Cdh Descent for Homotopy Hermitian K-Theory of Rings with Involution

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Abstract. We provide a geometric model for the classifying space of automorphism groups of Hermitian vector bundles over a ring with involution R such that $\frac{1}{2} \in R$; this generalizes a result of Schlichting-Tripathi [SST14]. We then prove a periodicity theorem for Hermitian K-theory and use it to construct an E_∞ motivic ring spectrum $\mathbf{KR}^{\mathrm{alg}}$ representing homotopy Hermitian K-theory. From these results, we show that $\mathbf{KR}^{\mathrm{alg}}$ is stable under base change, and cdh descent for homotopy Hermitian K-theory of rings with involution is a formal consequence.

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1 Introduction

Algebraic K-theory is an algebraic invariant introduced in the 1950s by Alexander Grothendieck where it served as the cornerstone of his reformulation of the Riemann-Roch theorem [Gro57]. Twenty years previously, Ernst Witt developed the notion of quadratic forms over arbitrary fields and introduced the Witt ring as an object to encapsulate the nature of all the quadratic forms over a given field [Wit37]. Combining the ideas of Grothendieck and Witt, Hyman Bass introduced a category of quadratic forms $\mathbf{Quad}(R)$ with isometries over a ring R and studied $K_1(\mathbf{Quad}(R))$ and $K_0(\mathbf{Quad}(R))$. $K_0(\mathbf{Quad}(R))$ is what we know today as the Grothendieck-Witt ring, and Bass was able to recover the Witt ring as a quotient of $K_0(\mathbf{Quad}(R))$ by the image of the hyperbolic quadratic forms. He went on to show that $K_1(\mathbf{Quad}(R))$ was related to the stable structure of the automorphisms of hyperbolic modules, which complemented the relationship between $K_1(R)$ and the group $GL(R)$. The K-theory of quadratic forms soon found applications to surgery theory where the periodic L-groups defined by Wall in 1966 [Wal66] served as obstructions to certain maps being cobordant to homotopy equivalences. When the means to define the higher algebraic K-groups via the + construction was discovered by Quillen in the 1970s, Karoubi applied it to the orthogonal groups BO in order to define the higher Hermitian K-theory of rings with involution as we know it today [Kar73].

Fast forward twenty years into the 1990s when Morel and Voevodsky developed the motivic homotopy category and proved that algebraic K-theory was representable in the stable motivic homotopy category [MV99]. The development of the stable motivic homotopy category not only gave a new domain to motivic cohomology, it also opened the door for applications of topological tools like obstruction theory to more algebraic objects. Several subsequent developments inspire our work here.

The first set of developments relates to Hermitian K-theory. In 2005 Hornbostel showed that Hermitian K-theory was representable in the stable motivic homotopy category on schemes [Hor05]. We note that Hornbostel defined Hermitian K-theory on schemes by extending the definition on rings using Jouanolou's trick. In 2011 Hu-Kriz-Ormsby showed that Hermitian K-theory on the category of C_2-schemes over a field is representable in the C_2-equivariant stable motivic homotopy category [HKO11]. Here they used a similar trick to Hornbostel in order to extend Hermitian K-theory from rings with involution to schemes with involution. In the meantime, Schlichting, building off of work of Thomason, Karoubi, and Balmer, defined the higher Hermitian K-theory of a dg-category with weak equivalences and duality and proved the analogues of the fundamental theorems of higher K-theory for these groups [Sch17]. Although some of Schlichting's theorems are stated only for schemes (rather than schemes with C_2 action), many of his proofs require only trivial modification to extend to Grothendieck-Witt groups of schemes with C_2 action. See also [Xie20] for the proofs of the equivariant version of some of the theorems together with a new transfer morphism. Another approach is taken by Hesselholt-Madsen, who define real algebraic K-theory of a category with weak equivalences and duality as a symmetric spectrum object in the monoidal category of pointed C_2-spaces. Schlichting's higher Grothendieck-Witt groups can be recovered from the Hesselholt-Madsen construction by taking homotopy groups of C_2-fixed points of deloopings of the real algebraic K-theory spaces with respect to the sign representation spheres. We note as well that the Ph.D. thesis of Alejo López-Ávila [Lv18] shows that the motivic spectrum representing Hermitian K-theory in the non-equivariant setting has an E_∞ structure.

Back in K-theory land, Cisinski proved that the six functor formalism in motivic homotopy theory developed by Ayoub [Ayo07] together with the fact that the motivic K-theory spectrum KGL is a cocartesian section of $\mathrm{SH}(-)$ yields a simple proof of cdh-descent (descent in the completely decomposed h topology) for homotopy K-theory [Cis13]. This in turn yields a short proof of Weibel's vanishing conjecture for homotopy K-theory, and inspired work of Kerz, Strunk, and Tamme who solved Weibel's conjecture by proving pro-cdh descent for ordinary K-theory [KST18]. Hoyois in [Hoy16] uses Cisinski's approach to show cdh descent for equivariant homotopy K-theory.

This paper, inspired by the above developments, shows cdh-descent for homotopy Hermitian K-theory of schemes with C_2 action. The techniques in [Hoy16] provide our pathway to descent. In order to show that Hermitian K-theory is a cocartesian section of $\mathrm{SH}^{C_2}(-)$, we need to show that the Hermitian K-theory space $\Omega^\infty GW$ can be represented by a certain Grassmannian, and we need a periodization theorem in order to pass from the Hermitian K-theory space $\Omega^\infty GW$ to the homotopy Hermitian K-theory motivic spectrum $L_{\mathbb{A}^1}\mathbb{G}W$. Schlichting and Tripathi [SST14] show that $\Omega^\infty GW$ is representable by a Grassmannian over schemes with trivial action over a regular base scheme with 2 invertible. Their techniques extend to the equivariant setting, and with slight modification provide a proof of representability over non-regular bases. The periodization techniques in [Hoy16] extend to Hermitian K-theory by investigating the Hermitian K-theory of T^ρ, the Thom space of the regular representation $\mathbb{A}^\rho$.

1.1 Outline

Section 2 begins with a review of G-equivariant motivic homotopy theory where G is a finite group scheme over a base S which is Noetherian of finite Krull dimension, has an ample family of line bundles, and has $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. First we review the definition of the equivariant étale and Nisnevich topologies, then we introduce the isovariant étale topology and give some examples of covers. For the reader familiar with non-equivariant motivic homotopy theory, the assumptions we make on G are strong enough so that structural results are mostly the same:

- the equivariant Nisnevich topology is generated by a nice cd-structure,
- equivariant schemes are locally affine in the equivariant Nisnevich topology, and
- to invert G-affine bundles $Y \to S$ it suffices to invert $\mathbb{A}^1_S$.

The content in this section is a selection of relevant content from [HKØ15]. We end this section with the definition of the unstable and stable equivariant motivic ∞-categories 2.4 a la Hoyois [Hoy17].
Section 3 reviews the definitions and results on Hermitian forms which will be necessary to work with the Grothendieck-Witt spectrum. Section 3.1 contains the basic definitions and examples, while section 3.3 contains the tools necessary to show that Hermitian forms are locally determined by rank in the isovariant or equivariant étale topologies. The final section 3.4 reviews the main definitions of [Sch17] to allow us to talk about the Grothendieck-Witt spectra of schemes with involution.
Section 4 is where the background material ends and the paper begins in earnest. We combine the techniques of [SST14] and [Hoy16] in order to show that classifying spaces of automorphism groups of Hermitian vectors bundles are representable in the C_2-equivariant motivic homotopy category.
This section culminates with the representability result, Theorem 4.14, which we note holds over non-regular base schemes:

Theorem 1.1. *Let S be a Noetherian scheme of finite Krull dimension with an ample family of line bundles and $\frac{1}{2} \in S$. There is an equivalence of motivic spaces on $\mathbf{Sm}^{C_2}_{S,qp}$*

$$L_{\mathrm{mot}}\mathbb{Z} \times \mathbb{R}\mathrm{Gr}_\bullet \xrightarrow{\sim} L_{\mathrm{mot}}\mathbb{Z} \times \operatorname*{colim}_n B_{isoEt}O(\mathbb{H}^n).$$

With a simple modification to remove the regularity hypothesis, one can follow [SST14] to show that

$$L_{\mathrm{mot}}\mathbb{Z}\times\operatorname*{colim}_n B_{isoEt}O(\mathbb{H}^n)\xrightarrow{\sim}L_{\mathrm{mot}}\Omega^\infty GW$$

but as this is unnecessary for proving cdh descent, we leave it out of this paper.
Section 5 provides a convenient way of passing from the presheaf of Grothendieck-Witt spectra to an E_∞-motivic spectrum in $\mathrm{SH}^{C_2}(S)$. The crucial fact is that the localizing version of Hermitian K-theory of rings with involution, denoted $\mathbb{G}W$, is the periodization of GW with respect to a certain Bott map derived from projective bundle formulas for $\mathbb{P}^1$ and $\mathbb{P}^\sigma$ (see Corollary 5.8). Here $\mathbb{P}^\sigma$ is a copy of $\mathbb{P}^1$ with action $[x:y]\mapsto[y:x]$. The fact that the periodization functor is monoidal together with Schlichting's results on monoidality of GW immediately give that the motivic spectrum $L_{\mathbb{A}^1}\mathbb{G}W\in\mathrm{SH}^{C_2}(S)$ is an E_∞ object 5.10.

THEOREM 1.2. *Let S be a Noetherian scheme of finite Krull dimension with an ample family of line bundles and $\frac{1}{2}\in S$. Then $L_{\mathbb{A}^1}\mathbb{G}W$ lifts to an E_∞ motivic spectrum, denoted $\mathbf{KR}^{\mathrm{alg}}$, over $\mathbf{Sm}^{C_2}_{S,qp}$.*

The final section 6 follows the recipe given by Cisinski and summarized in [Hoy16] to prove cdh descent for equivariant homotopy Hermitian K-theory on the category of quasi-projective S-schemes. After reviewing the K-theory case, the section culminates in theorem 6.2.

THEOREM 1.3. *Let S be a Noetherian scheme of finite Krull dimension with an ample family of line bundles and $\frac{1}{2}\in S$. Then the homotopy Hermitian K-theory spectrum of rings with involution $L_{\mathbb{A}^1}\mathbb{G}W$ satisfies descent for the equivariant cdh topology on $\mathbf{Sch}^{C_2}_{S,qp}$.*

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2 Equivariant Topologies and the Equivariant Motivic Homotopy Category

This section reviews the foundations of equivariant motivic homotopy theory. The key definitions are those of the equivariant étale and Nisnevich topologies – two topologies that play a crucial role in defining the equivariant motivic infinity category $\mathrm{H}^G(S)$ over a Noetherian base scheme S with finite Krull dimension, with an ample family of line bundles, and with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. Throughout we'll work with two categories of schemes. Let $\mathbf{Sch}^{C_2}_{S,qp}$ denote the category of quasi-projective C_2-schemes which are separated and finite type over S, and let $\mathbf{Sm}^{C_2}_{S,qp}$ be the full subcategory of schemes smooth over S.

Notation 2.1. Throughout this section, G will be either a finite group or the group scheme over S associated to a finite group. Recall that to pass between finite groups and group schemes over S, we form the scheme $\coprod_G S$ with multiplication (using that fiber products commute with coproducts in $\mathbf{Sch}_{S,qp}$):

$$\coprod_G S \times_S \coprod_G S \xrightarrow{\sim} \coprod_{(g_1,g_2)\in G\times G} S \xrightarrow{\mu} \coprod_G S$$

Whenever we write down a pullback square involving schemes, we'll tacitly be thinking of G as a group scheme, and $X \times Y$ will really mean $X \times_S Y$.

We introduce the background definitions from [HKØ15] which will allow us to define the isovariant étale topology. This is a topology which is slightly coarser than the equivariant étale topology, but whose points are still nice enough so that Hermitian vector bundles are locally determined by rank.

Definition 2.2. For a G-scheme X, the isotropy group scheme is a group scheme G_X over X defined by the cartesian square

$$\begin{array}{ccc} G_X & \longrightarrow & G \times X \\ \downarrow & & \downarrow{\scriptstyle (\mu_X, id_X)} \\ X & \xrightarrow{\Delta_X} & X \times X \end{array}$$

Definition 2.3. Let X be a G-scheme. The scheme-theoretic stabilizer of a point x in X is the pullback

$$\begin{array}{ccc} G_x & \longrightarrow & G_X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k(x) & \longrightarrow & X. \end{array}$$

By the pasting lemma, this is the same as the pullback

$$\begin{array}{ccc} G_x & \longrightarrow & G \times X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k(x) & \longrightarrow & X \times X \end{array}$$

DEFINITION 2.4. Let X be a G-scheme, and define the *set-theoretic* stabilizer S_x of $x \in X$ to be $\{g \in G \mid gx = x\}$.

REMARK 2.5. With notation as above, the underlying set of the scheme-theoretic stabilizer G_x can be described as

$$G_x = \{g \in S_x \mid \text{ the induced morphism } g : k(x) \to k(x) \text{ equals } id_{k(x)}\}.$$

The example below shows that set-theoretic and scheme-theoretic stabilizers need not agree.

EXAMPLE 2.6. (Herrmann [Her13]) Let k be a field, and consider the k-scheme given by a finite Galois extension $k \hookrightarrow L$. Let $G = Gal(L/k)$ be the Galois group. The set-theoretic stabilizer of the unique point in $\operatorname{Spec} L$ is G itself, while the scheme-theoretic stabilzer is $\{e\} \subset G$.

REMARK 2.7. Recall that if $Z \to X$ is a monomorphism of schemes, then the forgetful functor from schemes to sets preserves any pullback $Z \times_X Y$. The forgetful functor $\mathbf{Sch}^G_{S,qp} \to \mathbf{Sch}_{S,qp}$ is a right adjoint, hence preserves pullbacks. Since the inclusion of a point $\operatorname{Spec} k(x) \hookrightarrow X \times_S X$ will be a monomorphism, the difference between the set-theoretic and scheme-theoretic stabilizers is due to the fact that the underlying space of $X \times_S X$ is not necessarily the fiber product of the underlying spaces. Indeed, in the example above, $\operatorname{Spec} L \times_k \operatorname{Spec} L \cong \coprod_{g \in G} \operatorname{Spec} k$, whereas the pullback in spaces is just a single point.

2.1 THE EQUIVARIANT AND ISOVARIANT ÉTALE TOPOLOGIES

NOTATION 2.8. Let S be a G-scheme. The equivariant étale topology on $\mathbf{Sch}^G_{S,qp}$ will denote the site whose covers are étale covers whose component morphisms are equivariant.

DEFINITION 2.9. (Thomason) An equivariant map $f : Y \to X$ is said to be *isovariant* if it induces an isomorphism on isotropy groups $G_Y \cong G_X \times_X Y$. A collection $\{f_i : X_i \to X\}_{i \in I}$ of equivariant maps called an isovariant étale cover if it is an equivariant étale cover such that each f_i is isovariant.

REMARK 2.10. The isovariant topology is equivalent to the topology whose covers are equivariant, stabilizer preserving, étale maps. We'll use this notion more often in computations.

Remark 2.11. The points in the isovariant étale topology are schemes of the form $G \times^{G_x} \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}})$ where $\overline{x} \to x \to X$ is a geometric point, and $(-)^{sh}$ denotes strict henselization. See [HKØ15] for a proof.

The fact that the points in the isovariant étale topology are either strictly henselian local or hyperbolic rings will be crucial when we want to describe the isovariant étale sheafification of the category of Hermitian vector bundles. Fortunately Hermitian vector bundles over such rings are well understood, and we'll in fact show that Hermitian vector bundles are up to isometry determined by rank locally in the isovariant étale topology.

Remark 2.12. If $G = C_2$, then $G_x = \{e\}$ or $G_x = C_2$ for all $x \in X$. If $G_x = \{e\}$, then $G \times^{G_x} \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \cong C_2 \times \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \cong \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \coprod \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}})$ with a free action. If $G_x = C_2$, then $G \times^{G_x} \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) = \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}})$ where the induced action on the residue field is trivial.

2.2 The Equivariant Nisnevich Topology

Similarly to the non-equivariant case, the equivariant Nisnevich topology is defined by a particularly nice cd-structure. While there are a few different definitions of this topology in the literature which can give non Quillen equivalent model structures, we use the definition from [HKØ15].

Definition 2.13. A distinguished equivariant Nisnevich square is a cartesian square in $\mathbf{Sch}^G_{S,qp}$

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ A & \overset{i}{\hookrightarrow} & X \end{array}$$

where i is an open immersion, p is étale, and p restricts to an isomorphism $(Y - B)_{\mathrm{red}} \to (X - A)_{\mathrm{red}}$.

Definition 2.14. The equivariant Nisnevich cd-structure on $\mathbf{Sch}^G_{S,qp}$ is the collection of distinguished equivariant Nisnevich squares in $\mathbf{Sch}^G_{S,qp}$.

The next remark has the important consequence that to prove a map is an equivariant motivic equivalence, it suffices to check that it's an equivalence on affine G-schemes.

Remark 2.15. By Lemma 2.20 in [HKØ15], for finite groups G, any separated G-scheme of finite type over S is Nisnevich-locally affine.

2.3 The Equivariant cdh Topology

The completely decomposed h (cdh) topology is, roughly speaking, the coarsest topology satisfying Nisnevich excision and which allows for a theory of cohomology with compact support. Like the Nisnevich topology (and unlike the étale topology) it can be generated by a cd-structure, which gives a convenient way to check whether or not a presheaf is a cdh sheaf.

DEFINITION 2.16. An abstract blow-up square is a cartesian square in $\mathbf{Sch}^G_{S,qp}$

$$\begin{array}{ccc} \widetilde{Z} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow p \\ Z & \xhookrightarrow{i} & X \end{array}$$

where i is a closed immersion and p is a proper map which induces an isomorphism $(\widetilde{X}-\widetilde{Z}) \cong (X-Z)$.

DEFINITION 2.17. The cdh topology is the topology generated by the cd-structure whose distinguished squares are

- the equivariant Nisnevich distinguished squares;
- the abstract blowup squares.

One canonical example of a cdh cover is the map $X_{red} \to X$ for an equivariant scheme $X \to S$. Another example is given by resolution of singularities: given a proper birational map $p : X \to Y$, it's an isomorphism over some open set U in Y, so letting $Z = Y - U$ and $\widetilde{Z} = X - p^{-1}(U)$ we get an abstract blowup square

$$\begin{array}{ccc} \widetilde{Z} & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

2.4 THE EQUIVARIANT MOTIVIC HOMOTOPY CATEGORY

In this paper, we'll work with a Noetherian scheme of finite Krull dimension and a finite group scheme G over S. Equivariant motivic homotopy theory is developed in somewhat more generality by Hoyois in [Hoy17], though there's a price to be paid for allowing more general group schemes in that the motivic localization functor becomes more complicated.

DEFINITION 2.18. A presheaf F on $\mathbf{Sch}^G_{S,qp}$ is called *homotopy invariant* if the projection $\mathbb{A}^1_S \to S$ induces an equivalence $F(X) \simeq F(X \times \mathbb{A}^1)$ for each X in $\mathbf{Sch}^G_{S,qp}$. Denote by $\mathcal{P}_{\mathrm{htp}}(\mathbf{Sch}^G_{S,qp}) \subset \mathcal{P}(\mathbf{Sch}^G_{S,qp})$ the full subcategory spanned by the homotopy invariant presheaves. Denote by L_{htp} the corresponding localization endofunctor of $\mathcal{P}(\mathbf{Sch}^G_{S,qp})$. When restricting to $\mathbf{Sm}^G_{S,qp}$, we'll abuse notation and similarly let L_{htp} denote the corresponding localization endofunctor.

Now we give the usual definition of excision, the condition that guarantees that a presheaf is a Nisnevich sheaf.

Definition 2.19. A presheaf F on $\mathbf{Sch}^G_{S,qp}$ (or $\mathbf{Sm}^G_{S,qp}$) is called *Nisnevich excisive* if:

- $F(\emptyset)$ is contractible;
- for every equivariant Nisnevich square Q in $\mathbf{Sch}^G_{S,qp}$ (or $\mathbf{Sm}^G_{S,qp}$), $F(Q)$ is cartesian.

Denote by $\mathcal{P}_{\mathrm{Nis}}(\mathbf{Sm}^G_{S,qp}) \subset \mathcal{P}(\mathbf{Sm}^G_{S,qp})$ the full subcategory of Nisnevich excisive presheaves. Denote by L_{Nis} the corresponding localization endofunctor.

