Some Homological Properties of Category $\mathcal{O}$. VI

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Abstract. This paper explores various homological regularity phenomena (in the sense of Auslander) in category $\mathcal{O}$ and its several variations and generalizations. Additionally, we address the problem of determining projective dimension of twisted and shuffled projective and tilting modules.

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1 Introduction, motivation and description of the results

1.1 Category $\mathcal{O}$

Let $\mathfrak{g}$ be a semi-simple, finite dimensional Lie algebra over $\mathbb{C}$ with a fixed triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

Consider the Bernstein-Gelfand-Gelfand (BGG) *category* $\mathcal{O}$ associated to this decomposition. Category $\mathcal{O}$ plays an important role in modern representation theory and its applications. See e.g., [BGS, Hu, So1, St3] and references therein. Indecomposable blocks of $\mathcal{O}$ are described by finite dimensional algebras and possess a number of remarkable symmetries. For example, they have simple preserving duality and exhibit both Ringel self-duality and Koszul self-duality. See [So1, BGS, So2].

Category $\mathcal{O}$ has a number of interesting sub- and quotient- categories such as the *parabolic category* $\mathcal{O}$ associated with the choice of a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ (see [RC]) and the $\mathcal{S}$-*subcategories in* $\mathcal{O}$ associated with $\mathfrak{p}$ (see [FKM]). The latter categories are also known as the subcategories of $\mathfrak{p}$-*presentable modules*, see [MS1], and can be alternatively defined as certain Serre quotients of category $\mathcal{O}$.

1.2 Auslander regular algebras

A finite dimensional (associative) algebra A is called *Auslander-Gorenstein*, see [Iy, CIM], provided that the (left) regular module ${}_AA$ admits a finite injective coresolution

$$0 \to A \to Q_0 \to Q_1 \to \cdots \to Q_k \to 0,$$

such that $\text{proj.dim}(Q_i) \leq i$, for all $i = 0, 1, \dots, k$. An Auslander-Gorenstein algebra of finite global dimension is called an *Auslander regular* algebra.
Auslander regular algebras have a number of remarkable homological properties, see, for example, [Iy, Theorem 1.1] and [AR, Theorem 2.1].
We identify properties of algebras with that of their module categories, so, in an appropriate case, we can say that A-mod is Auslander regular, etc.

1.3 Motivation

This paper originates from a question which the second author received from René Marczinzik in July 2020. The question was whether blocks of category $\mathcal{O}$ are Auslander regular. It was motivated by the observations that the answer is positive in small ranks based on computer calculations using the quiver and relation presentations of blocks of category $\mathcal{O}$ from [St1].

1.4 The main result

The main result of the present paper is the following statement which, in particular, answers positively and vastly generalizes the question posed by René Marczinzik (see Theorem 3, Corollary 5, Theorem 8, Corollary 9, Theorem 11 and Theorem 12):

Theorem A. *All blocks of (parabolic) category $\mathcal{O}$ are Auslander regular. All blocks of $\mathcal{S}$-subcategories in $\mathcal{O}$ are Auslander-Gorenstein.*

The first two papers [Ma3] and [Ma4] of the "Some homological properties of category $\mathcal{O}$" series were devoted to the study of projective dimension of structural modules in category $\mathcal{O}$, with the main emphasis on the projective dimension of indecomposable tilting and injective modules. Our proof of Theorem A is heavily based on these results.

1.5 General setup for similar regularity phenomena

We observe that the condition used to define Auslander-Gorenstein and Auslander regular algebras makes perfect sense in the general setup of (generalized) tilting modules in the sense of Miyashita [Mi]. Let A be a finite-dimensional algebra and T an A-module. Recall, that T is called a *(generalized) tilting module* provided that it has the following properties:

- T has finite projective dimension;

- T is ext-self-orthogonal, that is, all extensions of positive degree from T to T vanish;
- the module ${}_AA$ has a finite coresolution by modules in $\mathrm{add}(T)$.

It is a standard fact that $\mathrm{proj.dim}(T)$ equals the length of a minimal coresolution of A by modules from $\mathrm{add}(T)$.
Now, given A and a (generalized) tilting A-module T, we say that A is T-regular provided that there is a coresolution

$$0 \to A \to Q_0 \to Q_1 \to \cdots \to Q_k \to 0,$$

such that $Q_i \in \mathrm{add}(T)$ and $\mathrm{proj.dim}(Q_i) \leq i$, for all $i = 0, 1, \dots, k$.
The notion of an Auslander-Gorenstein algebra corresponds to the situation when the injective cogenerator is a (generalized) tilting module.

1.6 Regularity phenomena for various generalized tilting modules in category $\mathcal{O}$

The bounded derived category of the principal block $\mathcal{O}_0$ of category $\mathcal{O}$ admits two different actions, by derived equivalences, of the braid group associated to (W, S) where W is the Weyl group of $\mathfrak{g}$ and S the set of simple reflections. These actions are given by the so-called *twisting functors*, see [AS, KM], and *shuffling functors*, see [MS1]. These actions can be used to define the following four classes of (generalized) tilting modules in $\mathcal{O}_0$:

- twisted projective modules;
- twisted tilting modules;
- shuffled projective modules;
- shuffled tilting modules.

In Sections 8 and 9 we explore the regularity phenomena in $\mathcal{O}_0$ with respect to these four families of (generalized) tilting modules. Each of these families contains $|W|$ (generalized) tilting modules with some overlap between the families.

Problem B. *For which of the above generalized tilting modules the category $\mathcal{O}_0$ has the regularity property?*

Here is a summary of our results, see Theorems 19 and 22, Propositions 24 and 26 and Examples in Subsections 8.5 and 9.4:

Theorem C.

(*a*) *The category $\mathcal{O}_0$ has the regularity property with respect to both projective and tilting modules twisted by the longest element in a parabolic subgroup of the Weyl group.*

(*b*) *The category $\mathcal{O}_0$ has the regularity property with respect to both projective and tilting modules shuffled by a simple reflection.*

(*c*) *There exist both twisted and shuffled projective and tilting modules, with respect to which the category $\mathcal{O}_0$ does not have the regularity property.*

1.7 Projective dimension of twisted and shuffled projective and tilting modules

Theorem C suggests that a complete answer to Problem B is non-trivial. One important step here is the following problem.

Problem D. *Determine the projective dimensions of twisted and shuffled projective and tilting modules in $\mathcal{O}_0$.*

We explore Problem D in Section 10. Since twisted projective modules coincide with translated Verma modules, while twisted tilting modules coincide with translated dual Verma modules, Problem D provides a nice connection to the more recent papers [CM, KMM] in the "Some homological properties of category $\mathcal{O}$" series. One of the main results of [KMM] determines projective dimension of translated simple modules in $\mathcal{O}$. In Section 10 we propose conjectures for projective dimension of twisted and shuffled projective and tilting modules in the spirit of the results of [KMM] and prove a number of partial results. All these conjectures and results are formulated in terms of Kazhdan-Lusztig combinatorics, namely, Lusztig's $\mathfrak{a}$-function from [Lu1, Lu2] and its various generalizations studied in [CM] and [KMM]. The case of shuffled modules seems at the moment to be significantly more difficult than the case of twisted modules. The main reason for this is the fact that, in contrast to twisting functors, shuffling functors do not commute with projective functors.

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2 Auslander-Ringel regular quasi-hereditary algebras

2.1 Quasi-hereditary algebras

Let $\Bbbk$ be an algebraically closed field and A a finite dimensional (associative) $\Bbbk$-algebra. Let $L_1, L_2, \dots, L_n$ be a complete and irredundant list of isomorphism classes of simple A-modules. Note that, by fixing this list, we have fixed a linear order on the isomorphism classes of simple A-modules, this will be an essential part of the structure we are going to define now.

For $i \in \{1, 2, \ldots, n\}$, we denote by P_i and I_i the indecomposable projective cover and injective envelope of L_i, respectively. Denote by Δ_i the quotient of P_i by the trace in P_i of all P_j with $j > i$. Denote by ∇_i the submodule of I_i defined as the intersection of the kernels of all homomorphisms from I_i to I_j with $j > i$. The modules Δ_i are called *standard* and the modules ∇_i are called *costandard*.

Recall from [CPS, DR], that A is said to be *quasi-hereditary* provided that

- the endomorphism algebra of each Δ_i is $\Bbbk$;
- the regular module ${}_AA$ has a filtration with standard subquotients.

According to [Ri], if A is quasi-hereditary, then, for each i, there is a unique indecomposable module T_i, called a *tilting module*, which has both, a filtration with standard subquotients and a filtration with costandard subquotients, and, additionally, such that $[T_i : L_i] \neq 0$ while $[T_i : L_j] = 0$ for $j > i$. The module $T = \bigoplus_{i=1}^{n} T_i$ is called the *characteristic tilting module* and (the opposite of) its endomorphism algebra is called the *Ringel dual* of A.

For each $M \in A$-mod, there is a unique minimal finite complex $\mathcal{T}_\bullet(M)$ of tilting modules which is isomorphic to M in the bounded derived category of A. We will denote by $\mathbf{r}(M)$ the maximal non-negative i such that $\mathcal{T}_i(M) \neq 0$ and by $\mathbf{l}(M)$ the maximal non-negative i such that $\mathcal{T}_{-i}(M) \neq 0$. Note that $\mathbf{l}(M) = 0$ if and only if M has a filtration with standard subquotients and $\mathbf{r}(M) = 0$ if and only if M has a filtration with costandard subquotients. We refer to [MO2] for further details.

2.2 Auslander-Ringel regular algebras

We say that a quasi-hereditary algebra A is *Auslander-Ringel regular* provided that there is a coresolution

$$0 \to A \to Q_0 \to Q_1 \to \cdots \to Q_k \to 0,$$

such that each $Q_i \in \mathrm{add}(T)$ and $\mathrm{proj.dim}(Q_i) \leq i$, for all $i = 0, 1, \ldots, k$. Note that, being quasi-hereditary, A has finite global dimension (see [CPS, DR]) and that the characteristic tilting module is a (generalized) tilting module. Thus, Auslander-Ringel regularity corresponds to T-regularity in the terminology of Subsection 1.5.

In Section 3, we will see that blocks of (parabolic) BGG category $\mathcal{O}$ are Auslander-Ringel regular.

3 Regularity phenomena in category $\mathcal{O}$

3.1 Category $\mathcal{O}$

We refer the reader to [Hu] for details and generalities about category $\mathcal{O}$.
We denote by $\mathcal{O}_0$ the principal block of $\mathcal{O}$, that is, the indecomposable direct summand of $\mathcal{O}$ containing the trivial $\mathfrak{g}$-module. The simple modules in $\mathcal{O}_0$ are simple highest weight modules, and their isomorphism classes are naturally indexed by elements of the Weyl group W. For $w \in W$, we denote by L_w the simple highest weight module in $\mathcal{O}_0$ with highest weight $w \cdot 0$, where 0 is the zero element in $\mathfrak{h}^*$ and $\cdot$ is the dot action of W.
We denote by P_w and I_w the indecomposable projective cover and injective envelope of L_w in $\mathcal{O}_0$, respectively. Let A be a basic, finite dimensional, associative algebra such that $\mathcal{O}_0$ is equivalent to A-mod. It is well-known that A is quasi-hereditary with respect to any linear order which extends the dominance order on weights. The latter is given by $\lambda \leq \mu$ if and only if $\mu - \lambda$ is a linear combination of positive roots with non-negative integer coefficients.
By [So1], the algebra A admits a Koszul $\mathbb{Z}$-grading. We denote by ${}^{\mathbb{Z}}\mathcal{O}_0$ the category of $\mathbb{Z}$-graded finite-dimensional A-modules. We denote by $\langle 1 \rangle$ the shift of grading which maps degree 0 to degree -1. We fix standard graded lifts of structural modules so that

- L_w is concentrated in degree zero;
- the top of P_w is concentrated in degree zero;
- the socle of I_w is concentrated in degree zero;
- the top of Δ_w is concentrated in degree zero;
- the socle of ∇_w is concentrated in degree zero;
- the canonical map $\Delta_w \hookrightarrow T_w$ is homogeneous of degree zero.

