# On a Torsion Analogue of the Weight-Monodromy Conjecture

Kazuhiro Ito

Received: October 7, 2020
Revised: July 30, 2021

Communicated by Takeshi Saito

Abstract. We formulate and study a torsion analogue of the weight-monodromy conjecture for a proper smooth scheme over a non-archimedean local field. We prove it for proper smooth schemes over equal characteristic non-archimedean local fields, abelian varieties, surfaces, varieties uniformized by Drinfeld upper half spaces, and set-theoretic complete intersections in projective smooth toric varieties. In the equal characteristic case, our methods rely on an ultraproduct variant of Weil II established by Cadoret.

2020 Mathematics Subject Classification: 11G25, 14F20, 14C25
Keywords and Phrases: Weight-monodromy conjecture, weight spectral sequence, ultraproduct

## 1 Introduction

Let $K$ be a non-archimedean local field with finite residue field $k$. Let $p > 0$ be the characteristic of $k$, and $q$ the number of elements in $k$. Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. Let $\ell \neq p$ be a prime number. Let $\overline{K}$ be an algebraic closure of $K$ and $K^{\mathrm{sep}}$ the separable closure of $K$ in $\overline{K}$. The absolute Galois group $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ naturally acts on the $\ell$-adic cohomology $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, where we put $X_{\overline{K}} := X \otimes_K \overline{K}$.
By Grothendieck's quasi-unipotence theorem, the action of an open subgroup of the inertia group $I_K$ of $K$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ defines the monodromy filtration

$$\{M_{i,\mathbb{Q}_\ell}\}_i$$

on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. It is an increasing filtration stable by the action of $G_K$. (See Section 3.1 for details.) The *weight-monodromy conjecture* due to Deligne states that the $i$-th graded piece

$$\mathrm{Gr}^M_{i,\mathbb{Q}_\ell} := M_{i,\mathbb{Q}_\ell}/M_{i-1,\mathbb{Q}_\ell}$$

of the monodromy filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is of weight $w+i$, i.e. every eigenvalue of a lift of the geometric Frobenius element $\mathrm{Frob}_k \in \mathrm{Gal}(\overline{k}/k)$ is an algebraic integer such that the complex absolute values of its conjugates are $q^{(w+i)/2}$. When $X$ has good reduction over the ring of integers $\mathcal{O}_K$ of $K$, it is nothing more than the Weil conjecture [12, 13]. In general, the weight-monodromy conjecture is still open. In this paper, we shall propose a torsion analogue of the weight-monodromy conjecture and prove it in some cases.

By the work of Rapoport-Zink [30] and de Jong's alteration [10], we can take an open subgroup $J \subset I_K$ such that the action of $J$ on the étale cohomology group with $\mathbb{F}_\ell$-coefficients $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ is unipotent for every $\ell \neq p$. By the same construction as in the $\ell$-adic case, we can define the monodromy filtration

$$\{M_{i,\mathbb{F}_\ell}\}_i$$

on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ for all but finitely many $\ell \neq p$, which is stable by the action of $G_K$; see Section 3.2 for details. We propose the following conjecture.

Conjecture 1.1 (A torsion analogue of the weight-monodromy conjecture, Conjecture 3.5). *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. Let $\mathrm{Frob} \in G_K$ be a lift of the geometric Frobenius element. For every $i$, there exists a non-zero monic polynomial $P_i(T) \in \mathbb{Z}[T]$ satisfying the following conditions:*

- *The roots of $P_i(T)$ have complex absolute values $q^{(w+i)/2}$.*
- *We have $P_i(\mathrm{Frob}) = 0$ on the $i$-th graded piece*

$$\mathrm{Gr}^M_{i,\mathbb{F}_\ell} := M_{i,\mathbb{F}_\ell}/M_{i-1,\mathbb{F}_\ell}$$

  *for all but finitely many $\ell \neq p$.*

Remark 1.2. The étale cohomology group with $\mathbb{Z}_\ell$-coefficients $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$ is torsion-free for all but finitely many $\ell \neq p$; see [14] and [36, Theorem 1.4]. (See also Remark 2.5.) When $X$ has good reduction over $\mathcal{O}_K$, Conjecture 1.1 follows from the Weil conjecture and this result.

The main theorem of this paper is as follows:

Theorem 1.3 (Theorem 3.7). *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. Assume that one of the following conditions holds:*

*(1) $K$ is of equal characteristic, i.e. the characteristic of $K$ is $p$.*

*(2) $X$ is an abelian variety.*

*(3) $w \leq 2$ or $w \geq 2\dim X - 2$.*

*(4) $X$ is uniformized by a Drinfeld upper half space.*

*(5) $K$ is of characteristic $0$, and $X$ is geometrically connected and is a set-theoretic complete intersection in a projective smooth toric variety.*

*Then the assertion of Conjecture 1.1 for $(X, w)$ is true.*

The weight-monodromy conjecture for $\mathbb{Q}_\ell$-coefficients is known to be true for $(X, w)$ if one of the above conditions (1)–(5) holds for $(X, w)$. However, it seems that the weight-monodromy conjecture for $\mathbb{Q}_\ell$-coefficients does not automatically imply Conjecture 1.1. The problem is that, in general, we do not know the torsion-freeness of the cokernel of the monodromy operator acting on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$ for all but finitely many $\ell \neq p$. (See Section 3.3 for details.)

Remark 1.4. As is the case for the weight-monodromy conjecture for $\mathbb{Q}_\ell$-coefficients, if $X$ has a proper strictly semi-stable model over $\mathcal{O}_K$, then Conjecture 1.1 is equivalent to a conjecture on the weight spectral sequence with $\mathbb{F}_\ell$-coefficients constructed by Rapoport-Zink; see Section 4 for details. In fact, the latter can be formulated for any Henselian discrete valuation field, and we will need to deal with a Henselian discrete valuation field whose residue field is the function field of some algebraic variety over $\mathbb{F}_p$ in the proof of Theorem 1.3 in the case (1).

Remark 1.5. There are other cases where the weight-monodromy conjecture is known to be true; see Remark 3.3. In this paper, we will restrict ourselves to the cases (1)–(5) for the sake of simplicity.

We shall give two applications of our results. The first one is an application to the finiteness of the Brauer group of a proper smooth scheme over $K$ for which the $\ell$-adic Chern class map for divisors is surjective; see Corollary 10.3. As the second application, we will show the finiteness of the $G_K$-fixed part of the prime-to-$p$ torsion part of the Chow group $\mathrm{CH}^2(X_{\overline{K}})$ of codimension two cycles on $X_{\overline{K}}$ if $(X, w = 3)$ satisfies one of the conditions (1)–(5); see Corollary 10.9.
The strategy of the proof of Theorem 1.3 is as follows. If $K$ is of equal characteristic and $X$ is defined over the function field of a smooth curve over a finite field, then Theorem 1.3 is a consequence of an ultraproduct variant of Weil II established by Cadoret [4]. The general case (1) can be deduced from this case by the same arguments as in [22].
As in [33], by using the tilting equivalence of Scholze, we will deduce the case (5) from the case (1) or from the results of Cadoret [4]. In his proof of the weight-monodromy conjecture in the case (5), Scholze used a theorem of Huber [18, Theorem 3.6] on étale cohomology of tubular neighborhoods of rigid analytic varieties. In our case, we use a uniform variant [19, Corollary 4.11] of Huber's theorem proved by the author. See Section 6 for details.
For the case (2), we prove that, for abelian varieties, the cokernels of the monodromy operators are torsion-free by using the theory of Néron models. Then the case (2) is deduced from the weight-monodromy conjecture for abelian varieties. For the proof in the remaining cases, we use the weight spectral sequence with $\mathbb{Z}_\ell$-coefficients. Since the weight-monodromy conjecture is known to be true under the assumptions, it suffices to prove that the cokernels of the monodromy operators are torsion-free for all but finitely many $\ell \neq p$. In our

settings, it basically follows from the torsion-freeness for all but finitely many $\ell \neq p$ of the cokernel of a homomorphism of one of the following types:

- The homomorphism $T_\ell A \to \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell A, \mathbb{Z}_\ell(1))$ induced by a polarization of an abelian variety $A$ over $\overline{k}$. Here $T_\ell A$ is the $\ell$-adic Tate module of $A$.