Finally we come to the definition of a motivic G-space, namely a presheaf that is both Nisnevich excisive and homotopy invariant.

Definition 2.20. Let S be a G-scheme. A *motivic G-space* over S is a presheaf on $\mathbf{Sm}^G_{S,qp}$ that is homotopy invariant and Nisnevich excisive. Denote by $\mathrm{H}^G(S) \subset \mathcal{P}(\mathbf{Sm}^G_{S,qp})$ the full subcategory of motivic G-spaces over S.
Let

$$L_{\mathrm{mot}} = \operatorname*{colim}_{n\to\infty}(L_{\mathrm{htp}} \circ L_{\mathrm{Nis}})^n(F)$$

denote the motivic localization functor, where the colimit is in the ∞-category of presheaves.

In order to form the stable equivariant motivic homotopy category, we also need to discuss pointed motivic G-spaces.

Definition 2.21. Let S be a G-scheme. A *pointed motivic G-space* over S is a motivic G-space X over S equipped with a global section $S \to X$. Denote by $\mathrm{H}^G_\bullet(S)$ the ∞-category of pointed motivic G-spaces.

The definition of stabilization can in general be complicated. With our assumptions however, we need only invert the Thom space of the regular representation T^ρ.

Definition 2.22. Let S be a G-scheme. The symmetric monoidal ∞-category of *motivic G-spectra* over S is defined by

$$\mathrm{SH}^G(S) = \mathrm{H}^G_\bullet(S)[(T^\rho_S)^{-1}] = \operatorname{colim}\left(\mathrm{H}^G_\bullet \xrightarrow{-\otimes T^\rho} \mathrm{H}^G_\bullet \xrightarrow{-\otimes T^\rho} \cdots\right)$$

where T^ρ is the Thom space of the regular representation $\mathbb{A}^\rho/\mathbb{A}^\rho - 0$ of G. The colimit is taken in the ∞-category of presentable ∞-categories.

2.5 Computations with Equivariant Spheres

Because we'll be using equivariant spheres to index our spectra, we'll record some of their basic properties here. These computations will be important when we investigate periodicity of GW in section 5. Though there are exotic elements of the Picard group even in non-equivariant stable motivic homotopy theory, we'll be concerned with the four building blocks $S^1, S^\sigma = \mathrm{colim}(* \leftarrow (C_2)_+ \to S^0), \mathbb{G}_m, \mathbb{G}_m^\sigma$. Here $\mathbb{G}_m^\sigma$ is the C_2 scheme corresponding to $S[T, T^{-1}]$ with action $T \mapsto T^{-1}$.

Lemma 2.23. *Let $\mathbb{P}^\sigma$ denote $\mathbb{P}^1$ with the action defined by $[x : y] \mapsto [y : x]$. There is an equivariant Nisnevich square*

$$\begin{array}{ccc} C_2 \times \mathbb{G}_m^\sigma & \longrightarrow & \mathbb{P}^1 - \{0\} \coprod \mathbb{P}^1 - \{\infty\} \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle f} \\ \mathbb{G}_m^\sigma & \xrightarrow{\quad i \quad} & \mathbb{P}^\sigma \end{array}$$

Proof. Here, we identify $\mathbb{G}_m^\sigma$ with $\mathbb{P}^\sigma - \{0, \infty\}$. The map i is clearly an open immersion. Its complement is $\{0, \infty\}$, and f maps $f^{-1}(\{0, \infty\})$ isomorphically onto $\{0, \infty\}$. Furthermore, f is a disjoint union of open immersions, and hence is (in particular) étale. $\square$

Lemma 2.24. *There is a homotopy pushout square, where f can be taken to map C_2 to $\{[1 : 1]\}$:*

$$\begin{array}{ccc} (C_2)_+ \wedge (\mathbb{G}_m^\sigma)_+ & \longrightarrow & (C_2)_+ \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle f} \\ (\mathbb{G}_m^\sigma)_+ & \xrightarrow{\quad i \quad} & \mathbb{P}^\sigma_+ \end{array}$$

Proof. The above square is equivalent to the square

$$\begin{array}{ccc} (C_2)_+ \wedge (\mathbb{G}_m^\sigma)_+ & \longrightarrow & (C_2)_+ \wedge \mathbb{A}^1_+ \, . \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle f} \\ (\mathbb{G}_m^\sigma)_+ & \xrightarrow{\quad i \quad} & \mathbb{P}^\sigma_+ \end{array}$$

By the lemma above,

$$\begin{array}{ccc} (C_2 \times \mathbb{G}_m^\sigma)_+ & \longrightarrow & (C_2 \times \mathbb{A}^1)_+ \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle f} \\ (G_m^\sigma)_+ & \xrightarrow{\quad i \quad} & \mathbb{P}^\sigma_+ \end{array}$$

is a homotopy pushout square. But adding a disjoint basepoint is a monoidal functor, so $X_+ \wedge Y_+ \cong (X \times Y)_+$ and this square is equivalent to the desired square. $\square$

Lemma 2.25. $\mathbb{P}^\sigma \approx S^\sigma \wedge \mathbb{G}_m^\sigma$.

Proof. Let Q denote the homotopy cofiber of $(C_2 \times \mathbb{G}_m^\sigma)_+ \to (\mathbb{G}_m^\sigma)_+$, and $\widetilde{Q}$ denote the homotopy cofiber of $(C_2 \times \mathbb{A}^1)_+ \to \mathbb{P}_+^\sigma$. Then the lemma above implies that $Q \approx \widetilde{Q}$.
Q is the homotopy cofiber of $(C_2)_+ \wedge (G_m^\sigma)_+ \to S^0 \wedge (G_m^\sigma)_+$, which is just $S^\sigma \wedge (G_m^\sigma)_+$. Recall that $\operatorname{colim}(* \leftarrow X \to X \wedge Y_+) \cong X \wedge Y$ since this is $X \wedge \operatorname{colim}(* \leftarrow S^0 \to Y_+)$. Thus the cofiber of $S^\sigma \to Q$ is $S^\sigma \wedge \mathbb{G}_m^\sigma$.
The diagram below in which the horizontal rows are cofiber sequences

$$\begin{array}{ccccc}
(C_2)_+ & \longrightarrow & S^0 & \longrightarrow & S^\sigma \\
\downarrow{\scriptstyle id} & & \downarrow & & \downarrow \\
(C_2)_+ & \longrightarrow & \mathbb{P}_+^\sigma & \longrightarrow & \widetilde{Q} \\
\downarrow & & \downarrow & & \downarrow \\
\star & \longrightarrow & \mathbb{P}^\sigma & \longrightarrow & T
\end{array}$$

implies that the cofiber of $S^\sigma \to \widetilde{Q}$ is $\mathbb{P}^\sigma$.
The result now follows from the commutativity of the following diagram and homotopy invariance of homotopy cofiber:

$$\begin{array}{ccc}
S^\sigma & \xrightarrow{id} & S^\sigma \\
\downarrow & & \downarrow \\
Q & \xrightarrow{\sim} & \widetilde{Q} \\
\downarrow & & \downarrow \\
S^\sigma \wedge \mathbb{G}_m^\sigma & \longrightarrow & \mathbb{P}^\sigma
\end{array}$$

$\square$

3 Hermitian Forms on Schemes

This section reviews the definitions and properties of Hermitian forms over schemes with involution from [Xie20]. After defining the proper notion of the dual of a quasi-coherent module over a scheme with involution, the definition of a Hermitian vector bundle finally appears in Definition 3.11 as a locally free $\mathcal{O}_X$-module with a well-behaved map to the dual module. Once the definitions are in place, we discuss in section 3.3 the structure of Hermitian forms over semilocal rings as this is the fundamental tool for showing that Hermitian forms are locally trivial in the isovariant étale topology. We prove this particular statement in Corollary 3.27. We end this section by recalling Schlichting's definition of a dg categoy with weak equivalence and duality and the Grothendieck-Witt groups of such an object.

3.1 DEFINITIONS

DEFINITION 3.1. Let R be a ring with involution $- : R \to R^{op}$. A *Hermitian module over R* is a finitely generated projective right R-module, M, together with a map

$$b : M \otimes_{\mathbb{Z}} M \to R$$

such that, for all $a \in R$,

1. $b(xa, y) = \overline{a}b(x,y)$,
2. $b(x, ya) = b(x,y)a$,
3. $b(x,y) = \overline{b(y,x)}$.

DEFINITION 3.2. Let R be a ring with involution $-$. Given a right R-module M, define a left R-module, denoted $\overline{M}$ as follows: $\overline{M}$ has the same underlying abelian group as M, and the action is given by $r \cdot m = m \cdot \overline{r}$. If R is commutative, we can promote $\overline{M}$ to an R-bimodule by introducing the right action $m \cdot r = m\overline{r}$.

REMARK 3.3. Let R be a commutative ring so that $R = R^{op}$. Given an involution $\sigma : R \to R$ and a right R-module M, we can identify $\overline{M}$ with $\sigma_* M$, where $\sigma_* M$ is the module M with R-action restricted through σ. Typically the pushforward would just take the right R-module M to another right R-module. Since we really view σ as landing in R^{op}, we use commutativity of R and the canonical identification of right R^{op} modules with left R-modules to think of $\sigma_* M$ as a left module. Indeed, $\sigma_* M$ is a left R-module via the rule $r \cdot m = m \cdot \sigma(r)$.

REMARK 3.4. Another way to define a Hermitian form over a ring R with involution σ is to give a finitely generated projective right R-module M together with an $R-R$-bimodule map

$$b : M \otimes_{\mathbb{Z}} M \to R$$

where we view R as a bimodule over itself by $r_1 \cdot r \cdot r_2 = r_1 r r_2$, M as a left R-module via the involution, and such that $b(x,y) = \sigma(b(y,x))$. If we remove the final condition, we obtain a sesquilinear form.

By the usual duality, we have a third definition:

Definition 3.5. A Hermitian module over a ring R with involution is a finitely generated projective right R-module M together with an R-linear map $b : M \to \overline{M}^{\vee} = M^*$ such that $b = b^* can_M$, where $b^* : M^{**} \to M^*$ is given by $(b^*(f))(m) = f(b(m))$. Here $can_M : M \to M^{**}$, $can_M(m)(f) = \overline{f(m)}$ is the canonical double dual isomorphism.

Now, we generalize the above definitions to schemes.

Definition 3.6. Let X be a scheme, and M a quasi-coherent $\mathcal{O}_X$-module. Define $\mathcal{O}_X{}^{\vee} = \mathbf{Hom}_{mod-\mathcal{O}_X}(M, \mathcal{O}_X)$.

Definition 3.7. Let X be a scheme with involution σ, and M a right $\mathcal{O}_X$-module. Note that there's an induced map $\sigma^{\#} : \mathcal{O}_X \to \sigma_* \mathcal{O}_X$. Define the right (note that we're working with sheaves of commutative rings, so we can do this) $\mathcal{O}_X$-module $\overline{M}$ to be $\sigma_* M$ with $\mathcal{O}_X$ action induced by the map $\sigma^{\#}$. That is, if $m \in \sigma_* M(U)$, and $c \in \mathcal{O}_X(U)$, then $m \cdot c = m \cdot \sigma^{\#}(c)$. Note that this last product is defined, because $m \in \sigma_* M(U) = M(\sigma^{-1}(U))$, $c \in \sigma_* \mathcal{O}_X(U) = \mathcal{O}_X(\sigma^{-1}(U))$, and M is a right $\mathcal{O}_X$-module.

Remark 3.8. We have two choices for the definition of the dual M^*. We can either define
$$M^* = \mathbf{Hom}_{mod-\mathcal{O}_X}(\sigma_* M, \mathcal{O}_X),$$
or we can define $M^* = \sigma_* \mathbf{Hom}_{mod-\mathcal{O}_X}(M, \mathcal{O}_X)$. We claim that these two choices of dual are naturally isomorphic.

Proof. Let $f : \sigma_* M|_U \to \mathcal{O}_X|_U$ be a map of right $\mathcal{O}_X|_U$-modules. Postcomposing with the map $\mathcal{O}_X|_U \to \sigma_* \mathcal{O}_X|_U$ yields a map $\overline{f} : \sigma_* M|_U \to \sigma_* \mathcal{O}_X|_U$, a.k.a. a map $M|_{\sigma^{-1}U} \to \mathcal{O}_X|_{\sigma^{-1}U}$. Note that
$\sigma_* \mathbf{Hom}_{mod-\mathcal{O}_X}(M, \mathcal{O}_X)(U) = \mathbf{Hom}_{mod-\mathcal{O}_X}(M, \mathcal{O}_X)(\sigma^{-1} U)$, so that $\overline{f} \in \sigma_* \mathbf{Hom}_{mod-\mathcal{O}_X}(M, \mathcal{O}_X)(U)$.
On the other hand, given $g \in \sigma_* \mathbf{Hom}_{mod-\mathcal{O}_X}(M, \mathcal{O}_X)(U)$, so that $g : \sigma_* M|_U \to \sigma_* \mathcal{O}_X|_U$, we can postcompose with $\sigma_*(\sigma^{\#})$ to get a map $\widetilde{g} : \sigma_* M|_* \to \sigma_* \sigma_* \mathcal{O}_X|_U = \mathcal{O}_X|_U$. Since $\sigma^2 = id$, this is clearly the inverse to the map above.
It's clear that these assignments are natural, since they're just postcomposition with a natural transformation. □

Definition 3.9. Define the adjoint module M^* to be $\mathbf{Hom}_{mod-\mathcal{O}_X}(\sigma_* M, \mathcal{O}_X)$. By the remark above, it doesn't really matter which of the two possible definitions we choose here. From this point, we will also use $\mathbf{Hom}_{mod-\mathcal{O}_X}$ synonymously with $\mathbf{Hom}_{\mathcal{O}_X}$.

Definition 3.10. Given a right $\mathcal{O}_X$-module M, we define the double dual isomorphism $\mathrm{can}_M : M \to M^{**}$ as follows: given an open $U \subseteq X$, we define a map
$$\begin{aligned} M(U) &\to \mathrm{Hom}_{\mathcal{O}_U}(\sigma_* \mathbf{Hom}_{\mathcal{O}_X}(\sigma_* M, \mathcal{O}_X)|_U, \mathcal{O}_U) \\ &= \mathrm{Hom}_{\mathcal{O}_U}(\mathbf{Hom}_{\mathcal{O}_X}(\sigma_* M|_{\sigma(U)}, \mathcal{O}_X|_{\sigma(U)}), \mathcal{O}_U) \end{aligned}$$

by $u \mapsto \eta_u$, where for an open $V \subseteq U$,

$$(\eta_u)_V(\gamma) = (\sigma^\#)_V^{-1}(\gamma_{\sigma(V)}(u|_V)).$$

Here $\gamma \in \mathrm{Hom}_{\mathcal{O}_{\sigma(U)}}(\sigma_* M|_{\sigma(U)}, \mathcal{O}_{\sigma(U)})$ and $\sigma^\#$ is the morphism of sheaves $\sigma^\# : \mathcal{O}_X \to \sigma_* \mathcal{O}_X$. Note that $\gamma_{\sigma(V)}(u|_V)$ makes sense because $\sigma_* M(\sigma(V)) = M(V)$.
More globally, there's an evaluation map

$$ev_\sigma : M \otimes \sigma_* \mathbf{Hom}_{\mathcal{O}_X}(\sigma_* M, \mathcal{O}_X) \to \mathcal{O}_X$$

defined by the composition

$$M \otimes \sigma_* \mathbf{Hom}_{\mathcal{O}_X}(\sigma_* M, \mathcal{O}_X) \cong M \otimes \mathbf{Hom}_{\mathcal{O}_X}(M, \sigma_* \mathcal{O}_X) \xrightarrow{ev} \sigma_* \mathcal{O}_X \xrightarrow{(\sigma^\#)^{-1}} \mathcal{O}_X$$

which under adjunction yields the above map.

DEFINITION 3.11. Let X be a scheme with involution $- : X \to X$. Let can_X be the double dual isomorphism of Definition 3.10. A *Hermitian vector bundle over X* is a locally free right $\mathcal{O}_X$-module V with an $\mathcal{O}_X$-module map $\phi : V \to V^*$ such that $\phi = \phi^* \mathrm{can}_V$. A Hermitian vector bundle is *non-degenerate* if ϕ is an isomorphism.

REMARK 3.12. Recall that there's an equivalence of categories between locally free coherent sheaves on X and geometric vector bundles given by $M \mapsto \mathbf{Spec}(\mathrm{Sym}(M^\vee))$ in one direction and the sheaf of sections in the other. For locally free sheaves, we have $M^\vee \otimes N^\vee \cong (M \otimes N)^\vee$ so that the functor is monoidal. We will use this to think of a Hermitian form as a map of schemes $V \otimes V \to \mathbb{A}^1$.

Below we give the key example of a Hermitian vector bundle.

EXAMPLE 3.13. Define (diagonal) hyperbolic n-space over a scheme $(S,-)$ with involution to be $\mathbb{A}_S^{2n}$ with the Hermitian form $(x_1,\dots,x_{2n},y_1,\dots,y_{2n}) \mapsto \sum_{i=1}^n \overline{x}_{2i-1}y_{2i-1} - \overline{x}_{2i}y_{2i}$. Denote this Hermitian form by h_{diag}.
As defined this way, the matrix of this Hermitian form is

$$\begin{bmatrix} 1 & 0 & \cdots & & \\ 0 & -1 & 0 & \cdots & \\ \vdots & \vdots & \vdots & & \\ 0 & \cdots & \cdots & \cdots & -1 \end{bmatrix}$$

the diagonal matrix $\mathrm{diag}(1,-1,1,\dots,-1)$. For this definition to give a Hermitian space isometric to other standard definitions of the hyperbolic form, it's crucial that 2 be invertible.
The isometries of $\mathbb{H}_\mathbb{R}$ (where we give it the hyperbolic form above) have the form

$$\begin{bmatrix} a & b \\ \pm b & \pm a \end{bmatrix}$$

with $a = \pm\sqrt{1+b^2}, b \in \mathbb{R}$ (or $a^2 - b^2 = 1$). The usual identification with $\mathbb{R}^\times \rtimes C_2$ follows by considering the decomposition $a^2 - b^2 = 1 \iff (a+b)(a-b) = 1$.

EXAMPLE 3.14. Similarly to above, we can define a hyperbolic form h by the matrix

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

This form is isometric to the above form, and we'll use both forms below.

3.2 PROPERTIES

We record two unsurprising structural results which will be useful when we define the Hermitian Grassmannian in section 4.

LEMMA 3.15. *Given a map of schemes with involution $f : (Y, i_Y) \to (X, i_X)$ and a (non-degenerate) Hermitian vector bundle (V, ω) on X, $f^*(V)$ is a (non-degenerate) Hermitian vector bundle on Y.*

Proof. The pullback of a locally free $\mathcal{O}_X$-module is a locally free $\mathcal{O}_Y$-module, so we just need to check that it's Hermitian. Given the map $\omega : V \to V^*$, we get an induced map $f^*V \to f^*(V^*)$ which is an isomorphism if ω is. Thus we just need to check that $f^*(V^*) \cong (f^*V)^*$. But pullback commutes with sheaf dual for locally free sheaves of finite rank, so we just need to check that changing the module structure via the involution commutes with pullback; that is, we need to check that $f^*(\overline{V}) = \overline{f^*(V)}$. However, this is clear since the structure map on $f^*(\overline{V})$ is given by

$$O_Y \times f^*V \cong f^*\mathcal{O}_X \times f^*V \xrightarrow{f^*(-)\times id} f^*(O_X) \times f^*(V) \to f^*(V).$$

□

LEMMA 3.16. *Let (V, ϕ) be a non-degenerate Hermitian vector bundle over a scheme with trivial involution X, and let $(M, \phi|_M)$ be a (possibly degenerate) subbundle. Given a map of schemes $g : Y \to X$, there is a canonical isomorphism $g^*(M^\perp) \cong (g^*M)^\perp$.*

Proof. Recall that, by definition, $M^\perp = \ker(V \xrightarrow{\phi} V^* \to M^*)$. Equivalently, $M^\perp$ is defined by the exact sequence

$$0 \to M^\perp \to V \to M^* \to 0.$$

It follows that the composite map $g^*(M^\perp) \to g^*V \to g^*(M^*)$ is zero, and hence by universal property of kernel there's a canonical map

$$g^*(M^\perp) \to \ker(g^*V \to g^*(M^*) \cong (g^*(M))^*) = (g^*(M))^\perp$$

where we've used the canonical isomorphism $g^*(M^*) \cong (g^*(M))^*$ for locally free sheaves.