For $w \in W$, we denote by θ_w the indecomposable projective endofunctor of $\mathcal{O}_0$, see [BG], uniquely defined by the property $\theta_w P_e \cong P_w$. By [St2], θ_w admits a natural graded lift normalized by the same condition.
We denote by $\geq_{\mathtt{L}}$, $\geq_{\mathtt{R}}$ and $\geq_{\mathtt{J}}$ the Kazhdan-Luszitg left, right and two-sided orders, respectively.

3.2 $\mathcal{O}_0$ is Auslander-Ringel regular

Theorem 1. *The category $\mathcal{O}_0$ is Auslander-Ringel regular.*

Proof. Consider the category $\mathcal{LT}(\mathcal{O}_0)$ of linear complexes of tilting modules in $\mathcal{O}_0$, see [Ma1, MO1]. The algebra A is a balanced quasi-hereditary algebra in the sense of [Ma2] and hence $\mathcal{LT}(\mathcal{O}_0)$ contains the tilting coresolution $\mathcal{T}_\bullet(P_e)$ of the dominant standard module $\Delta_e = P_e$.

Due to the Ringel-Koszul self-duality of $\mathcal{O}_0$, the category $\mathcal{LT}(\mathcal{O}_0)$ is equivalent to ${}^{\mathbb{Z}}\mathcal{O}_0$. This implies that the multiplicity of $T_w\langle i\rangle$ as a summand of $\mathcal{T}_i(P_e)$ coincides with the composition multiplicity of $L_{w_0w^{-1}w_0}\langle -i\rangle$ in Δ_e. The latter is given by Kazhdan-Lusztig combinatorics for (W,S). In particular, it is nonzero only if

$$\mathbf{a}(w)=\mathbf{a}(w_0w^{-1}w_0)\leq i\leq \ell(w_0w^{-1}w_0)=\ell(w),$$

where $\ell(w)$ is the length of w and $\mathbf{a}$ is Lusztig's $\mathbf{a}$-function from [Lu1, Lu2]. Consequently, T_w can appear (up to shift of grading) only in homological positions i such that $\mathbf{a}(w)\leq i\leq \ell(w)$. Taking into account that $\text{proj.dim.}(T_w)=\mathbf{a}(w)$ by [Ma3, Ma4], it follows that $\mathcal{T}_i(P_e)$ has projective dimension at most i. For $x\in W$, applying θ_x to $\mathcal{T}_\bullet(P_e)$ gives a tilting coresolution of P_x (not necessarily minimal or linear). Since θ_x is exact and sends projectives to projectives, it cannot increase the projective dimension. This means that

$$\text{proj.dim.}(\theta_x\mathcal{T}_i(P_e))\leq \text{proj.dim.}(\mathcal{T}_i(P_e))\leq i.$$

The claim of the theorem follows. □

Corollary 2.

(*i*) *Let* $\mathcal{P}_\bullet(T)$ *be a minimal projective resolution of* T*. Then* $\mathbf{r}(\mathcal{P}_{-i}(T))\leq i$*, for all* $i\geq 0$*.*

(*ii*) *Let* $\mathcal{T}_\bullet(I)$ *be a minimal tilting resolution of the basic injective cogenerator* I*. Then we have* $\text{inj.dim.}(\mathcal{T}_{-i}(I))\leq i$*, for all* $i\geq 0$*.*

(*iii*) *Let* $\mathcal{I}_\bullet(T)$ *be a minimal injective coresolution of* T*. Then* $\mathbf{l}(\mathcal{I}_i(T))\leq i$*, for all* $i\geq 0$*.*

Recall that the functions $\mathbf{r}$ and $\mathbf{l}$ were defined in Subsection 2.1.

Proof. Claim (ii) is obtained from Theorem 1 using the simple preserving duality on $\mathcal{O}$. Since $\mathcal{O}_0$ is Ringel self-dual, Claim (i) is the Ringel dual of Theorem 1 and, finally, Claim (iii) is the Ringel dual of Claim (ii). □

3.3 $\mathcal{O}_0$ is Auslander regular

Theorem 3. *The category* $\mathcal{O}_0$ *is Auslander regular.*

We will need the following auxiliary statement.

Lemma 4. *For* $w\in W$*, let* $\mathcal{I}_\bullet(T_{w_0w})$ *be a minimal injective coresolution of* T_{w_0w}*. Then we have:*

(*i*) *The maximal value of* i *such that* $\mathcal{I}_i(T_{w_0w})\neq 0$ *equals* $\mathbf{a}(w_0w)$*.*

(*ii*) *Each indecomposable direct summand of* $\mathcal{I}_\bullet(T_{w_0w})$ *is isomorphic, up to a graded shift, to* I_x*, for some* $x\geq_{\mathtt{J}} w$*.*

(*iii*) *If* $\mathcal{I}_i(T_{w_0w})$ *has a direct summand isomorphic, up to a graded shift, to* I_x*, for some* $x \in W$*, then* $i \geq \mathbf{a}(w_0x)$*.*

Proof. Using the simple preserving duality, Claim (i) is one of the main results of [Ma3, Ma4].
Since $T_{w_0w} \cong \theta_w T_{w_0}$, to prove Claim (ii), it is enough to prove the same statement for $\theta_w \mathcal{I}_\bullet(T_{w_0})$. But we have

$$\theta_w I_y = \theta_w\theta_y I_e = \bigoplus_{z\in W} \theta_z^{\oplus m_{w,y}^z} I_e = \bigoplus_{z\in W} I_z^{\oplus m_{w,y}^z}$$

and $m_{w,y}^z \neq 0$ only if $z \geq_{\mathtt{J}} w$.
Let us prove Claim (iii). We start with the case $w = e$. Due to Koszulity of $\mathcal{O}_0$, the minimal injective coresolution $\mathcal{I}_\bullet(T_{w_0})$ of the antidominant tilting=simple module $T_{w_0} = L_{w_0}$ is linear and hence is an object in the category $\mathcal{LI}(\mathcal{O}_0)$ of linear complexes of injective modules and is isomorphic to the dominant standard=projective object in this category.
Due to the Koszul self-duality of $\mathcal{O}_0$, the category $\mathcal{LI}(\mathcal{O}_0)$ is equivalent to ${}^{\mathbb{Z}}\mathcal{O}_0$. This implies that the multiplicity of $I_x\langle i\rangle$ as a summand of $\mathcal{I}_i(T_{w_0})$ coincides with the composition multiplicity of $L_{w_0x^{-1}}\langle -i\rangle$ in Δ_e. The latter is given by Kazhdan-Lusztig combinatorics. In particular, it is non-zero only if $\mathbf{a}(w_0x^{-1}) \leq i \leq \ell(w_0x^{-1})$. Consequently, the module I_x can appear (up to shift of grading) only in homological positions i such that $\mathbf{a}(w_0x) = \mathbf{a}(w_0x^{-1}) \leq i \leq \ell(w_0x^{-1})$. This proves Claim (iii) in the case $w = e$. The general case is obtained from this one applying θ_w. □

Proof of Theorem 3. Take the minimal tilting coresolution $\mathcal{T}_\bullet(P_e)$ of P_e considered in the proof of Theorem 1. We can take a minimal injective coresolution of each T_x, up to grading shift, appearing in $\mathcal{T}_\bullet(P_e)$ and glue these into an injective coresolution $\mathcal{I}_\bullet$ of P_e. Applying θ_w to $\mathcal{I}_\bullet$, gives an injective coresolution of P_w without increasing the projective dimensions of homological positions. By [Ma3, Ma4], the projective dimension of I_x is $2\mathbf{a}(w_0x)$. It is thus enough to show that any graded shift of I_x appearing in $\mathcal{I}_\bullet$ appears only in homological positions i such that $i \geq 2\mathbf{a}(w_0x)$.
By Lemma 4, I_x can only appear in homological position at least $\mathbf{a}(w_0x)$ when coresolving T_y. Furthermore, again by Lemma 4, I_x can only appear in coresolutions of T_{w_0y}, where $x \geq_{\mathtt{J}} y$. By Theorem 1, such T_{w_0y} appears in $\mathcal{T}_\bullet(P_e)$ in homological positions at least $\operatorname{proj.dim} T_{w_0y} = \mathbf{a}(w_0y)$. Adding these two estimates together, we obtain that I_x appears in $\mathcal{I}_\bullet$ in homological positions at least $\mathbf{a}(w_0y) + \mathbf{a}(w_0x) \geq 2\mathbf{a}(w_0x)$. This completes the proof. □

3.4 Singular blocks

Corollary 5. *All blocks of* $\mathcal{O}$ *are Auslander-Ringel regular, Auslander regular, and have the properties described in Corollary 2.*

Proof. Due to Soergel's combinatorial description of blocks of $\mathcal{O}$ from [So1], each block of category $\mathcal{O}$ is equivalent to an integral block of $\mathcal{O}$ (possibly for a different Lie algebra). Therefore we may restrict our attention to integral blocks.
Each regular integral block is equivalent to $\mathcal{O}_0$. Each singular integral block is obtained from a regular integral block using translation to the corresponding wall. These translation functors are exact, send projectives to projectives, injectives to injectives and tiltings to tiltings and do not increase projective dimension, injective dimension, nor the values of $\mathbf{l}$ and $\mathbf{r}$. Therefore the claim follows from Theorems 1 and 3 and Corollary 2 applying these translation functors. □

3.5 $\mathfrak{sl}_3$-EXAMPLE

For the Lie algebra $\mathfrak{sl}_3$, we have $W = \{e, s, t, st, ts, w_0 = sts = tst\}$. The projective dimensions of the indecomposable tilting and injective modules in $\mathcal{O}_0$ are given by:

w	e	s	t	st	ts	w_0
proj.dim(T_w)	0	1	1	1	1	3

w	e	s	t	st	ts	w_0
proj.dim(I_w)	6	2	2	2	2	0

The minimal (ungraded) tilting coresolutions of the indecomposable projectives in $\mathcal{O}_0$ are:

$$0 \to P_e \to T_e \to T_s \oplus T_t \to T_{st} \oplus T_{ts} \to T_{w_0} \to 0,$$

$$0 \to P_s \to T_e \to T_t \to 0,$$

$$0 \to P_t \to T_e \to T_s \to 0,$$

$$0 \to P_{st} \to T_e \to T_{ts} \to 0,$$

$$0 \to P_{ts} \to T_e \to T_{st} \to 0,$$

$$0 \to P_{w_0} \to T_e \to 0,$$

The minimal (ungraded) injective coresolutions of the indecomposable projectives in $\mathcal{O}_0$ are:

$$0 \to P_e \to I_{w_0} \to I_{w_0}^{\oplus 2} \to I_t \oplus I_s \oplus I_{w_0}^{\oplus 2} \to I_{ts} \oplus I_{st} \oplus I_{w_0} \to I_{st} \oplus I_{ts} \to I_s \oplus I_t \to I_e \to 0,$$

$$0 \to P_s \to I_{w_0} \to I_{w_0} \to I_s \to 0,$$

$$0 \to P_t \to I_{w_0} \to I_{w_0} \to I_t \to 0,$$

$$0 \to P_{st} \to I_{w_0} \to I_{w_0} \to I_{st} \to 0,$$

$$0 \to P_{ts} \to I_{w_0} \to I_{w_0} \to I_{ts} \to 0,$$

$$0 \to P_{w_0} \to I_{w_0} \to 0,$$

4 Regularity phenomena in parabolic category $\mathcal{O}^{\mathfrak{p}}$

4.1 Parabolic category $\mathcal{O}^{\mathfrak{p}}$

Fix a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{h} \oplus \mathfrak{n}_+$. Denote by $\mathcal{O}^{\mathfrak{p}}$ the full subcategory of $\mathcal{O}$ consisting of all objects the action of $U(\mathfrak{p})$ on which is locally finite, see [RC]. Then $\mathcal{O}^{\mathfrak{p}}$ is the Serre subcategory of $\mathcal{O}$ generated by all simple modules whose highest weights are (dot-)dominant (by which we mean it is the largest weight in its orbit under the dot action) and integral with respect to the Levi factor of $\mathfrak{p}$.

