Ratio Limits and Martin Boundary

Wolfgang Woess

Received: April 19, 2021
Revised: October 1, 2021

Communicated by Roland Speicher

Abstract. Consider an irreducible Markov chain which satisfies a ratio limit theorem, and let ρ be the spectral radius of the chain. We investigate the relation of the the ρ-Martin boundary with the boundary induced by the ρ-harmonic kernel which appears in the ratio limit. Special emphasis is on random walks on non-amenable groups, specifically, free groups and hyperbolic groups.

2020 Mathematics Subject Classification: 60J50, 60G50
Keywords and Phrases: Random walk, ratio limit, ρ-harmonic functions, Martin boundary

1 Introduction

Let $\mathcal{X}$ be the denumerable state space of a time-homogeneous Markov chain $(X_n)_{n\geq 0}$ with transition matrix P. We assume that it is *irreducible and aperiodic:* for any pair of points $x, y \in \mathcal{X}$, there is $k_{x,y} \in \mathbb{N}$ such that $p^{(n)}(x,y) > 0$ for all $n \geq k_{x,y}$, where $p^{(n)}(x,y)$ denotes the n-step transition probability from x to y. We will often (but not always) also assume that P has *finite range,* that is, for each $x \in \mathcal{X}$, there are only finitely many y with $p(x,y) > 0$. (This simplifies parts of the technicalities.) We choose and fix a *root* $e \in \mathcal{X}$. In case of a group, this will be the identity.

In a variety of cases, part of which will be displayed below, one knows that a *ratio limit theorem* holds, that is, there is a function (kernel) $h : \mathcal{X}^2 \to (0,\infty)$ such that

$$\lim_{n\to\infty} \frac{p^{(n)}(x,y)}{p^{(n)}(e,e)} = h(x,y) \quad \text{for all } \; x,y \in \mathcal{X}\,. \tag{1.1}$$

Now let $\rho = \rho(P) = \limsup p^{(n)}(x,y)^{1/n}$ be the so-called *spectral radius* of the Markov chain. It is independent of x and y by irreducibility. If (1.1) holds then in a large variety of cases (including finite range) one deduces that

$$Ph = hP = \rho \cdot h \tag{1.2}$$

in the sense of matrix products. In particular, for any fixed y, the function $x \mapsto h(x,y)$ is a positive ρ*-harmonic* function: more generally, for real $t \geq \rho$, a function $f : \mathcal{X} \to \mathbb{R}$ is t-harmonic, if

$$\sum_w p(x,w) f(w) = t\, f(x) \quad \text{for all} \;\; x \in \mathcal{X}. \tag{1.3}$$

When $t = 1$, one just speaks of a harmonic function. Let us write $\mathcal{H}^+(P,t)$ for the convex cone of positive t-harmonic functions.

For each x, we have $h(x,y) \leq C_x\, h(e,y)$ for all y, where $1/C_x = \rho^k\, p^{(k)}(e,x)$ with the smallest k such that this is non-zero. Therefore, we can consider the normalised *ratio limit kernel*

$$H(x,y) = h(x,y)/h(e,y),$$

and the function $y \mapsto H(x,y)$ is bounded by C_x for each y.

Definition 1. The *ratio limit compactification*[1] $\boldsymbol{\Delta}_{\rm ratio}(\mathcal{X})$ of $\mathcal{X}$ is the (up to homomorphism) unique compact Hausdorff space which contains $\mathcal{X}$ as a discrete, dense subset and has the following properties:

- each function $H(x,\cdot)$ extends continuously to $\boldsymbol{\Delta}_{\rm ratio}(\mathcal{X})$, and denoting the extended kernel also by H,
- if $\xi, \eta \in \mathcal{R}(\mathcal{X}) = \boldsymbol{\Delta}_{\rm ratio}(\mathcal{X}) \setminus \mathcal{X}$ are distinct, then there is $x \in \mathcal{X}$ such that $H(x,\xi) \neq H(x,\eta)$.

For the general construction of such a compactification, within the present context see e.g. [30, Thm. 7.13]. At least when P has finite range, each function $x \mapsto H(x,\xi)$, $\xi \in \boldsymbol{\Delta}_{\rm ratio}(\mathcal{X})$, is ρ-harmonic.

In relation with his current work on Cuntz algebras related to random walks, Adam Dor-On, in an exchange that lead to the "companion" paper [11], has asked how one can describe this compactification in terms of a given structure of the underlying state space, and how it relates with the ρ*-Martin compactification.* His questions were motivated by operator algebraic issues, and the answers provided here contribute to clarifying some of them. The Martin boundary will be recalled in the next section.

The main interest is in the case when $\mathcal{X} = \Gamma$ is a finitely generated, infinite group, and the Markov chain is a random walk induced by a finitely supported probability measure μ on Γ, that is,

$$p(x,y) = \mu(x^{-1}y)\,.$$

[1] [11] calls this the *full* ratio limit compactification.

Irreducibility & aperiodicity then amount to the property that the support of μ generates Γ as a semi-group and that it is not contained in a coset of a proper normal subgroup of Γ. Given the first of those two properties, the second will certainly hold if $\mu(e) > 0$, where e is the group identity. (Assuming this is not a crucial restriction.)
If in this case we have a ratio limit theorem, then clearly $h(gx, gy) = h(x, y)$ for all $x, y, g \in \Gamma$, so that the function $f(x) = h(x, e)$ is ρ-harmonic, and $h(x, y) = f(y^{-1}x)$. In this situation, the ratio limit boundary is a Γ-space, and the elements of Γ act continuously on $\mathbf{\Delta}_{\mathrm{ratio}}(\Gamma)$.

For random walks on groups, a famous ratio limit theorem is due to Avez:

Theorem 1. [2] *If the group Γ is amenable and μ is symmetric, i.e., $\mu(x^{-1}) = \mu(x)$, then*

$$\lim_{n\to\infty} \frac{p^{(n)}(x,y)}{p^{(n)}(e,e)} = 1 \quad \textit{for all} \;\; x, y \in \Gamma\,.$$

(Finite support is not needed here.) In this case, we see that $\mathbf{\Delta}_{\mathrm{ratio}}(\mathcal{X})$ is the one-point compactification. We mention here that there are symmetric random walks on amenable groups where the Martin boundary (and even the minimal one; see §2) is infinite. For example, for random walks on lamplighter groups over $\mathbb{Z}^d$ $(d \geq 3)$, the Poisson boundary is non-trivial, see Kaimanovich and Vershik [21], whence there are non-constant minimal harmonic functions. This is even true for certain lamplighter random walks over $\mathbb{Z}$, see [31].
For non-amenable groups, there is a variety of results which go beyond ratio limit theorems, namely, *local limit theorems* which provide an asymptotic evaluation of the n-step transition probabilities. Typically (but not necessarily), they are of the form

$$p^{(n)}(x,y) \sim C\;\beta(x,y)\,\rho^n\,n^{-\alpha} \quad \text{as} \;\; n \to \infty\,, \tag{1.4}$$

where $\sim$ means that the quotient of the left and right hand sides tends to 1, and $C, \beta(x,y), \alpha > 0$. For results up to 2000, see [29, Ch. III] and the references therein. The first of those results are due to Gerl [15] and Sawyer [27], concerning random walks on free groups, resp. regular trees. The latest results are by Gouëzel [17], [18] and Dussaule [12], for hyperbolic, resp. relatively hyperbolic groups. In between, there is a rather large body of work.

Whenever one has (1.4), this yields the ratio limit theorem with $h(x, y) = \beta(x, y)/\beta(e, e)$, and the kernel for the ratio limit compactification is $H(x, y) = \beta(x, y)/\beta(e, y)$, where e is the group identity.

In the present paper, we exhibit classes of random walks on non-amenable groups where the ratio limit compactifictation coincides with the ρ-Martin compactification, and one has a geometric description of the latter. Recall that $\rho < 1$ on those groups by Kesten [22].
After laying out some necessary preliminaries in §2, in §3 we consider free groups and regular trees, and instead of stating and proving right away the

most general result for general finite range random walks, we proceed step by step. Subsequently, in §4, we consider hyperbolic groups. In §5, we discuss how the ratio limit compactification interacts with taking direct or Cartesian products. In §6, we discuss the *reduced* ratio limit boundary and show that in the cases of §3, there is no reduction, while for hyperbolic groups, reduction may come only from a (finite) torsion subgroup. The final §7 contains a brief discussion and an outlook on work to be done in the future.
It is on purpose that three different methods are displayed for (1) isotropic random walks on trees, (2) finite range random walks on free groups, and (3) symmetric random walks on hyperolic groups (with finite range). In spite of the fact that the most modern approach is based on the cited work of Gouëzel [17], the older methods regarding (1), going back to Sawyer [27], resp. (2), going back to Derriennic [10] plus Lalley [23], are still very valid in the author's view. Also, for the time being, (3) does not cover (1) and (2) completely.

Acknowledgment

The author thanks Adam Dor-On for posing the question, for interesting exchanges on the subject, as well as for his comments on drafts of this paper. Also, the author is extremely grateful to Sébastien Gouëzel for helpful input that facilitated the elaboration of the proof of Theorem 12. Thanks also to the referee for very useful input. The author was supported by Austrian Science Fund projects FWF P31889-N35 and W1230.

2 Preliminaries; the Martin compactification

A. The t-Martin compactification

We let $t \geq \rho$ (focusing on the case when $t = \rho$). To describe the t-Martin compactification, consider for real $z > 0$ the *Green function*

$$G(x,y|z) = \sum_{n=0}^{\infty} p^{(n)}(x,y)\, z^n\,, \quad x,y \in \mathcal{X}\,.$$

The radius of convergence of this power series is $r = 1/\rho$. At that value, the series may either converge or diverge for all x, y. The latter is the *ρ-recurrent* case, the former the *ρ-transient* one. The quotient

$$F(x,y|z) = \frac{G(x,y|z)}{G(y,y,|z)} = \sum_{n=0}^{\infty} f^{(n)}(x,y)\, z^n$$

is also a generating function: $f^{(n)}(x,y)$ is the probability that the Markov chain starting at x first reaches y at time $n \geq 0$. It is known [30, Lemma 3.66] that $F(x,y|1/\rho)$ is always finite. In the case of a random walk on a group, it is known that one always has the ρ-transient case, except when that group is virtually $\mathbb{Z}$ or $\mathbb{Z}^2$, see e.g. Woess [29, Thm. 7.8]. For real $t \ge \rho$, the *t-Martin kernel* is

$$K(x,y|t) = \frac{F(x,y|1/t)}{F(e,y|1/t)}.$$

Here we shall mainly work with $t = \rho$. For fixed y, the function $x \mapsto K(x,y|\rho)$ satisfies (1.3) at every $x \in \mathcal{X} \setminus y$. Furthermore, for fixed x, we have $K(x,\cdot\, t) \le C_x$ with the same bound as above. Thus, we can construct the compactification $\mathbf{\Delta}_{\mathrm{Mart},\,t}(\mathcal{X})$ of $\mathcal{X}$, analogous to the one of Definition 1, for the t-Martin kernel $K(x,y|t)$ in the place of $H(x,y)$. This is the t-Martin compactification, and $\mathcal{M}_t(\mathcal{X}) = \mathbf{\Delta}_{\mathrm{Mart},\,t}(\mathcal{X}) \setminus \mathcal{X}$ is the *t-Martin boundary.* Each function $x \mapsto K(x,\xi|t)$, $\xi \in \mathcal{M}_t(\mathcal{X})$, is t-harmonic. The significance of $\mathcal{M}_t(\mathcal{X})$ lies in the fact that every positive t-harmonic function f on $\mathcal{X}$ has an integral representation

$$f(x) = \int_{\mathcal{M}_t(\mathcal{X})} K(x,\cdot\, t)\, d\nu^f\,,$$

where ν is a Borel measure on $\mathcal{M}_t(\mathcal{X})$. That measure is unique, if we require that $\nu^f\big(\mathcal{M}_t(\mathcal{X}) \setminus \mathcal{M}_{t,\min}(\mathcal{X})\big) = 0$, where $\mathcal{M}_{t,\min}(\mathcal{X})$ is the *minimal boundary:* a positive t-harmonic function f is called *minimal,* if $f(e) = 1$ and it cannot be written as a convex combination of two distinct positive t-harmonic functions with value 1 at e. The minimal boundary is $\{\xi \in \mathcal{M}_t(\mathcal{X}) : K(\cdot,\xi|t) \text{ is minimal}\}$; it consists of all minimal t-harmonic functions.

