Hearts for Commutative Noetherian Rings: Torsion Pairs and Derived Equivalences

Dedicated to Lidia Angeleri Hügel on the Occasion of Her 60th Birthday

Sergio Pavon and Jorge Vitória

Received: September 23, 2020
Revised: April 16, 2021

Communicated by Henning Krause

Abstract. Over a commutative noetherian ring R, the prime spectrum controls, via the assignment of support, the structure of both $\mathsf{Mod}(R)$ and $\mathsf{D}(R)$. We show that, just like in $\mathsf{Mod}(R)$, the assignment of support classifies hereditary torsion pairs in the heart of any nondegenerate compactly generated t-structure of $\mathsf{D}(R)$. Moreover, we investigate whether these t-structures induce derived equivalences, obtaining a new source of Grothendieck categories which are derived equivalent to $\mathsf{Mod}(R)$.

2020 Mathematics Subject Classification: 13D30, 13D09, 18E10, 18G80
Keywords and Phrases: Hereditary torsion pairs, support, HRS-tilting, derived equivalence, commutative noetherian ring

1 Introduction

Two common techniques to study the category of modules over a ring and its derived counterpart are: *decompose into simpler parts* and *replace by suitably equivalent category.* Concerning the former, the notion of a torsion pair is central. Modelled on the concept of torsion for abelian groups, these are pairs of subcategories that generate, by extensions, the whole category, while simultaneously being orthogonal under the Hom-pairing ([15]). This idea extends beyond abelian categories to triangulated categories, giving rise in particular to the important concepts of t-structures and recollements ([7]). Torsion pairs are closely linked to categorical localisations, another useful tool in representation theory. Concerning categorical equivalences, we have at our disposal the theory of Morita, studying equivalences of module categories, and its derived version, studying triangle equivalences between derived categories of modules ([40]). In recent years, derived Morita theory has been extended to include equivalences between derived categories of Grothendieck categories ([39],[46]). These include not only categories of modules over a ring but also, for example, categories of quasi-coherent sheaves over a scheme.

In this paper we look at Grothendieck categories naturally occurring in the derived category of a commutative noetherian ring as hearts of t-structures. We approach the problem of classifying their localising subcategories (or, equivalently, hereditary torsion classes) and the problem of determining whether they are derived equivalent to the module category. In the noncommutative setting, these questions are, in general, quite difficult even in the case of a finite-dimensional algebra. In fact, a complete classification of localising subcategories is not even available for the module category of the Kronecker algebra ([27]). Understandably, the same classification problem in the derived category is even more cumbersome. With regards to derived equivalences, there is often a triangle functor linking the bounded derived category of a given heart and that of the ring, and there are criteria to check whether such a functor is an equivalence (see [7, 12]). In general, however, they are not easy to apply.

A remarkable feature of the representation theory of a commutative noetherian ring R is that much of the structure of the module category or of its derived counterpart is controlled by the Zariski spectrum of the ring. In fact, some of the problems mentioned in the above paragraphs have been elegantly solved in the module category and in the derived category. On one hand, Gabriel classified localising subcategories of $\mathsf{Mod}(R)$ in terms of specialisation-closed subsets of the spectrum, and Neeman classified localising subcategories of $\mathsf{D}(R)$ in terms of arbitrary subsets of $\mathsf{Spec}(R)$. Even compactly generated t-structures are completely classified in such derived categories ([1]). On the other hand, it is well-known that if two commutative noetherian rings are derived equivalent, then they are isomorphic (see, for example, [36, Proposition 5.3] for a stronger statement). Our aim in this paper is to show that the prime spectrum also controls *localising subcategories* of Grothendieck hearts in the derived category of R and, moreover, that these hearts are often *derived equivalent* to $\mathsf{Mod}(R)$. In fact, we prove the following.

Theorem. *Let R be a commutative noetherian ring and let $\mathbb{T}$ be a nondegenerate compactly generated t-structure of $\mathsf{D}(R)$, with heart $\mathcal{H}_\mathbb{T}$. Then the following hold:*

(1) *(Proposition 4.1 and Theorem 4.5(1.a)) The hereditary torsion classes of $\mathcal{H}_\mathbb{T}$ are completely determined by their support in $\mathsf{Spec}(R)$. If $\mathbb{T}$ is intermediate (Definition 2.11), every specialisation-closed subset of $\mathsf{Spec}(R)$ is the support of a hereditary torsion class in $\mathcal{H}_\mathbb{T}$.*

(2) *If $\mathbb{T}$ is intermediate and restricts to $\mathsf{D}^b(\mathsf{mod}(R))$, then*

(a) (Theorem 6.16) $\mathbb{T}$ *induces a derived equivalence between* $\mathcal{H}_{\mathbb{T}}$ *and* $\mathsf{Mod}(R)$*;*

(b) (Theorem 4.5 and Corollary 6.18) There is a bijection between hereditary torsion pairs of finite type and specialisation-closed subsets of $\mathsf{Spec}(R)$*.*

In particular, the above two assertions hold for the HRS-tilt of the standard t*-structure at a hereditary torsion pair in* $\mathsf{Mod}(R)$*.*

For some hearts, we are able to give a complete description of the supports on hereditary torsion pairs (see Subsection 4.2). While the results obtained about torsion pairs rely on the well-developed theory of support, the results on derived equivalences instead rely on the study of HRS-tilts. Happel, Reiten and Smalø developed in [19] a way to create a new t-structure from an old one, provided we are given a torsion pair in the old heart. The properties of this new t-structure depend on the properties of the given torsion pair, and therefore one may say that studying HRS-tilts often can be reduced to studying the associated torsion pairs. However, HRS-tilts turn out to be an elementary operation that, when iterated, allows us obtain a large class of t-structures (see, for example, [16, 32]). Moreover, HRS-tilts turn out to play an important role in understanding Bridgeland's stability condition manifold (see, for example, [10], [11] and [38]). In [12], necessary and sufficient conditions for an HRS-tilt to induce a derived equivalence were studied, and we review this theorem in Section 5. We use this to prove the following result, that becomes a fundamental tool in our application to commutative rings in the theorem above.

Theorem (Theorems 5.6 and 5.10 and Corollary 5.11). *Let* $\mathcal{G}$ *be a Grothendieck abelian category with generator* G*, and let* $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ *be a hereditary torsion pair in* $\mathcal{G}$ *with torsion radical* $t\colon \mathcal{G} \to \mathcal{T}$*. Then* $\mathbf{t}$ *induces an equivalence of bounded derived categories if and only if* $G/\mathsf{tr}_{G/t(G)}(G)$ *lies in* $\mathcal{T}$*. Moreover, if* $\mathcal{G} = \mathsf{Mod}(R)$*, then* $\mathbf{t}$ *induces an equivalence of bounded derived categories if and only if* $R/\mathsf{Ann}({}_R t(R))$ *lies in* $\mathcal{T}$*. As a consequence, if* R *is commutative and noetherian, then every hereditary torsion pair induces a (bounded) derived equivalence.*

The results concerning derived equivalences between Grothendieck hearts and $\mathsf{Mod}(R)$, for R commutative noetherian, have implications on the level of silting theory. Namely, it allows us to show that any bounded cosilting complex in $\mathsf{D}(R)$ whose associated t-structure restricts to $\mathsf{D}^b(\mathsf{mod}(R))$ must in fact be a cotilting complex (Corollary 6.17). In particular, every two-term cosilting complex is, in this setting, necessarily cotilting (Corollary 5.12). These results may easily lead to the expectation that every bounded cosilting is cotilting. This is, however, not true, as shown in Example 6.19.

Structure of the paper. In Section 2 we recall some definitions and known results about torsion pairs and t-structures. In Section 3 we turn to the definition of support over a commutative noetherian ring and we collect some of the known classification results for various types of subcategories of $\mathsf{Mod}(R)$ and $\mathsf{D}(R)$. In Section 4 we prove that hereditary torsion pairs in the heart of a nondegenerate compactly generated t-structure of $\mathsf{D}(R)$ are determined by their supports (Proposition 4.1). We investigate the sets arising as support of hereditary torsion classes in such hearts. In Subsection 4.2 we are able to describe these subsets of $\mathsf{Spec}(R)$ for some specific hearts. In Section 5 we temporarily leave the commutative noetherian setting to address the question of when a hereditary torsion pair in a Grothendieck category $\mathcal{G}$ gives rise to a derived equivalent category via HRS-tilting. We then specialise the results to module categories. In Section 6 we return to the commutative noetherian case. We observe that the intermediate compactly generated t-structures of $\mathsf{D}(R)$ can be obtained from the standard one via a finite chain of HRS-tilting operations (Proposition 6.10), with respect to hereditary torsion pairs of finite type at each step. Combining the results of Sections 4 and 5 we show that if the intermediate compactly generated t-structure at the end of the chain is restrictable, then across each HRS-tilting step of the chain there is a derived equivalence (Theorem 6.16). This provides a class of t-structures in $\mathsf{D}(R)$ whose heart is derived equivalent to $\mathsf{Mod}(R)$. In particular, we conclude that a bounded cosilting objects whose associated t-structure is restrictable must be cotilting (Corollary 6.17).

Notation and conventions. All subcategories considered in this paper are strict and full. Given a class $\mathcal{S}$ of objects of a category $\mathcal{X}$, we denote by $\mathsf{Gen}(\mathcal{S})$ (respectively, $\mathsf{Cogen}(\mathcal{S})$) the subcategory of $\mathcal{X}$ formed by the epimorphic images of existing coproducts (respectively, subobjects of existing products) of objects in $\mathcal{S}$. If $\mathcal{X}$ is preadditive (e.g. abelian or triangulated) we write

$$\mathcal{S}^{\perp} := \{Z \in \mathcal{X}\colon\ \mathsf{Hom}_{\mathcal{X}}(S, Z) = 0\ \forall S \in \mathcal{S}\}$$
$$^{\perp}\mathcal{S} := \{Z \in \mathcal{X}\colon\ \mathsf{Hom}_{\mathcal{X}}(Z, S) = 0\ \forall S \in \mathcal{S}\}$$

If $\mathcal{X}$ is abelian and $I \subseteq \mathbb{N}_0$ is a set of naturals, we write

$$\mathcal{S}^{\perp_I} := \{Z \in \mathcal{X}\colon\ \mathsf{Ext}^k_{\mathcal{X}}(S, Z) = 0,\ \forall k \in I, \forall S \in \mathcal{S}\}.$$

If $\mathcal{X}$ is a triangulated category and $J \subseteq \mathbb{Z}$, we write

$$\mathcal{S}^{\perp_J} := \{Z \in \mathcal{X}\colon\ \mathsf{Hom}_{\mathcal{X}}(S, Z[k]) = 0\ \forall k \in J, S \in \mathcal{S}\}$$
$$^{\perp_J}\mathcal{S} := \{Z \in \mathcal{X}\colon\ \mathsf{Hom}_{\mathcal{X}}(Z, S[k]) = 0\ \forall k \in J, S \in \mathcal{S}\}.$$

Often we replace the subsets I and J above by expressions of the form > 0, ≥ 0, ≤ 0, < 0, or simply a list of integers, with the obvious meaning. If $\mathcal{X}$ is abelian (or triangulated) and $\mathcal{Y}$ and $\mathcal{Z}$ are subcategories of $\mathcal{X}$, we denote by $\mathcal{Y} * \mathcal{Z}$ the subcategory of $\mathcal{X}$ formed by the objects X for which there are Y in $\mathcal{Y}$, Z in $\mathcal{Z}$ and a short exact sequence (respectively, a triangle)

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$$

(respectively, $Y \longrightarrow X \longrightarrow Z \longrightarrow X[1]$).

If R is a ring, all modules considered are right modules. The category of R-modules is denoted by $\mathsf{Mod}(R)$, and its subcategory of finitely presented modules by $\mathsf{mod}(R)$. We denote by $\mathsf{D}(R)$ the unbounded derived category of $\mathsf{Mod}(R)$ and by $\mathsf{D}^+(R)$ (respectively, $\mathsf{D}^b(R)$) the bounded below (respectively, bounded) counterpart. The bounded derived category of finitely presented R-modules is denoted by $\mathsf{D}^b(\mathsf{mod}(R))$.

2 Preliminaries: torsion pairs and t-structures

Most statements in this section about abelian or triangulated categories hold under more general assumptions than those presented. For simplicity, we restrict ourselves to the settings of Grothendieck abelian categories and derived categories of module categories. The generality under which each result holds can be extracted by the references provided. An abelian category $\mathcal{G}$ is said to be *Grothendieck* if it admits arbitrary (set-indexed) coproducts, direct limits are exact in $\mathcal{G}$, and $\mathcal{G}$ has a generator. In a triangulated category with arbitrary (set-indexed) coproducts, an object X in $\mathcal{D}$ is said to be *compact* if $\mathsf{Hom}_{\mathcal{D}}(X, -)$ commutes with coproducts, and $\mathcal{D}$ is said to be *compactly generated* if the subcategory of compact objects, denoted by $\mathcal{D}^c$, is skeletally small and $(\mathcal{D}^c)^{\perp} = 0$. For a ring R, $\mathsf{D}(R)$ is compactly generated and $\mathsf{D}(R)^c$ is the subcategory of bounded complexes of finitely generated projective R-modules.

2.1 Torsion pairs

Torsion pairs are useful (orthogonal) decompositions of abelian categories.

Definition 2.1. A pair $(\mathcal{T}, \mathcal{F})$ of subcategories of an abelian category $\mathcal{A}$ is said to be a *torsion pair* if

1. $\mathsf{Hom}_{\mathcal{A}}(T, F) = 0$ for any T in $\mathcal{T}$ and any F in $\mathcal{F}$.
2. $\mathcal{A} = \mathcal{T} * \mathcal{F}$.

If $\mathbf{t} := (\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$, $\mathcal{T}$ is called its *torsion class* and $\mathcal{F}$ its *torsionfree class*. The pair $\mathbf{t}$ is said to be *hereditary* if $\mathcal{T}$ is closed under subobjects, and *cohereditary* if $\mathcal{F}$ is closed under quotient objects. A subcategory $\mathcal{V}$ of $\mathcal{A}$ is called a *torsion torsionfree class* (TTF class, for short) if there are torsion pairs $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$ in $\mathcal{A}$, i.e. if $\mathcal{V}$ is both a torsion class and a torsionfree class in $\mathcal{A}$.

A subcategory $\mathcal{X}$ of a Grothendieck category $\mathcal{A}$ is a torsion class if and only if it is closed under coproducts, quotient objects and extensions, and it is a torsionfree class if and only if it is closed under products, subobjects and extensions ([15]). Since the coproduct of a family of objects in $\mathcal{A}$ is a subobject of the product of that same family, $\mathcal{X}$ is a TTF class if and only if it is a cohereditary torsionfree class.

Example 2.2. Given a Grothendieck abelian category $\mathcal{G}$ and a set of objects $\mathcal{X}$ of $\mathcal{G}$, it follows from the description above of torsion and torsionfree classes that the pairs $(\mathcal{T}_{\mathcal{X}}, \mathcal{X}^{\perp})$ and $({}^{\perp}\mathcal{X}, \mathcal{F}_{\mathcal{X}})$ are torsion pairs. The first is said to be *generated by* $\mathcal{X}$ while the second is said to be *cogenerated by* $\mathcal{X}$.

Since a torsion class is closed under coproducts and quotients, it is always closed for direct limits; this in general is not the case for torsionfree classes.

Definition 2.3. A torsion pair in a Grothendieck abelian category is said to be *of finite type* if its torsionfree class is closed under direct limits.

An object X in a cocomplete abelian category $\mathcal{A}$ is said to be *finitely presented* if $\mathsf{Hom}_{\mathcal{A}}(X, -)$ commutes with direct limits. The subcategory of finitely presented objects of $\mathcal{A}$ will be denoted by $\mathsf{fp}(\mathcal{A})$. Recall from [14] that a Grothendieck category $\mathcal{G}$ is *locally finitely presented* provided that the subcategory $\mathsf{fp}(\mathcal{G})$ is skeletally small and every object of $\mathcal{G}$ can be expressed as the direct limit of a system of finitely presented objects. Moreover, $\mathcal{G}$ is *locally coherent* if it is locally finitely presented and the subcategory $\mathsf{fp}(\mathcal{G})$ is an abelian subcategory. For any ring R, $\mathsf{Mod}(R)$ is always locally finitely presented; the ring is said to be *coherent* precisely when $\mathsf{Mod}(R)$ is locally coherent. In particular, the category of modules over a noetherian ring is locally coherent (in the example of the noetherian ring, $\mathsf{fp}(\mathsf{Mod}(R)) = \mathsf{mod}(R)$ is the category of finitely generated modules).

Definition 2.4. If $\mathcal{G}$ is locally coherent, a torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{G}$ is said to be *restrictable* if $\mathbf{t} \cap \mathsf{fp}(\mathcal{G}) := (\mathcal{T} \cap \mathsf{fp}(\mathcal{G}), \mathcal{F} \cap \mathsf{fp}(\mathcal{G}))$ is a torsion pair in $\mathsf{fp}(\mathcal{G})$.

The following well-known result relates restrictable torsion pairs to those of finite type.

Lemma 2.5. *Let $\mathcal{G}$ be a locally coherent Grothendieck category, and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{G}$.*

1. *If $\mathbf{t}$ is generated by a set of finitely presented objects $\mathcal{S} \subseteq \mathsf{fp}(\mathcal{G})$, then $\mathbf{t}$ is of finite type.*
2. *[24, Lemma 2.3] If $\mathbf{t}$ is a hereditary torsion pair of finite type, then $\mathcal{T} = \varinjlim(\mathcal{T} \cap \mathsf{fp}(\mathcal{G}))$.*
3. *[14, Lemma 4.4] If $\mathbf{t}$ is restrictable, then it is of finite type if and only if*

$$\mathbf{t} = \varinjlim(\mathbf{t} \cap \mathsf{fp}(\mathcal{G})) := (\varinjlim(\mathcal{T} \cap \mathsf{fp}(\mathcal{G})), \varinjlim(\mathcal{F} \cap \mathsf{fp}(\mathcal{G}))).$$

Proof. We comment on statement (3). If $\mathcal{F} = \varinjlim(\mathcal{F} \cap \mathsf{fp}(\mathcal{G}))$, $\mathbf{t}$ is clearly of finite type. For the converse, by [14, §4.4], since $\mathbf{t} \cap \mathsf{fp}(\mathcal{G})$ is a torsion pair in $\mathsf{fp}(\mathcal{G})$, then $\varinjlim(\mathbf{t} \cap \mathsf{fp}(\mathcal{G}))$ is a torsion pair in $\mathcal{G}$. Now, if $\mathbf{t}$ is of finite type, we have both $\varinjlim(\mathcal{T} \cap \mathsf{fp}(\mathcal{G})) \subseteq \mathcal{T}$ and $\varinjlim(\mathcal{F} \cap \mathsf{fp}(\mathcal{G})) \subseteq \mathcal{F}$; for the converse inclusion, it suffices to consider the torsion decomposition sequences of objects of $\mathcal{T}$ and $\mathcal{F}$ with respect to $\varinjlim(\mathbf{t} \cap \mathsf{fp}(\mathcal{G}))$. □

Remark 2.6. We will prove in Lemma 3.11 the converse of item (1) when $\mathcal{G}$ is the heart of a compactly generated t-structure in the derived category of a commutative noetherian ring.

Remark 2.7. For a right noetherian ring R, item (1) of Lemma 2.5 guarantees that every hereditary torsion pair is of finite type. Indeed, recall that over any ring a module is the sum of its finitely generated submodules: hence a hereditary torsion pair is generated by the class of finitely generated torsion modules. If R is right noetherian, these are automatically also finitely presented. Note, furthermore, that any torsion pair over a right noetherian ring is restrictable.

The following lemma, which will be useful later on, states that a torsionfree class in a Grothendieck category admits a generator, in the sense below.