Similarly to Subsection 3.4, we can start with the integral regular situation. Let $W_{\mathfrak{p}}$ denote the Weyl group of the Levi factor of $\mathfrak{p}$ which we view as a parabolic subgroup of W. We denote by $w_0^{\mathfrak{p}}$ the longest element in $W_{\mathfrak{p}}$. The principal block $\mathcal{O}_0^{\mathfrak{p}}$ is the Serre subcategory of $\mathcal{O}_0^{\mathfrak{p}}$ generated by L_w, where w belongs to the set $\mathrm{short}({}_{W_{\mathfrak{p}}}\backslash W)$ of shortest coset representatives for cosets in ${}_{W_{\mathfrak{p}}}\backslash W$.

4.2 $\mathcal{O}_0^{\mathfrak{p}}$ is Auslander-Ringel regular

Theorem 6. *The category $\mathcal{O}_0^{\mathfrak{p}}$ is Auslander-Ringel regular.*

Proof. The proof is similar to the proof of Theorem 1, so we only emphasize the differences. By [BGS], the Koszul dual of $\mathcal{O}_0^{\mathfrak{p}}$ is the singular integral block $\mathcal{O}_\lambda$ of $\mathcal{O}$ where λ is chosen such that the dot-stabilizer of λ equals $W_{\mathfrak{p}'}$ where $\mathfrak{p}'$ is the w_0-conjugate of $\mathfrak{p}$. By [So2], the block $\mathcal{O}_\lambda$ is Ringel self-dual, and by [Ma2], the Ringel duality and the Koszul duality commute. Therefore, the category of linear complexes of tilting modules in $\mathcal{O}_0^{\mathfrak{p}}$ is equivalent to ${}^{\mathbb{Z}}\mathcal{O}_\lambda$.

By [Ma1], the tilting coresolution of the dominant projective (=standard) module in $\mathcal{O}_0^{\mathfrak{p}}$ is $\Delta(\lambda)$, the dominant standard object in ${}^{\mathbb{Z}}\mathcal{O}_\lambda$. Denoting by $T_0^\lambda : {}^{\mathbb{Z}}\mathcal{O}_0 \to {}^{\mathbb{Z}}\mathcal{O}_\lambda$ the graded translation functor to the λ-wall, we have $\Delta(\lambda) \cong T_0^\lambda \Delta_e \langle \ell(w_0^{\mathfrak{p}'}) \rangle$. This means that the degree i component of $\Delta(\lambda)$ consists of $T_0^\lambda L_u$ where L_u belongs to the degree $i + \ell(w_0^{\mathfrak{p}}) = i + \ell(w_0^{\mathfrak{p}'})$ component of Δ_e and such that $u \in \mathrm{long}(W/_{W_{\mathfrak{p}'}}) = w_0(\mathrm{long}({}_{W_{\mathfrak{p}}}\backslash W))^{-1} w_0$. It follows that the i-th component in the tilting coresolution contains only $T_x^{\mathfrak{p}}$ where $x \in \mathrm{short}({}_{W_{\mathfrak{p}}}\backslash W)$ is such that $\mathbf{a}(w_0(w_0^{\mathfrak{p}}x)^{-1}w_0) \geq i + \ell(w_0^{\mathfrak{p}}) = i + \mathbf{a}(w_0^{\mathfrak{p}})$.

It remains to check from [CM, Table 2] that

$$\mathrm{proj.\,dim}\, T_x^{\mathfrak{p}} = \mathbf{a}(w_0^{\mathfrak{p}}x) - \mathbf{a}(w_0^{\mathfrak{p}}) = \mathbf{a}(w_0(w_0^{\mathfrak{p}}x)^{-1}w_0) - \mathbf{a}(w_0^{\mathfrak{p}})$$

and compare with the condition in the previous paragraph. This proves the regularity property for the tilting coresolution of the dominant projective.

The regularity property for other projective modules in $\mathcal{O}_0^{\mathfrak{p}}$ is obtained by applying projective functors exactly as in Theorem 1. □

Let $P^{\mathfrak{p}}$ denote a projective generator, $I^{\mathfrak{p}}$ an injective cogenerator, and $T^{\mathfrak{p}}$ the characteristic tilting module in $\mathcal{O}_0^{\mathfrak{p}}$. Similarly to Corollary 2 (using that $\mathcal{O}_0^{\mathfrak{p}}$ is equivalent to its Ringel dual $\mathcal{O}_0^{\mathfrak{p}'}$), we have:

COROLLARY 7.

(*i*) *Let* $\mathcal{P}_\bullet(T^{\mathfrak{p}})$ *be a minimal projective resolution of* $T^{\mathfrak{p}}$ *in* $\mathcal{O}_0^{\mathfrak{p}}$. *Then* $\mathbf{r}(\mathcal{P}_{-i}(T^{\mathfrak{p}})) \leq i$, *for all* $i \geq 0$.

(*ii*) *Let* $\mathcal{T}_\bullet(I^{\mathfrak{p}})$ *be a minimal tilting resolution of the basic injective cogenerator* $I^{\mathfrak{p}}$ *in* $\mathcal{O}_0^{\mathfrak{p}}$. *Then* inj.dim.$(\mathcal{T}_{-i}(I^{\mathfrak{p}})) \leq i$, *for all* $i \geq 0$.

(*iii*) *Let* $\mathcal{I}_\bullet(T^{\mathfrak{p}})$ *be a minimal injective coresolution of* $T^{\mathfrak{p}}$ *in* $\mathcal{O}_0^{\mathfrak{p}}$. *Then* $\mathbf{l}(\mathcal{I}_i(T^{\mathfrak{p}})) \leq i$, *for all* $i \geq 0$.

4.3 $\mathcal{O}_0^{\mathfrak{p}}$ IS AUSLANDER REGULAR

THEOREM 8. *The category* $\mathcal{O}_0^{\mathfrak{p}}$ *is Auslander regular.*

Proof. Mutatis mutandis the proof of Theorem 3. Again, one could emphasize the $2\mathfrak{a}(w_0^{\mathfrak{p}}) = 2\ell(w_0^{\mathfrak{p}})$ shift for the projective dimension of injective modules in $\mathcal{O}_0^{\mathfrak{p}}$ in [CM, Table 2]. □

4.4 SINGULAR BLOCKS

COROLLARY 9. *All blocks of* $\mathcal{O}^{\mathfrak{p}}$ *are both Auslander-Ringel regular and Auslander regular and have the properties described in Corollary 7.*

Proof. Mutatis mutandis the proof of Corollary 5. □

4.5 $\mathfrak{sl}_3$-EXAMPLE

For the Lie algebra $\mathfrak{sl}_3$, we have $W = \{e, s, t, st, ts, w_0 = sts = tst\}$. Assume that $W_{\mathfrak{p}} = \{e, s\}$, then $\mathrm{short}({}_{W_{\mathfrak{p}}}\backslash W) = \{e, t, ts\}$. The projective dimensions of the indecomposable tilting and projective modules in $\mathcal{O}_0^{\mathfrak{p}}$ are given by:

w :	e	t	ts
proj.dim$(T_w^{\mathfrak{p}})$:	0	0	2

w :	e	t	ts
proj.dim$(I_w^{\mathfrak{p}})$:	4	0	0

The minimal (ungraded) tilting coresolutions of the indecomposable projectives in $\mathcal{O}_0^{\mathfrak{p}}$ are:

$$0 \to P_e^{\mathfrak{p}} \to T_e^{\mathfrak{p}} \to T_t^{\mathfrak{p}} \to T_{ts}^{\mathfrak{p}} \to 0,$$
$$0 \to P_t^{\mathfrak{p}} \to T_e^{\mathfrak{p}} \to 0,$$
$$0 \to P_{ts}^{\mathfrak{p}} \to T_t^{\mathfrak{p}} \to 0,$$

The minimal (ungraded) injective coresolutions of the indecomposable projectives in $\mathcal{O}_0^{\mathfrak{p}}$ are:

$$0 \to P_e^{\mathfrak{p}} \to I_t^{\mathfrak{p}} \to I_s^{\mathfrak{p}} \to I_s^{\mathfrak{p}} \to I_t^{\mathfrak{p}} \to I_e^{\mathfrak{p}} \to 0,$$
$$0 \to P_t^{\mathfrak{p}} \to I_t^{\mathfrak{p}} \to 0,$$
$$0 \to P_{ts}^{\mathfrak{p}} \to I_{ts}^{\mathfrak{p}} \to 0,$$

5 Auslander-Ringel-Gorenstein strongly standardly stratified algebras

5.1 Strongly standardly stratified algebras

In this section we return to the general setup of Subsection 2.1.

For $i \in \{1, 2, \dots, n\}$, we denote by $\overline{\Delta}_i$ the maximal quotient of Δ_i satisfying $[\overline{\Delta}_i : L_i] = 1$. Denote by $\overline{\nabla}_i$ the maximal submodule of ∇_i satisfying $[\overline{\nabla}_i : L_i] = 1$. The modules $\overline{\Delta}_i$ are called *proper standard* and the modules $\overline{\nabla}_i$ are called *proper costandard*.

Recall that A is said to be *standardly stratified* provided that the regular module ${}_AA$ has a filtration with standard subquotients and *strongly strandardly stratified* (see [Fr]) if, further, each standard module has a filtration with proper standard subquotients.

If A is a strongly standardly stratified algebra, then, by [AHLU], for each i, there is a unique indecomposable module T_i, called a *tilting module*, which has both a filtration with standard subquotients and a filtration with proper costandard subquotients, and, additionally, such that $[T_i : L_i] \neq 0$ while $[T_i : L_j] = 0$, for $j > i$. The module $T = \bigoplus_{i=1}^{n} T_i$ is called the *characteristic tilting module* and (the opposite of) its endomorphism algebra is called the *Ringel dual* of A. For each $M \in A$-mod, there is a unique minimal bounded from the right complex $\mathcal{T}_\bullet(M)$ of tilting modules which is isomorphic to M in the bounded derived category of A. We will denote by $\mathbf{r}(M)$ the maximal non-negative i such that $\mathcal{T}_i(M) \neq 0$. Note that $\mathbf{r}(M) = 0$ if and only if M has a filtration with proper costandard subquotients.

5.2 Auslander-Ringel-Gorenstein algebras

Let A be strongly standardly stratified. Then an A-module having a filtration with standard subquotients has a (finite) coresolution by modules in $\mathrm{add}(T)$. It is also well-known that T has finite projective dimension (see [Fr, AHLU]). We will say that A is *Auslander-Ringel-Gorenstein* provided that there is a coresolution

$$0 \to A \to Q_0 \to Q_1 \to \cdots \to Q_k \to 0,$$

such that each $Q_i \in \mathrm{add}(T)$ and $\mathrm{proj.dim}(Q_i) \leq i$, for all $i = 0, 1, \dots, k$.

Since the characteristic tilting module is a (generalized) tilting module, Auslander-Ringel-Gorenstein property agrees with T-regularity in the terminology of Subsection 1.5.