The finite range assumption simplifies a few things (for example, it yields that all extended Martin kernels are t-harmonic in the first variable), but is not needed for the construction and properties. Also, aperiodicity is not needed, irreducibility is sufficient, that is, for every pair $x,y \in \mathcal{X}$ there is $k = k_{x,y}$ such that $p^{(k)}(x,y) > 0$. (It even suffices to have this only for $x = e$ and every y.)

For all these facts, see e.g. [29, §24] or the references provided there. The largest part of the literature is on the ordinary 1-Martin compactification. There is a simple tool to pass from the t-Martin compactification to the latter. Namely, take one positive t-harmonic function f and use the *Doob transform:* define new transition probabilities by

$$p^f(x,y) = \frac{p(x,y)f(y)}{t\, f(x)} \tag{2.1}$$

Then the t-Martin compactification of the original Markov chain is the 1-Martin compactification of the Doob transform, and $K_{P^f}(x,\cdot\, 1) = K(x,\cdot\, t)/f(x)$.

B. Comparing compactifications

Let $\mathbf{\Delta}(\mathcal{X})$ and $\mathbf{\Delta}'(\mathcal{X})$ be two compactifications of the countable set $\mathcal{X}$, that is, compact Hausdorff spaces which contain $\mathcal{X}$ as open and dense subsets. Then we say that the former is *bigger* than the latter, if the identity map on $\mathcal{X}$ extends to a continuous surjection $\mathbf{\Delta}(\mathcal{X}) \to \mathbf{\Delta}'(\mathcal{X})$. Clearly, if also $\mathbf{\Delta}'(\mathcal{X})$ is bigger than $\mathbf{\Delta}(\mathcal{X})$ then that continuous mapping is a homeomorphism. In this case, we say that the two compactifications coincide *geometrically.*

In typical cases, one has a "natural" geometric compactification of $\mathcal{X}$ and then wants to show that it coincides geometrically with the Martin compactification, or that it is smaller than the Martin compactification. In particular, we want to compare the ratio limit compactification with the ρ-Martin compactification. In its geometric, resp. topological interpretation, this will mean that the latter is bigger than the former, or that they coincide in the above sense.
However, this is not the full picture; it does not yet fully reflect its analytic properties. One should also ask whether in that case the homoeomorphism $\tau : \mathbf{\Delta}_{\mathrm{Mart},\,\rho}(\mathcal{X}) \to \mathbf{\Delta}_{\mathrm{ratio}}(\mathcal{X})$, in case it exists, is such that

$$K(x,\xi|\rho) = H\big(x,\tau(\xi)\big) \quad \text{for all} \;\; x \in \mathcal{X}, \xi \in \mathcal{M}_\rho(\mathcal{X})\,. \tag{2.2}$$

When we have two geometrically equal compactifications constructed via two different kernels, let us say that they coincide *analytically,* when the extended kernels coincide on the boundary. ("Coincide" is of course meant in terms of a homeomorphism as above.)
No matter whether the ratio limit and ρ-Martin compactifications coincide geometrically or not: whenever ν is a Borel measure on the ratio limit boundary $\mathcal{R}$, the function

$$f(x) = \int_{\mathcal{R}} H(x,\cdot)\, d\nu$$

is positive ρ-harmonic, and the above question basically amounts to asking whether *every* positive ρ-harmonic function arises in this way.
In general, it is for example known that for any two finite range, irreducible random walks on a free group, the Martin compactifications coincide geometrically, but in general by no means analytically.

Proposition 2. [25, Prop. 1] *Let P, Q be two irreducible transition operators on $\mathcal{X}$ such that every positive harmonic function for P is also harmonic for Q. Suppose that*

- *the 1-Martin compactifications of P and Q coincide geometrically, and*
- *the 1-Martin boundary of P is Dirichlet-regular, that is, every continuous function on the boundary has a continuous extension to the entire compactification which is 1-harmonic for P on $\mathcal{X}$.*

Then the two compactifications coincide analytically, so that the positive 1-harmonic functions for P and Q coincide.

We shall apply this below.

3 Trees and free groups

Let $\mathbb{T}$ be the regular tree where each vertex has $q+1$ neighbours, with $q \geq 2$. If there is an edge between the vertices $x, y \in \mathbb{T}$ then we write $x \sim y$. For any pair of vertices x, y, there is a unique finite geodesic $\pi(x,y) = [x =$

$x_0\,, x_1\,, \dots, x_k = y]$ connecting the two, that is, $x_{i-1} \sim x_i$, and all x_i are distinct. The distance between x and y is then $d(x,y) = k$. An infinite *ray* is a one-sided infinite geodesic path $\pi = [x_0\,, x_1\,, x_2\,, \dots]$, that is, $x_k \sim x_{k-1}$ and all x_k are distinct. Two rays π and π' are called *equivalent,* if they differ only by finitely many initial vertices. An *end* of $\mathbb{T}$ is an equivalence class of rays. The *geometric boundary* $\partial\mathbb{T}$ is the set of all ends. For any $x \in \mathbb{T}$ and $\xi \in \partial\mathbb{T}$, there is a unique ray $\pi(x,\xi)$ starting at x which represents ξ. We set $\boldsymbol{\Delta}_{\text{ends}}(\mathbb{T}) = \mathbb{T} \cup \partial\mathbb{T}$.

We choose and fix a root vertex e. For $x \in \mathbb{T}$, we let $|x| = d(e,x)$. We define the *branch* of $\boldsymbol{\Delta}_{\text{ends}}(\mathbb{T})$ at x as

$$\boldsymbol{\Delta}_{\text{ends}}(\mathbb{T}_x) = \{w \in \boldsymbol{\Delta}_{\text{ends}}(\mathbb{T}) : x \in \pi(e,w)\}.$$

The topology on $\boldsymbol{\Delta}_{\text{ends}}(\mathbb{T})$ is discrete on $\mathbb{T}$, while a neighbourhood basis of $\xi \in \partial\mathbb{T}$ is given by the collection of all $\boldsymbol{\Delta}_{\text{ends}}(\mathbb{T}_x)$, where $x \in \pi(e,\xi)$. Each of those sets is open and compact. This turns $\boldsymbol{\Delta}_{\text{ends}}(\mathbb{T})$ into a totally disconnected, compact Hausdorff space in which $\mathbb{T}$ is open and dense. It can be metrisized as follows. For *distinct* $v, w \in \boldsymbol{\Delta}_{\text{ends}}(\mathbb{T})$, their *confluent* $v \wedge w$ is the vertex on $\pi(e,v) \cap \pi(e,w)$ furthest from the root e. Then

$$\vartheta(v,w) = \begin{cases} q^{-|v\wedge w|}\,, & \text{if } v \neq w,\\ 0\,, & \text{if } v = w \end{cases}$$

is an ultrametric which induces the above topology. Below, we shall also need the *Busemann function* or *horocycle index* with respect to an end $\xi \in \partial\mathbb{T}$. For a vertex $x \in \mathbb{T}$, this is

$$\mathfrak{h}(x,\xi) = d(x, x\wedge\xi) - |x\wedge\xi| = \lim_{\mathbb{T}\ni y\to\xi} d(x,y) - |y|.$$

A. Isotropic random walks on $\mathbb{T}$

A Markov chain with transition matrix P on $\mathbb{T}$ is called an *isotropic random walk,* if $p(x,y)$ depends only on $d(x,y)$. For $d \in \mathbb{N}$, let P_d be the stochastic transition matrix with entries

$$p_d(x,y) = \begin{cases} \dfrac{1}{(q+1)q^{d-1}}\,, & \text{if } d(x,y) = d\,,\\ 0\,, & \text{otherwise.} \end{cases}$$

Also, we set $P_0 = I$. For each d, there is a polynomial $\widehat{P}_d(t)$ of degree d such that $P_d = \widehat{P}_d(P_1)$. If P is isotropic then it can be written as a (possibly infinite) convex combination $P = \sum_{d=0}^{\infty} a_d\, P_d$. In order to guarantee irreducibility & aperiodicity, we assume that $a_d > 0$ for some odd and some even d. We now refer to the results explained in [29, §19.C], due to [27]. The *spherical transform* of P is

$$\widehat{P}(t) = \sum_{d=0}^{\infty} a_d\, \widehat{P}_d(t)\,.$$

Thus, $P = \widehat{P}(P_1)$. It is very well known that

$$\rho(P_1) = \frac{2\sqrt{q}}{q+1} \quad \text{and} \quad \rho = \rho(P) = \widehat{P}\big(\rho(P_1)\big). \tag{3.1}$$

Let

$$\varphi(n) = \Big(1 + \frac{q-1}{q+1} n\Big) q^{-n/2}\,, \quad n \in \mathbb{N}_0\,. \tag{3.2}$$

Set $\varPhi(x,y) = \varphi\big(d(x,y)\big)$. Thus, $\varPhi(x) = \varPhi(x,e)$ is the *spherical function* which satisfies $P_1 \varPhi(x) = \rho(P_1)\, \varPhi(x)$. ("Spherical" means that it is an eigenfunction of P_1 with value 1 at e that depends only on $|x|$.) We see that also $P\varPhi(x,y) = \rho\, \varPhi(x,y)$. The local limit theorem of [27] says that

$$p^{(n)}(x,y) \sim C\, \varPhi(x,y)\, \rho^n\, n^{-3/2}\,, \quad \text{as} \quad n \to \infty\,.$$

Thus, we get

$$H(x,y) = \frac{\varPhi(x,y)}{\varPhi(e,y)}.$$

Theorem 3. *For isotropic P as above, suppose that it has super-exponential moments:* $\limsup_{d\to\infty} a_d^{1/d} = 0$. *Then the ratio limit compactification coincides with the ρ-Martin compactification analytically. Geometrically, this is the end compactification, and for $\xi \in \partial\mathbb{T}$,*

$$K(x,\xi|\rho) = H(x,\xi) = q^{-\mathfrak{h}(x,\xi)/2}.$$

Proof. It is known from [18, Thm. 1.3] that under the super-exponential moment condition the ρ-Martin compactification coincides geometrically with the end compactification.

From (3.2), we compute for any end $\xi \in \partial\mathbb{T}$

$$\lim_{y\to\xi} H(x,y) = \lim_{y\to\xi} \frac{1 + \frac{q-1}{q+1} d(x,y)}{1 + \frac{q-1}{q+1}|y|} q^{-d(x,y)/2 + |y|/2} = q^{-\mathfrak{h}(x,\xi)/2}.$$

This shows that the ratio limit compactification coincides geometrically with the end compactification. Thus, what is left to show is that

$$K(x,y|\rho) = q^{-\mathfrak{h}(x,\xi)/2}. \tag{3.3}$$

For this purpose, we use Proposition 2. Since $P = \widehat{P}(P_1)$, equation 3.1 implies that $\mathcal{H}^+\big(P_1, \rho(P_1)\big) \subset \mathcal{H}^+(P,\rho)$. The Green function for P_1 is very well known, see e.g. [29, Lemma 1.12]. In particular,

$$G_{P_1}\big(x,y|1/\rho(P_1)\big) = \frac{2q}{q-1} q^{-d(x,y)/2},$$

so that (3.3) holds for P_1, whose Martin compactification is of course again the end compactification of $\mathbb{T}$.