We claim that this map is an isomorphism. It suffices to check on stalks, where the map can be identified with a map

$$M^{\perp}_{g(y)} \otimes \mathcal{O}_{Y,y} \to \ker(V_{g(y)} \otimes \mathcal{O}_{Y,y} \to M^{*}_{g(y)} \otimes \mathcal{O}_{Y,y}).$$

But $V_{g(y)} \cong M^{\perp}_{g(y)} \oplus M^{*}_{g(y)}$, so the sequence

$$0 \to M^{\perp}_{g(y)} \otimes \mathcal{O}_{Y,y} \to V_{g(y)} \otimes \mathcal{O}_{Y,y} \to M^{*}_{g(y)} \otimes \mathcal{O}_{Y,y} \to 0$$

is split exact, and the canonical map is an isomorphism. □

We record two incredibly useful results for working with Hermitian forms. The first implies that Hermitian forms over fields can be written as an orthogonal sum of rank 1 Hermitian forms, while the second gives a useful characterization of non-degenerate submodules of a Hermitian module.

THEOREM 3.17. *(Knus [Knu91] 6.2.4) Let (M,b) be a non-degenerate Hermitian vector bundle over a division ring D. Then (M,b) has an orthogonal basis in the following cases:*

1. *the involution of D is not trivial*
2. *the involution of D is trivial and char $D \neq 2$.*

LEMMA 3.18. *(Knus) Let (M,b) be a Hermitian module, and $(U,b|_U)$ be a non-degenerate finitely generated projective Hermitian submodule. Then $M = U \oplus U^{\perp}$.*

Proof. Since $b|_U : U \to U^*$ is an isomorphism, given an $m \in M$, there exists $u \in U$ such that $b(m,-)|_U = b(u,-)|_U$. But then $b(m-u,-)|_U = 0$, so that $m-u \in U^{\perp}$, and $m = u+m-u$. Thus $M = U+U^{\perp}$. Since $\phi|_U$ is non-degenerate, $U \cap U^{\perp} = 0$, so we're done. □

3.3 HERMITIAN FORMS ON SEMILOCAL RINGS

From here on out, all rings are assumed to be commutative. Many of the results of this section can be deduced from [FW17], though we include proofs in an effort to make the document self contained.
The following lemma is a slight generalization of a result from [Bae78] which will allow us to conclude that Hermitian forms diagonalize over semilocal rings with involution.

LEMMA 3.19. *Let (R,σ) be a commutative ring with involution, and let E be a Hermitian module over R. Let $I \subset Jac(R)$ be an ideal fixed by the involution. For every orthogonal decomposition $\overline{E} = \overline{F} \perp \overline{G}$ of $\overline{E} = E/IE$ over R/I, where $\overline{F}$ is a free non-degenerate subspace of $\overline{E}$, there exists an orthogonal decomposition $E = F \perp G$ of E with F free and non-degenerate, and $F/IF = \overline{F}, G/IG = \overline{G}$. Here R/I has the induced involution $\sigma(x+I) = \sigma(x)+\sigma(I) = \sigma(x)+I$.*

Proof. Write $\overline{F} = \langle \overline{x}_1 \rangle \oplus \cdots \oplus \langle \overline{x}_n \rangle$ with $\overline{x}_i \in \overline{F}$ and $\det(\overline{b}(\overline{x}_i, \overline{x}_j)) \in (R/I)^\times$. Choose representatives $x_i \in E$ of $\overline{x}_i$, and let $F = Rx_1 + \cdots + Rx_n$. We claim that the x_i are independent, so that F is free: indeed, if $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$, then we get n equations $\lambda_1 b(x_1, x_i) + \cdots + \lambda_n b(x_n, x_i) = 0$. We claim that $\det(b(x_i, x_j)) = t \in R^\times$. To wit, since $1 - st \in I$ for some s by assumption (because the determinant is a unit mod the ideal I), then st cannot be contained in any maximal ideal, so $st \in R^\times \implies t \in R^\times$. It follows that the λ_i are zero (otherwise we would have a non-zero vector in the kernel of an invertible matrix), so that the x_i are independent as desired. The determinant fact also shows that F is non-degenerate, so by the lemma above, it has an orthogonal summand G. By construction $F/I = \overline{F}$, so that $\overline{G} = (\overline{F})^\perp = (F/I)^\perp = F^\perp/I = G/I$. □

Lemma 3.20. *Hermitian forms over $R_1 \times R_2$ (with trivial involution) are in bijection with $Herm(R_1) \times Herm(R_2)$.*

Proof. First, recall that modules over $R_1 \times R_2$ correspond to a module over R_1 and a module over R_2. Indeed, consider the standard idempotents $(1,0) = e_1, (0,1) = e_2$. Fix a module M over $R_1 \times R_2$. Then $M = e_1 M \oplus e_2 M$. Now any $m \in M$ can be written as $e_1 m + e_2 m = (e_1 + e_2)m = m$. Furthermore, if $e_1 m_1 = e_2 m_2$, then $e_2 e_1 m_1 = e_2 e_2 m_2 \implies 0 = e_2 m_2$.
A Hermitian form $M \otimes M \to R_1 \times R_2$ is determined by two maps $M \otimes M \to R_1$ and $M \otimes M \to R_2$. Writing $M = e_1 M \oplus e_2 M$, we note that, by linearity, it must be the case that $e_1 M \otimes e_2 M \to R_1 \times R_2$ is the zero map; to wit, $b(e_1 m_1, e_2 m_2) = e_1 e_2 b(m_1, m_2) = 0$. Thus this Hermitian form is determined completely by the maps $e_1 M \otimes e_1 M \to R_1 \times R_2$ and $e_2 M \otimes e_2 M \to R_1 \times R_2$. Finally, note that, again by linearity, we see that $e_1 M \otimes e_1 M \to R_2$ is the zero map: $b(e_1 m_1, e_1 m_2) = b(e_1^2 m_1, e_1 m_2) = e_1 b(e_1 m_1, e_1 m_2)$, and $e_1 R_2 = 0$. Similarly for the other map. Hence the Hermitian form is completely determined by the maps $e_1 M \otimes e_1 M \to R_1$ and $e_2 M \otimes e_2 M \to R_2$. □

Corollary 3.21. Hermitian modules of constant rank diagonalize over commutative rings R with finitely many maximal ideals $m_1, \ldots, m_n$ (semi-local rings) and with $\frac{1}{2} \in R$.

Proof. By the Chinese Remainder Theorem, $R/(m_1 \cap \cdots \cap m_n) \cong R/m_1 \times \cdots \times R/m_n = F_1 \times \cdots \times F_n$. We claim that Hermitian forms over finite products of fields diagonalize, and then the result will follow from Lemma 3.19. By induction and Lemma 3.20, a Hermitian module M is determined by Hermitian modules M_i over F_i, $i = 1, \ldots, n$ as $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ with action $(f_1, \ldots, f_n) \cdot (m_1, \cdots, m_n) = (f_1 m_1, \ldots, f_n m_n)$. By Theorem 3.17, each M_i can be diagonalized into $M_i = \langle a_{1,i} \rangle \perp \cdots \perp \langle a_{m,i} \rangle$ (it's important to note here that the rank of each M_i is the same by assumption). Thus a diagonalization of M is given by $\langle (a_{1,1}, \ldots, a_{1,n}) \rangle \perp \cdots \perp \langle (a_{m,1}, \ldots, a_{m,n}) \rangle$. □

Corollary 3.22. Let R be a local ring with trivial involution and with $\frac{1}{2} \in R$. Then any Hermitian module (which is necessarily free) over R diagonalizes.

LEMMA 3.23. *Let R be a ring, and consider the ring $R \times R$ with the involution that switches factors. Then any module M can be written as $e_1 M \oplus e_2 M$ as in the proof of Lemma 3.20. A non-degenerate Hermitian form on this module is determined by a map $e_1 M \otimes e_2 M \to R \times R$. In other words, the matrix representing the map $e_1 M \oplus e_2 M \to e_1 M^* \oplus e_2 M^*$ has the form*

$$\begin{bmatrix} 0 & A \\ \overline{A}^t & 0 \end{bmatrix}.$$

where A is invertible.

Proof. The first claim is just that $b(e_1 x, e_1 y) = 0 = b(e_2 x, e_2 y)$ for any $x, y \in M$. This follows because $b(e_1 x, e_1 y) = b(e_1^2 x, e_1^2 y) = \overline{e_1} e_1 b(e_1 x, e_1 y) = e_2 e_1 b(e_1 x, e_1 y) = 0$. Similarly for $b(e_2 x, e_2 y)$. The statement about the matrix follows by identifying the map $M \otimes \overline{M} \to R \times R$ with an isomorphism $M \to \overline{M}^*$ and using the direct sum decomposition. □

COROLLARY 3.24. Let R, M be as in lemma 3.23 and such that $\frac{1}{2} \in R$. Then $M \cong H(e_1 M)$, where H denotes the hyperbolic module functor.

Proof. The assumption that 2 is invertible implies that M is an even Hermitian space in the notation of Knus. Now by Lemma 3.23 $b|_{e_1 M} = 0$, so M has direct summands $e_1 M, e_2 M$ such that $e_1 M = e_1 M^\perp$ and $M = e_1 M \oplus e_2 M$. Now [Knu91, Corollary 3.7.3] applies to finish the proof. □

COROLLARY 3.25. Let R be a semi-local ring with involution and with 2 invertible. Then any Hermitian module of constant rank over R diagonalizes.

Proof. Using Lemma 3.19 and reducing modulo the Jacobson radical (which is always stable under the involution), it suffices to prove the corollary for R a finite product of fields. Then $R = F_1 \times \cdots \times F_n$ is semi-simple, and hence we can index the fields in a particularly nice way (proof is by considering idempotents), writing $R = A_1 \times \cdots \times A_m \times B_1 \times \ldots B_{n-m}$ such that A_i is fixed setwise by the involution, and $\sigma(B_{2i}) = B_{2i+1}$, $\sigma(B_{2i+1}) = B_{2i}$. Now, any finitely generated module M can be written as a direct sum $M = \bigoplus_{i=1}^{m} M_i \bigoplus_{i=1}^{\frac{n-m}{2}} N_{2i} \oplus N_{2i-1}$. By Theorem 3.17 and Corollary 3.24, the form when restricted to each M_i or $N_{2i} \oplus N_{2i-1}$ is diagonalizable, so the form is diagonalizable (see the proof of Corollary 3.21). □

LEMMA 3.26. *Non-degenerate Hermitian vector bundles are determined by rank over strictly henselian local rings (R, m) with $\frac{1}{2} \in R$ such that the residue field R/m has trivial involution.*

Proof. By Corollary 3.25, any Hermitian vector bundle over R diagonalizes. Thus it suffices to prove that any two non-degenerate Hermitian vector bundles of rank 1 are isometric.

A non-degenerate rank 1 Hermitian vector bundle corresponds to a unit $x \in R^\times$ such that $x = \overline{x}$ (a one dimensional Hermitian matrix). Because R is strictly henselian, there is a square root c of x^{-1}. We claim that $c = \overline{c}$. Assume not. Then because the involution on R/m is trivial, $c-\overline{c} \in m$. Since 2 is invertible, we have $c = \frac{c+\overline{c}}{2} + \frac{c-\overline{c}}{2}$. It follows that $\frac{c+\overline{c}}{2}$ is a unit. Otherwise it would be contained in m which would imply that the unit c was contained in m.

However, we calculate $(c+\overline{c})(c-\overline{c}) = c^2 - \overline{c}^2$. But $(\overline{c})^2 = \overline{(c^2)} = \overline{x}^{-1} = x^{-1}$, so that $(c+\overline{c})(c-\overline{c}) = 0$. Because $c+\overline{c}$ is a unit, it follows that $c-\overline{c} = 0$.

This shows that given any one dimensional Hermitian matrix x, there's a unit c such that $cx\overline{c} = 1$ so that all one dimensional Hermitian forms are isometric to the form $\langle 1 \rangle$. □

Corollary 3.27. Non-degenerate Hermitian vector bundles are locally determined by rank in the isovariant étale topology.

Proof. The points in the isovariant étale topology are either strictly henselian local rings whose residue field has trivial involution or a ring of the form $\mathcal{O}^{sh}_{X,x} \times \mathcal{O}^{sh}_{X,x}$ with involution $(x,y) \mapsto (i(y), i(x))$. Via the map $(x,y) \mapsto (x, i(y))$, such rings are isomorphic to hyperbolic rings.

If the ring is a stricty henselian local ring whose residue field has trivial involution, Lemma 3.26 shows that non-degenerate Hermitian forms are determined by rank. If the ring is hyperbolic, then by Corollary 3.24 all non-degenerate Hermitian forms over the ring are hyperbolic forms of projective modules over a local ring. Since projective modules over a local ring are determined by rank, the corresponding hyperbolic forms are determined by rank. □

3.4 Higher Grothendieck-Witt Groups

In [Xie20], the author works with coherent Grothendieck-Witt groups on a scheme. Because the negative K-theory of the category of bounded complexes of quasi-coherent $\mathcal{O}_X$-modules with coherent cohomology vanishes (together with the pullback square relating the homotopy fixed points of K-theory to Grothendieck-Witt theory), there is no difference between the additive and localizing versions of Grothendieck-Witt spectra in this setting.

Therefore, we work instead with Grothendieck-Witt spectra of $\mathrm{sPerf}(X) = \mathrm{Ch}^b\mathrm{Vect}(X)$, the dg category of strictly perfect complexes on X. We review the relevant definitions from [Sch17] now.

Definition 3.28. A *pointed dg category with duality* is a triple $(\mathcal{A}, \vee, \mathrm{can})$ where $\mathcal{A}$ is a pointed dg category, $\vee : \mathcal{A}^{op} \to \mathcal{A}$ is a dg functor called the duality functor, and $\mathrm{can} : 1 \to \vee \circ \vee^{op}$ is a natural transformation of dg functors called the double dual identification such that $\mathrm{can}^{\vee}_A \circ \mathrm{can}_{A^\vee} = 1_{A^\vee}$ for all objects A in $\mathcal{A}$.

Remark 3.29. A dg category with duality has an underlying exact category with duality $(Z^0\mathcal{A}^{\mathrm{ptr}}, \vee, \mathrm{can})$, where $Z^0\mathcal{A}^{\mathrm{ptr}}$ has the same objects as $\mathcal{A}^{\mathrm{ptr}}$ but the morphism sets are the zero cycles in the morphism complexes of $\mathcal{A}^{\mathrm{ptr}}$. Here $\mathcal{A}^{\mathrm{ptr}}$ is the pretriangulated hull of $\mathcal{A}$ (see [Sch17] definition 1.7).

DEFINITION 3.30. A *dg category with weak equivalences* is a pair $(\mathcal{A}, w)$ where $\mathcal{A}$ is a pointed dg category and $w \subseteq Z^0\mathcal{A}^{\text{ptr}}$ is a set of morphisms which saturated in $\mathcal{A}$. A map f in w is called a weak equivalence.

DEFINITION 3.31. Given a pointed dg category with duality $(\mathcal{A}, \vee, \text{can})$, a Hermitian object in $\mathcal{A}$ is a pair (X, ϕ) where $\phi : X \to X^\vee$ is a morphism in $\mathcal{A}$ satisfying $\phi^\vee \text{can}_X = \phi$.

DEFINITION 3.32. A *dg category with weak equivalences and duality* is a quadruple $\mathscr{A} = (\mathcal{A}, w, \vee, \text{can})$ where $(\mathcal{A}, w)$ is a dg category with weak equivalences and $(\mathcal{A}, \vee, \text{can})$ is a dg category with duality such that the dg subcategory $\mathcal{A}^w \subset \mathcal{A}$ of w-acyclic objects is closed under the duality functor $\vee$ and $\text{can}_A : A \to A^{\vee\vee}$ is a weak equivalence for all objects A of $\mathcal{A}$.

DEFINITION 3.33. Let $\mathscr{A} = (\mathcal{A}, w, \vee, \text{can})$ be a dg category with weak equivalences and dualiy. A symmetric space in $\mathscr{A}^{\text{ptr}}$ is a Hermitian object A whose dual map $\phi : A \to A^\vee$ is a weak equivalence in $\mathscr{A}^{\text{ptr}}$. The Grothendieck-Witt group $GW_0(\mathscr{A})$ of $\mathscr{A}$ is the abelian group generated by symmetric spaces $[X, \phi]$ in the underlying category with weak equivalences and duality $(Z^0\mathcal{A}^{ptr}, w, \vee, \text{can})$, subject to the following relations:

1. $[X, \phi] + [Y, \psi] = [X \oplus Y, \phi \oplus \psi]$
2. if $g : X \to Y$ is a weak equivalence, then $[Y, \psi] = [X, g^\vee \psi g]$, and
3. if $(E_\bullet, \phi_\bullet)$ is a symmetric space in the category of exact sequences in $Z^0\mathcal{A}^{\text{ptr}}$, that is, a map

$$\begin{array}{ccccccc} E_\bullet : & & E_{-1} & \xrightarrow{i} & E_0 & \xrightarrow{p} & E_1 \\ \sim\downarrow \phi_\bullet & & \sim\downarrow \phi_{-1} & & \sim\downarrow \phi_0 & & \sim\downarrow \phi_1 \\ E_\bullet^\vee : & & E_1^\vee & \xrightarrow{p^\vee} & E_0^\vee & \xrightarrow{i^\vee} & E_1^\vee \end{array}$$

   of exact sequences with $(\phi_{-1}, \phi_0, \phi_1) = (\phi_1^\vee \text{can}, \phi_0^\vee \text{can}, \phi_{-1}^\vee \text{can})$ a weak equivalence, then

$$[E_0, \phi_0] = \left[E_{-1} \oplus E_1, \begin{pmatrix} 0 & \phi_1 \\ \phi_{-1} & 0 \end{pmatrix} \right].$$

DEFINITION 3.34. Given a dg-category with weak equivalences and duality $\mathscr{A} = (\mathcal{A}, w, \vee, \text{can})$, Schlichting defines [Sch17, Section 4.1] a functorial monoidal symmetric spectrum $GW(\mathscr{A})$ using a modified version of the Waldhausen $\mathcal{S}_\bullet$ construction. For the sake of brevity, we don't reproduce his construction here.

Noting in general that GW doesn't sit in a localization sequence, Schlichting defines a localizing variant, $\mathbb{G}W$ in [Sch17, Section 8.1] as a bispectrum. The reason Schlichting defines $\mathbb{G}W$ as an object in bispectra rather than spectra is to get a monoidal structure on $\mathbb{G}W$. We provide an alternative approach to producing $\mathbb{G}W$ via periodization in section 5.

Definition 3.35. Let X be a Noetherian scheme of finite Krull dimension with an ample family of line bundles, and let $\sigma : X \to X$ be an involution on X. Let sPerf(X) denote the category of strictly perfect complexes on X with the weak equivalences being the quasi-isomorphisms. Define a family of dualities on sPerf(X) indexed by $i \in \mathbb{N}$ by

$$*^i : E \mapsto \mathbf{Hom}_{\mathrm{sPerf}(X)}(\sigma_* E, \mathcal{O}_X[i]).$$

Note that because σ is an involution, $\sigma_* E$ is a strictly perfect complex. Define the canonical isomorphim can as in Definition 3.10 as the adjoint of the evaluation map

$$ev : E \otimes \sigma_* \mathbf{Hom}_{\mathrm{sPerf}(X)}(\sigma_* E, \mathcal{O}_X[i]) \to \mathcal{O}_X[i].$$

Combining all this data we get a collection of dg categories with weak equivalences and duality

$$(\mathrm{sPerf}(X), \mathrm{q.\ iso}, *^i, \mathrm{can}).$$

The ith shifted Grothendieck-Witt spectrum of (X, σ) is defined as

$$GW^{[i]}(X, \sigma) = GW(\mathrm{sPerf}(X), \mathrm{q.\ iso}, *^i, \mathrm{can}).$$

If Z is an invariant closed subset of X, then the duality on sPerf(X) restricts to a duality on the subcategory of complexes supported on Z, $\mathrm{sPerf}_Z(X)$. We define

$$GW^{[i]}(X \text{ on } Z) = GW(\mathrm{sPerf}_Z(X), \mathrm{q.\ iso}, *^i, \mathrm{can}).$$

4 Representability of Automorphism Groups of Hermitian Forms

Representability of K-theory in the stable motivic homotopy category allows one to check that K-theory pulls back nicely. In particular, given $f : X \to S$ a map of schemes over S, one can use ind-representability of KGL to show that $f^*(\mathrm{KGL}_S) = \mathrm{KGL}_X$. Together with the formalism of six operations in motivic homotopy theory, one obtains rather formally cdh descent for algebraic K-theory, see [Cis13].