6 Regularity phenomena in $\mathcal{S}$-subcategories in $\mathcal{O}$

6.1 $\mathcal{S}$-subcategories in $\mathcal{O}$

We again fix a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{h} \oplus \mathfrak{n}_+$ and restrict our attention to the integral part $\mathcal{O}_{\text{int}}$ of $\mathcal{O}$.
Let $\mathcal{X}$ denote the Serre subcategory of $\mathcal{O}_{\text{int}}$ generated by all simple highest weight modules whose highest weights λ are not anti-dominant with respect to $W_\mathfrak{p}$, that is, $w \cdot \lambda < \lambda$ for some $w \in W_\mathfrak{p}$. Denote by $\mathcal{S}^\mathfrak{p}$ the Serre quotient category $\mathcal{O}_{\text{int}}/\mathcal{X}$, see [FKM, MS1]. From [FKM], we know that blocks of $\mathcal{S}^\mathfrak{p}$ correspond to strongly standardly stratified algebras.
Let $\mathcal{S}_0^\mathfrak{p}$ be the principal block of $\mathcal{S}^\mathfrak{p}$.

6.2 $\mathcal{S}_0^\mathfrak{p}$ is Auslander-Ringel-Gorenstein

Theorem 10. *The category $\mathcal{S}_0^\mathfrak{p}$ is Auslander-Ringel-Gorenstein.*

Proof. By [FKM], the indecomposable projectives in $\mathcal{S}_0^\mathfrak{p}$ are exactly the images of P_w, where w belongs to $\text{long}({}_{W_\mathfrak{p}}\backslash W)$ (the set of longest coset representatives in ${}_{W_\mathfrak{p}}\backslash W$). Furthermore, the indecomposable tilting objects in $\mathcal{S}_0^\mathfrak{p}$ are exactly the images of T_w, where $w \in \text{short}({}_{W_\mathfrak{p}}\backslash W)$.
Note that the above objects in $\mathcal{O}$ are exactly those indecomposable projective (resp. tilting) objects which are admissible in the sense of [MPW, Lemma 14]. From [MPW, Lemma 14 and Theorem 15] it follows that the minimal projective resolution (in $\mathcal{O}$) of any T_w as above contains only P_x as above. The Ringel dual of this property is that a minimal tilting coresolution (in $\mathcal{O}$) of any P_x as above contains only T_w as above. Since the projection functor $\mathcal{O}_0 \twoheadrightarrow \mathcal{S}_0^\mathfrak{p}$ is exact and preserves the projective dimension for the involved projective and tilting modules, see [MPW, Theorem 15], the claim of our theorem follows from Theorem 1. □

6.3 $\mathcal{S}_0^\mathfrak{p}$ is Auslander-Gorenstein

Theorem 11. *The category $\mathcal{S}_0^\mathfrak{p}$ is Auslander-Gorenstein.*

Proof. The indecomposable injectives in $\mathcal{S}_0^\mathfrak{p}$ are exactly the images of I_w for $w \in \text{long}({}_{W_\mathfrak{p}}\backslash W)$ and these $I_w \in \mathcal{O}$ are admissible in the sense of [MPW]. Thus, the claim follows from Theorem 3 similarly to the proof of Theorem 10. □

6.4 Singular blocks

Theorem 12. *All blocks of $\mathcal{S}^\mathfrak{p}$ are both Auslander-Ringel-Gorenstein and Auslander-Gorenstein.*

Proof. Mutatis mutandis the proof of Corollary 5 □

6.5 $\mathfrak{sl}_3$-example

For the Lie algebra $\mathfrak{sl}_3$, we have $W = \{e, s, t, st, ts, w_0 = sts = tst\}$. Assume that $W_\mathfrak{p} = \{e, s\}$, then $\mathrm{long}({}_{W_\mathfrak{p}}\backslash W) = \{s, st, w_0\}$. The projective dimensions of the indecomposable tilting and injective modules in $\mathcal{S}_0^\mathfrak{p}$ are given by:

$w:$	e	t	ts
proj.dim(T_w) :	0	1	1

$w:$	s	st	w_0
proj.dim(I_w) :	2	2	0

The minimal (ungraded) tilting coresolutions of the indecomposable projectives in $\mathcal{S}_0^\mathfrak{p}$ are:

$$0 \to P_s \to T_e \to T_t \to 0,$$

$$0 \to P_{st} \to T_e \to T_{ts} \to 0,$$

$$0 \to P_{w_0} \to T_e \to 0,$$

The minimal (ungraded) injective coresolutions of the indecomposable projectives in $\mathcal{S}_0^\mathfrak{p}$ are:

$$0 \to P_s \to I_{w_0} \to I_{w_0} \to I_s \to 0,$$

$$0 \to P_{st} \to I_{w_0} \to I_{w_0} \to I_{st} \to 0,$$

$$0 \to P_{w_0} \to I_{w_0} \to 0,$$

7 Applications to the cohomology of twisting and Serre functors

7.1 Twisting and Serre functors on $\mathcal{O}$

For a simple reflection s, we denote by $\top_s$ the corresponding twisting functor on $\mathcal{O}$, see [AS]. For $w \in W$, with a fixed reduced expression $w = s_1 s_2 \cdots s_k$, we denote by $\top_w$ the composition $\top_{s_1}\top_{s_2}\cdots\top_{s_k}$ and note that it does not depend on the choice of a reduced expression by [KM].
All functors $\top_w$ are right exact, functorially commute with projective functors, acyclic on Verma modules and the corresponding derived functors are self-equivalences of the derived category of $\mathcal{O}$. Furthermore, we have $\top_{w_0} P_x \cong T_{w_0 x}$ and $\top_{w_0} T_x \cong I_{w_0 x}$, for all $x \in W$. We refer to [AS, KM] for all details.
The functor $\mathcal{L}\top_{w_0}^2$ is a Serre functor on $\mathcal{D}^b(\mathcal{O}_0)$, see [MS2].

7.2 Auslander regularity via Serre functors

Let A be a finite dimensional associative algebra of finite global dimension over an algebraically closed field $\Bbbk$. Then the left derived $\mathcal{L}\mathbf{N}$ of the Nakayama functor $\mathbf{N} = A^* \otimes_A {}_-$ for A is a Serre functor on $\mathcal{D}^b(A)$.
Recall that L_i, where $i = 1, 2, \ldots, k$, is a complete and irredundant list of simple A-modules, P_i denotes the indecomposable projective cover of L_i and I_i denotes the indecomposable injective envelope of L_i. Let P be a basic projective generator of A-mod and I a basic injective cogenerator of A-mod.

Lemma 13. *For $M \in A$-mod, $j \in \{1, 2, \dots, k\}$ and $i \in \mathbb{Z}_{\geq 0}$, we have*

$$\dim \mathrm{Ext}^i_A(M, P_j) = (\mathcal{L}_i\mathbf{N}(M) : L_j).$$

Proof. Being a Serre functor, $\mathcal{L}\mathbf{N}$ is a self-equivalence of $\mathcal{D}^b(A)$. Therefore, we have

$$\begin{aligned}\mathrm{Ext}^i_A(M, P_j) &= \mathrm{Hom}_{\mathcal{D}^b(A)}(M, P_j[i]) \\ &= \mathrm{Hom}_{\mathcal{D}^b(A)}(\mathcal{L}\mathbf{N}(M), \mathcal{L}\mathbf{N}(P_j[i])) \\ &= \mathrm{Hom}_{\mathcal{D}^b(A)}(\mathcal{L}\mathbf{N}(M), I_j[i]).\end{aligned}$$

The claim of the lemma follows. □

The above observation has the following consequence:

Proposition 14. *The algebra A is Auslander regular if and only if, for any simple A-module L_j, we have $\mathcal{L}_i\mathbf{N}(L_j) = 0$, for all $i < \mathrm{proj.dim}(I_j)$.*

Proof. By definition, A is Auslander regular if and only if, for any simple A-module L_j, we have $\mathrm{Ext}^i(L_j, A) = 0$ unless $i \geq \mathrm{proj.dim}(I_j)$. Now the necessary claim follows from Lemma 13. □

7.3 Cohomology of twisting and Serre functors for category $\mathcal{O}_0$

Corollary 15. *For $w \in W$, we have $\mathcal{L}_i\top^2_{w_0}L_w = 0$ for all $0 \leq i < 2\mathbf{a}(w_0w)$.*

Proof. By Theorem 3, $\mathcal{O}_0$ is Auslander regular. By the main results of [Ma3, Ma4], the projective dimension of I_w equals $2\mathbf{a}(w_0w)$. Therefore the claim follows from Proposition 14. □

Corollary 15 admits the following refinement.

Proposition 16. *For $w \in W$, we have $\mathcal{L}_i\top_{w_0}L_w = 0$, for all $0 \leq i < \mathbf{a}(w_0w)$.*

Proof. The injective resolution $\mathcal{I}_\bullet(L_{w_0})$ of $T_{w_0} = L_{w_0}$ is linear and is a dominant standard object in the category of linear complexes of injective modules in $\mathcal{O}_0$, by the Koszul self-duality of $\mathcal{O}_0$, see [So1]. Therefore, for $x \in W$, the module I_x can only appear as a summand of $\mathcal{I}_i(L_{w_0})$, for $\mathbf{a}(w_0x^{-1}) = \mathbf{a}(w_0x) \leq i$. This means that

$$\mathrm{Ext}^i_{\mathcal{O}}(L_w, T_{w_0}) = 0, \text{ for all } i < \mathbf{a}(w_0w).$$

Note that, for any projective functor θ, all simple subquotients L_x of the module θL_w satisfy $\mathbf{a}(w_0x) \geq \mathbf{a}(w_0w)$. Therefore, for the adjoint θ' of θ, the previous paragraph implies that

$$\mathrm{Ext}^i_{\mathcal{O}}(L_w, \theta T_{w_0}) = \mathrm{Ext}^i_{\mathcal{O}}(\theta' L_w, T_{w_0}) = 0, \text{ for all } i < \mathbf{a}(w_0w).$$

To sum up, for any tilting module T, we have

$$\mathrm{Ext}^i_{\mathcal{O}}(L_w, T) = 0, \text{ for all } i < \mathbf{a}(w_0w).$$

Applying the equivalence $\mathcal{L}\top_{w_0}$ and noting that it sends tilting modules to injective, we obtain the claim of the proposition. □

Now we prove a result "in the opposite direction". Let I be an injective cogenerator of $\mathcal{O}_0$.