The spherical function $\Phi(x) = \Phi(x,e)$ is in $\mathcal{H}^+\big(P_1, \rho(P_1)\big)$. We now consider the Doob transforms $Q_1 = P_1^\Phi$ and $Q = P^\Phi$, that is, their respective matrix elements are

$$q_1(x,y) = \frac{p_1(x,y)\,\Phi(y)}{\rho(P_1)\;\Phi(x)} \quad \text{and} \quad q(x,y) = \frac{p(x,y)\,\Phi(y)}{\rho\;\Phi(x)}$$

Then $f \in \mathcal{H}(P,\rho)$ if and only if $f/\Phi \in \mathcal{H}(Q,1)$, and analogously for P_1 and Q_1. This implies that $\mathcal{H}\big(Q_1,1\big) \subset \mathcal{H}(Q,1)$. In view of Proposition 2, we need to verify that the 1-Martin boundary $\partial\mathbb{T}$ is Dirichlet regular for Q_1. By Cartwright et al. [8], for this it is necessary and sufficient that the Green kernel of Q_1 vanishes at infinity, that is,

$$\lim_{|x|\to\infty} G_{Q_1}(x,e|1) = 0.$$

Now, $G_{Q_1}(x,y|1) = G_{P_1}\big(x,y|1/\rho(P_1)\big)\,\Phi(y)/\Phi(x)$. Thus,

$$G_{Q_1}(x,e|1) = \frac{2q}{q-1}\bigg/\Big(1+\frac{q-1}{q+1}|x|\Big) \to 0\,, \quad \text{as } \ |x|\to\infty\,.$$

This concludes the proof. □

Remark. It seems likely that the super-exponential moment condition may be relaxed here. Cartwright and Sawyer [6] have shown that $\mathcal{H}^+(P,1) = \mathcal{H}^+(P_1\,,1)$ for arbitrary isotropic P, as long as it is irreducible. However, the 1-Martin compactification is the end compactification (and thus coincides analytically with the one of P_1) only under additional hypotheses, such as first moment, i.e., $\sum_d d\,a_d < \infty$. Otherwise, the 1-Martin boundary can contain further, non-minimal elements. [6] contains no analogous result at the critical value ρ, where also the Martin boundary might behave more "critically". On the other hand, isotropic random walks on $\mathbb{T}$ are a rather special case, where more general results may hold.

B. Nearest neighbour random walk on free groups

Let $\mathbb{F} = \mathbb{F}_s$ be the free group on s free generators $a_1\,,\dots,a_s$, and write $a_{-i} = a_i^{-1}$. Let $I = \{\pm1,\dots,\pm s\}$ and $S = \{a_i : i \in I\}$. Recall that every element x of $\mathbb{F}$ can be written as a reduced word over S,

$$x = a_{i_1}a_{i_2}\cdots a_{i_k}\,, \quad i_l \in I\,,\ i_l \neq -i_{l-1}\,. \tag{3.4}$$

The length $|x|$ of x is k, and when $k=0$, this is the empty word e, which is the group identity. The Cayley graph of $\mathbb{F}$ with respect to S is the tree $\mathbb{T} = \mathbb{T}_{2s-1}$: its vertex set is $\mathbb{F}$, and $x,y \in \mathbb{F}$ are connected by an edge whenever $x^{-1}y \in S$. Thus, the natural geometric compactification $\widehat{F}$ of $\mathbb{F}$ is the end compactification of $\mathbb{T}$, with boundary $\partial\mathbb{F} = \partial\mathbb{T}$.

We now let μ be a probability measure on $\mathbb{F}$ whose support is $\mathsf{supp}(\mu) = \{e\}\cup S$. This nearest neighbour case serves as a warm-up for the next sub-section;

it might be skipped but may be instructive. The local limit theorem and the involved ρ-harmonic function have been studied in detail by Gerl and Woess [16]. One has (1.4) with $\alpha = 3/2$, and $\rho < 1$ by non-amenability of $\mathbb{F}$. We subsume those facts which are needed here. (The notation is slightly modified.) By group invariance, $G(x,y|z) = G(e,x^{-1}y|z)$. The function $G(z) = G(x,x|z)$ is solution of an implicit equation, which leads to a formula for ρ. Set $F_i(z) = F(e,a_i|z)$ for $i \in I$. If $x \in \mathbb{F}$ has the reduced representation (3.4) then

$$F(e,x|z) = F_{i_1}(z) \cdots F_{i_k}(z).$$

The ends of the tree $\mathbb{T}$ which is the Cayley graph of $\mathbb{F}$ can be written as infinite words

$$\xi = a_{j_1} a_{j_2} a_{j_3} \cdots, \quad j_l \in I\,,\ j_l \neq -j_{l-1}\,, \tag{3.5}$$

and the n^{th} vertex on the geodesic $\pi(e,\xi)$ is $x_n = a_{j_1} a_{j_2} \cdots a_{j_n}$. For any $t \geq \rho$, the t-Martin compactification is $\partial\mathbb{T}$. This goes back to Dynkin and Malyutov [13] and Cartier [4].

With ξ as above, if $x \in \mathbb{F}$ has reduced representation (3.4), then there is a maximal index $m = m(x,\xi) \leq k$ such that $j_1 = i_1\,, \dots, j_m = i_m$. Then $x_m = x \wedge \xi$ in the above description of confluents in the geometry of the tree. Then the Martin kernel at ξ is

$$K(x,\xi|t) = \frac{F_{-i_k}(1/t) F_{-i_{k-1}}(1/t) \cdots F_{-i_{m+1}}(1/t)}{F_{i_1}(1/t) F_{i_2}(1/t) \cdots F_{i_m}(1/t)}$$

It is always minimal. The analysis of [16] yields that in a neighbourhood of the principal singularity $r = 1/\rho$, for $z \in \mathbb{C} \setminus [r\,,\infty)$, one has Puiseux series expansions of the form

$$G(z) = \alpha_0 - \beta_0\sqrt{r-z} + h.o.t. \quad \text{and} \quad F_i(z) = \alpha_i - \beta_i\sqrt{r-z} + h.o.t.\,, \tag{3.6}$$

where $\alpha_i, \beta_i > 0$ for $i \in I \cup \{0\}$, and *h.o.t.* stands for series of "higher order terms" of the form $C\cdot(r-z)^q$, where C is a constant and q is a rational number with $q > 1/2$, and the appearing exponents form a discrete subset of $\mathbb{Q}$. As a matter of fact, in the present case, q is always an integer multiple of $1/2$. From this, one gets with x as in (3.4), ξ as in (3.5) and $m = m(x,\xi)$ as above that

$$K(x,\xi|\rho) = \frac{\alpha_{-i_k}\alpha_{-i_{k-1}} \cdots \alpha_{-i_{m+1}}}{\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_m}}. \tag{3.7}$$

On the other hand, again for x as in (3.4), we get for z near r as above the Puiseux series expansion

$$G(e,x|z) = F_{i_1}(z) \cdots F_{i_k}(z) G(z) = \alpha(x) - \beta(x)\sqrt{r-z} + h.o.t. \quad \text{with}$$

$$\alpha(x) = \alpha_0\,\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_k}\,, \quad \beta(x) = \alpha(x)\,\gamma(x)\,, \quad \text{and} \quad \gamma(x) = \frac{\beta_0}{\alpha_0} + \sum_{l=1}^{k} \frac{\beta_{i_l}}{\alpha_{i_l}}. \tag{3.8}$$

Corollary 4. *For an aperiodic nearest neighbour random walk on the free group $\mathbb{F}$, the ratio limit kernel is $H(x,y) = \beta(x^{-1}y)/\beta(y)$. The ratio limit compactification coincides analytically with the ρ-Martin compactification, which geometrically is the end compactification $\widehat{\mathbb{F}} = \mathbf{\Delta}_{ends}(\mathbb{T})$.*

Proof. The expansion (3.8) yields the following local limit theorem, see [16, Thm. 2]:

$$p^{(n)}(e,x) \sim \frac{1}{2\sqrt{\rho\,\pi}}\,\beta(x)\,\rho^n\,n^{-3/2}\,.$$

This yields the stated form of the ratio limit kernel. We need to show that for every $x \in \mathbb{F}$ and every end ξ,

$$\lim_{y\to\xi} \frac{\beta(x^{-1}y)}{\beta(y)} = K(x,\xi|\rho).$$

Again, suppose that x is as in (3.4). Let $m = m(x,\xi) \in \{0,\dots,k\}$. We now write the reduced representation of y as $y = a_{j_1}a_{j_2}\cdots a_{j_n}$. In principle, the indices $j_1,\dots,j_n$ vary with y, but $y \to \xi$ means that $\bar m = m(y,\xi) \to \infty$, so that the initial piece $a_{j_1}a_{j_2}\cdots a_{j_{\bar m}}$ coincides with the initial word of ξ of the same length. Furthermore, we will have $\bar m \ge m(x,\xi)$ when y is close to ξ in the geometric compactification $\mathbf{\Delta}_{\text{ends}}(\mathbb{T})$. For such y, recalling that x_m is the m^{th} element on $\pi(e,x)$,

$$x^{-1}y = \underbrace{a_{-i_k}a_{-i_{k-1}}\cdots a_{-i_{m+1}}}_{x^{-1}x_m}\,\underbrace{a_{j_{m+1}}a_{j_{m+2}}\cdots a_{j_n}}_{x_m^{-1}y}\,.$$

Note that by (3.7) we then have $\alpha(x^{-1}y)/\alpha(y) = K(x,\xi|\rho)$. Therefore, when $\bar m \ge m$,

$$H(x,y) = K(x,\xi|\rho)\,\frac{\gamma(x^{-1}y)}{\gamma(y)}.$$

We then get $\gamma(x^{-1}y) - \gamma(y) = \gamma(x^{-1}x_m) - \gamma(x_m)$, while $\gamma(y) \to \infty$. Thus,

$$\frac{\gamma(x^{-1}y)}{\gamma(y)} \to 1\,, \quad \text{as} \quad y \to \xi.$$

This concludes the proof. $\square$

C. Bounded range random walk on free groups

The result of this sub-section generalises the previous one.

Theorem 5. *Suppose that the probability measure μ on $\mathbb{F}$ has finite support S which generates $\mathbb{F}$ as a semi-group and contains the group identity. Then the ratio limit compactification coincides with ρ-Martin compactification analytically. Geometrically, it is the end compactification.*

This needs some preparation. From Lalley [23], it is known that the random walk satisfies again a local limit theorem (1.4) with $\alpha = 3/2$, and $\rho < 1$ by non-amenability of $\mathbb{F}$. We shall use a mix of the methods of Derrienic [10] and its extensions by Picardello and Woess [24], and of [23], compare with [29, §19.B and §26.A].

For $n \in \mathbb{N}$, let $B_n = \{x : |x| \le n\}$ be the ball of radius n around the identity (root) with respect to the metric of the tree $\mathbb{T}$ which is the Cayley graph of the group. For any $y \in \mathbb{F}$, the set yB_n is the ball of radius n centred at y. Let $R = \max\{|x| : x \in S\}$, and let $B = B_R$. For any set $A \subset \mathbb{F}$, we consider the stopping time plus associated probability generating function

$$\mathbf{s}^A = \inf\{n \ge 0 : X_n \in yA\} \quad \text{and}$$
$$F^A(u,v|z) = \sum_{n=0}^{\infty} \mathsf{Pr}[\mathbf{s}^A = n\,,\ X_n = v \mid X_0 = u]\, z^n\,,$$

where $u, v \in \mathbb{F}$. For the simple proof of the following, see the above references.