- The base change $M_1 \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to M_2 \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ of a homomorphism $M_1 \to M_2$ of finitely generated $\mathbb{Z}$-modules.

The outline of this paper is as follows. In Section 2, we define a notion of weight for a family $\{H_\ell\}_{\ell \neq p}$ of $G_K$-representations over $\mathbb{F}_\ell$ and prepare some elementary lemmas used in this paper. In Section 3, we define the monodromy filtration with $\mathbb{F}_\ell$-coefficients for all but finitely many $\ell \neq p$ and propose a torsion analogue of the weight-monodromy conjecture (Conjecture 1.1). We also discuss a relation between the weight-monodromy conjecture and Conjecture 1.1. In Section 4, we discuss some torsion-freeness properties of the weight spectral sequence and their relation to Conjecture 1.1. We also include some standard techniques used to study the weight-monodromy conjecture. In Sections 5–9, we prove Theorem 1.3. In Section 10, as applications of Theorem 1.3, we discuss some finiteness properties of the Brauer group and the codimension two Chow group of a proper smooth scheme over $K$.

### Notation

Throughout this paper, we will use the following notation. For a field $F$, let $\overline{F}$ be an algebraic closure of $F$ and $F^{\mathrm{sep}}$ the separable closure of $F$ in $\overline{F}$. Let $G_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$ be the absolute Galois group of $F$. Let $\mathrm{char}(F)$ denote the characteristic of $F$. We call a finitely generated $\mathbb{Z}_\ell$-module endowed with a continuous action of $G_F$ a *$G_F$-module over $\mathbb{Z}_\ell$* for simplicity.

Along this paper $K$ denotes a Henselian discrete valuation field. The ring of integers of $K$ is denoted by $\mathcal{O}_K$ and the residue field of $\mathcal{O}_K$ is denoted by $k$. Let $I_K \subset G_K$ be the inertia group of $K$. In Sections 2–3 and Sections 6–10, we will assume that $K$ is a non-archimedean local field. In that case, let $p > 0$ denote the characteristic of $k$ and let $q$ denote the number of elements in $k$. In Sections 4–5 we will work not only with non-archimedean local fields, but also with general Henselian discrete valuation fields. (See also Remark 1.4.)

## 2 Preliminaries

### 2.1 Weights

Let $p$ be a prime number. In this subsection, we fix a finitely generated field $F$ over $\mathbb{F}_p$.

Let $\ell \neq p$ be a prime number. For a finite dimensional representation of $G_F$ over $\mathbb{Q}_\ell$, there is a notion of weight; see [22, Section 2.2] for example. In this

paper, we will use the following notion of weight for a family $\{H_\ell\}_{\ell\neq p}$ of $G_F$-modules over $\mathbb{Z}_\ell$. (See Notation in Section 1 for the definition of $G_F$-modules over $\mathbb{Z}_\ell$.)
Let $\mathcal{L}$ be an infinite set of prime numbers $\ell \neq p$. Let $w$ be an integer.

- Let $q$ be a power of $p$. For a non-zero monic polynomial $P(T) \in \mathbb{Z}[T]$, we say that $P(T)$ is a *Weil $q$-polynomial* if the complex absolute value of every root of $P(T)$ is $q^{1/2}$.
- Let $U$ be an integral scheme of finite type over $\mathbb{F}_p$ with function field $F$. We say that a family $\{\mathcal{F}_\ell\}_{\ell\in\mathcal{L}}$ of locally constant constructible $\mathbb{Z}_\ell$-sheaves on $U$ is *of weight $w$* if, for every closed point $x \in U$, there is a Weil $(q_x)^w$-polynomial $P_x(T) \in \mathbb{Z}[T]$ such that, for all but finitely many $\ell \in \mathcal{L}$, we have $P_x(\mathrm{Frob}_x) = 0$ on $\mathcal{F}_{\ell,\overline{x}}$. Here $q_x$ is the number of elements in the residue field $\kappa(x)$ of $x$, $\overline{x}$ is a geometric point of $U$ above $x$, and $\mathrm{Frob}_x \in G_{\kappa(x)}$, $a \mapsto a^{1/q_x}$ is the geometric Frobenius element.
- We say that a family $\{H_\ell\}_{\ell\in\mathcal{L}}$ of $G_F$-modules over $\mathbb{Z}_\ell$ is *of weight $w$* if there is an integral scheme $U$ of finite type over $\mathbb{F}_p$ with function field $F$ such that the family $\{H_\ell\}_{\ell\in\mathcal{L}}$ comes from a family $\{\mathcal{F}_\ell\}_{\ell\in\mathcal{L}}$ of locally constant constructible $\mathbb{Z}_\ell$-sheaves on $U$ of weight $w$.

When there is no possibility of confusion, we will omit $\mathcal{L}$ from the notation and write $\{H_\ell\}_\ell$ in place of $\{H_\ell\}_{\ell\in\mathcal{L}}$.

LEMMA 2.1. *Let $\{H_{1,\ell}\}_{\ell\in\mathcal{L}}$ and $\{H_{2,\ell}\}_{\ell\in\mathcal{L}}$ be families of $G_F$-modules over $\mathbb{Z}_\ell$ of weight $w_1$ and $w_2$, respectively. We assume $w_1 \neq w_2$. Then, for all but finitely many $\ell \in \mathcal{L}$, every map $H_{1,\ell} \to H_{2,\ell}$ of $G_F$-modules over $\mathbb{Z}_\ell$ is zero.*

