

TRANSFER MAPS IN GENERALIZED GROUP HOMOLOGY
VIA SUBMANIFOLDS

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ABSTRACT. Let $N \subset M$ be a submanifold embedding of spin manifolds of some codimension $k \geq 1$. A classical result of Gromov and Lawson, refined by Hanke, Pape and Schick, states that M does not admit a metric of positive scalar curvature if $k = 2$ and the Dirac operator of N has non-trivial index, provided that suitable geometric conditions on $N \subset M$ are satisfied. In the cases $k = 1$ and $k = 2$, Zeidler and Kubota, respectively, established more systematic results: There exists a transfer $\mathrm{KO}_*(C^*\pi_1 M) \rightarrow \mathrm{KO}_{*-k}(C^*\pi_1 N)$ which maps the index class of M to the index class of N . The main goal of this article is to construct analogous transfer maps $E_*(B\pi_1 M) \rightarrow E_{*-k}(B\pi_1 N)$ for different generalized homology theories E and suitable submanifold embeddings. The design criterion is that it is compatible with the transfer $E_*(M) \rightarrow E_{*-k}(N)$ induced by the inclusion $N \subset M$ for a chosen orientation on the normal bundle. Under varying restrictions on homotopy groups and the normal bundle, we construct transfers in the following cases in particular: In ordinary homology, it works for all codimensions. This slightly generalizes a result of Engel and simplifies his proof. In complex K-homology, we achieve it for $k \leq 3$. For $k \leq 2$, we have a transfer on the equivariant KO-homology of the classifying space for proper actions.

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1 INTRODUCTION

Our starting point is the following result of Gromov and Lawson [8] and Hanke, Pape, and Schick [9]. All manifolds we consider are smooth.

THEOREM 1.1. *Let M be a closed m -dimensional spin manifold and $N \subset M$ a compact submanifold without boundary of codimension two with trivial normal bundle. Assume that the inclusion $N \hookrightarrow M$ induces an injection $\pi_1(N) \rightarrow \pi_1(M)$ and a surjection $\pi_2(N) \rightarrow \pi_2(M)$.*

If the Rosenberg index $\alpha(N) \in \mathrm{KO}_{m-2}(\mathrm{C}^\pi_1(N))$ is non-zero, then M does not admit a Riemannian metric of positive scalar curvature.*

The “standard” obstruction to the existence of a metric with positive scalar curvature on M is the Rosenberg index $\alpha(M) \in \mathrm{KO}_m(\mathrm{C}^*\pi_1(M))$ of M . The surprise is that, here, we can use $\alpha(N)$ as a “submanifold obstruction”. However, the Rosenberg index of M is conjectured to be the “universal” index obstruction to positive scalar curvature on M , see [20, Conjecture 1.5]. Thus in the situation of Theorem 1.1, one expects the non-vanishing of $\alpha(N)$ to imply the non-vanishing of $\alpha(M)$. This was recently proved by Kubota for the Rosenberg index in the maximal group C^* -algebra [11, Section 3.3]. A somewhat simpler proof of this theorem is given in [12], where it is restated as [12, Theorem 1.2].

Moreover, Zeidler [25, Theorem 1.7] showed the analog assertion for submanifolds of codimension one, for the reduced C^* -algebra and without a condition on π_2 . Engel [6] and Zeidler [25] also prove variants of this for other codimensions under strong additional hypotheses.

It is our goal to conceptually illuminate the submanifold obstruction results à la Theorem 1.1 by constructing *transfer maps*: We think of the Rosenberg index of a spin manifold M as the image of the spin bordism fundamental class under the sequence of transformations

$$\Omega_n^{\mathrm{spin}}(M) \xrightarrow{T_{\mathrm{ABS}}} \mathrm{KO}_n(M) \xrightarrow{c_{\Gamma^*}} \mathrm{KO}_n(\mathrm{B}\Gamma) \xrightarrow{\iota_*} \mathrm{KO}_n^{\Gamma}(\underline{\mathbb{E}}\Gamma) \xrightarrow{\mu} \mathrm{KO}_n(\mathrm{C}^*\Gamma), \quad (1.2)$$

where $\Gamma = \pi_1(M)$, c_{Γ} is classifying the principal Γ -bundle $\widetilde{M} \rightarrow M$, T_{ABS} is the Atiyah–Bott–Shapiro orientation, ι is the classifying map for proper actions and μ is the Baum–Connes assembly map. On purpose, we leave it open which C^* -completion $\mathrm{C}^*\Gamma$ of $\mathbb{R}\Gamma$ we use: if we use $\mathrm{C}_{\max}^*\Gamma$ we get a priori a stronger invariant, if we use $\mathrm{C}_{\mathrm{red}}^*\Gamma$ the Baum–Connes conjecture predicts that μ is an isomorphism.

When $N \subset M$ is a codimension k submanifold (satisfying certain conditions) and $q_{\pi}: \pi_1(N) \rightarrow \pi := i_{N*}(\pi_1(N)) < \pi_1(M)$ is the induced epimorphism on the fundamental group, we —ideally— want to construct transfer maps fitting

into a commutative diagram

$$\begin{array}{ccccccccc}
 \Omega_*^{\text{spin}}(M) & \xrightarrow{c_{\Gamma^*}} & \Omega_*^{\text{spin}}(B\Gamma) & \xrightarrow{T_{\text{ABS}}} & \text{KO}_*(B\Gamma) & \xrightarrow{\iota_*} & \text{KO}_*^{\Gamma}(\underline{E}\Gamma) & \xrightarrow{\mu} & \text{KO}_*(C^*\Gamma) \\
 \downarrow \tau_{M,N} & & \downarrow \tau_{\Gamma,\pi} & & \downarrow \tau_{\Gamma,\pi} & & \downarrow \underline{\tau}_{\Gamma,\pi} & & \downarrow \sigma_{\Gamma,\pi} \\
 \Omega_{*-k}^{\text{spin}}(N) & \xrightarrow{q_{\pi_* \circ c_{\pi^*}}} & \Omega_{*-k}^{\text{spin}}(B\pi) & \xrightarrow{T_{\text{ABS}}} & \text{KO}_{*-k}(B\pi) & \xrightarrow{\iota_*} & \text{KO}_{*-k}^{\pi}(\underline{E}\pi) & \xrightarrow{\mu} & \text{KO}_{*-k}(C^*\pi),
 \end{array} \tag{1.3}$$

such that $\tau_{M,N}$ maps the fundamental class of M to that of N . The usual choice for $\tau_{M,N}$ is the *classical transfer map* coming from the Pontryagin–Thom collapse of the normal bundle of N (see (1.5) below). The existence of the complete diagram would make it most transparent how $\alpha(N)$ is an obstruction to positive scalar curvature on M . But already part of the diagram would suffice to show this under some additional assumptions:

When we use $C^*\Gamma = C_{\text{red}}^*\Gamma$ and Γ satisfies the Baum–Connes Conjecture, i.e. when μ is an isomorphism, the rightmost extension can be abstractly defined as $\mu \circ \underline{\tau}_{\Gamma,\pi} \circ \mu^{-1}$ provided $\underline{\tau}_{\Gamma,\pi}$ exists. When μ is at least injective, existence of $\underline{\tau}_{\Gamma,\pi}$ is still enough to conclude that $\alpha(N) \neq 0$ implies $\alpha(M) \neq 0$. Similarly, the existence of $\tau_{\Gamma,\pi}$ suffices for the same conclusion when Γ satisfies the Strong Novikov Conjecture, i.e. when $\mu \circ \iota_*$ is injective. Note that the Baum–Connes Conjecture for real K-Theory is equivalent to the one for complex K-Theory, and after inverting 2 this is also true for injectivity of μ [19]. The map ι_* is always rationally injective and an isomorphism if Γ is torsion-free.

Our first and most complete result deals with codimension one. Zeidler already proved that if $k = 1$ and the map $\pi_1(N) \rightarrow \pi_1(M)$ is injective, then a suitable transformation on the level of the group C^* -algebras exists [25, Remark 1.8]. In Section 3, we reprove this in a way which obtains the complete picture as in Diagram 1.3. However, we do not prove that our new approach agrees with Zeidler’s original construction even though we strongly expect that to be the case. We also generalize the result slightly by dropping the π_1 -injectivity requirement and instead work with the image of the induced map $\pi_1(N) \rightarrow \pi_1(M)$.

THEOREM 1.4 (see Theorem 3.1). *Let M be a closed connected manifold and $N \subset M$ a closed connected submanifold of codimension one with trivial normal bundle. Let $\Gamma := \pi_1(M)$, and let $\pi \leq \Gamma$ be the image of the homomorphism $\pi_1(N) \rightarrow \pi_1(M) = \Gamma$ which is induced by the inclusion $N \hookrightarrow M$. Then the classical transfer map $\tau_{M,N}$ can be extended as in Diagram 1.3. This holds for C^* either C_{red}^* or C_{max}^* .*

In particular, if M is spin and $\alpha(N) \neq 0 \in \text{KO}_{m-k}(C^\pi)$, then $\alpha(M) \neq 0 \in \text{KO}_m(C^*\Gamma)$ and M does not admit a metric of positive scalar curvature.*

The ultimate goal would be to obtain Diagram 1.3 also for submanifolds of higher codimension (and suitable assumptions on the homotopy type). Unfortunately, in complete generality we do not achieve this. However, we obtain various partial results for different generalized homology theories.

The obvious choice for $\tau_{M,N}$ is the *classical transfer map*, which can be defined

for generalized homology theories: Let E be some multiplicative cohomology theory and suppose that both M and N admit an E -orientation. Then these orientations induce a Thom class $\theta \in E^k(D\nu, S\nu)$, where $D\nu$ and $S\nu$ denote the disk- and sphere bundle of the normal bundle for $N \subset M$. Let

$$\tau_{M,N} = \tau_\theta: E_*(M) \rightarrow E_*(M, M \setminus D\nu) \cong E_*(D\nu, S\nu) \xrightarrow{\cap \theta} E_{*-k}(N). \quad (1.5)$$

In Section 4, which is partly based on the first-named author’s PhD thesis [16], we introduce for an equivariant generalized cohomology theory E the notion of an *lf-restriction* operation, and we show that singular homology, K-theory and the bordism theories all allow such an operation. Building on this, we construct transfer maps in the equivariant setting,

$$\begin{aligned} E_*(B\Gamma) &\cong E_*^\Gamma(E\Gamma) \rightarrow E_{*-k}^\pi(E\Gamma) \cong E_{*-k}(B\pi) \\ \text{and} \quad E_*^\Gamma(\underline{E}\Gamma) &\rightarrow E_{*-k}^\pi(\underline{E}\Gamma) \cong E_{*-k}^\pi(\underline{E}\pi), \end{aligned}$$

between the classifying spaces for free and for proper actions. Here, the isomorphisms on the right-hand side come from the fact that $E\Gamma$ and $\underline{E}\Gamma$, with the restricted action, are models for $E\pi$, respectively $\underline{E}\pi$. We obtain a general (but technical) criterion for when this construction can be used to extend the classical transfer map:

$$\begin{array}{ccccc} E_*(M) & \longrightarrow & E_*(B\Gamma) & \longrightarrow & E_*^\Gamma(\underline{E}\Gamma) \\ \downarrow \tau_{M,N} & & \downarrow & & \downarrow \\ E_{*-k}(N) & \longrightarrow & E_{*-k}(B\pi) & \longrightarrow & E_{*-k}^\pi(\underline{E}\pi) \end{array} \quad (1.6)$$

Our construction is inspired by the ideas of Engel [6], who constructed an extension of the classical transfer map for singular homology in the framework of uniformly finite homology. It can be seen as a generalization and simplification of Engel’s arguments. Specializing in suitable situations to spin bordism and real K-homology, the method gives rise to Diagram 1.3 except for the transfer on the level of C^* -algebras, which so far has not been constructed in the general setting.

In Section 5 we apply the general transfer construction to different generalized cohomology theories, always with the goal to obtain information about positive scalar curvature on M .

Our only result which yields the desired transformation in all codimensions is for ordinary homology. It slightly generalizes the result of Engel [6].

THEOREM 1.7 (see Theorem 5.1). *Let M be a closed connected manifold and $N \subset M$ a closed connected submanifold of codimension k with A -oriented normal bundle for some commutative ring A . Let $\Gamma := \pi_1(M)$, and $\pi = i_{N*}(\pi_1(N)) < \pi_1(M)$. Let $E_* = H_*(-; A)$ be singular homology with coefficients in A . Assume that*

- (1) *if $k > 1$, then $\pi_k(N) \rightarrow \pi_k(M)$ is surjective;*

- (2) if $k > 2$, then $\pi_j(M) = 0$ for $j = 2, \dots, k - 1$;
- (3) there exists a subset $S \subseteq \pi_k(N)$ which generates $\pi_k(M)$ such that the composition of the Hurewicz homomorphism $\pi_k(N) \rightarrow H_k(N; \mathbb{Z})$ with the evaluation on the Euler class of ν vanishes on S . In particular, this is satisfied if the normal bundle is trivial or $\pi_k(M) = 0$ or $\dim N < \frac{\dim M}{2}$.

Then the classical transfer map can be extended to $H_*(B\Gamma; A) \rightarrow H_{*-k}(B\pi; A)$.

For the π_1 -injective case this result appeared in the article of Engel [6, Main Theorem] but without the condition (3). However, the counterexample described in Example 5.5 below shows that this condition is indeed necessary—even in the π_1 -injective case. Engel informed us that his proof nonetheless works under the additional assumption that $\pi_k(M)$ is zero or that the normal bundle is trivial, see the correction [7]. These hypotheses are both covered by our condition (3).

Theorem 1.7 already has interesting consequences for the positive scalar curvature problem provided that the rational analytic Novikov conjecture holds for Γ . The following applications were also observed by Engel [6, p. 425].

COROLLARY 1.8 (see Corollary 5.6). *Assume that we are in the situation of Theorem 1.7 (with $A = \mathbb{Z}$), that the assembly map $\mu \otimes \mathbb{Q}: KO_*(B\Gamma) \otimes \mathbb{Q} \rightarrow KO_*(C^*\Gamma) \otimes \mathbb{Q}$ is injective, and that M is spin. Moreover, assume that either*

- (1) *the push forward of the homological fundamental class of N does not vanish in $H_*(B\pi; \mathbb{Q})$, or*
- (2) *the normal bundle of $N \subset M$ is trivial and the push forward of the KO-fundamental class of N (with the induced spin structure) in $KO_*(B\pi) \otimes \mathbb{Q}$ does not vanish.*¹

Then $\alpha(M) \neq 0$ and hence M does not admit a metric of positive scalar curvature.

For codimension two or three, we also have results for certain generalized cohomology theories. The crucial observation which makes the next theorem work is that an orientation on a two-dimensional real vector bundle defines a complex structure on it. This, in turn, induces an orientation in every complex oriented cohomology theory such as spin^c bordism and complex K-theory (and up to inverting 2, also spin bordism and real K-theory).