Lemma 6. *If $x_0\,, x_1 \in \mathbb{F}$ are distinct and $y \in \pi(x_0\,, x_1)$ then the random walk starting at x_0 must pass through yB in order to reach x_1. Thus,*

$$F(x_0\,, x_1|z) = \sum_{v \in yB} F^{yB}(x_0\,, v|z) F(v, x_1|z).$$

Next, for any $A \subset \mathbb{F}$, let

$$F_A(u,u'|z) = \sum_{n=0}^{\infty} \mathsf{Pr}[\mathbf{s}^{\mathbb{F}\setminus A} > n\,,\ \mathbf{s}^{\{u'\}} = n \mid X_0 = u]\, z^n\,,$$

where $u, u' \in A$. There is $N \ge R$ such that

$$F_{B_N}(u,u'|z) > 0 \quad \text{for all}\ \ u,u' \in B = B_R \ \text{ and all }\ z \in (0\,, r]\,,$$

where, as above, $r = 1/\rho$. (This is a simple observation: there must be a sequence $u = u_0\,, u_1\,\dots, u_k = v$ such that $p(u_{i-1}\,, u_i) > 0$ for all i. We take N large enough such that for any choice of $u, v \in B$ there is such a sequence which is entirely contained in B_N.)

For $x, y \in \mathbb{F}$, we define the square matrix, resp. (column) vectors

$$\begin{gathered} \mathbf{F}(x,y|z) = \big(F^{yB}(xu, yv|z)\big)_{u,v \in B}\,, \\ \mathbf{f}(x,y|z) = \big(F^{yB}(x, yu|z)\big)_{u \in B} \quad \text{and} \quad \mathbf{g}(x,y|z) = \big(G(xu, y|z)\big)_{v \in B}. \end{gathered} \tag{3.9}$$

We now let $D = N + 2R + 1$ and consider the set $W_D = \{w \in \mathbb{F} : |w| = D\}$ of all elements (words) in $\mathbb{F}$ with length D. We observe that when $d(x,y) = D$ then $w = x^{-1}y \in W_D$ and $\mathbf{F}(x,y|z) = \mathbf{F}(e,w|z) =: \mathbf{F}(w|z)$. Then the following is a consequence of Lemma 6, see [10], [24].

Lemma 7. *Let x, y in $\mathbb{F}$ and $u_0, u_1, \ldots, u_n \in \pi(x,y)$ such that $d(u_0, x) > R$, $d(u_n, y) > R$ and $d(u_k, x) = d(u_0, x) + k\,D$, so that $w_k = u_{k-1}^{-1}u_k \in W_D$ for $k = 1, \ldots, n$. Then for $z \in (0, r]$ with $r = 1/\rho$,*

$$G(x,y|z) = \big\langle \mathbf{f}(x,u_0|z)\,,\, \underbrace{\mathbf{F}(u_0\,,u_1|z)\cdots\mathbf{F}(u_{n-1}\,,u_n|z)}_{\mathbf{F}(w_1|z)\cdots\mathbf{F}(w_n|z)}\,\mathbf{g}(u_n,y|z)\big\rangle.$$

Here, $\mathbf{F}(w_1|z)\cdots\mathbf{F}(w_n|z)\mathbf{g}(u_n,y|z)$ is the product of n square matrices applied to the column vector $\mathbf{g}(u_n,y|z)$, and $\langle\cdot,\cdot\rangle$ is the ordinary inner product of column vectors indexed by B. Now set

$$\lambda_z = \min\{F_{B_N}(u,u'|z) : u,u' \in B\}$$

Then, for all $u, u', v \in B$, $w \in W_D$ and $z \in (0, r]$,

$$F^{wB}(u,wv|z) \geq F_{B_N}(u,u'|z)F^{wB}(u',wv|z) \geq \lambda_z F^{wB}(u',wv|z)$$

In words, the first of the two inequalities comes from the fact that the random walk starting at $u \in B$ can reach $u' \in B$ with positive probability *before* entering wB at wv. In potential theoretic terms, this can be interpreted as "balayage" or as a Harnack inequality. What is important for us is that it tells us that all the matrices $\mathbf{F}(w|z)$, $w \in W_D$, $z \in (0, r]$, have their zeros disposed in columns, and that in each non-zero column, the ratio of any two entries is bounded below by λ_z when $z \in (0, r]$.
For any vector $\mathbf{v} \in (0, \infty)^B$, let

$$\mathsf{Proj}\,\mathbf{v} = \frac{1}{\langle \mathbf{v}, \mathbf{1}\rangle}\,\mathbf{v}$$

be its projection onto the standard simplex over B (all non-negative vectors whose coordinates sum up to 1). Then the above yields the following, see [10], [24] or [29, §26.A].

Proposition 8. *Let $z \in (0, r]$ and $\underline{w} = (w_n)_{n\in\mathbb{N}}$ be a sequence in W_D. Then there is a vector $\mathbf{w}_\infty = \mathbf{w}_\infty(\underline{w}, z) \in (0, \infty)^B$ with $\langle \mathbf{w}_\infty, \mathbf{1}\rangle = 1$ such that uniformly for any sequence of non-zero vectors $\mathbf{a}_n \in [0, \infty)^B$,*

$$\lim_{n\to\infty} \mathsf{Proj}\,\mathbf{F}(w_1|z)\cdots\mathbf{F}(w_n|z)\,\mathbf{a}_n = \mathbf{w}_\infty.$$

The reason is that for each $w \in W_D$, the mapping $\mathbf{a} \mapsto \mathsf{Proj}\,\mathbf{F}(w|z)\,\mathbf{a}$ is a contraction of the standard simplex over B with Lipschitz constant $\ell(\lambda_z) < 1$ that maps the simplex into its interior.
It is known from the cited references that the λ-Martin compactification for $\lambda \geq \rho$ is always the end compactification of the tree, and that each end is a minimal boundary element. For our purpose, we need the above material in order to identify the Martin kernels for $\lambda = \rho$, i.e., $z = r$, as follows. Let $\xi \in \partial\mathbb{T}$, and let $\pi(e,\xi) = [e = x_0, x_1, x_2, \ldots]$. Let $u_n = x_{nD}$ and, with the

group operation of $\mathbb{F}$, $w_n = u_{n-1}^{-1} u_n$. We obtain a sequence $\underline{w} = \underline{w}(\xi)$ in W_D. Following Proposition 8, we let for $k \in \mathbb{N}$

$$\mathbf{w}_{k,\infty} = \mathbf{w}_{k,\infty}(\xi) = \lim_{n\to\infty} \mathsf{Proj}\, \mathbf{F}(w_{k+1}|r) \cdots \mathbf{F}(w_n|r)\, \mathbf{a}_n\,,$$

which is independent of the specific choice of the positive vectors $\mathbf{a}_n$. Using Lemma 7 and Proposition 8, we now obtain the following.

Corollary 9. *With the foregoing notation, in particular $\pi(e,\xi) = [e = x_0\,, x_1\,, \dots]$ for $\xi \in \partial\mathbb{T}$, let $x \in \mathbb{F} \equiv \mathbb{T}$, and let k be such that $u_k = x_{kD} \in \pi(x \wedge \xi, \xi)$ and $d(u_k, x) > R$. Then*

$$\lim_{\mathbb{T}\ni y\to\xi} K(x,y|\rho) = \frac{\big\langle \mathbf{f}(x,u_k|r), \mathbf{w}_{k,\infty}\big\rangle}{\big\langle \mathbf{f}(e,u_k|r), \mathbf{w}_{k,\infty}\big\rangle} = K(x,\xi|\rho).$$

Proof of Thm. 5. We need to describe the kernel $H(x,y)$ arising from the local limit theorem of [23] at least when $d(x,y)$ is large, and we need to show that $H(x,y) \to K(x,\xi|\rho)$ when $y \to \xi \in \partial\mathbb{T}$.

The main point is that by [23], for z near r, there is once more a Puiseux expansion

$$G(x,y|z) = G(x,y|r) - \beta(x,y)\,\sqrt{r-z} + h.o.t.,$$

where $\beta(x,y) = \beta(e,x^{-1}y) > 0$. Then $H(x,y) = \beta(x,y)/\beta(e,y)$. It follows from [23] (see also the exposition in [29, §26.A]) that also the non-vanishing entries of all the matrices $\mathbf{F}(w|z)$, as well as the entries of the vectors $\mathbf{f}(x,w|z)$, where $d(x,w) > R$, have Puiseux expansions of the same form. That is, in multidimensional notation we can expand

$$\begin{aligned}
\mathbf{F}(w|z) &= \mathbf{F}(w|r) \; - \sqrt{r-z}\; \mathbf{B}(w) \; + h.o.t., \quad w \in W_D\,,\\
\mathbf{f}(x,y|z) &= \mathbf{f}(x,y|r) - \sqrt{r-z}\, \mathbf{b}(x,y) + h.o.t., \quad x,y \in \mathbb{F}\,,\ d(x,y) > R\,,\\
\mathbf{g}(x,y|z) &= \mathbf{g}(x,y|r) - \sqrt{r-z}\, \widetilde{\mathbf{b}}(x,y) + h.o.t, \quad x,y \in \mathbb{F}\,.
\end{aligned}$$

Here, $\mathbf{B}(w)$ is a non-negative matrix indexed by $B \times B$, and $\mathbf{B}(w)$ is strictly positive in the same entries as $\mathbf{F}(w|r)$. Furthermore, the non-negative B-indexed vectors $\mathbf{f}(x,y|r)$ and $\mathbf{b}(x,y)$ are strictly positive in the same entries, while $\mathbf{g}(x,y|r)$ and $\widetilde{\mathbf{b}}(x,y)$ are positive in all entries.

We now use the same notation as in Corollary 9. If $y \to \xi$ then $n = n_y \to \infty$, where n_y is the largest integer such that $u_n = x_{nD} \in \pi(e,y)$ and $d(u_n\,,y) > R$. In particular, we shall have $n > k$. Using Lemma 7, we obtain

$$\begin{aligned}
\beta(x,y) = {}& \big\langle \mathbf{f}(x,u_k|r)\,,\ \mathbf{F}(w_{k+1}|r) \cdots \mathbf{F}(w_n|r)\, \widetilde{\mathbf{b}}(u_n,y)\big\rangle\\
&+ \big\langle \mathbf{b}(x,u_k)\,,\ \mathbf{F}(w_{k+1}|r) \cdots \mathbf{F}(w_n|r)\, \mathbf{g}(u_n,y|r)\big\rangle\\
&\quad + \sum_{i=k+1}^{n} \big\langle \mathbf{f}(x,u_k|r)\,,\ \mathbf{F}(w_{k+1}|r) \cdots \mathbf{B}(w_i) \cdots \mathbf{F}(w_n|r)\, \mathbf{g}(u_n,y|r)\big\rangle\,,
\end{aligned} \tag{3.10}$$

where more precisely, $\mathbf{F}(w_{k+1}|r)\cdots\mathbf{B}(w_i)\cdots\mathbf{F}(w_n|r)$ is obtained from the matrix product $\mathbf{F}(w_{k+1}|r)\cdots\mathbf{F}(w_n|r)$ by replacing the i^{th} factor $\mathbf{F}(w_i|r)$ by $\mathbf{B}(w_i)$. There is the analogous formula for $\beta(e,y)$, where one just needs to replace x by e. By a slight abuse of notation, we write $\mathsf{Proj}\,\beta(x,y)$ for the expression where in all the inner products of (3.10), the vectors appearing in the second variable are replaced by their projection onto the standard simplex. We choose $m(n) \geq k$ such that $m(n) \to \infty$ and $m(n)/n \to 0$ (for example, $m(n) = \max\{\lfloor \log n \rfloor, k\}$). Then for all $i > m(n)$

$$\mathsf{Proj}\,\mathbf{F}(w_{k+1}|r)\cdots\mathbf{F}(w_{m(n)}|r)\cdots\mathbf{B}(w_i)\cdots\mathbf{F}(w_n|r)\,\mathbf{g}(u_n,y|r) \to \mathbf{w}_{k+1,\infty}\,.$$