PROOF. We may assume that $\{H_{1,\ell}\}_\ell$ and $\{H_{2,\ell}\}_\ell$ come from families $\{\mathcal{F}_{1,\ell}\}_\ell$ and $\{\mathcal{F}_{2,\ell}\}_\ell$ of locally constant constructible $\mathbb{Z}_\ell$-sheaves on $U$ of weight $w_1$ and $w_2$, respectively. Here $U$ is an integral scheme of finite type over $\mathbb{F}_p$ with function field $F$. Take a closed point $x \in U$. Let $P_{1,x}(T) \in \mathbb{Z}[T]$ be a Weil $(q_x)^{w_1}$-polynomial such that, for all but finitely many $\ell \in \mathcal{L}$, we have $P_{1,x}(\mathrm{Frob}_x) = 0$ on $(\mathcal{F}_{1,\ell})_{\overline{x}}$. Let $P_{2,x}(T) \in \mathbb{Z}[T]$ be a Weil $(q_x)^{w_2}$-polynomial which satisfies the same condition for $\{\mathcal{F}_{2,\ell}\}_\ell$. The polynomials $P_{1,x}(T)$ and $P_{2,x}(T)$ are relatively prime. For $\ell \in \mathcal{L}$ such that $P_{1,x}(\mathrm{Frob}_x) = 0$ on $(\mathcal{F}_{1,\ell})_{\overline{x}}$ and $P_{2,x}(\mathrm{Frob}_x) = 0$ on $(\mathcal{F}_{2,\ell})_{\overline{x}}$, we have $P_{1,x}(\mathrm{Frob}_x) = P_{2,x}(\mathrm{Frob}_x) = 0$ on the stalk of the image of any map $\mathcal{F}_{1,\ell} \to \mathcal{F}_{2,\ell}$ at $\overline{x}$. Therefore, the assertion follows from Lemma 2.2 below. □

LEMMA 2.2. *Let $P_1(T), P_2(T) \in \mathbb{Q}[T]$ be two relatively prime polynomials. For all but finitely many prime numbers $\ell$, every $\mathbb{Z}_\ell[T]$-module $H_\ell$ such that $P_1(T) = P_2(T) = 0$ on $H_\ell$ is zero.*

PROOF. There exist polynomials $Q_1(T), Q_2(T) \in \mathbb{Q}[T]$ satisfying

$$P_1(T)Q_1(T) + P_2(T)Q_2(T) = 1$$

in $\mathbb{Q}[T]$ since $P_1(T)$ and $P_2(T)$ are relatively prime. Thus, for all but finitely many prime numbers $\ell$, we have $P_1(T), P_2(T) \in \mathbb{Z}_\ell[T]$, and they generate the unit ideal of $\mathbb{Z}_\ell[T]$. The assertion follows from this fact. □

We need the following theorem to define monodromy filtrations with $\mathbb{F}_\ell$-coefficients for all but finitely many $\ell$ and to prove main results in this paper.

Theorem 2.3 (Gabber [14], Suh [36, Theorem 1.4]). *Let $X$ be a proper smooth scheme over a separably closed field of characteristic $p \geq 0$. For all but finitely many $\ell \neq p$, the $\mathbb{Z}_\ell$-module $H^w_{\text{ét}}(X, \mathbb{Z}_\ell)$ is torsion-free for every $w$. In particular, for all but finitely many $\ell \neq p$, the natural map $H^w_{\text{ét}}(X, \mathbb{Z}_\ell) \to H^w_{\text{ét}}(X, \mathbb{F}_\ell)$ gives an isomorphism*

$$H^w_{\text{ét}}(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\sim} H^w_{\text{ét}}(X, \mathbb{F}_\ell)$$

*for every $w$.*

Proof. If $X$ is projective, this is a theorem of Gabber [14, Theorem]. By using de Jong's alteration [10, Theorem 4.1], the general case can be deduced from the projective case; see the proof of [36, Theorem 1.4] for details. (See also Lemma 4.11 in Section 4.2.) □

Corollary 2.4. *Let $X$ be a proper smooth scheme over $F$. Then $\{H^w_{\text{ét}}(X_{\overline{F}}, \Lambda_\ell)\}_{\ell \neq p}$ is a family of $G_F$-modules of weight $w$, where $\Lambda_\ell$ is either $\mathbb{Z}_\ell$ or $\mathbb{F}_\ell$.*

Proof. This follows from the Weil conjecture [13, Corollaire (3.3.9)] and Theorem 2.3. □

Remark 2.5. An alternative proof of Theorem 2.3 using ultraproduct Weil cohomology theory was obtained by Orgogozo; see [29, Théorème 6.2.2]. Moreover, Cadoret also gave a new proof of Theorem 2.3 without using the pgcd theorem [13, Théorème (4.5.1)] (contrary to the proofs of Gabber and Orgogozo); see [4, Corollary 12.1.2]. In fact, Cadoret first proved Corollary 2.4 and then obtained Theorem 2.3 as a consequence. It is worth to mention that the results of Cadoret in [4] (especially [4, Theorem 3.6.3]) are based on [29, Théorème 3.1.1], which is the main theorem of [29].

Let $K$ be a non-archimedean local field. We will use the following notion of weight for representations of $G_K$. Let $\{H_\ell\}_{\ell \in \mathcal{L}}$ be a family of $G_K$-modules over $\mathbb{Z}_\ell$. We assume that there is an open subgroup $J \subset I_K$ such that the action of $J$ on $H_\ell$ is trivial for all but finitely many $\ell \in \mathcal{L}$.

Definition 2.6. We say that the family $\{H_\ell\}_{\ell \in \mathcal{L}}$ is *of weight $w$* if, for every lift $\mathrm{Frob} \in G_K$ of the geometric Frobenius element $\mathrm{Frob}_k \in G_k$, $a \mapsto a^{1/q}$, there is a Weil $q^w$-polynomial $P(T) \in \mathbb{Z}[T]$ such that, for all but finitely many $\ell \in \mathcal{L}$, we have $P(\mathrm{Frob}) = 0$ on $H_\ell$.

Lemma 2.7.

*(i) Assume that for* one *lift* $\mathrm{Frob} \in G_K$ *of the geometric Frobenius element there is a Weil $q^w$-polynomial $P(T) \in \mathbb{Z}[T]$ such that, for all but finitely many $\ell \in \mathcal{L}$, we have $P(\mathrm{Frob}) = 0$ on $H_\ell$. Then the family $\{H_\ell\}_{\ell \in \mathcal{L}}$ is of weight $w$.*

*(ii) Let $L$ be a finite extension of $K$. Then $\{H_\ell\}_{\ell \in \mathcal{L}}$ is of weight $w$ as a family of $G_K$-modules over $\mathbb{Z}_\ell$ if and only if $\{H_\ell\}_{\ell \in \mathcal{L}}$ is of weight $w$ as a family of $G_L$-modules over $\mathbb{Z}_\ell$.*

Proof. (i) Let $\mathrm{Frob}' \in G_K$ be a lift of the geometric Frobenius element. We want to show that there exists a Weil $q^w$-polynomial $Q(T) \in \mathbb{Z}[T]$ such that, for all but finitely many $\ell \in \mathcal{L}$, we have $Q(\mathrm{Frob}') = 0$ on $H_\ell$. Since the action of $J$ on $H_\ell$ is trivial for all but finitely many $\ell \in \mathcal{L}$, there is a positive integer $n$ such that, for all but finitely many $\ell \in \mathcal{L}$, the action of $\mathrm{Frob}^n$ on $H_\ell$ coincides with that of $(\mathrm{Frob}')^n$ on $H_\ell$. We write $P(T)$ in the form $P(T) = \prod_i (T - \alpha_i)$ with $\alpha_i \in \overline{\mathbb{Q}}$ and put

$$Q(T) := P^{(n)}(T) := \prod_i (T^n - \alpha_i^n) \in \mathbb{Z}[T],$$

which is a Weil $q^w$-polynomial. Then we have, for all but finitely many $\ell \in \mathcal{L}$, $Q(\mathrm{Frob}') = P^{(n)}(\mathrm{Frob}) = 0$ on $H_\ell$.