THEOREM 1.9 (see Theorems 5.11 and 5.12). *Let M be a closed connected manifold and $N \subset M$ a closed connected submanifold of codimension $k \in \{2, 3\}$ with oriented normal bundle ν . Let $\Gamma := \pi_1(M)$, and $\pi := i_{N*}(\pi_1(N)) < \pi_1(M)$. Moreover, assume the following.*

- *For $k = 2$: The induced map $\pi_2(N) \rightarrow \pi_2(M)$ is surjective. Also, there exists a subset $S \subseteq \pi_2(N)$ which generates $\pi_2(M)$ such that the composition of the Hurewicz homomorphism $\pi_2(N) \rightarrow H_2(N; \mathbb{Z})$ with the first*

¹This can be rephrased as the condition that there exists a non-vanishing higher \hat{A} -genus of N which—via the map $\pi_1 N \rightarrow \pi$ —comes from a cohomology class of π .

Chern class of ν vanishes on S , where we use the orientation to view ν as a complex line bundle.

- For $k = 3$: The induced map $\pi_1(N) \rightarrow \pi$ is an isomorphism, $\pi_2(N) = 0 = \pi_2(M)$ and $\pi_3(N) \rightarrow \pi_3(M)$ is surjective. Also, ν is trivial and the Thom class coincides with the pullback of the three times suspended unit $x \in E^3(\mathbb{D}^3, S^2)$ under the trivialization map.

Then we have a commutative diagram

$$\begin{array}{ccccc}
 \Omega_*^{\text{spin}^c}(M) & \longrightarrow & \Omega_*^{\text{spin}^c}(\text{B}\Gamma) & \longrightarrow & \text{K}_*(\text{B}\Gamma) \\
 \downarrow \tau_{M,N} & & \downarrow \tau_{\Gamma,\pi} & & \downarrow \tau_{\Gamma,\pi} \\
 \Omega_{*-k}^{\text{spin}^c}(N) & \longrightarrow & \Omega_{*-k}^{\text{spin}^c}(\text{B}\pi) & \longrightarrow & \text{K}_{*-k}(\text{B}\pi).
 \end{array}$$

Moreover, we obtain the analogous diagram for spin-bordism and KO-theory up to inverting 2.

In Section 5.3, which is again based on [16], we consider once more the most general case of a generalized cohomology theory. We give a sufficient condition for extending the transfer map to the classifying space for proper actions in terms of group cohomology. Except for very low codimension, however, the condition depends on the group $\pi_1(M)$. The best result that we get independent of $\pi_1(M)$ is in codimension 2 when the normal bundle of $N \subset M$ is trivial. Specialized to spin bordism and real K-Theory, Theorem 5.15 gives:

THEOREM 1.10. *Let M be a closed connected spin manifold and $N \subset M$ a closed connected submanifold of codimension $k = 2$ with trivial normal bundle. Let $\Gamma := \pi_1(M)$ and $\pi := \pi_1(N)$. Moreover, assume that $\pi_1(N) \rightarrow \pi_1(M)$ is injective and that $\pi_2(N) \rightarrow \pi_2(M)$ is surjective.*

Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
 \Omega_*^{\text{spin}}(M) & \longrightarrow & \Omega_*^{\text{spin}}(\text{B}\Gamma) & \longrightarrow & \text{KO}_*(\text{B}\Gamma) & \longrightarrow & \text{KO}_*^\Gamma(\underline{\text{E}}\Gamma) \longrightarrow \text{KO}_*(\text{C}^*\Gamma) \\
 \downarrow \tau_{M,N} & & \downarrow \tau_{\Gamma,\pi} & & \downarrow \tau_{\Gamma,\pi} & & \downarrow \tau_{\Gamma,\pi} \\
 \Omega_{*-k}^{\text{spin}}(N) & \longrightarrow & \Omega_{*-k}^{\text{spin}}(\text{B}\pi) & \longrightarrow & \text{KO}_{*-k}(\text{B}\pi) & \longrightarrow & \text{KO}_{*-k}^\pi(\underline{\text{E}}\pi) \longrightarrow \text{KO}_{*-k}(\text{C}^*\pi).
 \end{array}$$

Note that the assumptions in the preceding theorem precisely match those of Theorem 1.1. It seems very likely that the resulting transfer maps are compatible with Kubota’s map on the K-Theory of the maximal group C*-algebras $\text{KO}_*(\text{C}_{\max}^*\Gamma) \rightarrow \text{KO}_{*-2}(\text{C}_{\max}^*\pi)$ [11, proof of Theorem 3.7], since both Kubota’s and our maps arise from a quite “natural” construction. The problem with comparing these maps is that our general transfer construction is defined in a very geometric way, and in a setting that is much more general than K-theory. Kubota’s construction, on the other hand, relies on tools involving operator algebras such as K-theory with coefficients in the Calkin algebra. Another obvious line of inquiry would be to further relax the conditions under which the transfer map can be extended in various generalized cohomology

theories. We note, however, that some assumptions on the homotopy type of M and N and on the generalized cohomology theory will always be necessary. One cannot expect, for example, that the mere existence of a submanifold inclusion was enough to obtain the transfer on group homology. This is only conceivable if the manifolds resemble to a suitable degree the classifying spaces of the groups. Furthermore, we do not expect a version of Theorem 5.1 for arbitrary cohomology theories and arbitrary prescribed orientations on the normal bundle. A —perhaps artificial— counterexample in KO-theory is an embedding $S^1 \subset S^{k+1}$, where we endow the normal bundle with the KO-orientation which induces the “antiperiodic” spin structure on S^1 . However, it is not clear where the precise frontier lies, and hence it would be of interest to find more (counter-)examples.

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2 GEOMETRIC SETUP

In this short section, we describe the general geometric setup that will be used in the technical sections of this article.

SETUP 2.1. Fix a codimension $k \in \mathbb{N}$. Let M and N be finite CW-complexes and let ν be a real vector bundle of fiber dimension k over N , and let there be an embedding $\mathfrak{t}: \nu \hookrightarrow M$ of the total space of ν as an open subset of M . Usually, we will use the map \mathfrak{t} implicitly and consider ν as a subset of M .

Let $S\nu \subset D\nu \subset \nu \subset M$ be the unit sphere and disk bundles associated to ν , respectively. Fix a base-point $x_0 \in N \subset \nu \subset M$. Let $\Gamma := \pi_1(M, x_0)$ and let $\pi \leq \Gamma$ be the image of the map $\pi_1(N, x_0) \rightarrow \pi_1(M, x_0)$ induced by the restriction of the embedding $\nu \hookrightarrow M$ to the zero section. Let $\tilde{p}: \tilde{M} \rightarrow M$ denote the universal covering of M . Let $\tilde{M} = \pi \backslash \tilde{M}$, and $\underline{p}: \tilde{M} \rightarrow \tilde{M}$ and $p: \tilde{M} \rightarrow M$ be the corresponding covering maps. Fix a base-point $\tilde{x}_0 \in \tilde{p}^{-1}(x_0)$ and let $\bar{x}_0 = \underline{p}(\tilde{x}_0) \in \tilde{M}$. Then the embedding $\mathfrak{t}: \nu \hookrightarrow M$ uniquely lifts to an embedding $\bar{\mathfrak{t}}: \nu \hookrightarrow \tilde{M}$ taking x_0 to \bar{x}_0 . Then we let $\tilde{D}\nu \rightarrow D\nu$ be the restriction of the covering $\tilde{M} \rightarrow \tilde{M}$ via $\bar{\mathfrak{t}}: D\nu \hookrightarrow \tilde{M}$. We obtain an embedding $\tilde{\mathfrak{t}}: \tilde{D}\nu \hookrightarrow \tilde{M}$ as a π -compact subset. Similarly, define $\tilde{S}\nu$ and \tilde{N} to be the restrictions to $S\nu$ and N , respectively.

Remark 2.2. The covering $\tilde{N} \rightarrow N$ is the connected normal covering of N with $\pi_1 \tilde{N} = \ker(\pi_1 N \rightarrow \pi)$. In particular, if the map $\pi_1 N \rightarrow \pi$ is an isomorphism,

then \tilde{N} is the universal covering of N .

Remark 2.3. The typical situation we consider is that $N \subset M$ is a submanifold embedding of codimension k with normal bundle ν which is embedded into M as a tubular neighborhood of N .

3 CODIMENSION ONE TRANSFER VIA KK-THEORY

In this section, we state and prove the precise version of Theorem 1.4.

THEOREM 3.1. *Let M be a closed spin manifold and $N \subset M$ a submanifold of codimension 1 with trivial normal bundle, both connected. Set $\pi := \text{im}(\pi_1(N) \rightarrow \pi_1(M) = \Gamma)$. Then the dashed homomorphisms in the following diagram exist and make it commutative.*

$$\begin{array}{ccccccccc}
 \Omega_*^{\text{spin}}(M) & \xrightarrow{c_*} & \Omega_*^{\text{spin}}(\text{B}\Gamma) & \xrightarrow{\alpha} & \text{KO}_*(\text{B}\Gamma) & \xrightarrow{t} & \text{KO}_*^\Gamma(\underline{\text{E}}\Gamma) & \xrightarrow{\mu^\Gamma} & \text{KO}_*(\text{C}^*\Gamma) \\
 \downarrow \tau_{M,N} & & \downarrow \tau_{\Gamma,\pi} & & \downarrow \tau_{\Gamma,\pi} & & \downarrow \tau_{\Gamma,\pi} & & \downarrow \sigma_{\Gamma,\tau} \\
 \Omega_{* - 1}^{\text{spin}}(N) & \xrightarrow{\bar{c}_*} & \Omega_{* - 1}^{\text{spin}}(\text{B}\pi) & \xrightarrow{\alpha} & \text{KO}_{* - 1}(\text{B}\pi) & \xrightarrow{t} & \text{KO}_{* - 1}^\pi(\underline{\text{E}}\pi) & \xrightarrow{\mu^\pi} & \text{KO}_{* - 1}(\text{C}^*\pi).
 \end{array} \tag{3.2}$$

The remaining arrows are defined as follows. The map $\tau_{M,N}$ on the left-hand side is induced by the Pontryagin–Thom collapse $M \rightarrow \Sigma N$ (via the trivial normal bundle). We use the map $c: M \rightarrow \text{B}\Gamma$ inducing the identity on fundamental groups, and $\bar{c}: N \rightarrow \text{B}\pi$ the composition of $N \rightarrow \text{B}\pi_1(N)$ which is the identity on the fundamental group and $\text{B}\pi_1(N) \rightarrow \text{B}\pi$ induced by $\pi_1(N) \twoheadrightarrow \pi$. The map α is the Atiyah orientation from spin bordism to real K-homology. Furthermore, t is the composition of the transfer isomorphism $\text{KO}_*(\text{B}\Gamma) \rightarrow \text{KO}_*^\Gamma(\underline{\text{E}}\Gamma)$ with the map induced by the unique homotopy class of Γ -equivariant maps $\underline{\text{E}}\Gamma \rightarrow \underline{\text{E}}\Gamma$ (using that $\underline{\text{E}}\Gamma$ is a universal space for proper Γ -actions). Finally, μ^Γ and μ^π are the Baum–Connes assembly maps corresponding to Γ and π , respectively.

Our definitions of the maps $\tau_{\Gamma,\pi}$ and $\sigma_{\Gamma,\pi}$ are based on a direct construction of a suitable boundary class in equivariant KK-theory similar to the work of Oyono–Oyono [17] on groups acting on trees. This is not surprising because the situation of Theorem 3.1 implies that Γ is an amalgamated product or an HNN-extension [24, Corollaries 1 and 2]. The previous construction of Zeidler [25, Remark 1.8] on the level of the group C*-algebra is based on a different method: it uses coarse index theory and the partitioned manifold index theorem. We do not prove that these two approaches yield the same map but nevertheless strongly expect that to be the case.

To prove the theorem, we consider a slightly more abstract setup based on Setup 2.1.

SETUP 3.3. Suppose that we are in the situation of Setup 2.1 with $k = 1$ and where ν is the trivial line bundle. In this section, we will keep the embedding $\mathfrak{t}: N \times \mathbb{R} \hookrightarrow M$ explicit in our notation. Hence the chosen base-point $x_0 \in M$ is of the form $x_0 = \mathfrak{t}(y_0, 0)$ for some $y_0 \in N$. Since $\pi_1(M, \bar{x}_0) = \pi = \mathfrak{t}_*(\pi_1(N \times \{0\}, y_0))$, codimension one and a Mayer–Vietoris argument imply that

$\bar{M} \setminus \bar{\mathfrak{t}}(N \times \{0\})$ consists of two connected components. We denote the closures of these components by \bar{M}_\pm , where the sign in the notation is determined by the requirement $\bar{\mathfrak{t}}(N \times \{\pm 1\}) \subseteq \bar{M}_\pm$.

We assume Setup 3.3 in the following. Fix an odd function $\chi_0: \mathbb{R} \rightarrow [-1, 1]$ with $\chi_0(\pm t) = \pm 1$ for $t \geq 1$. Then we define a function $\chi: \bar{M} \rightarrow [-1, 1]$ by

$$\chi(\bar{x}) = \begin{cases} \chi_0(t) & \bar{x} = \bar{\mathfrak{t}}(y, t) \in \bar{\mathfrak{t}}(N \times [-1, 1]), \\ \pm 1 & \bar{x} \in \bar{M}_\pm \setminus \bar{\mathfrak{t}}(N \times [-1, 1]). \end{cases} \tag{3.4}$$

Let $\tilde{\chi}: \tilde{M} \rightarrow [-1, 1]$ denote the pullback of χ to \tilde{M} .