Recall that this convergence is uniform in whatever non-negative vector appears on the right of $\mathbf{F}(w_{m(n)}|r)$. Therefore, as $y \to \xi$, i.e., $n \to \infty$,

$$\begin{aligned}\frac{1}{n}\sum_{i=m(n)+1}^{n}&\big\langle \mathbf{f}(x,u_k|r)\,,\,\mathsf{Proj}\,\mathbf{F}(w_{k+1}|r)\cdots\mathbf{B}(w_i)\cdots\mathbf{F}(w_n|r)\,\mathbf{g}(u_n,y|r)\big\rangle\\ &\sim \frac{n-m(n)}{n}\big\langle \mathbf{f}(x,u_k|r)\,,\,\mathbf{w}_{k+1,\infty}\big\rangle \to \big\langle \mathbf{f}(x,u_k|r)\,,\,\mathbf{w}_{k+1,\infty}\big\rangle\end{aligned}$$

On the other hand, since $m(n)/n \to 0$,

$$\frac{1}{n}\sum_{i=k+1}^{m(n)}\big\langle \mathbf{f}(x,u_k|r)\,,\,\mathsf{Proj}\,\mathbf{F}(w_{k+1}|r)\cdots\mathbf{B}(w_i)\cdots\mathbf{F}(w_n|r)\,\mathbf{g}(u_n,y|r)\big\rangle \to 0.$$

The separate first two terms of $\mathsf{Proj}\,\beta(x,y)$ divided by n also tend to 0. All this is also valid for e in the place of x. We see that

$$\begin{aligned}&\tfrac{1}{n}\,\mathsf{Proj}\,\beta(x,y) \to \big\langle \mathbf{f}(x,u_k|r)\,,\,\mathbf{w}_{k+1,\infty}\big\rangle \quad \text{and}\\ &\tfrac{1}{n}\,\mathsf{Proj}\,\beta(e,y) \to \big\langle \mathbf{f}(e,u_k|r)\,,\,\mathbf{w}_{k+1,\infty}\big\rangle\,.\end{aligned}$$

Taking quotients and comparing with Corollary 9, we see that $H(x,y) \to K(x,\xi|r)$ as $y \to \xi$. $\square$

We remark here that the hypothesis that $\mathsf{supp}(\mu)$ contains the identity can be replaced without substantial change by aperiodicity. Recall that this means that $p^{(n)}(e,e) > 0$ for all but finitely many n, or in group theoretic terms, that $\mathsf{supp}(\mu)$ is not contained in a coset of a proper normal subroup of Γ (in our case, $\mathbb{F}$). Also, Theorem 5 holds without substantial change of the proof for virtually free groups.

4 Hyperbolic groups

We briefly recall the basic definition of hyperbolicity in the sense of Gromov [20]. Let $(\mathcal{X}, d)$ be a geodesic metric space, i.e., for any pair of points $x, y \in \mathcal{X}$, there is a (not necessarily unique) geodesic $\pi(x,y)$, that is, an isometric embedding $[0\,,\,d(x,y)] \hookrightarrow \mathcal{X}$ which maps 0 to x and $d(x,y)$ to y. In our situation, $\mathcal{X}$ will carry the structure of a locally finite, connected graph and d will be the graph metric. In this case, we replace the real interval $[0\,,\,d(x,y)]$ by its integer counterpart $[0\,,\,d(x,y)]_{\mathbb{Z}} = \{0, 1, \ldots, d(x,y)\}$.

The metric space is called *hyperbolic,* if it is δ-hyperbolic for some $\delta \geq 0$ (possibly large): if $a, b, c \in \mathcal{X}$ and $\pi(a,b)$, $\pi(b,c)$, $\pi(c,a)$ are geodesics between the respective points (the sides of a triangle with vertices a, b, c) then for every $x \in \pi(a,b)$ there is $y \in \pi(b,c) \cup \pi(c,a)$ such that $d(x,y) \leq \delta$. The most basic examples are provided by trees, where $\delta = 0$. For all our purposes, it will be convenient to assume without loss of generality that $\delta \in \mathbb{N}_0$ (non-negative integer).

We remark here that this implies, among many other facts, that any two geodesics connecting the same two points x and y are at Hausdorff distance at most δ, and we let $\Pi(x,y)$ denote the union of all those geodesics, a kind of "slim sausage".

A finitely generated group Γ is called hyperbolic, if its Cayley graph with respect to some finite, symmetric set of generators is hyperbolic. This does not depend on the specific generating set, up to a change of δ. Basic examples are free groups and co-compact Fuchsian groups. The entire theory will not be repeated here. For the present purpose, the exposition in [29, §22] plus the references given there will suffice. Besides very basic cases (virtually cyclic groups), all infinite hyperbolic groups are non-amenable.

A locally finite hyperbolic graph $\mathcal{X}$, resp. finitely generated hyperbolic group Γ has its hyperbolic compactification $\mathbf{\Delta}_{\mathrm{hyp}}(\mathcal{X})$, resp. $\mathbf{\Delta}_{\mathrm{hyp}}(\Gamma)$. It was shown by Ancona [1] that under natural assumptions on P on a hyperbolic graph $\mathcal{X}$ (bounded range, uniform irrecducibility; see [29, §27]), $\mathbf{\Delta}_{\mathrm{hyp}}(\mathcal{X})$ is a geometric realisation of the t-Martin compactification for positive $t > \rho$. In the group case this has been progressively strengthened in papers by Gouëzel and Lalley:

Theorem 10. [19], [17] *Let Γ be a non-amenable, finitely generated hyperbolic group, and μ a probability measure which induces an irreducible random walk. If μ is symmetric and is finitely supported, then the t-Martin compactification coincides with the $\mathbf{\Delta}_{hyp}(\Gamma)$ for every $t \geq \rho$.*

If in addition, μ is aperiodic, then the random walk satisfies a local limit theorem

$$p^{(n)}(x,y) \sim C\ \beta(x,y)\,\rho^n\,n^{-3/2} \quad \textit{as}\ \ n \to \infty\,.$$

Regarding the Martin compactification, the noteworthy part is that it is also valid at the critical value $t = \rho$. The crucial tool for this is the following, whose part (a) was again first proved in [1] for $z < r = 1/\rho$ without requiring group-invariance, and then extended to $z = r$ in the group case in [19], [17], and finally [18], including the strong inequality (b).

Proposition 11. [Ancona inequalities] *Suppose that Γ and μ are as in Theorem 10, and consider a Cayley graph of Γ with respect to a finite, symmetric set of generators. Then there are constants $C_{\mathrm{Anc}} \geq 1$ and $0 \leq \alpha < 1$ such that* *the following holds for all* $z \in [1\,,\,r]$.

(a) *For any geodesic path* $\pi(x,y)$ *in the graph and any* $w \in \pi(x,y)$*, one has*

$$C_{\mathrm{Anc}}^{-1}\, G(x,w|z)G(w,y|z) \le G(x,y|z) \le C_{\mathrm{Anc}}\, G(x,w|z)G(w,y|z)\,.$$

(b) *For any quadruple of points* x, x', y, y' *such that* $d\big(\Pi(x,x'),\Pi(y,y')\big) = n > 2\delta$*, one has*

$$\left|\frac{G(x,y|z)G(x',y'|z)}{G(x,y'|z)G(x',y|z)} - 1\right| \le C_{\mathrm{Anc}}\,\alpha^n.$$

In order to get a feeling for the last inequality, observe that for a nearest neighbour random walk on a tree, the left hand side vanishes. As a matter of fact, it is proved in [18] that Proposition 11 and Theorem 10 are also valid when instead of finite support, one assumes that μ has super-exponential moments, that is, $\sum_x a^{|x|}\,\mu(x) < \infty$ for all $a > 1$.
Irreducibilty plus group-invariance of the random walk yield a local Harnack inequality: there is a constant $C_{\mathrm{Har}} > 1$ such that

$$G(x',y|z) \le C_{\mathrm{Har}}^{d(x,x')} G(x,y|z) \quad \text{for all} \quad x,x',y \in \Gamma \quad \text{and} \quad z \in [1\,,\,r]\,. \tag{4.1}$$

Even without symmetry, the same also holds for $G(y,x|z)$ and $G(y,x'|z)$. This leads to the following generalisation of the first Ancona inequality. For $x,y,w \in \Gamma$ with $\ell = d\big(w,\Pi(x,y)\big)$ and $z \in [1\,,\,r]$,

$$\begin{gathered} C_\ell^{-1}\, G(x,w|z)G(w,y|z) \le G(x,y|z) \le C_\ell\, G(x,w|z)G(w,y|z)\,, \\ \text{where} \quad C_\ell = C_{\mathrm{Anc}}\, C_{\mathrm{Har}}^{2\ell}\,. \end{gathered} \tag{4.2}$$

Theorem 12. *Let* Γ *be a non-amenable, finitely generated hyperbolic group, and* μ *a finitely supported symmetric probability measure which induces an irreducible & aperiodic random walk. Then the ratio limit compactification coincides with* ρ*-Martin compactification analytically.*

Proof. Let $G'(x,y|z)$ be the derivative of the Green function with respect to z, where $|z| < r$. It is proved in [19] and [17] that for all $x,y \in \Gamma$, there is $\beta(x,y) > 0$ such that

$$G'(x,y|z) \sim \beta(x,y)\big/\sqrt{r-z} \quad \text{as} \quad z \to r\,,\ 0 < z < r\,.$$

By working through the "Tauberian" last part of [19], one learns that symmetry and aperiodicity yield the asymptotics of Theorem 10 above, with constant $C = \sqrt{r/\pi}$. Therefore the ratio limit kernel is the left-sided limit

$$H(x,y) = \lim_{z\to r-} \frac{G'(x,y|z)}{G'(e,y|z)}.$$

It is a well-known consequence of the *resolvent equation* that for $z \in (0\,,\,r)$ one has

$$G'(x,y|z) = G^{(2)}(x,y|z)/z^2\,, \quad \text{where} \quad G^{(2)}(x,y|z) = \sum_{v\in\mathcal{X}} G(x,v|z)G(v,y|z)\,.$$

We now set

$$\Phi(x,y|z) = \frac{G^{(2)}(x,y|z)}{G(x,y|z)}\,, \quad \text{so that} \quad \frac{H(x,y)}{K(x,y|\rho)} = \lim_{z\to r-} \frac{\Phi(x,y|z)}{\Phi(e,y|z)} =: \frac{\Phi(x,y|r-)}{\Phi(e,y|r-)}.$$

Here, we always assume that z is real, $z \in [1\,,\,r)$. The proof will be complete once we have shown the following.