(ii) Let $f$ be the residue degree of the extension $L/K$. Let $\mathrm{Frob} \in G_K$ and $\mathrm{Frob}' \in G_L$ be lifts of the geometric Frobenius elements. As in the proof of (i), there is a positive integer $n$ such that, for all but finitely many $\ell \in \mathcal{L}$, the action of $\mathrm{Frob}^{fn}$ on $H_\ell$ coincides with that of $(\mathrm{Frob}')^n$ on $H_\ell$.

We assume that $\{H_\ell\}_{\ell \in \mathcal{L}}$ is of weight $w$ as a family of $G_L$-modules and let $P(T) \in \mathbb{Z}[T]$ be a Weil $q^{fw}$-polynomial satisfying the condition in Definition 2.6 for $\mathrm{Frob}'$. Then $Q(T) := P^{(n)}(T^f) \in \mathbb{Z}[T]$ is a Weil $q^w$-polynomial and we have

$$Q(\mathrm{Frob}) = P^{(n)}(\mathrm{Frob}^f) = P^{(n)}(\mathrm{Frob}') = 0$$

on $H_\ell$ for all but finitely many $\ell \in \mathcal{L}$. Therefore $\{H_\ell\}_{\ell \in \mathcal{L}}$ is of weight $w$ as a family of $G_K$-modules by (i).

The converse can be proved in a similar way. □

## 2.2 Some elementary lemmas on nilpotent operators

We collect some elementary lemmas on nilpotent operators, which will be used in the sequel.

Lemma 2.8. *Let $\ell$ be a prime number. Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be a complex of free $\mathbb{Z}_\ell$-modules of finite rank. The reduction modulo $\ell$ of $f$ and $g$ will be denoted by $\overline{f}$ and $\overline{g}$, respectively. Hence we have the following complex of $\mathbb{F}_\ell$-vector spaces:*

$$M_1 \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\overline{f}} M_2 \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\overline{g}} M_3 \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell.$$

*Then we have*

$$\mathrm{rank}_{\mathbb{Z}_\ell}(\mathrm{Ker}\, g/\,\mathrm{Im}\, f) \leq \dim_{\mathbb{F}_\ell}(\mathrm{Ker}\, \overline{g}/\,\mathrm{Im}\, \overline{f}).$$

*The equality holds if and only if the $\mathbb{Z}_\ell$-modules* $\mathrm{Coker}\, f$ *and* $\mathrm{Coker}\, g$ *are torsion-free. If this is the case, then we have* $(\mathrm{Ker}\, g/\,\mathrm{Im}\, f) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \overset{\sim}{\to} \mathrm{Ker}\, \overline{g}/\,\mathrm{Im}\, \overline{f}$.

Proof. By the theory of elementary divisors, we have

$$\mathrm{rank}_{\mathbb{Z}_\ell}(\mathrm{Ker}\, g/\,\mathrm{Im}\, f) \leq \dim_{\mathbb{F}_\ell} (\mathrm{Ker}\, g/\,\mathrm{Im}\, f) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell,$$

and the equality holds if and only if $\mathrm{Ker}\, g/\,\mathrm{Im}\, f$ is torsion-free. Since $M_3$ is torsion-free, we see that $\mathrm{Ker}\, g/\,\mathrm{Im}\, f$ is torsion-free if and only if $\mathrm{Coker}\, f$ is torsion-free. Moreover, we have inclusions $\mathrm{Im}\, \overline{f} \subset (\mathrm{Ker}\, g) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \subset \mathrm{Ker}\, \overline{g}$. Hence we have

$$\dim_{\mathbb{F}_\ell} (\mathrm{Ker}\, g/\,\mathrm{Im}\, f) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \leq \dim_{\mathbb{F}_\ell} (\mathrm{Ker}\, \overline{g}/\,\mathrm{Im}\, \overline{f}),$$

and the equality holds if and only if $(\mathrm{Ker}\, g) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell = \mathrm{Ker}\, \overline{g}$. It is easy to see that $(\mathrm{Ker}\, g) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell = \mathrm{Ker}\, \overline{g}$ if and only if $\mathrm{Coker}\, g$ is torsion-free. This fact completes the proof of the lemma. □

Lemma 2.9. *Let $V$ be a vector space of dimension $n$ over a field $\mathbb{F}$ of positive characteristic $\ell$. We assume that $\ell \geq n$. For a unipotent operator $U$ on $V$, we define*

$$\log(U) := \sum_{1 \leq i \leq n-1} \frac{(-1)^{i+1}}{i}(U-1)^i.$$

*For a nilpotent operator $N$ on $V$, we define*

$$\exp(N) := \sum_{0 \leq i \leq n-1} \frac{1}{i!} N^i.$$

*Then the following assertions hold.*

*(i)* $\log(-)$ *defines a bijection from the set of unipotent operators on $V$ to the set of nilpotent operators on $V$ with inverse map* $\exp(-)$.

*(ii) For two unipotent operators $U, U'$ (resp. two nilpotent operators $N, N'$) such that they commute, we have* $\log(UU') = \log(U) + \log(U')$ *(resp.* $\exp(N + N') = \exp(N)\exp(N')$*).*

Proof. Although this lemma is well known, we recall the proof for the reader's convenience.
(i) Let $\mathbb{Z}_{(\ell)}$ be the localization of $\mathbb{Z}$ at the prime ideal $(\ell)$. It suffices to prove that the homomorphism $\mathbb{Z}_{(\ell)}[S]/(S-1)^n \to \mathbb{Z}_{(\ell)}[T]/(T)^n$, $S \mapsto \exp(T)$ and the homomorphism $\mathbb{Z}_{(\ell)}[T]/(T)^n \to \mathbb{Z}_{(\ell)}[S]/(S-1)^n$, $T \mapsto \log(S)$ are inverse to each other, where $\exp(-)$ and $\log(-)$ are defined by the same formulas

as above. Since both rings are torsion-free over $\mathbb{Z}_{(\ell)}$, it suffices to prove the claim after tensoring with $\mathbb{Q}$. Then it follows from the fact that the map $\mathbb{Q}[[S-1]] \to \mathbb{Q}[[T]]$, $S-1 \mapsto \exp(T)-1$ and the map $\mathbb{Q}[[T]] \to \mathbb{Q}[[S-1]]$, $T \mapsto \log(S)$ are inverse to each other, where $\exp(-)$ and $\log(-)$ are defined in the usual way.
(ii) By (i), we only need to prove that, for two nilpotent operators $N, N'$ such that they commute, we have $\exp(N+N') = \exp(N)\exp(N')$. We have $N^i(N')^j = 0$ on $V$ for $i, j \geq 0$ with $i + j \geq n$. Thus, it suffices to prove $\exp(T+T') = \exp(T)\exp(T')$ in $\mathbb{Z}_{(\ell)}[T,T']/(T,T')^n$, where $\exp(-)$ is defined by the same formula as above. As in (i), this can be deduced from an analogous statement for $\mathbb{Q}[[T,T']]$. $\square$

Let $R$ be a principal ideal domain and $F$ its field of fractions. Let $H$ be a free $R$-module of finite rank. Let $N\colon H \to H$ be a nilpotent homomorphism. By [13, Proposition (1.6.1)], the nilpotent homomorphism $N_F := N \otimes_R F$ on $H_F := H \otimes_R F$ determines a unique increasing, separated, exhaustive filtration $\{M_{i,F}\}_i$ on $H_F$ characterized by the following properties:

- $N_F(M_{i,F}) \subset M_{i-2,F}$ for every $i$.
- For every integer $i \geq 0$, the $i$-th iterate $N_F^i$ induces an isomorphism $\mathrm{Gr}^M_{i,F} \xrightarrow{\sim} \mathrm{Gr}^M_{-i,F}$. Here we put $\mathrm{Gr}^M_{i,F} := M_{i,F}/M_{i-1,F}$.