The goal of this section is to define a sheaf on $\mathbf{Sm}^{C_2}_{S,qp}$, denoted $\mathbb{R}\mathrm{Gr}$, which represents Hermitian K-theory in the motivic homotopy category $\mathbf{H}^{C_2}_S$. We first check that over a regular base S with 2 invertible (e.g. $\mathbb{Z}[\frac{1}{2}]$), Hermitian K-theory is representable in the category of C_2-schemes over S, $\mathbf{Sm}^{C_2}_{S,qp}$. To extend this result to non-regular bases S, we utilize the Morel-Voevodsky approach to classifying spaces and obtain representability of homotopy Hermitian K-theory in the motivic homotopy category $\mathbf{H}^{C_2}_S$.

By analogy with the K theory case, the equivariant scheme representing Hermitian K-theory on $\mathbf{Sm}^{C_2}_{S,qp}$ will be a colimit of schemes which parametrize non-degenerate Hermitian sub-bundles of a given Hermitian vector bundle V.

The new results here are mostly the definitions, as the proofs in this section are either minor modifications or identical to the proofs in [SST14]. The main difference which might cause concern is that stalks in the isovariant étale topology are now semi-local (rather than local) rings.

We combine the techniques of [SST14] with a Morel-Voevodsky style argument to compare $\mathbb{R}\mathrm{Gr}_{2d}(\mathbb{H}^\infty)$ to the isovariant étale classifying space $B_{isoEt}O(\mathbb{H}^d)$ of the group of automorphisms of hyperbolic d-space. The key to the comparison is that locally in the isovariant étale topology, Hermitian vector bundles are determined by rank. This will utilize some of the analysis of Hermitian forms over semi-local rings from section 3.3. Note that this is a key difference from the K-theory case where one must pass only to local (rather than strictly henselian local) rings in order for K-theory to be determined by rank.

A straightforward generalization of the techniques in [SST14] allows one to compare $\mathrm{colim}_n B_{isoEt}O(\mathbb{H}^n)(\Delta R)$ to the Grothendieck-Witt space defined in section 3.4 by viewing them both as group completions and comparing their homology. This approach is inspired by the Karoubi-Villamayor definition of higher algebraic K-theory. We don't carry out this comparison here as it is unnecessary for proving cdh descent.

4.1 THE DEFINITION OF THE HERMITIAN GRASSMANNIAN $\mathbb{R}\mathrm{Gr}$

The definition here describes the sections of the underlying scheme of $\mathbb{R}\mathrm{Gr}$ over a scheme $X \to S$. We advise the hurried reader to skip to section 4.2.

LEMMA 4.1. *Let $\mathcal{F}$ be a presheaf on $\mathbf{Sm}_{S,qp}$ and let $a : \mathcal{F} \implies \mathcal{F}$ be a natural transformation such that $a \circ a = id_{\mathcal{F}}$. Then there's an associated presheaf on $\mathbf{Sm}^{C_2}_{S,qp}$ defined by the formula $(X, \sigma : X \to X) \mapsto \mathcal{F}(X)^{C_2}$ where the action of C_2 on $\mathcal{F}(X)$ is defined by $f \mapsto a_X \mathcal{F}(\sigma)(f)$.*

Proof. Note that this is indeed a C_2-action, since $a_X\mathcal{F}(\sigma)(a_X\mathcal{F}(\sigma)(f)) = \mathcal{F}(\sigma)a_X(a_X\mathcal{F}(\sigma)(f)) = \mathcal{F}(\sigma)(\mathcal{F}(\sigma)(f)) = f$ using naturality. □

Fix a (possibly degenerate) Hermitian vector bundle (V, ϕ) over a base scheme S with 2 invertible and with trivial involution. The canonical example of such a base scheme is $S = \mathrm{Spec}\,\mathbb{Z}[\frac{1}{2}]$.

We'll define a presheaf $\mathbb{R}\mathrm{Gr} : (\mathbf{Sm}^{C_2}_{S,qp})^{op} \to \mathbf{Set}$ by first defining a presheaf on $\mathbf{Sm}_{S,qp}$, showing that it's representable, equipping with an action, then taking the corresponding representable functor on $\mathbf{Sm}^{C_2}_{S,qp}$. We can then extend to an arbitrary equivariant base T with 2 invertible by pulling back along the unique map $T \to \mathbb{Z}[\frac{1}{2}]$.

- On objects, $\mathbb{R}\mathrm{Gr}(V)(f : X \to S)$ for an S-scheme $f : X \to S$ is a split surjection (p, s)

$$f^*V \underset{p}{\overset{s}{\rightleftarrows}} W\ ,$$

where W is locally free.

Here by an isomorphism of split surjections we mean a diagram

$$\begin{array}{ccc} f^*V & \underset{p}{\overset{s}{\rightleftarrows}} & W \\ \| & & \downarrow \phi \\ f^*V & \underset{p'}{\overset{s'}{\rightleftarrows}} & W' \end{array}$$

such that ϕ is an isomorphism satisfying $\phi \circ p = p'$ and $s = s' \circ \phi$.

- Given a morphism

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ & {\scriptstyle h} \searrow \quad \swarrow {\scriptstyle f} & \\ & S & \end{array}$$

over S, define

$$\mathbb{R}\mathrm{Gr}_V(g)(\ f^*V \underset{p}{\overset{s}{\rightleftarrows}} W\ ) = \ h^*V \xrightarrow{can} g^*f^*V \underset{g^*p}{\overset{g^*s}{\rightleftarrows}} g^*W\ .$$

There's a natural action of C_2 on $\mathbb{R}\mathrm{Gr}_V$ whose non-trivial natural transformation will be denoted η. Define η as follows:
Fix an object $X \in \mathbf{Sm}_{S,qp}$. Define

$$\eta_X(\ f^*V \underset{p}{\overset{s}{\rightleftarrows}} W\ ) = \ f^*V \underset{q}{\overset{t}{\rightleftarrows}} (\ker p)^{\perp}\ .$$

We'll define the maps q and t now. Let t' denote the canonical map $\ker p \to f^*V$, let q' be the map

$$q' :\ f^*V \xrightarrow{\mathrm{id}-(s\circ p)} f^*V \xrightarrow{\mathrm{im}} \mathrm{im}(t') \xrightarrow{(t')^{-1}} \ker p$$

where we've used the identification $\mathrm{im}\, t' = \mathrm{im}(\mathrm{id} - (s \circ p))$.
Recall that

$$W^{\perp} = \ker(f^*V \xrightarrow{f^*\phi} f^*(V^*) \xrightarrow{can} (f^*V)^* \xrightarrow{s^*} W^*)$$

and similarly for $(\ker p)^{\perp}$.
Leaving out the *can* map for convenience, we get split exact sequences

$$0 \longrightarrow W^{\perp} \longrightarrow f^*V \underset{s^*}{\overset{p^*}{\rightleftarrows}} W^* \longrightarrow 0$$

and

$$0 \longrightarrow (\ker p)^{\perp} \longrightarrow f^*V \underset{(t')^*}{\overset{(q')^*}{\rightleftarrows}} (\ker p)^* \longrightarrow 0\,.$$

By the splitting lemma for abelian categories, $f^*V \cong W^{\perp} \oplus W^*$, and there's a (canonical) split surjection $f^*V \twoheadrightarrow W^{\perp}$ with $W^{\perp}$ locally free. Similarly we obtain a canonical surjection $q : f^*V \twoheadrightarrow (\ker p)^{\perp}$ split by a map t.
Given an isomorphism

$$\begin{array}{ccc} f^*V & \underset{p}{\overset{s}{\rightleftarrows}} & W \\ \| & & \downarrow \psi \\ f^*V & \underset{p'}{\overset{s'}{\rightleftarrows}} & W' \end{array}$$

we get an isomorphism of (split) diagrams

$$\begin{array}{ccccc} f^*V & \xrightarrow{f^*\phi} & (f^*V)^* & \xrightarrow{s^*} & W^* \\ \| & & \| & & \downarrow (\psi^{-1})^* \\ f^*V & \xrightarrow{f^*\phi} & (f^*V)^* & \xrightarrow[(s')^*]{} & (W')^* \end{array}$$

and hence an isomorphism of split surjections

$$\begin{array}{ccc} f^*V & \underset{q}{\overset{t}{\rightleftarrows}} & W^{\perp} \\ \| & & \downarrow \delta \\ f^*V & \underset{q'}{\overset{t'}{\rightleftarrows}} & (W')^{\perp} \end{array} \quad ,$$

so that η_X is a well-defined map of sets. Given a map of schemes $g : Y \to X$, such that $f \circ g = h$ and an element

$$f^*V \underset{p}{\overset{s}{\rightleftarrows}} W$$

in $\mathbb{R}\mathrm{Gr}_V(X)$,

$$\mathbb{R}\mathrm{Gr}(g)\circ\eta_X\big(\, f^*V \underset{p}{\overset{s}{\rightleftarrows}} W\,\big) = \mathbb{R}\mathrm{Gr}(g)\big(\, f^*V \underset{q}{\overset{t}{\rightleftarrows}} (\ker p)^{\perp}\,\big)$$

$$= h^*V \xrightarrow{can} g^*f^*V \underset{g^*q}{\overset{g^*t}{\rightleftarrows}} g^*((\ker(p))^{\perp})$$

while

$$\eta_Y\circ\mathbb{R}\mathrm{Gr}(g)\big(\, f^*V \underset{p}{\overset{s}{\rightleftarrows}} W\,\big)\big) = h^*V \xrightarrow{can} g^*f^*V \underset{q'}{\overset{t'}{\rightleftarrows}} (g^*(\ker(p)))^{\perp}\,.$$

By Lemma 3.16, there's a canonical isomorphism $g^*((\ker(p)^{\perp})) \to (g^*(\ker(p)))^{\perp}$, and under this isomorphism q' and t' correspond to g^*q, and g^*t, respectively. This concludes the check of naturality.

Now by Lemma 4.1, there's a presheaf $\mathbb{R}\mathrm{Gr} : \mathbf{Sm}^{C_2}_{S,qp} \to \mathbf{Set}$. To determine its values on a C_2-scheme (X,σ), we note that a fixed point of the action of Lemma 4.1 is determined by an isomorphism of split surjections

$$\begin{array}{ccc} f^*V & \underset{q}{\overset{t}{\rightleftarrows}} & \sigma^*(\ker(p)^{\perp}) \\ \Big\| & & \big\downarrow \psi \\ f^*V & \underset{p}{\overset{s}{\rightleftarrows}} & \ker(p) \end{array}$$

Note that because σ is an involution, for any $\mathcal{O}_X$-module M, there's a canonical isomorphism of $\mathcal{O}_X$-modules $\sigma_*M \cong \sigma^*M$. Thus there's a natural isomorphism

$$\mathrm{Hom}_{mod-\mathcal{O}_X}(\sigma_*f^*V,-) \cong \mathrm{Hom}_{mod-\mathcal{O}_X}(\sigma^*f^*V,-) \cong \mathrm{Hom}_{mod-\mathcal{O}_X}(f^*V,-).$$

It follows that any Hermitian form

$$\phi : f^*V \to \mathrm{Hom}_{mod-\mathcal{O}_X}(f^*V,\mathcal{O}_X)$$

can be promoted to a Hermitian form

$$\widetilde{\phi} : f^*V \to \mathrm{Hom}_{mod-\mathcal{O}_X}(\sigma_*f^*V,\mathcal{O}_X)$$

compatible with an involution σ on X.

Let $(M,\phi|_M)$ be a Hermitian sub-bundle of f^*V over the scheme X with trivial involution. We claim that $\sigma^*(M^{\perp})$ is the orthogonal complement of M viewed as a Hermitian sub-bundle of f^*V with the promoted form $\widetilde{\phi}$. Said differently, we claim that

$$\sigma^*(\ker(f^*V \xrightarrow{\phi|_M} \mathrm{Hom}(M,\mathcal{O}_X))) \cong \ker(f^*V \xrightarrow{\widetilde{\phi}|_M} \mathrm{Hom}(\sigma_*M,\mathcal{O}_X)).$$

But using the natural isomorphism between σ^* and σ_*, together with the natural isomorphisms

$$\sigma^* \operatorname{Hom}(M, \mathcal{O}_x) \cong \operatorname{Hom}(M, \mathcal{O}_X)$$

and $\sigma^* f^* V \cong f^* V$, this becomes a question of whether σ^* is left exact. In general it isn't, but because σ is an involution, σ^* is naturally isomorphic to σ_* which is left exact. The claim follows.

4.2 Representability of $\mathbb{R}$Gr

Fix a Hermitian vector bundle (V, ϕ) over S where $\dim(V) = n$ and S is a scheme with trivial involution. Then the underlying scheme of $\mathbb{R}\mathrm{Gr}(V)$ is the pullback

$$\begin{array}{ccc} \mathbb{R}\mathrm{Gr}(V) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V,V) \times \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V,V) \\ \downarrow & & \downarrow {\scriptstyle \circ, id} \\ \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V,V) & \xrightarrow{\Delta} & \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V,V) \times \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V,V) \end{array}$$

where the right vertical map sends $p \mapsto (p \circ p, p)$. In other words, the underlying scheme is the scheme of idempotent endomorphisms of V. The action corresponds to the map $p \mapsto p^\dagger$, where $p^\dagger$ is the adjoint of p with respect to the form ϕ.

Note that using this description, an equivariant map $(X, \sigma) \to \mathbb{R}\mathrm{Gr}(V)$ corresponds to an idempotent $p : V_X \to V_X$ such that $\phi^{-1}(\gamma^{-1}(\sigma^* p)\gamma)^* \phi = p$, where we're being cavalier and using $*$ to denote both dual (on the outside) and pullback (by σ). Here γ is the canonical isomorphism $V_X \xrightarrow{\gamma} \sigma^* V_X$; if the structure map of X is $f : X \to S$, then γ arises from the equality $\sigma \circ f = f$.

Note that the form on $V_{(X,\sigma)}$ is by definition the composite

$$\widetilde{\phi} : V_X \xrightarrow{\phi} V_X^* \xrightarrow{(\gamma^*)^{-1}} \sigma^* V_X^* \xrightarrow{(\eta^*)^{-1}} \sigma_* V_X^*,$$

and the adjoint of p is given by $\widetilde{\phi}^{-1}(\sigma_* p)^* \widetilde{\phi}$. Expanding, this is

$$\phi^{-1}(\gamma^*)(\eta^*)(\eta^*)^{-1}(\sigma^* p)^*(\eta^*)(\eta^*)^{-1}(\gamma^*)^{-1}\phi = \phi^{-1}(\gamma^{-1}(\sigma^* p)\gamma)^* \phi,$$

and so we recover the condition that $p^\dagger = p$, which corresponds to the fact that $V_X = \ker p \perp \operatorname{im} p$, and hence the restriction of the form on V_X to $\operatorname{im} p$ (and $\ker p$) is non-degenerate.

To summarize, the underlying scheme of $\mathbb{R}\mathrm{Gr}(V)$ represents idempotents, and equivariant maps pick out those idempotents which correspond to orthogonal projections.

DEFINITION 4.2. Now fix a dimension d and a non-degenerate Hermitian vector bundle (V, ϕ) over S. Define $\mathbb{R}\mathrm{Gr}_d(V)$ to be the closed subscheme of $\mathbb{R}\mathrm{Gr}(V)$ cut out by $\mathrm{rk}(p) = d$, where rk is the rank map. In other words, $\mathbb{R}\mathrm{Gr}_d(V)$ is the pullback

$$\begin{array}{ccc} \mathbb{R}\mathrm{Gr}_d(V) & \longrightarrow & \mathbb{R}\mathrm{Gr}(V) \\ \downarrow & & \downarrow \mathrm{rk} \\ \{d\} & \longrightarrow & \mathbb{Z} \end{array}$$

The requirement that V be non-degenerate is necessary so that the action on $\mathbb{R}\mathrm{Gr}(V)$ sends rank d subspaces to rank d subspaces and hence induces an action on $\mathbb{R}\mathrm{Gr}_d(V)$.

Remark 4.3. Denote by $g : \mathbb{R}\mathrm{Gr}_d(V) \to S$ the structure map of $\mathbb{R}\mathrm{Gr}_d(V)$. Because $\mathbb{R}\mathrm{Gr}_d(V)$ is representable by a C_2-scheme, there's an idempotent $g^*(V) \to g^*(V)$ corresponding to the identity map $id : \mathbb{R}\mathrm{Gr}_d(V) \to \mathbb{R}\mathrm{Gr}_d(V)$. This idempotent is simply the idempotent which, over a point of $\mathbb{R}\mathrm{Gr}_d(V)$ represented by an idempotent $p : V \to V$, restricts to p. There's an action σ on $\mathbb{R}\mathrm{Gr}_d(V) \times_S V$ induced by the action on $\mathbb{R}\mathrm{Gr}_d(V)$, and using the fact that $\sigma p \sigma = p^\dagger$ one can see that this idempotent is non-degenerate with respect to the promoted Hermitian form on $g^*(V)$ compatible with the involution on $\mathbb{R}\mathrm{Gr}_d(V)$.

Remark 4.4. Since we've shown that $\mathbb{R}\mathrm{Gr}(V)$ represents non-degenerate Hermitian subbundles of V, at this point we'll move away from explicitly referring to split surjections and just represent the sections of $\mathbb{R}\mathrm{Gr}(V)$ by non-degenerate subbundles.

Definition 4.5. Let $\mathbb{H}_S$ denote the hyperbolic space 3.13 over the base scheme S. Let $\mathbb{H}^\infty = \mathrm{colim}_n \mathbb{H}^n_S$. Similarly given a non-degenerate Hermitian vector bundle V, let $V \perp \mathbb{H}^\infty = \mathrm{colim}_n V \perp \mathbb{H}^n$. For $V \subset \mathbb{H}^\infty$ a constant rank non-degenerate subbundle, let $|V|$ denote the rank of V. Order such subbundles of $\mathbb{H}^\infty$ by inclusion, and denote the resulting poset by P. Given an inclusion $V \hookrightarrow V'$ of non-degenerate subbundles, denote by $V' - V$ the complement of V in V'. Let $\mathcal{H} : P \to \mathrm{Fun}(\mathbf{Sm}^{C_2,op}_{S,qp}, \mathrm{Set})$ be the functor which on objects sends a subbundle V to $\mathbb{R}\mathrm{Gr}_{|V|}(V \perp \mathbb{H}^\infty) = \mathrm{colim}_n \mathbb{R}\mathrm{Gr}_{|V|}(V \perp \mathbb{H}^n)$. Given an inclusion $V \hookrightarrow V'$, the induced map $\mathbb{R}\mathrm{Gr}_{|V|}(V \perp \mathbb{H}^\infty) \to \mathbb{R}\mathrm{Gr}_{|V'|}(V' \perp \mathbb{H}^\infty)$ is given by $E \mapsto E \perp (V' - V)$. Note that because V is non-degenerate, $V \perp (V' - V) = V'$. Define

$$\mathbb{R}\mathrm{Gr}_\bullet = \mathrm{colim}\, \mathcal{H}. \tag{1}$$

4.3 The Étale Classifying Space

The content of this section is a straightforward generalization of the work of [SST14] to the C_2-equivariant setting. Fix a scheme S with 2 invertible, and let (V, ϕ) be a (possibly degenerate) Hermitian vector bundle over S. For a

C_2-scheme $f : X \to S$ over S, let

$$\mathcal{S}(V,\phi)(X)$$

be the category of non-degenerate Hermitian sub-bundles of f^*V. A morphism in this category from E_0 to E_1 is an isometry not necessarily compatible with the embeddings $E_0, E_1 \subseteq f^*V$. Using pullbacks of quasi-coherent modules, we turn $\mathcal{S}$ into a presheaf of categories on $\mathbf{Sm}^{C_2}_{S,qp}$. For integer $d \geq 0$, define

$$\mathcal{S}_d(V,\phi) \subset \mathcal{S}(V,\phi)$$

to be the presheaf which on a C_2-scheme $f : X \to S$ assigns the full subcategory of non-degenerate Hermitian sub-bundles of $(f^*V, f^*\phi)$ which have constant rank d. The associated presheaf of objects is $\mathbb{R}\mathrm{Gr}_d(V,\phi)$.
Note that the object $V = (V,0) \in \mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty)$ has automorphism group $O(V)$. Thus we get an inclusion $O(V,\phi) \to \mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty)$, where $O(V)$ is the isometry group considered as a category on one object. After isovariant étale sheafification, this inclusion becomes an equivalence; this follows from Corollary 3.27 that on the points in the isovariant étale topology, Hermitian vector bundles are determined by rank.
Upon applying the nerve, we get maps of simplicial presheaves $BO(V) \to B\mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty)$ which is a weak equivalence in the isovariant étale topology. Abusing notation, let $B_{isoEt}O(V)$ denote a fibrant replacement of $B\mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty)$ in the isovariant étale topology so that we get a sequence of maps

$$BO(V) \to B\mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty) \to B_{isoEt}O(V).$$

which are weak equivalences in the isovariant étale topology.

