\rho}\right)^k}{1 + \left(\frac{\sigma(\rho)}{\rho}\right)^k} \rightarrow 1.$$

From this we get that

$$\frac{f(w_k)}{w_{k,1}^2} = \frac{\beta_k^2 - \sqrt{5}\alpha_k^2 + O(\beta_k)}{\beta_k^2} = 1 - \frac{\sqrt{5}\alpha_k^2}{\beta_k^2} + o(1) \rightarrow 1,$$

since

$$\frac{\alpha_k^2}{\beta_k^2} = \frac{1}{2\sqrt{5}} \frac{u_k - \sqrt{5}v_k + O(\sqrt{u_k})}{u_k + O(\sqrt{u_k})} = \frac{1}{2\sqrt{5}} \left(1 - \frac{\sqrt{5}v_k}{u_k + o(u_k)}\right) \rightarrow 0.$$

Thus $\frac{w_{k,1}^2}{f(w_k)} \rightarrow 1$.

It is clear that $f(w_k) = \rho^k > 0$. To show that $w_{k,1}^2 > f(w_k)$, observe first that

$$\frac{\sigma(\rho)^{2k}}{u_k} \geq \frac{\sigma(\rho)^{2k}}{\rho^k} = \left(\frac{\sigma(\rho)^2}{\rho}\right)^k \rightarrow \infty,$$

since $\sigma(\rho)^2 > \rho$ by hypothesis. It follows that

$$f(w_k) - w_{k,1}^2 = f(w_k) - \beta_k^2 = -\sqrt{5}\alpha_k^2 + O(\beta_k)$$

is negative for large enough k , since

$$\frac{\alpha_k^4}{\beta_k^2} = \frac{1}{10} \frac{\sigma(\rho^{2k}) + O(u_k^{3/2})}{u_k + O(u_k^{1/2})} \rightarrow +\infty.$$

We have proven that equation (1) holds for w_k , for k large enough. It now follows from Lemma 1.1 in [9] that the trace ring A_k of M_k satisfies

$$A_k \subset \mathcal{O}_K \left[\frac{1}{f(w_k)} \right] = \mathcal{O}_K \left[\frac{1}{\rho^k} \right] = \mathcal{O}_K \left[\frac{1}{\rho} \right].$$

As there are only finitely many integrally closed subrings of $\mathcal{O}_K[\frac{1}{\rho}]$, we conclude as in the proof of Theorem 2.1. \square

REMARK 3.2. The same proof idea works over any quadratic number field. However, there is no analogue of Maass' Theorem here (in fact, Siegel showed that \mathbb{Q} and $\mathbb{Q}(\sqrt{5})$ are the only totally real fields whose totally positive integers are sums of squares of integers, see [6, Th. 14.3.3]).

To palliate this, one can consider instead quadratic forms of the shape

$$-\sqrt{d}x_0^2 + x_1^2 + (\lceil \sqrt{d} \rceil - \sqrt{d})(x_2^2 + \cdots + x_5^2) + x_6^2 + \cdots + x_n^2.$$

Here for $n \geq 9$, the above error ϵ can be compensated by using the Four Squares Theorem twice, once with variables x_2, \dots, x_5 and once with x_6, \dots, x_9 . From there, essentially the same method works to produce compact examples.

Also, it is to be noted that there is nothing special about the case $n = 3$ in our context, and with more precise calculations one could probably include it in the compact version.

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