Erratum for "Stability of Equivariant Vector Bundles over Toric Varieties"

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Abstract. We correct the proof of [2, Proposition 3.1.1].

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The proof of [2, Proposition 3.1.1] contains a gap. In this note we rectify the proof and modify the statement of this proposition accordingly. Note that this also leads to a modification in the statement of [2, Corollary 3.1.2]. It does not affect any other part of the paper [2] as to study stability of torsion-free sheaves it suffices to consider only saturated subsheaves.

Proposition 0.0.1 ([2, Proposition 3.1.1]). *Let $X = X(\Delta)$ be a complex toric variety and $\mathcal{E}$ be a torsion-free equivariant sheaf on X corresponding to a family of multifiltrations $\{E^\sigma_m\}_{\sigma\in\Delta,\ m\in M}$ of the vector space $\mathbf{E}^0$. There is a one-to-one correspondence between equivariant subsheaves of $\mathcal{E}$ and family of submultifiltrations $\{F^\sigma_m\}_{\sigma\in\Delta,\ m\in M}$ of the vector space $\mathbf{F}^0$, where $\mathbf{F}^0$ is a subspace of $\mathbf{E}^0$ and $\{F^\sigma_m\}_{\sigma\in\Delta,\ m\in M}$ being a submultifiltrations we mean*

$$F^\sigma_m \subseteq E^\sigma_m \cap \mathbf{F}^0.$$

Moreover, equivariant saturated subsheaves of $\mathcal{E}$ corresponds to family of submultifiltrations $\{F^\sigma_m\}_{\sigma\in\Delta,\ m\in M}$ of the vector space $\mathbf{F}^0$ which further satisfies

$$F^\sigma_m = E^\sigma_m \cap \mathbf{F}^0.$$

Proof. The proof remains the same till [2, Equation (3.4)]. We continue with the same notations as used in [2]. By the diagrams (3.3) and (3.4) in the proof of [2, Proposition 3.1.1], for $e \in E^\sigma_m \cap \mathbf{F}^0 = E^\sigma_m \cap \mathbf{F}^\sigma$, we have $[e, m] = [e', m'] \in \mathbf{F}^\sigma \subset \mathbf{E}^\sigma$, where $e' \in F^\sigma_{m'}$. Then there exists $m'' \in M$ such that $m, m' \leq_\sigma m''$ and $\chi^\sigma_{m,m''}(e) = \chi^\sigma_{m',m''}(e') \in F^\sigma_{m''}$. Now $\chi^\sigma_{m,m''} : E^\sigma_m \to E^\sigma_{m''}$ induces a map $\tilde{\chi}^\sigma_{m,m''} : E^\sigma_m/F^\sigma_m \to E^\sigma_{m''}/F^\sigma_{m''}$. Note that torsion-freeness of $\mathcal{E}/\mathcal{F}$ implies that E^σ/F^σ is also torsion-free $\mathbb{C}[S_\sigma]$-module, which further implies that the map $\tilde{\chi}^\sigma_{m,m''}$ is injective. We have $\tilde{\chi}^\sigma_{m,m''}(\bar{e}) = 0$ in $E^\sigma_{m''}/F^\sigma_{m''}$, where $\bar{e}$ denotes the image of e in E^σ_m/F^σ_m. This implies that $e \in F^\sigma_m$. Thus we have $F^\sigma_m = E^\sigma_m \cap \mathbf{F}^0$.
Conversely, suppose a family of submultifiltrations $\{F^\sigma_m\}_{\sigma\in\Delta,\ m\in M}$ of the vector space $\mathbf{F}^0 (\subseteq \mathbf{E}^0)$ is given. Then using the compatibility condition in Definition 2.2.6 (v), there exists an integer i^τ_m such the the following diagram commutes:

$$\begin{array}{ccc} F^\sigma_{m+i^\tau_m m_\tau} & \xrightarrow{=} & F^\tau_m \\ \big\downarrow & & \big\downarrow \\ E^\sigma_{m+i^\tau_m m_\tau} & \xrightarrow[=]{} & E^\tau_m. \end{array} \tag{0.1}$$

Then using the natural isomorphism $(i^*_{\tau\sigma}\widehat{E}^\sigma)_m \cong E^\sigma_{m+i^\tau_m m_\tau}$ (see [5, Proposition 5.15]), the diagram (0.1) essentially becomes the commutative diagram (3.1). Thus the gluing data arising from the family of submultifiltrations $\{F^\sigma_m\}_{\sigma\in\Delta,\ m\in M}$ is induced from the gluing data of $\mathcal{E}$. Hence the given family of submultifiltrations corresponds to an equivariant subsheaf $\mathcal{F}$ of $\mathcal{E}$ on X.
Now, additionally assume that $F^\sigma_m = E^\sigma_m \cap \mathbf{F}^0$. Note that $\Gamma(U_\sigma, \mathcal{E}/\mathcal{F}) = E^\sigma/F^\sigma$. It is enough to show that E^σ/F^σ is torsion-free as $\mathbb{C}[S_\sigma]$-module. To see this, let $\bar{e} \in E^\sigma_m/F^\sigma_m$ be such that

$$\chi^{m'} \cdot \bar{e} = 0$$

in $E^\sigma_{m+m'}/F^\sigma_{m+m'}$, where $e \in E^\sigma_m$, $\chi^{m'} \in \mathbb{C}[S_\sigma]$ and $\bar{e}$ denote the image of e in E^σ_m/F^σ_m. So

$$\chi^{m'} \cdot e \in F^\sigma_{m+m'}$$

which implies that

$$[\chi^{m'} \cdot e, m + m'] \in \mathbf{F}^\sigma \subseteq \mathbf{E}^\sigma.$$

Then we have

$$[\chi^{m'} \cdot e, m + m'] = [e, m]$$

in $\mathbf{E}^\sigma$. This implies that

$$[e, m] \in \mathbf{F}^\sigma = \mathbf{F}^0.$$

Since $F^\sigma_m = E^\sigma_m \cap \mathbf{F}^0$, we have that $e \in F^\sigma_m$. Hence the sheaf $\mathcal{E}/\mathcal{F}$ is torsion-free. □

Recall that given a filtration $(V, \{F^p V\})$ on a vector space V and a subspace $W \subseteq V$, there is an induced subfiltration on W by setting $F^p(W) := W \cap F^p(V)$. As an immediate corollary of Proposition 0.0.1 we can characterize saturated reflexive subsheaves in terms of induced subfiltrations (see also [3, Remark 2.4]).

Corollary 0.0.2 ([2, Corollary 3.1.2]). *Let $\mathcal{E}$ be an equivariant reflexive sheaf on X with associated filtrations $\left(\mathbf{E}^0, \{E^\rho(i)\}_{\rho\in\Delta(1)}\right)$. Then equivariant saturated subsheaves of $\mathcal{E}$ are in one-to-one correspondence with induced subfiltrations $\left(\mathbf{F}^0, \{F^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}}\right)$ of $\left(\mathbf{E}^0, \{E^\rho(i)\}_{\rho\in\Delta(1),\ i\in Z}\right)$, where $\mathbf{F}^0$ is a subspace of $\mathbf{E}^0$.*

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