LEMMA 4.6. *Let (V,ϕ) be a non-degenerate Hermitian vector bundle over a scheme S with trivial involution and $\frac{1}{2} \in \Gamma(S,\mathcal{O}_S)$. Then for any affine C_2-scheme* $\operatorname{Spec} R$ *over S, the map*

$$B\mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty)(R) \to B_{isoEt}O(V)(R)$$

is a weak equivalence of simplicial sets. In particular, the map

$$B\mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty) \to B_{isoEt}O(V)$$

is a weak equivalence in the equivariant Nisnevich topology, and hence an equivalence after C_2 motivic localization.

Proof. Each Hermitian vector bundle $W \in S_{|V|}(V \perp \mathbb{H}^\infty)(R)$ gives rise to an $O(V)$-torsor via $W \mapsto Isom(V,W)$. Note that this is an $O(V)$-torsor because locally in the isovariant étale topology, $W \cong V$, so that locally $Isom(V,W) \cong Isom(V,V) \cong O(V)$. Because Hermitian vector bundles are isovariant étale locally determined by rank, the same proof as the ordinary vector bundle case shows that the category of $\mathcal{O}(V)$ torsors is equivalent to the category of Hermitian vector bundles. Because over an affine scheme, every Hermitian vector bundle is a summand of a hyperbolic module, it follows that $S_{|V|}(V \perp \mathbb{H}^\infty)(R)$ is equivalent to the category of isovariant étale $\mathcal{O}(V)$ torsors.

Let $\mathcal{F} : \mathbf{Sm}^{C_2}_{S,qp} \to Gpd$ be the sheaf which assigns to $f : X \to S$ the groupoid of $O(f^*V)$-torsors. The construction $W \mapsto Isom(f^*V, W)$ described above defines a functor $S_{|V|}(V \perp \mathbb{H}^\infty) \to \mathcal{F}$ which is an equivalence when evaluated at affine C_2-schemes. It follows that there's a sequence

$$BS_{|V|}(V \perp \mathbb{H}^\infty) \to B\mathcal{F} \to B_{isoEt}O(V)$$

where the first map is a weak equivalence of simplicial sets when evaluated at affine C_2-schemes, and by [Jar01] Theorem 6, the second map is a weak equivalence of simplicial sets when evaluated at any C_2-scheme. □

DEFINITION 4.7. Following [SST14], let

$$\mathcal{S}_\bullet = \operatorname*{colim}_{V \subset \mathbb{H}^\infty_S} \mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty)$$

where similarly to the definition of $\mathbb{R}\mathrm{Gr}$, for $V \subset V'$ the functor

$$\mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty) \to \mathcal{S}_{|V'|}(V' \perp \mathbb{H}^\infty)$$

is defined on objects by $E \mapsto E \perp V' - V$ and on morphisms by $f \mapsto f \perp 1_{V'-V}$.

DEFINITION 4.8. Define the infinite orthogonal group

$$O(\mathbb{H}^\infty_S) = \operatorname*{colim}_{W \subseteq \mathbb{H}^\infty_S} O(W)$$

where the colimit is over non-degenerate subbundles of $\mathbb{H}^\infty$. If V is a Hermitian vector bundle, define

$$O(V \perp \mathbb{H}^\infty_S) = \operatorname*{colim}_{W \subseteq V \perp \mathbb{H}^\infty_S} O(W)$$

where W is a non-degenerate subbundle of $V \perp \mathbb{H}^\infty$.

DEFINITION 4.9. Let R be a commutative ring. Define ΔR to be the simplicial ring with involution $[n] \mapsto R[x_0, \ldots, x_n]/(\sum x_i - 1)$, where the involution is inherited from the involution on R.

LEMMA 4.10. *([SST14]) Let V be a non-degenerate Hermitian vector bundle over a commutative ring with involution (R, σ) such that $\frac{1}{2} \in R$. Then the inclusion $\mathbb{H}^\infty \subset V \perp \mathbb{H}^\infty$ induces a homotopy equivalence of simplicial groups*

$$O(\mathbb{H}^\infty)(\Delta R) \to O(V \perp \mathbb{H}^\infty)(\Delta R) \qquad A \mapsto 1_V \perp A.$$

Proof. First, assume that $V = \mathbb{H}$. Consider the map $j : O(\mathbb{H}^n) \to O(\mathbb{H}^{2n+2})$ sending A to $1_{\mathbb{H}} \perp A \perp 1_{\mathbb{H}^{n+1}}$. We claim that this is naïvely $\mathbb{A}^1$ homotopic to the inclusion $i : O(\mathbb{H}^n) \to O(\mathbb{H}^{2n+2})$, $i(A) = A \perp 1_{\mathbb{H}^{n+2}}$ which defines the colimit $O(\mathbb{H}^\infty)$. Let $g = \begin{pmatrix} 0 & I_{2n} & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & I_{2n+2} \end{pmatrix}$ where I_n denotes an $n \times n$ identity matrix. Then $i = gjg^{-1} = gjg^t$. Because g corresponds to an even permutation matrix, it can be written as a product of elementary matrices, each of which is naïvely $\mathbb{A}^1$ homotopic to the identity. It follow that g is naïvely $\mathbb{A}^1$ homotopic to the identity, and hence the induced maps $i, j : O(\mathbb{H}^n)(\Delta R) \to O(\mathbb{H}^{2n+2})(\Delta R)$ are simplicially homotopic via a base-point preserving homotopy. It follows that i, j induce the same map on homotopy groups, so that $j_* = i_* : \pi_k O(\mathbb{H}^\infty)(\Delta R) = \mathrm{colim}_n \pi_k O(\mathbb{H}^n)(\Delta R) \to \pi_k O(\mathbb{H}^\infty)(\Delta R)$ is the colimit of a map corresponding to a cofinal inclusion of diagrams, and hence is an isomorphism on all simplicial homotopy groups. Because simplicial groups are Kan complexes, it follows that j is a homotopy equivalence, and the claim is proved when $V = \mathbb{H}$.

Now a trivial induction shows that the lemma holds when $V = \mathbb{H}^n$. In general, choose an embedding $V \subseteq \mathbb{H}^n$, and consider the sequence of maps

$$O(\mathbb{H}^\infty)(\Delta R) \to O(V \perp \mathbb{H}^\infty)(\Delta R) \to O(\mathbb{H}^n \perp \mathbb{H}^\infty)(\Delta R)$$
$$\to O(\mathbb{H}^n \perp V \perp \mathbb{H}^\infty)(\Delta R).$$

The composites $O(\mathbb{H}^\infty)(\Delta R) \to O(\mathbb{H}^n \perp \mathbb{H}^\infty)$ and $O(V \perp \mathbb{H}^\infty)(\Delta R) \to O(\mathbb{H}^n \perp V \perp \mathbb{H}^\infty)(\Delta R)$ are weak equivalences, so by 2 out of 6 the first map is a weak equivalence. Because it is a map of simplicial groups it is a homotopy equivalence. ∎

For non-degenerate Hermitian vector bundles $(V, \phi_V), (W, \phi_W)$ and a commutative R-algebra with involution (A, σ), let

$$\mathrm{St}(V, W)(A)$$

be the set of A-linear isometric embeddings $f : V_A \to W_A$. Given a map $A \to B$ of commutative R-algebras with involution, tensoring over R with B makes $\mathrm{St}(V, W)(-)$ a presheaf on commutative R-algebras with involution. There's a transitive left action of $O(V \perp \mathbb{H}^\infty)$ on $\mathrm{St}(V, V \perp \mathbb{H}^\infty)$ given by $(f, g) \mapsto f \circ g$. Let i_V denote the isometric embedding $V \hookrightarrow V \perp \mathbb{H}^\infty : v \mapsto (v, 0)$. The stabilizer of i_V is the subgroup $O(\mathbb{H}^\infty) \subset O(V \perp \mathbb{H}^\infty)$ where the inclusion map is $A \mapsto 1_V \perp A$.

It follows that there's an isomorphism of presheaves of sets

$$O(\mathbb{H}^\infty)\backslash O(V \perp \mathbb{H}^\infty) \cong \mathrm{St}(V, V \perp \mathbb{H}^\infty) \qquad f \mapsto f \circ i_V.$$

Now Lemma 4.10 shows that the map $O(\mathbb{H}^\infty)(\Delta R) \to O(V \perp \mathbb{H}^\infty)(\Delta R)$ is an equivariant map which is a non-equivariant homotopy equivalence. The simplicial group $O(\mathbb{H}^\infty)(\Delta R)$ acts freely on both the domain and codomain, so

that the quotients $O(\mathbb{H}^\infty)(\Delta R)\backslash O(V \perp \mathbb{H}^\infty)(\Delta R)$ and $O(\mathbb{H}^\infty)(\Delta R)\backslash O(\mathbb{H}^\infty)(\Delta R)$ are homotopy equivalent.
Together with the isomorphism of simplicial sets

$$O(\mathbb{H}^\infty)(\Delta R)\backslash O(V \perp \mathbb{H}^\infty)(\Delta R) \cong \mathrm{St}(V, V \perp \mathbb{H}^\infty)(\Delta R)$$

it follows that $\mathrm{St}(V, V \perp \mathbb{H}^\infty)(\Delta R)$ is contractible for a commutative ring (R, σ) with involution and $\frac{1}{2} \in R$. Morever, this simplicial set is fibrant because G/H is fibrant for a simplicial group G and subgroup H. We have thus proved:

Lemma 4.11. *Let R be a commutative ring with $\frac{1}{2} \in R$. Then*

$$\mathrm{St}(V, V \perp \mathbb{H}^\infty)(\Delta R)$$

is a contractible Kan set.

Now we move to identifying $\mathbb{R}\mathrm{Gr}_V$ as a quotient of a contractible space by a free group action. Let V be a non-degenerate Hermitian vector bundle over a ring R with involution. Then the group $O(V)$ acts on the right on $\mathrm{St}(V, U)$ by precomposition. The map $\mathrm{St}(V, U) \to \mathbb{R}\mathrm{Gr}_V(U) : f \mapsto \mathrm{im}(f)$ factors through the quotient $\mathrm{St}(V, U)/O(V)$. The map is clearly surjective, and hence furnishes an isomorphism of sets

$$\mathrm{St}(V, U)/O(V) \cong \mathbb{R}\mathrm{Gr}_V(U) \qquad f \mapsto \mathrm{im}(f).$$

In particular, there's an isomorphism of presheaves of sets

$$\mathrm{St}(V, V \perp \mathbb{H}^\infty)/O(V) \cong \mathbb{R}\mathrm{Gr}_V(V \perp \mathbb{H}^\infty) \tag{2}$$

Now, let V be a non-degenerate Hermitian vector bundle over a ring with involution R and let U be a possibly degenerate Hermitian form over R. Define $\mathcal{E}_V(U)$ to be the category whose objects are R-linear maps $V \to U$ of Hermitian forms (aka isometric embeddings), and whose morphisms from two objects $a : V \to U$ and $b : V \to U$ are maps $c : \mathrm{im}(a) \to \mathrm{im}(b)$ making the diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & \mathrm{im}(a) \\ & \searrow^{b} & \downarrow{c} \\ & & \mathrm{im}(b) \end{array}$$

commute.
There's a natural right action of $O(V)$ on $\mathcal{E}_V(U)$ which on objects sends

$$\mathcal{E}_V(U) \times O(V) \to \mathcal{E}_V(U) : (a, g) \mapsto ag$$

and which on morphisms is the trivial action. Then clearly there's an isomorphism

$$\mathcal{E}_V(U)/O(V) \cong \mathcal{S}_V(U) \qquad a \mapsto \mathrm{im}(a).$$

LEMMA 4.12. *The category $\mathcal{E}_V(V \perp \mathbb{H}^\infty)$ is contractible.*

Proof. The category is nonempty and every object is initial. ☐

The map of simplicial sets

$$\mathrm{St}(V, V \perp \mathbb{H}^\infty)(\Delta R) \to \mathcal{E}_V(V \perp \mathbb{H}^\infty)(\Delta R)$$

is $O(V)(\Delta R)$ equivariant and a weak equivalence after forgetting the action. Furthermore, $O(V)(\Delta R)$ acts freely on both sides, so that the induced map on quotients

$$\mathbb{R}\mathrm{Gr}_V(V \perp \mathbb{H}^\infty)(\Delta R) \to \mathcal{S}_V(V \perp \mathbb{H}^\infty)(\Delta R) \tag{3}$$

is also a weak equivalence. As an aside, the inclusion $BO(V) \subset B\mathcal{S}_V(V \perp \mathbb{H}^\infty)$ is a weak equivalence since $\mathcal{S}_V(V \perp \mathbb{H}^\infty)$ is a connected groupoid.
We now show that there's a motivic equivalence $\mathbb{R}\mathrm{Gr}_\bullet \to \mathrm{colim}_n B_{isoEt}O(\mathbb{H}^n)$ over possibly non-regular Noetherian base rings.
Let $X \to S$ be an affine C_2-scheme over S, and let W be a non-degenerate Hermitian vector bundle over X. Given an isovariant étale $O(W)$ torsor $\pi : T \to X$, and an isovariant étale torsor U, let U_π denote the twisted sheaf $(U \times T)/O(W)$.
Our goal is to appy Lemma 2.1 from [Hoy16], which we restate below:

LEMMA 4.13. *(Hoyois) Let Γ be an isovariant étale sheaf of groups on $\mathbf{Sm}^{C_2}_{S,qp}$ acting on an isovariant étale sheaf U. Suppose that, for every $X \in \mathbf{Sm}^{C_2}_{S,qp}$ and every isovariant étale torsor $\pi : T \to X$ under Γ, $U_\pi \to X$ is a motivic equivalence on $\mathbf{Sm}^{C_2}_{S,qp}$. Then the map*

$$L_{isoEt}(U/\Gamma) \to B_{isoEt}\Gamma$$

induced by $U \to *$ is a motivic equivalence on $\mathbf{Sm}^{C_2}_{S,qp}$.

Given an $O(W)$-torsor $\pi : T \to X$, we want to check that $\mathrm{St}(W, \mathbb{H}^\infty)_\pi = (\mathrm{St}(W, \mathbb{H}^\infty) \times T)/O(W) \to X$ is a motivic equivalence on $\mathbf{Sm}^{C_2}_{S,qp}$. Leting $V = W_\pi$, this is equivalent to checking that $\mathrm{St}(V, \mathbb{H}^\infty_X)$ is motivically contractible over $\mathbf{Sm}^{C_2}_X$. To wit, because X is affine there's an embedding $V \hookrightarrow \mathbb{H}^m$, and we have $V \perp \mathbb{H}^\infty_X \cong \mathbb{H}^\infty_X$ since $\mathbb{H}^\infty_X = \mathrm{colim}_{W \subset \mathbb{H}^\infty_X} W$. It follows that $\mathrm{St}(V, \mathbb{H}^\infty_X) \cong \mathrm{St}(V, V \perp \mathbb{H}^\infty_X)$, and Lemma 4.11 (which didn't assume regularity of the base) shows that $\mathrm{St}(V, V \perp \mathbb{H}^\infty_X)$ is motivically contractible over $\mathbf{Sm}^{C_2}_X$.
It's a direct consequence of the above lemma that

$$L_{isoEt}(\mathrm{St}(W, W \perp \mathbb{H}^\infty)/O(W)) \to B_{isoEt}O(W)$$

is a motivic equivalence. However, we've already shown (2) that

$$\mathrm{St}(W, W \perp \mathbb{H}^\infty)/O(W) \cong \mathbb{R}\mathrm{Gr}_W(W \perp \mathbb{H}^\infty),$$

so that

$$L_{isoEt}\mathrm{St}(W, W \perp \mathbb{H}^\infty)/O(W) \cong L_{isoEt}\mathbb{R}\mathrm{Gr}_W(W \perp \mathbb{H}^\infty) \cong \mathbb{R}\mathrm{Gr}_{|W|}(W \perp \mathbb{H}^\infty)$$

which after taking colimits gives the desired result.

Theorem 4.14. *Let S be a Noetherian scheme of finite Krull dimension with $\frac{1}{2} \in S$. There are equivalences of motivic spaces on $\mathbf{Sm}^{C_2}_{S,qp}$*

$$\mathbb{Z} \times \mathbb{R}\mathrm{Gr}_\bullet \xrightarrow{\sim} \mathbb{Z} \times \operatorname*{colim}_n B_{isoEt} O(\mathbb{H}^n)$$

5 Periodicity in the Hermitian K-Theory of Rings with Involution

5.1 A Projective Bundle Formula for $\mathbb{P}^\sigma$

Let $\mathbb{P}^\sigma$ denote $\mathbb{P}^1$ with involution σ defined by $[x : y] \mapsto [y : x]$. When necessary we'll point it at the point $[1 : 1]$. Throughout this section, we'll fix the notation $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1}$.

Consider the square of $\mathcal{O}$-modules

$$\begin{array}{ccc} \mathcal{O}(-1) & \xrightarrow{\frac{T+S}{2}} & \mathcal{O} \\ \downarrow{\scriptstyle \frac{T-S}{2}} & & \downarrow{\scriptstyle \frac{T-S}{2}} \\ \mathbf{Hom}(\sigma_*\mathcal{O}, \mathcal{O}) & \xrightarrow{\frac{T+S}{2}} & \mathbf{Hom}(\sigma_*\mathcal{O}(-1), \mathcal{O}) \end{array} \tag{4}$$

where the map $\frac{T-S}{2} : \mathcal{O}(-1) \to \mathcal{O}$ is induced via the tensor-hom adjunction by the composition

$$\begin{aligned} \mathcal{O}(-1) \otimes \left\{\frac{T-S}{2}\right\} \otimes \sigma_*\mathcal{O} &\xrightarrow{id\otimes i \otimes id} \mathcal{O}(-1) \otimes \mathcal{O}(1) \otimes \sigma_*\mathcal{O} \\ &\xrightarrow{id\otimes id \otimes (\sigma^\#)^{-1}} \mathcal{O}(-1) \otimes \mathcal{O}(1) \otimes \mathcal{O} \xrightarrow{\mu \otimes id} \mathcal{O} \otimes \mathcal{O} \xrightarrow{\mu} \mathcal{O} \end{aligned} \tag{5}$$

and the map $\frac{T-S}{2} : \mathcal{O} \to \sigma_*\mathcal{O}(-1)$ is induced via the tensor-hom adjunction by the composition

$$\begin{aligned} \mathcal{O} \otimes \left\{\frac{T-S}{2}\right\} \otimes \sigma_*\mathcal{O}(-1) &\xrightarrow{\sigma^\#\otimes\sigma^\#\circ i \otimes id} \sigma_*\mathcal{O} \otimes \sigma_*\mathcal{O}(1) \otimes \sigma_*\mathcal{O}(-1) \\ &\xrightarrow{id\otimes\sigma_*(\mu)} \sigma_*\mathcal{O} \otimes \sigma_*\mathcal{O} \xrightarrow{\sigma_*\mu} \sigma_*\mathcal{O} \xrightarrow{(\sigma^\#)^{-1}} \mathcal{O} \end{aligned} \tag{6}$$

where μ denotes multiplication. We're abusing notation in the map (6) and using $\sigma^\#$ to denote both the maps $\mathcal{O} \to \sigma_*\mathcal{O}$ and $\mathcal{O}(1) \to \sigma_*\mathcal{O}(1)$ induced by the graded automorphism of $k[S,T]$ given by $f(S,T) \mapsto f(T,S)$. The image of $1 \in \mathcal{O}$ under the adjoint of the map (6) yields the element $\frac{S-T}{2}$ as a global section of $\sigma_*\mathcal{O}(1)$. The map $\frac{T+S}{2} : \mathbf{Hom}(\sigma_*\mathcal{O},\mathcal{O}) \to \mathbf{Hom}(\sigma_*\mathcal{O}(-1),\mathcal{O})$ is induced by precomposition with $\sigma_*(\frac{T+S}{2})$, which is just multiplication by the global section $\frac{T+S}{2}$. Equation (5) can be understood similarly.