We call $\{M_{i,F}\}_i$ the filtration on $H_F$ associated with $N_F$. Let $\{M_i\}_i$ be the filtration on the $R$-module $H$ defined by $M_i := H \cap M_{i,F}$ for every $i$. The $R$-module $\mathrm{Gr}^M_i := M_i/M_{i-1}$ is torsion-free for every $i$.

LEMMA 2.10. *Let the notation be as above. The cokernel of the $i$-th iterate $N^i\colon H \to H$ of $N$ is torsion-free for every $i \geq 0$ if and only if $N^i$ induces an isomorphism $\mathrm{Gr}^M_i \xrightarrow{\sim} \mathrm{Gr}^M_{-i}$ for every $i \geq 0$.*

PROOF. Assume that the cokernel of $N^i\colon H \to H$ is torsion-free for every $i \geq 0$. Let $d \geq 0$ be the smallest integer such that $N^{d+1} = 0$. The cokernel of the $i$-th iterate of the homomorphism

$$\operatorname{Ker} N^d / \operatorname{Im} N^d \to \operatorname{Ker} N^d / \operatorname{Im} N^d$$

induced by $N$ is torsion-free for every $i \geq 0$. Thus, by the same argument as in the proof of [13, Proposition (1.6.1)], we can construct inductively an increasing, separated, exhaustive filtration $\{M'_i\}_i$ on $H$ satisfying the following properties:

- $\mathrm{Gr}^{M'}_i := M'_i/M'_{i-1}$ is torsion-free for every $i$.
- $N(M'_i) \subset M'_{i-2}$ for every $i$.
- For every integer $i \geq 0$, the $i$-th iterate $N^i$ induces an isomorphism $\mathrm{Gr}^{M'}_i \xrightarrow{\sim} \mathrm{Gr}^{M'}_{-i}$.

By uniqueness, the filtration $\{M_{i,F}\}_i$ coincides with $\{M'_i \otimes_R F\}_i$. Since both $\mathrm{Gr}_i^M$ and $\mathrm{Gr}_i^{M'}$ are torsion-free for every $i$, the filtration $\{M_i\}_i$ coincides with $\{M'_i\}_i$ and we have an isomorphism $N^i \colon \mathrm{Gr}_i^M \xrightarrow{\sim} \mathrm{Gr}_{-i}^M$ for every $i \geq 0$.

Conversely, we assume that $N^i$ induces an isomorphism $\mathrm{Gr}_i^M \xrightarrow{\sim} \mathrm{Gr}_{-i}^M$ for every $i \geq 0$. We fix an integer $i \geq 0$. For every $j \leq i$, the $i$-th iterate $N^i \colon \mathrm{Gr}_j^M \to \mathrm{Gr}_{j-2i}^M$ is surjective. It follows that $N^i \colon M_j \to M_{j-2i}$ is surjective for every $j \leq i$ since it is surjective for sufficiently small $j$. For every $j \geq i$, the $i$-th iterate $N^i \colon \mathrm{Gr}_j^M \to \mathrm{Gr}_{j-2i}^M$ is a split injection. It follows that the cokernel of $N^i \colon M_j \to M_{j-2i}$ is torsion-free for every $j \geq i$ since we have shown that it is zero for $j = i$. Hence the cokernel of $N^i \colon H \to H$ is torsion-free. □

## 3 A torsion analogue of the weight-monodromy conjecture

In this section, let $K$ be a non-archimedean local field. For a prime number $\ell \neq p$, the group of $\ell^n$-th roots of unity in $\overline{K}$ is denoted by $\mu_{\ell^n}$. Let

$$t_\ell \colon I_K \to \mathbb{Z}_\ell(1) := \varprojlim_n \mu_{\ell^n}$$

be the map defined by $g \mapsto \{g(\varpi^{1/\ell^n})/\varpi^{1/\ell^n}\}_n$ for a uniformizer $\varpi \in \mathcal{O}_K$. This map is independent of the choice of $\varpi$ and gives the maximal pro-$\ell$ quotient of $I_K$. Let $X$ be a proper smooth scheme over $K$ and $w$ an integer.

### 3.1 The weight-monodromy conjecture

We shall recall the definition of the monodromy filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ for every $\ell \neq p$, where $X_{\overline{K}} := X \otimes_K \overline{K}$. The absolute Galois group $G_K$ naturally acts on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ via the natural isomorphism $\mathrm{Aut}(\overline{K}/K) \xrightarrow{\sim} G_K$.

By Grothendieck's quasi-unipotence theorem, there is an open subgroup $J$ of $I_K$ such that the action of $J$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unipotent and factors through $t_\ell$. Take an element $\sigma \in J$ such that $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a non-zero element. We define

$$N_\sigma := \log(\sigma) := \sum_{1 \leq i} \frac{(-1)^{i+1}}{i}(\sigma - 1)^i \colon H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \to H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell).$$

Let $\{M_{i,\mathbb{Q}_\ell}\}_i$ be the filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ associated with $N_\sigma$; see [13, Proposition (1.6.1)]. The filtration $\{M_{i,\mathbb{Q}_\ell}\}_i$ is independent of $J$ and $\sigma \in J$. It is called the *monodromy filtration*. We have $\chi_{\mathrm{cyc}}(g) N_\sigma g = g N_\sigma$ for every $g \in G_K$, where $\chi_{\mathrm{cyc}} \colon G_K \to \mathbb{Z}_\ell^\times$ is the $\ell$-adic cyclotomic character. It follows from the uniqueness of the monodromy filtration that $\{M_{i,\mathbb{Q}_\ell}\}_i$ is stable by the action of $G_K$. We note that the filtration associated with $\sigma - 1$ coincides with $\{M_{i,\mathbb{Q}_\ell}\}_i$. We put

$$\mathrm{Gr}^M_{i,\mathbb{Q}_\ell} := M_{i,\mathbb{Q}_\ell}/M_{i-1,\mathbb{Q}_\ell}.$$

We recall the weight-monodromy conjecture due to Deligne.

Conjecture 3.1 (Deligne [11]). *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. Let $\ell \neq p$ be a prime number. Then the $i$-th graded piece $\mathrm{Gr}^M_{i,\mathbb{Q}_\ell}$ of the monodromy filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is of weight $w+i$, i.e. every eigenvalue of every lift* $\mathrm{Frob} \in G_K$ *of the geometric Frobenius element is an algebraic integer such that the complex absolute values of its conjugates are $q^{(w+i)/2}$.*

When $X$ has good reduction over $\mathcal{O}_K$, it is nothing more than the Weil conjecture. Conjecture 3.1 is known to be true in the following cases.