LEMMA 3.5. *Let $K \subseteq \tilde{M}$ be a compact subset. Then there are only finitely many different cosets $\gamma\pi \in \Gamma/\pi$ such that*

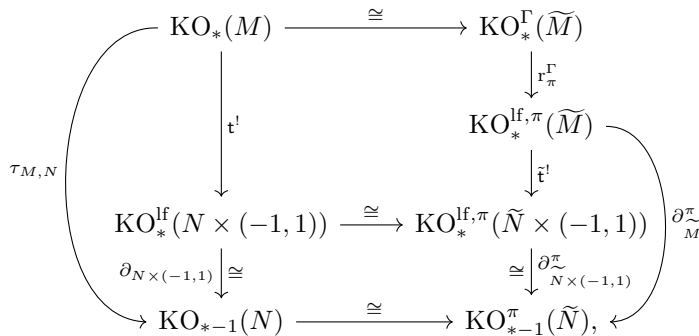
$$\sup_{\tilde{x}_1, \tilde{x}_2 \in K} |\tilde{\chi}(\gamma^{-1}\tilde{x}_1) - \tilde{\chi}(\gamma^{-1}\tilde{x}_2)| \neq 0.$$

Proof. Since \tilde{M} is a path-connected CW complex, every compact subset is contained in a path-connected compact subset. Hence we may assume without loss of generality that K is path-connected. By construction, the set $\bar{\mathfrak{t}}(\tilde{N} \times [-1, 1])$ is π -compact. Thus, since the action of Γ on \tilde{M} is proper, there are only finitely many $\gamma\pi \in \Gamma/\pi$ such that $\gamma^{-1}K \cap \bar{\mathfrak{t}}(\tilde{N} \times [-1, 1])$ is non-empty. The set $\tilde{M} \setminus \bar{\mathfrak{t}}(\tilde{N} \times [-1, 1])$ consists of two connected components on each of which the function $\tilde{\chi}$ is constant. Using that K is connected, we conclude that for all but finitely many cosets $\gamma\pi \in \Gamma/\pi$, the set $\gamma^{-1}K$ is contained in one of the components. Hence the function $\tilde{\chi}$ is constant on $\gamma^{-1}K$ for all but finitely $\gamma\pi \in \Gamma/\pi$. This proves the lemma. \square

In the following, we work with equivariant KK-theory for Real C*-algebras. We use the following picture of $\text{KKO}_{-1}^\Gamma(A, B)$ for separable Real Γ -C*-algebras A and B (compare [3, 20.2 and 17.5.2]): elements are represented by triples (E, ϕ, T) , where E is a countably generated Real Hilbert B -module with an action of Γ , ϕ is an equivariant representation of A on E by adjointable operators, T is a self-adjoint operator on E such that $[\phi(a), T]$, $\phi(a)(T^2 - 1)$ and $\phi(a)(g \cdot F - F)$ are compact operators (in the Hilbert B -module sense) for all $a \in A$ and $g \in \Gamma$.

The collapse induces a map $\partial_M^\pi: \text{KO}_*^{\text{lf}, \pi}(\tilde{M}) \xrightarrow{\mathfrak{t}^!} \text{KO}_*^{\text{lf}, \pi}(\tilde{N} \times (-1, 1)) \cong \text{KO}_{*-1}^\pi(\tilde{N})$, where $\mathfrak{t}^!$ is induced from the inclusion $C_0(\tilde{N} \times (-1, 1)) \hookrightarrow C_0(\tilde{M})$ coming from the embedding $\mathfrak{t}: N \times \mathbb{R} \hookrightarrow M$. The map ∂_M^π is given by Kasparov product with an element in $\text{KKO}_{-1}^\pi(C_0(\tilde{N}), C_0(\tilde{M}))$, which we will also denote by ∂_M^π and which can be explicitly described as follows: Let $T_0 := 1_{\tilde{N}} \otimes \chi_0$. This represents a π -equivariant self-adjoint multiplier of $C_0(\tilde{N} \times (-1, 1))$ with $f \otimes 1 (T_0^2 - 1) \in C_0(\tilde{N} \times (-1, 1))$ for all $f \in C_0(\tilde{N})$. Thus T_0 represents a class $\partial_{\tilde{N} \times (-1, 1)}^\pi \in \text{KKO}_{-1}^\pi(C_0(\tilde{N}), C_0(\tilde{N} \times (-1, 1)))$. Then the desired class is $\partial_M^\pi = \mathfrak{t}_! \partial_{\tilde{N} \times (-1, 1)}^\pi$.

Remark 3.6. We have the commutative diagram



where \mathfrak{r}_π^Γ is the forgetful map restricting equivariance to π and $\partial_{N \times (-1, 1)}^\pi$ is defined similarly as $\partial_{\widetilde{N} \times (-1, 1)}^\pi$. The vertical composition on the left-hand side is by definition the map $\tau_{M,N}$ which is induced by the Pontryagin–Thom collapse. Next we discuss induction (compare [10, §5]). Let A be a π - C^* -algebra. Then the induced Γ - C^* -algebra $\text{Ind}_\pi^\Gamma A$ consists of all those functions $f \in C_b(\Gamma, A)$ such that $f(\gamma h) = h^{-1}f(\gamma)$ for all $\gamma \in \Gamma$, $h \in \pi$ and the function $\Gamma/\pi \rightarrow \mathbb{R}$, $\gamma\pi \mapsto \|f(\gamma)\|$ vanishes at infinity. For two π - C^* -algebras there is Kasparov’s induction homomorphisms

$$i_\pi^\Gamma: \text{KKO}^\pi(A, B) \rightarrow \text{KKO}^\Gamma(\text{Ind}_\pi^\Gamma A, \text{Ind}_\pi^\Gamma B).$$

Note that if $A = C_0(Z)$ for some proper locally compact π -space Z , then $\text{Ind}_\pi^\Gamma A = C_0(\Gamma \times_\pi Z)$. If Z is a π -invariant subspace of a proper locally compact Γ -space W , then there is a proper Γ -equivariant map $\Gamma \times_\pi Z \rightarrow W$ which takes $[\gamma, z]$ to $\gamma \cdot z \in W$. Composing this map with induction yields a map

$$\begin{aligned}
 \text{Ind}_\pi^\Gamma: \text{KKO}^\pi(C_0(Z), B) &\xrightarrow{i_\pi^\Gamma} \text{KKO}^\Gamma(C_0(\Gamma \times_\pi Z), \text{Ind}_\pi^\Gamma B) \\
 &\rightarrow \text{KKO}^\Gamma(C_0(W), \text{Ind}_\pi^\Gamma B).
 \end{aligned}
 \tag{3.7}$$

If $\underline{E}\Gamma$ is a universal space for proper Γ -actions, then it is also a universal space for proper π -actions if endowed with the restricted action. Moreover, every π -invariant π -compact subset $Z \subseteq \underline{E}\Gamma$ is contained in a Γ -invariant Γ -compact subset $W \subseteq \underline{E}\Gamma$. Thus passing to the colimit over all such subsets $Z \subseteq \underline{E}\Gamma$ yields the induction map

$$\text{Ind}_\pi^\Gamma: \text{KO}_*^\pi(\underline{E}\Gamma; B) \rightarrow \text{KO}_*^\Gamma(\underline{E}\Gamma; \text{Ind}_\pi^\Gamma B). \tag{3.8}$$

THEOREM 3.9 ([5, 17]). *The induction map (3.8) is an isomorphism.*

Proof. For complex K-homology, this result is due to Oyono–Oyono [17]. See also Chabert–Echterhoff [5] for a generalization to the case of locally compact groups. The real case can be obtained from the complex case via the technique of [19]. \square

After this interlude on induction we return to our setup:

DEFINITION 3.10. Assume Setup 3.3. We let $\partial_{\Gamma/\pi} \in \text{KKO}_{-1}^{\Gamma}(\mathbb{R}, C_0(\Gamma/\pi))$ be the class defined as follows. Let $i: \Gamma/\pi \hookrightarrow \widetilde{M}$, $i(\gamma\pi) := \underline{p}(\gamma^{-1}\tilde{x}_0) \in \widetilde{M} = \pi \backslash \widetilde{M}$. Set $\chi_{\Gamma/\pi}: \Gamma/\pi \rightarrow [-1, 1]$, $\chi_{\Gamma/\pi}(z) = \chi(i(z))$, where χ is defined in (3.4). Then $\chi_{\Gamma/\pi}$ is a self-adjoint multiplier of $C_0(\Gamma/\pi)$ which satisfies $\chi_{\Gamma/\pi}^2 = 1$ modulo $C_0(\Gamma/\pi)$. Furthermore, $\chi_{\Gamma/\pi}$ is Γ -invariant modulo $C_0(\Gamma/\pi)$: For each fixed $g \in \Gamma$, Lemma 3.5 with $K := \{\tilde{x}_0, g \cdot \tilde{x}_0\}$ implies that the function $z \mapsto \chi_{\Gamma/\pi}(z) - \chi_{\Gamma/\pi}(g^{-1} \cdot z)$ has finite support. Thus $\chi_{\Gamma/\pi}$ represents a class $\partial_{\Gamma/\pi} \in \text{KKO}_{-1}^{\Gamma}(\mathbb{R}, C_0(\Gamma/\pi))$.

If W is a proper Γ -space, then we in fact have a Γ -equivariant homeomorphism

$$\Gamma/\pi \times W \cong \Gamma \times_{\pi} W, \quad (\gamma\pi, x) \mapsto [\gamma, \gamma^{-1}x].$$

In particular, in Setup 3.3, there is a canonical isomorphism of Γ - C^* -algebras

$$\text{Ind}_{\pi}^{\Gamma}(C_0(\widetilde{M})) = C_0(\Gamma \times_{\pi} \widetilde{M}) \cong C_0(\Gamma/\pi \times \widetilde{M}) = C_0(\Gamma/\pi) \otimes C_0(\widetilde{M}). \quad (3.11)$$

PROPOSITION 3.12. Assume Setup 3.3. Let $\partial_{\Gamma/\pi} \in \text{KKO}_{-1}^{\Gamma}(\mathbb{R}, C_0(\Gamma/\pi))$ be as in Definition 3.10. Then

$$\text{Ind}_{\pi}^{\Gamma}(\partial_{\widetilde{M}}^{\pi}) = \partial_{\Gamma/\pi} \otimes_{\mathbb{R}} 1_{C_0(\widetilde{M})} \in \text{KKO}_{-1}^{\Gamma}(C_0(\widetilde{M}), C_0(\Gamma/\pi) \otimes C_0(\widetilde{M})).$$

This identity implicitly uses the canonical isomorphism (3.11) applied to the right-hand side argument of KKO as well as the map $C_0(\widetilde{M}) \rightarrow \text{Ind}_{\pi}^{\Gamma}(C_0(\widetilde{N}))$ used in (3.7) on the left-hand side.

Proof. Recall that $\partial_{\widetilde{M}}^{\pi}$ is represented by the operator $T_0 := 1_{\widetilde{N}} \otimes \chi_0$ acting as a multiplier of $C_0(\widetilde{N} \times (-1, 1))$. Then $\text{Ind}_{\pi}^{\Gamma} \partial_{\widetilde{M}}^{\pi} \in \text{KKO}_{-1}^{\Gamma}(C_0(\widetilde{M}), \text{Ind}_{\pi}^{\Gamma} C_0(\widetilde{M}))$ is represented by the triple

$$\Xi := (C_0(\Gamma \times_{\pi} \widetilde{N} \times (-1, 1)), \phi, 1_{\Gamma \times_{\pi} \widetilde{N}} \otimes \chi_0),$$

where we view $C_0(\Gamma \times_{\pi} \widetilde{N} \times (-1, 1))$ as a Hilbert $C_0(\Gamma \times_{\pi} \widetilde{M})$ -module via the inclusion $\Gamma \times_{\pi} \mathfrak{t}_l: C_0(\Gamma \times_{\pi} \widetilde{N} \times (-1, 1)) \hookrightarrow C_0(\Gamma \times_{\pi} \widetilde{M})$ and ϕ is the multiplication representation given as the composition

$$\phi: C_0(\widetilde{M}) \rightarrow C_0(\Gamma \times_{\pi} \widetilde{N}) \rightarrow C_b(\Gamma \times_{\pi} \widetilde{N} \times (-1, 1)),$$

where the first map is induced by $\Gamma \times_{\pi} \widetilde{N} \rightarrow \widetilde{M}$, $(\gamma, y) \mapsto \gamma \cdot \mathfrak{t}(\tilde{y}, 0)$.

We first observe that we may change the representation ϕ by a straightforward homotopy to

$$\phi': C_0(\widetilde{M}) \xrightarrow{\psi} C_b(\Gamma \times_{\pi} \widetilde{M}) \xrightarrow{(\Gamma \times_{\pi} \mathfrak{t})^* \otimes 1} C_b(\Gamma \times_{\pi} \widetilde{N} \times (-1, 1)),$$

where the first map is induced by $\Gamma \times_{\pi} \widetilde{M} \rightarrow \widetilde{M}$, $(\gamma, x) \mapsto \gamma \cdot x$. Then the triple

$$\Xi' := (C_0(\Gamma \times_{\pi} \widetilde{N} \times (-1, 1)), \phi', 1_{\Gamma \times_{\pi} \widetilde{N}} \otimes \chi_0),$$

represents the same class as Ξ . Next the triple Ξ' may be extended to

$$\Xi'' := (C_0(\Gamma \times_\pi \widetilde{M}), \psi, 1_\Gamma \otimes \tilde{\chi}),$$

where $(1_\Gamma \otimes \tilde{\chi})(\gamma, x) = \tilde{\chi}(x)$. This still represents the same KK-class because $\tilde{\chi}$ agrees with $1 \otimes \chi_0$ on $\widetilde{N} \times (-1, 1)$ and is invertible outside. Finally, applying the homeomorphism $\Gamma \times_\pi \widetilde{M} \cong \Gamma/\pi \times \widetilde{M}$, we see that Ξ'' is isomorphic to

$$\Xi''' := (C_0(\Gamma/\pi) \otimes C_0(\widetilde{M}), 1 \otimes \mu, \omega),$$

where $\omega: \Gamma/\pi \times \widetilde{M} \rightarrow [-1, 1]$, $\omega(\gamma\pi, x) = \tilde{\chi}(\gamma^{-1}x)$ and μ is simply the multiplication representation of $C_0(\widetilde{M})$ on itself. To be more precise, the isomorphism between Ξ'' and Ξ''' is an isomorphism of Kasparov modules over the isomorphism of Γ - C^* -algebras (3.11) in the right-hand side argument.

The element $\partial_{\Gamma/\pi} \otimes_{\mathbb{R}} 1_{C_0(\widetilde{M})}$ is represented by the triple

$$\Upsilon := (C_0(\Gamma/\pi) \otimes C_0(\widetilde{M}), 1 \otimes \mu, \chi_{\Gamma/\pi} \otimes 1).$$

We claim that Ξ''' and Υ represent the same KK-class. Indeed, for each compactly supported element $f \in C_0(\widetilde{M})$, Lemma 3.5 implies that

$$(\gamma\pi, x) \mapsto f(x)(\tilde{\chi}(\gamma^{-1}x) - \tilde{\chi}(\gamma^{-1}x))$$

is compactly supported on $\Gamma/\pi \times \widetilde{M}$. Thus $1 \otimes \mu(f)(1 \otimes \chi_{\Gamma/\pi} - \omega) \in C_0(\Gamma/\pi) \otimes C_0(\widetilde{M})$ for each $f \in C_0(\widetilde{M})$. This proves the claim and hence finishes the proof of the proposition. \square

LEMMA 3.13. *Let $\pi \leq \Gamma$ be a subgroup of a discrete group Γ . Let $\xi \in \text{KKO}_{-k}^\Gamma(\mathbb{R}, C_0(\Gamma/\pi))$. Then we have a commutative diagram,*

$$\begin{array}{ccc} \text{KO}_*^\Gamma(\underline{E}\Gamma) & \xrightarrow{\mu^\Gamma} & \text{KO}_*(C^*\Gamma) \\ \downarrow \xi & & \downarrow j_\Gamma(\xi) \\ \text{KO}_{*-k}^\Gamma(\underline{E}\Gamma, C_0(\Gamma/\pi)) & \xrightarrow{\mu_{C_0(\Gamma/\pi)}^\Gamma} & \text{KO}_{*-k}(C_0(\Gamma/\pi) \rtimes \Gamma) \\ \text{Ind}_\pi^\Gamma \uparrow & & \cong \uparrow \\ \text{KO}_{*-k}^\pi(\underline{E}\Gamma) & \xrightarrow{\mu^\pi} & \text{KO}_{*-k}(C^*\pi) \end{array}$$

where $j_\Gamma(\xi) \in \text{KKO}_{-k}(C^*\Gamma, C_0(\Gamma/\pi) \rtimes \Gamma)$ is the descent homomorphism applied to ξ . This holds for either reduced group C^* -algebras and crossed products as well as for maximal ones.