Lemma 2.8. *Let $\mathcal{G}$ be a Grothendieck category with generator G, and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{G}$ with torsion radical $t\colon \mathcal{G} \to \mathcal{T}$. Then every object in $\mathcal{F}$ is a quotient of a coproduct of copies of $G/t(G)$, i.e. $\mathcal{F} \subseteq \mathsf{Gen}(G/t(G))$.*

Proof. Let X be an object of $\mathcal{F}$ and let $f\colon G^{(I)} \to X$ be an epimorphism. Since both $\mathcal{T}$ and $\mathcal{F}$ are closed under coproducts, it is easy to see that $t(G^{(I)}) \simeq t(G)^{(I)}$ and that $G^{(I)}/t(G^{(I)}) \simeq (G/t(G))^{(I)}$. Hence, since $\mathsf{Hom}_{\mathcal{G}}(t(G^{(I)}), X) = 0$, the morphism f factors through an epimorphism $\overline{f}\colon (G/t(G))^{(I)} \longrightarrow X$. ☐

2.2 t-structures

The role of t-structures in triangulated categories, as first defined in [7], is analogous to that of torsion pairs in abelian categories. Recall that, as set up in the Introduction, given subcategories $\mathcal{U}$ and $\mathcal{V}$ in a triangulated category $\mathcal{T}$, $\mathcal{U} * \mathcal{V}$ stands for the subcategory given by extensions of $\mathcal{U}$ by $\mathcal{V}$.

Definition 2.9. A pair of subcategories $\mathbb{T} := (\mathcal{U}, \mathcal{V})$ in a triangulated category $\mathcal{D}$ is a *t-structure* if

1. $\mathsf{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V}) = 0$;
2. $\mathcal{U}[1] \subseteq \mathcal{U}$;
3. $\mathcal{U} * \mathcal{V} = \mathcal{D}$.

In that case, we call $\mathcal{U}$ the *aisle*, $\mathcal{V}$ the *coaisle* and $\mathcal{H}_{\mathbb{T}} = \mathcal{V}[1] \cap \mathcal{U}$ the *heart* of $\mathbb{T}$. The t-structure $\mathbb{T}$ is said to be *nondegenerate* if $\cap_{n\in\mathbb{Z}}\mathcal{U}[n] = 0 = \cap_{n\in\mathbb{Z}}\mathcal{V}[n]$ and *bounded* if $\cup_{n\in\mathbb{Z}}\mathcal{U}[n] = \mathcal{D} = \cup_{n\in\mathbb{Z}}\mathcal{V}[n]$.

It is well-known from [7] that the heart of a t-structure $\mathbb{T}$ in $\mathcal{D}$ is an abelian category and that there is an associated cohomological functor $H^0_{\mathbb{T}}\colon \mathcal{D} \to \mathcal{H}_{\mathbb{T}}$ which restricts to the identity functor in $\mathcal{H}_{\mathbb{T}}$.

Example 2.10. For a ring R, let $H^0\colon \mathsf{D}(R) \longrightarrow \mathsf{Mod}(R)$ denote the standard cohomology functor of the derived category of R. This functor arises from a t-structure, also called *standard*. This is the pair $\mathbb{D} = (\mathbb{D}^{\leq 0}, \mathbb{D}^{\geq 1})$ where each of the subcategories of $\mathsf{D}(R)$ in the pair are made of the complexes whose non-zero cohomologies are concentrated in the degrees indicated in superscript. Note that shifts of t-structures are also t-structures; we write $\mathbb{D}^{\leq k} = \mathbb{D}^{\leq 0}[-k]$ and $\mathbb{D}^{\geq k} = \mathbb{D}^{\geq 1}[1-k]$ for shifts of the standard aisle and coaisle. Note that the standard t-structure in $\mathsf{D}(R)$ is nondegenerate but not bounded, while its restriction to the subcategory of bounded complexes $\mathsf{D}^b(R)$ is both nondegenerate and bounded.

In the derived category of a ring R we may consider a useful notion of *directed homotopy colimit*: this is the derived functor of the direct limit functor. Since direct limits are exact in any Grothendieck category (and, hence, in $\mathsf{Mod}(R)$), the directed homotopy colimit of a directed system of complexes of R-modules is the object of $\mathsf{D}(R)$ obtained by applying the direct limit functor of $\mathsf{Mod}(R)$ componentwise. We now recall some properties of t-structures.

Definition 2.11. For a ring R, a t-structure $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ in $\mathsf{D}(R)$ is said to be

- *intermediate* if there are integers $a < b$ such that $\mathbb{D}^{\leq a} \subseteq \mathcal{U} \subseteq \mathbb{D}^{\leq b}$;
- *smashing* if $\mathcal{V}$ is closed under coproducts;
- *homotopically smashing* if $\mathcal{V}$ is closed under directed homotopy colimits;
- *compactly generated* if there is a set of compact objects $\mathcal{S}$ in $\mathcal{U}$ such that $\mathcal{V} = \mathcal{S}^{\perp}$.
- *cosilting* if $\mathbb{T} = ({}^{\perp_{\leq 0}}C, {}^{\perp_{>0}}C)$ for some object C of $\mathsf{D}(R)$ (then called a *cosilting* object);
- *cotilting* if it is cosilting and C^I lies in ${}^{\perp_{\neq 0}}C$ for all sets I (in this case C is said to be *cotilting*).

If R is coherent, we say that $\mathbb{T}$ is *restrictable* if the pair $(\mathcal{U} \cap \mathsf{D}^b(\mathsf{mod}(R)), \mathcal{V} \cap \mathsf{D}^b(\mathsf{mod}(R)))$ is a t-structure of the triangulated subcategory $\mathsf{D}^b(\mathsf{mod}(R))$ of $\mathsf{D}(R)$.

Compactly generated triangulated categories $\mathcal{D}$ are often studied through the help of the category of additive functors $(\mathcal{D}^c)^{op} \to \mathsf{Mod}(\mathbb{Z})$. This category, usually denoted by $\mathsf{Mod}(\mathcal{D}^c)$, is a locally coherent Grothendieck category and the functor $\mathbf{y}$ sending an object X of $\mathcal{D}$ to the functor $\mathbf{y}X := \mathsf{Hom}_{\mathcal{D}}(-, X)_{|\mathcal{D}^c}$ is a cohomological functor. Properties of the Grothendieck category $\mathsf{Mod}(\mathcal{D}^c)$ are reflected on properties of $\mathcal{D}$ via $\mathbf{y}$, and this allows us to distinguish objects of $\mathcal{D}$ by the way they relate to the subcategory of compact objects (see [8] and [25] for more details). An important class of objects of $\mathcal{D}$ is given by those X whose corresponding functor $\mathbf{y}X$ is injective in the category $\mathsf{Mod}(\mathcal{D}^c)$ – these objects are called *pure-injective*. It turns out that, in many contexts, cosilting (and cotilting) objects are automatically pure-injective (see [5, 45] and [30]). It is

not known whether every cosilting object in a compactly generated triangulated category is necessarily pure-injective.
The following theorems relate properties of the t-structures with properties of their hearts. The first one is a particular case of [28, Theorem 4.6], adding to a sequence of previous results in [3] and [43].

Theorem 2.12. *The following are equivalent for a nondegenerate t-structure $\mathbb{T}$ of $\mathsf{D}(R)$, for a ring R.*

1. *$\mathbb{T}$ is homotopically smashing;*
2. *$\mathbb{T}$ is smashing and its heart is a Grothendieck category;*
3. *$\mathbb{T}$ is cosilting, for a pure-injective cosilting object.*

Moreover, a compactly generated t-structure has the above properties.

For intermediate t-structures with the properties of the theorem above, the question of whether or not they are restrictable boils down to a property of the heart, as follows.

Theorem 2.13. *[31] Let R be a noetherian ring and let $\mathbb{T}$ be an intermediate smashing t-structure in $\mathsf{D}(R)$ with a Grothendieck heart $\mathcal{H}_{\mathbb{T}}$. Then $\mathbb{T}$ is restrictable if and only if $\mathcal{H}_{\mathbb{T}}$ is locally coherent and $\mathsf{fp}(\mathcal{H}_{\mathbb{T}}) = \mathcal{H}_{\mathbb{T}} \cap \mathsf{D}^b(\mathsf{mod}(R))$.*

We conclude this subsection by recalling an important and much studied source of t-structures, which will be central to our paper.

Theorem 2.14 ([19]). *Let $\mathcal{D}$ be a triangulated category and $\mathbb{T}$ a t-structure in $\mathcal{D}$, with heart $\mathcal{H}_{\mathbb{T}}$. Given a torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}_{\mathbb{T}}$, there is a t-structure $\mathbb{T}_{\mathbf{t}} := (\mathbb{T}_{\mathbf{t}}^{\leq 0}, \mathbb{T}_{\mathbf{t}}^{\geq 1})$ in $\mathcal{D}$ defined as follows:*

$$\begin{aligned}\mathbb{T}_{\mathbf{t}}^{\leq 0} &= \{X \in \mathcal{D} : H^0_{\mathbb{T}}(X) \in \mathcal{T},\ H^0_{\mathbb{T}}(X[k]) = 0, \forall k > 0\}\\ \mathbb{T}_{\mathbf{t}}^{\geq 1} &= \{X \in \mathcal{D} : H^0_{\mathbb{T}}(X) \in \mathcal{F},\ H^0_{\mathbb{T}}(X[k]) = 0, \forall k < 0\}.\end{aligned}$$

This t-structure is called the HRS-tilt of $\mathbb{T}$ with respect to $\mathbf{t}$. Its heart is

$$\mathcal{H}_{\mathbf{t}} := \{X \in \mathcal{D} : H^0_{\mathbb{T}}(X[-1]) \in \mathcal{F},\ H^0_{\mathbb{T}}(X) \in \mathcal{T},\ H^0_{\mathbb{T}}(X[k]) = 0\ \forall k \neq -1, 0\}.$$

The pair $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in $\mathcal{H}_{\mathbf{t}}$.

Remark 2.15. In the theorem above, if $\mathcal{H}_{\mathbf{t}}$ is a Grothendieck category and $\mathcal{T}$ is a hereditary torsion class in $\mathcal{H}_{\mathbb{T}}$, it can moreover be shown that $\mathcal{T}$ is in fact a TTF class in $\mathcal{H}_{\mathbf{t}}$. Since $\mathcal{H}_{\mathbf{t}}$ is Grothendieck and $\mathcal{T}$ is a torsionfree class, $\mathcal{T}$ is closed under coproducts and it only remains to see that it is closed under quotients. If $X \in \mathcal{T}$ and $f\colon X \to Z$ is an epimorphism in $\mathcal{H}_{\mathbf{t}}$, $H^{-1}_{\mathbb{T}}(Z)$ lies in $\mathcal{F}$ (because Z lies $\mathcal{H}_{\mathbf{t}}$) and, simultaneously, $H^{-1}_{\mathbb{T}}(Z)$ is a subobject (in $\mathcal{H}_{\mathbb{T}}$) of $\mathsf{Ker}(f)$, which is an object of $\mathcal{T}$. Thus, we have $H^{-1}_{\mathbb{T}}(Z) = 0$ and Z lies in $\mathcal{T}$.

There are many results in the literature linking the properties of a torsion pair $\mathbf{t}$ and those of the HRS-tilt and its associated heart. We recall the following.

Theorem 2.16. *Let R be a ring and let $\mathbb{T}$ be a nondegenerate t-structure in* $\mathsf{D}(R)$ *satisfying the equivalent conditions (1–3) of Theorem 2.12. Suppose that* $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ *is a torsion pair in the heart* $\mathcal{H}_\mathbb{T}$ *of* $\mathbb{T}$*, and let* $\mathbb{T}_\mathbf{t}$ *be the associated HRS-tilt, with heart* $\mathcal{H}_\mathbf{t}$*. Then:*

1. *[43, Proposition 6.1]* $\mathbb{T}_\mathbf{t}$ *satisfies the equivalent conditions of Theorem 2.12 if and only if* $\mathbf{t}$ *is of finite type in* $\mathcal{H}_\mathbb{T}$*.*
2. *[43, Proposition 6.4] If* $\mathbb{T}_\mathbf{t}$ *is compactly generated, then* $\mathbf{t}$ *is generated by a set of finitely presented objects in* $\mathcal{H}_\mathbb{T}$*.*
3. *If R is noetherian and $\mathbb{T}$ is intermediate and restrictable, then the following are equivalent:*
   (a) $\mathbf{t}$ *is of finite type and it is restrictable;*
   (b) $\mathbb{T}_\mathbf{t}$ *is restrictable and* $\mathcal{H}_\mathbf{t}$ *is Grothendieck;*
   (c) $\mathcal{H}_\mathbf{t}$ *is a locally coherent Grothendieck category with* $\mathsf{fp}(\mathcal{H}_\mathbf{t}) = \mathcal{H}_\mathbf{t} \cap \mathsf{D}^b(\mathsf{mod}(R))$

Proof. We comment on item (3). This is a small generalisation of [41, Theorem 5.2]. The arguments comparing between the restrictability of $\mathbf{t}$ and the restrictability of $\mathbb{T}$ follow analogously to those in the proof of [41, Proposition 5.1]. The equivalence with item (c) follows from Theorem 2.13. ☐

2.3 Derived equivalences

It is quite natural to ask whether the derived category of a heart $\mathcal{H}$ in a triangulated category $\mathcal{D}$ is (triangle) equivalent to $\mathcal{D}$. This issue was first addressed in [7], where a functor between the bounded derived category of $\mathcal{H}$ and $\mathcal{D}$ was built (provided that $\mathcal{D}$ is *nice enough* — which $\mathsf{D}(R)$ certainly is, for any ring R). A similar construction for the unbounded derived category is the subject of [46]; the second statement of the following theorem has been translated from the language of derivators, and specialised to our setting.

Theorem 2.17. *Let R be a ring and $\mathbb{T}$ an intermediate t-structure in* $\mathsf{D}(R)$ *with heart* $\mathcal{H}_\mathbb{T}$*.*

1. *[7, §3.1] The inclusion* $\mathcal{H}_\mathbb{T} \hookrightarrow \mathsf{D}^b(R)$ *extends to a triangle functor* $\mathsf{real}^b_\mathbb{T} \colon \mathsf{D}^b(\mathcal{H}_\mathbb{T}) \to \mathsf{D}^b(R)$*, called a bounded realisation functor. This functor induces isomorphisms*

   $$\mathsf{Hom}_{\mathcal{H}_\mathbb{T}}(X,Y) \simeq \mathsf{Hom}_{\mathsf{D}(R)}(X,Y) \quad \textit{and} \quad \mathsf{Ext}^1_{\mathcal{H}_\mathbb{T}}(X,Y) \simeq \mathsf{Hom}_{\mathsf{D}(R)}(X,Y[1])$$

   for all X, Y of $\mathcal{H}_\mathbb{T}$*. Moreover,* $\mathsf{real}^b_\mathbb{T}$ *is an equivalence if and only if for every $n > 1$ we also have*

   $$\mathsf{Ext}^n_{\mathcal{H}_\mathbb{T}}(X,Y) \simeq \mathsf{Hom}_{\mathsf{D}(R)}(X,Y[n]).$$

2. *[46, Theorem B] If $\mathbb{T}$ is smashing and $\mathcal{H}_{\mathbb{T}}$ is Grothendieck (in other words, if $\mathbb{T}$ is a cosilting t-structure), then the inclusion $\mathcal{H}_{\mathbb{T}} \hookrightarrow \mathsf{D}(R)$ extends to a triangle functor $\mathsf{real}_{\mathbb{T}}\colon \mathsf{D}(\mathcal{H}_{\mathbb{T}}) \to \mathsf{D}(R)$, called an (unbounded) realisation functor, which restricts to a bounded realisation functor $\mathsf{real}^b_{\mathbb{T}}\colon \mathsf{D}^b(\mathcal{H}_{\mathbb{T}}) \to \mathsf{D}^b(R)$. Moreover, if $\mathsf{real}^b_{\mathbb{T}}$ is a triangle equivalence, then so is $\mathsf{real}_{\mathbb{T}}$.*

Proof. In addition to the references provided, we would like to point out that [46, Theorem B] applies to our context precisely because the fact that the t-structure is intermediate guarantees that products in the heart have finite homological dimension (see [46, Lemma 7.6]). Finally we comment on the fact that a bounded equivalence induces an unbounded equivalence in the setting described above. Indeed, by [39, Corollary 5.2], an intermediate cosilting t-structure inducing a bounded derived equivalence is cotilting. The result then follows by [46, Theorem 7.9]. □

DEFINITION 2.18. We say that an intermediate t-structure $\mathbb{T}$ in $\mathsf{D}(R)$ with heart $\mathcal{H}_{\mathbb{T}}$ *induces a (bounded) derived equivalence* if $\mathsf{real}_{\mathbb{T}}$ (respectively, $\mathsf{real}^b_{\mathbb{T}}$) is an equivalence. In that case we say that $\mathcal{H}_{\mathbb{T}}$ and $\mathsf{Mod}(R)$ are *(bounded) derived equivalent.*

3 PRELIMINARIES: COMMUTATIVE NOETHERIAN RINGS

In this section, R denotes a commutative noetherian ring. The set of prime ideals of R, partially ordered by inclusion, will be denoted by $\mathsf{Spec}(R)$. For an ideal $I \leq R$, we write

$$\vee(I) := \{\mathfrak{p} \in \mathsf{Spec}(R)\colon I \subseteq \mathfrak{p}\} \quad \text{and} \quad \wedge(I) := \{\mathfrak{p} \in \mathsf{Spec}(R)\colon \mathfrak{p} \subseteq I\}$$

The set $\mathsf{Spec}(R)$ has a natural topology, whose closed subsets are the $\vee(I)$ for all ideals $I \leq R$. This is called the *Zariski topology* on $\mathsf{Spec}(R)$. This topological space turns out to encode significant information concerning the representation theory of R.

DEFINITION 3.1. A subset $\mathcal{P}$ of $\mathsf{Spec}(R)$ is said to be *specialisation-closed* if for any $\mathfrak{p}$ in $\mathcal{P}$ we have that $\vee(\mathfrak{p})$ is contained in $\mathcal{P}$. Dually, the subset $\mathcal{P}$ is called *generalisation-closed* if for any $\mathfrak{p}$ in $\mathcal{P}$ we have that $\wedge(\mathfrak{p})$ is contained in $\mathcal{P}$.

Note that the complement of a specialisation-closed subset is generalisation-closed and vice-versa. We will denote the complement of a subset $\mathcal{P} \subseteq \mathsf{Spec}(R)$ by $\mathcal{P}^{\mathsf{c}}$. From their definition, specialisation-closed subsets are (possibly infinite) unions of Zariski-closed subsets, and thus, generalisation-closed subsets are (possibly infinite) intersections of Zariski-open subsets. For a family $\mathcal{P} \subseteq \mathsf{Spec}\, R$, its *specialisation closure* is the smallest specialisation-closed set containing $\mathcal{P}$, namely $\vee(\mathcal{P}) := \bigcup_{\mathfrak{p}\in\mathcal{P}} \vee(\mathfrak{p})$.

3.1 Supports

Given a prime ideal $\mathfrak{p}$ of R, let $R_\mathfrak{p}$ denote the localisation of R at the complement of $\mathfrak{p}$ and $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ the residue field of R at $\mathfrak{p}$. Consider the two left derived functors

$$-_\mathfrak{p} := - \otimes_R R_\mathfrak{p} \colon \mathsf{D}(R) \longrightarrow \mathsf{D}(R) \qquad \text{and} \qquad - \otimes^{\mathbb{L}} k(\mathfrak{p}) \colon \mathsf{D}(R) \longrightarrow \mathsf{D}(R),$$

Since $R_\mathfrak{p}$ is a flat R-module, for a complex X, $X_\mathfrak{p} := X \otimes_R R_\mathfrak{p}$ is the componentwise localisation of X as an object of $\mathsf{D}(R)$. In particular, we have $H^i(X_\mathfrak{p}) \simeq H^i(X)_\mathfrak{p}$ for all i in $\mathbb{Z}$.

Definition 3.2. Let R be a commutative noetherian ring. Given a complex X in $\mathsf{D}(R)$, we define the following subsets of $\mathsf{Spec}(R)$:

- $\mathsf{supp}(X) := \{\mathfrak{p} \in \mathsf{Spec}(R) \colon X \otimes^{\mathbb{L}}_R k(\mathfrak{p}) \neq 0\}$, the (small) *support* of X;
- $\mathsf{Supp}(X) := \{\mathfrak{p} \in \mathsf{Spec}(R) \colon X_\mathfrak{p} \neq 0\}$, the *big support* of X.

The (big) support of a subcategory $\mathcal{X}$ of $\mathsf{D}(R)$ is the union of the (big) supports of the objects in $\mathcal{X}$. Since localisation at $\mathfrak{p}$ commutes with standard cohomology, as noticed above, $\mathsf{Supp}(X) = \mathsf{Supp}(\coprod H^i(X))$. This set is therefore also called the *homological support* of X.