Claim. $$\lim_{|y|\to\infty} \frac{\Phi(x,y|r-)}{\Phi(e,y|r-)} = 1\,.$$

Proof of the Claim[2]. Let $\pi(e,y) = [e = y_0\,, y_1\,, \dots, y_n = y]$ be a geodesic from e to y in our Cayley graph, so that $n = |y|$. For $v \in \Gamma$, We set

$$k(v) = k_y(v) = \max\bigl\{k : d\bigl(y_k, \Pi(e,v)\bigr) \le \delta\bigr\}.$$

If $k(v) < |y|$ then we must have $d\bigl(y_{k(v)+1}\,, \pi(v,y)\bigr) \le \delta$. Thus (even when $k(v) = |y|$) we have

$$d(y_{k(v)}\,, \bar v) \le \delta + 1 \quad \text{for some } \ \bar v \in \pi(v,y). \tag{4.3}$$

Let $w \in \Pi(e,v)$ be such that $d(y_{k(v)}, w) = d\bigl(y_{k(v)}, \Pi(e,v)\bigr)$, and let $u \in \Pi(y,w)$. Then, for any $y_k \in \pi(e,y)$, using the triangle inequality 3 times,

$$\begin{aligned}
2d(e,u) &\ge d(e,y) - d(y,u) + d(e,v) - d(v,u)\\
&= d(e,y) + d(e,v) - d(y,v)\\
&\ge d(e,y) + d(e,v) - \bigl(d(y,y_k) + d(y_k\,,w) + d(w,v)\bigr)\\
&= d(e,y_k) + d(e,w) - d(y_k\,,w)\\
&\ge 2d(e,y_k) - 2d(y_k\,,w) = 2k - 2d(y_k\,,w)\,.
\end{aligned}$$

The first equality holds because u lies on a geodesic $\pi(y,v)$, and the second because $y_k \in \pi(e,y)$ and w lies on a geodesic $\pi(e,v)$. In particular, for $k = k(v)$ we get for every $x \in \Gamma$

$$d\bigl(\Pi(e,x), \Pi(y,v)\bigr) \ge d\bigl(\Pi(y,v), e\bigr) - |x| \ge k(v) - \delta - |x|\,.$$

Thus, when $k(v) > |x| + 3\delta$, the strong Ancona inequality of Proposition 11(b) yields

$$\left|\frac{G(x,v|z)G(e,y|z)}{G(e,v|z)G(x,y|z)} - 1\right| \le C_{\mathrm{Anc}}\,\alpha^{k(v)-|x|-\delta}. \tag{4.4}$$

On the other hand, when $k(v) \le |x| + 3\delta$ then (4.1) yields

$$\left|\frac{G(x,v|z)G(e,y|z)}{G(e,v|z)G(x,y|z)} - 1\right| \alpha^{-k(v)} \le (C_{\mathrm{Har}}^{2|x|} + 1)\,\alpha^{-|x|-3\delta} \tag{4.5}$$

[2] This was facilitated significantly by input from Sébastien Gouëzel, and simplified by a suggestion of the referee.

Combining (4.4) with (4.5), we get for the given y and geodesic $\pi(e,y)$ that

$$\left|\frac{G(x,v|z)G(e,y|z)}{G(e,v|z)G(x,y|z)} - 1\right| \le C(x)\,\alpha^{k(v)} \quad \text{for all } \ x,v \in \Gamma \ \text{ and } \ z \in [1\,,\,r]\,, \tag{4.6}$$

where $C(x)$ depends only on x.
We now write for $z \in [1,r)$

$$\Phi(x,y|z) - \Phi(e,y|z) = \sum_{v\in\Gamma} \frac{G(e,v|z)G(v,y|z)}{G(e,y|z)} \left(\frac{G(x,v|z)G(e,y|z)}{G(e,v|z)G(x,y|z)} - 1\right).$$

Then (4.6) yields

$$\big|\Phi(x,y|z) - \Phi(e,y|z)\big| \le C(x) \sum_{k=0}^{|y|} \alpha^k \sum_{v:k(v)=k} \frac{G(e,v|z)G(v,y|z)}{G(e,y|z)}$$

By Proposition 11(a),

$$G(e,y|z) \ge C_{\mathrm{Anc}}^{-1} G(e,y_{k(v)}|z)G(y_{k(v)}\,,y|z).$$

By (4.2) and, for the second inequality, (4.3)

$$\begin{aligned} &G(e,v|z) \le C_\delta\, G(e,y_{k(v)}|z)G(y_{k(v)},v|z) \quad \text{and}\\ &G(v,y|z) \le C_{\delta+1}\, G(v,y_{k(v)}|z)G(y_{k(v)},y|z). \end{aligned}$$

We get for any $k \le |y|$ that with $\overline{C} = C_{\mathrm{Anc}}\, C_\delta\, C_{\delta+1}$,

$$\begin{aligned} \sum_{v:k(v)=k} \frac{G(e,v|z)G(v,y|z)}{G(e,y|z)} &\le \overline{C} \sum_{v:k(v)=k} G(y_k\,,v|z)G(v,y_k|z)\\ &\le \overline{C}\, G^{(2)}(y_k\,,y_k|z) = \overline{C}\, G^{(2)}(e,e|z).\ [3] \end{aligned}$$

With $\overline{C}(x) = C(x)\,\overline{C}/(1-\alpha)$, we obtain

$$\left|\frac{\Phi(x,y|z)}{\Phi(e,y|z)} - 1\right| \le \overline{C}(x) \frac{G^{(2)}(e,e|z)}{\Phi(e,y|z)}.$$

Now the crucial point is that following [17, Lemma 3.20 and equation (3.15)],

$$\Phi(e,y|z) \asymp |y|\, G^{(2)}(e,e|z) \quad \text{uniformly for } \ z \in [1\,,\,r) \ \text{ as } \ |y| \to \infty\,,$$

i.e., the ratio is bounded above and below by uniform positive constants A and $1/A$, respectively, when $|y|$ is large enough. Thus, for large $|y|$,

$$\left|\frac{\Phi(x,y|r-)}{\Phi(e,y|r-)} - 1\right| \le \overline{C}(x)\,\frac{A}{|y|},$$

which tends to 0 as $|y| \to \infty$. This concludes the proof of the Claim and the Theorem. $\square$

[3] We remark that [17] uses the notation $G^{(2)}(e,e|z) = \eta(r)$, with $z \leftrightarrow r$, while our $r = 1/\rho$ is denoted R.

We note here that the last theorem does not fully cover the results on free groups and trees of §3. Theorem 3 does not need finite range, and Theorem 5 does not need symmetry, while symmetry of μ is a crucial tool for the local limit theorem on hyperbolic groups stated in Theorem 10. In any case, it may be interesting to watch out how certain features of the respective proofs show up in different "disguise" in each of them.

5 Direct and Cartesian products

The ratio limit compactification adapts quite well to direct products. In general, let $\mathbf{\Delta}(\mathcal{X}_1)$ and $\mathbf{\Delta}(\mathcal{X}_2)$ be compactifications of the two discrete state spaces $\mathcal{X}_1$ and $\mathcal{X}_2$, with respective boundaries $\partial\mathcal{X}_1$ and $\partial\mathcal{X}_2$. Then $\mathbf{\Delta}(\mathcal{X}_1) \times \mathbf{\Delta}(\mathcal{X}_2)$ is the natural associated compactification of $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$. In this case, the boundary is

$$\partial\mathcal{X} = \big(\partial\mathcal{X}_1 \times \partial\mathcal{X}_2\big) \cup \big(\partial\mathcal{X}_1 \times \mathcal{X}_2\big) \cup \big(\mathcal{X}_1 \times \partial\mathcal{X}_2\big). \tag{5.1}$$

We write elements of the product space as w_1w_2, where $w_i \in \mathbf{\Delta}(\mathcal{X}_i)$. In the resulting topology, let $\big(y_1(n)y_2(n)\big)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{X}$. Then, as $n \to \infty$,

$$\begin{aligned}
y_1(n)y_2(n) \to \xi_1\xi_2 \in \partial\mathcal{X}_1 \times \partial\mathcal{X}_2 &\iff y_i(n) \to \xi_i \text{ in } \mathbf{\Delta}(\mathcal{X}_i) \text{ for } i = 1,2\,,\\
y_1(n)y_2(n) \to \xi_1 w_2 \in \partial\mathcal{X}_1 \times \mathcal{X}_2 &\iff y_1(n) \to \xi_1 \text{ in } \mathbf{\Delta}(\mathcal{X}_1)\,, \text{ and}\\
&\qquad y_2(n) = w_2 \text{ for all but finitely many } n,\\
y_1(n)y_2(n) \to w_1\xi_2 \in \mathcal{X}_1 \times \partial\mathcal{X}_2 &\iff y_2(n) \to \xi_2 \text{ in } \mathbf{\Delta}(\mathcal{X}_2)\,, \text{ and}\\
&\qquad y_1(n) = w_1 \text{ for all but finitely many } n.
\end{aligned}$$

Let us call this the *product compactification* of the given compactifications of $\mathcal{X}_1$ and $\mathcal{X}_2$, with the *product boundary* (5.1). Now let $(\mathcal{X}_1\,, P_1)$ and $(\mathcal{X}_2\,, P_2)$ be the state spaces plus irreducible and aperiodic transition matrices of two respective Markov chains. Suppose that for $i = 1,2$

$$\lim_{n\to\infty} \frac{p_i^{(n)}(x_i\,, y_i)}{p_i^{(n)}(e_i\,, e_i)} = h_i(x_i\,, y_i) \quad \text{for all } x_i, y_i \in \mathcal{X}\,.$$

The direct product is the Markov chain on $\mathcal{X}_1 \times \mathcal{X}_2$ with transition matrix $P = P_1 \otimes P_2$, where

$$p(x_1x_2\,, y_1y_2) = p_1(x_1\,, y_1)p_2(x_2\,, y_2)\,.$$

It is clear that it has a ratio limit (1.1) with

$$h(x_1x_2\,, y_1y_2) = h_1(x_1\,, y_1)h_2(x_2\,, y_2),$$

so that also the kernel $H(x_1x_2\,, y_1y_2)$ splits in the same way. The associated ratio limit compactification of $\mathcal{X}_1 \times \mathcal{X}_2$ is not always the product compactification of the two ratio limit compactifications, but it is a factor thereof. The following is quite obvious; see e.g. the "preamble" on compactifications in [30, §7.B], and recall that the ratio limit compactification of $\mathcal{X}_1 \times \mathcal{X}_2$ is the minimal one which provides continuous extensions of the functions $H(x_1x_2\,,\cdot)$, where $x_1x_2 \in \mathcal{X}_1 \times \mathcal{X}_2$.

LEMMA 13. *Consider the extensions of $H_i(x_i\,,\cdot)$ to the ratio limit boundary $\mathcal{R}_i$ of $\mathcal{X}_i$, $i = 1, 2$. For $\eta = \eta_1\eta_2$ and $\zeta = \zeta_1\zeta_2 \in \big(\mathcal{R}_1 \times \mathcal{R}_2\big) \cup \big(\mathcal{X}_1 \times \mathcal{R}_2\big) \cup \big(\mathcal{R}_1 \times \mathcal{X}_2\big)$, let*

$$\eta \approx \zeta \iff H_1(x_1\,,\eta_1)H_2(x_2,\eta_2) = H_1(x_1\,,\zeta_1)H_2(x_2,\zeta_2) \quad \forall\, x_1x_2 \in \mathcal{X}_1 \times \mathcal{X}_2\,.$$

Then the ratio limit boundary $\mathcal{R}$ of $P = P_1 \otimes P_2$ is the image of the product boundary of the two ratio limit compactifications with respect to the factor map of the equivalence relation $\approx$. In particular, the extension of $H(x_1x_2\,,\cdot)$ to $\mathcal{R}$ is given by $H(x_1x_2,\xi) = H_1(x_1\,,\eta_1)H_2(x_2,\eta_2)$, where $\eta_1\eta_2$ is a representative of the $\approx$-equivalence class ξ.

In a certain sense more natural than direct products are *Cartesian products.* Given P_i in $\mathcal{X}_i$ for $i = 1, 2$, write I_i for the identity operator (or matrix) over $\mathcal{X}_i$ and choose a parameter $s \in (0\,,1)$. On $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$, let

$$P = P_s = s \cdot P_1 \otimes I_2 + (1-s) \cdot I_2 \otimes P_2\,.$$

That is, the new Markov chain is such that first a coin is tossed where "heads" comes up with probability s, and in that case, a step is performed according to P_1 in the first coordinate, while the second coordinate remains unchanged. And with probability $1-s$, a step is performed according to P_2 in the second coordinate, while the first one remains unchanged.
For example, simple random walk on $\mathbb{Z}^{d_1+d_2}$ arises as a Cartesian (and not direct) product of the simple random walks on $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$ with $s = d_1/(d_1 + d_2)$.
In general, it does not appear to be completely straightforward that a ratio limit for each P_i also implies one for the Cartesian products P_s. However, the following was proved by CARTWRIGHT AND SOARDI [7].