Theorem 3.2. *Conjecture 3.1 for $(X, w)$ is true in the following cases:*

*(1) $K$ is of equal characteristic ([13, 40, 22]).*

*(2) $X$ is an abelian variety ([44, Exposé IX]).*

*(3) $w \leq 2$ or $w \geq 2\dim X - 2$ ([30, 10, 32]).*

*(4) $X$ is uniformized by a Drinfeld upper half space ([21, 9]).*

*(5)* $\mathrm{char}(K) = 0$*, and $X$ is geometrically connected and is a set-theoretic complete intersection in a projective smooth toric variety ([33]).*

Proof. See the references given above. ☐

We will prove a torsion analogue of Conjecture 3.1 in each of the above cases.

Remark 3.3. There are other cases in which Conjecture 3.1 is known to be true. For example, in [20], it is proved for a certain projective threefold with strictly semistable reduction, and in [24], it is proved for a variety which is uniformized by a product of Drinfeld upper half spaces. We will not discuss a torsion analogue of Conjecture 3.1 for these varieties in this paper for the sake of simplicity.

Remark 3.4. Assume that $\mathrm{char}(K) = 0$. Let $X$ be a proper smooth geometrically connected scheme over $K$. Of course, if $X$ is a set-theoretic complete intersection in the projective space $\mathbb{P}^n$, then $X$ satisfies the condition (5), and the weight-monodromy conjecture holds for $X$ by [33]. This was already a new result (even for smooth hypersurfaces in $\mathbb{P}^n$). However, the condition (5) that $X$ is a set-theoretic complete intersection in a projective smooth toric variety (rather than in $\mathbb{P}^n$) might be more interesting. To the best of our knowledge, there are no known examples of $X$ that do not seem to satisfy this condition (5). (See also [34, p.12].) We remark that, if $X$ is a set-theoretic complete intersection in $\mathbb{P}^n$ and $\dim X \geq 2$, then $X_{\overline{K}}$ is simply connected (see [1, Theorem 1.5] for example), however this does not hold for a set-theoretic complete intersection in a projective smooth toric variety in general; for example, a hypersurface $\mathbb{P}^1 \times E$ in $\mathbb{P}^1 \times \mathbb{P}^2$ over an algebraically closed field, where $E \subset \mathbb{P}^2$ is an elliptic curve, is not simply connected.

### 3.2 A torsion analogue of the weight-monodromy conjecture

Let $\{H_\ell\}_{\ell \neq p}$ be a family of finite dimensional $G_K$-representations over $\mathbb{F}_\ell$. We define the monodromy filtrations when the family $\{H_\ell\}_{\ell \neq p}$ satisfies the following two conditions:

- There is an open subgroup $J$ of $I_K$ such that, for every $\ell \neq p$, the action of $J$ on $H_\ell$ is unipotent (i.e. $\sigma$ is a unipotent operator on $H_\ell$ for every $\sigma \in J$).
- $n := \sup_{\ell \neq p} \dim_{\mathbb{F}_\ell} H_\ell < \infty$.

The action of $J$ factors through $t_\ell$ for every $\ell \neq p$. Take an element $\sigma \in J$ such that, for all but finitely many $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator. For a prime number $\ell \neq p$ with $\ell \geq n$, we define

$$N_\sigma := \log(\sigma) := \sum_{1 \leq i \leq n-1} \frac{(-1)^{i+1}}{i}(\sigma - 1)^i \colon H_\ell \to H_\ell.$$

(See also Lemma 2.9.) Let

$$\{M_{i,\mathbb{F}_\ell}\}_i$$

be the filtration on $H_\ell$ associated with $N_\sigma$. The filtration $\{M_{i,\mathbb{F}_\ell}\}_i$ is independent of $J$ and $\sigma \in J$ up to excluding finitely many $\ell \neq p$. Moreover, for all but finite many $\ell \neq p$, we have

$$\overline{\chi_{\mathrm{cyc}}(g)} N_\sigma g = g N_\sigma$$

for every $g \in G_K$, where $\overline{\chi_{\mathrm{cyc}}(g)}$ is the reduction modulo $\ell$ of $\chi_{\mathrm{cyc}}(g)$, and $\{M_{i,\mathbb{F}_\ell}\}_i$ is stable by the action of $G_K$. We note that the filtration induced by $\sigma - 1$ coincides with $\{M_{i,\mathbb{F}_\ell}\}_i$ up to excluding finitely many $\ell \neq p$. We call $\{M_{i,\mathbb{F}_\ell}\}_i$ the *monodromy filtration with* $\mathbb{F}_\ell$*-coefficients* on $H_\ell$. For all but finitely many $\ell \neq p$, the action of $J$ is trivial on $M_{i,\mathbb{F}_\ell}/M_{i-1,\mathbb{F}_\ell}$ for every $i$, and we can ask whether the family $\{M_{i,\mathbb{F}_\ell}/M_{i-1,\mathbb{F}_\ell}\}_\ell$ of $G_K$-representations over $\mathbb{F}_\ell$ is of weight $w$ for some integer $w$ in the sense of Definition 2.6.

Now let us come back to our original setting. By the work of Rapoport-Zink [30] and de Jong's alteration [10, Theorem 6.5], there is an open subgroup $J$ of $I_K$ such that, for every $\ell \neq p$, the action of $J$ on $H^w_{\text{ét}}(X_{\overline{K}}, \Lambda_\ell)$ is unipotent and factors through $t_\ell$, where $\Lambda_\ell$ is $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$, or $\mathbb{F}_\ell$. (See also [2, Proposition 6.3.2].) By Theorem 2.3, we have

$$\sup_{\ell \neq p} \dim_{\mathbb{F}_\ell} H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell) < \infty.$$

(Alternatively, this fact can be proved by using the argument in [29, Section 6.2.4].) Therefore, the family $\{H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)\}_{\ell \neq p}$ satisfies the above two conditions, and we have the monodromy filtration $\{M_{i,\mathbb{F}_\ell}\}_i$ with $\mathbb{F}_\ell$-coefficients on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ for all but finitely many $\ell \neq p$. We put

$$\mathrm{Gr}^M_{i,\mathbb{F}_\ell} := M_{i,\mathbb{F}_\ell}/M_{i-1,\mathbb{F}_\ell}.$$

Here we omit $X$ and $w$ from the notation. This will not cause any confusion in the context.
A torsion analogue of Conjecture 3.1 can be formulated as follows:

Conjecture 3.5. *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. The family $\{\mathrm{Gr}^M_{i,\mathbb{F}_\ell}\}_\ell$ of finite dimensional $G_K$-representations over $\mathbb{F}_\ell$ defined above is of weight $w+i$ for every $i$ in the sense of Definition 2.6.*

Remark 3.6. When $X$ has good reduction over $\mathcal{O}_K$, Conjecture 3.5 is a consequence of the Weil conjecture and Theorem 2.3; see Corollary 2.4.

The main theorem of this paper is as follows.

Theorem 3.7. *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. We assume that $(X, w)$ satisfies one of the conditions (1)–(5) in Theorem 3.2. Then the assertion of Conjecture 3.5 for $(X, w)$ is true.*

We will prove Theorem 3.7 in Sections 5–9.

### 3.3 Torsion-freeness of monodromy operators

In this subsection, we discuss a relation between Conjecture 3.1 and Conjecture 3.5.
Let $J$ be an open subgroup of $I_K$ such that the action of $J$ on $H^w_{\text{ét}}(X_{\overline{K}}, \Lambda_\ell)$ is unipotent for every $\ell \neq p$, where $\Lambda_\ell$ is $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$, or $\mathbb{F}_\ell$. Take an element $\sigma \in J$ such that $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator for all but finitely many $\ell \neq p$.