Proof. The fact that the bottom square in the diagram exists and commutes is a standard fact on the induction isomorphism, see for instance [5, Proposition 2.3].

To see that the top square commutes, recall the definition of the Baum–Connes assembly map: Let $X \subseteq \underline{E}\Gamma$ be a Γ -compact subset and $x \in \text{KKO}_*^\Gamma(C_0(X), \mathbb{R})$.

Then $j_\Gamma(x) \in \text{KKO}_*(C_0(X) \rtimes \Gamma, C^*\Gamma)$ and $\mu^\Gamma(x) = [p_{X,\Gamma}] \otimes_{C_0(X) \rtimes \Gamma} j_\Gamma(x)$, where $[p_{X,\Gamma}] \in \text{KKO}(\mathbb{R}, C_0(X) \rtimes \Gamma)$ is the K-theory class determined by a cutoff function for (X, Γ) . Similarly, $\mu_{C_0(\Gamma/\pi)}^\Gamma(x \otimes_{\mathbb{R}} \xi) = [p_{X,\Gamma}] \otimes_{C_0(X) \rtimes \Gamma} j_\Gamma(x \otimes_{\mathbb{R}} \xi)$. Moreover, since Kasparov’s descent homomorphism is compatible with the composition product (see [10, §6, Theorem 1]), we have $j_\Gamma(x \otimes_{\mathbb{R}} \xi) = j_\Gamma(x) \otimes_{C^*\Gamma} j_\Gamma(\xi)$. Thus

$$\begin{aligned} \mu_{C_0(\Gamma/\pi)}^\Gamma(x \otimes_{\mathbb{R}} \xi) &= [p_{X,\Gamma}] \otimes_{C_0(X) \rtimes \Gamma} j_\Gamma(x \otimes_{\mathbb{R}} \xi) \\ &= [p_{X,\Gamma}] \otimes_{C_0(X) \rtimes \Gamma} j_\Gamma(x) \otimes_{C^*\Gamma} j_\Gamma(\xi) \\ &= \mu^\Gamma(x) \otimes_{C^*\Gamma} j_\Gamma(\xi). \end{aligned} \quad \square$$

LEMMA 3.14. *Let $\partial_{\Gamma/\pi}$ be as in Proposition 3.12. Then the following diagram commutes:*

$$\begin{array}{ccccccc} \text{KO}_*(M) & \xrightarrow{\cong} & \text{KO}_*^\Gamma(\widetilde{M}) & \xlongequal{\quad} & \text{KKO}_*^\Gamma(C_0(\widetilde{M}), \mathbb{R}) & \xrightarrow{\quad} & \text{KO}_*^\Gamma(\underline{E}\Gamma) \\ \downarrow \tau_{M,N} & & \downarrow r_\pi^\Gamma & & \downarrow -\otimes_{\mathbb{R}} \partial_{\Gamma/\pi} & & \downarrow -\otimes_{\mathbb{R}} \partial_{\Gamma/\pi} \\ & & \text{KO}_*^{\text{lf},\pi}(\widetilde{M}) \otimes & \text{KKO}_{*-1}^\Gamma(C_0(\widetilde{M}), C_0(\Gamma/\pi)) & \longrightarrow & \text{KO}_{*-1}^\Gamma(\underline{E}\Gamma; C_0(\Gamma/\pi)) \\ & & \downarrow \partial_M^\pi & \uparrow \text{Ind}_\pi^\Gamma & & \uparrow \text{Ind}_\pi^\Gamma \\ \text{KO}_{*-1}(N) & \xrightarrow{\cong} & \text{KO}_{*-1}^\pi(\widetilde{N}) & \xlongequal{\quad} & \text{KKO}_{*-1}^\pi(C_0(\widetilde{N}), \mathbb{R}) & \xrightarrow{\quad} & \text{KO}_{*-1}^\pi(\underline{E}\Gamma). \end{array}$$

Proof. The rectangle on the left-hand side was already discussed in Remark 3.6. The commutativity of the squares on the right-hand side is due to naturality of the Kasparov product and the induction map. It remains to show that the rectangle \otimes commutes. Indeed, let $\xi \in \text{KKO}_*^\Gamma(C_0(\widetilde{M}), \mathbb{R})$. Then by [10, §5, Theorem 1] and Proposition 3.12, we obtain

$$\begin{aligned} \text{Ind}_\pi^\Gamma \left(\partial_{\widetilde{M}}^\pi \otimes_{C_0(\widetilde{M})} r_\pi^\Gamma(\xi) \right) &= \text{Ind}_\pi^\Gamma(\partial_{\widetilde{M}}^\pi) \otimes_{\text{Ind}_\pi^\Gamma(C_0(\widetilde{M}))} i_\pi^\Gamma r_\pi^\Gamma(\xi) \\ &= \left(\partial_{\Gamma/\pi} \otimes_{\mathbb{R}} 1_{C_0(\widetilde{M})} \right) \otimes_{C_0(\Gamma/\pi) \otimes C_0(\widetilde{M})} (1_{C_0(\Gamma/\pi)} \otimes_{\mathbb{R}} \xi) \\ &= \partial_{\Gamma/\pi} \otimes_{\mathbb{R}} \xi = \xi \otimes_{\mathbb{R}} \partial_{\Gamma/\pi}. \end{aligned}$$

We have again implicitly used the isomorphism (3.11). □

THEOREM 3.15. *Suppose Setup 3.3. Then we have a commutative diagram:*

$$\begin{array}{ccccc} \text{KO}_*(M) & \longrightarrow & \text{KO}_*^\Gamma(\underline{E}\Gamma) & \longrightarrow & \text{KO}_*(C^*\Gamma) \\ \downarrow & & \downarrow & & \downarrow \\ \text{KO}_{*-1}(N) & \longrightarrow & \text{KO}_{*-1}^\pi(\underline{E}\pi) & \longrightarrow & \text{KO}_{*-1}(C^*\pi). \end{array}$$

This holds for the reduced as well as maximal group C-algebras. Moreover, this diagram is natural in the data of Setup 3.3 for fixed subgroup inclusions $\pi \leq \Gamma$.*

Proof. The left square follows from Lemma 3.14 and the fact that the induction map $\text{Ind}_\pi^\Gamma: \text{KO}_*^\pi(\underline{E}\Gamma) \rightarrow \text{KO}_*^\Gamma(\underline{E}\Gamma, C_0(\Gamma/\pi))$ is an isomorphism (see Theorem 3.9). The right square follows from Lemma 3.13.

Next we explain naturality. Let $\pi \leq \Gamma$ be fixed and let $\mathfrak{t}: N \times \mathbb{R} \hookrightarrow M$ and $\mathfrak{t}': N' \times \mathbb{R} \hookrightarrow M'$ both as in Setup 3.3. Suppose that there are continuous maps $f_M: M \rightarrow M'$ and $f_N: N \rightarrow N'$ such that $f_M \circ \mathfrak{t} = \mathfrak{t}' \circ (f_N \times \text{id})$ and f_M induces the identity on π_1 . Then f_M and f_N induce equivariant maps $f_{\widetilde{M}}$ and $f_{\widetilde{N}}$ between the corresponding covering spaces and hence equivariant $*$ -homomorphisms between the associated function algebras. By abuse of notation, we also denote the resulting KK-elements by $f_{\widetilde{M}} \in \text{KKO}_0^\pi(C_0(\widetilde{M}'), C_0(\widetilde{M}))$ and $f_{\widetilde{N}} \in \text{KKO}_0^\pi(C_0(\widetilde{N}'), C_0(\widetilde{N}))$. The fact that the Mayer–Vietoris boundary classes are natural can then be expressed as:

$$f_{\widetilde{N}} \otimes_{C_0(\widetilde{N})} \partial_{\widetilde{M}}^\pi = \partial_{\widetilde{M}'}^\pi \otimes_{C_0(\widetilde{M}')} f_{\widetilde{M}}. \tag{3.16}$$

By construction, the functions $\tilde{\chi}: \widetilde{M} \rightarrow \mathbb{R}$ and $\tilde{\chi}': \widetilde{M}' \rightarrow \mathbb{R}$ satisfy $\tilde{\chi}' \circ f_{\widetilde{M}} = \tilde{\chi}$. This shows that the element $\partial_{\Gamma/\pi}$ from Definition 3.10 is in fact the same for \widetilde{M} and \widetilde{M}' . This together with (3.16) implies that the commutative diagram is natural in this situation. \square

Remark 3.17. With little extra work, the naturality assertion of Theorem 3.15 extends to the case where the groups change. We leave the details to the reader.

Proof of Theorem 3.1. We first observe that the pair (M, N) precisely fits into Setup 3.3. Thus Theorem 3.15 already yields the resulting diagram except for the two transformations in the middle which involve the spin bordism and KO-homology of $B\Gamma$. To construct these, we will see that $B\pi$ and $B\Gamma$ can be constructed in such a way that $B\pi$ is a subspace of $B\Gamma$ and Setup 3.3 becomes applicable. This uses the special structure of Γ in this codimension 1 situation. We have two cases:

The first case is that N separates M into two connected components. That is, $M = M_+ \cup_N M_-$, with codimension 0 submanifolds M_+, M_- , both with boundary N . Set

$$\Gamma_+ := \text{im}(\pi_1(M_+) \rightarrow \pi_1(M)) \subset \Gamma; \quad \Gamma_- := \text{im}(\pi_1(M_-) \rightarrow \pi_1(M)) \subset \Gamma.$$

By the van Kampen theorem we then have an amalgamated free product $\Gamma = \Gamma_+ *_\pi \Gamma_-$.

The second case is that N is non-separating. Then let W be the manifold with boundary obtained by cutting M open along N . W has two boundary pieces N_+, N_- both identified with N , and we have $M = W / \sim$, where \sim is the equivalence relation identifying N_+ with N_- . Let $H := \text{im}(\pi_1(W) \rightarrow \pi_1(M)) \subset \Gamma$ be the image under the collapse map, where we place the basepoint in N_+ . The inclusion $N = N_+ \hookrightarrow W$ then induces an inclusion $\pi \hookrightarrow H$ (injective because $\pi \subset \Gamma$). Let $\gamma: [0, 1] \rightarrow W$ be any path from the basepoint in N_+ to the corresponding point in N_- ; conjugation with it allows to define

a second embedding (as $\gamma^{-1}\pi\gamma \subset \Gamma$) induced by the inclusion $N = N_- \subset W$. The van Kampen theorem now implies that we get Γ as an HNN extension $\Gamma = H *_{\pi=\gamma^{-1}\pi\gamma}$, see e.g. [21, Proposition 1.2]. Then one can construct a model of $B\Gamma$ out of $B\pi$ and $B\Gamma_+$, $B\Gamma_-$ and BH . In the separating case, we have

$$B\Gamma = B\Gamma_+ \cup_{B\pi \times \{-1\}} B\pi \times [-1, 1] \cup_{B\pi \times \{1\}} B\Gamma_-,$$

where we construct $B\Gamma_+, B\Gamma_-$ such that they contain copies of $B\pi$ which induce the inclusion maps on fundamental groups. More specifically, one can construct the classifying spaces in question as CW-complexes starting with M : first attach cells to N to construct $B\pi$. Taking the product with $[-1, 1]$ and glueing it into M along a trivialization of a tubular neighborhood of N in M produces $M \cup_{N \times [-1, 1]} B\pi \times [-1, 1]$. Now, attach further cells to $M_+ \cup_{N \times \{1\}} B\pi$ to obtain $B\Gamma_+$ and to $M_- \cup_{N \times \{-1\}}$, to obtain $B\Gamma_-$. Here, for convenience we slightly change notation and write $M = M_+ \cup N \times [-1, 1] \cup M_-$, glued along the two boundary components of $N \times [-1, 1]$.

The construction in the second case, where N is not separating, is precisely the same. We then use the fact that the classifying space of an HNN-extension $H *_{\pi=\gamma^{-1}\pi\gamma}$ can be obtained from a classifying space of H by glueing in $B\pi \times [-1, 1]$, attaching the two ends according to the two inclusions of π and $\gamma^{-1}\pi\gamma$ into H ; and this can be modelled exactly as before, starting with $N \times [-1, 1] \subset M$.

In either case, we obtain a diagram of embeddings

$$\begin{array}{ccc} N \times [-1, 1] & \hookrightarrow & B\pi \times [-1, 1] \\ \downarrow & & \downarrow \\ M & \hookrightarrow & B\Gamma. \end{array}$$

By definition, the collapse map $M \rightarrow \Sigma N$ maps every point inside $N \times (-1, 1)$ of N to the corresponding point in $N \times (-1, 1) \subset \Sigma N$, and every point outside this tubular neighborhood to the (collapsed) base point in ΣN . By applying the same construction to $B\pi \times (-1, 1) \subset B\Gamma$ we get a map $B\Gamma \rightarrow \Sigma B\pi$ and a commutative diagram

$$\begin{array}{ccc} M & \hookrightarrow & B\Gamma \\ \downarrow & & \downarrow \\ \Sigma N & \longrightarrow & \Sigma B\pi. \end{array}$$

From this we obtain the commutative diagram

$$\begin{array}{ccccc} \Omega_*^{\text{spin}}(M) & \xrightarrow{c_*} & \Omega_*^{\text{spin}}(B\Gamma) & \longrightarrow & \text{KO}_*(B\Gamma) \\ \downarrow \tau_{M,N} & & \downarrow \tau_{\Gamma,\pi} & & \downarrow \tau_{\Gamma,\pi} \\ \Omega_{*-1}^{\text{spin}}(N) & \xrightarrow{\bar{c}_*} & \Omega_{*-1}^{\text{spin}}(B\pi) & \longrightarrow & \text{KO}_{*-1}(B\pi), \end{array} \tag{3.18}$$

which consists precisely of the first three columns in (3.2).

Finally, if the space $B\Gamma$ constructed above is compact (that is, we only had to add finitely many cells), then we are in Setup 3.3. The diagram (3.2) is completed by combining (3.18) with the diagram from Theorem 3.15 (for $M = B\Gamma$ and $N = B\pi$).