Recall that there is a bijection between $\mathsf{Spec}(R)$ and the set of isoclasses of indecomposable injective R-modules (Matlis' Theorem). This assignment is given by sending a prime ideal $\mathfrak{p}$ to the injective envelope of $R/\mathfrak{p}$ — which we denote by $E(R/\mathfrak{p})$. Moreover, since R is noetherian, every injective R-module is a coproduct of copies of such indecomposable injectives. The following lemma gathers some well-known statements about support that we will use later.

Lemma 3.3. *Let $\mathfrak{p}$ be a prime ideal of a commutative noetherian ring R.*

1. *(a)* $\mathsf{supp}(k(\mathfrak{p})) = \{\mathfrak{p}\} = \mathsf{supp}(E(R/\mathfrak{p}))$;
   (b) $\mathsf{supp}(R/\mathfrak{p}) = \vee(\mathfrak{p})$;
   (c) $\mathsf{supp}(R_\mathfrak{p}) = \wedge(\mathfrak{p})$;
2. *For any X in $\mathsf{D}(R)$, $\mathsf{Supp}(X)$ is specialisation closed;*
3. *[17, pag. 158] For any X in $\mathsf{D}(R)$, $\mathsf{supp}(X) \subseteq \mathsf{Supp}(X)$;*
4. *[17, Proposition 2.8/Remark 2.9][13, Proposition 2.1/Remark 2.2] For any bounded below X in $\mathsf{D}^+(R)$, $\mathsf{supp}(X)$ coincides with the set of prime ideals $\mathfrak{p}$ for which the module $E(R/\mathfrak{p})$ is a summand of a module appearing in the minimal K-injective resolution of X;*
5. *For any bounded below X in $\mathsf{D}^+(R)$, we have $\vee(\mathsf{supp}(X)) = \mathsf{Supp}(X)$.*

Proof. We prove items (2) and (5) for the sake of completeness (see [6, Lemma 2.2] for the analogous statements for modules). For (2), note that the claim follows from [6, Lemma 2.1], taking into account that the big support of a complex coincides with the union of the big supports of its cohomologies.
For (5), notice that by (2) and (3) we already have that $\bigvee(\mathsf{supp}(X)) \subseteq \mathsf{Supp}(X)$. For the converse inclusion, let $X \to E(X)$ denote a minimal K-injective resolution of X in $\mathsf{D}(R)$ and let $\mathfrak{q}$ be a prime ideal of R. Then $\mathfrak{q}$ is not in $\bigvee(\mathsf{supp}\, X)$ if and only if $\mathfrak{q}$ is not in $\bigvee(\mathfrak{p})$ for any $\mathfrak{p}$ in $\mathsf{supp}(X)$, i.e. if and only if, by [6, Lemma 2.2], $\mathfrak{q}$ is not in $\bigvee(\mathfrak{p}) = \mathsf{Supp}(E(R/\mathfrak{p}))$ for any $\mathfrak{p}$ in $\mathsf{supp}(X)$. Hence, by item (4), if $\mathfrak{q}$ does not lie in $\bigvee(\mathsf{supp}\, X)$ then $E(X)_{\mathfrak{q}} = 0$ (or, equivalently, $X_{\mathfrak{q}} = 0$) in $\mathsf{D}(R)$. □

For a subset $\mathcal{P} \subseteq \mathsf{Spec}(R)$, we write $\mathsf{supp}^{-1}(\mathcal{P})$ (respectively, $\mathsf{Supp}^{-1}(\mathcal{P})$) for the subcategory of $\mathsf{D}(R)$ whose objects have small (respectively, big) support contained in $\mathcal{P}$. By item (3) above, $\mathsf{Supp}^{-1}(\mathcal{P})$ is contained in $\mathsf{supp}^{-1}(\mathcal{P})$. If $\mathcal{P}$ is specialisation-closed, then item (5) of the lemma above guarantees that a bounded below complex belongs to $\mathsf{supp}^{-1}(\mathcal{P})$ if and only if it belongs to $\mathsf{Supp}^{-1}(\mathcal{P})$, i.e.

$$\mathcal{P} = \bigvee(\mathcal{P}) \quad \Longrightarrow \quad \mathsf{supp}^{-1}(\mathcal{P}) \cap \mathsf{D}^+(R) = \mathsf{Supp}^{-1}(\mathcal{P}) \cap \mathsf{D}^+(R) \tag{3.1}$$

3.2 Localising subcategories

Hereditary torsion classes of $\mathsf{Mod}(R)$ are also called *localising subcategories*. There is a well-known bijection (holding, in fact, for any ring) between localising subcategories of $\mathsf{Mod}(R)$ and *Giraud subcategories* of $\mathsf{Mod}(R)$, i.e. subcategories whose inclusion functor admits an exact left adjoint ([44]). This bijection associates a localising subcategory $\mathcal{T}$ of $\mathsf{Mod}(R)$ to the Giraud subcategory $\mathcal{T}^{\perp_{0,1}}$, whose objects (we recall) are those R-modules X such that for any T in $\mathcal{T}$

$$\mathsf{Hom}_R(T, X) = 0 = \mathsf{Ext}^1_R(T, X).$$

Localising subcategories of $\mathsf{Mod}(R)$ are completely characterised by their support as follows.

Theorem 3.4. *Let R be a commutative noetherian ring. Then the following statements hold.*

1. *[18] The assignment of support yields a bijection between localising subcategories of $\mathsf{Mod}(R)$ and specialisation-closed subsets of $\mathsf{Spec}(R)$.*
2. *For a localising subcategory $\mathcal{T}$ of $\mathsf{Mod}(R)$ we have*
   - *(a) a prime $\mathfrak{p}$ lies in $\mathsf{supp}(\mathcal{T})$ if and only if $k(\mathfrak{p})$ lies in $\mathcal{T}$;*
   - *(b) for any $\mathfrak{p}$ in $\mathsf{Spec}(R)$, then $k(\mathfrak{p})$ lies in either $\mathcal{T}$ or in $\mathcal{T}^{\perp_{0,1}}$;*
   - *(c)* $$\mathcal{T}^{\perp} = \{M \in \mathsf{Mod}(R) \colon \operatorname{Ass}(M) \cap \mathsf{supp}(\mathcal{T}) = \emptyset\} = \mathsf{Cogen}(\mathsf{supp}^{-1}(\mathsf{supp}(\mathcal{T})^{\mathsf{c}}) \cap \mathsf{Mod}(R)).$$

3. *[2, Lemma 4.2] A torsion pair in* $\mathsf{Mod}(R)$ *is hereditary if and only if it is of finite type.*

Proof. These are well-known statements; we comment on those for which we could not find a direct reference, for sake of readability. The assertion (2.a) follows from (1) and Lemma 3.3. For (2.b), one needs to check that if $\mathfrak{p}$ does not lie in $\mathsf{supp}(\mathcal{T})$, then $\mathsf{Hom}_R(T, k(\mathfrak{p})) = \mathsf{Ext}^1_R(T, k(\mathfrak{p})) = 0$ for all T in $\mathcal{T}$. By Lemma 3.3, every injective module in the minimal injective resolution of $k(\mathfrak{p})$ is a coproduct of copies of $E(R/\mathfrak{p})$. The statement now follows from the fact that T has only maps to injective modules of the form $E(R/\mathfrak{q})$, with $\mathfrak{q}$ in $\mathsf{supp}(\mathcal{T})$. Finally, (2.c) follows from the fact that, since $\mathcal{T}$ is localising, $\mathcal{T}^\perp$ is closed under injective envelopes and, thus, the torsion pair is cogenerated by a set of injectives. Clearly, these are the ones not supported in $\mathsf{supp}(\mathcal{T})$, as wanted. □

In the derived category, we can also parametrise certain subcategories in terms of their supports.

Definition 3.5. Let $\mathcal{D}$ be a triangulated category admitting arbitrary (set-indexed) coproducts. A subcategory $\mathcal{L}$ of $\mathcal{D}$ is said to be *localising* if $\mathcal{L}$ is a triangulated subcategory closed under coproducts. A localising subcategory $\mathcal{L}$ is furthermore said to be *smashing* if $\mathcal{L}^\perp$ is closed under coproducts.

Theorem 3.6. *[33] Let R be a commutative noetherian ring. Then, the following statements hold.*

1. *The assignment of support yields a bijection between localising subcategories of* $\mathsf{D}(R)$ *and the power set of* $\mathsf{Spec}(R)$. *Moreover, this bijection restricts to a bijection between smashing subcategories of* $\mathsf{D}(R)$ *and the set of specialisation-closed subsets of* $\mathsf{Spec}(R)$.
2. *For a localising subcategory $\mathcal{L}$ of* $\mathsf{D}(R)$ *we have that:*
   (a) *a prime $\mathfrak{p}$ lies in* $\mathsf{supp}(\mathcal{L})$ *if and only if $k(\mathfrak{p})$ lies in $\mathcal{L}$;*
   (b) *for any $\mathfrak{p}$ in* $\mathsf{Spec}(R)$, *then $k(\mathfrak{p})$ lies either in $\mathcal{L}$ or in $\mathcal{L}^\perp$;*
   (c) *$\mathcal{L}$ is the smallest localising subcategory containing* $\{k(\mathfrak{p}) \colon \mathfrak{p} \in \mathsf{supp}(\mathcal{L})\}$*;*
   (d) $(\mathcal{L}, \mathcal{L}^\perp)$ *is a t-structure.*

Notice the parallel, albeit with some subtle differences, between the abelian and the derived classification results. The following result summarises the relation between the two theorems above.

Proposition 3.7. *Let R be a commutative noetherian ring, V a specialisation-closed subset of* $\mathsf{Spec}(R)$ *and* $\mathcal{L} = \mathsf{supp}^{-1}(V)$ *the associated smashing subcategory of* $\mathsf{D}(R)$. *Then the localising subcategory $\mathcal{T}$ of* $\mathsf{Mod}(R)$ *associated to V is* $\mathcal{L} \cap \mathsf{Mod}(R)$ *and, moreover,* $\mathcal{T}^{\perp_{\geq 0}} = \mathcal{L}^\perp \cap \mathsf{Mod}(R)$.

Remark 3.8. Note that for a hereditary torsion class in $\mathsf{Mod}(R)$, it can be shown that $\mathcal{T}^{\perp_{\geq 0}} = \mathcal{T}^{\perp_{0,1}}$ if and only if $\mathcal{T}$ is a perfect torsion class, i.e. if and only if the associated Gabriel topology is perfect. In §4.2 we will prove this fact and make use of it to obtain a complete classification of hereditary torsion pairs in the HRS-tilt of $\mathsf{Mod}(R)$ with respect to a perfect torsion pair.

3.3 t-structures

There is a wealth of knowledge about some classes of t-structures in the derived category of a commutative noetherian ring. Once again subsets of the spectrum play a crucial role in the characterisation of (compactly generated) t-structures.

Definition 3.9. Let R be a commutative noetherian ring. A function φ from $\mathbb{Z}$ to the power set of $\mathsf{Spec}(R)$ is said to be an *sp-filtration of* $\mathsf{Spec}(R)$ if φ is a decreasing function between posets (i.e. if for all integers n, $\varphi(n) \supseteq \varphi(n+1)$) and $\varphi(n)$ is specialisation-closed, for all n.

The following theorem concerns the classification and properties of compactly generated t-structures over commutative noetherian rings.

Theorem 3.10. *[1, Theorem 4.10] [22, Theorem 1.1] Let R be a commutative noetherian ring. The following are equivalent for a nondegenerate t-structure $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ in $\mathsf{D}(R)$.*

1. *$\mathbb{T}$ is compactly generated;*

2. *$\mathbb{T}$ is smashing with Grothendieck heart (and the equivalent conditions of Theorem 2.12);*

3. *There is an sp-filtration of $\mathsf{Spec}(R)$ for which*

$$\mathcal{U} = \{X \in \mathsf{D}(R)\colon \mathsf{Supp}(H^0(X[n])) \subseteq \varphi(n), \forall n \in \mathbb{Z}\}$$
$$\mathcal{V} = \{X \in \mathsf{D}(R)\colon \mathbb{R}\Gamma_{\varphi(n)}(X) \in \mathbb{D}^{\geq n+1}, \forall n \in \mathbb{Z}\},$$

where Γ_V denotes the (left exact) torsion radical of the hereditary torsion pair $(\mathsf{Supp}^{-1}(V), \mathcal{F}_V)$ of $\mathsf{Mod}(R)$, for a specialisation-closed set V (see Theorem 3.4(1)).

It is easy to see that a t-structure as in the theorem above is nondegenerate if and only if the associated sp-filtration φ satisfies $\cup_{n\in\mathbb{Z}}\varphi(n) = \mathsf{Spec}(R)$ and $\cap_{n\in\mathbb{Z}}\varphi(n) = \emptyset$. Moreover, such a t-structure is intermediate if and only if there are integers $a < b$ such that $\varphi(a) = \mathsf{Spec}(R)$ and $\varphi(b) = \emptyset$. In this case φ will therefore be called *intermediate*.
Combining the results above, we observe the following useful statement.

Corollary 3.11. *Let R be a commutative notherian ring, and $\mathcal{H}_{\mathbb{T}}$ be the heart of a nondegenerate compactly generated t-structure $\mathbb{T}$ in $\mathsf{D}(R)$. Then, a torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}_{\mathbb{T}}$ is of finite type if and only if it is generated by a set of finitely presented objects of $\mathcal{H}_{\mathbb{T}}$.*

Proof. By Lemma 2.5(1), we only need to prove one implication. If $\mathbf{t}$ is of finite type in $\mathcal{H}_{\mathbb{T}}$, Theorem 2.16(1) shows that the HRS-tilt of $\mathbb{T}$ with respect to $\mathbf{t}$ is a smashing t-structure with a Grothendieck heart and, thus, by Theorem 3.10 it is compactly generated. The result then follows from 2.16(2). □

Remark 3.12. Note that in the above corollary, $\mathcal{H}_{\mathbb{T}}$ is not necessarily locally coherent — although, as shown in [42], it is locally finitely presented.

4 Hereditary torsion pairs in Grothendieck hearts

In this section we discuss hereditary torsion pairs in a given Grothendieck heart in the derived category of a commutative noetherian ring. Throughout, once again R will denote a commutative noetherian ring.

4.1 A characterisation by support

We begin by showing that hereditary torsion classes in the heart of a smashing nondegenerate t-structure of $\mathsf{D}(R)$ are completely determined by their support in $\mathsf{Spec}(R)$.

Proposition 4.1. *Let R be a commutative noetherian ring and let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be a nondegenerate t-structure in $\mathsf{D}(R)$ with heart $\mathcal{H}_{\mathbb{T}}$ and cohomological functor $H^0_{\mathbb{T}}\colon \mathsf{D}(R) \to \mathcal{H}_{\mathbb{T}}$. If $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion pair in $\mathcal{H}_{\mathbb{T}}$, then:*

1. *for each $\mathfrak{p}$ in $\mathsf{Spec}(R)$, there is an integer $n_{\mathfrak{p}}$ for which $k(\mathfrak{p})[n_{\mathfrak{p}}]$ lies in $\mathcal{H}_{\mathbb{T}}$;*
2. *$\mathsf{supp}(\mathcal{H}_{\mathbb{T}}) = \mathsf{Spec}(R)$;*
3. *if $\mathbb{T}$ is, in addition, smashing, then*
   - *(a) $\mathcal{L}_{\mathbf{t}} := \{X \in \mathsf{D}(R)\colon H^0_{\mathbb{T}}(X[i]) \in \mathcal{T} \text{ for every } i \in \mathbb{Z}\}$ is a localising subcategory of $\mathsf{D}(R)$;*
   - *(b) $\mathsf{supp}(\mathcal{L}_{\mathbf{t}}) = \mathsf{supp}(\mathcal{T}) = \{\mathfrak{p} \in \mathsf{Spec}(R)\colon k(\mathfrak{p})[n_{\mathfrak{p}}] \in \mathcal{T}\}$*
   - *(c) $\mathcal{L}_{\mathbf{t}}$ is the smallest localising subcategory containing $\mathcal{T}$;*
   - *(d) $\mathcal{L}_{\mathbf{t}}^{\perp} \cap \mathcal{H}_{\mathbb{T}} \subseteq \mathcal{T}^{\perp_{0,1}}$;*
   - *(e) for each $\mathfrak{p}$ in $\mathsf{Spec}(R)$, $k(\mathfrak{p})[n_{\mathfrak{p}}]$ lies in $\mathcal{T}$ or $k(\mathfrak{p})[n_{\mathfrak{p}}]$ lies in the Giraud subcategory $\mathcal{T}^{\perp_{0,1}}$;*
   - *(f) $\mathsf{supp}^{-1}(\mathsf{supp}(\mathcal{T})) \cap \mathcal{H}_{\mathbb{T}} = \mathcal{T}$, i.e. $\mathcal{T}$ is completely determined by its support.*

Proof. (1) It is shown in [22, Lemma 2.7] that the following two subsets form a partition of $\mathbb{Z}$:

$$A(\mathfrak{p}) := \{a \in \mathbb{Z}\colon k(\mathfrak{p}) \in \mathcal{U}[a]\} \qquad \text{and} \qquad B(\mathfrak{p}) := \{b \in \mathbb{Z}\colon k(\mathfrak{p}) \in \mathcal{V}[b]\}.$$

Since $\mathbb{T}$ is nondegenerate, this is a nontrivial partition. Moreover, since $\mathcal{U}$ (respectively, $\mathcal{V}$) is closed under positive (respectively, negative) shifts, if $m \leq n \in A(\mathfrak{p})$ then $m \in A(\mathfrak{p})$ (respectively, if $m \geq n \in B(\mathfrak{p})$ then $m \in B(\mathfrak{p})$). Hence, $A(\mathfrak{p})$ has a maximum, say α, and $B(\mathfrak{p})$ has a minimum, say β. Since these sets form a partition of $\mathbb{Z}$, we conclude that $\beta = \alpha + 1$ and, thus, $k(\mathfrak{p})$ lies in $\mathcal{U}[\alpha] \cap \mathcal{V}[\alpha+1] = \mathcal{H}_{\mathbb{T}}[\alpha]$. In other words, we have that $k(\mathfrak{p})[-\alpha]$ lies in $\mathcal{H}_{\mathbb{T}}$, so it suffices to take $n_{\mathfrak{p}} = -\alpha$.
(2) Since $\mathsf{supp}(k(\mathfrak{p})[n]) = \{\mathfrak{p}\}$ for any integer n, it follows from (1) that $\mathsf{supp}(\mathcal{H}_{\mathbb{T}}) = \mathsf{Spec}(R)$.
(3.a) Since $\mathcal{T}$ is closed under subobjects, extensions and quotient objects, it is easy to see that $\mathcal{L}_{\mathbf{t}}$ is a triangulated subcategory. Furthermore, since $\mathbb{T}$ is smashing, $H^0_{\mathbb{T}}$ commutes with coproducts and, thus, the fact that $\mathcal{T}$ is closed under coproducts allows us to conclude that $\mathcal{L}_{\mathbf{t}}$ is a localising subcategory.
(3.b) Let us denote by $\mathcal{P}$ the subset of $\mathsf{Spec}(R)$ consisting of the prime ideals $\mathfrak{p}$ for which $k(\mathfrak{p})[n_{\mathfrak{p}}]$ lies in $\mathcal{T}$. We prove our statement by showing that

$$\mathsf{supp}(\mathcal{L}_{\mathbf{t}}) \subseteq \mathcal{P} \subseteq \mathsf{supp}(\mathcal{T}) \subseteq \mathsf{supp}(\mathcal{L}_{\mathbf{t}}).$$

Let $\mathfrak{p}$ be a prime ideal of R. If $\mathfrak{p}$ lies in $\mathsf{supp}(\mathcal{L}_{\mathbf{t}})$ then $k(\mathfrak{p})$ lies in $\mathcal{L}_{\mathbf{t}}$ (see Theorem 3.6) and, thus, $k(\mathfrak{p})[n_{\mathfrak{p}}]$ lies in $\mathcal{T}$, i.e. $\mathfrak{p}$ lies in $\mathcal{P}$. If $\mathfrak{p}$ lies in $\mathcal{P}$, since $\mathsf{supp}(k(\mathfrak{p})[n_{\mathfrak{p}}]) = \{\mathfrak{p}\}$, then $\mathfrak{p}$ lies in $\mathsf{supp}(\mathcal{T})$. Finally, if $\mathfrak{p}$ lies in $\mathsf{supp}(\mathcal{T})$, since $\mathcal{T}$ is contained in $\mathcal{L}_{\mathbf{t}}$, it follows that $\mathfrak{p}$ lies in $\mathsf{supp}(\mathcal{L}_{\mathbf{t}})$.
(3.c) Since $\mathcal{T}$ is contained in $\mathcal{L}_{\mathbf{t}}$, the smallest localising subcategory containing $\mathcal{T}$ must be contained in $\mathcal{L}_{\mathbf{t}}$. Conversely, if $\mathcal{L}$ is an arbitrary localising subcategory containing $\mathcal{T}$, then $\mathsf{supp}(\mathcal{L})$ must contain $\mathsf{supp}(\mathcal{T}) = \mathsf{supp}(\mathcal{L}_{\mathbf{t}})$ and, thus, $\mathcal{L}$ must contain $\mathcal{L}_{\mathbf{t}}$.
(3.d) Given X in $\mathcal{L}_{\mathbf{t}}^{\perp} \cap \mathcal{H}_{\mathbb{T}}$ and T in $\mathcal{T}$ (and, thus, in $\mathcal{L}_{\mathbf{t}}$), we have that $\mathsf{Hom}_{\mathsf{D}(R)}(T, X) = 0$ and $\mathsf{Ext}^1_{\mathcal{H}_{\mathbb{T}}}(T, X) \simeq \mathsf{Hom}_{\mathsf{D}(R)}(T[-1], X) = 0$ since $T[-1]$ lies also in $\mathcal{L}_{\mathbf{t}}$. Thus, X lies in $\mathcal{T}^{\perp_{0,1}}$.
(3.e) Let $\mathfrak{p}$ be an arbitrary prime ideal of R and consider the object $k(\mathfrak{p})[n_{\mathfrak{p}}]$ of $\mathcal{H}_{\mathbb{T}}$. By Theorem 3.6, this object either lies in $\mathcal{L}_{\mathbf{t}} \cap \mathcal{H}_{\mathbb{T}} = \mathcal{T}$ or in $\mathcal{L}_{\mathbf{t}}^{\perp} \cap \mathcal{H}_{\mathbb{T}} \subseteq \mathcal{T}^{\perp_{0,1}}$.
(3.f) From (3.c) and Theorem 3.6, it follows that that $\mathsf{supp}^{-1}(\mathsf{supp}(\mathcal{T}))) = \mathcal{L}_{\mathbf{t}}$. By definition of $\mathcal{L}_{\mathbf{t}}$, we have $\mathcal{L}_{\mathbf{t}} \cap \mathcal{H}_{\mathbb{T}} = \mathcal{T}$, thus proving our claim. □

Note that, in particular, it follows that the hereditary torsion classes of the heart of a nondegenerate smashing t-structure form a set. Item (3.f) of the previous proposition motivates the following definition.