Proposition 17. *For $w \in W$, we have $\mathcal{L}_i \top_{w_0} L_w = 0$, for all $i > \ell(w_0 w)$.*

Proof. We want to prove that $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathcal{L}\top_{w_0} L_w, I[i]) = 0$, for all $i > \ell(w_0 w)$. Applying the adjoint of the equivalence $\mathcal{L}\top_{w_0}$, we get an equivalent statement that $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(L_w, T[i]) = 0$, for all $i > \ell(w_0 w)$, where T is the characteristic tilting module in $\mathcal{O}_0$.
Consider the linear complex $\mathcal{T}_\bullet(L_w)$ of tilting modules which represents L_w. By [Ma1], it is a tilting object in the category of linear complexes of tilting modules. Combining the Ringel and Koszul self-dualities of $\mathcal{O}_0$, we obtain that the absolute value of the minimal non-zero component of $\mathcal{T}_\bullet(L_w)$ equals the maximal degree of a non-zero component of $T_{w_0 w^{-1} w_0}$. The latter is equal to $\ell(w_0 w)$. Now the necessary claim follows from [Ha, Chapter III(2), Lemma 2.1]. □

Proposition 18. *For $w \in W$, we have $[\mathcal{L}_i \top_{w_0} I : L_w] \neq 0$ only if $i \leq \mathbf{a}(w_0 w)$.*

Proof. Applying projective functors, the statement reduces to the special case when I is substituted by $I_e = \nabla_e$. Note that $[\mathcal{L}_i \top_{w_0} \nabla_e : L_w]$ equals the dimension of $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathcal{L}\top_{w_0} \nabla_e, I_w[i])$.
Now, we write $\nabla_e = \mathcal{L}\top_{w_0} T_{w_0}$. Moving $(\mathcal{L}\top_{w_0})^2$ from the first argument to the second using adjunction, we arrive to the space $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(T_{w_0}, P_w[i])$. Now the necessary claim follows from the observation that $\mathbf{r}(P_w) = \mathbf{a}(w_0 w)$, which is the Ringel dual of the main results of [Ma3, Ma4]. □

7.4 $\mathfrak{sl}_3$-example

In the case of $\mathfrak{sl}_3$, we have $W = \{e, s, t, st, ts, w_0\}$. In Figure 7.4, we give an explicit $\mathbb{Z}$-graded description of composition factors of the tilting resolution

$$T_{w_0} \hookrightarrow T_{st} \oplus T_{ts} \to T_s \oplus T_t \to T_e$$

of ∇_e and its image after applying $\mathcal{L}\top_{w_0}$. The original resolution is in magenta and black with ∇_e being the magenta part. The simple subquotients added during the application of $\mathcal{L}\top_{w_0}$ are blue. The resulting cohomology in negative positions is boxed. The module L_w is denoted by w. The values of the **a**-function are as follows: $\mathbf{a}(e) = 0$, $\mathbf{a}(s) = \mathbf{a}(t) = \mathbf{a}(st) = \mathbf{a}(ts) = 1$, $\mathbf{a}(w_0) = 3$.

8 Regularity phenomena with respect to twisted projective and tilting modules

8.1 Twisted projective modules

Let P be a projective generator of $\mathcal{O}_0$. For $w \in W$, the module $\top_w P$ is a (generalized) tilting module in $\mathcal{O}_0$ because $\top_w$ is a derived self-equivalence which is acyclic on modules with Verma flag. A question is, for which w is the category $\mathcal{O}_0$ $\top_w P$-regular. Below we show that the answer is non-trivial.

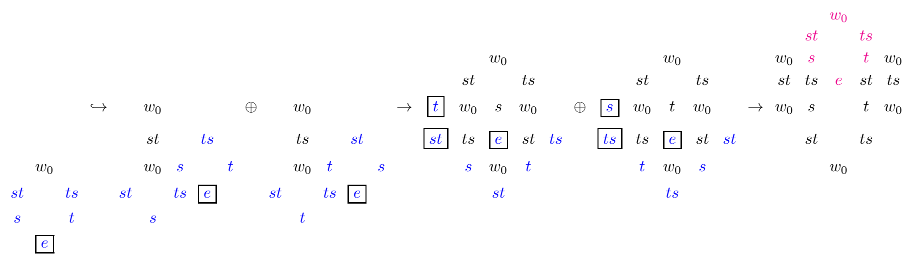

Figure 1: $\mathcal{L}\top_{w_0}\nabla_e$ and its cohomology for $\mathfrak{sl}_3$

8.2 Regularity with respect to twisted projectives

Theorem 19. *If $w = w_0^{\mathfrak{p}}$, for some parabolic subalgebra $\mathfrak{p}$ in $\mathfrak{g}$, then $\mathcal{O}_0$ is $\top_w P$-regular.*

Proof. Let $w = w_0^{\mathfrak{p}}$ as above. Since twisting functors functorially commute with projective functors, we only need to show that Δ_e has a coresolution by modules in $\mathrm{add}(\top_w P)$ satisfying the regularity condition.
By construction, twisting functors commute with parabolic induction. For the category $\mathcal{O}$ associated to the Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$, the claim of our Theorem coincides with the claim of Theorem 1. The parabolic induction from $\mathfrak{l}$ to $\mathfrak{g}$ is exact and sends the indecomposable projective $P_x^{\mathfrak{l}}$ (for $x \in W_{\mathfrak{p}}$) to the indecomposable projective P_x. It also sends (indecomposable) tiltings to our twisted projective modules. To see this, write the indecomposable tilting module for $\mathfrak{l}$ corresponding to $x \in W_{\mathfrak{p}}$ as $T_x^{\mathfrak{l}} \cong \top_w P_{wx}^{\mathfrak{l}}$ and use that the parabolic induction commutes with $\top_w$ to conclude that $T_x^{\mathfrak{l}}$ is sent to $\top_w P_{wx}$. Therefore a tilting coresolution of the dominant projective for $\mathfrak{l}$ is sent to a coresolution of the dominant projective for $\mathfrak{g}$ by our twisted projective modules. The claim follows. □

Corollary 20. *If $w = w_0^{\mathfrak{p}}$, for some parabolic subalgebra $\mathfrak{p}$ in $\mathfrak{g}$, then all blocks of $\mathcal{O}$ are $\top_w P$-regular.*

Proof. Since twisting functors functorially commute with projective functors, we can use translations to walls to extend Theorem 19 to singular blocks. □

Remark 21. The module Δ_w admits a (linear) coresolution by tilting modules, which starts with T_w. Applying the inverse of $\mathcal{L}\top_w$ to this coresolution, we obtain a coresolution of Δ_e by modules in $\mathrm{add}(\top_{w^{-1}w_0}P)$. We note that, by [AS], the inverse of $\mathcal{L}\top_w$ is $\mathcal{R}(\star \circ \top_{w^{-1}} \circ \star)$, where $\star$ is the simple preserving duality, and the claim in the previous sentence follows by using the acyclicity results in [AS]. Hence, a necessary condition for $\mathcal{O}_0$ to be $\top_{xw_0}P$-regular is that the module $(\mathcal{L}\top_w)^{-1}T_w$, which starts this coresolution, is projective. In case the multiplicity of Δ_{w_0} in a standard filtration of T_w is greater than 1, the module $(\mathcal{L}\top_w)^{-1}T_w$ will have Δ_e appearing with multiplicity 1 (as Δ_w appears in T_w with multiplicity 1) while some standard module will have higher multiplicity, by assumption. Therefore, in this case, $(\mathcal{L}\top_w)^{-1}T_w$ is not a projective module. This shows that the condition $[T_w : \Delta_{w_0}] = 1$ is necessary for $\mathcal{O}_0$ to be $\top_{xw_0}P$-regular. This implies that examples of $w \in W$ such that $\mathcal{O}_0$ is not $\top_w P$-regular exist already in type A_3. We will see in Subsection 8.5 below that $\top_w P$-regularity can fail already in $\mathcal{O}_0$ of type A_2.

8.3 Twisted tilting modules

Let T be a characteristic tilting module for $\mathcal{O}_0$. For $w \in W$, the module $\top_w T$ is a (generalized) tilting module in $\mathcal{O}_0$ because $\top_w$ is a derived self-equivalence which is acyclic on modules with Verma flag.
This raises an interesting problem, namely, to determine for which w the category $\mathcal{O}_0$ is $\top_w T$-regular. We show below that the answer is non-trivial.

8.4 Regularity with respect to twisted tiltings

Theorem 22. *If $w = w_0^{\mathfrak{p}}$, for some parabolic subalgebra $\mathfrak{p}$ in $\mathfrak{g}$, then $\mathcal{O}_0$ is $\top_w T$-regular.*

Proof. As usual, we use that the projective functors commutes with twisting functors to reduce the claim to finding a desired coresolution for $P_e = \Delta_e$.
Let $\mathfrak{l}$ be the Levi subalgebra of $\mathfrak{p}$ and take a coresolution of $\Delta_e^{\mathfrak{l}} = P_e^{\mathfrak{l}}$ by injectives for $\mathfrak{l}$ with the regularity property, guaranteed by Theorem 3. Then just like in the proof of Theorem 19, the parabolic induction produces a coresolution of Δ_e in $\mathrm{add}(\top_w T)$ with the regularity condition. In fact, the $w_0^{\mathfrak{p}}$-twists of tiltings are obtained by the parabolic induction from injective modules over $\mathfrak{l}$, which are the $w_0^{\mathfrak{p}}$-twists of tiltings over $\mathfrak{l}$. The proof is complete. □

Corollary 23. *If $w = w_0^{\mathfrak{p}}$, for some parabolic subalgebra $\mathfrak{p}$ in $\mathfrak{g}$, then all blocks of $\mathcal{O}$ are $\top_w T$-regular.*

Proof. Since twisting functors functorially commute with projective functors, we can use translations to walls to extend the statement in Theorem 22 to singular blocks. □

We will see in Subsection 8.5 below that $\top_w T$-regularity can fail already in $\mathcal{O}_0$ of type A_2.

8.5 $\mathfrak{sl}_3$-example

For the Lie algebra $\mathfrak{sl}_3$, we have $W = \{e, s, t, st, ts, w_0 = sts = tst\}$.
The left of the two tables below describes the projective dimensions of the twisted projective modules $\top_x P_y$. The right table below describes the projec-

tive dimensions of the twisted tilting modules $\top_x T_y$.

$x\backslash y$	e	s	t	st	ts	w_0
e	0	0	0	0	0	0
s	1	0	1	0	1	0
t	1	1	0	1	0	0
st	2	1	1	1	1	0
ts	2	1	1	1	1	0
w_0	3	1	1	1	1	0

$x\backslash y$	e	s	t	st	ts	w_0
e	0	1	1	1	1	3
s	0	2	1	2	1	4
t	0	1	2	1	2	4
st	0	2	2	2	2	5
ts	0	2	2	2	2	5
w_0	0	2	2	2	2	6

Here are the graded characters of the modules $\top_s P_x$ (with the characters of the tilting cores displayed in magenta):

$\deg\backslash x$	e	s	t	st	ts	w_0
-1		s		st		w_0
0	s	st e ts	st	s w_0 t	w_0 s	st ts
1	st ts	s w_0 t	s w_0	st ts e st ts	st e ts st	w_0 s t w_0
2	w_0	st ts	st ts	w_0 s w_0 t	w_0 t w_0 s	st ts e st ts
3		w_0	w_0	ts st	st ts	w_0 s t w_0
4				w_0	w_0	st ts
5						w_0

Here are the graded characters of the modules $\top_{ts} P_x$ (with the characters of the tilting cores displayed in magenta):

$\deg\backslash x$	e	s	t	st	ts	w_0
-2						w_0
-1		ts		w_0 t	w_0	st ts
0	ts	t w_0	w_0	ts e st ts	st ts	w_0 s t w_0
1	w_0	ts st	ts	w_0 s w_0 t	w_0 t w_0	st ts e st ts
2		w_0	w_0	st ts	st ts	w_0 s t w_0
3				w_0	w_0	st ts
4						w_0

Here are the graded characters of the modules $\top_s T_x$ (with the characters of the tilting cores displayed in magenta):

$\deg\backslash x$	w_0	st	ts	s	t	e
-4						w_0
-3				w_0	w_0	st ts
-2		w_0	w_0	st ts	st ts	w_0 s t w_0
-1	w_0	st ts	ts	w_0 s w_0 t	w_0 t w_0	st ts e st ts
0	ts	w_0 t	w_0	st ts ts e	st ts	w_0 s t w_0
1		ts		w_0 t	w_0	st ts
2						w_0

Here are the graded characters of the modules $\top_{ts}T_x$ (with the characters of the tilting cores displayed in magenta):

deg\\x	w_0	st	ts	s	t	e
-5						w_0
-4				w_0	w_0	st ts
-3		w_0	w_0	st ts	st st	w_0 s t w_0
-2	w_0	st ts	st ts	w_0 s w_0 t	w_0 t w_0 s	st ts e st ts
-1	st ts	w_0 t s	s w_0	st ts ts e st	st e ts st	w_0 s t w_0
0	s	ts st e	st	w_0 t s	w_0 s	st ts
1		s		st		w_0