PROPOSITION 14. *Suppose that each $\mathcal{X}_i$ satisfies a local limit theorem of the form*

$$p_i^{(n)}(x_i, y_i) \sim \beta_i(x_i, y_i)\, \rho_i^n\, n^{-\alpha_i} \quad \text{as} \;\; n \to \infty\,,$$

where $\rho_i = \rho(P_i)$ for $i = 1, 2$. Then one has for $P = P_s$ that $\rho = \rho(P) = s\,\rho_1 + (1-s)\,\rho_2$,

$$p^{(n)}(x_1x_2\,, y_1y_2) \sim C\,\beta_1(x_1, y_1)\beta_2(x_2, y_2)\,\rho^n\, n^{-\alpha_1-\alpha_2}\,,$$

where $C = \theta^{\alpha_1}(1-\theta)^{\alpha_2}$, with $\theta = s\rho_1/\rho$.
In particular, in this case the ratio limit kernel for the Cartesian product is also $H(x_1x_2\,, y_1y_2) = H_1(x_1\,, y_1)H_2(x_2\,, y_2)$, and the ratio limit boundary is the same as for the direct product.

At this point, there is a natural question. Suppose that the ρ_i-Martin boundary of P_i coincides with the respective ratio limit boundary for $i = 1, 2$. Is this then also true for the ρ-Martin boundary for the direct, resp. Cartesian products? So far, there is no general answer; the general problem lies in the possible presence of ρ-Martin kernels of the product which are not minimal ρ-harmonic functions. Some examples are known. We present three of them, which illustrate different situations regarding the equivalence relation $\approx$ that appears in Lemma 13. All three examples are valid for direct as well as Cartesian products, and the answer to the above question is "yes".

Example 1. Suppose that $\mathcal{X}_i = \mathbb{Z}^{d_i}$ and that the respective irreducible, aperiodic random walk is induced by a finitely supported probability measure μ_i. Let $h_i(x) = \exp\langle c_i\,, x\rangle$ be the unique ρ_i-harmonic exponential on $\mathbb{Z}^{d_i}$. That is, the vector c_i minimises $c \mapsto \sum_x \mu_i(x)\exp\langle c\,, x\rangle$, where $c \in \mathbb{R}^{d_i}$, and the value of that minimum is the corresponding spectral radius ρ_i. Every positive ρ_i-harmonic function on $\mathbb{Z}^{d_i}$ is a constant multiple of h_i, and in the ratio limit theorem, one has by a slight abuse of notation $h_i(x, y) = h_i(x - y)$. Thus, $H_i(x, y) = h_i(x)$ for $x \in \mathbb{Z}^{d_i}$, and the ratio limit kernel $H(x_1x_2\,, y_1y_2) = h_1(x_1)h_2(x_2)$ of the direct product also does not depend on y_1y_2. That is, the equivalence relation $\approx$ of Lemma 13 has a single equivalence class.

In this case, one also knows that all the involved compactifications coincide analytically with the Martin compactifications at the respective spectral radii. $\square$

Example 2. Let $\mathcal{X}_1 = \mathbb{T}$, the regular tree with degree $q + 1 \geq 3$, and $\mathcal{X}_2 = \mathbb{Z}$. On each of the two, we consider "lazy" simple random walk, that is,

$$p_1(x_1\,, x_1) = \tfrac{1}{2} \;\text{ and }\; p_1(x_1\,, y_1) = \tfrac{1}{2q+2} \;\text{ when }\; x_1 \sim y_1\,,$$
$$p_2(x_2\,, x_2) = \tfrac{1}{2} \;\text{ and }\; p_2(x_2\,, y_2) = \tfrac{1}{4} \;\text{ when }\; x_2 \sim y_2\,,$$

while all other transition probabilities are 0. Then $H_1(x_1\,, y_1) = \Phi(x_1\,, y_1)$, see §3.A, and $H_2(x_2\,, y_2) = 1$. The ratio limit compactification of $\mathbb{Z}$ is the one-point compactification $\mathbb{Z} \cup \{\infty\}$, and the ratio limit boundary of the product space $\mathbb{T} \times \mathbb{Z}$ is $\mathbf{\Delta}_{\text{ends}}\mathbb{T}$. Convergence to the boundary of a sequence $\bigl(y_1(n)y_2(n)\bigr)_{n\in\mathbb{N}}$ is as follows, where $K_1(\cdot, \cdot\,\rho_1)$ is the Martin kernel on $\mathbb{T}$ given by (3.3)

$$y_1(n) \to \xi_1 \in \partial\mathbb{T}\,, y_2(n) \text{ arbitrary} \Rightarrow H\bigl(x_1x_2\,, y_1(n)y_2(n)\bigr) \to K_1(x_1\,, \xi_1|\rho_1)\,,$$
$$y_1(n) = y_1 \in \mathbb{T} \text{ for all } \; n \geq n_0\,,\; |y_2(n)| \to \infty$$
$$\Rightarrow H\bigl(x_1x_2\,, y_1(n)y_2(n)\bigr) \to \frac{\Phi(x_1\,, y_1)}{\Phi(o_1\,, y_1)}.$$

In particular, by Crotti [9], the ratio limit compactification coincides analytically with the ρ-Martin compactification; see [29, Thm. 28.8]. In that reference, the result is stated for Cartesian products; the proof carries over to direct products with some obvious modifications. $\square$

Example 3. Let $\mathcal{X}_i = \mathbb{T}_i$ be two regular trees with respective degrees $q_i+1 \ge 3$. On each of the two, we consider "lazy" simple random walk as above, that is,

$$p_i(x_i\,, x_i) = \tfrac{1}{2} \quad \text{and} \quad p_i(x_i\,, y_i) = \tfrac{1}{2q_i+2} \quad \text{when} \quad x_i \sim y_i\,,$$

while all other transition probabilities are 0. Then $H_i(x_i\,, y_i) = \Phi_i(x_i\, y_i)$, the spherical function on the respective tree as in §3.A. The ratio limit compactification of the direct or any Cartesian product of the two random walks is the product compactification of the ratio limit compactifications of the two trees. Convergence to the boundary of a sequence $\bigl(y_1(n)y_2(n)\bigr)_{n\in\mathbb{N}}$ is as follows, where $K_i(\cdot,\cdot\,\rho_i)$ is the Martin kernel on $\mathbb{T}_i$ given by (3.3)

$$\begin{aligned}
&y_1(n) \to \xi_1 \in \partial\mathbb{T}_1\,,\; y_2(n) \to \xi_2 \in \partial\mathbb{T}_2\\
&\qquad\qquad \Rightarrow H\bigl(x_1x_2\,, y_1(n)y_2(n)\bigr) \to K_1(x_1\,, \xi_1|\rho_1)K_2(x_2\,, \xi_2|\rho_2)\,,\\
&y_1(n) = y_1 \in \mathbb{T}_1 \text{ for all } \; n \ge n_0\,,\; y_2(n) \to \xi_2 \in \partial\mathbb{T}_2\\
&\qquad\qquad \Rightarrow H\bigl(x_1x_2\,, y_1(n)y_2(n)\bigr) \to \frac{\Phi_1(x_1\,, y_1)}{\Phi_1(o_1\,, y_1)}\, K_2(x_2, \xi_2|\rho_2)\,,\\
&y_1(n) \to \xi_1 \in \partial\mathbb{T}_1\,,\; y_2(n) = y_2 \in \mathbb{T}_2 \text{ for all } \; n \ge n_0\\
&\qquad\qquad \Rightarrow H\bigl(x_1x_2\,, y_1(n)y_2(n)\bigr) \to K_1(x_1, \xi_1|\rho_1)\, \frac{\Phi_2(x_2\,, y_2)}{\Phi_2(o_2\,, y_2)}\,.
\end{aligned}$$

By [26], the ratio limit compactification coincides once more analytically with the ρ-Martin compactification. Again, this holds for direct products in the same way as for Cartesian products. □

6 Reduced ratio limit compactification

The companion paper [11] makes crucial use of the following variant of the ratio limit compactification. Let $\sim$ be the equivalence relation on $\mathcal{X}$ such that

$$y \sim y' \iff H(x,y) = H(x,y') \quad \text{for all} \;\; x \in \mathcal{X}\,. \tag{6.1}$$

We denote by $\widetilde{\mathcal{X}}$ the set of equivalence classes, and by $\tilde{y}$ the equivalence class of $y \in \mathcal{X}$. Then the ratio limit kernel decends to a kernel on $\mathcal{X} \times \widetilde{\mathcal{X}}$ by

$$H_{\mathrm{red}}(x, \tilde{y}) = H(x, y).$$

Definition 2. The *reduced ratio limit compactification*[4] $\mathbf{\Delta}_{\mathrm{ratio}}(\widetilde{\mathcal{X}})$ associated with $\mathcal{X}$ and P satisfying (1.1) is the (up to homomorphism) unique compact Hausdorff space which contains $\widetilde{\mathcal{X}}$ as a discrete, dense subset and has the following properties:

- for each $x \in \mathcal{X}$, the function $H_{\mathrm{red}}(x,\cdot)$ extends continuously to $\mathbf{\Delta}_{\mathrm{ratio}}(\widetilde{\mathcal{X}})$, and denoting the extended kernel also by H_{red},

[4] [11] calls this the (ordinary) ratio limit compactification.

- if $\xi, \eta \in \mathcal{R}(\widetilde{\mathcal{X}}) = \mathbf{\Delta}_{\text{ratio}}(\widetilde{\mathcal{X}}) \setminus \widetilde{\mathcal{X}}$ are distinct, then there is $x \in \mathcal{X}$ such that $H_{\text{red}}(x,\xi) \neq H_{\text{red}}(x,\eta)$.

If $\widetilde{\mathcal{X}}$ is finite then it is already compact, and there is no ratio limit boundary added to that space. The proofs of the following facts are easy exercises.

Lemma 15. (i) *If the equivalence relation* (6.1) *is extended to all of* $\mathbf{\Delta}_{ratio}(\mathcal{X})$ *via the extended ratio limit kernel on* $\mathcal{X} \times \mathbf{\Delta}_{ratio}(\mathcal{X})$*, then the resulting factor space is* $\mathbf{\Delta}_{ratio}(\widetilde{\mathcal{X}})$*.*

(ii) *The factor map* $\mathcal{X} \to \widetilde{\mathcal{X}}$ *extends to a continuous surjection* $\mathbf{\Delta}_{ratio}(\mathcal{X}) \to \mathbf{\Delta}_{ratio}(\widetilde{\mathcal{X}})$ *which is one-to-one from* $\mathcal{R}(\mathcal{X})$ *into the reduced ration limit compactification.*

(iii) *If an equivalence class* $\tilde{y}$ *is infinite, then it has a unique accumulation point* $\xi \in \mathcal{R}(\mathcal{X})$*, and* $H(x,y) = H(x,\xi)$ *for all* $y \in \tilde{y}$*.*

If $\mathcal{X} = \Gamma$ is a countable group and P is a random walk induced by a probability measure μ, then

$$R_\mu = \{y \in \Gamma : H(x,y) = H(x,e) \ \text{ for every } \ x \in \Gamma\} \tag{6.2}$$

is a subgroup of Γ, see [11]. Since the ratio limit kernel $h(x,y)$ in (1.1) satisfies $h(x,y) = f(x^{-1}y)$, where $f(x) = h(x,e)$, one gets that

$$\widetilde{\mathcal{X}} \ \bigl(= \Gamma_{\text{red}}\bigr) = \Gamma/R_\mu \,.$$

Elder and Rogers [14] have extended Avez' Theorem 1: they show that for a symmetric, aperiodic random walk on an arbitrary finitely generated group, the set A_μ of all $y \in \Gamma$ for which $p^{(n)}(e,y)/p^{(n)}(e,e) \to 1$ is an amenable subgroup of Γ. This implies at least in the symmetric case that $R_\mu \subset A_\mu$ is amenable.