Lemma 3.8. *By pulling back the monodromy filtration $\{M_{i,\mathbb{Q}_\ell}\}_i$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, we define a filtration $\{M_{i,\mathbb{Z}_\ell}\}_i$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$. Then the following two statements for $(X, w)$ are equivalent:*

*(i) For all but finitely many $\ell \neq p$, the reduction modulo $\ell$ of $\{M_{i,\mathbb{Z}_\ell}\}_i$ coincides with the monodromy filtration $\{M_{i,\mathbb{F}_\ell}\}_i$ with $\mathbb{F}_\ell$-coefficients via the isomorphism $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\sim} H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ (see Theorem 2.3).*

*(ii) The cokernel of*

$$(\sigma - 1)^i \colon H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell) \to H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$$

*is torsion-free for all but finitely many $\ell \neq p$ and every $i \geq 0$.*

Proof. Use Theorem 2.3, Lemma 2.10 and Nakayama's lemma. □

Definition 3.9. If the two equivalent statements in Lemma 3.8 hold for $(X, w)$, then we say that $(X, w)$ *satisfies the property* (t-f).

Let $G$ be a group and let $M$ be an abelian group equipped with an action of $G$. Let $M^G$ denote the $G$-fixed part of $M$. Let $M_G$ denote be the group of $G$-coinvariants of $M$.

Proposition 3.10.

*(i) If* $(X, w)$ *satisfies the property* (t-f), *then for all but finitely many* $\ell \neq p$, *we have*
$$H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)^{I_K} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\sim} H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)^{I_K}$$
*and the* $\mathbb{Z}_\ell$*-module* $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)_{I_K}$ *is torsion-free.*

*(ii) If Conjecture 3.5 for* $(X, w)$ *is true, then* $(X, w)$ *satisfies the property* (t-f).

*(iii) Assume that Conjecture 3.1 for* $(X, w)$ *is true and* $(X, w)$ *satisfies the property* (t-f). *Then Conjecture 3.5 for* $(X, w)$ *is true.*

Proof. In the proof, we will use the weight filtration $\{W_{i,\mathbb{Q}_\ell}\}_i$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, which we will recall in Remark 4.16.
(i) In order to prove the assertion, we may assume that $J$ is a normal open subgroup of $I_K$. We recall that for a finite group $G$ of order $m$ and a finitely generated $\mathbb{Z}_\ell$-module $M$, if $m$ is not divisible by $\ell$, then the $G$-fixed part $M^G$ is the image of the idempotent
$$e \colon M \to M, \quad x \mapsto \frac{1}{m} \sum_{g \in G} gx$$
and $e$ induces an isomorphism $M_G \xrightarrow{\sim} M^G$. Using this fact, we see that it suffices to prove the same statement for $J$-fixed parts and $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)_J$. This follows from the torsion-freeness of the cokernel of $\sigma - 1 \colon H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell) \to H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$ and Lemma 2.8. (To be more precise, we only need the condition (ii) of Lemma 3.8 for $i = 1$ here.)
(ii) Let $\{W_{i,\mathbb{Q}_\ell}\}_i$ be the weight filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. By pulling back $\{W_{i,\mathbb{Q}_\ell}\}_i$ to $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$, we have a filtration $\{W_{i,\mathbb{Z}_\ell}\}_i$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$. We have $(\sigma - 1)(W_{i,\mathbb{Z}_\ell}) \subset W_{i-2,\mathbb{Z}_\ell}$ and the $i$-th graded piece $\mathrm{Gr}^W_{i,\mathbb{Z}_\ell} := W_{i,\mathbb{Z}_\ell}/W_{i-1,\mathbb{Z}_\ell}$ is torsion-free for every $i$. By Theorem 2.3, for all but finitely many $\ell \neq p$, we can define a filtration $\{W_{i,\mathbb{F}_\ell}\}_i$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ by taking the reduction modulo $\ell$ of the filtration $\{W_{i,\mathbb{Z}_\ell}\}_i$. We define $\mathrm{Gr}^W_{i,\mathbb{F}_\ell} := W_{i,\mathbb{F}_\ell}/W_{i-1,\mathbb{F}_\ell}$. Then the family $\{\mathrm{Gr}^W_{i,\mathbb{F}_\ell}\}_\ell$ is of weight $w + i$; see Proposition 4.15 (ii) in Section 4.
Now we assume that Conjecture 3.5 for $(X, w)$ is true. Then $\{W_{i,\mathbb{F}_\ell}\}_i$ coincides with the monodromy filtration $\{M_{i,\mathbb{F}_\ell}\}_i$ with $\mathbb{F}_\ell$-coefficients for all but finitely many $\ell \neq p$ by Lemma 2.2. Thus, the $i$-th iterate $(\sigma - 1)^i$ of $\sigma - 1$ induces an isomorphism
$$(\sigma - 1)^i \colon \mathrm{Gr}^W_{i,\mathbb{F}_\ell} \xrightarrow{\sim} \mathrm{Gr}^W_{-i,\mathbb{F}_\ell}$$
for every $i \geq 0$ and all but finitely many $\ell \neq p$. By Nakayama's lemma, we have
$$(\sigma - 1)^i \colon \mathrm{Gr}^W_{i,\mathbb{Z}_\ell} \xrightarrow{\sim} \mathrm{Gr}^W_{-i,\mathbb{Z}_\ell}$$
for every $i \geq 0$ and all but finitely many $\ell \neq p$. It follows that the weight filtration $\{W_{i,\mathbb{Q}_\ell}\}_i$ coincides with the monodromy filtration $\{M_{i,\mathbb{Q}_\ell}\}_i$ for all but finitely many $\ell \neq p$, and the condition (i) in Lemma 3.8 is satisfied.

(iii) Assume that Conjecture 3.1 for $(X, w)$ is true. Then the weight filtration $\{W_{i,\mathbb{Q}_\ell}\}_i$ coincides with the monodromy filtration $\{M_{i,\mathbb{Q}_\ell}\}_i$ for every $\ell \neq p$. Assume further that $(X, w)$ satisfies the property (t-f). Then it follows that the monodromy filtration $\{M_{i,\mathbb{F}_\ell}\}_i$ coincides with $\{W_{i,\mathbb{F}_\ell}\}_i$ for all but finitely many $\ell \neq p$. Thus, Conjecture 3.5 for $(X, w)$ is true. □

For later use, we state the following result as a corollary.