The cases $x = e$ and w_0 are already discussed in the previous sections. To prove regularity, we only need to consider the coresolution of P_e. Up to the symmetry of the Dynkin diagram, it is enough to consider the four cases $\top_sP$, $\top_{ts}P$, $\top_sT$ and $\top_{ts}T$. The first two are given as follows:

$$0 \to P_e \to \top_sP_s \to \top_sP_e \to 0$$

$$0 \to P_e \to \top_{ts}P_{st} \to \top_{ts}P_s \oplus \top_{ts}P_t \to \top_{ts}P_e \to 0$$

Here we see that the first coresolution is regular, while in the second one, $\top_{ts}P_{st}$ is not projective and hence we do not have regularity with respect to $\top_{ts}P$. The case of $\top_sT$ is regular and given as follows:

$$0 \to P_e \to \top_sT_e \to \top_sT_t \oplus \top_sT_{ts} \oplus \top_sT_e \to \top_sT_{ts} \oplus \top_sT_t \oplus \top_sT_s \to \top_sT_{ts} \oplus \top_sT_{st} \to \top_sT_{w_0} \to 0$$

Finally, we claim that we do not have the regularity in the case of $\top_{ts}T$. Indeed, in order not to fail already in position zero, we must start with $0 \to P_e \to \top_{ts}T_e \to \mathrm{Coker}$. Further, in order not to fail on the next step, we again must embed Coker into $\top_{ts}T_e \oplus \top_{ts}T_e$. The new cokernel will necessarily have both L_s and L_t in the socle. However, L_t does not appear in the socle of $\top_{ts}T$ and hence the coresolution cannot continue. This implies that one of the first two steps requires correction by adding non-projective summands of $\top_{ts}T$, which implies the failure of the regularity.

9 Regularity phenomena with respect to shuffled projective and tilting modules

9.1 Shuffled projective modules

For $w \in W$, we denote by C_w the corresponding shuffling functor on $\mathcal{O}_0$, see [MS1, Section 5]. Let P be a projective generator of $\mathcal{O}_0$. For $w \in W$, the module C_wP is a (generalized) tilting module in $\mathcal{O}_0$ because C_w is a derived self-equivalence.

Thus, a problem is to determine for which w the category $\mathcal{O}_0$ is C_wP-regular. This problem looks much harder than the one involving the twisting functors, due to the fact that shuffling functors do not commute with projective functors.

9.2 Regularity with respect to shuffled projectives

PROPOSITION 24. *If s is a simple reflection, then $\mathcal{O}_0$ is C_sP-regular.*

Proof. The functor C_s is defined as the cokernel of the adjunction morphism $\mathrm{adj}_s : \theta_e \to \theta_s$. If $x \in W$ is such that $xs < x$, then $\mathrm{C}_sP_x \cong P_x$. If $x \in W$ is such that $xs > x$, then C_sP_x has projective dimension 1 and a minimal projective resolution of the following form:

$$0 \to P_x \to \theta_sP_x \to \mathrm{C}_sP_x \to 0, \tag{1}$$

where any summand P_y of θ_sP_x satisfies $ys < y$ and hence $\mathrm{C}_sP_y = P_y$.
The latter implies that (1) can be viewed as a coresolution of P_x by modules in $\mathrm{add}(\mathrm{C}_sP)$ and it is manifestly regular. The claim follows. □

Proposition 24 and Theorem 19 motivate the following:

CONJECTURE 25. *If $w_0^{\mathfrak{p}}$ is the longest element in some parabolic subgroup of W, then $\mathcal{O}_0$ is $\mathrm{C}_{w_0^{\mathfrak{p}}}P$-regular.*

Similarly to Subsection 8.5 one can show that $\mathcal{O}_0$ is not $\mathrm{C}_{st}P$-regular for $\mathfrak{g} = \mathfrak{sl}_3$.

9.3 Shuffled tilting modules

Let T be a characteristic tilting module for $\mathcal{O}_0$. For $w \in W$, the module C_wT is a (generalized) tilting module in $\mathcal{O}_0$ because C_w induces a derived self-equivalence which is acyclic on tilting modules (the latter follows by combining [MS1, Proposition 5.3] and [MS1, Theorem 5.16]).
It seems to be an interesting problem to determine, for which w, the category $\mathcal{O}_0$ is C_wT-regular. Again, this problem looks much harder than the one involving the twisting functors due to the fact that shuffling functors do not commute with projective functors.

9.4 Regularity with respect to shuffled tiltings

PROPOSITION 26. *If s is a simple reflection, then $\mathcal{O}_0$ is C_sT-regular.*

Proof. This is very similar to the proof of Proposition 24. If $x \in W$ is such that $xs > x$, then $\mathrm{C}_sT_x \cong T_x$. If $x \in W$ is such that $xs < x$, then C_sT_x has a tilting resolution of the following form:

$$0 \to T_x \to \theta_sT_x \to \mathrm{C}_sT_x \to 0, \tag{2}$$

where any summand T_y of θ_sT_x satisfies $ys > y$ and hence $\mathrm{C}_sT_y = T_y$. Also, since θ_s is exact, the projective dimension of θ_sT_x does not exceed that of T_x. Consequently, the projective dimension of C_sT_x is bounded by the projective dimension of T_x plus 1.
We can now take a minimal tilting coresolution of P, which we know has the regularity property, and coresolve each summand T_x, for $xs < x$, in this resolution using (2). The outcome is a regular coresolution of P by modules in $\mathrm{add}(\mathrm{C}_sT)$. This completes the proof. □

Proposition 26 motivates the following:

Conjecture 27. *If $w_0^{\mathfrak{p}}$ is the longest element in some parabolic subgroup of W, then $\mathcal{O}_0$ is $\mathrm{C}_{w_0^{\mathfrak{p}}}T$-regular.*

Similarly to Subsection 8.5 one can show that $\mathcal{O}_0$ is not $\mathrm{C}_{st}T$-regular for $\mathfrak{g} = \mathfrak{sl}_3$.

9.5 $\mathfrak{sl}_3$-example

Let $\mathfrak{g} = \mathfrak{sl}_3$. Denote $W = \{e, s, t, st, ts, w_0 = sts = tst\}$ as before.
The left of the two tables below describes the projective dimensions of the twisted projective modules $\mathrm{C}_x P_y$. The right table below describes the projective dimensions of the twisted tilting modules $\mathrm{C}_x T_y$.

$x \backslash y$	e	s	t	st	ts	w_0
e	0	0	0	0	0	0
s	1	0	1	1	0	0
t	1	1	0	0	1	0
st	2	1	1	1	1	0
ts	2	1	1	1	1	0
w_0	3	1	1	1	1	0

$x \backslash y$	e	s	t	st	ts	w_0
e	0	1	1	1	1	3
s	0	2	1	1	2	4
t	0	1	2	2	1	4
st	0	2	2	2	2	5
ts	0	2	2	2	2	5
w_0	0	2	2	2	2	6

In the examples below, we note the following difference with the case of twisting functors: we do not know whether the notion of a “tilting core” makes sense for shuffled projective and tilting modules. Here are the graded characters of the modules $\mathrm{C}_s P_x$:

deg$\backslash x$	e	s	t	st	ts	w_0
-1		s			ts	w_0
0	s	st e ts	ts	w_0 s	t w_0 s	st ts
1	st ts	s w_0 t	s w_0	ts e st ts	ts st e ts st	w_0 s t w_0
2	w_0	st ts	ts st	w_0 s w_0 t	w_0 t w_0 s	st ts e st ts
3		w_0	w_0	ts st	st ts	w_0 s t w_0
4				w_0	w_0	st ts
5						w_0

Here are the graded characters of the modules $\mathrm{C}_{st} P_x$:

deg$\backslash x$	e	s	t	st	ts	w_0
-2						w_0
-1		st		w_0	w_0 t	st ts
0	st	t w_0	w_0	st ts	st e ts st	w_0 s t w_0
1	w_0	st ts	st	w_0 t w_0	w_0 s w_0 t	st ts e st ts
2		w_0	w_0	st ts	st ts	w_0 s t w_0
3				w_0	w_0	st ts
4						w_0

Here are the graded characters of the modules $\mathrm{C}_s T_x$:

$\deg\backslash x$	w_0	st	ts	s	t	e
-4						w_0
-3				w_0	w_0	st ts
-2		w_0	w_0	st ts	st ts	w_0 s t w_0
-1	w_0	st	ts st	w_0 s w_0 t	w_0 t w_0	st ts e st st
0	ts	w_0	w_0 t	st ts st e	st ts	w_0 s t w_0
1			st	w_0 t	w_0	st ts
2						w_0

Here are the graded characters of the modules $\mathrm{C}_{st}T_x$:

$\deg\backslash x$	w_0	st	ts	s	t	e
-5						w_0
-4				w_0	w_0	st ts
-3		w_0	w_0	st ts	st st	w_0 s t w_0
-2	w_0	ts st	st ts	w_0 s w_0 t	w_0 t w_0 s	st ts e st ts
-1	st ts	w_0 s	s w_0 t	st ts ts e st	ts e ts st	w_0 s t w_0
0	s	st	st e ts	w_0 t s	w_0 s	st ts
1			s	ts		w_0

The non-trivial (ungraded) coresolutions of projectives using C_sP are:

$$0 \to P_e \to \mathrm{C}_sP_s \to \mathrm{C}_sP_e \to 0,$$
$$0 \to P_t \to \mathrm{C}_sP_{ts} \to \mathrm{C}_sP_t \to 0,$$
$$0 \to P_{st} \to \mathrm{C}_sP_s \oplus \mathrm{C}_sP_s \to \mathrm{C}_sP_{st} \to 0.$$

These all are, clearly, regular.

Next we claim that P_e does not have a regular coresolution using $\mathrm{C}_{st}P$. Indeed, to have a chance at the zero step, we must embed P_e into $\mathrm{C}_{st}P_{w_0}$. Let Coker be the cokernel. In order to embed Coker, in the next step we need a copy of $\mathrm{C}_{st}P_{st}$ or $\mathrm{C}_{st}P_{w_0}$ and another copy of $\mathrm{C}_{st}P_{ts}$ or $\mathrm{C}_{st}P_{w_0}$. Either way, the new cokernel will have a copy of L_t in the socle, while it is easy to see that no module in $\mathrm{add}(\mathrm{C}_{st}P)$ has L_t in the socle, a contradiction.

The non-trivial (ungraded) coresolutions of projectives using C_sT are:

$$0 \to P_e \to \mathrm{C}_sT_e \to \mathrm{C}_sT_t \oplus \mathrm{C}_sT_e \oplus \mathrm{C}_sT_{st} \to \mathrm{C}_sT_s \oplus \mathrm{C}_sT_{st} \oplus \mathrm{C}_sT_t \to \mathrm{C}_sT_{ts} \oplus \mathrm{C}_sT_{st} \to \mathrm{C}_sT_{w_0} \to 0,$$
$$0 \to P_s \to \mathrm{C}_sT_e \to \mathrm{C}_sT_t \to 0,$$
$$0 \to P_t \to \mathrm{C}_sT_e \to \mathrm{C}_sT_e \oplus \mathrm{C}_sT_{st} \to \mathrm{C}_sT_s \to 0,$$
$$0 \to P_{st} \to \mathrm{C}_sT_e \to \mathrm{C}_sT_t \to \mathrm{C}_sT_{ts} \to 0$$
$$0 \to P_{ts} \to \mathrm{C}_sT_e \to \mathrm{C}_sT_{st} \to 0.$$

These all are, clearly, regular.