Definition 4.2. Let R be a commutative noetherian ring and $\mathbb{T}$ a nondegenerate smashing t-structure in $\mathsf{D}(R)$, with heart $\mathcal{H}_{\mathbb{T}}$. A set $U \subseteq \mathsf{Spec}(R)$ is called a $\mathcal{H}_{\mathbb{T}}$-*support* if it is the support of a hereditary torsion class in $\mathcal{H}_{\mathbb{T}}$. This torsion class will then be $\mathsf{supp}^{-1}(U) \cap \mathcal{H}_{\mathbb{T}}$.

As a side corollary of the proposition above, we deduce a relation between the support of a complex and that of its cohomologies with respect to a smashing t-structure.

Corollary 4.3. *Let R be a commutative noetherian ring, and let $\mathbb{T}$ be a nondegenerate smashing t-structure in $\mathsf{D}(R)$, with heart $\mathcal{H}_\mathbb{T}$ and cohomology functor $H^0_\mathbb{T}$. Let $U \subseteq \mathsf{Spec}(R)$ be a $\mathcal{H}_\mathbb{T}$-support: then, for every object X of $\mathsf{D}(R)$, we have*

$$\mathsf{supp}(X) \subseteq U \quad \textit{if and only if} \quad \mathsf{supp}(H^0_\mathbb{T}(X[n])) \subseteq U \ \forall n \in \mathbb{Z}$$

Proof. This is direct consequence of items (3.a) and (3.b) of Theorem 4.1. ☐

Remark 4.4. Note that this corollary recovers and extends the known relation (see [6, Corollary 5.3])

$$\mathsf{V}(\mathsf{supp}(X)) = \mathsf{V}(\mathsf{supp}(\bigoplus\nolimits_{n\in\mathbb{Z}} H^0(X[n])))$$

by taking $\mathbb{T} = \mathbb{D}$ the standard t-structure, and using that both sides of the equation are $\mathsf{Mod}(R)$-supports (i.e. specialisation-closed subsets).

The following theorem provides some examples of $\mathcal{H}_\mathbb{T}$-supports, for some particular kinds of hearts.

Theorem 4.5. *Let R be a commutative noetherian ring and let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be an intermediate compactly generated t-structure in $\mathsf{D}(R)$ with heart $\mathcal{H}_\mathbb{T}$. The following statements hold.*

1. *If V is specialisation closed, then*
   - *(a) $\mathcal{T}_V := \mathsf{supp}^{-1}(V) \cap \mathcal{H}_\mathbb{T}$ is a hereditary torsion class in $\mathcal{H}_\mathbb{T}$;*
   - *(b) if, additionally, $\mathbb{T}$ induces a derived equivalence, then $\mathbf{t}_V = (\mathcal{T}_V, \mathcal{T}_V^\perp)$ is a torsion pair of finite type and $\mathcal{T}_V^\perp = \mathsf{Cogen}(\mathsf{supp}^{-1}(V^\mathsf{c}) \cap \mathcal{H}_\mathbb{T})$.*
2. *If $\mathbb{T}$ is restrictable, then for any hereditary torsion pair of finite type $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}_\mathbb{T}$ we have that $\mathsf{supp}(\mathcal{T})$ is specialisation closed.*

Proof. (1.a) We first show that $\mathcal{T}_V := \mathsf{supp}^{-1}(V) \cap \mathcal{H}_\mathbb{T}$ is a hereditary torsion class in $\mathcal{H}_\mathbb{T}$, whenever V is a specialisation closed subset of $\mathsf{Spec}(R)$. First note that, since $\mathbb{T}$ is intermediate, $\mathcal{H}_\mathbb{T}$ is contained in $\mathsf{D}^b(R)$, and since V is specialisation closed, it follows from Lemma 3.3 that $\mathsf{supp}^{-1}(V) \cap \mathcal{H}_\mathbb{T} = \mathsf{Supp}^{-1}(V) \cap \mathcal{H}_\mathbb{T}$. Since $\mathbb{T}$ is compactly generated, it is homotopically smashing and, thus, both $\mathcal{U}$ and $\mathcal{V}$ are closed under directed homotopy colimits. From [22, Lemma 2.11] it follows that both $\mathcal{U}$ and $\mathcal{V}$ are closed under $- \otimes_R R_\mathfrak{p}$ and, therefore, $- \otimes_R R_\mathfrak{p}$ is exact in $\mathcal{H}_\mathbb{T}$, for any $\mathfrak{p}$ in $\mathsf{Spec}(R)$. This shows that given a short exact sequence in $\mathcal{H}_\mathbb{T}$ of the form

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

we have that $Y \otimes_R R_\mathfrak{p} = 0$ if and only if $X \otimes_R R_\mathfrak{p} = 0 = Z \otimes_R R_\mathfrak{p}$. In other words, we have that $\mathsf{Supp}(Y) = \mathsf{Supp}(X) \cup \mathsf{Supp}(Z)$ and, thus $\mathsf{Supp}(Y)$ is contained in V if and only if both $\mathsf{Supp}(X)$ and $\mathsf{Supp}(Z)$ are contained in V.

This shows that $\mathcal{T}_V$ is closed under extensions, subobjects and quotient objects. Since it is also clearly closed under coproducts, $\mathcal{T}_V$ is a hereditary torsion class.
(1.b) Suppose now that $\mathbb{T}$ induces a derived equivalence. In this case we know that there is an isomorphism $\mathsf{Ext}^k_{\mathcal{H}_\mathbb{T}}(X,Y) \simeq \mathsf{Hom}_{\mathsf{D}(R)}(X,Y[k])$ for any X and Y in $\mathcal{H}_\mathbb{T}$ and $k \geq 0$. In particular, for a subcategory $\mathcal{S}$ of $\mathcal{H}_\mathbb{T}$, there is no ambiguity when calculating the orthogonal $\mathcal{S}^{\perp_J}$: this Ext-orthogonal subcategory in $\mathcal{H}_\mathbb{T}$ coincides with the intersection with $\mathcal{H}_\mathbb{T}$ of the orthogonal computed in $\mathsf{D}(R)$.
We first show that $\mathcal{T}_V^{\perp_{\geq 0}} = \mathsf{supp}^{-1}(V^\mathsf{c}) \cap \mathcal{H}_\mathbb{T}$. It follows from [33] that $\mathcal{L}_V := \mathsf{supp}^{-1}(V)$ is a smashing subcategory of $\mathsf{D}(R)$ and, thus, $\mathcal{B}_V := \mathcal{L}_V^\perp$ is also localising with $\mathsf{supp}(\mathcal{B}_V) = V^\mathsf{c}$. Since, from Proposition 4.1, $\mathcal{L}_V$ is the smallest localising subcategory containing $\mathcal{T}_V$ and since $\mathbb{T}$ induces a derived equivalence, it follows that $\mathcal{B}_V \cap \mathcal{H}_\mathbb{T} = \mathcal{T}_V^{\perp_{\geq 0}}$. Finally, note that since $\mathcal{T}_V$ is hereditary, we have that $(\mathcal{T}_V, \mathcal{T}_V^\perp) = ({}^\perp E_V, \mathsf{Cogen}(E_V))$ for an injective object E_V in $\mathcal{H}$, and E_V lies in $\mathcal{T}^{\perp_{\geq 0}}$. It then follows that $\mathcal{T}_V^\perp = \mathsf{Cogen}(\mathcal{B}_V \cap \mathcal{H}_\mathbb{T}) = \mathsf{Cogen}(\mathsf{supp}^{-1}(V^\mathsf{c}) \cap \mathcal{H}_\mathbb{T})$.
We now show that this torsion pair $\mathbf{t} = (\mathcal{T}_V, \mathcal{T}_V^\perp)$ is of finite type. Recall that since $\mathcal{L}_V$ is smashing, it is well-known (see [25, Theorem 4.2] and [29, Proposition 6.3]) that $\mathcal{B}_V$ is a definable subcategory and, thus, closed under directed homotopy colimits [28, Theorem 3.11]. By [43, Corollary 5.8], since $\mathbb{T}$ is homotopically smashing, every direct limit in $\mathcal{H}_\mathbb{T}$ is a directed homotopy colimit in $\mathsf{D}(R)$. Hence, $\mathcal{B}_V \cap \mathcal{H}_\mathbb{T} = T^{\perp_{\geq 0}}$ is closed under direct limits in $\mathcal{H}_\mathbb{T}$. Since direct limits are exact in $\mathcal{H}_\mathbb{T}$ and $\mathcal{T}_V^\perp = \mathsf{Cogen}(\mathcal{T}^{\perp_{\geq 0}} \cap \mathcal{H}_\mathbb{T})$, we get that $\mathcal{T}_V^\perp$ is closed under direct limits.
(2) By Theorem 2.13, since $\mathbb{T}$ is restrictable, $\mathcal{H}_\mathbb{T}$ is locally coherent and $\mathsf{fp}(\mathcal{H}_\mathbb{T}) = \mathcal{H}_\mathbb{T} \cap D^b(\mathsf{mod}(R))$. Since $\mathbf{t}$ is a hereditary torsion pair of finite type, it follows from Lemma 2.5(2) that $\mathcal{T} = \varinjlim(\mathcal{T} \cap \mathsf{fp}(\mathcal{H}_\mathbb{T}))$. Let $\mathcal{L}$ be the smallest localising subcategory of $\mathsf{D}(R)$ containing $\mathcal{T} \cap \mathsf{fp}(\mathcal{H}_\mathbb{T})$. Clearly, $\mathcal{L}$ is contained in the smallest localising subcategory containing $\mathcal{T}$, which we denote by $\mathcal{L}_\mathbf{t}$. Since $\mathcal{L}$ is the aisle of a t-structure (namely $(\mathcal{L}, \mathcal{L}^\perp)$), $\mathcal{L}$ is closed under directed homotopy colimits. As above, since $\mathbb{T}$ is homotopically smashing, directed limits in $\mathcal{H}_\mathbb{T}$ are directed homotopy colimits in $\mathsf{D}(R)$ and, thus, $\mathcal{T}$ is contained in $\mathcal{L}$, showing that $\mathcal{L} = \mathcal{L}_\mathbf{t}$. Therefore, we have that $\mathsf{supp}(\mathcal{T}) = \mathsf{supp}(\mathcal{L}_\mathbf{t}) = \mathsf{supp}(\mathcal{L})$. Now, by assumption, the t-structure $(\mathcal{L}, \mathcal{L}^\perp)$ is generated by all shifts of $\mathcal{T} \cap \mathsf{fp}(\mathcal{H}_\mathbb{T})$ which, by assumption is made of complexes in $\mathsf{D}^b(\mathsf{mod}(R))$. Now by [1, Theorem 3.10] this means that $\mathcal{L}$ is compactly generated and, therefore, smashing. This shows, by Theorem 3.6, that $\mathsf{supp}(\mathcal{T}) = \mathsf{supp}(\mathcal{L})$ is specialisation closed. □

Notice that the theorem above provides an immediate generalisation of Theorem 3.4, which we will further simplify in Corollary 6.18.

Corollary 4.6. *Let R be a commutative noetherian ring and let $\mathbb{T}$ be a restrictable and intermediate compactly generated t-structure in $\mathsf{D}(R)$ such that $\mathbb{T}$ induces a derived equivalence. Then there is a bijection between hereditary torsion pairs of finite type in $\mathcal{H}_\mathbb{T}$ and specialisation closed subsets of $\mathsf{Spec}(R)$.*

In fact, we can be more precise about the support of a hereditary torsion pair of finite type for nondegenerate compactly generated t-structures.

Proposition 4.7. *Let R be a commutative noetherian ring and let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be a nondegenerate compactly generated t-structure in $\mathsf{D}(R)$ with heart $\mathcal{H}_\mathbb{T}$ and associated sp-filtration φ.*

1. *For each $\mathfrak{p}$ in $\mathsf{Spec}(R)$, let $\varphi_{max}(\mathfrak{p})$ denote the largest integer n for which $\mathfrak{p}$ belongs to $\varphi(n)$. Then we have $n_\mathfrak{p} = -\varphi_{max}(\mathfrak{p})$ and, in particular, if $\mathfrak{p}$ is contained in a prime $\mathfrak{q}$, then $n_\mathfrak{p} \geq n_\mathfrak{q}$.*

2. *If $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion pair of finite type in $\mathcal{H}_\mathbb{T}$, then there is an sp-filtration ψ such that $\varphi(j+1) \subseteq \psi(j) \subseteq \varphi(j)$ for all j in $\mathbb{Z}$ and*

$$\mathsf{supp}(\mathcal{T}) = \bigcup_{j \in \mathbb{Z}} [\psi(j) \setminus \varphi(j+1)] .$$

Proof. (1) Given the cohomological description of $\mathcal{U}$ (see Theorem 3.10), we know that the stalk complex $k(\mathfrak{p})[-\varphi_{max}(\mathfrak{p})]$ lies in $\mathcal{U}$ but $k(\mathfrak{p})[-(\varphi_{max}(\mathfrak{p})+1)]$ does not. By [22, Lemma 2.7], this means that $k(\mathfrak{p})[-\varphi_{max}(\mathfrak{p}) - 1]$ lies in $\mathcal{V}$ and, therefore, $k(\mathfrak{p})[-\varphi_{max}(\mathfrak{p})]$ lies in $\mathcal{V}[1] \cap \mathcal{U} = \mathcal{H}_\mathbb{T}$, as wanted. Since an sp-filtration is a decreasing sequence of specialisation-closed subsets, we have that if $\mathfrak{p}$ is contained in a prime $\mathfrak{q}$, then $\varphi_{max}(\mathfrak{p}) \leq \varphi_{max}(\mathfrak{q})$ and, thus, $n_\mathfrak{p} \geq n_\mathfrak{q}$.
(2) Let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair of finite type in $\mathcal{H}_\mathbb{T}$. The t-structure obtained by HRS-tilting $\mathbb{T}$ with respect to $\mathbf{t}$ is compactly generated, since $\mathbf{t}$ is of finite type (see Theorems 2.16 and 3.10). Thus, it is determined by an sp-filtration ψ satisfying $\varphi(j+1) \subseteq \psi(j) \subseteq \varphi(j)$. Let $\mathcal{U}_\psi$ be the aisle of the t-structure associated to ψ. Clearly, we have that $\mathcal{T} = \mathcal{U}_\psi \cap \mathcal{H}_\mathbb{T}$. We need to check which shifted residue fields belong to $\mathcal{T}$ (following Proposition 4.1(3.b)). For any $\mathfrak{p}$ in $\mathsf{Spec}(R)$, by (1) $k(\mathfrak{p})[-j]$ lies in $\mathcal{H}_\mathbb{T}$ if and only if $\mathfrak{p}$ lies in $\varphi(j) \setminus \varphi(j+1)$. Now, $k(\mathfrak{p})[-j]$ lies in $\mathcal{T}$ if and only if $k(\mathfrak{p})[-j]$ lies in $\mathcal{H}_\mathbb{T} \cap \mathcal{U}_\psi$, i.e. $\mathfrak{p}$ lies in $\psi(j) \setminus \varphi(j+1)$. Thus $\mathsf{supp}(\mathcal{T})$ coincides with the union of all such sets $\psi(j) \setminus \varphi(j+1)$. □

4.2 A complete classification of hereditary torsion pairs in a special case

The previous section shows that when trying to classify the hereditary torsion pairs in the heart $\mathcal{H}_\mathbb{T}$ of a nondegenerate compactly generated t-structure, one may equivalently describe the corresponding $\mathcal{H}_\mathbb{T}$-supports. For example, by Theorem 4.5 we know that specialisation-closed sets are often $\mathcal{H}_\mathbb{T}$-supports. While the problem of classifying all $\mathcal{H}_\mathbb{T}$-supports remains, in general, open, we are able to provide a complete classification for some hearts. These occur as HRS-tilts of $\mathsf{Mod}(R)$ at a perfect torsion pair.

Remark 4.8. Notice that the HRS-tilt of $\mathsf{Mod}(R)$ at a hereditary torsion pair, corresponding to a specialisation-closed subset $V \subseteq \mathsf{Spec}(R)$, is always compactly generated. Indeed, its aisle corresponds to the sp-filtration $\cdots = \mathsf{Spec}(R) \supseteq V \supseteq \emptyset = \cdots$, with V in degree 0.

Definition 4.9. A hereditary torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in a module category is said to be *perfect* if the associated Giraud subcategory $\mathcal{T}^{\perp_{0,1}}$ admits a right adjoint to the inclusion functor.

Note that, by definition, the inclusion functor of a Giraud subcategory admits a left adjoint. Therefore, the adjective *perfect* applied to a hereditary torsion pair guarantees that the associated Giraud subcategory is, in fact, a (extension-closed) bireflective subcategory of $\mathsf{Mod}(R)$. We learned the following useful property of perfect torsion pairs from a private communication with Lidia Angeleri Hügel and Ryo Takahashi. We include a proof for sake of completion.