In the general case, where $B\Gamma$ is non-compact, $KO_*(B\Gamma)$ is identified with the colimit of $KO_*(X)$, where X runs over compact subsets of $B\Gamma$. Then the colimit can be restricted to the directed set of those compact subsets X such that $X \cap (B\pi \times [-1, 1]) = Y \times [-1, 1]$ for some compact $Y \subseteq B\pi$. Since $\Gamma = \pi_1(M)$ is finitely presented and $\pi = \text{im}(\pi_1(N) \rightarrow \pi_1(M))$ is finitely generated, we can arrange it so that we can further restrict the colimit to those pairs (X, Y) with $\pi_1(X) = \Gamma$ and $\pi_1(Y) \twoheadrightarrow \pi$. Then we may apply Theorem 3.15 to each of these pairs (X, Y) and the naturality statement of Theorem 3.15 implies that this passes to the colimit and fits together with (3.18), thereby completing (3.2). \square

4 GENERAL TRANSFER CONSTRUCTION

In this section, we describe a general transfer construction which is used in the proof of the remaining results. Suppose that we are in the situation of Setup 2.1 for some codimension k . Let E be a multiplicative equivariant generalized homology theory (see, e.g., [14]). Assume that ν is E -oriented and let $\theta \in E^k(D\nu, S\nu)$ be the corresponding Thom class. Then the *classical transfer map* (see [18, Chapter V.2]) is given by

$$\tau_\theta : E_*(M) \rightarrow E_*(M, M \setminus D\nu^\circ) \cong E_*(D\nu, S\nu) \xrightarrow{\cap \theta} E_{*-k}(N).$$

If M and N are E -oriented manifolds, then —fixing orientations on M and N — we obtain an orientation on the normal bundle such that the corresponding transfer sends the fundamental class of M to the fundamental class of N .

It is our goal to find conditions under which τ_θ can be extended to the classifying spaces for free actions, or even the classifying spaces for proper actions, such that the following diagram commutes, where the horizontal arrows are induced by classifying maps:

$$\begin{array}{ccccc} E_*(M) & \longrightarrow & E_*(B\Gamma) & \longrightarrow & E_*^\Gamma(\underline{E}\Gamma) \\ \downarrow \tau_\theta & & \downarrow & & \downarrow \\ E_{*-k}(N) & \longrightarrow & E_{*-k}(B\pi) & \longrightarrow & E_{*-k}^\pi(\underline{E}\pi). \end{array}$$

To do so we will work in the equivariant setting. Conceptually, our method uses “locally finite equivariant generalized homology theories”. But there does not seem to be an established set of axioms for this concept, and it would take us too far to introduce one. Instead, we give an ad-hoc definition for an “lf-restriction” operation (the “lf” stands for locally finite).

DEFINITION 4.1. Let Γ be a countable discrete group, let π be a subgroup, and let Y be a proper Γ -CW-complex. A π -invariant subspace $K \subset Y$ is called Γ -locally π -precompact if for every Γ -cell $X \subset Y$ the intersection $K \cap X$ is contained in only finitely many of the π -cells that together form X .

DEFINITION 4.2. A generalized equivariant homology theory E has *lf-restrictions* if for every inclusion of groups $\pi < \Gamma$, for every proper Γ -CW-complex Y and for every Γ -locally π -precompact subspace $K \subset Y$ such that $(Y, Y \setminus K)$ is a pair of π -CW-complexes, there is a homomorphism

$$r_K : E_*^\Gamma(Y) \rightarrow E_*^\pi(Y, Y \setminus K).$$

These maps must be natural with respect to cellular equivariant maps of CW-pairs, and they must be compatible with the induction isomorphisms in the following sense: Suppose that Γ acts freely on Y , that \overline{K} is a π -subcomplex of Y , and that $\Gamma \setminus \Gamma \cdot \overline{K} = \pi \setminus \overline{K}$. The latter condition means $\gamma \cdot \overline{K} \cap \overline{K} \neq \emptyset$ is possible only for $\gamma \in \pi \subset \Gamma$. Then the following diagram makes sense

$$\begin{array}{ccccc} E_*^\Gamma(Y) & \xrightarrow{r_K} & E_*^\pi(Y, Y \setminus K) & \xrightarrow[\text{exc}]{\cong} & E_*^\pi(\overline{K}, \overline{K} \cap (Y \setminus K)) \\ \text{ind} \downarrow \cong & & & & \text{ind} \downarrow \cong \\ E_*(\Gamma \setminus Y) & \longrightarrow & E_*(\Gamma \setminus Y, (\Gamma \setminus Y) \setminus (\Gamma \setminus \Gamma \cdot K)) & \xrightarrow[\text{exc}]{\cong} & E_*(\pi \setminus \overline{K}, \pi \setminus (\overline{K} \cap (Y \setminus K))) \end{array}$$

where *ind* denotes induction maps, *exc* denotes excision, and the bottom left map is inclusion induced. We require that the diagram commutes.

NOTATION 4.3. In Setup 2.1 we assume that $D\nu$, $M \setminus D\nu^\circ$ and $S\nu$ are subcomplexes of M , and that the Γ -CW-structure on \widetilde{M} is the one lifted from M .

We will show in Section 4.1 that real K-homology, the most relevant example for E_* , has lf-restrictions.

Assume now that E_* is a generalized equivariant multiplicative homology theory with lf-restrictions, $\pi < \Gamma$ and Y is a proper Γ -space. Given an element $\theta \in E_\pi^k(Y, Y \setminus K)$, with K Γ -locally π -precompact, we define the equivariant transfer map

$$\tau_{\theta, K}^{\text{eq}} : E_*^\Gamma(Y) \rightarrow E_{*-k}^\pi(Y), \quad y \mapsto r_K(y) \cap \theta.$$

Because both the cap product and the lf-restriction are natural with respect to maps of pairs $(Y, Y \setminus K)$, the same is true for $\tau_{\theta, K}^{\text{eq}}$. In particular, if $i : Y \setminus K' \rightarrow Y \setminus K$ is the inclusion of the complement of a larger Γ -locally π -precompact subspace K' , then $\tau_{i^*(\theta), K'}^{\text{eq}} = \tau_{\theta, K}^{\text{eq}}$. Thus, an element $\theta \in \text{colim}_K E_\pi^*(Y, Y \setminus K)$ defines a map τ_θ^{eq} , where the colimit runs over all π -subcomplexes $Y \setminus K$ with Γ -locally π -precompact complement.

If $f : X \rightarrow Y$ is a Γ -equivariant cellular map between proper Γ -CW-complexes, then for any Γ -locally π -precompact subset $K \subset Y$ one can form the subcomplex $L \subset X$ consisting of all cells whose image under f does not intersect K . It follows directly from Definition 4.1 that $X \setminus L$ is Γ -locally π -precompact.

This means that the elements in $\operatorname{colim}_K E_\pi^*(Y, Y \setminus K)$ can be pulled back to X via f . Furthermore, the pullback only depends on the homotopy class of f . Indeed, if $H: X \times [0, 1] \rightarrow Y$ is a homotopy from f to f' , we can restrict ourselves to the smaller subcomplex $L_H \subset X$ of all cells whose image under f does not intersect K during the whole homotopy. Then $X \setminus L_H$ is still Γ -locally π -precompact.

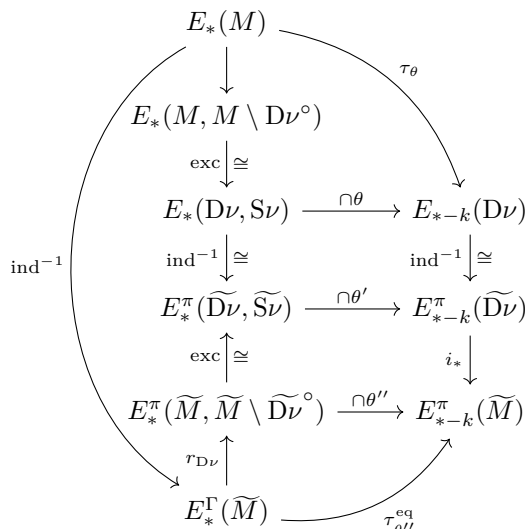
We now establish a general-purpose result that will allow us to extend the classical transfer map in many cases.

THEOREM 4.4. *Assume Setup 2.1. Let $\theta \in E^k(D\nu, S\nu) \cong E_\pi^k(\widetilde{D\nu}, \widetilde{S\nu}) \cong E_\pi^k(\widetilde{M}, \widetilde{M} \setminus \widetilde{D\nu}^\circ)$ and let $\tau_\theta: E_*(M) \rightarrow E_{*-k}(N)$ be the associated classical transfer map. Consider the following conditions:*

- (1) *There exist Γ -locally π -precompact subspaces $K \supset \widetilde{D\nu}$ and $K_E \subset E\Gamma$ such that the classifying map $\widetilde{M} \rightarrow E\Gamma$ is a map of π -CW-pairs $(\widetilde{M}, \widetilde{M} \setminus K) \rightarrow (E\Gamma, E\Gamma \setminus K_E)$, and such that the restriction $i_{D\nu, K}^*(\theta) \in E_\pi^k(\widetilde{M}, \widetilde{M} \setminus K)$ can be lifted to $\theta_E \in E_\pi^k(E\Gamma, E\Gamma \setminus K_E)$.*
- (2) *There exist Γ -locally π -precompact subspaces $K_E \subset K'_E \subset E\Gamma$ and $K_E \subset \underline{E}\Gamma$ such that the classifying map $E\Gamma \rightarrow \underline{E}\Gamma$ is a map of π -CW-pairs $(E\Gamma, E\Gamma \setminus K'_E) \rightarrow (\underline{E}\Gamma, \underline{E}\Gamma \setminus K_E)$, and such that the restriction $i_{K_E, K'_E}^*(\theta_E) \in E_\Gamma^k(E\Gamma, E\Gamma \setminus K'_E)$ can be lifted to $\theta_{\underline{E}} \in E_\pi^k(\underline{E}\Gamma, \underline{E}\Gamma \setminus K_E)$.*

If condition (1) is satisfied, τ_θ can be extended to $E_*(B\Gamma) \rightarrow E_{*-k}(B\pi)$. If, in addition, condition (2) is satisfied, τ_θ can be extended further to $E_*^\Gamma(\underline{E}\Gamma) \rightarrow E_{*-k}^\pi(\underline{E}\pi)$.

Proof. Consider first the following diagram where θ' and θ'' are the images of θ under the isomorphisms $E^k(D\nu, S\nu) \cong E_\pi^k(\widetilde{D\nu}, \widetilde{S\nu}) \cong E_\pi^k(\widetilde{M}, \widetilde{M} \setminus \widetilde{D\nu}^\circ)$:



The left part of the diagram commutes by the compatibility of lf-restrictions with induction (setting $K = \widetilde{D\nu}^\circ$). The top and bottom triangles commute by the definition of τ_θ and $\tau_{\theta''}^{\text{eq}}$, respectively. The top square commutes because the cap product is compatible with induction, the bottom square commutes because the cap product is natural.

Consider next the following two diagrams:

$$\begin{array}{ccc}
 E_*(M) & \xrightarrow{\tau_\theta} & E_{*-k}(N) \\
 \text{ind}^{-1} \downarrow & & \downarrow i_* \circ \text{ind}^{-1} \\
 E_*^\Gamma(\widetilde{M}) & \xrightarrow{\tau_{\theta''}^{\text{eq}}} & E_{*-k}^\pi(\widetilde{M}) \\
 \downarrow & & \downarrow \\
 E_*^\Gamma(\text{E}\Gamma) & \xrightarrow{\tau_{\theta_E}^{\text{eq}}} & E_{*-k}^\pi(\text{E}\Gamma) \\
 \text{ind}^{-1} \downarrow \cong & & \cong \downarrow \text{ind}^{-1} \\
 E_*(\text{B}\Gamma) & \dashrightarrow & E_{*-k}(\text{B}\pi)
 \end{array}
 \qquad
 \begin{array}{ccc}
 E_*(M) & \xrightarrow{\tau_\theta} & E_{*-k}(N) \\
 \text{ind}^{-1} \downarrow & & \downarrow i_* \circ \text{ind}^{-1} \\
 E_*^\Gamma(\widetilde{M}) & \xrightarrow{\tau_{\theta''}^{\text{eq}}} & E_{*-k}^\pi(\widetilde{M}) \\
 \downarrow & & \downarrow \\
 E_*^\Gamma(\text{E}\Gamma) & \xrightarrow{\tau_{\theta_E}^{\text{eq}}} & E_{*-k}^\pi(\text{E}\Gamma) \\
 \downarrow & & \downarrow \\
 E_*^\Gamma(\underline{\text{E}}\Gamma) & \xrightarrow{\tau_{\underline{\theta}_E}^{\text{eq}}} & E_{*-k}^\pi(\underline{\text{E}}\pi).
 \end{array}$$

In both diagrams the top square is given by the outer arrows of the previous diagram. Hence, it commutes. The middle squares in both diagrams, as well as the bottom square in the second diagram, commute by naturality of the equivariant transfer map. The dashed morphism in the left diagram is defined to make the bottom square commutative.

We note that $\text{E}\Gamma$ and $\underline{\text{E}}\Gamma$ with the group action restricted to π are models for $\text{E}\pi$ and $\underline{\text{E}}\pi$, respectively, and the composition along the left and right columns in both diagrams are classifying maps. Hence the claims follow. \square

Remark 4.5. Under certain conditions there is another canonical way to extend τ_θ to $E_*(\text{B}\Gamma) \rightarrow E_{*-k}(\text{B}\pi)$: Assume that we have models $\text{B}\pi \subset \text{B}\Gamma$ such that a neighborhood of $\text{B}\pi$ is homeomorphic to a k -disk bundle $D\nu_{\text{B}}$. Then any Thom class $\theta_{\text{B}} \in E^k(D\nu_{\text{B}}, S\nu_{\text{B}})$ gives rise to a transfer map $\tau_{\theta_{\text{B}}} : E_*(\text{B}\Gamma) \rightarrow E_{*-k}(\text{B}\pi)$. Assume further that the classifying map $M \rightarrow \text{B}\Gamma$ restricts to a bundle map $D\nu \rightarrow D\nu_{\text{B}}$ and sends $M \setminus D\nu^\circ$ into $\text{B}\Gamma \setminus D\nu_{\text{B}}^\circ$, and that the Thom class θ is the pullback of θ_{B} under the classifying map. Then $\tau_{\theta_{\text{B}}}$ extends τ_θ . This is precisely how the maps in (3.18) are obtained in the proof of Theorem 3.1.