10 Projective dimension of indecomposable twisted and shuffled projectives and tiltings

10.1 Projective dimension of twisted projectives

The results of Subsection 8.2 motivate the problem to determine the projective dimension of twisted projective modules in $\mathcal{O}$. Since twisting functors commute with projective functors, twisted projective modules are exactly the modules obtained by applying projective functors to Verma modules:

$$\top_x P_y \cong \top_x \theta_y \Delta_e \cong \theta_y \top_x \Delta_e \cong \theta_y \Delta_x. \qquad (3)$$

This allows us to reformulate the problem as follows:

Problem 28. *For $x, y \in W$, determine the projective dimension of the module $\theta_x \Delta_y$.*

Here are some basic observations about this problem:

- If $y = e$, the module $\theta_x \Delta_e$ is projective and hence the answer is 0.
- If $y = w_0$, the module $\theta_x \Delta_{w_0}$ is a tilting module and hence the answer is $\mathbf{a}(w_0 x)$, see [Ma3, Ma4].
- If $x = e$, the answer is $\ell(y)$, see [Ma3].
- If $x = w_0$, we have $\theta_{w_0} \Delta_y = P_{w_0}$ and the answer is 0.
- For a fixed y, the answer is weakly monotone in x, with respect to the right Kazhdan-Lusztig order, in particular, the answer is constant on the right Kazhdan-Lusztig cell of x.
- For a simple reflection s, we have $\theta_x \Delta_y = \theta_x \Delta_{ys}$ provided that $\ell(sx) < \ell(x)$, in particular, it is enough to consider the situation where x is a Duflo involution and y is a shortest (or longest) element in a coset from W/W', where W' is the parabolic subgroup of W generated by all simple reflections in the left descent set of x.
- If $x = w_0^{\mathfrak{p}}$, for some parabolic $\mathfrak{p}$, then the projective dimension of $\theta_{w_0^{\mathfrak{p}}} \Delta_y$ coincides with the projective dimension of the singular Verma module obtained by translating Δ_y to the wall corresponding to $w_0^{\mathfrak{p}}$. This can be computed in therms of a certain function $\mathtt{d}_\lambda$, see [CM, Table 2] (see also [CM, Formula (1.2)] and [KMM, Remark 6.9]).

The last observation suggest that Problem 28 might be not easy. Also, note that, by Koszul duality, the problem to determine the projective dimension of a singular Verma module is equivalent to the problem to determine the graded length of a parabolic Verma module. The latter is certainly "combinatorial" in the sense that the answer can be formulated purely in terms of Kazhdan-Lusztig combinatorics.

Let $\mathbf{H}$ denote the Hecke algebra of W (over $\mathbb{A} = \mathbb{Z}[v, v^{-1}]$ and in the normalization of [So3]) with standard basis $\{H_w : w \in W\}$ and Kazhdan-Lusztig basis $\{\underline{H}_w : w \in W\}$. Consider the structure constants $h^z_{x,y} \in \mathbb{A}$ with respect to the KL-basis, that is

$$\underline{H}_x \underline{H}_y = \sum_{z \in W} h^z_{x,y} \underline{H}_z.$$

In [KMM, Subsection 6.3], for $x, y \in W$, we defined the function $\mathbf{b} : W \times W \to \mathbb{Z}_{\geq 0} \sqcup \{-\infty\}$ as follows:

$$\mathbf{b}(x, y) := \max\{\deg(h^y_{z,x^{-1}}) : z \in W\}.$$

(By our convention the degree of the zero polynomial is $-\infty$.) The value $\mathbf{b}(x, y)$ is, if not $-\infty$, equal to the maximal degree of a non-zero graded component of $\theta_x L_y$, and also to the maximal non-zero position in the minimal complex of tilting modules representing $\theta_{y^{-1}w_0} L_{w_0 x^{-1}}$.
Here is an upper bound for the projective dimension of $\theta_x \Delta_y$ expressed in terms of the $\mathbf{b}$-function.

PROPOSITION 29. *For all $x, y \in W$, we have:*

i. $\text{proj.dim}\, \theta_x \Delta_y \leq \max\{\mathbf{b}(w_0 a^{-1} w_0, x^{-1} w_0) : a \leq y\}$.

ii. If the maximum in (i) *coincides with* $\mathbf{b}(w_0 y^{-1} w_0, x^{-1} w_0)$*, then the latter value is equal to* $\text{proj.dim}\, \theta_x \Delta_y$.

Proof. For $x, y, z \in W$ and $k \in \mathbb{Z}_{\geq 0}$, by adjunction, we have

$$\text{Ext}^k_{\mathcal{O}}(\theta_x \Delta_y, L_z) \cong \text{Ext}^k_{\mathcal{O}}(\Delta_y, \theta_{x^{-1}} L_z).$$

By [Ma4], the module $\theta_{x^{-1}} L_z$ can be represented by a linear complex of tilting module. Moreover, the multiplicity of $T_a\langle k\rangle[-k]$ in this complex coincides with the composition multiplicity of $L_{w_0 a^{-1} w_0}\langle k\rangle$ in $\theta_{z^{-1} w_0} L_{w_0 x}$.
A costandard filtration of $T_a\langle k\rangle[-k]$ can contain ∇_y only when $a \leq y$, and hence only such summand $T_a\langle k\rangle[-k]$ in the tilting complex can, potentially, give rise to a non-zero element in $\text{Ext}^k_{\mathcal{O}}(\Delta_y, \theta_{x^{-1}} L_z)$. Here we use the fact that standard and costandard modules are homologically orthogonal and hence derived homomorphisms can be constructed already on the level of the homotopy category. This implies claim (i).
To prove claim (ii), assume

$$k := \mathbf{b}(w_0 y^{-1} w_0, x^{-1} w_0) = \max\{\mathbf{b}(w_0 a^{-1} w_0, x^{-1} w_0) : a \leq y\}.$$

The canonical map $\Delta_y \to T_y$ gives rise to a homomorphism of $\Delta_y\langle k\rangle$ to the k-th homological position of the linear complex of tilting modules representing $\theta_{x^{-1}} L_z$. Because of the maximality assumption on k, there are no homomorphisms from Δ_y to the $k+1$-st homological position. This means that the map from the previous sentence is a homomorphism of complexes. It is not homotopic to zero since since the complex representing $\theta_{x^{-1}} L_z$ is linear and $T_y\langle k\rangle[-k]$ is in a diagonal position in this complex. The corresponding level at the position $k-1$ does not contain any socles of any costandard modules since all indecomposable tilting summands there are shifted by one in the positive direction of the grading. This means that the map we constructed gives a non-zero extension. Hence claim (ii) now follows from claim (i). □

Corollary 30. *For any parabolic* $\mathfrak{p}$*, in case* $x \leq_{\mathtt{R}} w_0^{\mathfrak{p}} w_0$*, we have* $\mathrm{proj.dim}\, \theta_x \Delta_{w_0^{\mathfrak{p}}} = \ell(w_0^{\mathfrak{p}})$*.*

Proof. If $x \leq_{\mathtt{R}} w_0^{\mathfrak{p}} w_0$, then [KMM, Proposition 6.8] implies $\mathbf{b}(w_0 w_0^{\mathfrak{p}} w_0, x^{-1} w_0) = \ell(w_0^{\mathfrak{p}})$. For any $a \leq w_0^{\mathfrak{p}}$, we also have

$$\mathbf{b}(w_0 a w_0, x^{-1} w_0) \leq \ell(a) \leq \ell(w_0^{\mathfrak{p}}) = \mathbf{b}(w_0 w_0^{\mathfrak{p}} w_0, x^{-1} w_0),$$

also using [KMM, Proposition 6.8]. Hence the claim follows from Proposition 29(ii). □

10.2 Projective dimension of twisted tiltings

The results of Subsection 8.4 motivate the problem to determine the projective dimension of twisted tilting modules in $\mathcal{O}$. By

$$\top_x T_{w_0 y} \cong \top_x \theta_y T_{w_0} \cong \top_x \theta_y \nabla_{w_0} \cong \theta_y \top_x \nabla_{w_0} \cong \theta_y \nabla_{x w_0}, \tag{4}$$

we reformulate the problem as follows:

Problem 31. *For* $x, y \in W$*, determine the projective dimension of the module* $\theta_x \nabla_y$*.*

Here are some basic observations about this problem:

- If $y = w_0$, the module $\theta_x \nabla_{w_0}$ is tilting and hence the answer is $\mathbf{a}(w_0 x)$, see [Ma3, Ma4].
- If $y = e$, the module $\theta_x \nabla_e$ is an indecomposable injective module and hence the answer is $2\mathbf{a}(w_0 x)$, see [Ma3, Ma4].
- If $x = e$, the answer is $2\ell(w_0) - \ell(y)$, see [Ma3].
- If $x = w_0$, we have $\theta_{w_0} \nabla_y = P_{w_0}$ and the answer is 0.
- For a fixed y, the answer is weakly monotone in x, with respect to the right Kazhdan-Lusztig order, in particular, the answer is constant on the right Kazhdan-Lusztig cell of x.
- For a simple reflection s, we have $\theta_x \nabla_y = \theta_x \nabla_{ys}$ provided that $\ell(sx) < \ell(x)$, in particular, it is enough to consider the situation where x is a Duflo involution and y is a shortest (or longest) element in a coset from W/W', where W' is the parabolic subgroup of W generated by all simple reflections in the left descent set of x.

- If $x = w_0^{\mathfrak{p}}$, for some parabolic $\mathfrak{p}$, then the projective dimension of $\theta_{w_0^{\mathfrak{p}}}\nabla_y$ coincides with the projective dimension of the singular dual Verma module obtained by translating ∇_y to the wall corresponding to $w_0^{\mathfrak{p}}$. This can be computed in therms of a certain function $\mathtt{d}_\lambda$, see [CM, Table 2] (see also [CM, Formula (1.2)] and [KMM, Remark 6.9]).

Let us now observe that $\nabla_y \cong \top_{w_0}\Delta_{w_0y}$ and that $\theta_x\nabla_y \cong \top_{w_0}\theta_x\Delta_{w_0y}$ since twisting and projective functors commute. We conjecture the following connection between Problems 28 and 31.

CONJECTURE 32. *For $x, y \in W$, we have* $\text{proj.dim}\,\theta_x\nabla_y = \mathbf{a}(w_0x) + \text{proj.dim}\,\theta_x\Delta_{w_0y}$.

Below we present some evidence for Conjecture 32.

PROPOSITION 33. *For $x, y \in W$, we have* $\text{proj.dim}\,\theta_x\nabla_y \leq \mathbf{a}(w_0x) + \text{proj.dim}\,\theta_x\Delta_{w_0y}$.

Proof. Assume that $\text{proj.dim}\,\theta_x\Delta_{w_0y} = k$ and let $\mathcal{P}_\bullet$ be a minimal projective resolution of $\theta_x\Delta_{w_0y}$. Applying $\top_{w_0}$ to $\mathcal{P}_\bullet$, we get a minimal tilting resolution of $\theta_x\nabla_y$ (of length k). To obtain a projective resolution of $\theta_x\nabla_y$, we need to projectively resolve each indecomposable tilting module T_u appearing in $\top_{w_0}\mathcal{P}_\bullet$ and glue all these resolutions together. In particular, $\text{proj.dim}\,\theta_x\nabla_y$ is bounded by k plus the maximal value of $\text{proj.dim}\,T_u$, for T_u appearing in $\top_{w_0}\mathcal{P}_\bullet$.
Note that any indecomposable projective P_v appearing in $\mathcal{P}_\bullet$ satisfies $v \geq_{\mathrm{L}} x$, because it is a summand of θ_xP_w, for some w. Therefore $u = w_0v$ satisfies $u \leq_{\mathrm{L}} w_0x$. In particular, we have $\mathbf{a}(u) \leq \mathbf{a}(w_0x)$. By [Ma3, Ma4], the projective dimension of T_u equals $\mathbf{a}(u)$. The claim of the proposition follows. □

COROLLARY 34. *For $x, y \in W$, let* $\text{proj.dim}\,\theta_x\Delta_{w_0y} = k$. *Assume that there exists $v \in W$ such that $v \sim_{\mathrm{L}} x$ and $\mathrm{Ext}^k_{\mathcal{O}}(\theta_x\Delta_{w_0y}, L_v) \neq 0$. Then* $\text{proj.dim}\,\theta_x\nabla_y = \mathbf{a}(w_0x) + \text{proj.dim}\,\theta_x\Delta_{w_0y}$.