Lemma 4.10. *Let R be a ring and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a perfect torsion pair in $\mathsf{Mod}(R)$. Then $\mathcal{T}^{\perp_{0,1}} = \mathcal{T}^{\perp_{\geq 0}}$.*

Proof. Since $\mathbf{t}$ is hereditary, $\mathcal{F} = \mathcal{T}^{\perp}$ is closed under injective envelopes, so $\mathcal{T}^{\perp_{0,1}}$ is as well. Recall that a Giraud subcategory of $\mathsf{Mod}(R)$ is always an abelian subcategory ([44, Proposition 1.3]). Since $\mathbf{t}$ is perfect, the inclusion functor $\mathcal{T}^{\perp_{0,1}} \hookrightarrow \mathsf{Mod}(R)$ has a right adjoint: therefore, cokernels taken in $\mathcal{T}^{\perp_{0,1}}$ coincide with those in $\mathsf{Mod}(R)$. This shows that $\mathcal{T}^{\perp_{0,1}}$ is closed under cokernels in $\mathsf{Mod}(R)$. It follows that $\mathcal{T}^{\perp_{0,1}}$ is closed under cosygyzies, and one concludes by dimension shifting. □

The following lemma gives examples of hearts $\mathcal{H}$ with $\mathcal{H}$-supports that are not specialisation-closed.

Lemma 4.11. *Let R be a ring and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a perfect torsion pair in $\mathsf{Mod}(R)$. Denote the corresponding Giraud subcategory by $\mathcal{C} := \mathcal{T}^{\perp_{0,1}}$. Let $\mathcal{H}_{\mathbf{t}}$ be the heart of the HRS-tilt at $\mathbf{t}$. The following statements hold:*

1. *There is a TTF triple $(\mathcal{F}[1], \mathcal{T}, \mathcal{C}[1])$ in $\mathcal{H}_{\mathbf{t}}$;*
2. *$\mathcal{C}[1]$ itself is a hereditary torsion class in $\mathcal{H}_{\mathbf{t}}$;*
3. *If R is commutative noetherian, then $\mathsf{supp}(\mathcal{C}[1]) = V^{\mathsf{c}}$. Hence, V^{c} is a (generalisation-closed) $\mathcal{H}_{\mathbf{t}}$-support.*

Proof. (1) The fact that $\mathcal{T}$ is TTF class in $\mathcal{H}_{\mathbf{t}}$ follows from Remark 2.15. We only need to verify that the corresponding torsionfree class in $\mathcal{H}_{\mathbf{t}}$, denoted by $\mathcal{F}'$, coincides with $\mathcal{C}[1]$. Since $\mathsf{Hom}_{\mathcal{H}_{\mathbf{t}}}(T, C[1]) = \mathsf{Hom}_{\mathsf{D}(R)}(T, C[1]) \simeq \mathsf{Ext}^1_R(T, C) = 0$, for all T in $\mathcal{T}$ and C in $\mathcal{C}$, we have $\mathcal{C}[1] \subseteq \mathcal{F}'$. For the converse, let X be an object in $\mathcal{F}'$, and consider the triangle

$$F_X[1] \longrightarrow X \longrightarrow T_X \xrightarrow{w} F_X[2]$$

which corresponds to the short exact sequence in $\mathcal{H}_{\mathbf{t}}$ given by the torsion pair $(\mathcal{F}[1], \mathcal{T})$. Since $\mathcal{F}'$ is closed under subobjects in $\mathcal{H}_{\mathbf{t}}$, for every T in $\mathcal{T}$ we have $0 = \mathsf{Hom}_{\mathcal{H}_{\mathbf{t}}}(T, F_X[1]) = \mathsf{Hom}_{\mathsf{D}(R)}(T, F_X[1]) \simeq \mathsf{Ext}^1_R(T, F_X)$. This shows that in fact F_X lies in $\mathcal{C}$. Since the torsion pair $\mathbf{t}$ is perfect, $\mathcal{C} = \mathcal{T}^{\perp_{\geq 0}}$ and, therefore, $0 = \mathsf{Ext}^2_R(T_X, F_X) \simeq \mathsf{Hom}_{\mathsf{D}(R)}(T_X, F_X[2])$. Since w lies in the latter Hom-space, we conclude that $w = 0$ and that the triangle above splits; hence $T_X = 0$ and $X = F_X[1]$ lies in $\mathcal{C}[1]$.
(2) Since $\mathcal{C}[1]$ is a torsionfree class in $\mathcal{H}_{\mathbf{t}}$, it suffices to show that $\mathcal{C}[1]$ is closed under cokernels of monomorphisms in $\mathcal{H}_{\mathbf{t}}$. Consider then C and C' in $\mathcal{C}$ and a triangle

$$C[1] \longrightarrow C'[1] \longrightarrow X \longrightarrow C[2]$$

with X in $\mathcal{H}_{\mathbf{t}}$. Applying the functor $\mathsf{Hom}_{\mathsf{D}(R)}(T, -)$ for every T in $\mathcal{T}$, since $\mathsf{Hom}_{\mathsf{D}(R)}(T, C'[1]) \simeq \mathsf{Ext}^1_R(T, C') = 0$ and $\mathsf{Hom}_{\mathsf{D}(R)}(T, C[2]) \simeq \mathsf{Ext}^2_R(T, C) = 0$ (given that $\mathbf{t}$ is a perfect torsion pair), we have that $\mathsf{Hom}_{\mathsf{D}(R)}(T, X) = 0$. This shows that X indeed lies in $\mathcal{F}' = \mathcal{C}[1]$.
(3) If R is commutative noetherian, since $\mathsf{supp}(\mathcal{C}[1]) = \mathsf{supp}(\mathcal{C})$, it suffices to observe that $\mathsf{supp}(\mathcal{C}) = V^{\mathsf{c}}$ (because $E(R/\mathfrak{p})$ lies in $\mathcal{C}$ for all $\mathfrak{p}$ not in V). The last assertion then follows from part (2). □

In fact, using this generalisation-closed $\mathcal{H}_{\mathbf{t}}$-support one can construct many more which are not specialisation-closed. Recall that torsion pairs can be ordered by inclusion of torsion classes. Under this partial order, they form a lattice, where the meet is given by intersecting torsion classes and the join is given by intersecting torsionfree classes. The following lemma recalls that this structure restricts to hereditary torsion classes, and shows that it translates well via the assignment of support.

Lemma 4.12. *Let R be a commutative noetherian ring and let $\mathcal{H}$ be the heart of a nondegenerate compactly generated t-structure in $\mathsf{D}(R)$. Let $\mathbf{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$, $\mathbf{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ be hereditary torsion pairs in $\mathcal{H}$ with $\mathsf{supp}(\mathcal{T}_1) = V_1$, $\mathsf{supp}(\mathcal{T}_2) = V_2$. Then:*

1. *$\mathbf{t}_3 := \mathbf{t}_1 \wedge \mathbf{t}_2 := (\mathcal{T}_3 := \mathcal{T}_1 \cap \mathcal{T}_2, \mathcal{F}_3)$ is a hereditary torsion pair in $\mathcal{H}$ with $\mathsf{supp}(\mathcal{T}_3) = V_1 \cap V_2$;*
2. *$\mathbf{t}_4 := \mathbf{t}_1 \vee \mathbf{t}_2 := (\mathcal{T}_4, \mathcal{F}_4 := \mathcal{F}_1 \cap \mathcal{F}_2)$ is a hereditary torsion pair in $\mathcal{H}$ with $\mathsf{supp}(\mathcal{T}_4) = V_1 \cup V_2$.*

In particular, intersections and unions of $\mathcal{H}$-supports are again $\mathcal{H}$-supports.

Proof. (1) If $\mathcal{T}_1, \mathcal{T}_2$ are closed under subobjects, then clearly $\mathcal{T}_3$ is as well, so $\mathbf{t}_3$ is hereditary. The claim on the support follows from the second equality of item (3.b) of Proposition 4.1.
(2) Recall that $\mathcal{H}$ is a Grothendieck category. Since both $\mathcal{F}_1$ and $\mathcal{F}_2$ are closed under injective envelopes, then so is $\mathcal{F}_4$: thus, $\mathbf{t}_4$ is hereditary. Now using items (3.b) and (3.e) of Proposition 4.1 one proves the claim about support. □

We are now able to classify supports in the heart $\mathcal{H}_{\mathbf{t}}$ of an HRS-tilt at a perfect torsion pair.

PROPOSITION 4.13. *Let R be a commutative noetherian ring, $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a perfect torsion pair in $\mathsf{Mod}(R)$ with $\mathsf{supp}(\mathcal{T}) = V$, and $\mathcal{H}_{\mathbf{t}}$ the associated heart by HRS-tilt. Then the $\mathcal{H}_{\mathbf{t}}$-supports are all the sets of the form $(W \cap V) \cup (W' \cap V^{\mathsf{c}})$, for $W, W' \subseteq \mathsf{Spec}(R)$ specialisation closed.*

Proof. We have already noted that $\mathcal{H}_{\mathbf{t}}$ is the heart of an intermediate compactly generated t-structure, so it follows from Theorem 4.5(1.a) that specialisation-closed subsets are $\mathcal{H}_{\mathbf{t}}$-supports. Since both V and V^{c} are $\mathcal{H}_{\mathbf{t}}$-supports (see Lemma 4.11), it follows from Lemma 4.12 that the subsets presented in the statement are indeed $\mathcal{H}_{\mathbf{t}}$-supports. To prove the converse, let U be a $\mathcal{H}_{\mathbf{t}}$-support and let $\mathbf{t}' = (\mathcal{T}', \mathcal{F}')$ be the hereditary torsion pair in $\mathcal{H}_{\mathbf{t}}$ with $\mathsf{supp}(\mathcal{T}') = U$. We first show that

1. If $U \subseteq V$, then U is specialisation-closed;

2. If $U \subseteq V^{\mathsf{c}}$, then $U = \vee(U) \cap V^{\mathsf{c}}$.

(1) If $U \subseteq V$, then we have that $\mathcal{T}' \subseteq \mathcal{T} \subseteq \mathsf{Mod}(R)$. We show that $\mathcal{T}'$ is a hereditary torsion class in $\mathsf{Mod}(R)$ as well, and so U is specialisation-closed. Clearly $\mathcal{T}'$ is closed under extensions and coproducts in $\mathsf{D}(R)$, and thus it is so in $\mathsf{Mod}(R)$ as well. Let now

$$(*)\colon \qquad 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

be a short exact sequence in $\mathsf{Mod}(R)$, with Y in $\mathcal{T}' \subseteq \mathcal{T}$. Since $\mathcal{T}$ is a hereditary torsion class in $\mathsf{Mod}(R)$, both X and Z belong to $\mathcal{T} \subseteq \mathcal{H}_{\mathbf{t}}$, so that $(*)$ is a short exact sequence in $\mathcal{H}_{\mathbf{t}}$ as well. Now since $\mathcal{T}'$ is a hereditary torsion class in $\mathcal{H}_{\mathbf{t}}$ we conclude that X and Z belong to $\mathcal{T}'$, as wanted.
(2) The non-trivial inclusion is $U \supseteq \vee(U) \cap V^{\mathsf{c}}$. Let $\mathcal{C}$ denote the Giraud subcategory associated to $\mathbf{t}$, i.e. $\mathcal{C} = \mathcal{T}^{\perp_{0,1}}$. Given $\mathfrak{p}$ in $U \subseteq V^{\mathsf{c}}$ and $\mathfrak{q}$ in V^{c} such that $\mathfrak{p} \subseteq \mathfrak{q}$, we will show that $\mathfrak{q}$ lies in U as well. Translating this in terms of objects of $\mathcal{H}_{\mathbf{t}}$, consider the stalk complexes $E(R/\mathfrak{p})[1]$ and $E(R/\mathfrak{q})[1]$ in $\mathcal{C}[1] \subseteq \mathcal{H}_{\mathbf{t}}$. By assumption we have $E(R/\mathfrak{p})[1]$ lies in $\mathcal{T}'$, and we want to prove that $E(R/\mathfrak{q})[1]$ lies in $\mathcal{T}'$ as well. Denote by $(**)$ the torsion sequence of $E(R/\mathfrak{q})[1]$ with respect to $\mathbf{t}'$. Since, by Lemma 4.11, $\mathcal{C}[1]$ is a hereditary torsion class in $\mathcal{H}_{\mathbf{t}}$, we deduce that $(**)$ has all its terms in $\mathcal{C}[1]$. Therefore, applying a shift to it, we obtain the exact sequence of modules (in solid arrows) with T in $\mathcal{T}'[-1]$ and F in $\mathcal{F}'[-1]$.

$$(**)[-1]\colon \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & E(R/\mathfrak{q}) & \longrightarrow & F & \longrightarrow & 0 \\ & & & \searrow & \uparrow\downarrow & & & & \\ & & & & E(T) & & & & \end{array}$$

In $\mathsf{Mod}(R)$, consider then the injective envelope $E(T)$ of T: by injectivity, we get the two dotted vertical arrows in the diagram above. Moreover, since the morphism $T \to E(T)$ is left minimal, we conclude that $E(T)$ is a direct summand of the indecomposable module $E(R/\mathfrak{q})$. Now, we use our hypothesis that $\mathfrak{p} \subseteq \mathfrak{q}$ to notice that there is a nonzero morphism $E(R/\mathfrak{p})[1] \to E(R/\mathfrak{q})[1]$. Since the source of this morphism is in $\mathcal{T}'$, the target cannot be in $\mathcal{F}'$, and therefore $T \neq 0$. Then, we must have an isomorphism $0 \neq E(T) \simeq E(R/\mathfrak{q})$, which means that

$$\{\mathfrak{q}\} = \mathsf{supp}(E(R/\mathfrak{q})) = \mathsf{supp}(E(T)) \subseteq \mathsf{supp}(T) = \mathsf{supp}(T[1]) \subseteq U.$$

Returning to the general case of an arbitrary $\mathcal{H}_{\mathbf{t}}$-support U, note that by Lemma 4.12 and Lemma 4.11, both $U \cap V$ and $U \cap V^{\mathsf{c}}$ are $\mathcal{H}_{\mathbf{t}}$-supports. Set $W := U \cap V$ and $W' := \bigvee(U \cap V^{\mathsf{c}})$: the first is specialisation-closed by item (1) above, while the second is specialisation-closed by definition. Now by item (2) above it follows that $U = (U \cap V) \cup (U \cap V^{\mathsf{c}}) = (W \cap V) \cup (W' \cap V^{\mathsf{c}})$. ☐

Remark 4.14. Item (3) of Theorem 3.4 states that in $\mathsf{Mod}(R)$ hereditary torsion pairs coincide with those of finite type. We will see in the next section (see Corollary 5.10) that the hearts $\mathcal{H}_{\mathbf{t}}$ considered in this subsection are all derived equivalent to $\mathsf{Mod}(R)$. Then by Corollary 4.6 (since every torsion pair $\mathbf{t}$ in $\mathsf{Mod}(R)$ is restrictable and, thus, so is the t-structure $\mathbb{T}_{\mathbf{t}}$, by Theorem 2.16), we conclude that in $\mathcal{H}_{\mathbf{t}}$ hereditary torsion pairs of finite type correspond bijectively to specialisation closed subsets of $\mathsf{Spec}(R)$. Proposition 4.13 then shows that if $\mathbf{t}$ is perfect then, in general, not every hereditary torsion pair is of finite type (since not all $\mathcal{H}_{\mathbf{t}}$-supports are specialisation-closed).

We conclude this subsection with an illustrating example.

Example 4.15. Let R be a commutative noetherian ring of Krull dimension 1. In this case, every hereditary torsion pair is perfect (see [26, Corollary 4.3] and [4, Corollary 4.10]). Let V denote the set of maximal ideals of R. It is, of course, a specialisation-closed subset of $\mathsf{Spec}(R)$; denote by $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ the associated hereditary torsion pair in $\mathsf{Mod}(R)$. Let $\mathcal{H}_{\mathbf{t}} := \mathcal{F}[1] * \mathcal{T}$ be the heart of the HRS-tilt of the standard t-structure with respect to $\mathbf{t}$. Following Proposition 4.13, the $\mathcal{H}_{\mathbf{t}}$-supports are the sets of primes of the form $(W \cap V) \cup (W' \cap V^{\mathsf{c}})$, for specialisation-closed subsets W and W' of $\mathsf{Spec}(R)$. However, it is quite easy to see that, since R has Krull dimension 1, any subset of $\mathsf{Spec}(R)$ is of this form. Interestingly, this means that in this case hereditary torsion pairs in $\mathcal{H}_{\mathbf{t}}$ are in bijection with localising subcategories of $\mathsf{D}(R)$ (not only the smashing ones, as it happens with hereditary torsion pairs in $\mathsf{Mod}(R)$). Concretely, following items (3.a) and (3.b) of Theorem 4.1, the bijection is

$$\begin{array}{rcl} \{\textit{hereditary torsion pairs in } \mathcal{H}_{\mathbf{t}}\} & \overset{1:1}{\longleftrightarrow} & \{\textit{localising subcategories of } \mathsf{D}(R)\} \\ (\mathcal{T}, \mathcal{F}) & \longmapsto & \{X \in \mathsf{D}(R) \colon H^0_{\mathbf{t}}(X[i]) \in \mathcal{T}\ \forall i \in \mathbb{Z}\} \\ \mathcal{H}_{\mathbf{t}} \cap \mathcal{L} & \longleftarrow & \mathcal{L} \end{array}$$

where $H^0_{\mathbf{t}}\colon \mathsf{D}(R) \to \mathcal{H}_{\mathbf{t}}$ is the cohomology functor. In particular, all localising subcategories of $\mathsf{D}(R)$ admit a cohomological description with respect to $\mathcal{H}_{\mathbf{t}}$. We will later prove that $\mathcal{H}_{\mathbf{t}}$ is derived equivalent to $\mathsf{Mod}(R)$ (as a consequence of Corollary 5.11). This means that we get different insights on the triangulated structure of this derived category, depending on the abelian category that we start with.

5 Torsion pairs inducing derived equivalences

For this section we leave the commutative noetherian setting to explore the following general notion for an abelian category $\mathcal{A}$ that admits a derived category $\mathsf{D}(\mathcal{A})$.

Definition 5.1. We say that a torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ *induces a (bounded) derived equivalence* if the associated HRS-tilt in $\mathsf{D}(\mathcal{A})$ induces a (bounded) derived equivalence (as in Definition 2.18).

In order to understand when a torsion pair induces (bounded) derived equivalence, we will make use of a criterion by [12] (see §5.2). Before doing that, we will recall some homological tools.

5.1 Zero Yoneda extensions

Fix two objects X and Y in an abelian category $\mathcal{A}$. Recall that the elements of the Yoneda group $\mathsf{Ext}^n_{\mathcal{A}}(X,Y)$ (i.e. *Yoneda n-extensions*) are exact sequences of the form

$$\varepsilon\colon \qquad 0 \to Y \to Z_1 \to \cdots \to Z_n \to X \to 0,$$

which we call *n-sequences*, up to the equivalence relation generated by pairs of exact sequences $(\varepsilon, \varepsilon')$ such that there is a morphism

$$\begin{array}{ccccccccccccc}
\varepsilon\colon & & 0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & \cdots & \longrightarrow & Z_n & \longrightarrow & X & \longrightarrow & 0 \\
\downarrow & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
\varepsilon'\colon & & 0 & \longrightarrow & Y & \longrightarrow & Z_1' & \longrightarrow & \cdots & \longrightarrow & Z_n' & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

making the diagram commute. This is called a *morphism of n-sequences.* In other words, two n-sequences are equivalent if and only if there exists a zigzag of morphisms of n-sequences linking them. In particular, if $n = 1$, two 1-sequences are equivalent if and only if they are isomorphic. For $n > 1$, on the other hand, n-sequences are clearly in bijection with complexes

$$Z := 0 \to Z_1 \to \cdots \to Z_n \to 0$$

with Z_n in degree zero and which have $H^{-n+1}(Z) = Y$, $H^0(Z) = X$, and are acyclic in all other degrees. It is then easy to observe that a morphism of n-sequences translates to a quasi-isomorphism of the associated complexes, and viceversa. Since isomorphisms in the derived category $\mathsf{D}(\mathcal{A})$ are zigzags of quasi-isomorphisms of complexes, it follows that two n-sequences are equivalent (and therefore, they represent the same Yoneda n-extension) if and only if the associated complexes are isomorphic in $\mathsf{D}(\mathcal{A})$.

Lemma 5.2. *Let X and Y be objects of an abelian category $\mathcal{A}$. Let $n > 1$ and consider an exact sequence*

$$\varepsilon\colon \qquad 0 \to Y \to Z_1 \to Z_2 \to \cdots \to Z_n \to X \to 0.$$

Let Z be the associated complex as above. Then ε represents the zero element in $\mathsf{Ext}^n_{\mathcal{A}}(X,Y)$ if and only if the truncation triangle in $\mathsf{D}(\mathcal{A})$

$$\tau^{\leq -1}Z \longrightarrow Z \longrightarrow \tau^{\geq 0}Z \longrightarrow (\tau^{\leq -1}Z)[1]$$

where $\tau^{\leq -1}$ and $\tau^{\geq 0}$ are the truncation functors for the standard t-structure in $\mathsf{D}(\mathcal{A})$, is a split triangle.

