Erratum for "The Minimal Exact Crossed Product"

Cf. Documenta Mathematica, vol. 23 (2018), p. 2043-2077.
DOI: 10.25537/dm.2018v23.2043-2077

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Received: May 4, 2021
Revised: August 22, 2021

Communicated by Wilhelm Winter

Abstract. We point out some mistakes in our paper [2]. The most severe one is a gap in our proof that the smallest exact crossed product functor, as constructed in [2], is automatically Morita compatible. We were not able to close this gap so far. So, at this time, we cannot conclude that the C^*-group algebra $C^*_{\epsilon_M}(G) = \mathbb{C} \rtimes_{\epsilon_M} G$ corresponding to the smallest exact *Morita compatible* crossed-product functor $\rtimes_{\epsilon_M}$ is equal to the reduced group C^*-algebra $C^*_r(G)$.

2020 Mathematics Subject Classification: 46L55, 46L80
Keywords and Phrases: Exotic crossed products, Baum-Connes conjecture, exact groups, exotic group algebras

Unfortunately, there are two severe gaps in the paper [2]. We are very grateful to Yosuke Kubota who pointed these out to us. The first of these can be fixed under a mild additional hypothesis; we sketch how to do this below. The second seems much more serious, and we are currently unable to fix it.

There is also a third mistake in a side-remark that we record below; this third error does not affect the rest of the paper at all.

The first problem appears in the proof of [2, Proposition 2.4]. The first part of the proof of this proposition, dealing with the general case, is correct. The problem arises with the additional argument dealing with the unital case. Specifically, when trying to prove exactness of the left column of the diagram (5) on page 2051, we construct a map $E : \tilde{C} \rtimes_\mu G \to \tilde{C} \rtimes_\mu G$ and claim at the bottom of that page that it maps $C_c(G, J)$ into $C_c(G, I)$. As pointed out to us by Yosuke Kubota, this is not true in general. However, under the mild extra assumption that the crossed-product functor $\rtimes_\mu$ satisfies the ideal property the proof of Proposition 2.4 can be fixed; in particular, this applies when $\rtimes_\mu$ is the reduced crossed product, which is the most important special instance of Proposition 2.4. The ideal property means that for every G-invariant ideal $I \subseteq A$ the crossed product $I \rtimes_\mu G$ injects into $A \rtimes_\mu G$. The following lemma is due to Narutaka Ozawa and was communicated to us by Yosuke Kubota.

Lemma 1. *Suppose that $\rtimes_\mu$ has the ideal property. Then every short exact sequence of G-algebras $0 \to I \to A \to \mathbb{C} \to 0$ induces a short exact sequence*

$$0 \to I \rtimes_\mu G \to A \rtimes_\mu G \to \mathbb{C} \rtimes_\mu G \to 0.$$

Proof. Consider the generalized homomorphism $\mathbb{C} \to M(A); \lambda \mapsto \lambda 1_{M(A)}$. By the ideal property, this descends to a nondegenerate morphism $\mathbb{C} \rtimes_\mu G \to M(A \rtimes_\mu G)$ (see [1, Lemma 3.3]) and therefore uniquely extends to a $*$-homomorphism $\phi : M(\mathbb{C} \rtimes_\mu G) \to M(A \rtimes_\mu G)$ which splits the extension $M(A \rtimes_\mu G) \to M(\mathbb{C} \rtimes_\mu G)$ of the quotient map $q : A \rtimes_\mu G \to \mathbb{C} \rtimes_\mu G$. Assume now that $x \in A \rtimes_\mu G$ is mapped to 0 in $\mathbb{C} \rtimes_\mu G$ and let $(x_n)_n$ be a sequence in $C_c(G, A)$ which converges to x in norm. Let $(e_i)_{i \in I}$ be a bounded approximate unit for $A \rtimes_\mu G$ which lies in $C_c(G, A)$. Then $x_{n,i} := \big(x_n - \phi \circ q(x_n)\big) e_i$ is an element of $C_c(G, A)$ such that $q(x_{n,i}) = 0$, and hence $x_{n,i} \in C_c(G, I)$ for all $(n, i) \in \mathbb{N} \times I$. On the other hand we have $x_{n,i} \to x$ in norm, hence $x \in I \rtimes_\mu G$, which, by the ideal property, coincides with the norm-closure of $C_c(G, I)$ in $A \rtimes_\mu G$. □

Now, if $\rtimes_\mu$ satisfies the ideal property, the above lemma implies that the two left columns of diagram (5) on p. 2051 of [2] are exact. The proof of Proposition 2.4 then follows as given. As a result of this, Proposition 2.6 should also assume the ideal property in its statement. No other results in Sections 2 or 3 are affected by this mistake. From Section 5, the following results are affected: the parts of Proposition 5.2, Proposition 5.5, and Corollary 5.6 dealing with the unital case. All hold if we assume the functor used has the ideal property. The results of Section 6 are not affected.