Proof. Let us look closely at the proof of Proposition 33. From [KMM, Section 6], it follows that there exists $w \in W$ such that T_{w_0v} appears in position $\mathbf{a}(w_0x)$ of a minimal tilting complex $\mathcal{T}_\bullet$ representing L_w and, moreover, this position $\mathbf{a}(w_0x)$ is a maximal non-zero position in $\mathcal{T}_\bullet$.
The module T_{w_0v} appears as a summand in $\top_{w_0}\mathcal{P}_{-k}$ and in $\mathcal{T}_{\mathbf{a}(w_0x)}$. Similarly to the proof of [MO2, Theorem 1], the identity map on T_{w_0v} induces a non-zero map from $\top_{w_0}\mathcal{P}_\bullet$ to $\mathcal{T}_\bullet[\mathbf{a}(w_0x) + k]$ in the homotopy category and hence gives rise to a non-zero extension fro $\theta_x\nabla_y$ to L_w of degree $\mathbf{a}(w_0x) + k$, by construction. Therefore $\text{proj.dim}\,\theta_x\nabla_y \geq \mathbf{a}(w_0x) + \text{proj.dim}\,\theta_x\Delta_{w_0y}$ and the claim of the corollary follows from Proposition 33. □

We note that the condition "there exists $v \in W$ such that $v \sim_{\mathrm{L}} x$ and $\mathrm{Ext}^k_{\mathcal{O}}(\theta_x\Delta_{w_0y}, L_v) \neq 0$" in Corollary 34 is very similar to [KMM, Conjecture 1.3] proved in [KMM, Theorem A]. We suspect that this condition is always satisfied.

10.3 Projective dimension of shuffled projectives

The results of Subsection 9.2 motivate the problem to determine the projective dimension of shuffled projective modules in $\mathcal{O}$.

Problem 35. *For $x, y \in W$, determine the projective dimension of the module $\mathrm{C}_x P_y$.*

This problems looks much harder than the one for the twisted projective modules, mostly because twisting functors do not commute with projective functors, in general.

Here are some basic observations about this problem:

- If $x = e$, the module $\mathrm{C}_e P_y$ is projective and hence the answer is 0.
- If $x = w_0$, the module $\mathrm{C}_{w_0} P_y$ is the tilting module T_{yw_0} (this follows from [MS2, Proposition 2.2, Proposition 4.4] by a character argument). Hence the answer is $\mathbf{a}(yw_0)$ by [Ma3, Ma4].
- If $y = e$, we have $\mathrm{C}_x P_e \cong \mathrm{C}_x \Delta_e \cong \Delta_x$ and the answer is $\ell(x)$, see [MS1, Ma3].
- If $y = w_0$, we have $\mathrm{C}_x P_{w_0} = P_{w_0}$ and the answer is 0.
- The projective dimension of $\mathrm{C}_x P_y$ is at most $\ell(x)$, since each C_s, where s is a simple reflection, has derived length 1.
- For $x = s$, a simple reflection, we have $\mathrm{C}_s P_y \cong P_y$ if $ys < y$, in which case the answer is 0. In case $ys > y$, the module $\mathrm{C}_s P_y$ is not projective and the answer is 1, see the proof of Proposition 24.

In the spirit of Subsection 7.3, we can reformulate Problem 35 in terms of the cohomology of certain functors. For $w \in W$, we denote by K_w the right adjoint of C_w, called the *coshuffling* functor, see [MS1, Section 5]. Note that, for a reduced expression $w = rs \dots t$, we have $\mathrm{C}_w = \mathrm{C}_t \dots \mathrm{C}_s \mathrm{C}_r$ and $\mathrm{K}_w = \mathrm{K}_r \mathrm{K}_s \dots \mathrm{K}_t$. Also, we have $\mathrm{K}_w = \star \circ \mathrm{C}_w \circ \star$. We denote by L the direct sum of all simple modules in $\mathcal{O}_0$.

Proposition 36. *For $x, y \in W$, the projective dimension of $\mathrm{C}_x P_y$ coincides with the maximal $k \geq 0$ such that $[\mathcal{L}_k \mathrm{C}_x\, L : L_y] \neq 0$.*

Proof. The projective dimension of a module coincides with the maximal degree of a non-vanishing extension to a simple module. Since $\mathcal{L}\mathrm{C}_x$ is a derived equivalence with inverse $\mathcal{R}\mathrm{K}_x$ by [MS1, Theorem 5.7], for $i \geq 0$, we have

$$\begin{aligned}
\dim \mathrm{Ext}^i_{\mathcal{O}}(\mathrm{C}_x P_y, L) &= \dim \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathrm{C}_x P_y, L[i]) \\
&= \dim \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathcal{L}\mathrm{C}_x P_y, L[i]) \\
&= \dim \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(P_y, \mathcal{R}\mathrm{K}_x L[i]) \\
&= [\mathcal{R}^i \mathrm{K}_x\, L : L_y] \\
&= [\mathcal{L}_i \mathrm{C}_x\, L^\star : L_y^\star] \\
&= [\mathcal{L}_i \mathrm{C}_x\, L : L_y]
\end{aligned}$$

and the claim follows. □

10.4 Projective dimension of shuffled tiltings

The results of Subsection 9.4 motivate the problem to determine the projective dimension of shuffled projective modules in $\mathcal{O}$.

Problem 37. *For $x, y \in W$, determine the projective dimension of the module $\mathrm{C}_x T_y$.*

This problems looks much harder than the one for the twisted tilting modules, mostly because twisting functors do not commute with projective functors, in general.
Here are some basic observations about this problem:

- If $x = e$, the module $\mathrm{C}_e T_y$ is tilting and hence the answer is $\mathbf{a}(y)$, see [Ma3, Ma4].

- If $x = w_0$, the module $\mathrm{C}_{w_0} T_y$ is the injective module I_{yw_0}. In fact, we have

$$\mathrm{C}_{w_0} T_y \cong \mathrm{C}_{w_0} \top_{w_0} P_{w_0 y} \cong \top_{w_0} \mathrm{C}_{w_0} P_{w_0 y} \cong \top_{w_0} T_{w_0 y w_0} \cong I_{y w_0}.$$

  Hence the answer is $2\mathbf{a}(w_0 y w_0)$ by [Ma3, Ma4].

- If $y = e$, we have $\mathrm{C}_x T_e \cong \mathrm{C}_x \top_{w_0} P_{w_0} \cong \top_{w_0} \mathrm{C}_x P_{w_0} \cong \top_{w_0} P_{w_0} \cong P_{w_0}$ and the answer is 0.

- If $y = w_0$, we have $\mathrm{C}_x T_{w_0} \cong \mathrm{C}_x \nabla_{w_0} \cong \nabla_{w_0 x}$ and the answer is $\ell(w_0) + \ell(x)$, see [Ma3].

- The projective dimension of $\mathrm{C}_x T_y$ is at most $\ell(x) + \mathbf{a}(y)$, since the projective dimension of T_y is $\mathbf{a}(y)$ by [Ma3, Ma4] and each C_s, where s is a simple reflection, has derived length 1.

- For $x = s$, a simple reflection, we have $\mathrm{C}_s T_y \cong T_y$ if $ys > y$, in which case the answer is $\mathbf{a}(y)$ by [Ma3, Ma4]. In case $ys < y$, the module $\mathrm{C}_s P_y$ is no longer tilting and the answer is $\mathbf{a}(y) + 1$ because the minimal tilting resolution of $\mathrm{C}_s P_y$ has T_y in position -1.

In the spirit of Subsection 10.2, we make the following conjecture:

Conjecture 38. *For $x, y \in W$, we have* $\mathrm{proj.dim}\, \mathrm{C}_x T_y = \mathbf{a}(y) + \mathrm{proj.dim}\, \mathrm{C}_x P_{w_0 y}$.

Below we present some evidence for Conjecture 38.

Proposition 39. *For $x, y \in W$, we have* $\mathrm{proj.dim}\, \mathrm{C}_x T_y \leq \mathbf{a}(y) + \mathrm{proj.dim}\, \mathrm{C}_x P_{w_0 y}$.

Proof. Assume that $\mathrm{proj.dim}\, \mathrm{C}_x P_{w_0 y} = k$ and let $\mathcal{P}_\bullet$ be a minimal projective resolution of $\mathrm{C}_x P_{w_0 y}$. Applying $\top_{w_0}$ to $\mathcal{P}_\bullet$, and using that twisting and shuffling functors commute (e.g. because twisting functors commute with projective functors and natural transformations between them and shuffling functors are defined in terms of (co)kernels of such natural transformations), we get a minimal tilting resolution of $\mathrm{C}_x T_y$ (of length k). To obtain a projective resolution of $\mathrm{C}_x T_y$, we need to projectively resolve each indecomposable tilting module T_u appearing in $\top_{w_0}\mathcal{P}_\bullet$ and glue all these resolutions together. In particular, $\mathrm{proj.dim}\, \mathrm{C}_x T_y$ is bounded by k plus the maximal value of $\mathrm{proj.dim}\, T_u$, for T_u appearing in $\top_{w_0}\mathcal{P}_\bullet$.
Note that a projective resolution of $\mathrm{C}_s P_w$, for any $w \in W$ and $s \in S$, has the following form: $0 \to P_w \to \theta_x P_w \to 0$ and a projective resolution of $\mathrm{C}_x P_{w_0 y}$ is obtained by gluing such resolutions inductively along a reduced decomposition of x. Thus, an indecomposable projective P_v appearing in $\mathcal{P}_\bullet$ is a summand of $\theta P_{w_0 y}$ for some projective functor θ and satisfies $v \geq_{\mathtt{R}} w_0 y$. Therefore, $u = w_0 v$ satisfies $u \leq_{\mathtt{R}} y$. In particular, we have $\mathbf{a}(u) \leq \mathbf{a}(y)$. By [Ma3, Ma4], the projective dimension of T_u equals $\mathbf{a}(u)$. The claim of the proposition follows. □

Corollary 40. *For $x, y \in W$, let* $\mathrm{proj.dim}\, \mathrm{C}_x P_{w_0 y} = k$. *Assume that there exists $v \in W$ such that $v \sim_{\mathtt{R}} w_0 y$ and $\mathrm{Ext}^k_{\mathcal{O}}(\mathrm{C}_x P_{w_0 y}, L_v) \neq 0$. Then* $\mathrm{proj.dim}\, \mathrm{C}_x T_y = \mathbf{a}(y) + \mathrm{proj.dim}\, \mathrm{C}_x P_{w_0 y}$.

Proof. Follows from Proposition 39 by a line of arguments analogous to the ones in the proof of Corollary 34. □

Again, we suspect that the above assumption "there exists $v \in W$ such that $v \sim_{\mathtt{R}} w_0 y$ and $\mathrm{Ext}^k_{\mathcal{O}}(\mathrm{C}_x P_{w_0 y}, L_v) \neq 0$" in Corollary 40 is always satisfied.

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