Proof. The zero element of $\mathsf{Ext}^n_{\mathcal{A}}(X,Y)$ is the equivalence class of the n-sequence

$$0 \longrightarrow Y \xrightarrow{1_Y} Y \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow X \xrightarrow{1_X} X \longrightarrow 0$$

and its associated complex is, clearly, the direct sum $X \oplus Y[n-1]$. By the considerations above, ε represents the zero Yoneda n-extension if and only if the associated complex Z is isomorphic to $X \oplus Y[n-1]$. This occurs if and only if, for any degree $-n+1 \leq i < 0$, the truncation triangle

$$\begin{array}{ccccccc} \tau^{\leq i}Z & \longrightarrow & Z & \longrightarrow & \tau^{\geq i+1}Z & \xrightarrow{w} & (\tau^{\leq i}Z)[1] \\ \downarrow{\simeq} & & & & \downarrow{\simeq} & & \downarrow{\simeq} \\ Y[n-1] & & & & X & & Y[n] \end{array}$$

splits or, equivalently, if and only if w vanishes. As a confirmation, note that w is the morphism corresponding to the equivalence class of ε under the isomorphism $\mathsf{Ext}^n_{\mathcal{A}}(X,Y) \simeq \mathsf{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y[n])$. ☐

5.2 CHZ-sequences

We recall the main result of [12], using the point of view expressed above.

Proposition 5.3 ([12, Theorem A]). *Let $\mathcal{A}$ be an abelian category admitting a derived category $\mathsf{D}(\mathcal{A})$, and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. The following are equivalent:*

1. $\mathbf{t}$ *induces a bounded derived equivalence;*

2. *Every object M of $\mathcal{A}$ admits an exact sequence of the form*

$$\varepsilon_M\colon \qquad 0 \to F^0_M \to F^1_M \to M \to T^0_M \to T^1_M \to 0,$$

with T^0_M, T^1_M in $\mathcal{T}$ and F^0_M, F^1_M in $\mathcal{F}$, which represents the zero element of $\mathsf{Ext}^3_{\mathcal{A}}(T^1_M, F^0_M)$;

3. *For every object M of $\mathcal{A}$, there is a complex $Z_M := F^1_M \to M \to T^0_M$ with $T^0_M \in \mathcal{T}$ in degree zero and $F^1_M \in \mathcal{F}$, such that the truncation triangle*

$$\Delta_M\colon \qquad \tau^{\leq -1} Z_M \longrightarrow Z_M \longrightarrow \tau^{\geq 0} Z_M \longrightarrow (\tau^{\leq -1} Z_M)[1]$$

splits, i.e. that $Z_M \simeq \tau^{\leq -1} Z_M \oplus \tau^{\geq 0} Z_M$.

Sequences as in (2) and complexes as in (3) will be called CHZ-sequences and CHZ-complexes.

Proof. The proof that (1) is equivalent to (2) is essentially the cited result of [12]. The proof that (2) and (3) are equivalent is an immediate consequence of Lemma 5.2 and of the fact that $\mathcal{F}$ is closed under subobjects and $\mathcal{T}$ is closed under quotient objects. □

Note that, in some cases, we can avoid checking the vanishing of the Yoneda class of a candidate CHZ-sequence. For example, if the abelian category $\mathcal{A}$ has global dimension less or equal than two, the Yoneda bifunctor $\mathsf{Ext}^3_{\mathcal{A}}(-,-)$ is identically zero, and thus the Yoneda-vanishing condition is automatic. Similar phenomena happen for certain torsion pairs. For example, a complete analysis for cohereditary torsion pairs in Grothendieck categories can be deduced from [12, Corollary 4.1]. We will later focus on the class of hereditary torsion pairs. Firstly, however, let us observe that the criterion given by the proposition above can be simplified for Grothendieck abelian categories.

Proposition 5.4. *Let $\mathcal{G}$ be a Grothendieck abelian category with a generator G and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{G}$. Then $\mathbf{t}$ induces a bounded derived equivalence if and only if there is an exact sequence*

$$0 \longrightarrow F^0_G \xrightarrow{a} F^1_G \xrightarrow{b} G \xrightarrow{c} T^0_G \xrightarrow{d} T^1_G \longrightarrow 0$$

with T^0_G, T^1_G in $\mathcal{T}$ and F^0_G, F^1_G in $\mathcal{F}$, which represents the zero element of $\mathsf{Ext}^3_R(T^1_G, F^0_G)$.

Proof. As in Proposition 5.3, we translate the existence of such a sequence to the existence of a complex

$$Z_G := \qquad F^1_G \xrightarrow{b} G \xrightarrow{c} T^0_G$$

with T^0_G in degree zero for which the truncation triangle (Δ_G) splits. First observe that since coproducts of split triangles are again split and since both $\mathcal{T}$

and $\mathcal{F}$ are closed under coproducts, the existence of a CHZ-complex is also guaranteed for any coproduct $M = G^{(I)}$, for any set I. In order to show that such complexes exist for any M in the category $\mathcal{G}$, we may, therefore, suppose, without loss of generality, that M is a quotient of G itself (otherwise, we could replace G by $G^{(I)}$ for a suitable set I).
Let then $f\colon G \to M$ be an epimorphism. Denote by $\overline{f}\colon \operatorname{Coker}(b) \to \operatorname{Coker}(fb)$ the epimorphism induced by f. We define the following complex concentrated in degrees $-2, -1$ and 0,

$$Z_M := \qquad F_G^1 \xrightarrow{fb} M \xrightarrow{\overline{c}} T_G^0/K$$

where $K = \operatorname{Ker}(\overline{f})$ and $\overline{c}$ is the composition of the projection $M \to \operatorname{Coker}(fb)$ and the inclusion of $\operatorname{Coker}(fb) = \operatorname{Coker}(b)/K$ into T_G^0/K. Note that, by construction, $\operatorname{Ker}(\overline{c}) = \operatorname{Im}(fb)$ and, thus, we have $H^{-1}(Z_M) = 0$. Moreover, note that $\operatorname{Ker}(fb)$ is a subobject of F_G^1 and that

$$\operatorname{Coker}(\overline{c}) = (T_G^0/K)/(\operatorname{Coker}(b)/K) \simeq T_G^0/\operatorname{Coker}(b) \simeq T_G^1. \qquad (*)$$

This shows that $H^{-2}(Z_M)$ is torsionfree and $H^0(Z_M)$ is torsion. Finally, consider the map of complexes

$$\begin{array}{cccccc} Z_G\colon & F_G^1 & \xrightarrow{b} & G & \xrightarrow{c} & T_G^0 \\ {\scriptstyle\varphi}\downarrow & \Vert & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle\pi} \\ Z_M\colon & F_G^1 & \xrightarrow{fb} & M & \xrightarrow{\overline{c}} & T_G^0/K \end{array}$$

where π is the canonical projection. If we now consider the truncation triangles of Z_G and Z_M, as in Proposition 5.3, we get a morphism of triangles in $\mathsf{D}(\mathcal{G})$ as follows:

$$\begin{array}{ccccccc} F_G^0[2] & \xrightarrow{a} & Z_G & \xrightarrow{d} & T_G^1 & \longrightarrow & F_G^0[3] \\ {\scriptstyle \tau^{\leq -2}\varphi = H^{-2}(\varphi)[2]}\downarrow & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle \tau^{\geq -1}\varphi = H^0(\varphi)}\downarrow & & \downarrow \\ \operatorname{Ker}(fb)[2] & \xrightarrow{\overline{a}} & Z_M & \xrightarrow{\overline{d}} & T_G^1 & \longrightarrow & \operatorname{Ker}(fb)[3] \end{array}$$

Finally, since d is a split epimorphism and $H^0(\varphi)$ is an isomorphism (as seen in $(*)$), we conclude that also $\overline{d}$ is a split epimorphism, finishing our proof. □

5.3 Hereditary torsion pairs inducing bounded derived equivalences

We now focus on hereditary torsion pairs. The following lemma provides a simplification of the criterion given in Proposition 5.3 for such torsion pairs.

LEMMA 5.5. *Let* $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ *be a hereditary torsion pair in an abelian category* $\mathcal{A}$. *If* $\mathbf{t}$ *is hereditary, then a CHZ-sequence for an object* M *of* $\mathcal{A}$ *exists if and only if there exists a sequence of the form*

$$F_M \longrightarrow M \longrightarrow T_M \longrightarrow 0,$$

with T_M *in* $\mathcal{T}$ *and* F_M *in* $\mathcal{F}$. *Such sequences will be called short CHZ-sequences.*

Proof. If $\mathbf{t}$ is hereditary, any CHZ-sequence

$$0 \longrightarrow F_M^0 \longrightarrow F_M^1 \longrightarrow M \xrightarrow{b} T_M^0 \longrightarrow T_M^1 \longrightarrow 0,$$

with T_M^0, T_M^1 in $\mathcal{T}$ and F_M^0, F_M^1 in $\mathcal{F}$, gives rise to a short CHZ-sequence

$$F_M^1 \longrightarrow M \longrightarrow \mathrm{Im}(b) \longrightarrow 0,$$

where $\mathrm{Im}(b) \leq T_M^0$ lies in $\mathcal{T}$ because $\mathcal{T}$ is closed under subobjects. Conversely, since $\mathcal{F}$ is closed under subobjects, a short CHZ-sequence can be completed with a torsion-free kernel on the left. Note that the class of the obtained exact sequence as a 3-extension is zero because such a sequence is naturally a 2-extension; therefore, it is a CHZ-sequence. $\square$

Given two objects X and M in a Grothendieck abelian category $\mathcal{G}$, define the *trace of* M *in* X to be

$$\mathsf{tr}_M(X) := \sum_{f \in \mathsf{Hom}_{\mathcal{A}}(M,X)} \mathrm{Im}(f).$$

Notice that the trace of M in X is the biggest subobject of X which is generated by M, in the sense that it is a quotient of a coproduct of copies of M.

THEOREM 5.6. *Let* $\mathcal{G}$ *be a Grothendieck abelian category with generator* G, *and let* $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ *be a hereditary torsion pair in* $\mathcal{G}$ *with torsion radical* $t\colon \mathcal{G} \to \mathcal{T}$. *Then* $\mathbf{t}$ *induces a bounded derived equivalence if and only if* $G/\mathsf{tr}_{G/t(G)}(G)$ *lies in* $\mathcal{T}$.

Proof. If $G/\mathsf{tr}_{G/t(G)}(G)$ lies in $\mathcal{T}$, let $I = \mathsf{Hom}_{\mathcal{G}}(G/t(G), G)$ and $f\colon (G/t(G))^{(I)} \to G$ be the canonical morphism, whose image is precisely $\mathsf{tr}_{G/t(G)}(G)$. Then the sequence

$$(G/t(G))^{(I)} \xrightarrow{\ f\ } G \longrightarrow G/\,\mathsf{tr}_{G/t(G)}(G) \longrightarrow 0$$

is a short CHZ-sequence for G and, thus, by Lemma 5.5 and Lemma 5.4, we have that $\mathbf{t}$ induces a bounded derived equivalence. Conversely, if $\mathbf{t}$ induces a bounded derived equivalence, then again by Lemma 5.4 and Lemma 5.5 we have that there is an exact sequence of the form

$$F_G \xrightarrow{\ h\ } G \longrightarrow T_G \longrightarrow 0$$

with F_G in $\mathcal{F}$ and T_G in $\mathcal{T}$. Now, since by Lemma 2.8 we know that F_G is a quotient of a coproduct of copies of $G/t(G)$, it follows that $\mathrm{Im}(h)$ is contained in $\mathsf{tr}_{G/t(G)}(G)$. Thus, $G/\mathsf{tr}_{G/t(G)}(G)$ is a quotient of T_G and therefore it lies in $\mathcal{T}$, as wanted. □

Corollary 5.7. *Let $\mathcal{G}$ be a Grothendieck category and $\mathbf{t}_1 := (\mathcal{T}_1, \mathcal{F}_1)$, $\mathbf{t}_2 := (\mathcal{T}_2, \mathcal{F}_2)$ be hereditary torsion pairs in $\mathcal{G}$ inducing bounded derived equivalence. Then their meet induces a bounded derived equivalence.*

Proof. Let $\mathbf{t}_3 := (\mathcal{T}_3 := \mathcal{T}_1 \cap \mathcal{T}_2, \mathcal{F}_3)$ be the meet of $\mathbf{t}_1$ and $\mathbf{t}_2$. It is again hereditary, so we can apply Theorem 5.6. Let G be a generator of $\mathcal{G}$, and $t_i(G)$ the torsion part of G with respect to $\mathbf{t}_i$, for $i = 1, 2, 3$.
Since $t_3(G)$ lies in $\mathcal{T}_3 \subseteq \mathcal{T}_1$, it is a subobject of $t_1(G)$, so that $G/t_3(G) \twoheadrightarrow G/t_1(G)$. It follows that $\mathsf{tr}_{G/t_1(G)}(G) \hookrightarrow \mathsf{tr}_{G/t_3(G)}(G)$, and $G/\mathsf{tr}_{G/t_1(G)}(G) \twoheadrightarrow G/\mathsf{tr}_{G/t_3(G)}(G)$. Now the source of this epimorphism is in $\mathcal{T}_1$, so the target is as well; a similar argument for $t_2(G)$ and $t_3(G)$ shows then that $G/\mathsf{tr}_{G/t_3(G)}(G)$ belongs to $\mathcal{T}_1 \cap \mathcal{T}_2 = \mathcal{T}_3$. □

Let us now restrict our setting to the Grothendieck abelian category $\mathsf{Mod}(R)$ of right R-modules over a ring R. Recall that given a right R-module M, its *(right) annihilator* is the ideal of R

$$\mathsf{Ann}(M_R) := \{r \in R : Mr = 0\}.$$

Similarly, we define the annihilator of a left module ${}_RM$. Notice, however, that the annihilator of a module, be it right or left, is always a two-sided ideal.

Proposition 5.8. *Let R be a ring and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathsf{Mod}(R)$. Let $K_\mathbf{t}$ be the two-sided ideal obtained as the intersection of all the annihilators of modules in $\mathcal{F}$. The following statements hold:*

1. *$R/K_\mathbf{t}$ lies in $\mathcal{F}$;*
2. *$\mathcal{F}$ is a subcategory of $\mathsf{Mod}(R/K_\mathbf{t})$, where $\mathsf{Mod}(R/K_\mathbf{t})$ is identified with the subcategory of R-modules annihilated by $K_\mathbf{t}$;*
3. *$t(R)$ coincides with $K_\mathbf{t}$.*

Proof. (1) For every element r in $R \setminus K_\mathbf{t}$, there is an object F_r in $\mathcal{F}$ and an element f_r in F_r such that $f_r r \neq 0$. Consider the torsionfree module $F := \prod_{r \in R\setminus K} F_r$, and the morphism $R \to F$ defined by $1 \mapsto (f_r)_r$. Its kernel is clearly $K_\mathbf{t}$, so it induces an embedding (of right R-modules) $R/K_\mathbf{t} \hookrightarrow F$. Hence $R/K_\mathbf{t}$ lies in $\mathcal{F}$.
(2) The claim follows from the fact that $K_\mathbf{t}$ annihilates every module in $\mathcal{F}$ by definition.

(3) Consider the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & t(R) & \longrightarrow & R & \xrightarrow{p_1} & R/t(R) & \longrightarrow & 0 \\
 & & & & \Big\| & & c \big\downarrow \big\uparrow d & & \\
0 & \longrightarrow & K_{\mathbf{t}} & \longrightarrow & R & \xrightarrow{p_2} & R/K_{\mathbf{t}} & \longrightarrow & 0
\end{array}$$

Since $R/t(R)$ lies in $\mathcal{F}$, which is contained in $\mathsf{Mod}(R/K_{\mathbf{t}})$ by (2), the map p_1 factors through $R/K_{\mathbf{t}}$, yielding the dotted epimorphism d. On the other hand, $R/K_{\mathbf{t}}$ is torsionfree (by (1)), and so the map p_2 factors through $R/t(R)$, giving the dotted epimorphism c. Finally it is an easy verification that c and d are inverse to each other, showing that indeed $K_{\mathbf{t}} = t(R)$ as ideals of R. □

REMARK 5.9. Combining items (2) and (3) of the previous proposition one recovers Lemma 2.8.

We can finally apply Theorem 5.6 to the case of a module category.

THEOREM 5.10. *Let R be a ring and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair in $\mathsf{Mod}(R)$. Then $\mathbf{t}$ induces a bounded derived equivalence if and only if $R/\mathsf{Ann}({}_Rt(R))$ lies in $\mathcal{T}$. This holds in particular whenever ${}_Rt(R)$ is finitely generated (as a left R-module) and $\mathsf{Ann}({}_Rt(R)) = \mathsf{Ann}(t(R)_R)$.*

Proof. The first assertion follows immediately from Theorem 5.6 once we observe that $\mathsf{tr}_{R/t(R)}(R) = \mathsf{Ann}({}_Rt(R))$. Indeed, by definition an element r of R lies in $\mathsf{tr}_{R/t(R)}(R)$ if and only if it lies in the image of a morphism $f\colon R/t(R) \to R$: and this happens if and only if $rt(R) = 0$, i.e. if r lies in $\mathsf{Ann}({}_Rt(R))$.

For the second statement, let $x_1, ..., x_n$ be elements in R such that ${}_Rt(R) = \sum_{k=1}^n Rx_k$, and consider the morphism of right R-modules $\alpha\colon R \to t(R)^{\oplus n}$ defined by $1 \mapsto (x_1, \dots, x_n)$. Then its kernel is $\mathsf{Ann}(t(R)_R)$. Indeed, if r is in $\mathsf{Ann}(t(R)_R)$ then $x_k r = 0$ for all $1 \leq k \leq n$, and so $\alpha(r) = (x_1 r, \dots, x_n r) = 0$. Conversely, if $r \in \ker \alpha$, then $x_k r = 0$ for all $1 \leq k \leq n$. Now every x in $t(R)$ can be written as $\sum_{k=1}^n r_k x_k$, so $xr = \sum_{k=1}^n (r_k x_k) r = 0$. By our hypothesis, then, we also have $\ker \alpha = \mathsf{Ann}({}_Rt(R))$. Therefore, α induces a monomorphism of right R-modules

$$R/\mathsf{Ann}({}_Rt(R)) = R/\mathsf{Ann}(t(R)_R) \hookrightarrow t(R)^{\oplus n}.$$

Since the latter is a torsion module and $\mathbf{t}$ is hereditary, we conclude that $R/\mathsf{Ann}({}_Rt(R))$ is torsion. □

COROLLARY 5.11. *Let R be a noetherian ring and suppose it is either commutative or semiprime. Then every hereditary torsion pair induces an unbounded derived equivalence.*

Proof. The fact that a hereditary torsion pair $\mathbf{t}$ of $\mathsf{Mod}(R)$ induces bounded derived equivalence follows immediately from Theorem 5.10; indeed, for commutative or semiprime rings, left and right annihilators of ideals coincide and, furthermore, since R is left noetherian, ${}_Rt(R)$ is finitely generated.
Now, the t-structure obtained by HRS-tilting $\mathsf{Mod}(R)$ at $\mathbf{t}$ is smashing with Grothendieck heart by Remark 2.7 and item (1) of Theorem 2.16; therefore by applying item (2) of Proposition 2.17 we get that $\mathbf{t}$ induces an unbounded derived equivalence. □

The corollary above has a direct implication in silting theory for commutative noetherian rings.

Corollary 5.12. *Every two-term cosilting complex over a commutative noetherian ring is cotilting.*

Proof. If R is a commutative noetherian ring, then the HRS-tilting t-structure at any hereditary torsion pair in $\mathsf{Mod}(R)$ is a cosilting t-structure associated with a two-term cosilting complex ([2, Corollary 4.1, Lemma 4.2]). Since, by Corollary 5.11, this t-structure induces a derived equivalence, the two-term cosilting complex must be cotilting ([39, Corollary 5.2]). □

6 t-structures inducing derived equivalences for commutative noetherian rings

We turn now our attention back to compactly generated t-structures in the derived category of a commutative noetherian ring R. In particular, we aim to find sufficient conditions for a given intermediate compactly generated t-structure to induce a derived equivalence.