We claim that diagram 4 commutes. Fix an open $U \subseteq \mathbb{P}^\sigma$ which need not be invariant, and open $V \subseteq U$. Going down then right yields the composite map

$$u \mapsto (v \mapsto \frac{T-S}{2} \cdot u \cdot (\sigma^\#)^{-1}\Big(\frac{T+S}{2} \cdot v\Big)).$$

Going right first then down yields the composite

$$u \mapsto (v \mapsto (\sigma^\#)^{-1}(\sigma^\#(\frac{T+S}{2} \cdot u) \cdot \frac{S-T}{2} \cdot v))$$

These are equal since $\frac{T+S}{2}$ is an invariant global section. Note that the diagram 4 is a map in
$\mathrm{Fun}([1],\mathrm{Vect}(\mathbb{P}^\sigma))$ from

$$\mathcal{O}(-1) \xrightarrow{\frac{T+S}{2}} \mathcal{O}$$

to its dual,

$$\mathbf{Hom}(\sigma_*\mathcal{O},\mathcal{O}) \xrightarrow{\frac{T+S}{2}} \mathbf{Hom}(\sigma_*\mathcal{O}(-1),\mathcal{O}).$$

Thus this diagram defines a (not necessarily non-degenerate) form, which we denote by ϕ.

In order to show that this ϕ is symplectic, we have to check that $\phi^* \circ (-\mathrm{can}) = \phi$. To spell this out in detail, the dual and double dual are functors. Applying these two functors, we get the two objects

$$O^* \xrightarrow{\frac{T+S}{2}^*} O(-1)^*$$

and

$$O(-1)^{**} \xrightarrow{\frac{T+S}{2}^{**}} O^{**}$$

in $\mathrm{Fun}([1],\mathrm{Ch}^b\,\mathrm{Vect}(\mathbb{P}^\sigma))$.

Because can is a natural transformation $id \to **$, there's a commutative diagram

$$\begin{array}{ccc} O(-1) & \xrightarrow{\frac{T+S}{2}} & O \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\ O(-1)^{**} & \xrightarrow{\frac{T+S}{2}^{**}} & O^{**} \\ \downarrow{\scriptstyle \frac{T-S}{2}^*} & & \downarrow{\scriptstyle \frac{T-S}{2}^*} \\ O^* & \xrightarrow{\frac{T+S}{2}^*} & O(-1)^* \end{array}$$

The goal is to show that the vertical maps in the large rectangle are the negative of the vertical maps in diagram 4. Tracing through the definitions, we see that can is the map which sends $u \in \mathcal{O}(-1)(U)$ to the natural transformation

$$\gamma \mapsto (\sigma^{\#})^{-1}(\gamma(u|_V)),$$

and $\phi^* \circ \mathrm{can}(u)$ is the natural transformation

$$v \mapsto (\sigma^{\#})^{-1}\left(\frac{T-S}{2} \cdot v \cdot (\sigma^{\#})^{-1}(u)\right)$$

which is the same thing as

$$v \mapsto \left(-\frac{T-S}{2} \cdot (\sigma^{\#})^{-1}(v) \cdot u\right).$$

On the other hand, $\frac{T-S}{2} : \mathcal{O}(-1) \to \mathcal{O}^*$ is the map

$$u \mapsto (v \mapsto \frac{T-S}{2} \cdot u \cdot (\sigma^{\#})^{-1}(v))$$

which is by what we calculated above equal to $-(\phi^* \circ \mathrm{can}) = \phi^* \circ (-\mathrm{can})$.
Now just as in [Sch17], taking the mapping cone of ϕ via the functor

$$\mathrm{Cone} : \mathrm{Fun}([1], \mathrm{Ch}^b\,\mathrm{Vect}(\mathbb{P}^\sigma))^{[0]} \to \left(\mathrm{Ch}^b\,\mathrm{Vect}(\mathbb{P}^\sigma)\right)^{[1]}$$

yields a symplectic form $\beta^\sigma = \mathrm{Cone}(\phi)$.
We claim that there's an exact sequence

$$\mathcal{O}(-1) \xrightarrow{\begin{pmatrix} \frac{T+S}{2} \\ \frac{T-S}{2} \end{pmatrix}} \mathcal{O} \oplus \mathcal{O}^* \xrightarrow{\begin{pmatrix} \frac{T+S}{2} & -\frac{T-S}{2} \end{pmatrix}} \mathcal{O}(-1)^*$$

where the maps are the maps in diagram 4. The fact that the composite is zero follows from commutativity of that 4. To show that the kernel equals the image, note that any permutation of $(\frac{T+S}{2}, \frac{S-T}{2})$ is a regular sequence on $k[S,T]$. Thus if $\frac{T+S}{2}x + \frac{S-T}{2}y = 0$, reducing mod $\frac{T+S}{2}$ we see that $y \in (\frac{T+S}{2})$ and reducing mod $\frac{S-T}{2}$ we see that $x \in (\frac{S-T}{2})$. It follows that the square defining ϕ is a pushout, and hence the induced map on mapping cones is a quasi isomorphism. Hence β^σ is a well-defined, non-degenerate symplectic form in $\left(\mathrm{Ch}^b\,\mathrm{Vect}(\mathbb{P}^\sigma)\right)^{[1]}$.

THEOREM 5.1. *Let X be a scheme with trivial involution with an ample family of line bundles and $\frac{1}{2} \in X$, and denote by $p : \mathbb{P}^\sigma \to X$ the structure map of the equivariant projective line over X, with action $[x:y] \mapsto [y:x]$. Then for all $n \in \mathbb{Z}$, the following are natural stable equivalences of (bi-) spectra*

$$GW^{[n]}(X) \oplus GW^{[n-1]}(X, -\mathrm{can}) \xrightarrow{\sim} GW^{[n]}(\mathbb{P}^\sigma_X)$$
$$\mathbb{G}W^{[n]}(X) \oplus \mathbb{G}W^{[n-1]}(X, -\mathrm{can}) \xrightarrow{\sim} \mathbb{G}W^{[n]}(\mathbb{P}^\sigma_X)$$
$$(x,y) \mapsto p^*(x) + \beta^\sigma \cup p^*(y).$$

Proof. The proof of Theorem 9.10 in [Sch17] can be easily adapted. Note that our Bott element β^σ is a linear change of coordinates from the standard Bott element on $\mathbb{P}^1$. Keeping in mind that the involution only affects the duality and not the underlying derived category with weak equivalences, it's still true that $\beta^\sigma \otimes : \mathcal{T}\mathrm{sPerf}(X) \to \mathcal{T}\mathrm{sPerf}(\mathbb{P}^1_X)/p^*\mathcal{T}\mathrm{sPerf}(X)$ is an equivalence of triangulated categories. As in *loc. cit.*, if we denote by w the set of morphisms in $\mathrm{sPerf}(\mathbb{P}^1_X)$ which are isomorphisms in $\mathcal{T}\mathrm{sPerf}(\mathbb{P}^1_X)/p^*\mathcal{T}\mathrm{sPerf}(X)$, we get a sequence

$$(\mathrm{sPerf}(X), \mathrm{quis}) \xrightarrow{p^*} (\mathrm{sPerf}(\mathbb{P}^1_X), \mathrm{quis}) \longrightarrow (\mathrm{sPerf}(\mathbb{P}^1_X), w)$$

which is a Morita exact sequence of categories with duality. That is, the maps are maps of categories with duality, and the underlying sequence of categories is Morita exact. It follows that this sequence induces a homotopy fibration of $GW^{[n]}$ and $\mathbb{G}W^{[n]}$ spectra. As remarked above, these fibration sequences split via the exact dg form functors

$$(\mathrm{sPerf}(X), \mathrm{quis}) \xrightarrow{\beta^\sigma \otimes} (\mathrm{sPerf}(\mathbb{P}^1_X), \mathrm{quis}) \longrightarrow (\mathrm{sPerf}(\mathbb{P}^1_X), w)$$

so that the composite induces an equivalence of triangulated categories. Finally, using that GW and $\mathbb{G}W$ are invariant under derived equivalences [Sch17, Theorem 6.5] [Sch17, Theorem 8.9], we conclude the theorem. □

Considering GW as a presheaf of spectra on $\mathbf{Sch}^{C_2}_{S,qp}$ it follows from Theorem 5.1 that $GW^{[n]}(\mathbb{P}^\sigma, [1:1]) \cong GW^{[n-1]}(X, -\mathrm{can}) \cong GW^{[n+1]}(X)$, recovering one of the results of [Xie20]. Hence

$$\mathbf{Hom}(\Sigma^\infty(\mathbb{P}^\sigma, [1:1]), GW^{[n]}) \cong GW^{[n+1]}$$

as presheaves of spectra on $\mathbf{Sch}^{C_2}_{S,qp}$. In particular, by the projective bundle formula from [Sch17] and the usual cofiber sequence

$$([1:1] \times \mathbb{P}^\sigma) \vee (\mathbb{P}^1 \times [1:1]) \to \mathbb{P}^\sigma \times \mathbb{P}^1 \to \mathbb{P}^\sigma \wedge \mathbb{P}^1$$

we obtain the periodicity isomorphism

$$\mathbf{Hom}((\mathbb{P}^1, [1:1]) \wedge (\mathbb{P}^\sigma, [1:1]), GW^{[n]}) \cong GW^{[n]}$$

induced by the map

$$\begin{aligned} GW^{[n]}(X) \to GW^{[n+1]}(\mathbb{P}^1_X) &\to GW^{[n]}(\mathbb{P}^\sigma_{\mathbb{P}^1_X}) \\ x \mapsto \beta \cup p^*(x) &\mapsto \beta^\sigma \cup q^*(\mathcal{O}_X[-1] \otimes \beta \cup p^*(x)) \end{aligned}$$

where p is the projection $\mathbb{P}^1_X \to X$, and q is the projection $\mathbb{P}^\sigma_{\mathbb{P}^1_X} \to \mathbb{P}^1_X$. The analogous statements hold for the presheaf of spectra $\mathbb{G}W$.
As notation for later, let $\beta^{1+\sigma}$ denote the induced map

$$\beta^{1+\sigma} : (\mathbb{P}^1, [1:1]) \wedge (\mathbb{P}^\sigma, [1:1]) \to GW. \tag{7}$$

Lemma 5.2. *The Bott element β^σ restricts to zero in $C_2 \times \mathbb{A}^\sigma = \mathbb{P}^\sigma - [1:0] \coprod \mathbb{P}^\sigma - [0:1]$.*

Proof. As in [Sch17], because the Bott element is natural it suffices to prove that the bott element β^σ in $\mathbb{P}^\sigma_{\mathbb{Z}[\frac{1}{2}]}$ restricts to zero. From the definition of the Bott element, it's clear that it's supported on $[1:-1]$. There's a commutative diagram

$$\begin{array}{ccc} GW^{[n]}(C_2 \times \mathbb{A}^\sigma \text{ on } [1:-1]\coprod[1:-1]) & \xrightarrow{k} & GW^{[n]}(C_2 \times \mathbb{A}^\sigma) \\ & \swarrow^{g} & \downarrow f \\ GW^{[n]}(C_2 \times \mathbb{A}^\sigma - [1:-1]) & \xrightarrow{h} & GW^{[n]}(C_2 \times \operatorname{Spec}(\mathbb{Z}[\tfrac{1}{2}])) \end{array}$$

where f and h are induced by inclusion of the point $[1:1]$. Because $\mathbb{Z}[\frac{1}{2}]$ is regular and $C_2 \times \mathbb{A}^\sigma$ is equivariantly isomorphic to $C_2 \times \mathbb{A}^1$, [Xie20, Theorem 7.5] shows that f is an isomorphism, hence g is an injection. By localization [Sch17, Theorem 6.6], the maps k and g compose to form an exact sequence, and it follows that k is the zero map. □

5.2 The Periodization of GW

The idea behind the Bass construction in algebraic K-theory is that as a consequence of satisfying localization, there is a Bass exact sequence ending in

$$\cdots \to K_n(\mathbb{G}_m) \xrightarrow{\partial} K_{n-1}(X) \to 0$$

for all n. This comes from applying K-theory to the pushout square manifesting the usual cover of $\mathbb{P}^1$ together with the projective bundle formula. The map ∂ is split by $x \mapsto [T] \cup p^*(x)$ where p is the projection to the base scheme $p : \mathbb{G}_m \to X$. It follows that if K exhibits an exact Bass sequence in all degrees n, then $K_{n-1}(X)$ can be identified with the image of $\partial([T]) \cup -$ (i.e. this map is an automorphism of $K_{n-1}(X)$. In fact, $\partial([T]) \cup -$ is the idempotent endomorphism $(0,1)$ of $K_0(\mathbb{P}^1) \cong K_0(X) \oplus K_0(X)$). The Bass construction can be thought of as defining $K^B_n(X)$ so that there's an exact sequence $K^B_n(\mathbb{A}^1) \oplus K^B_n(\mathbb{A}^1) \to K^B_n(\mathbb{G}_m) \to K^B_{n-1}(X)$, then identifying $K^B_{n-1}(X)$ with $(0,1) \cdot K^B_{n-1}(\mathbb{P}^1)$. In other words, it can be constructed as the colimit

$$K^B = \operatorname{colim}(K \to \mathbf{Hom}(\mathbb{A}^1 \coprod_{\mathbb{G}_m} \mathbb{A}^1, K) \to \dots)$$

where the pushouts are taken in presheaves and the maps are induced by applying $\mathbf{Hom}(-,K)$ in the category of K-modules to the composite

$$\mathbb{A}^1 \coprod_{\mathbb{G}_m} \mathbb{A}^1 \to \Sigma \mathbb{G}_m \xrightarrow{T} K.$$

Here, loosely speaking, the first map in the composite represents the boundary in the long Bass exact sequence ∂ while the second represents $[T]$, so that in the category of K modules this map represents cup product with $\partial([T])$.
We'll spell out an example a bit more explicitly to give a flavor for the constructions to come. Let $W = \mathbb{A}^1 \coprod_{\mathbb{G}_m} \mathbb{A}^1$, where we emphasize again that the pushout is in the category of presheaves. Because this is a (homotopy) pushout in the category of presheaves, applying $\mathbf{Hom}(-,K)$ gives us a homotopy pullback square, and hence a Mayer-Vietoris long exact sequence. In particular, it gives us a map of presheaves of spectra (which can be promoted to a map of K-modules) $\Omega K(\mathbb{G}_m) \to K(W)$, where we abuse notation and write $K(W)$ for the internal hom of W into K. Because K^B satisfies Nisnevich descent, and $K_i(-) = K_i^B(-)$ for $i \geq 0$, it follows by the 5-lemma that $K_0(W) \cong K_0(\mathbb{P}^1) \cong K_0(X) \oplus K_0(X)$, and that the element $\partial([T]) \cup -$ represents projection onto the second factor as an endomorphism of $K_0(W)$.
Now, we want to explain why $\partial([T]) \cup K_{-1}(W) \cong K_{-1}^B(X)$. We'll use the fact that $K^B(W) \cong K^B(\mathbb{P}^1)$ and that $K_{-1}^B(X) = \partial([T]) \cup K_{-1}^B(\mathbb{P}^1) = \partial(K_0^B(\mathbb{G}_m))$.
To begin, because $\partial([T])$ is zero in $K_0(\mathbb{A}^1)$, the image of $\partial([T]) \cup K_{-1}(W)$ in $K_{-1}(\mathbb{A}^1) \oplus K_{-1}(\mathbb{A}^1)$ is zero. By exactness, it follows that $\partial([T]) \cup K_{-1}(W) \subseteq \partial K_0(\mathbb{G}_m)$.
There's a map $\phi : K_{-1}(W) \to K_{-1}^B(W) \cong K_{-1}^B(\mathbb{P}^1)$ and a commutative diagram

$$\begin{array}{ccccc} K_0(\mathbb{A}^1) \oplus K_0(\mathbb{A}^1) & \longrightarrow & K_0(\mathbb{G}_m) & \xrightarrow{\partial} & K_{-1}(W) \\ \downarrow & & \downarrow & & \downarrow \phi \\ K_0(\mathbb{A}^1) \oplus K_0(\mathbb{A}^1) & \longrightarrow & K_0(\mathbb{G}_m) & \xrightarrow{\partial} & K_{-1}^B(W) \end{array}$$

which shows that ϕ restricts to an isomorphism $\partial(K_0(\mathbb{G}_m)) \cong \partial(K_0^B(\mathbb{G}_m))$, and in particular that $\phi(\partial(K_0(\mathbb{G}_m))) = \partial(K_0^B(\mathbb{G}_m))$. Now

$$\phi(\partial([T]) \cup \partial(K_0(\mathbb{G}_m)) = \partial([T]) \cup \phi(\partial K_0(\mathbb{G}_m)) = \partial([T]) \cup \partial K_0^B(\mathbb{G}_m) = \partial K_0^B(\mathbb{G}_m)$$

where we've crucially used that for Bass K-theory, $\partial([T]) \cup \partial K_0^B(\mathbb{G}_m) = \partial([T]) \cup \partial K_0^B(\mathbb{P}^1) = \partial K_0^B(\mathbb{G}_m)$.
But as remarked above, the fact that $\partial([T])$ is trivial in $K_0(\mathbb{A}^1)$ implies that $\partial([T]) \cup \partial(K_0(\mathbb{G}_m)) \subseteq \partial(K_0(\mathbb{G}_m))$, and we know that $\phi|_{\partial K_0(\mathbb{G}_m)}$ is an isomorphism. Since $\phi|_{\partial([T]) \cup \partial(K_0(\mathbb{G}_m))}$ is surjective by the chain of equalities above, it follows that $\partial K_0(\mathbb{G}_m) = \partial([T]) \cup K_{-1}(W)$. We've shown that

$$\partial([T]) \cup K_{-1}(W) = \partial K_0(\mathbb{G}_m) \cong \partial K_0^B(\mathbb{G}_m) \cong K_{-1}^B(X).$$

If we take pointed versions of the above sequences by pointing all the schemes in question at $[1:1]$ everything goes through as above with the extra benefit that $\partial([T]) \cup K_{-1}^B(W,1) = K_{-1}^B(W,1)$, and the map $K^B(X) \to K^B(W,1)$, $x \mapsto p^*(x) \cup \partial([T])$ is an isomorphism by the projective bundle formula. Now the map $p : (W,1) \to 1$ is split by inclusion of the base point, and thus $p^* : K(W,1) \to K((W,1) \otimes (W,1))$ is injective. Furthermore, $p^*(x \cup \partial([T])) = \partial([T]) \cup p^*(x)$, so that the image of $K_{-1}(W,1)$ in $K_{-1}((W,1) \otimes (W,1))$ under the map $x \mapsto \partial([T]) \cup p^*(x)$ is, by what we showed above, isomorphic to $K^B_{-1}(X)$. This shows that

$$\pi_{-1} K^B = \pi_{-1} \operatorname{colim}(K \to \mathbf{Hom}(\mathbb{A}^1 \coprod_{\mathbb{G}_m} \mathbb{A}^1, K) \to \cdots).$$

This argument is mostly formal given a few pieces of structural information:

- A map $K \to K^B$ which respects cup products,
- Nisnevich descent for K^B, and
- A Bass exact sequence split by cup product with an element in $K_1(\mathbb{G}^m)$.

The remainder of this section will show that these three pieces of structure are present for Grothendieck-Witt groups, which will allow us to repeat essentially the same argument to give a construction of the localizing $\mathbb{G}W$ as a periodization of GW. When the base scheme is a perfect field, a similar construction of GW as a periodic spectrum was given in [HKO11].
First, equivariant Nisnevich descent for $\mathbb{G}W$ is a consequence of results from [Sch17].