The extension $\tau_{\theta_{\text{B}}}$ coincides with the one obtained from Theorem 4.4, with θ_{E} the image of θ_{B} under the isomorphisms $E^k(D\nu_{\text{B}}, S\nu_{\text{B}}) \cong E_\pi^k(\widetilde{D\nu_{\text{B}}}, \widetilde{S\nu_{\text{B}}}) \cong E_\pi^k(\widetilde{\text{B}\Gamma}, \widetilde{\text{B}\Gamma} \setminus \widetilde{D\nu_{\text{B}}^\circ})$. Indeed, the first diagram of the preceding proof, applied to the pair $D\nu_{\text{B}} \subset \text{B}\Gamma$, shows that $\text{ind}^{-1} \circ \tau_{\theta_{\text{E}}}^{\text{eq}} \circ \text{ind} = i_* \circ \tau_{\theta_{\text{B}}}$, where $i : D\nu_{\text{B}} \rightarrow \pi \setminus \text{E}\Gamma$ is a homotopy equivalence.

In codimension ≥ 2 it is generally not clear when the above conditions can be satisfied. This is discussed in [16, 5.3.9–11] for codimension 2.

Remark 4.6. If we assume that the lf-restrictions are transitive with respect to inclusions of subgroups $\pi' < \pi < \Gamma$, the equivariant transfer map will also be

transitive in the sense that $\tau_{\theta'}^{\text{eq}} \circ \tau_{\theta}^{\text{eq}} = \tau_{\theta' \cup \theta}^{\text{eq}}$. We will not make use of this in the following.

4.1 HOMOLOGY THEORIES WITH LF-RESTRICTIONS

The first equivariant homology theory for which we construct lf-restrictions is Bredon homology (see [15]). However, we only deal with the case of a constant coefficient module M with the trivial group action. In this case $H_*^\Gamma(X) = H_*^\Gamma(X; M)$ is canonically isomorphic to the ordinary homology of the quotient, $H_*(\Gamma \backslash X; M)$.

LEMMA 4.7. *Bredon homology with constant coefficients and trivial group action on the coefficients has lf-restrictions.*

Proof. Let π, Γ, Y, K be as in Definition 4.2. An element $x \in H_*^\Gamma(Y)$ can be represented by a cycle $\sum_i \lambda_i c_i^\Gamma$, where the λ_i lie in the coefficient module, c_i^Γ are Γ -cells of Y and the sum is finite. We denote by C_i^π the formal sum of π -cells of Y that have non-empty intersection with $c_i^\Gamma \cap K$. This sum is finite because K is Γ -locally π -precompact. Let now $r_K(x) = [\sum_i \lambda_i C_i^\pi] \in H_*^\pi(Y, Y \setminus K)$. One can easily check that the map r_K , thus defined, satisfies the conditions of Definition 4.2. \square

To show that real K-homology has lf-restrictions we use its geometric description. For a detailed definition and a proof of equivalence to the analytic description see Baum–Higson–Schick, [2] (They consider the complex case, but the real case is analogous, see their Remark 4.1).

DEFINITION 4.8. Let (X, A) be a pair of topological spaces and $\Gamma \curvearrowright (X, A)$ a proper action of a discrete group. Elements of $\text{KO}_n^\Gamma(X, A)$ are represented by quadruples (M, s, E, f) , where M is a Γ -compact, proper Γ -manifold of dimension $n \bmod 8$, s is a Γ -spin-structure on M , $E \rightarrow M$ a real Γ -vector bundle and $f: M \rightarrow X$ a continuous Γ -equivariant map such that $f(\partial M) \subset A$. The equivalence relation is generated by:

- (1) direct sum of vector bundles equals disjoint union,
- (2) equivariant bordism,
- (3) equivariant vector bundle modification.

LEMMA 4.9. *Real K-homology has lf-restrictions.*

Proof. Let π, Γ, Y, K be as in Definition 4.2 and let $[M, s, E, f] \in \text{KO}_*^\Gamma(Y)$. To define the lf-restriction map we have to find any π -invariant π -compact submanifold with boundary $M' \subset M$ of codimension 0, such that $f(M \setminus M') \subset Y \setminus K$. The restricted K-homology cycle $(M', s|_{M'}, E|_{M'}, f_{M'})$ with an action of only π then represents an element in $\text{KO}_*^\pi(Y, Y \setminus K)$ that does not depend on the concrete choice of M' . The construction is compatible with taking disjoint unions, with vector bundle addition and with vector bundle modification. Because the method given below to find M' can also be applied to bordisms,

the construction is compatible with the bordism relation and therefore defines a group homomorphism from $KO_*^\Gamma(Y)$ to $KO_*^\pi(Y, Y \setminus K)$.

To construct the submanifold $M' \subset M$, we first check that $L := \overline{f^{-1}(K)}$ is π -compact. Indeed, if $F \subset M$ is the closure of a Γ -fundamental domain, then F is compact. Hence, $\overline{\gamma} \cdot F \cap f^{-1}(K) \neq \emptyset$ only for finitely many cosets $\overline{\gamma} \in \overline{\Gamma}_F \subset \Gamma/\pi$, and it follows that $f^{-1}(K) \subset \bigcup_{\overline{\gamma} \in \overline{\Gamma}_F} \overline{\gamma} \cdot F$ is π -precompact. We choose a Γ -invariant Riemannian metric on M and consider the function $d: M \rightarrow \mathbb{R}$ that assigns to $m \in M$ the distance from m to L . This function is π -equivariant. The induced function $\overline{d}: \pi \backslash M \rightarrow \mathbb{R}$ is the distance function (in the metric induced from M) to the compact set $\pi \backslash L$. Hence \overline{d} is proper.

Next, we need a π -invariant smooth approximation of d . Because the action $\Gamma \curvearrowright M$ is proper, every point $m \in M$ has a Γ -invariant neighborhood of the form $\bigsqcup_{[\gamma] \in \Gamma/\Gamma_m} W_{m, [\gamma]}$, where Γ_m is the stabilizer of m . Because M is Γ -compact, it is covered by a finite set of such neighborhoods. Because all stabilizer groups are finite, one can construct a Γ -invariant smooth partition of unity subordinate to the covering. Using the partition of unity we can now smoothen d separately on each neighborhood. This works by choosing a smoothing on one $W_{m, \overline{\gamma}}$ for each coset $\overline{\gamma} \in \pi \backslash \Gamma/\Gamma_m$, averaging these over the action of π_m and extending them π -equivariantly to the whole Γ -neighborhood. Finally, pick any regular value $r > 0$ of d such that $f(d^{-1}([r, \infty))) \subset Y \setminus K$. The preimage $M' = \overline{d^{-1}((-\infty, r])} \subset M$ is a π -invariant submanifold with boundary. Because \overline{d} is still a proper function after smooth approximation, M' is π -compact.

The lf-restriction map, thus constructed, is clearly natural with respect to maps of CW-pairs. To see that it is compatible with induction, assume now that Γ acts freely on Y and that \overline{K} is a subcomplex with $\Gamma \backslash \Gamma \cdot \overline{K} = \pi \backslash \overline{K}$, as in Definition 4.2. We can construct a π -invariant neighborhood $U \supset \overline{K}$ such that $\Gamma \backslash \Gamma \cdot U = \pi \backslash U$ and such that there is a π -equivariant deformation retraction H_t from U to \overline{K} . Then $M \backslash f^{-1}(U)$ has a positive distance to L , and so we may arrange that $f(M') \subset U$. The image $\text{ind} \circ \text{exc} \circ r_K([M, s, E, f])$ is represented by the quotient of $(M', s|_{M'}, E|_{M'}, H_1 \circ f_{M'})$ by π . But the same representative can be obtained by first taking the quotient by Γ , then cutting $\Gamma \backslash M$ off to $\Gamma \backslash \Gamma \cdot M' = \pi \backslash M'$ and applying the deformation retraction induced by H_t on the quotient. Therefore, one sees on the level of cycles that the diagram of Definition 4.2 is commutative. \square

Remark 4.10. The above proof works, word-for-word, also for complex K-homology, and if one simply forgets the vector bundle, it also works for all bordism theories. Therefore, these generalized homology theories all have lf-restrictions.

5 APPLYING THE GENERAL TRANSFER CONSTRUCTION

We apply Theorem 4.4 to specific scenarios where we put certain restrictions on the homology theory E , on the codimension of the subspace $N \subset M$ or on

the normal bundle of N .

In each case we will describe the cohomology class θ in terms of a pullback of a fixed cohomology class $x \in E^*(K(\mathbb{Z}, n))$ along some classifying map. The extensions of θ are then obtained as the pullback of x under extensions of the classifying map. This approach has the additional benefit that it is natural with respect to transformations of homology theories.

5.1 TRANSFER IN SINGULAR HOMOLOGY

The following is a restatement of Theorem 1.7.

THEOREM 5.1. *Suppose that we are in Setup 2.1 with $N \subset M$ an embedding of codimension k with normal bundle ν . Let $E_* = H_*(-; A)$ be singular homology with coefficients in a commutative ring A and let ν be oriented with Thom class $\theta \in H^k(D\nu, S\nu; A)$.*

In addition, we assume that

- (1) *if $k > 1$, then $\pi_k(N) \rightarrow \pi_k(M)$ is surjective;*
- (2) *if $k > 2$, then $\pi_j(M) = 0$ for $j = 2, \dots, k - 1$;*
- (3) *there exists a subset $S \subseteq \pi_k(N)$ which generates $\pi_k(M)$ such that the composition of the Hurewicz homomorphism $\pi_k(N) \rightarrow H_k(N; \mathbb{Z})$ with the Euler class of ν vanishes on S . In particular, this is satisfied if the normal bundle is trivial or $\pi_k(M) = 0$ or $\dim N < \frac{\dim M}{2}$.*

Then the classical transfer map can be extended to the classifying space for free actions.

Proof. We have to extend θ from $H_\pi^k(\widetilde{D\nu}, \widetilde{S\nu})$ to $H_\pi^k(E\Gamma, E\Gamma \setminus K)$ for a suitable Γ -locally π -precompact subspace $K \subseteq E\Gamma$. Then the result follows from Theorem 4.4.

We start with $K = \widetilde{D\nu}$ and build $E\Gamma$ out of \widetilde{M} by inductively attaching free Γ -cells of dimension $> k$. Each new Γ -cell decomposes into π -cells. Because K is Γ -locally π -precompact and the attaching maps are Γ -equivariant, all but finitely many of the π -cells do not attach to K . The interior of the other π -cells has to be added to K , but since these are only finitely many, K stays Γ -locally π -precompact.

To complete the inductive step, θ has to be extended to the added cells. If the new π -cells have dimension $> k + 1$, this extension is possible because singular homology satisfies the dimension axiom. In dimension $k + 1$ we may assume that the attaching map represents an element of $S \subset \pi_k(N)$. The extension is then possible by the condition on the Euler class. \square

Remark 5.2. For singular homology our extension of the transfer map coincides with the map constructed by Engel in the setting of rough homology in [6]: Without recapitulating all the definitions, we note that, conceptually, Engel's map is given by cap product with a certain cohomology class that arises from

the Thom class (see [6, Proof of Theorem 2.16]), and the same is true for our construction.

To verify that the two maps are the same one has to retrace the proof of [6, Theorem 3.10]. Without loss of generality, $E\pi_1(M)$ is also used as a model for $E\pi_1(N)$, and $B\pi_1(M)$ contains M as a subcomplex. To evaluate Engel's map on a homology class $x \in H_*(B\pi_1(M))$ we first represent x by a cycle, such that the vertices of all singular simplices map to a fixed base point in M . Because M is highly connected, we can further assume that the q -faces of every simplex are mapped to M . Now we trace through all the isomorphisms involved in Engel's construction. Whenever $\pi_1(M)$ is embedded into $E\pi_1(M)$ as the orbit of a point, we let this point be a fixed lift of the base point. Whenever the simplicial operator Δ of [6, Proof of Proposition 2.8] is applied to the vertex set \bar{x} of a singular simplex, we may assume that $\Delta(\bar{x})$ is the original simplex. In the end, the image of x agrees with the image under our own extended transfer map.

Although the hypotheses of Theorem 1.7 appear elaborate, the following examples demonstrate why they are necessary.

Example 5.3. Let N be an oriented aspherical manifold and $k > 1$. Then the codimension k submanifold inclusion $N \cong N \times * \subset N \times S^k =: M$ with $\pi = \pi_1 N = \pi_1 M = \Gamma$ satisfies all conditions of Theorem 1.7 except (1). However, the desired commutative diagram cannot exist because the fundamental class of N does not vanish in the homology of $B\pi = N$ but the fundamental class of $N \times S^k$ does vanish in $H_*(B\pi)$.

Example 5.4. To see why (2) is necessary, consider the standard embedding $N := S^2 = \mathbb{C}P^1 \subset \mathbb{C}P^3 =: M$. Then N has codimension four in M , $\pi_1 N = \pi_1 M = 1$ and the normal bundle is orientable because $\mathbb{C}P^1$ and $\mathbb{C}P^3$ are. Since $\pi_4(\mathbb{C}P^3) = 0$, condition (1) is satisfied. So is (3) due to $H^4(\mathbb{C}P^1) = 0$. However, (2) fails because $\pi_2(\mathbb{C}P^3) \cong \mathbb{Z} \neq 0$. Since $\mathbb{C}P^2$ and $\mathbb{C}P^1$ have intersection number 1 in $\mathbb{C}P^3$ and $H_4(\mathbb{C}P^3)$ is generated by $\mathbb{C}P^2$, in degree 4 the transfer map of the inclusion $\mathbb{C}P^1 \subset \mathbb{C}P^3$ yields an isomorphism $H_4(\mathbb{C}P^3) \xrightarrow{\cong} H_0(\mathbb{C}P^1) \cong H_0(*) \cong \mathbb{Z}$. But H_4 of the trivial group is zero. Hence the diagram

$$\begin{array}{ccc} H_4(\mathbb{C}P^3) & \longrightarrow & H_4(*) = 0 \\ \downarrow \cong & & \vdots \\ H_0(\mathbb{C}P^1) & \xrightarrow{\cong} & H_0(*) \cong \mathbb{Z} \end{array}$$

cannot be completed with a dashed arrow to make it commutative.

A priori, a plausible alternative to condition (2) might be to require that the induced maps $\pi_j(N) \rightarrow \pi_j(M)$ are surjective (or isomorphisms) for $j = 2, \dots, k - 1$. For instance, this is suggested by the case of fiber bundles over aspherical manifolds which have been studied in [25, Theorem 1.5]. But Example 5.4 would satisfy such a hypothesis and hence this is not appropriate for our purposes.

Example 5.5. A variation of Example 5.4 shows that condition (3) cannot be dropped either. Indeed, consider the standard embedding $N := S^2 = \mathbb{C}P^1 \subset \mathbb{C}P^2 =: M$. Then N has codimension two in M , $\pi_1 N = \pi_1 M = 1$ and the Hurewicz theorem implies that the inclusion $N \hookrightarrow M$ induces an isomorphism on π_2 because it does so on H_2 . Now this example satisfies all hypotheses of Theorem 1.7 except (3). Since $\mathbb{C}P^1$ has self-intersection number 1 in $\mathbb{C}P^2$ and $H_2(\mathbb{C}P^2)$ is generated by $\mathbb{C}P^1$, in degree 2 the transfer map of the inclusion $\mathbb{C}P^1 \subset \mathbb{C}P^2$ yields again an isomorphism $H_2(\mathbb{C}P^2) \xrightarrow{\cong} H_0(\mathbb{C}P^1) \cong H_0(*) \cong \mathbb{Z}$. Since $H_2(*) = 0$, this again obstructs the existence of the desired commutative diagram.