Remark 6.1. Recall that by Theorem 2.12, intermediate compactly generated t-structures in $\mathsf{D}(R)$ satisfy the hypotheses of item (2) of Proposition 2.17. Therefore, their heart is derived equivalent to $\mathsf{Mod}(R)$ if and only if it is bounded derived equivalent. This includes the case of the HRS-tilt of an intermediate compactly generated t-structure at a torsion pair of finite type (see item (1) of Theorem 2.16 and Theorem 3.10). In the following we will use this fact without an explicit mention.

6.1 Sufficient conditions for derived equivalence

Let R be a commutative noetherian ring, and consider a hereditary torsion pair $\mathbf{t}$ in $\mathsf{Mod}(R)$. If $\mathbb{T}_\mathbf{t}$ denotes the t-structure obtained by HRS-tilting $\mathsf{Mod}(R)$ at $\mathbf{t}$, with heart $\mathcal{H}_\mathbf{t}$, we know that:

- $\mathbb{T}_\mathbf{t}$ is an intermediate compactly generated t-structure (Remark 4.8);
- $\mathcal{H}_\mathbf{t}$ is a locally coherent Grothendieck category (Theorem 2.16, Remark 2.7 and item (3) of Theorem 3.4);

- $\mathcal{H}_{\mathbf{t}}$ is derived equivalent to $\mathsf{Mod}(R)$ (Corollary 5.11).

This subsection takles the question of whether we can proceed with a chain of HRS-tilts at suitable torsion pairs, so that all the obtained t-structures will retain these properties. For this purpose, we need to first extend the scope of Theorem 5.6 in order to apply it not only to HRS-tilts of the heart of the standard t-structure but also to hearts of t-structures inducing derived equivalences.

Lemma 6.2. *Let R be a ring and $\mathcal{A}$ the heart of an intermediate t-structure $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ in $\mathsf{D}(R)$ such that $\mathbb{T}$ induces a derived equivalence. Given a torsion pair $\mathbf{t}$ in $\mathcal{A}$, let $\mathbb{S}$ be the t-structure in $D(\mathcal{A})$ produced by HRS-tilting the standard t-structure in $D(\mathcal{A})$ with respect to $\mathbf{t}$, and let $\mathbb{T}_{\mathbf{t}}$ be the t-structure in $\mathsf{D}(R)$ obtained by HRS-tilting the t-structure $\mathbb{T}$ in $\mathsf{D}(R)$. Then $\mathbb{S}$ induces a derived equivalence if and only if so does $\mathbb{T}_{\mathbf{t}}$.*

Proof. Let $\rho_1\colon \mathsf{D}^b(\mathcal{A}) \longrightarrow \mathsf{D}^b(R)$ be a realisation functor for the t-structure $\mathbb{T}$ (which is assumed to be a triangle equivalence). It can easily be checked, using elementary properties of realisation functors, that the t-structure $\mathbb{S} \cap \mathsf{D}^b(\mathcal{A})$ in $\mathsf{D}^b(\mathcal{A})$ is sent, via the equivalence ρ_1, to the t-structure $\mathbb{T}_{\mathbf{t}} \cap \mathsf{D}^b(R)$ in $\mathsf{D}^b(R)$. In particular, ρ_1 induces an equivalence between $\mathcal{H}_{\mathbb{S}}$ and $\mathcal{A}_{\mathbf{t}}$, the heart of $\mathbb{T}_{\mathbf{t}}$ in $\mathsf{D}(R)$.
Suppose now that $\mathbb{S}$ induces a derived equivalence, i.e. the realisation functor $\rho_2\colon \mathsf{D}^b(\mathcal{H}_{\mathbb{S}}) \longrightarrow \mathsf{D}^b(\mathcal{A})$ is a triangle equivalence. Hence, it follows that for X and Y in $\mathcal{A}_{\mathbf{t}}$, with $X \simeq \rho_1 X'$ and $Y = \rho_1 Y'$ for X' and Y' in $\mathcal{H}_{\mathbb{S}}$, we have $\mathsf{Ext}^n_{\mathcal{A}_{\mathbf{t}}}(X,Y) \simeq \mathsf{Ext}^n_{\mathcal{H}_{\mathbb{S}}}(X',Y')$ and, since ρ_2 is a derived equivalence, the latter is isomorphic to $\mathsf{Hom}_{\mathsf{D}^b(\mathcal{A})}(X', Y'[n])$. Hence, we have

$$\begin{aligned}\mathsf{Ext}^n_{\mathcal{A}_{\mathbf{t}}}(X,Y) &\simeq \mathsf{Hom}_{\mathsf{D}^b(\mathcal{A})}(X', Y'[n]) \\ &\simeq \mathsf{Hom}_{\mathsf{D}^b(\mathcal{A})}(\rho_1 X', \rho_1 Y'[n]) \simeq \mathsf{Hom}_{\mathsf{D}^b(R)}(X, Y[n]),\end{aligned}$$

proving that $\mathbb{T}_{\mathbf{t}}$ induces a derived equivalence. The converse implication is completely analogous. □

As a consequence of the lemma above, we can use the criteria in Proposition 5.3 and, consequently, in Theorem 5.6 to test whether HRS-tilts of hearts of t-structures that induce derived equivalences still induce derived equivalences.

Theorem 6.3. *Let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be an intermediate compactly generated t-structure in $\mathsf{D}(R)$ with heart $\mathcal{H}_{\mathbb{T}}$, and suppose that $\mathbb{T}$ induces a derived equivalence. Let $\mathbf{t} = (\mathcal{T}_V, \mathcal{F}_V)$ be the hereditary torsion pair of finite type in $\mathcal{H}_{\mathbb{T}}$ associated to a specialisation closed subset V, and let $t\colon \mathcal{H}_{\mathbb{T}} \to \mathcal{T}$ denote the corresponding torsion radical. If there is a set of generators $\{G_\lambda : \lambda \in \Lambda\}$ of $\mathcal{H}_{\mathbb{T}}$ such that $G_\lambda/t(G_\lambda)$ is finitely presented, then the HRS-tilted t-structure associated to $\mathbf{t}$ induces a derived equivalence.*

Proof. We use an adaptation of Theorem 5.6 (see also Lemma 6.2). In fact, we will check that for each λ in Λ, the quotient $G_\lambda/\mathsf{tr}_{G_\lambda/t(G_\lambda)}(G_\lambda)$ is torsion;

this provides us with a sequence such as in Proposition 5.4 by considering the coproduct over the set Λ of the resulting sequences. This will show that, indeed, the HRS-tilted t-structure associated to $\mathbf{t}$ induces a derived equivalence (see Remark 6.1).

First observe that since V is specialisation closed and $\mathcal{H}_{\mathbb{T}}$ is contained in $D^b(R)$ ($\mathbb{T}$ is intermediate), we have from Equation (3.1) that $\mathsf{supp}^{-1}(V) \cap \mathcal{H}_{\mathbb{T}} = \mathsf{Supp}^{-1}(V) \cap \mathcal{H}_{\mathbb{T}}$ and, therefore, all we want to show is that $G_\lambda/\mathsf{tr}_{G_\lambda/t(G_\lambda)}(G_\lambda) \otimes_R R_\mathfrak{p} = 0$ for all primes $\mathfrak{p}$ that do not lie in V. Let then $\mathfrak{p}$ be a prime not in V. Since $R_\mathfrak{p}$ is flat, we can write it as a direct limit of free R-modules, i.e. $R_\mathfrak{p} = \varinjlim_{j\in J} R^{(n_j)}$. Since directed homotopy colimits in $\mathsf{D}(R)$ are computed as componentwise direct limits, one sees that

$$G_\lambda \otimes R_\mathfrak{p} = G_\lambda \otimes \varinjlim_J R^{(n_j)} = \mathrm{Hocolim}_J\, G_\lambda^{(n_j)}$$

(see the proof of [21, Lemma 4.1] for the details). Now, by [43, Corollary 5.8], this directed homotopy colimit of objects of $\mathcal{H}_{\mathbb{T}}$ is a direct limit in $\mathcal{H}_{\mathbb{T}}$. We can therefore use the hypothesis that $G_\lambda/t(G_\lambda)$ is finitely presented in $\mathcal{H}_{\mathbb{T}}$ to write

$$\begin{aligned}\mathsf{Hom}_{\mathsf{D}(R)}(G_\lambda/t(G_\lambda), G_\lambda) \otimes_R R_\mathfrak{p} &\simeq \varinjlim_J \mathsf{Hom}_{\mathsf{D}(R)}(G_\lambda/t(G_\lambda), G_\lambda)^{(n_j)} \\ &\simeq \mathsf{Hom}_{\mathsf{D}(R)}(G_\lambda/t(G_\lambda), G_\lambda \otimes_R R_\mathfrak{p}).\end{aligned}$$

Since $\mathsf{D}(R_\mathfrak{p})$ is a bireflective subcategory of $\mathsf{D}(R)$ with reflection functor $- \otimes_R R_\mathfrak{p}$, it follows that

$$\mathsf{Hom}_{\mathsf{D}(R)}(G_\lambda/t(G_\lambda), G_\lambda) \otimes_R R_\mathfrak{p} \simeq \mathsf{Hom}_{\mathsf{D}(R)}(G_\lambda/t(G_\lambda) \otimes_R R_\mathfrak{p}, G_\lambda \otimes_R R_\mathfrak{p}).$$

Now, observe that since $\mathfrak{p}$ does not lie in V and since $\mathcal{T}_V$ is supported in V, we have $t(G_\lambda) \otimes_R R_\mathfrak{p} = 0$. Since $- \otimes_R R_\mathfrak{p}$ is an exact functor in $\mathcal{H}_{\mathbb{T}}$, it then follows that $G_\lambda \otimes_R R_\mathfrak{p} \simeq G_\lambda/t(G_\lambda) \otimes_R R_\mathfrak{p}$. Therefore, the isomorphism of Hom-spaces in the last equation above allows us to conclude that there is a morphism $f\colon G_\lambda/t(G_\lambda) \to G_\lambda$ such that $f \otimes_R R_\mathfrak{p}$ is an isomorphism. Since $\mathsf{tr}_{G_\lambda/t(G_\lambda)}(G_\lambda) \otimes_R R_\mathfrak{p}$ contains $\mathrm{Im}(f) \otimes_R R_\mathfrak{p}$, we have that $\mathsf{tr}_{G_\lambda/t(G_\lambda)}(G_\lambda) \otimes_R R_\mathfrak{p} = G_\lambda \otimes_R R_\mathfrak{p}$. We thus conclude that, indeed, $G_\lambda/\,\mathsf{tr}_{G_\lambda/t(G_\lambda)}(G_\lambda) \otimes_R R_\mathfrak{p} = 0$, as wanted. ☐

Corollary 6.4. *Let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be an intermediate compactly generated t-structure in $\mathsf{D}(R)$ with a locally coherent heart $\mathcal{H}_{\mathbb{T}}$, and suppose that $\mathbb{T}$ induces a derived equivalence. Let $\mathbf{t} = (\mathcal{T}_V, \mathcal{F}_V)$ be the hereditary torsion pair of finite type in $\mathcal{H}_{\mathbb{T}}$ associated to a specialisation-closed V and suppose that $(\mathcal{T}_V, \mathcal{F}_V)$ is restrictable. Then the HRS-tilted t-structure associated to $\mathbf{t}$ induces a derived equivalence.*

Proof. Under the assumption that $\mathbf{t}$ is restrictable, for any set $\{G_\lambda \colon \lambda \in \Lambda\}$ of finitely presented generators of $\mathcal{H}_{\mathbb{T}}$, $G_\lambda/t(G_\lambda)$ will also be finitely presented. Hence, the result follows from Theorem 6.3. ☐

Remark 6.5. Since we know that for a commutative noetherian ring R, every hereditary torsion pair in $\mathsf{Mod}(R)$ is restrictable, note that Corollary 5.11 follows immediately from the corollary above.

Remark 6.6. If $\mathbb{T}$ is as in Corollary 6.4, $\mathbf{t}$ is any torsion pair in $\mathcal{H}_{\mathbb{T}}$ and we happen to know that $\mathcal{H}_{\mathbf{t}}$ is locally coherent, then the torsion pair $\mathbf{t}$ is restrictable if and only if there is a set $\{G_\lambda \colon \lambda \in \Lambda\}$ of finitely presented generators of $\mathcal{H}_{\mathbb{T}}$ such that $G_\lambda/t(G_\lambda)$ is finitely presented (see [37, Remark 6.3(3)]). Therefore, knowing this information about $\mathcal{H}_{\mathbf{t}}$, the hypothesis of Corollary 6.4 is minimal to apply Theorem 6.3.

6.2 Intermediate compactly generated t-structures via iterated HRS-tilts

In this subsection, we show that any intermediate compactly generated t-structure of $\mathsf{D}(R)$ is obtained from the standard t-structure by a sequence of HRS-tilts at hereditary torsion pairs of finite type. The following proposition gives us some TTF classes in hearts of compactly generated t-structure (compare with Remark 2.15).

Proposition 6.7. *Let φ be an intermediate sp-filtration, with $\varphi(0) \neq \varphi(1) = \emptyset$, and denote by $\mathcal{H}_\varphi$ the heart of the associated compactly generated t-structure $\mathbb{T}_\varphi = (\mathcal{U}_\varphi, \mathcal{V}_\varphi)$. Then $\mathcal{T}_0 := \mathcal{H}_\varphi \cap \mathsf{Mod}(R)$ is a TTF class in $\mathcal{H}_\varphi$. In particular, we have that $\mathcal{T}_0 = \mathsf{supp}^{-1}(\varphi(0)) \cap \mathcal{H}_\varphi = \mathsf{Supp}^{-1}(\varphi(0)) \cap \mathcal{H}_\varphi$.*

Proof. First note that

$$\mathcal{H}_\varphi \cap \mathbb{D}^{\geq 0} \overset{(1)}{=} \mathcal{T}_0 \overset{(2)}{=} \mathcal{U}_\varphi \cap \mathsf{Mod}(R) \overset{(3)}{=} \mathsf{Supp}^{-1}(\varphi(0)) \cap \mathsf{Mod}(R).$$

Indeed, equality (1) follows from $\mathcal{H}_\varphi \subseteq \mathcal{U}_\varphi \subseteq \mathbb{D}^{\leq 0}$, (2) follows from $\mathsf{Mod}(R) \subseteq \mathbb{D}^{\geq 0} \subseteq \mathcal{V}_\varphi[1]$, and (3) follows by definition of $\mathcal{U}_\varphi$. In particular, (3) shows that $\mathsf{supp}(\mathcal{T}_0) = \mathsf{Supp}(\mathcal{T}_0) = \varphi(0)$ by Theorem 3.4.
We now show that $(\mathcal{H}_\varphi \cap \mathbb{D}^{\leq -1}, \mathcal{T}_0) = (\mathcal{H}_\varphi \cap \mathbb{D}^{\leq -1}, \mathcal{H}_\varphi \cap \mathbb{D}^{\geq 0})$ is a torsion pair in $\mathcal{H}_\varphi$. First, $\mathsf{Hom}_{\mathcal{H}_\varphi}(\mathcal{H}_\varphi \cap \mathbb{D}^{\leq -1}, \mathcal{T}_0) = 0$ is clear. Now, let X be an object of $\mathcal{H}_\varphi$ and consider the truncation triangle with respect to the standard t-structure

$$\tau^{\leq -1}X \longrightarrow X \longrightarrow H^0(X) \longrightarrow (\tau^{\leq -1}X)[1].$$

By definition of $\mathcal{U}_\varphi$, it is closed under standard truncations, so all the vertices belong to $\mathcal{U}_\varphi$. Moreover, it is also clear that $H^0(X)$ lies in $\mathbb{D}^{\geq 0} \subseteq \mathcal{V}_\varphi[1]$, and therefore $H^0(X)$ lies in $\mathcal{H}_\varphi$. Lastly, since $\mathcal{V}_\varphi[1]$ is closed under taking co-cones, also $\tau^{\leq -1}X$ belongs to $\mathcal{V}_\varphi[1]$, and hence $\tau^{\leq -1}X$ lies also in $\mathcal{H}_\varphi$. The triangle above is then a short exact sequence in $\mathcal{H}_\varphi$ and it is the torsion decomposition of X with respect to the torsion pair $(\mathcal{H}_\varphi \cap \mathbb{D}^{\leq -1}, \mathcal{T}_0)$.
It remains to show that $\mathcal{T}_0$ is also a torsion class in $\mathcal{H}_\varphi$ or, equivalently, that $\mathcal{T}_0$ is closed under quotients in $\mathcal{H}_\varphi$. Consider a short exact sequence in $\mathcal{H}_\varphi$

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

where Y lies in $\mathcal{T}_0$. Since $\mathcal{T}_0$ is a torsionfree class in $\mathcal{H}_\varphi$, it follows that X is also in $\mathcal{T}_0$. Since $\mathsf{supp}(Z) \subseteq \mathsf{supp}(X[1]) \cup \mathsf{supp}(Y)$, it follows that $\mathsf{supp}(Z) \subseteq \varphi(0)$. It remains to show that Z lies in $\mathsf{Mod}(R)$. Applying the standard cohomology functor to the triangle induced by the short exact sequence above, we observe that, since $\mathcal{T}_0$ is a hereditary torsion class in $\mathsf{Mod}(R)$, $H^{-1}(Z)$ lies in $\mathcal{T}_0 \subseteq \mathcal{H}_\varphi$. Moreover, the above paragraph has also shown that $\tau^{\leq -1} Z \simeq H^{-1}(Z)[1]$ lies in $\mathcal{H}_\varphi$. But this means that $H^{-1}(Z) = 0$ and Z must then lie in $\mathcal{T}_0$.
Finally, the last statement follows from the fact that $\varphi(0)$ is specialisation closed, $\mathcal{H}_\varphi$ is contained in $\mathsf{D}^b(R)$ and hereditary torsion classes in $\mathcal{H}_\varphi$ are determined by their support (see Proposition 4.1). □

Lemma 6.8. *Let φ and ψ be intermediate sp-filtrations such that $\varphi(1) = \psi(1) = \emptyset$ and such that $\psi(i) = \varphi(i+1)$ for every $i < 0$. Then the compactly generated t-structure $\mathbb{T}_\psi$ associated to ψ is obtained by HRS-tilting $\mathbb{T}_\varphi = (\mathcal{U}_\varphi, \mathcal{V}_\varphi)$ (with heart $\mathcal{H}_\varphi$) with respect to a hereditary torsion pair of finite type whose torsion class is $\mathsf{Supp}^{-1}(\psi(0)) \cap \mathcal{H}_\varphi$.*

Proof. Let $\mathcal{T}_i$ denote the hereditary torsion class in $\mathsf{Mod}(R)$ supported on $\varphi(i)$, for any integer i. Since $\psi(0) \subseteq \psi(-1) = \varphi(0)$, we have that $\mathcal{T}' := \mathcal{H}_\varphi \cap \mathsf{Supp}^{-1}(\psi(0)) \subseteq \mathcal{H}_\varphi \cap \mathsf{Supp}^{-1}(\varphi(0)) = \mathcal{T}_0$, the last equality following from Proposition 6.7. We know from Theorem 4.5 that $\mathcal{T}'$ is a hereditary torsion class in $\mathcal{H}_\varphi$. If we tilt $\mathcal{H}_\varphi$ with respect to $\mathcal{T}'$ we obtain a t-structure having aisle

$$\overline{\mathcal{U}} := \mathcal{U}_\varphi[1] * \mathcal{T}'.$$

Now, since we have that $\mathcal{U}_\varphi[1] \subseteq \mathbb{D}^{\leq -1}$ and that $\mathcal{T}' \subseteq \mathcal{T}_0 \subseteq \mathbb{D}^{\geq 0}$, this aisle $\overline{\mathcal{U}}$ consists of the objects X such that the standard truncation $\tau^{\leq -1}X$ lies in $\mathcal{U}_\varphi[1]$ and the standard truncation $\tau^{\geq 0}X$ lies in $\mathcal{T}'$, i.e.

$$\overline{\mathcal{U}} = \{X \in \mathsf{D}(R)\colon \mathsf{Supp}(H^iX) \subseteq \varphi(i+1) = \psi(i),\ \forall i < 0,\ \mathsf{Supp}(H^0X) \subseteq \psi(0)\}$$

In other words, we have that $\overline{\mathcal{U}}$ is the aisle of the t-structure determined by ψ. Moreover, since this is also a compactly generated t-structure, it follows that the hereditary torsion pair we have tilted at is of finite type (see item (1) of Theorem 2.16). □

Notation 6.9. Note that in the above lemma, the sp-filtration φ can be recovered from ψ. We will denote this operation on sp-filtrations by writing $\varphi = \psi^{\langle 1 \rangle}$. In the notation of [1, §5.3], we have that $\psi^{\langle 1 \rangle}$ is a shift of ψ', i.e. $\psi^{\langle 1 \rangle}(i) = \psi'(i-1)$. Moreover, starting with an sp-filtration ψ such that $\psi(1) = \emptyset$, we will denote the iterations of this process by $\psi^{\langle n \rangle}$, for $n \geq 1$:

$$\psi^{\langle n \rangle}(i) = \begin{cases} \emptyset & \text{if } i > 0 \\ \psi(i-n) & \text{if } i \leq 0. \end{cases}$$

Proposition 6.10. *Let φ be an intermediate sp-filtration such that $\varphi(1) = \emptyset$. Then the compactly generated t-structure $\mathbb{T}_\varphi$ associated to φ can be built from* *the standard t-structure by an iteration of HRS-tilts at hereditary torsion pairs of finite type having specialisation-closed support.*

Proof. Since φ is intermediate, we have $\mathsf{Spec}(R) = \varphi(-n) \supsetneq \varphi(-n+1)$ for some $n \geq 0$. The statement then follows by induction on n, using Lemma 6.8. ☐

6.3 Restrictable t-structures and derived equivalences

We now turn to restrictable, intermediate and compactly generated t-structures, with the aim of establishing that they induce derived equivalences. We begin by reviewing what is known about how to characterise the sp-filtrations associated to the restrictable compactly generated t-structures (see [1]). The following condition turns out to play a significant role in that characterisation for some commutative rings.