Lemma 5.3. *$\mathbb{G}W$ is Nisnevich excisive on the category of schemes with an ample family of line bundles over S.*

Proof. Recall that the distinguished squares defining the equivariant Nisnevich cd-structure are cartesian squares in $\mathbf{Sch}^G_{S,qp}$

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ A & \xrightarrow{j} & X \end{array}$$

where j is an open immersion, p is étale, and $(Y - B)_{\mathrm{red}} \to (X - A)_{\mathrm{red}}$ is an isomorphism.
As in [Sch17, Theorem 9.6], a result of Thomason [TT07, Theorem 2.6.3] tells us that the map p induces a quasi-equivalence of dg categories

$$p^* : \mathrm{sPerf}_Z(X) \to \mathrm{sPerf}_Z(Y).$$

Because $\mathbb{G}W$ is invariant under derived equivalences [Sch17, Theorem 8.9], it follows that p^* induces an isomorphism on Grothendieck-Witt groups. Noting that U and the closed subset $Z = X - A$ are G-invariant, the localization sequence [Sch17, Theorem 9.5] generalizes to our setting and identifies $\mathbb{G}W(\mathrm{sPerf}_Z(X))$ and $\mathbb{G}W(\mathrm{sPerf}_Z(Y))$ as the horizontal homotopy fibers. This allows us to conclude the result. □

Next, we identify the analogues of the Bass sequence and the splittings therein. From [Sch17, Theorem 9.13], we know that there's a Bass sequence

$$0 \longrightarrow \mathbb{G}W_i^{[n]}(X) \longrightarrow \mathbb{G}W_i^{[n]}(\mathbb{A}^1_X) \oplus \mathbb{G}W_i^{[n]}(\mathbb{A}^1_X)$$

$$\longrightarrow \mathbb{G}W_i^{[n]}(X[T,T^{-1}]) \longrightarrow \mathbb{G}W_{i-1}^{[n-1]}(X) \longrightarrow 0$$

where the last non-trivial map is split by cup product with (the pullback of) $[T]$ in $\mathbb{G}W_1^{[1]}(\mathbb{Z}[\frac{1}{2}][T,T^{-1}])$. This gives us a candidate map $\mathbb{A}^1 \coprod_{\mathbb{G}_m} \mathbb{A}^1 \to \Sigma\mathbb{G}_m \xrightarrow{[T]} GW^{[1]}$.

Now, we want to find a candidate map $\Sigma^\sigma \mathbb{G}_m^\sigma \to GW^{[-1]}$ so that we can eventually invert

$$\Sigma^\sigma \mathbb{G}_m^\sigma \otimes \Sigma\mathbb{G}_m \to GW^{[-1]} \otimes GW^{[1]} \to GW^{[0]}.$$

Define W_σ by the pushout square in the category of presheaves

$$\begin{array}{ccc} (C_2 \times \mathbb{G}_m^\sigma)_+ & \longrightarrow & (C_2 \times \mathbb{A}^\sigma)_+ \\ \downarrow & & \downarrow \\ (\mathbb{G}_m^\sigma)_+ & \longrightarrow & W_\sigma \end{array}$$

There's an associated homotopy pushout square

$$\begin{array}{ccc} (C_2 \times \mathbb{G}_m^\sigma)_+/(C_2)_+ & \longrightarrow & (C_2 \times \mathbb{A}^\sigma)_+/(C_2)_+ \\ \downarrow & & \downarrow \\ (\mathbb{G}_m^\sigma)_+/S^0 & \longrightarrow & W_\sigma/S^0 \end{array}$$

and taking the homotopy cofiber of the left vertical map yields $S^\sigma \wedge \mathbb{G}_m^\sigma$. It follows that the homotopy cofiber of the right vertical map is equivalent to $S^\sigma \wedge \mathbb{G}_m^\sigma$, and that there's a long exact sequence

$$\cdots \longrightarrow GW_i^{[n]}(S^\sigma \wedge \mathbb{G}_m^\sigma) \longrightarrow GW_i^{[n]}(W_\sigma/S^0) \tag{8}$$

$$\longrightarrow GW_i^{[n]}((C_2 \times \mathbb{A}^\sigma)_+/(C_2)_+) \longrightarrow \cdots$$

Here if $\mathbb{A}^\sigma_S \cong S$, then $(C_2 \times \mathbb{A}^\sigma)_+/(C_2)_+ \cong (C_2)_+ \wedge \mathbb{A}^\sigma$ is contractible and $W/S^0 \cong S^\sigma \wedge \mathbb{G}_m^\sigma$. Working over the regular ring $\mathbb{Z}[\frac{1}{2}]$, $GW(W_\sigma/S^0) \cong GW(\mathbb{P}^\sigma/S^0)$, and

$$GW_i^{[n]}(W_\sigma/S^0) \cong GW_i^{[n]}(\mathbb{P}^\sigma/S^0) \cong GW_i^{[n+1]}(S)$$

by the projective bundle formula 5.1.

The maps in the sequence (8) are maps of $GW_*^{[0]}$-modules, and the sequence is natural in the base scheme. The induced map

$$GW_0^{[-1]}(S^\sigma \wedge \mathbb{G}_m^\sigma) \to GW_0^{[0]}(\mathbb{Z}[\frac{1}{2}])$$

is an isomorphism of $GW_0^{[0]}(\mathbb{Z}[\frac{1}{2}])$-modules, and hence the inverse is uniquely determined by a lift of the element $\langle 1\rangle \in GW_0^{[0]}(\mathbb{Z}[\frac{1}{2}])$ to $GW_0^{[-1]}(S^\sigma \wedge \mathbb{G}_m^\sigma)$. We stress that this element $\langle 1\rangle$ maps to $\beta^\sigma \cup \mathcal{O}_{\mathbb{Z}[\frac{1}{2}]}[-1] \cup \langle 1\rangle$ in $GW(\mathbb{P}^\sigma)$, and in particular it isn't the unit of multiplication in $GW(\mathbb{P}^\sigma)$. We'll denote this element by $[T^\sigma]$ in analogy with the non-equivariant case.

Over an arbitrary base scheme X, we denote by $[T^\sigma]$ the pullback of $[T^\sigma]$ to $GW_0^{[-1]}(S^\sigma \wedge \mathbb{G}_m^\sigma \times_{\mathbb{Z}[\frac{1}{2}]} X)$ using functoriality of GW. We summarize in the definition below.

Definition 5.4. Let $[T]$ denote the class of the element T in $\mathbb{G}W_1^{[1]}(\mathbb{Z}[\frac{1}{2}][T,T^{-1}])$. Let $\partial([T])$ denote the image of $[T]$ under the connecting map in the Bass sequence

$$\partial : GW_1^{[1]}(\mathbb{Z}[\frac{1}{2}][T,T^{-1}]) \to GW_0^{[1]}(\mathbb{P}^1_{\mathbb{Z}[\frac{1}{2}]}).$$

Let $[T^\sigma]$ denote the lift of the element $\langle 1\rangle \in GW_0^{[0]}(\mathbb{Z}[\frac{1}{2}])$ to $GW_0^{[-1]}(S^\sigma \wedge \mathbb{G}_m^\sigma)$. Let $\partial([T^\sigma])$ denote the image of $[T^\sigma]$ under the connecting map in the long exact sequence 8

$$\partial :\ GW_1^{[-1]}(S^\sigma \wedge \mathbb{G}_m^\sigma) \longrightarrow GW_0^{[-1]}(W_\sigma/S^0)\ .$$

Over an arbitrary scheme S with $\frac{1}{2} \in S$, let $[T]$ and $[T^\sigma]$ denote the pullbacks $f^*([T])$, $f^*([T^\sigma])$ under the unique map $f : S \to \mathbb{Z}[\frac{1}{2}]$, and similarly for $\partial([T])$ and $\partial([T^\sigma])$.

Let $W = (\mathbb{A}^1 \coprod_{\mathbb{G}_m} \mathbb{A}^1)_+$. Now (by taking the pointed version of everything) we have a candidate map

$$\gamma : W_\sigma/S^0 \otimes W/S^0 \to S^\sigma \wedge \mathbb{G}_m^\sigma \otimes S^1 \wedge \mathbb{G}_m \xrightarrow{[T^\sigma]\otimes[T]} GW^{[-1]} \otimes GW^{[1]} \to GW \quad (9)$$

to invert.

Given a presentably symmetric monoidal ∞-category and a morphism $\alpha : x \to \mathbf{1}$ to the monoidal unit, define

$$Q_\alpha E = \operatorname{colim}(E \xrightarrow{\alpha} \mathbf{Hom}(x,E) \xrightarrow{\alpha} \mathbf{Hom}(x^{\otimes 2},E) \xrightarrow{\alpha} \dots).$$

In general $Q_\alpha E$ is not the periodization of E with respect to α, one obstruction being that the cyclic permutation of α^3 can fail to be homotopic to the identity. This matters because checking periodicity requires permuting $\alpha \otimes id$ to $id \otimes \alpha$, and these can fail to be homotopic.

LEMMA 5.5. *The canonical map $\mathbb{G}W \to Q_\gamma \mathbb{G}W$ is an equivalence of (pre)sheaves of spectra on $\mathbf{Sch}^{C_2}_{S,qp}$.*

Proof. We know by the projective bundle formulas that

$$\begin{aligned}\mathbb{G}W(\mathbb{P}^\sigma \times \mathbb{P}^1 = \mathbb{P}^\sigma_{\mathbb{P}^1}) &\cong \mathbb{G}W(\mathbb{P}^1) \oplus \mathbb{G}W^{[1]}(\mathbb{P}^1)\\ &\cong \mathbb{G}W(X) \oplus \mathbb{G}W^{[-1]}(X) \oplus \mathbb{G}W^{[1]}(X) \oplus \mathbb{G}W(X).\end{aligned}$$

We claim that under this isomorphism, cup product with $\partial[T^\sigma]$ is projection onto $\mathbb{G}W^{[1]}(\mathbb{P}^1)$ and cup product with $\partial[T]$ on $GW^{[1]}(\mathbb{P}^1)$ is projection onto $\mathbb{G}W(X)$. The latter statement is already known from [Sch17, Theorem 9.10], so we show the former. It suffices to show that cup product with $\partial[T^\sigma_X] \cup - :$ $\mathbb{G}W^{[n]}(X) \oplus \mathbb{G}W^{[n+1]}(X) \to \mathbb{G}W(X) \oplus \mathbb{G}W^{[n+1]}(X)$ is projection onto the second factor. But this is precisely how $[T^\sigma]$ is defined: it's a lift under ∂ of a generator of $\mathbb{G}W^{[1]}(X)$, so cup product with it is cup product with $\langle 1 \rangle$ on $GW^{[n+1]}(X)$ and it's necessarily zero on the other factor because it gives a well-defined element on the pointed $\mathbb{G}W^{[-1]}(\mathbb{P}^\sigma, [1:1])$.
Because $\mathbb{G}W$ satisfies equivariant Nisnevich descent, $\mathbb{G}W(W/S^0) \cong \mathbb{G}W(\mathbb{P}^1, [1:1])$, and $\mathbb{G}W(W_\sigma/S^0) \cong \mathbb{G}W(\mathbb{P}^\sigma, [1:1])$. Now we're essentially done. The maps in the colimit defining $Q_\gamma \mathbb{G}W$ first identify $\mathbb{G}W_i^{[n]}(X)$ with $\partial([T]) \cup \mathbb{G}W_i^{[n]}(\mathbb{P}^1_X, [1:1])$, then identify $\mathbb{G}W_i^{[n]}(\mathbb{P}^1_X)$ with $\partial([T^\sigma]) \cup \mathbb{G}W_i^{[n]}(\mathbb{P}^\sigma_{\mathbb{P}^1_X}, [1:1])$. As we noted above, the projective bundle formulas imply that the image of $\mathbb{G}W_i^{[n]}(X)$ under these identifications is isomorphic to $\mathbb{G}W_i^{[n]}(X)$, and hence $Q_\gamma \mathbb{G}W_i^{[n]}(X) \simeq \mathbb{G}W_i^{[n]}(X)$ as desired. □

LEMMA 5.6. *The canonical map $Q_\gamma GW^{[m]} \to Q_\gamma \mathbb{G}W^{[m]} \cong \mathbb{G}W^{[m]}$ induces isomorphisms $\pi_n Q_\gamma GW^{[m]} \cong \pi_n \mathbb{G}W^{[m]}$ for $n \geq 0$ and for all m.*

Proof. This follows from two out of three and the proof of lemma 5.5 since $\pi_n GW^{[m]} \cong \pi_n \mathbb{G}W^{[m]}$ for $n \geq 0$ and for all m. □

LEMMA 5.7. *The canonical map $Q_\gamma GW^{[m]} \to Q_\gamma \mathbb{G}W^{[m]} \cong \mathbb{G}W$ induces an isomorphism $\pi_n Q_\gamma GW^{[m]} \cong \pi_n \mathbb{G}W^{[m]}$ for $n <= 0$ and for all m.*

Proof. Because homotopy groups commute with filtered (homotopy) colimits of spectra

$$\pi_n Q_\gamma GW^{[m]} = \operatorname{colim}(\pi_n GW \xrightarrow{\alpha} \pi_n \mathbf{Hom}(W/S^0 \otimes W_\sigma/S^0, GW) \xrightarrow{\alpha^{\otimes 2}} \cdots).$$

Fix $[m]$ for now and denote by F^i_n the image of the map of groups

$$\gamma^* : GW_n^{[m]}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i}) \to GW_n^{[m]}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+1})$$

and note that $F_n^0 \cong GW_n^{[m]}$. Denote by FB_n^i the same construction as above with GW replaced by $\mathbb{G}W$.

For $i \geq -n$, we claim that there are exact sequences

$$F_n^i(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \oplus F_n^i(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \to F_n^i(\mathbb{G}_m/1 \otimes W_\sigma/S^0) \xrightarrow{\partial} F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)$$

such that $\partial(F_n^i(\mathbb{G}_m/1 \otimes W_\sigma/S^0)) = \partial([T]) \cup \partial([T^\sigma]) \cup F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)$. We prove this in conjunction with the statement that, for each n, $F_n^i \cong \mathbb{G}W_n^{[m]}$ for $i \geq -n$. The proof is induction in i, and we must show that $\partial(F_n^i(\mathbb{G}_m/1 \otimes W_\sigma/S^0)) = \partial([T]) \cup F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)$. For $n \geq 0$, the same argument that we gave for K-theory together with lemma 5.6 works. In more detail, there's an exact sequence

$$GW_n^{[m]}(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \oplus GW_n^{[m]}(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \to GW_n^{[m]}(\mathbb{G}_m/1 \otimes W_\sigma/S^0) \xrightarrow{\partial} GW_{n-1}^{[m]}(W/S^0 \otimes W_\sigma/S^0)$$

and because $n \geq 0$, the same argument we gave for K-theory above identifies $\partial(GW_n^{[m]}(\mathbb{G}_m/1 \otimes W_\sigma/S^0))$ with $\partial([T]) \cup \partial([T^\sigma]) \cup GW_{n-1}^{[m]}(W/S^0 \otimes W_\sigma/S^0)$ and in turn with $\mathbb{G}W^{[m]}(X)$. Then we just use the fact that p^* is injective and a module map to conclude that $\partial([T]) \cup \partial([T^\sigma]) \cup p^*(GW_{n-1}^{[m]}(W/S^0 \otimes W_\sigma/S^0))$ is isomorphic to $\mathbb{G}W^{[m]}(X)$.

Now fix an i, and assume by induction that our claim holds for all $-n \leq i$. Then there's an exact sequence

$$\mathbb{G}W_n^{[m]}(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \oplus \mathbb{G}W_n^{[m]}(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \to \mathbb{G}W_n^{[m]}(\mathbb{G}_m/1 \otimes W_\sigma/S^0) \xrightarrow{\partial} F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)$$

which identifies $\partial(\mathbb{G}W_n^{[m]}(\mathbb{G}_m/1 \otimes W_\sigma/S^0))$ with $\partial([T]) \cup \partial([T^\sigma]) \cup F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)$, but we know that $\partial(\mathbb{G}W_n^{[m]}(\mathbb{G}_m/1 \otimes W_\sigma/S^0))$ is equal to $\mathbb{G}W_{n-1}^{[m]}(W/S^0 \otimes W_\sigma/S^0) \cong \mathbb{G}W_{n-1}^{[m]}(X)$. Thus, letting p denote the projection $W/S^0 \otimes W_\sigma/S^0 \to X$ to the basepoint,

$$\begin{aligned}\mathbb{G}W_{n-1}^{[m]}(X) &\cong p^*(\partial([T]) \cup \partial([T^\sigma]) \cup F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)) \\ &= \partial([T]) \cup \partial([T^\sigma]) \cup p^*(F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)) \\ &= F_{n-1}^{i+1}\end{aligned}$$

since p^* is split injective.

The meatier part of the argument is producing the exact sequence for F_{n-1}^{i+1}, though the proof is essentially the same as the proof of the base case.

First note that for all i and n, there's a chain complex

$$F_n^i(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \oplus F_n^i(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \to F_n^i(\mathbb{G}_m/1 \otimes W_\sigma/S^0) \xrightarrow{\partial} F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)$$

which is just the image of the usual long exact sequence for GW under the map γ^*. Depending on n, this sequence may or may not be exact, as the image of an exact sequence is in general not exact.
Consider the commutative diagram

$$\begin{array}{ccc}
F_{n-1}^{i+1}(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \oplus F_{n-1}^{i+1}(\mathbb{A}^1/1 \otimes W_\sigma/S^0) & \longrightarrow & \mathbb{G}W(\mathbb{A}^1/1 \otimes W_\sigma/S^0 \otimes (W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2})^{\oplus 2} \\
\downarrow & & \downarrow \\
F_{n-1}^{i+1}(\mathbb{G}_m/1 \otimes W_\sigma/S^0) & \longrightarrow & \mathbb{G}W(\mathbb{G}_m/1 \otimes W_\sigma/S^0 \otimes (W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2}) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial^B} \\
F_{n-2}^{i+1}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2}) & \xrightarrow{\phi} & \mathbb{G}W_{n-2}^{[m]}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+3})
\end{array}$$

where the upper two horizontal maps are isomorphisms by what we've already shown. We claim that the left column is exact. The composite is zero since it's a chain complex, and if $x \in \ker(\partial)$, then using the fact that the middle and top maps are isomorphisms we produce a lift of x.
Now it remains only to check that the image of ∂ coincides with $\partial([T]) \cup \partial([T^\sigma]) \cup F_{n-2}^{i+1}$. This is the part of the proof we adapt from the K-theory case. First, it's clear that $\partial([T]) \cup \partial([T^\sigma]) \cup F_{n-2}^{i+1} \subseteq \mathrm{im}(\partial)$, since $\partial([T])$ restricts to zero in $\mathbb{A}^1$. For the other containment, by exactness and the fact that the left two vertical arrows are isomorphisms, we know that $\mathrm{im}(\partial) \cong \mathrm{im}(\partial^B)$. Now since $\partial([T]) \cup \partial([T^\sigma]) \cup p^*(F_{n-2}^{i+1}) \subseteq \mathrm{im}(\partial)$, it is isomorphic to its image in $\mathbb{G}W_{n-2}^{[m]}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2})$. But ϕ is a map of modules, so that

$$\phi(\partial([T]) \cup \partial([T^\sigma]) \cup F_{n-2}^{i+1}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2})) \cong \partial([T]) \cup \partial([T^\sigma]) \cup \mathrm{im}(\phi)$$

But ϕ is necessarily surjective, and cup product with $\partial([T]) \cup \partial([T^\sigma])$ is an automorphism of $\mathbb{G}W$. It follows that

$$\partial([T]) \cup \partial([T^\sigma]) \cup F_{n-2}^{i+1}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2}) \cong \mathrm{im}(\phi) = \mathrm{im}(\partial^B) \cong \mathrm{im}(\partial)$$

so that $\partial([T]) \cup \partial([T^\sigma]) \cup F_{n-2}^{i+1}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2}) = \mathrm{im}(\partial)$.
We've shown that if the inductive statement holds for i, n, then it holds for $i+1, n-1$. The fact that it holds for $i+1, m$ for any $m < n+1$ is clear by appealing to results for $\mathbb{G}W$. Now, the lemma follows from the explicit description for filtered colimits of groups. □

Corollary 5.8. Let γ be the map (9). Then there are weak equivalences of presheaves of spectra

$$Q_\gamma GW \xrightarrow{\sim} Q_\gamma \mathbb{G}W \simeq \mathbb{G}W.$$

Proof. Combining Lemma 5.6 and Lemma 5.7 we see that $Q_\gamma GW \to Q_\gamma \mathbb{G}W$ induces an isomorphism on stable homotopy groups. Lemma 5.5 shows that $Q_\gamma \mathbb{G}W \simeq \mathbb{G}W$. □

Recall the definition of $\beta^{1+\sigma}$ from equation (7).

Definition 5.9. A GW-module E is called *Bott periodic* if the map

$$\mathbf{Hom}(\beta^{1+\sigma}, E) : E \to \mathbf{Hom}((\mathbb{P}^1,[1:1]) \wedge (\mathbb{P}^\sigma,[1:1]), E)$$

is an equivalence.