COROLLARY 5.6. *Assume that we are in the situation of Theorem 5.1 (with $A = \mathbb{Z}$), that the assembly map $\mu \otimes \mathbb{Q}: KO_*(B\Gamma) \otimes \mathbb{Q} \rightarrow KO_*(C^*\Gamma) \otimes \mathbb{Q}$ is injective, and that M is spin. Moreover, assume that either*

- (1) *the push forward of the homological fundamental class of N does not vanish in $H_*(B\pi; \mathbb{Q})$, or*
- (2) *the normal bundle of $N \subset M$ is trivial and the push forward of the KO-fundamental class of N (with the induced spin structure) in $KO_*(B\pi) \otimes \mathbb{Q}$ does not vanish.*

Then $\alpha(M) \neq 0$ and hence M does not admit a metric of positive scalar curvature.

Proof. On complex K-homology the Chern character is a transformation

$$\text{ch}: K_n(X) \rightarrow \bigoplus_{l \in \mathbb{Z}} H_{n+2l}(X; \mathbb{Q})$$

which becomes an isomorphism after taking the tensor product with \mathbb{Q} . Precomposing the Chern character with the complexification transformation on real K-homology yields the Pontryagin character

$$\text{ph}: KO_n(X) \rightarrow \bigoplus_{l \in \mathbb{Z}} H_{n+4l}(X; \mathbb{Q})$$

which again becomes an isomorphism after taking the tensor product with \mathbb{Q} . The composition $\Omega_n^{\text{spin}}(X) \rightarrow KO_n(X) \xrightarrow{\text{ph}} \bigoplus_{l \in \mathbb{Z}} H_{n+4l}(X; \mathbb{Q})$ agrees in degree n with the composition $\Omega_n^{\text{spin}}(X) \rightarrow \Omega_n^{\text{SO}}(X) \rightarrow H_n(X; \mathbb{Z}) \rightarrow H_n(X; \mathbb{Q})$ which assigns to an oriented manifold its homological fundamental class. However, in general they are not equal because the former contains information determined by the \hat{A} -class of the tangent bundle in the lower degree summands. Similarly, if $N \subset M$ is a codimension k submanifold embedding with KO-oriented normal bundle, then the diagram

$$\begin{CD} KO_n(M) @>\text{ph}>> \bigoplus_{l \in \mathbb{Z}} H_{n+4l}(M; \mathbb{Q}) \\ @V\tau_{M,N}VV @VV\tau_{M,N}V \\ KO_{n-k}(N) @>\text{ph}>> \bigoplus_{l \in \mathbb{Z}} H_{n-k+4l}(N; \mathbb{Q}) \end{CD} \tag{5.7}$$

does not commute in general because of the Chern character defect. But if M is spin, then the classes $\tau_{M,N} \text{ph}([M]_{\text{KO}})$ and $\text{ph} \tau_{M,N}([M]_{\text{KO}})$ agree in the top degree because they are both equal to the homological fundamental class of N . This observation together with Theorem 5.1 proves (1).

Finally, in the special case of a trivial normal bundle, the diagram (5.7) commutes. Together with the fact that the Pontryagin character is a rational isomorphism and Theorem 5.1, this proves (2). \square

5.2 TRANSFER IN A COMPLEX ORIENTED HOMOLOGY THEORY

In this section, we apply our construction in codimension two and three to complex oriented cohomology theories. A classic textbook treatment of this concept is [1, Part II]. Alternatively, see for instance [13, Lectures 4–6].

DEFINITION 5.8. Let E be a generalized multiplicative cohomology theory. A *complex orientation* for E is a class $x \in E^2(\mathbb{C}P^\infty, \text{pt})$ which restricts to the twice-suspended unit in $E^2(\mathbb{C}P^1, \text{pt}) = E^2(S^2, \text{pt})$. E is called *complex orientable* if such an element exists.

Remark 5.9 ([1, Part II, Lemma 4.6], [13, Lecture 6, Theorem 8]). Complex bordism is the universal example of a complex oriented cohomology theory. Indeed, complex orientations on a multiplicative cohomology theory E are in bijection with multiplicative transformations of cohomology theories $f: \text{MU} \rightarrow E$. There is a universal complex orientation $x^{\text{MU}} \in \text{MU}^2(\mathbb{C}P^\infty, \text{pt})$ such that this bijection is implemented by taking f to $f(x^{\text{MU}})$.

Example 5.10. The transformations $\text{MU} \rightarrow \text{MSpin}^c \rightarrow \text{MSO}$ show that the spin^c - and oriented bordism theories are complex oriented. The Atiyah–Bott–Shapiro orientation $\text{MSpin}^c \rightarrow \text{K}$ makes complex K-theory complex oriented.

On the other hand, spin-bordism and real K-theory are both *not* complex oriented. But they become complex oriented after inverting 2. For spin bordism, this is the case because the map $\text{MSpin} \rightarrow \text{MSO}$ is an equivalence after inverting 2 and MSO is complex oriented. Then it also follows for KO via the Atiyah–Bott–Shapiro orientation $\text{MSpin} \rightarrow \text{KO}$.

An ordinary orientation on a two-dimensional vector bundle ν over N determines a complex structure on the bundle. Consequently, there is a homotopy class of classifying maps $f: (\text{D}\nu, \text{S}\nu) \rightarrow (\text{D}\gamma, \text{S}\gamma) \simeq (\mathbb{C}P^\infty, \text{S}^\infty)$, where γ is the universal complex line bundle over $\mathbb{C}P^\infty$. Using that S^∞ is contractible, we can identify f with a map $f: (\text{D}\nu, \text{S}\nu) \rightarrow (\mathbb{C}P^\infty, \text{pt})$. The condition that $x \in E^2(\mathbb{C}P^\infty, \text{pt})$ is a complex orientation implies that $f^*(x) \in E^2(\text{D}\nu, \text{S}\nu)$ is a Thom class. Because $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, the homotopy class $[f]$ corresponds to an element in $\text{H}^2(\text{D}\nu, \text{S}\nu)$. This is the Thom class in ordinary cohomology for the given choice of ordinary orientation. The restriction $\text{H}^2(\text{D}\nu, \text{S}\nu) \rightarrow \text{H}^2(\text{D}\nu) \rightarrow \text{H}^2(N)$ maps $[f]$ to the (ordinary) Chern class of ν .

THEOREM 5.11. *Let E be a multiplicative equivariant homology theory furnished with a complex orientation $x \in E^2(\mathbb{C}P^\infty, \text{pt})$ and lf -restrictions.*

Suppose that we are in Setup 2.1 with $N \subset M$ an embedding of codimension $k = 2$. Let the normal bundle ν be oriented with ordinary Thom class $[f] \in H^2(D\nu, S\nu)$, viewed as a homotopy class of maps $f: (D\nu, S\nu) \rightarrow (\mathbb{C}P^\infty, \text{pt})$. Define the corresponding E -Thom class by $\theta := f^*(x) \in E^2(D\nu, S\nu)$.

Assume that the induced map $\pi_2(N) \rightarrow \pi_2(M)$ is surjective and that there exists a subset $S \subseteq \pi_2(N)$ which generates $\pi_2(M)$ such that the composition of the Hurewicz homomorphism with the Chern class of ν vanishes on S .

Then the classical transfer map in E -homology defined by θ can be extended to the classifying space for free actions. The extension is natural with respect to transformations of homology theories (that preserve all relevant structure, including the complex orientation).

Proof. The same proof as for Theorem 5.1 with $k = 2$ shows that we can extend $[f] \in H^2(D\nu, S\nu) \cong H_\pi^2(\widetilde{M}, \widetilde{M} \setminus \widetilde{D\nu})$ to $[f_E] \in H_\pi^2(E\Gamma, E\Gamma \setminus K)$ for some Γ -locally π -precompact subspace $K \subset E\Gamma$. Then we view $[f_E]$ as a homotopy class of π -invariant maps of pairs $f_E: (E\Gamma, E\Gamma \setminus K) \rightarrow (\mathbb{C}P^\infty, \text{pt})$, and we set $\theta_E = f_E^*(x)$. This is the extension of θ required to apply Theorem 4.4. \square

Under more restrictive hypotheses, we also obtain a codimension three transfer for complex orientable cohomology theories.

THEOREM 5.12. *Let E be a complex orientable multiplicative equivariant homology theory with lf-restrictions.*

Suppose that we are in Setup 2.1 with $N \subset M$ an embedding of codimension $k = 3$. Let the normal bundle ν be trivialized. Using the chosen trivialization, let $\theta \in E^3(D\nu, S\nu) \cong E^3(N \times D^3, N \times S^2)$ be the pullback of the three times suspended unit in $E^3(D^3, S^2)$.

Assume that the induced map $\pi_1(N) \rightarrow \pi_1(M)$ is injective, $\pi_2(N) = 0$, $\pi_2(M) = 0$, and that the induced map $\pi_3(N) \rightarrow \pi_3(M)$ is surjective.

Then the classical transfer map in E -homology defined by θ can be extended to the classifying space for free actions. The extension is natural with respect to transformations of homology theories (that preserve all relevant structure, including the complex orientation).

Proof. Let $f: S\nu \rightarrow \mathbb{C}P^\infty$ the map obtained as the composition of the projection $S\nu \rightarrow S^2$ defined by the trivialization and the inclusion $S^2 = \mathbb{C}P^1 \subset \mathbb{C}P^\infty$. Since E is complex oriented, the twice suspended unit in $E^2(S^2)$ is the restriction of a class $x \in E^2(\mathbb{C}P^\infty)$. This implies that $\theta = \delta f^*(x)$, where $\delta: E^2(S\nu) \rightarrow E^3(D\nu, S\nu)$ is the coboundary map associated to the pair $(D\nu, S\nu)$. Now consider the lifted embedding $D\nu \subset \bar{M} = \pi \setminus \widetilde{M}$. From the long exact sequence of homotopy groups and our assumptions on π_1, π_2 and π_3 it follows that $\pi_i(\bar{M}, D\nu) = 0$ for $i \leq 3$. The relative Hurewicz Theorem then implies that $H_i(\bar{M}, D\nu) = 0$ for $i \leq 3$. Applying the universal coefficient theorem and excision yields $H^i(\bar{M} \setminus D\nu^\circ, S\nu) \cong H^i(\bar{M}, D\nu) = 0$ for $i \leq 3$. From the long exact sequence of cohomology it follows that the inclusion $S\nu \subset \bar{M} \setminus D\nu^\circ$ induces an isomorphism $H^2(\bar{M} \setminus D\nu^\circ) \xrightarrow{\cong} H^2(S\nu)$. This proves that $f: S\nu \rightarrow \mathbb{C}P^\infty$ can be

extended to a map $f_{\widetilde{M}}: \widetilde{M} \setminus D\nu^\circ \rightarrow \mathbb{C}P^\infty$. Lifting this to \widetilde{M} yields a π -invariant map $f_{\widetilde{M}}: \widetilde{M} \setminus \widetilde{D\nu}^\circ \rightarrow \mathbb{C}P^\infty$ which represents a class $[f_{\widetilde{M}}] \in H_\pi^2(\widetilde{M} \setminus \widetilde{D\nu}^\circ)$. Next, we build $E\Gamma$ by attaching free Γ -cells of dimension ≥ 4 to \widetilde{M} . During this process, we also build a suitable Γ -locally π -precompact subset $K \subseteq E\Gamma$ and extend $[f_{\widetilde{M}}] \in H_\pi^2(\widetilde{M} \setminus \widetilde{D\nu}^\circ)$ to $[f_E] \in H_\pi^2(E\Gamma \setminus K)$. This works exactly as in Theorem 5.1 —note that there is no obstruction to extending a degree 2 cohomology class along the added cells because they have dimension at least 4. Finally, we apply Theorem 4.4, where the required extension of θ is given by $\theta_E := \delta f_E^*(x) \in E_\pi^3(E\Gamma, E\Gamma \setminus K)$ using $\delta: E_\pi^2(E\Gamma \setminus K) \rightarrow E_\pi^3(E\Gamma, E\Gamma \setminus K)$ associated to the pair $(E\Gamma, E\Gamma \setminus K)$. \square

5.3 EXTENDING THE EQUIVARIANT TRANSFER MAP TO $\underline{E}\Gamma$

Finally, we study the question when the equivariant transfer map can be further extended to the classifying space for proper actions. In all previous applications we have reduced the construction of the transfer map to the task of extending a class in ordinary cohomology. Now we want to extend this class to $\underline{E}\Gamma$. As usual, this is done by a spectral sequence argument. But we have to be careful with the choice of our models for the classifying spaces.

By the discussion preceding Theorem 4.4 the equivariant transfer map does not rely on a particular choice of model for $E\Gamma$ or $\underline{E}\Gamma$. For $E\Gamma$ we take Milnor’s infinite join construction (i.e. the “fat model”) with the usual Γ -CW-structure, where n -simplices correspond to $n + 1$ -tuples in Γ . For $\underline{E}\Gamma$ we take the “geometric model”, whose n -simplices correspond to finite subsets of Γ of cardinality $n + 1$ (see Mislin’s appendix to [23]). After passing to the barycentric subdivision this becomes a Γ -CW-complex. The canonical map $q: E\Gamma \rightarrow \underline{E}\Gamma$ is obtained by extending the obvious map on the 0-skeleton affinely to all higher cells. Crucially, it is proper when restricted to the skeleta $E\Gamma^{(k)}$.

LEMMA 5.13. *Let $E\Gamma$ and $\underline{E}\Gamma$ be represented by the models above, let $E\Gamma \setminus K_E$ be a subcomplex with Γ -locally π -precompact complement, and let $n \in \mathbb{N}$. Then there exists a π -subcomplex $L \subset \underline{E}\Gamma$ with Γ -locally π -precompact complement, such that $q^{-1}(L) \subset E\Gamma$ is a π -subcomplex with Γ -locally π -precompact complement, and such that the restriction maps*

$$\begin{aligned} H_\pi^n((E\Gamma \setminus K_E) \cup q^{-1}(L); \mathbb{Z}) &\longrightarrow H_\pi^n(E\Gamma \setminus K_E; \mathbb{Z}), \\ H_\pi^{n+1}(E\Gamma, (E\Gamma \setminus K_E) \cup q^{-1}(L); \mathbb{Z}) &\longrightarrow H_\pi^{n+1}(E\Gamma, E\Gamma \setminus K_E; \mathbb{Z}) \end{aligned}$$

are surjective.