Definition 6.11. An sp-filtration φ is said to satisfy the *weak Cousin condition* if whenever $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals such that $\mathfrak{p} \subsetneq \mathfrak{q}$ and $\mathfrak{p}$ is maximal under $\mathfrak{q}$ (i.e. there is no prime ideal $\mathfrak{t}$ such that $\mathfrak{p} \subsetneq \mathfrak{t} \subsetneq \mathfrak{q}$), then we have

$$\forall j \in \mathbb{Z},\ \mathfrak{q} \in \varphi(j) \Rightarrow \mathfrak{p} \in \varphi(j-1)$$

Theorem 6.12. *[1, Theorem 3.10 and 6.9, Corollary 4.5][41, Theorem 6.3] Let R be a commutative noetherian ring, $\mathscr{B}$ the set of t-structures in $\mathsf{D}^b(\mathsf{mod}(R))$ and $\mathscr{T}$ the set of compactly generated t-structures in $\mathsf{D}(R)$. There is an assignment $\Theta\colon \mathscr{B} \longrightarrow \mathscr{T}$, sending a t-structure $\mathbb{B} := (\mathcal{X}, \mathcal{Y})$ in $\mathsf{D}^b(\mathsf{mod}(R))$ to the t-structure generated by $\mathcal{X}$, namely $\Theta(\mathbb{B}) := ({}^{\perp}(\mathcal{X}^{\perp}), \mathcal{X}^{\perp})$. Moreover, for every $\mathbb{B}$ in $\mathscr{B}$, we have*

1. *$\Theta(\mathbb{B}) \cap \mathsf{D}^b(\mathsf{mod}(R)) = \mathbb{B}$ (and, in particular, Θ is injective);*
2. *The sp-filtration associated to $\Theta(\mathbb{B})$ satisfies the weak Cousin condition;*
3. *The heart of $\Theta(\mathbb{B})$ is locally coherent and its subcategory of finitely presented objects coincides with the heart of $\mathbb{B}$.*

The image of Θ is, then, the set of restrictable compactly generated t-structures. Moreover, if R admits a dualising complex, then the t-structures in the image of Θ are those whose associated sp-filtrations satisfy the weak Cousin condition.

Definition 6.13. Let R be a commutative noetherian ring. We say that an sp-filtration φ in $\mathsf{Spec}(R)$ is *restrictable* if the associated compactly generated t-structure is restrictable (in other words, the associated t-structure is in the image of the assignment Θ).

Note that it follows easily from the definition that if φ is an sp-filtration with $\varphi(1) = \emptyset$ satisfying the weak Cousin condition, then $\varphi^{\langle n\rangle}$ also satisfies the weak Cousin condition, for every $n \geq 1$. In fact, the following related statement holds.

Proposition 6.14. *[1, Lemma 5.7] If an intermediate sp-filtration φ is restrictable, then $\varphi^{\langle n\rangle}$ is also (intermediate and) restrictable, for any $n \geq 1$.*

Corollary 6.15. *Let φ be an intermediate sp-filtration with $\varphi(1) = \emptyset$. If φ is restrictable, then the torsion pairs involved in the HRS-tilts of Proposition 6.10 are restrictable.*

Proof. By Proposition 6.10, we know that $\mathbb{T}_\varphi$ is obtained from the standard t-structure by iterated HRS-tilts at hereditary torsion pairs of finite type. Moreover, by Proposition 6.14, each of the t-structures involved in this process is restrictable and, in particular, their hearts are locally coherent with finitely presented objects given by the intersection with $\mathsf{D}^b(\mathsf{mod}(R))$, by Theorem 6.12. Finally, Theorem 2.16 completes the argument. ☐

Theorem 6.16. *Let R be a commutative noetherian ring and let $\mathbb{T}$ be an intermediate restrictable compactly generated t-structure in $\mathsf{D}(R)$. Then $\mathbb{T}$ induces a derived equivalence.*

Proof. Up to shifting $\mathbb{T}$, we may assume that the associated sp-filtration φ has $\varphi(1) = \emptyset$. Proposition 6.10 then shows that $\mathbb{T}$ can be obtained from the standard t-structure via an iterated HRS-tilting process involving hereditary torsion pairs of finite type. Moreover, Corollary 6.15 shows that these torsion pairs are restrictable. Applying Corollary 6.4 to these tilts, we obtain a chain of triangle equivalences linking $\mathsf{D}(\mathcal{H}_\mathbb{T})$ and $\mathsf{D}(R)$, as wanted. ☐

Corollary 6.17. *Let R be a commutative noetherian ring. Every bounded cosilting object of $\mathsf{D}(R)$ whose t-structure is restrictable is cotilting.*

Proof. It follows from [30, Proposition 3.10] that every bounded cosilting object is pure-injective. The associated t-structure is then compactly generated by [22, Corollary 2.14]. Since the complex is bounded, the associated t-structure is an intermediate t-structure. The result then follows from Theorem 6.16 and from the fact that a cosilting t-structure induces a derived equivalence if and only if it is cotilting ([39, Corollary 5.2]). ☐

Taking Theorem 6.16 into account, the assumption that $\mathbb{T}$ induces a derived equivalence in Corollary 4.6 is redundant, therefore leading to the following simplification.

Corollary 6.18. *Let R be a commutative noetherian ring and $\mathbb{T}$ an intermediate restrictable compactly generated t-structure in $\mathsf{D}(R)$ with heart $\mathcal{H}_\mathbb{T}$. Then there is a bijection between hereditary torsion pairs of finite type in $\mathcal{H}_\mathbb{T}$ and specialisation-closed subsets of $\mathsf{Spec}(R)$.*

At this point, one might speculate whether every intermediate compactly generated t-structure leads to a derived equivalence. We instead show an example of such a t-structure that does not induce a derived equivalence. In other words, this provides (implicitly) an example of a bounded (3-term) pure-injective cosilting complex which is not cotilting over a commutative noetherian ring. Note that by Corollary 5.12 such an example cannot be found among 2-term cosilting complexes.

Example 6.19. Recall the situation considered in Example 4.15 and assume, furthermore, that R is connected, i.e. that it has no non-trivial idempotent elements. With the same notation, $\mathcal{H}_\mathbf{t}$ is the heart of the t-structure corresponding to the sp-filtration $\mathsf{Spec}\, R \supseteq V \supseteq \emptyset$. By Lemma 4.11 we know that the set V also corresponds to a hereditary torsion pair (of finite type, by Theorem 4.5) in $\mathcal{H}_\mathbf{t}$, namely $\mathbf{s} = (\mathcal{T}, \mathcal{C}[1])$, where $\mathcal{C}$ is the Giraud subcategory associated to $\mathcal{T}$ in $\mathsf{Mod}(R)$. Consider the heart $\mathcal{H}_\mathbf{s}$ of the HRS-tilt of the t-structure with heart $\mathcal{H}_\mathbf{t}$ with respect to $\mathbf{s}$. The corresponding t-structure, by Lemma 6.8 is associated to the intermediate sp-filtration $\mathsf{Spec}(R) \supseteq V \supseteq V \supseteq \emptyset$. Notice that this filtration does not satisfy the weak Cousin condition and, hence, this t-structure is not restrictable.
By construction, we have $\mathcal{H}_\mathbf{s} = \mathcal{C}[2] * \mathcal{T}$. Notice that since $\mathbf{t}$ is perfect, for all objects T in $\mathcal{T}$ and C in $\mathcal{C}$ we have $\mathsf{Hom}_{\mathsf{D}(R)}(T, C[3]) \simeq \mathsf{Ext}^3_R(T, C) = 0$ and hence all triangles

$$C[2] \longrightarrow X \longrightarrow T \longrightarrow C[3]$$

split. In other words, the torsion pair $(\mathcal{C}[2], \mathcal{T})$ in $\mathcal{H}_\mathbf{s}$ is a split torsion pair. Moreover, the same argument shows that $\mathsf{Hom}_{\mathsf{D}(R)}(T, C[2]) \simeq \mathsf{Ext}^2_R(T, C) = 0$ and, thus, we have that in fact also $(\mathcal{T}, \mathcal{C}[2])$ is a torsion pair in $\mathcal{H}_\mathbf{s}$. In other words, $\mathcal{C}[2]$ and $\mathcal{T}$ are abelian subcategories of $\mathcal{H}_\mathbf{s}$ and $\mathcal{H}_\mathbf{s} \simeq \mathcal{C}[2] \times \mathcal{T}$.
Now, since R is connected, it follows that $\mathsf{D}(R)$ is an indecomposable triangulated category, i.e. it is not the product of two triangulated subcategories (see [9, Example 3.2]). However, it is clear that $\mathsf{D}(\mathcal{H}_\mathbf{s})$ is not indecomposable, as it is equivalent to the product $\mathsf{D}(\mathcal{C}[2]) \times \mathsf{D}(\mathcal{T})$. As a consequence, $\mathcal{H}_\mathbf{s}$ cannot be derived equivalent to $\mathsf{Mod}(R)$. Note that, in particular, this provides an example of a cosilting (3-term) object of $\mathsf{D}(R)$ which is not cotilting.

We conclude the paper exploring some consequences for the hearts of t-structures of $\mathsf{D}^b(\mathsf{mod}(R))$.

Proposition 6.20. *Let R be a commutative noetherian ring, and $\mathbb{B}$ a bounded t-structure of $\mathsf{D}^b(\mathsf{mod}(R))$, with heart $\mathcal{B}$. Then $\mathcal{B}$ is the category of finitely presented objects of a locally coherent Grothendieck category which is derived equivalent to $\mathsf{Mod}(R)$. Moreover, Serre subcategories of $\mathcal{B}$ are in bijection with specialisation-closed subsets of $\mathsf{Spec}(R)$.*

Proof. Consider the compactly generated t-structure $\Theta(\mathbb{B})$. It is intermediate because so is $\mathbb{B}$, and it is restrictable by construction. Hence, its heart $\mathcal{H}_\mathbb{T}$ is derived equivalent to $\mathsf{Mod}(R)$, by Theorem 6.16. Now, by Theorem 2.13, $\mathcal{H}_\mathbb{T}$ is a locally coherent Grothendieck category with $\mathsf{fp}(\mathcal{H}_\mathbb{T}) = \mathcal{H}_\mathbb{T} \cap \mathsf{D}^b(\mathsf{mod}(R)) = \mathcal{B}$. Finally, since Serre subcategories of $\mathcal{B}$ are in bijection with hereditary torsion pairs of $\mathcal{H}_\mathbb{T}$ (see [21, 24]), and therefore with specialisation closed subsets of $\mathsf{Spec}(R)$ by Corollary 6.18. □

Acknowledgments

The authors are grateful to Frederik Marks and Alexandra Zvonareva for sharing with them results of their work in progress [31], and to Lidia Angeleri Hügel and Ryo Takahashi for Lemma 4.10. The authors acknowledge financial support from the Fondazione Cariverona under the programme *Ricerca Scientifica di Eccellenza 2018*, project: *Reducing complexity in algebra, logic, combinatorics* — REDCOM. The first-named author was also partially supported by the Department of Mathematics Tullio Levi-Civita of the University of Padova (Project BIRD163492/16 *Categorical homological methods in the study of algebraic structures* and the Research programme DOR1828909 *Anelli e categorie di moduli*).

References

[1] L. Alonso Tarrio, A. Jeremias López and M. Saorín, *Compactly generated t-structures on the derived category of a noetherian ring*, J. Algebra 324, 313–346.

[2] L. Angeleri Hügel and M. Hrbek, *Silting modules over commutative rings*, Int. Math. Res. Not. IMRN 2017 (2017), no. 3, 4131–4151.

[3] L. Angeleri Hügel, F. Marks and J. Vitória, *Torsion pairs in silting theory*, Pacific J. Math. 291 (2017), 257–278.

[4] L. Angeleri Hügel, F. Marks, J. Šťovíček, R. Takahashi and J. Vitória, *Flat ring epimorphisms and universal localisations of commutative rings*, Q. J. Math. 71 (2020), no. 4, 1489–1520.

[5] S. Bazzoni, *Cotilting modules are pure-injective*, Proc. Amer. Math. Soc. 131 (2003), no. 12, 3665–3672.

[6] D. Benson, S. B. Iyengar and H. Krause, *Local cohomology and support for triangulated categories*, Ann. Sci. Éc. Norm. Supér. 41 (2009), 575–621.

[7] A. Beilinson, J. Bernstein and P. Deligne, *Analyse et topologie sur les espaces singuliers*, Astérisque, 100 (1982), Soc. Math. France.

[8] A. Beligiannis, *Relative homological algebra and purity in triangulated categories*, J. Algebra 227 (2000), 268–361.

[9] T. Bridgeland, *Equivalences of triangulated categories and Fourier-Mukai transforms*, Bull. London Math. Soc. 31 (1999), 25–34.

[10] T. Bridgeland, *Stability conditions on a non-compact Calabi-Yau threefold*, Comm. Math. Phys. 266 (2006), 715–733.

[11] N. Broomhead, D. Pauksztello and D. Ploog, *Derived discrete categories II: the silting pairs CW complex and the stability manifold*, J. London Math. Soc. 93 (2016), 273–300.

[12] X.-W. Chen, Z. Han and Y. Zhou, *Derived equivalences via HRS-tilting*, Adv. Math. 354 (2019), 106749, 26 pp.

[13] X.-W. Chen and S. B. Iyengar, *Support and injective resolutions of complexes over commutative rings*, Homology Homotopy Appl. 12 (2010), no. 1, 39–44.

[14] W. Crawley-Boevey, *Locally finitely presented additive categories*, Comm. Algebra 22 (1994), no. 5, 1641–1674.

[15] S. E. Dickson, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc. 121 (1966), 223–235.

[16] L. Fiorot, F. Mattiello and A. Tonolo, *A classification theorem for t-structures*, J. Algebra 465 (2016), 214–258.

[17] H.-B. Foxby, *Bounded complexes of flat modules*, J. Pure Appl. Algebra 15 (1979), 149–172.

[18] P. Gabriel, *Des catégories abéliennes*, Bulletin de la S.M.F. 90 (1962), 323–448.

[19] D. Happel, I. Reiten and S. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 120 (1996), no. 575, viii+ 88.

[20] I. Herzog, *The Ziegler spectrum of a locally coherent Grothendieck category*, Proc. London Math. Soc. 74 (1997), no. 3, 503–558.

[21] M. Hrbek, *Compactly generated t-structures in the derived category of a commutative ring*, Math. Z. 295 (2020), 47–72.

[22] M. Hrbek and T. Nakamura, *Telescope conjecture for homotopically smashing t-structures over commutative noetherian rings*, J. Pure Appl. Algebra 225 (2021), no. 4, 106571, 13 pp.

[23] B. Keller, *Deriving DG categories*, Ann. Scient. Ec. Norm. Sup. 27 (1994), 63–102.

[24] H. Krause, *The spectrum of a locally coherent category*, J. Pure Appl. Algebra 114 (1997), 259–271.

[25] H. Krause, *Smashing subcategories and the telescope conjecture – an algebraic approach*, Inv. Math. 139 (2000), 99–133.

[26] H. Krause, *Thick subcategories of modules over commutative noetherian rings. With an appendix by S. Iyengar*, Math. Ann. 340 (2008), 733–747.

[27] H. Krause and G. Stevenson, *The derived category of the projective line*, in: Spectral Structures and Topological Methods in Mathematics, EMS Publishing House (2019) 275–297.

[28] R. Laking, *Purity in compactly generated derivators and t-structures with Grothendieck hearts*, Math. Z. 295 (2020), 1615–1641.

[29] R. Laking and J. Vitória, *Definability and approximations in triangulated categories*, Pacific J. Math. 306 (2020), 557–586.

[30] F. Marks and J. Vitória, *Silting and cositling classes in derived categories*, J. Algebra 501 (2019), 526–544.

[31] F. Marks and A. Zvonareva, *Lifting and restricting t-structures*, in preparation.

[32] F. Mattiello, S. Pavon and A. Tonolo, *Tilting modules and tilting torsion pairs*, In: Facchini A., Fontana M., Geroldinger A., Olberding B. (eds) Advances in Rings, Modules and Factorizations. Springer Proceedings in Mathematics & Statistics 321 (2018), 317–339.

[33] A. Neeman, *The chromatic tower for* $\mathsf{D}(R)$. *With an appendix by Marcel Bökstedt*, Topology 31 (1992), 519–532.

[34] C. E. Parra and M. Saorín, *Direct limits in the heart of a t-structure: the case of a torsion pair*, J. Pure App. Algebra 219 (2015), 4117–4143.

[35] C. E. Parra and M. Saorín, *Addendum to Direct limits in the heart of a t-structure: the case of a torsion pair*, J. Pure App. Algebra 220 (2016), 2467–2469.

[36] C. E. Parra and M. Saorín, *Hearts of t-structures in the derived category of a commutative noetherian ring*, Trans. Amer. Math. Soc. 369 (2017), 7789–7827.

[37] C. E. Parra, M. Saorín and S. Virili, *Locally finitely presented and coherent hearts*. Preprint (2019), arXiv:1908.00649.

[38] D. Pauksztello, M. Saorín and A. Zvonareva, *Contractibility of the stability manifold for silting-discrete algebras*, Forum Math. 30 (2018), 1255–1263.

[39] C. Psaroudakis and J. Vitória, *Realisation functors in tilting theory*, Math. Z. 288 (2018), no. 3-4, 965–1028.

[40] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. s2-39 (1989), no. 3, 436–456.

[41] M. Saorín, *On locally coherent hearts*, Pacific J. Math. 287 (2017), no. 1, 199–221.

[42] M. Saorín and J. Šťovíček, *t-structure with Grothendieck hearts via functor categories*. Preprint (2020), arXiv:2003.01401.

[43] M. Saorín, J. Šťovíček and S. Virili, *t-structures on stable derivators and Grothendieck categories*. Preprint (2017), arXiv:1708.07540.

[44] B. Stenstrøm, *Rings of quotients*, Springer-Verlag (1975).

[45] J. Šťovíček, *All n-cotilting modules are pure-injective*, Proc. Amer. Math. Soc. 134 (2006), no. 7, 1891–1897.

[46] S. Virili, *Morita theory for stable derivators.* Preprint (2018), arXiv:1807.01505.

Sergio Pavon
Dipartimento di Matematica
Università di Padova
Via Trieste 63
35121 Padova
Italy
sergio.pavon@math.unipd.it

Jorge Vitória
Dipartimento di Matematica e Informatica
Università di Cagliari
Via Ospedale 72
09124 Cagliari
Italy
jorge.vitoria@unica.it