Proposition 16. *Consider a probability measure* μ *on the finitely generated group* Γ *which induces an irreducible & aperiodic random walk satisfying* (1.1) *and* (1.2)*. Suppose that* R_μ *is infinite, and that the associated element* $\xi \in \mathcal{R}(\mathcal{X})$ *according to Lemma 15(iii) is such that* $x \mapsto H(x,\xi)$ *is a* <u>*minimal*</u> ρ*-harmonic function.*
Then Γ *fixes the boundary point* ξ*.*

Proof. Like on the t-Martin compactification, the group acts continuously on the ratio limit compactifictation, and the extended ratio limit kernel satisfies the cocycle identity

$$H(gx,\xi) = H(x,g^{-1}\xi)H(g,\xi) \quad \text{for all } \ x,g \in \Gamma$$

(and of course for every boundary element, not only the ξ of the statement).

By our assumptions, $H(x,y) = H(x,e) = h(x,e) = f(x)$ for all $y \in R_\mu$, and as $|y| \to \infty$, we have $y \to \xi$ in the topology of the ratio limit compactification. Thus, $f(x) = H(x,\xi)$ for all $x \in \Gamma$. By (1.2), the function

$$\check{f}(x) = f(x^{-1}) = h(e,x)$$

satisfies $fP = \rho \cdot f$, where P is the transition matrix of the random walk. Using the cocycle identity, this can be rewritten as

$$H(x,\xi) = \check{f}(x) = \frac{1}{\rho}\sum_{g\in\Gamma}\mu(g)H(gx,\xi) = \sum_{g\in\Gamma}\underbrace{\frac{\mu(g)H(g,\xi)}{\rho}}_{=:\,c_g}H(x,g^{-1}\xi)$$

for every $x \in \Gamma$. Since $\sum_g c_g = 1$, the minimality assumption on $H(\cdot,\xi)$ yields that $H(x,g^{-1}\xi) = H(x,\xi)$ for every $x \in \Gamma$ and every g in the support of μ. Therefore $g\xi = \xi$ for every $g \in \mathsf{supp}(\mu)$, and consequently for every $g \in \Gamma$. □

Corollary 17. (a) *For random walks on trees and free groups as considered in theorems 3 and 5, the subgroup R_μ is trivial.*

(b) *For random walks on non-amenable hyperbolic groups as considered in Theorem 12, the subgroup R_μ is finite. It is trivial when the group is torsion-free.*

Statement (a) can of course also be seen directly from the respective form of the extended ratio limit kernel, in particlar for the isotropic case. However, in the non-isotropic case, the present method is more convenient. Note that in all cases of the corollary, the ρ-Martin boundary coincides with the minimal one. Furthermore, in all those cases, the entire group cannot fix a single boundary element: this has several different proofs, among which the author is best acquainted with [28, Prop. 4].
In examples 2 and 3, think $\mathbb{T}$ (even degree) as the free group. The repective random walks are of course induced be probabiliy measure on the respective product groups. In Example 2, we have $R_\mu = \{e_1\} \times \mathbb{Z}$, and the reduced ratio limit compactification is $\boldsymbol{\Delta}_{\text{ends}}(\mathbb{T})$. In Example 3, there is no reduction; R_μ is trivial.

7 Final remarks

The above material should be seen as a collection of first answers to the question stated in the Introduction. The next step in the same direction would concern relatively hyperbolic groups, see the local limit theorem of [12]. Under the assumptions of that work, there is again a local limit theorem of the same form as for hyperbolic groups (see Theorem 10 above). It is not so easy to lay hands on the ratio limit kernel $H(x,y)$ in those cases, but one would expect that one also has that the ratio limit compactification coincides analytically with the ρ-Martin compactification.

Questions. (a) Under which general conditions is it true that the ratio limit kernel is the left-sided limit $H(x,y) = G^{(2)}(x,y|r-)/G^{(2)}(e,y|r-)$?

(b) Under which general conditions does the corresponding compactification coincide analytically (or only geometrically) with the ρ-Martin compactification ?

For relatively hyperbolic groups, in particular for free products of groups, there also are local limit theorems of the form (1.4), but with $\alpha > 2$, see Cartwright [5] and Candellero and Gilch [3]. This may be more challenging.

In the last decades there has not been much work on ratio limit theorems for random walks on groups; [14] is one of the interesting exceptions. On non-amenable groups, local limit theorems have prevailed. In a certain sense, the cases considered here might also be referred to as the "local limit boundary". Indeed, apart from very few exceptions (e.g. isotropic random walks on trees), the author does not know of methods which provide a ratio limit theorem in the non-amenable environment without first proving a local limit theorem. This may indicate possibilities for future research.

More generally, let $t \ge \rho = \rho(P)$, and suppose that we have a positive t-harmonic kernel $h(x,y)$, that is, $Ph(\cdot,y) = t\cdot h(\cdot,y)$. In our situation, harmonicity is two-sided, i.e., we also have $h(x,\cdot)P = t\cdot h(x,\cdot)$. For example, if P is the transition matrix of a random walk on a group Γ induced by the probability measure μ, then we may look for a solution of the convolution equation $\mu * \sigma = t\cdot\sigma$ (or the two-sided version) and set $h(x,y) = \sigma(x^{-1}y)$. Then we can normalise by setting $H(x,y) = h(x,y)/h(e,y)$ and try to understand the corresponding compactification. If $K(\cdot,\cdot t)$ is the t-Martin kernel of P, then for each $y \in \mathcal{X}$ there is a probability measure ν^y on $\mathcal{M}_t(\mathcal{X})$ such that

$$H(x,y) = \int K(x,\xi|t)\,d\nu^y(\xi)\,.$$

In the case when $\mathcal{M}_t(\mathcal{X})$ has only minimal boundary elements, the compactification induced by H will coincide analytically with the t-Martin compactification when for every $\xi \in \mathcal{M}_t(\mathcal{X})$, one has that $\nu^y \to \delta_\xi$ weakly as $y \to \xi$ in $\mathbf{\Delta}_{\mathrm{Mart},\,t}(\mathcal{X})$.

References

[1] Ancona, A.: *Positive harmonic functions and hyperbolicity,* Lecture Notes in Math. 1344 (1988) 1–23.

[2] Avez, A.: *Limite de quotients pour des marches aléatoires sur des groupes,* C. R. Acad. Sci. Paris Sér. A-B 276 (1973) A317–A320.

[3] Candellero, E. and Gilch, L. A.: *Phase transitions for random walk asymptotics on free products of groups,* Random Structures Algorithms 40 (2012) 150–181.

[4] Cartier, P.: *Fonctions harmoniques sur un arbre,* Symposia Mathematica 9 (1972) 203–270.

[5] Cartwright, D. I.: *On the asymptotic behaviour of convolution powers of probabilities on discrete groups,* Monatsh. Math. 107 (1989) 287–290.

[6] Cartwright, D. I. and Sawyer, S.: *The Martin boundary for general isotropic random walks in a tree,* J. Theoret. Probab. 4 (1991) 111–136.

[7] Cartwright, D. I. and Soardi, P. M.: *A local limit theorem for random walks on the Cartesian product of discrete groups,* Boll. Un. Mat. Ital. 1-A (1987) 107–115.

[8] Cartwright, D. I., Soardi, P. M., and Woess, W.: *Martin and end compactifications for non-locally finite graphs,* Trans. Amer. Math. Soc. 338 (1993) 679–693.

[9] Crotti, S.: *La frontiera di Martin del prodotto Cartesiano di un albero e di una retta,* Tesi di Laurea (Master thesis, in Italian), Univ. Milano, 1995.

[10] Derriennic, Y.: *Marche aléatoire sur le groupe libre et frontière de Martin,* Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 32 (1975) 261–276.

[11] Dor-On, A.: *Toeplitz quotient C^*-algebras and ratio limits for random walks,* Doc. Math. 26 (2021) 1529–1556.

[12] Dussaule, M.: *Local limit theorems in relatively hyperbolic groups II: the non-spectrally degenerate case,* preprint (2020), `arXiv:2004.13986`.

[13] Dynkin, E. B. and Maljutov, M. B.: *Random walk on groups with a finite number of generators* (Russian), Dokl. Akad. Nauk SSSR 137 (1961) 1042–1045.

[14] Elder, M. and Rogers, C.: *On a theorem of Avez,* J. Group Theory 22 (2019), 383–395.

[15] Gerl, P.: *Irrfahrten auf F_2,* Monatsh. Math. 84 (1977) 29–35.

[16] Gerl, P. and Woess, W.: *Local limits and harmonic functions for non-isotropic random walks on free groups,* Probab. Theory Relat. Fields 71 (1986) 341–355.

[17] Gouëzel, S.: *Local limit theorem for symmetric random walks in Gromov-hyperbolic groups,* J. Amer. Math. Soc. 27 (2014) 893–928.

[18] Gouëzel, S.: *Martin boundary of random walks with unbounded jumps in hyperbolic groups,* Ann. Probab. 43 (2015) 2374–2404.

[19] Gouëzel, S. and Lalley, S. P.: *Random walks on co-compact Fuchsian groups,* Ann. Sci. Éc. Norm. Supér. 46 (2013) 129–173.

[20] Gromov, M.: *Hyperbolic groups,* in *Essays in Group Theory* (ed. S. M. Gersten), pp. 75–263, Math. Sci. Res. Inst. Publ. 8, Springer, New York, 1987.

[21] Kaimanovich, V. A. and Vershik, A. M.: *Random walks on discrete groups: boundary and entropy,* Ann. Probab. 11 (1983) 457–490.

[22] Kesten, H.: *Full Banach mean values on countable groups,* Math. Scand. 7 (1959) 146–156.

[23] Lalley, S. P.: *Finite range random walk on free groups and homogeneous trees,* Ann. Probab. 21 (1993) 2087–2130.

[24] Picardello, M. A. and Woess, W.: *Martin boundaries of random walks: ends of trees and groups,* Trans. Amer. Math. Soc. 302 (1987) 185–205.

[25] Picardello, M. A., and Woess, W.: *A converse to the mean value property on homogeneous trees,* Trans. Amer. Math. Soc. 311 (1989) 209–225.

[26] Picardello, M. A. and Woess, W.: *The full Martin boundary of the bi-tree,* Ann. Probab. 22 (1994) 2203–2222.

[27] Sawyer, S.: *Isotropic random walks in a tree,* Z. Wahrsch. Verw. Gebiete 42 (1978) 279–292.

[28] Woess, W.: *Fixed sets and free subgroups of groups acting on metric spaces,* Math. Zeitschr. 214 (1993) 425–440.

[29] Woess, W.: *Random Walks on Infinite Graphs and Groups*, Cambridge Tracts in Mathematics 138, Cambridge University Press, Cambridge, 2000.

[30] Woess, W.: *Denumerable Markov Chains. Generating Functions, Boundary Theory, Random Walks on Trees,* European Math. Soc. Publishing House, 2009.

[31] Woess, W.: *Lamplighters, Diestel-Leader graphs, random walks, and harmonic functions,* Combin. Probab. Comput. 14 (2005) 415–433.

Wolfgang Woess
Institut für Diskrete Mathematik
Technische Universität Graz
Steyrergasse 30
8010 Graz
Austria
woess@tugraz.at