Proof. First, we enlarge $E\Gamma \setminus K_E$ to a bigger subcomplex $E\Gamma \setminus K'_E$ such that every π -cell of dimension $> n + 1$ whose boundary completely lies in $E\Gamma \setminus K'_E$ does itself belong to $E\Gamma \setminus K'_E$. We do this by induction over the skeleta of $E\Gamma$, in each step adding all necessary π -cells to $E\Gamma \setminus K'_E$. If a cohomology class in $H_\pi^n(E\Gamma, E\Gamma \setminus K_E; \mathbb{Z})$ or $H_\pi^{n+1}(E\Gamma \setminus K_E; \mathbb{Z})$ is given, it can, in each induction step, be simultaneously extended to all added π -cells. In fact, the extension is

unique in each step, and by the \lim^1 -short exact sequence and the Mittag-Leffler condition it is also unique in the limit.

Let now $L = \underline{E}\Gamma \setminus K_{\underline{E}}$ be the π -subcomplex of $\underline{E}\Gamma$ consisting of all those π -cells whose preimage under q does not intersect $K'_{\underline{E}}$. We claim that the complement $K_{\underline{E}}$ is Γ -locally π -precompact. Indeed, let a non-equivariant cell of $\underline{E}\Gamma$ be given and let $S \subset \Gamma$ be the subset to which the cell belongs under the barycentric subdivision. The cell can only intersect $K_{\underline{E}}$ if there is some tuple in S such that the corresponding non-equivariant cell in $E\Gamma$ intersects $K'_{\underline{E}}$. By the construction of $K'_{\underline{E}}$ this can only happen if there is a tuple in S of length $\leq n + 1$ such that the corresponding cell intersects $K'_{\underline{E}}$. But there are only finitely many such tuples, and for each cell corresponding to one of them there are only finitely many π -cells in its Γ -orbit that intersect $K'_{\underline{E}}$. Hence, only finitely many π -cells in the Γ -orbit of the original cell intersect $K_{\underline{E}}$.

Finally, $q^{-1}(L)$ is a π -subcomplex of $E\Gamma$. Its complement, being the preimage of a Γ -locally π -precompact set, is Γ -locally π -precompact. The restriction maps are surjective because $(E\Gamma \setminus K_{\underline{E}}) \cup q^{-1}(L) \subset E\Gamma \setminus K'_{\underline{E}}$, and we have already seen that cohomology classes can be extended to $H^n_{\pi}(E\Gamma \setminus K'_{\underline{E}}; \mathbb{Z})$ and $H^{n+1}_{\pi}(E\Gamma, E\Gamma \setminus K'_{\underline{E}}; \mathbb{Z})$. \square

LEMMA 5.14. *Let $E\Gamma$ and $\underline{E}\Gamma$ be represented by the models above. Let $\underline{A} \subset \underline{X} \subset \underline{E}\Gamma$ be any π -invariant subcomplexes (possibly $\underline{X} = \underline{E}\Gamma$ or $\underline{A} = \emptyset$), let $X = q^{-1}(\underline{X})$ and $A = q^{-1}(\underline{A})$. Assume that $n \geq 1$ and $H^k(BG; \mathbb{Z}) = 0$ for all finite subgroups $G < \pi$ and all $0 < k < n$.*

Then there is a short exact sequence

$$0 \longrightarrow H^n_{\pi}(\underline{X}, \underline{A}; \mathbb{Z}) \xrightarrow{q^*} H^n_{\pi}(X, A; \mathbb{Z}) \longrightarrow Z \longrightarrow 0,$$

where Z is the group of sections from $\pi \setminus X$ into a certain sheaf \mathcal{A} , all of whose germs are given by $H^n(BG; \mathbb{Z})$ for some finite subgroup $G < \pi$.

Proof. Because the action of π on the coefficient module \mathbb{Z} is trivial, $H^*_{\pi}(X, A; \mathbb{Z})$ and $H^*_{\pi}(\underline{X}, \underline{A}; \mathbb{Z})$ are isomorphic to $H^*(\pi \setminus X, \pi \setminus A; \mathbb{Z})$ and $H^*(\pi \setminus \underline{X}, \pi \setminus \underline{A}; \mathbb{Z})$, respectively. Let $pr^{\pi}_{\underline{E}}: E\Gamma \rightarrow \pi \setminus E\Gamma$ and $pr^{\pi}_{\underline{E}}: \underline{E}\Gamma \rightarrow \pi \setminus \underline{E}\Gamma$ denote the quotient maps. We apply the Leray spectral sequence for sheaf cohomology to the induced map on the quotient $\bar{q}: (\pi \setminus X, \pi \setminus A) \rightarrow (\pi \setminus \underline{X}, \pi \setminus \underline{A})$.

Using the terminology of Bredon [4] we let the “families of supports” Ψ and Φ be the families of closed sets on $\pi \setminus X$ and $\pi \setminus \underline{X}$, respectively (in [16] compact subsets were used erroneously).

By [4, Definition IV.6.1] there is a spectral sequence of sheaf cohomology groups

$$E_2^{p,r} = H^p_{\Phi}(\pi \setminus \underline{X}; \mathcal{A}^r) \Rightarrow H^{p+r}_{\Phi(\Psi)}(\pi \setminus X, \pi \setminus A; \mathbb{Z}),$$

where $\mathcal{A}^r = \mathcal{H}^r_{\Psi}(\bar{q}, \bar{q}|(\pi \setminus A); \mathbb{Z})$ is the Leray sheaf, obtained from the presheaf that assigns to each open set $U \subset \underline{X}$ the abelian group

$$H^r_{\Psi \cap \bar{q}^{-1}(U)}(\bar{q}^{-1}(U), \bar{q}^{-1}(U) \cap (\pi \setminus A); \mathbb{Z}).$$

All families of supports occurring in the above sheaf cohomology groups are simply the families of closed sets on the respective spaces. Because CW-complexes are paracompact, these families are paracompactifying in the sense of [4, Theorem I.6.1], and it follows by [4, Theorem III.1.1] that $H_{\Phi(\Psi)}^{p+r}(\pi \setminus X, \pi \setminus A; \mathbb{Z})$ and $H_{\Psi \cap \bar{q}^{-1}(U)}^r(\bar{q}^{-1}(U), \bar{q}^{-1}(U) \cap (\pi \setminus A); \mathbb{Z})$ coincide with the usual singular cohomology groups.

We claim that the set of germs of \mathcal{A}^r at a point $pr_{\underline{E}}^\pi(y) \in \pi \setminus \underline{X}$ is zero if $pr_{\underline{E}}^\pi(y) \in \pi \setminus \underline{A}$, and otherwise given by $H^r(B\pi_y; \mathbb{Z})$, the group cohomology of the stabilizer of y in π . To prove the claim we first find a neighborhood $pr_{\underline{E}}^\pi(y) \in U \subset \pi \setminus \underline{X}$ such that $(pr_{\underline{E}}^\pi)^{-1}(U) = \bigsqcup_{\bar{g} \in \pi/\pi_y} V_{\bar{g}}$. This is possible because π acts properly on \underline{X} . Unless $pr_{\underline{E}}^\pi(y) \in \pi \setminus \underline{A}$, we also assume $U \cap \pi \setminus \underline{A} = \emptyset$. Next, we construct, via cellular induction, a neighborhood basis $\{V'_i\}_{i \in \mathbb{N}}$ of y such that each V'_i completely lies in $(pr_{\underline{E}}^\pi)^{-1}(U)$ and is star-shaped around y . Then $\{U'_i\} = \{pr_{\underline{E}}^\pi(V'_i)\}$ is a neighborhood basis of $pr_{\underline{E}}^\pi(y)$. It suffices to show that $q^{-1}(y)$ and all $q^{-1}(V'_i)$ are (non-equivariantly) contractible because from this it follows that all the preimages $\bar{q}^{-1}(pr_{\underline{E}}^\pi(V'_i)) = \pi_y \setminus q^{-1}(V'_i)$ are models for $B\pi_y$ and that $\varinjlim H^r(\bar{q}^{-1}(U'_i); \mathbb{Z}) = H^r(\bar{q}^{-1}(pr_{\underline{E}}^\pi(y)); \mathbb{Z}) = H^r(B\pi_y; \mathbb{Z})$.

To construct the contracting homotopies, recall that the points in the infinite join model $E\Gamma$ can be written in coordinate form $x = (t_j \gamma_j)_{\mathbb{N}}$ with $t_j \in [0, 1]$, $\gamma_j \in \Gamma$, $\sum t_j = 1$ and $t_j = 0$ for all but finitely many entries. The image $q(x)$ is obtained by adding together those coefficients t_j that precede the same group elements and forgetting the ordering of the group elements.

The first step is to apply the homotopy described in [22, 14.4.4], from the identity map on $E\Gamma$ to the map that sends $(t_1 \gamma_1, t_2 \gamma_2, \dots)$ to $(t_1 \gamma_1, 0, t_2 \gamma_2, 0, \dots)$. The reverse of this homotopy is obtained by stacking together an infinite number of homotopies of the form

$$(t_1 \gamma_1, \dots, t_j \gamma_j, tt_{j+1} \gamma_{j+1}, (1-t)t_{j+1} \gamma_{j+1}, tt_{j+2} \gamma_{j+2}, (1-t)t_{j+2} \gamma_{j+2}, \dots).$$

The homotopy preserves the fibers of q . The second step is to choose any preimage of y under q of the form $(0, t'_1 \gamma'_1, 0, t'_2 \gamma'_2, 0, \dots)$ and to apply the homotopy $((1-t)t_1 \gamma_1, tt'_1 \gamma'_1, (1-t)t_2 \gamma_2, tt'_2 \gamma'_2, \dots)$. Because V'_i is star-shaped, this homotopy is well-defined on $q^{-1}(V'_i)$. This finishes the proof of the claim. Now, by assumption, $H^k(B\pi_y; \mathbb{Z}) = 0$ for $0 < k < n$. Hence, there are only two non-vanishing entries on the n -th diagonal of the E_∞ page of the spectral sequence, and the extension problem becomes

$$0 \longrightarrow E_\infty^{n,0} \longrightarrow H^n(\pi \setminus X, \pi \setminus A; \mathbb{Z}) \longrightarrow E_\infty^{0,n} \longrightarrow 0.$$

The group $E_\infty^{n,0} = E_2^{n,0}$ is equal to $H_\Phi^n(\pi \setminus \underline{X}; \mathcal{A}^0)$, where the sheaf \mathcal{A}^0 is zero over $\pi \setminus \underline{A}$ and the trivial \mathbb{Z} -sheaf everywhere else. By [4, Proposition II.12.3], and again [4, Theorem III.1.1], it follows that $E_\infty^{n,0} = H^n(\pi \setminus \underline{X}, \pi \setminus \underline{A}; \mathbb{Z})$. The group $E_\infty^{0,n} = E_2^{0,n}$ is, by definition, the group of global sections from $\pi \setminus \underline{X}$ into the sheaf \mathcal{A}^n that is zero over $\pi \setminus \underline{A}$ and has germs $\mathcal{A}_{pr_{\underline{E}}^\pi(y)}^n = H^n(B\pi_y; \mathbb{Z})$ elsewhere.

The first homomorphisms in the short exact sequence is the induced map \bar{q}^* , the second one is pointwise induced by the restriction to the respective fiber (see [4, IV.6.3 and Exercise 5]). \square

In principle, we can try to use the preceding two lemmas in order to extend the equivariant transfer map for all the application scenarios considered previously. Unfortunately, the conditions on Γ occurring in Lemma 5.14 are very restrictive. In the special case of a codimension 2 submanifold with trivial normal bundle we obtain the following.

THEOREM 5.15. *Let E be a multiplicative equivariant homology theory with lf-restrictions. Suppose that we are in Setup 2.1, with $N \subset M$ an embedding of codimension $k = 2$.*

In addition, assume that the induced map $\pi_1(N) \rightarrow \pi_1(M)$ is injective and $\pi_2(N) \rightarrow \pi_2(M)$ surjective, and that the normal bundle of N is trivialized by a map $(D\nu, S\nu) \rightarrow (D^2, S^1)$. Let $\theta \in E^2(D\nu, S\nu)$ be the pullback of the twice suspended unit in $E^2(D^2, S^1)$ under the trivialization.

Then the transfer map τ_θ can be extended to the classifying space for free actions and further to the classifying space for proper actions. The extensions are natural with respect to transformations of homology theories (that preserve all structure).

Proof. The twice suspended unit in $E^2(D^2, S^1)$ is the coboundary of the once suspended unit $x \in E^1(S^1)$. Hence, $\theta = \delta \circ f^*(x)$, where $f: S\nu \rightarrow S^1$ is the restriction of the trivialization map. The homotopy class of f defines an element in $H^1(S\nu; \mathbb{Z}) \cong H_\pi^1(\widetilde{S\nu})$. As in the proof of Theorem 5.12, we can extend $[f]$ to $[f_{\widetilde{M}}] \in H_\pi^1(\widetilde{M} \setminus \widetilde{D\nu}^\circ)$ and to $[f_E] \in H_\pi^1(E\Gamma \setminus K_E)$, where K_E is Γ -locally π -precompact.

Now we switch to the infinite join model for $E\Gamma$ and invoke Lemma 5.13 to replace $[f_E]$ with $(i_{q^{-1}(L)})^*[f'_E] \in H_\pi^1(q^{-1}(L); \mathbb{Z})$. Note that by naturality of the equivariant transfer map, the generalized cohomology classes $\delta \circ f_E^*(x)$, $\delta \circ (f'_E)^*(x)$ and $\delta \circ ((i_{q^{-1}(L)})^* f'_E)^*(x)$ all give rise to the same transfer map.

Now, the conditions of Lemma 5.14 are satisfied for $\underline{X} = L$, $\underline{A} = \emptyset$ and $n = 1$. Because $H^1(BG; \mathbb{Z}) = 0$ for all finite groups G , it follows that the induced map $q^*: H_\pi^1(L; \mathbb{Z}) \rightarrow H_\pi^1(q^{-1}(L); \mathbb{Z})$ is an isomorphism. Hence, we can extend $(i_{q^{-1}(L)})^*[f'_E]$ to $[f_E] \in H_\pi^1(L; \mathbb{Z})$.

Finally, the extensions of the transfer map are obtained by applying Theorem 4.4, where the extensions of θ are given by $\delta \circ f_E^*(x)$ and $\delta \circ f'_E^*(x)$. \square

Theorem 5.15 applies to all the generalized homology theories for which we have shown in Section 4.1 that lf-restrictions exist: ordinary homology, real and complex K-homology, and the cobordism theories. As discussed in the introduction, the most interesting application is the case where $E = \text{KO}$ is real K-homology, where our extension provides some context for Theorem 1.1. We expect that the extension is compatible with the transfer map constructed by Kubota [11] on the K-theory of the maximal group C^* -algebras, but we do not know whether this is true.

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