Let $\mathbb{A}^-$ and $\mathbb{G}_m^-$ denote $\mathbb{A}^1$ and $\mathbb{G}_m$ with the sign action $x \mapsto -x$. There are zigzags

$$\mathbb{A}^1/\mathbb{G}_m \hookrightarrow \mathbb{P}^1/(\mathbb{P}^1 - [-1:1]) \twoheadleftarrow \mathbb{P}^1/[1:1]$$

and

$$\mathbb{A}^-/\mathbb{G}_m^- \hookrightarrow \mathbb{P}^\sigma/(\mathbb{P}^\sigma - [-1:1]) \twoheadleftarrow \mathbb{P}^\sigma/[1:1].$$

The maps $\beta : \mathbb{P}^1/[1:1] \to GW^{[1]}$ and $\beta^\sigma : \mathbb{P}^\sigma/[1:1] \to GW^{[-1]}$ lift to $\mathbb{P}^1/(\mathbb{P}^1 - [-1:1])$ and $\mathbb{P}^\sigma/(\mathbb{P}^\sigma - [-1:1])$ respectively by results analogous to Lemma 5.2, and hence there are induced maps

$$\beta' : \mathbb{A}^1/\mathbb{G}_m \to GW^{[1]}$$

$$(\beta^\sigma)' : \mathbb{A}^-/\mathbb{G}_m^- \to GW^{[-1]}.$$

Taking smash products and using that $\mathbb{A}^1 \oplus \mathbb{A}^- \cong \mathbb{A}^\rho$, we get a map

$$\beta' \otimes (\beta^\sigma)' : \mathbb{A}^\rho/(\mathbb{A}^\rho - 0) \to GW^{[1]} \otimes GW^{[-1]} \to GW. \tag{10}$$

When working over a scheme other than the base scheme S, we'll let β'_X and $(\beta^\sigma)'_X$ denote the analogous constructions with $\mathbb{A}^1$ and $\mathbb{G}_m$ replaced by $\mathbb{A}^1_X$ and $(\mathbb{G}_m)_X$. For a vector bundle E, let $\mathbb{V}_0(E)$ denote $E/(E-0)$, the quotient by the complement of the zero section.

Theorem 5.10. *Let S be a Noetherian scheme of finite Krull dimension with an ample family of line bundles and $\frac{1}{2} \in S$. Then $L_{\mathbb{A}^1}\mathbb{G}W$ lifts to an E_∞ motivic spectrum, denoted $\mathbf{KR}_S^{\mathrm{alg}}$, over $\mathbf{Sm}_{S,qp}^{C_2}$.*

Proof. GW is an E_∞ object in presheaves of spectra (it's a commutative monoid in the category of presheaves of symmetric spectra) on $\mathbf{Sch}_{S,qp}^{C_2}$ via the cup product defined in [Sch17, Remark 5.1]. By [Hoy16] Lemma 3.3, together with corollary 5.8 above, $\mathbb{G}W$ is the periodization of GW with respect to γ. Let T^ρ denote the Thom space of the regular representation $\mathbb{A}^\rho$. Now $\mathbb{G}W$ is Nisnevich excisive, so that $\mathbb{G}W(W/S^0 \otimes W_\sigma/S^0) \cong \mathbb{G}W(\mathbb{P}^1 \wedge \mathbb{P}^\sigma)$, and $\mathbb{G}W$ is γ periodic if and only if it is Bott periodic. Because $L_{\mathbb{A}^1}$ preserves Nisnevich sheaves of spectra and E_∞-objects, $L_{\mathbb{A}^1}\mathbb{G}W$ is an E_∞ object in the category of homotopy invariant Nisnevich sheaves of spectra. In the notation of

Hoyois, let $L_{\mathbb{A}^1}\mathbb{G}W_X$ denote the restriction of $L_{\mathbb{A}^1}\mathbb{G}W$ to $X \in \mathbf{Sch}^{C_2}_{S,qp}$. Then $L_{\mathbb{A}^1}\mathbb{G}W_X \in \mathbf{Sp}(\mathrm{H}^{C_2}(X))$, where $\mathbf{Sp}(\mathrm{H}^{C_2}(X))$ is the ∞-category of homotopy invariant Nisnevich sheaves of spectra on $\mathbf{Sm}^{C_2}_{X,qp}$. Let $(L_{\mathbb{A}^1}\mathbb{G}W_X)_{mod}$ denote the category of modules over $L_{\mathbb{A}^1}\mathbb{G}W_X$ in $\mathbf{Sp}(\mathrm{H}^{C_2}(X))$. By [Hoy16], proposition 3.2, $L_{\mathbb{A}^1}\mathbb{G}W$ lifts to an E_∞ object $\mathbf{KR}^{\mathrm{alg}}_X$ in $(L_{\mathbb{A}^1}\mathbb{G}W_X)_{mod}[(T^\rho)^{-1}]$ and by forgetting the module structure an E_∞-algebra in $\mathrm{SH}^{C_2}(X)$. ☐

LEMMA 5.11. *The $\mathbb{A}^1$-localization of the Bott element $L_{\mathbb{A}^1}(\beta'_X \otimes (\beta^\sigma)'_X)$: $L_{\mathbb{A}^1}\mathbb{V}_0(\mathbb{A}^\rho) \to L_{\mathbb{A}^1}GW_X$, viewed as an element of $\mathbf{Sp}(\mathcal{P}_{\mathbb{A}^1}(\mathbf{Sch}^{C_2}_{S,qp}))_{/L_{\mathbb{A}^1}GW_X}$ is 3-symmetric.*

Proof. The proof is identical to Lemma 4.8 in [Hoy16]. The main idea is that the identity and the cyclic permutation σ_3 are both induced by matrices in $SL_{3.2}(\mathbb{Z})$ acting on $\mathbb{A}^{3\rho}$, and any two such matrices are (naïvely) $\mathbb{A}^1$-homotopic so that there's a map $h : \mathbb{A}^1 \times \mathbb{A}^{3\rho} \to \mathbb{A}^{3\rho}$ witnessing the homotopy. We can extend this to a map

$$\phi : \mathbb{A}^1 \times \mathbb{A}^{3\rho} \xrightarrow{\pi_1 \times h} \mathbb{A}^1 \times \mathbb{A}^{3\rho}.$$

Letting $p : \mathbb{A}^1 \times S \to S$ denote the projection, ϕ is an automorphism of the vector bundle $p^*(\mathbb{A}^{3\rho})$.
Now we claim that the automorphisms ϕ_0, ϕ_1 of $\mathbb{V}_0(\mathbb{A}^{3\rho})$ induced by the restrictions of ϕ to 1 and 0 are $\mathbb{A}^1$-homotopic over $L_{\mathbb{A}^1}GW$. Let $\beta'_{\mathbb{A}^{3\rho}}$ denote $(\beta'_X \otimes (\beta^\sigma)'_X)^{\otimes 3}$. Let $\beta'_{\mathbb{A}^{3\rho}}$ denote $(\beta'_{\mathbb{A}^1 \times X} \otimes (\beta^\sigma)'_{\mathbb{A}^1 \times X})^{\otimes 3}$.
To prove the claim, any automorphism ϕ as above induces a commutative triangle

$$\begin{array}{ccc} \mathbb{V}_0(p^*(\mathbb{A}^{3\rho})) & \xrightarrow{\phi} & \mathbb{V}_0(p^*(\mathbb{A}^{3\rho})) \\ & {\scriptstyle \beta'_{p^*(\mathbb{A}^{3\rho})}} \searrow \quad \swarrow {\scriptstyle \beta'_{p^*(\mathbb{A}^{3\rho})}} & \\ & L_{\mathbb{A}^1}GW_{\mathbb{A}^1 \times X} & \end{array}$$

or presheaves of spectra on $\mathbf{Sch}^{C_2}_{\mathbb{A}^1 \times X}$. As in [Hoy16], the diagram ultimately comes from our construction of the Bott elements via the projective bundle formula and the functoriality of the Proj(Sym -) construction with respect to automorpshims of the underlying vector bundle (in particular the fact that there are induced isomorphisms on each twisting sheaf $\mathcal{O}(d)$). By adjunction, this is equivalent to a triangle

$$\begin{array}{ccc} \mathbb{A}^1_+ \otimes \mathbb{V}_0(\mathbb{A}^{3\rho}) & \longrightarrow & \mathbb{V}_0(\mathbb{A}^{3\rho}) \\ & {\scriptstyle \beta'_{\mathbb{A}^{3\rho}}} \searrow \quad \swarrow {\scriptstyle \beta'_{\mathbb{A}^{3\rho}}} & \\ & L_{\mathbb{A}^1}GW_X & \end{array}$$

which is an $\mathbb{A}^1$-homotopy between ϕ_0 and ϕ_1 over $L_{\mathbb{A}^1}GW$ as desired. ☐

We've shown that $\mathbb{G}W$ is Bott periodic and Nisnevich excisive. Since it's the γ periodization of GW and γ periodicity is equivalent to Bott periodicity for Nisnevich excisive sheaves, $\mathbb{G}W$ is in fact the reflection of GW in the subcategory of Nisnevich excisive and Bott periodic GW-modules. Thus by definition, $L_{\mathbb{A}^1}\mathbb{G}W$ is the reflection of GW in the subcategory of homotopy invariant, Nisnevich excisive, and Bott periodic GW-modules.

Corollary 5.12. The canonical map $GW \to Q_\beta L_{\mathrm{mot}} GW$ is the universal map to a homotopy invariant, Nisnevich excisive, and Bott periodic GW-module. In particular

$$L_{\mathbb{A}^1}\mathbb{G}W = Q_\beta L_{\mathrm{mot}} GW$$

Proof. Given Lemma 5.11, the proof is identical to Proposition 4.9 in [Hoy16]. □

Replacing GW with its connective cover $GW_{\geq 0}$, the same reasoning yields:

Porism 5.13. The canonical map $GW_{\geq 0} \to Q_\beta L_{\mathrm{mot}} GW_{\geq 0}$ is the universal map to a homotopy invariant, Nisnevich excisive, and Bott periodic $GW_{\geq 0}$-module. In particular

$$L_{\mathbb{A}^1}\mathbb{G}W = Q_\beta L_{\mathrm{mot}} GW_{\geq 0}$$

Proof. In short, the reason the result extends to the connective cover $GW_{\geq 0}$ is that we have at no point used the negative homotopy groups of GW in our arguments. We'll spell this out more explicitly now.

The connective cover construction is monoidal, and the canonical map $GW^{[m]}_{\geq 0} \to GW^{[m]}$ is a ring map. The Bott elements β and β^σ live in the zeroth homotopy groups of $GW^{[1]}(\mathbb{P}^1)$ and $GW^{[1]}(\mathbb{P}^\sigma, -\mathrm{can})$. It follows that these Bott elements restrict to well defined elements in the zeroth homotopy groups of $GW^{[1]}_{\geq 0}(\mathbb{P}^1)$ and $GW^{[1]}_{\geq 0}(\mathbb{P}^\sigma, -\mathrm{can})$. The definition of the map γ in (9) extends without modification to $GW_{\geq 0}$, as all the elements involved in the discussion prior to (9) were in the non-negative homotopy groups of GW.

In particular, there's a canonical map $GW^{[m]}_{\geq 0} \to \mathbb{G}W^{[m]}$ which exhibits $\mathbb{G}W$ as a $GW_{\geq 0}$ module and is an isomorphism on non-negative homotopy groups, and Lemma 5.6 remains true replacing GW with $GW_{\geq 0}$. Lemma 5.7, which is just an analogue of the Bass construction, is an inductive argument which at no point uses any facts about the negative homotopy groups of GW. The exact sequence involving GW in the proof of Lemma 5.7 is just a formal consequence of the definition of W and W_σ (more precisely, that they're pushouts of presheaves of spectra), and remains exact replacing GW with $GW_{\geq 0}$. Finally, the proof of Lemma 5.11 holds without modification when we replace each instance of GW by $GW_{\geq 0}$. □

6 cdh Descent for Homotopy Hermitian K-theory

Recall from Definition 2.16 that the cdh topology is the topology generated by the Nisnevich and abstract blow-up squares. Fix a Noetherian scheme of

finite Krull dimension S, and a scheme X over S.
Let $\mathrm{H}^{C_2}(S)$ denote the motivic ∞-category on $\mathbf{Sm}^{C_2}_{S,qp}$. Just as in [Hoy16] section 5, we let $\underline{\mathrm{H}}$ and $\underline{\mathrm{SH}}$ denote the "big" versions of H^{C_2} and SH^{C_2}: they can be identified with the ∞-categories of sections of $\mathbf{Sp}(\mathrm{H}^{C_2}(-))$ and $\mathrm{SH}^{C_2}(-)$ over $\mathbf{Sch}^{C_2}_{S,qp}$ that are cocartesian over smooth morphisms. By the results of the previous section, homotopy Hermitian K-theory, $L_{\mathbb{A}^1}\mathbb{G}W$, is a Bott-periodic E_∞-algebra in $\mathbf{Sp}(\underline{\mathrm{H}})$, and thus by [Hoy16, Proposition 3.2], there is a unique Bott periodic E_∞-algebra $\underline{\mathbf{KR}}^{\mathrm{alg}}$ in $\underline{\mathrm{SH}}$ such that $\Omega^\infty\underline{\mathbf{KR}}^{\mathrm{alg}} \simeq L_{\mathbb{A}^1}\mathbb{G}W$. More explicitly, by Porism 5.13, we can write $\underline{\mathbf{KR}}^{\mathrm{alg}}$ as the image under the localization functor

$$QL_{\mathrm{mot}} : \mathrm{Stab}^{\mathrm{lax}}_{T^\rho}\mathbf{Sp}(\mathcal{P}(\mathbf{Sch}^{C_2}_{S,qp})) \to \mathrm{Stab}_{T^\rho}\mathbf{Sp}(\underline{\mathrm{H}}) \simeq \underline{\mathrm{SH}}$$

of the "constant" T^ρ-spectrum $c_{\beta'\otimes(\beta^\sigma)'}GW_{\geq 0}$, where the maps $T^\rho \wedge GW_{\geq 0} \to GW_{\geq 0}$ are induced by adjunction after applying $\mathbf{Hom}_{GW_{\geq 0}-mod}(-, GW_{\geq 0})$ to the map

$$\beta' \otimes (\beta^\sigma)' : T^\rho \to GW_{\geq 0}$$

with $\beta' \otimes (\beta^\sigma)'$ the map (10) restricted to the connective cover.

DEFINITION 6.1. For $X \in \mathbf{Sch}^{C_2}_{S,qp}$, let $\mathbf{KR}^{\mathrm{alg}}_X$ denote the restriction of $\underline{\mathbf{KR}}^{\mathrm{alg}}$ to $\mathbf{Sm}^{C_2}_{X,qp}$. Note that this agrees with the notation of Theorem 5.10.

We want to show that $L_{\mathbb{A}^1}\mathbb{G}W$ is a cdh sheaf on $\mathbf{Sch}^{C_2}_{S,qp}$. By first checking that the formalism of six operations holds in equivariant motivic homotopy theory and following the same recipe as the K-theory case, [Hoy17, Corollary 6.25] proves that it suffices to show that for each $f : D \to X \in \mathbf{Sch}^{C_2}_{S,qp}$, the restriction map

$$f^*(\mathbf{KR}^{\mathrm{alg}}_X) \to \mathbf{KR}^{\mathrm{alg}}_D$$

in $\mathrm{SH}^{C_2}(D)$ is an equivalence. We show this now.
By [Sch17, Appendix A], there's a map

$$\mathrm{Herm}(X)^+ \to \Omega^\infty GW(X)$$

where $\mathrm{Herm}(X)$ is the E_∞ space of non-degenerate Hermitian vector bundles over X and $(-)^+$ denotes group completion. If X is an affine C_2-scheme, the category of vector bundles is a split exact category with duality, and the above map is an equivalence. It follows that

$$\mathrm{Herm}^+ \to \Omega^\infty GW|\mathbf{Sm}^{C_2}_{X,qp}$$

is a motivic equivalence in $\mathcal{P}(\mathbf{Sm}^{C_2}_{X,qp})$.
Just as in [Hoy16] we note that

$$\coprod_{n\geq 0} B_{isoEt}O(\langle 1\rangle^{\perp n}) \to \mathrm{Herm}$$

exhibits Herm as the equivariant Zariski sheafification of the subgroupoid of non-degenerate Hermitian vector bundles of constant rank (in other words, it "corrects" the sections over non-connected or hyperbolic rings). Since L_{Zar} preserves finite products, by [Hoy16, Lemma 5.5], the map remains a Zariski equivalence after group completion yielding a motivic equivalence

$$\left(\coprod_{n\geq 0} B_{isoEt}O(\langle 1\rangle^{\perp n})\right)^{+} \to \Omega^{\infty} GW|\mathbf{Sm}^{C_2}_{X,qp}.$$

Fix a map $f : D \to X$ in $\mathbf{Sch}^{C_2}_{S,qp}$. Again by [Hoy16, Lemma 5.5], since the pullback $f^* : \mathcal{P}(\mathbf{Sch}^{C_2}_{X,qp}) \to \mathcal{P}(\mathbf{Sch}^{C_2}_{X,qp})$ preserves finite products, it commutes with group completion of E_∞-monoids. The same is true for L_{mot}. It follows that there are motivic equivalences

$$f^*(\Omega^{\infty} GW|\mathbf{Sm}^{C_2}_{X}) \to f^*\left(\coprod_{n\geq 0} B_{isoEt}O(\langle 1\rangle^{\perp n})\right)^{+} \to \left(\coprod_{n\geq 0} f^*B_{isoEt}O(\langle 1\rangle^{\perp n})\right)^{+}$$

Because $B_{isoEt}O(\langle 1\rangle^{\perp n})$ is representable by the results of Section 4, [Hoy16, Proposition 2.9] yields a motivic equivalence

$$\left(\coprod_{n\geq 0} f^*B_{isoEt}O(\langle 1\rangle^{\perp n})\right)^{+} \to \left(\coprod_{n\geq 0} B_{isoEt}f^*O(\langle 1\rangle^{\perp n})\right)^{+}.$$

But $f^*O(\langle 1\rangle^{\perp n})|\mathbf{Sm}^{C_2}_{X,qp} = O(\langle 1\rangle^{\perp n})|\mathbf{Sm}^{C_2}_{D,qp}$ since $f^*\langle 1\rangle_X = \langle 1\rangle_D$. It follows that there's a motivic equivalence

$$\left(\coprod_{n\geq 0} B_{isoEt}f^*O(\langle 1\rangle^{\perp n})\right)^{+} \to \Omega^{\infty} GW|\mathbf{Sm}^{C_2}_{D,qp},$$

and combining everything we get that the restriction map

$$f^*(\Omega^{\infty} GW|\mathbf{Sm}^{C_2}_{X,qp}) \to \Omega^{\infty} GW|\mathbf{Sm}^{C_2}_{D,qp}$$

is a motivic equivalence in the ∞-category of grouplike E_∞-monoids in $\mathcal{P}(\mathbf{Sm}^{C_2}_{D,qp})$. Moving to the category of connective spectra, It follows that pullback agrees with restriction for $GW_{\geq 0}$. Because the localization functor QL_{mot} is also compatible with the base change f^*, it follows that each arrow

$$f^*(QL_{\mathrm{mot}}c_{\beta'\otimes(\beta^\sigma)'}GW_{\geq 0}|\mathbf{Sm}^{C_2}_{X,qp}) \to QL_{\mathrm{mot}}(f^*c_{\beta'\otimes(\beta^\sigma)'}GW_{\geq 0}|\mathbf{Sm}^{C_2}_{X,qp})$$
$$\to QL_{\mathrm{mot}}(c_{\beta'\otimes(\beta^\sigma)'}GW_{\geq 0}|\mathbf{Sm}^{C_2}_{D,qp})$$

is a motivic equivalence. Finally, Porism 5.13 tells us that $c_{\beta'\otimes(\beta^\sigma)'}GW_{\geq 0}|\mathbf{Sm}^{C_2}_{X,qp} \simeq \mathbf{KR}^{\mathrm{alg}}_X$, so we've proved

THEOREM 6.2. *Let S be a Noetherian scheme of finite Krull dimension with an ample family of line bundles and $\frac{1}{2} \in S$. Then the homotopy Hermitian K-theory spectrum of rings with involution $L_{\mathbb{A}^1}\mathbb{G}W$ satisfies descent for the equivariant cdh topology on $\mathbf{Sch}^{C_2}_{S,qp}$.*

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