The second, and most severe, problem in our paper [2] comes from Lemma 4.3, which was an important step in our proof of [2, Proposition 4.4]. Proposition 4.4 says that the smallest exact crossed-product $\rtimes_{\epsilon(\mu)}$ which dominates a Morita compatible functor $\rtimes_\mu$ with the ideal property is automatically Morita compatible. We do not know if [2, Proposition 4.4] is correct or not, but [2, Lemma 4.3] is definitely false. The following counterexample to [2, Lemma 4.3] resulted from a discussion with Timo Siebenand.

Example 2. Following a construction of Brown and Guentner, we construct a crossed-product functor for a group G as the completion of $C_c(G, A)$ by the norm

$$\|f\|_\mu := \sup\{\|\pi \rtimes U(f)\| : (\pi, U) \text{ covariant rep. of } (A, G, \alpha),\ U \prec \lambda_G\},$$

where "$\prec$" denotes weak containment of representations. Now, up to unitary equivalence, every nondegenerate covariant representation (π, U) of $(\mathcal{K}(L^2(G)), G, \mathrm{Ad}\lambda_G)$ is given as $(\pi, U) = (\mathrm{id}_{\mathcal{K}(L^2(G))} \otimes 1_{\mathcal{H}}, \lambda_G \otimes V)$ acting on $L^2(G) \otimes \mathcal{H}$ for some Hilbert space $\mathcal{H}$. Therefore $U \prec \lambda_G$ by Fell's trick. It follows that

$$\mathcal{K}(L^2(G)) \rtimes_{\mathrm{Ad}\lambda_G,\mu} G = \mathcal{K}(L^2(G)) \rtimes_{\mathrm{Ad}\lambda_G,\max} G \cong \mathcal{K}(L^2(G)) \otimes C^*_{\max}(G),$$

while $\mathcal{K}(L^2(G)) \otimes (\mathbb{C} \rtimes_\mu G) = \mathcal{K}(L^2(G)) \otimes C^*_r(G)$. Since $\rtimes_\mu$ satisfies the ideal property, this example contradicts [2, Lemma 4.3] whenever G is not amenable.

A version of Lemma 4.3 could still be true under the extra assumption that $\rtimes_\mu$ is Morita compatible itself. Note that the functor $\rtimes_\mu$ of the above example is not. Since the map

$$\Phi_{\mathcal{E}(\mu)} : (A \otimes \mathcal{K}) \rtimes_{\mathcal{E}(\mu)} G \to (A \rtimes_{\mathcal{E}(\mu)} G) \otimes \mathcal{K}.$$

as constructed in the proof of Proposition 4.4 still exists, this would save [2, Proposition 4.4] and all its consequences. Unfortunately, so far we were not able to give a proof of the lemma even under this extra assumption. Therefore [2, Proposition 4.4] together with [2, Corollaries 4.5 – 4.8] are still open. In particular it is still open, whether the group algebra $C^*_{\epsilon_M}(G) = \mathbb{C} \rtimes_{\epsilon_M} G$ for the smallest *Morita compatible* exact crossed-product functor $\rtimes_{\epsilon_M}$ coincides with the reduced group algebra $C^*_r(G)$.
The results in [2, §5] and [2, §6] appear unaffected by these problems. In particular, Theorem 6.3 holds true with $\rtimes_{\epsilon_G}$ and $\rtimes_{\epsilon_H}$ denoting either the smallest exact crossed-product functors or the smallest exact Morita compatible exact crossed-product functors for G and H.

We finally would like to point out a third mistake, which appears in the paragraph before [2, Theorem 3.5]. There we claim that for a G-algebra A the canonical map

$$\iota^{**} : A^{**} \to (A \rtimes_\mu G)^{**}$$

is always injective, where A^{**} denotes the double dual of A. This is indeed true if G is discrete, but it does not hold in general. We did not use this statement anywhere in the paper.
To see a counterexample, let G be any locally compact group acting on itself by translation. Then $C_0(G) \rtimes G \cong \mathcal{K}(L^2(G))$, hence $(C_0(G) \rtimes G)^{**} = \mathcal{B}(L^2(G))$ and then ι^{**} maps $C_0(G)^{**}$ onto $C_0(G)'' \cong L^\infty(G) \subseteq \mathcal{B}(L^2(G))$, where we represented $C_0(G)$ into $\mathcal{B}(L^2(G))$ via multiplication operators. But for general locally compact groups the map from $C_0(G)^{**}$ (which contains all characteristic functions $\chi_{\{g\}}$ of single points $g \in G$) to $L^\infty(G)$ is not injective. We should point out that if ι^{**} is not faithful on A^{**}, then it is also not faithful on the continuous part $A^{**}_c = \{a \in A^{**} : g \mapsto \alpha^{**}_g(a) \text{ norm continuous}\}$. Indeed, since $N := \ker \iota^{**}$ is a von Neumann subalgebra of A^{**} the unit 1_N of N clearly lies in A^{**}_c.

Acknowledgement

The authors were supported by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany 's Excellence Strategy – EXC 2044 – 390685587, Mathematics Münster: Dynamics – Geometry - Structure, by CNPq/CAPES/Humboldt - Brazil, and by the US NSF (DMS 1401126 and DMS 1564281).

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[2] Alcides Buss, Siegfried Echterhoff, and Rufus Willett, *The minimal exact crossed product*, Documenta Math. 23 (2018), 2043–2077.

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