Corollary 3.11. *Assume that $(X, w)$ satisfies one of conditions (1)–(5) in Theorem 3.2. Then, for all but finitely many $\ell \neq p$, we have $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)^{I_K} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\sim} H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)^{I_K}$ and the $\mathbb{Z}_\ell$-module $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)_{I_K}$ is torsion-free.*

Proof. Use Theorem 3.7 and Proposition 3.10. □

Remark 3.12. Let $Z$ be a proper smooth scheme over a finitely generated field $F$ over $\mathbb{F}_p$. Cadoret-Hui-Tamagawa proved that the natural map $H^w_{\text{ét}}(Z_{\overline{F}}, \mathbb{Z}_\ell) \to H^w_{\text{ét}}(Z_{\overline{F}}, \mathbb{F}_\ell)$ gives an isomorphism

$$H^w_{\text{ét}}(Z_{\overline{F}}, \mathbb{Z}_\ell)^{\mathrm{Gal}(F^{\mathrm{sep}}/F.\overline{\mathbb{F}}_p)} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\sim} H^w_{\text{ét}}(Z_{\overline{F}}, \mathbb{F}_\ell)^{\mathrm{Gal}(F^{\mathrm{sep}}/F.\overline{\mathbb{F}}_p)}$$

for all but finitely many $\ell \neq p$; see [5, Theorem 4.5]. Corollary 3.11 is a local analogue of this result.

## 4 The weight spectral sequence and preliminary reductions

In this section, we discuss some properties of the weight spectral sequence which are related to Conjecture 3.1 and Conjecture 3.5. We also review some standard reductions used in the proof of Theorem 3.7.

### 4.1 Torsion-freeness of the weight spectral sequence

Let $K$ be a Henselian discrete valuation field. For a prime number $\ell \neq \mathrm{char}(k)$, let $t_\ell \colon I_K \to \mathbb{Z}_\ell(1)$ be the map defined in the same way as in Section 3. Let $\varpi \in \mathcal{O}_K$ be a uniformizer.

Let $\mathfrak{X}$ be a proper scheme over $\mathcal{O}_K$. We assume that $\mathfrak{X}$ is *strictly semi-stable* over $\mathcal{O}_K$ purely of relative dimension $d$, i.e. it is, Zariski locally on $\mathfrak{X}$, étale over

$$\operatorname{Spec} \mathcal{O}_K[T_0, \ldots, T_d]/(T_0 \cdots T_r - \varpi)$$

for an integer $r$ with $0 \leq r \leq d$.

Let $X$ and $Y$ be the generic fiber and the special fiber of $\mathfrak{X}$, respectively. Let $D_1, \ldots, D_m$ be the irreducible components of $Y$. We equip each $D_i$ with the reduced induced subscheme structure. Following [32], we introduce some notation. Let $v$ be a non-negative integer. For a non-empty subset $I \subset \{1, \ldots, m\}$ of cardinality $v + 1$, we define $D_I := \cap_{i \in I} D_i$ (scheme-theoretic intersection). If $D_I$ is non-empty, then it is purely of codimension $v$ in $Y$. Moreover, we put

$$Y^{(v)} := \coprod_{I \subset \{1, \ldots, m\},\, \mathrm{Card}\, I = v+1} D_I.$$

Theorem 4.1 (Rapoport-Zink [30, Satz 2.10], Saito [32, Corollary 2.8]). *Let the notation be as above. Let $\ell \neq \mathrm{char}(k)$ be a prime number. Let $\Lambda_\ell$ be $\mathbb{Z}/\ell^n\mathbb{Z}$, $\mathbb{Z}_\ell$, or $\mathbb{Q}_\ell$.*

*(i) We have a spectral sequence*

$$E_{1,\Lambda_\ell}^{v,w} = \bigoplus_{i \geq \max(0,-v)} H_{\text{ét}}^{w-2i}(Y_{\overline{k}}^{(v+2i)}, \Lambda_\ell(-i)) \Rightarrow H_{\text{ét}}^{v+w}(X_{\overline{K}}, \Lambda_\ell),$$

*which is compatible with the action of $G_K$. Here $(-i)$ denotes the Tate twist.*

*(ii) Let $\sigma \in I_K$ be an element such that $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator. There exists the following homomorphism of spectral sequences:*

$$\begin{array}{ccc} E_{1,\Lambda_\ell}^{v,w} = \bigoplus_{i\geq \max(0,-v)} H^{w-2i}(Y^{v+2i})(-i) & \Longrightarrow & H^{v+w}(X) \\ \big\downarrow{\scriptstyle 1\otimes t_\ell(\sigma)} & & \big\downarrow{\scriptstyle \sigma-1} \\ E_{1,\Lambda_\ell}^{v+2,w-2} = \bigoplus_{i-1\geq \max(0,-v-2)} H^{w-2i}(Y^{v+2i})(-i+1) & \Longrightarrow & H^{v+w}(X). \end{array}$$

*Here we write*

$$H^w(Y^v)(i) := H_{\text{ét}}^w(Y_{\overline{k}}^{(v)}, \Lambda_\ell(i)) \quad \text{and} \quad H^w(X) := H_{\text{ét}}^w(X_{\overline{K}}, \Lambda_\ell)$$

*for short.*

Proof. For (i), see [30, Satz 2.10] and [32, Corollary 2.8 (1)]. We remark that the spectral sequence constructed in [30] coincides with that constructed in [32] up to signs; see [32, p.613]. In this paper, we use the spectral sequence constructed in [32]. The assertion (ii) follows from [32, Corollary 2.8 (2)]. ☐

The spectral sequence in Theorem 4.1 is called the *weight spectral sequence with $\Lambda_\ell$-coefficients.*

Remark 4.2. The boundary map $d_1^{v,w} \colon E_{1,\Lambda_\ell}^{v,w} \to E_{1,\Lambda_\ell}^{v+1,w}$ of the weight spectral sequence is of the form $\sum_{i\geq\max(0,-v)} \delta_{v+2i,*} + \delta_{v+2i}^*$, where

$$\delta_{v+2i,*} \colon H_{\text{ét}}^{w-2i}(Y_{\overline{k}}^{(v+2i)}, \Lambda_\ell(-i)) \to H_{\text{ét}}^{w-2i+2}(Y_{\overline{k}}^{(v+2i-1)}, \Lambda_\ell(-i+1))$$

and

$$\delta_{v+2i}^* \colon H_{\text{ét}}^{w-2i}(Y_{\overline{k}}^{(v+2i)}, \Lambda_\ell(-i)) \to H_{\text{ét}}^{w-2i}(Y_{\overline{k}}^{(v+2i+1)}, \Lambda_\ell(-i))$$

can be described as follows. For subsets $J \subset I \subset \{1, \dots, m\}$ with $\mathrm{Card}\, I = \mathrm{Card}\, J + 1$, let $i_{JI} \colon D_I \to D_J$ be the inclusion. Let

$$i_{JI,*} \colon H_{\text{ét}}^{w-2i}((D_I)_{\overline{k}}, \Lambda_\ell(-i)) \to H_{\text{ét}}^{w-2i+2}((D_J)_{\overline{k}}, \Lambda_\ell(-i+1))$$

denote the Gysin map. Then we have

$$\delta_{v+2i,*} = \sum_{J\subset I\subset\{1,\dots,m\},\mathrm{Card}\, I=\mathrm{Card}\, J+1=v+2i+1} \epsilon(J,I) i_{JI,*}.$$

Here $\epsilon(J,I) \in \{1,-1\}$ is an integer which only depends on $J \subset I$ (and is in particular independent of $\ell \neq \mathrm{char}(k)$); see [32, p.610] for the precise definition. Similarly, let

$$i^*_{JI} \colon H^{w-2i}_{\text{ét}}((D_J)_{\overline{k}}, \Lambda_\ell(-i)) \to H^{w-2i}_{\text{ét}}((D_I)_{\overline{k}}, \Lambda_\ell(-i))$$

denote the restriction map. Then we have

$$\delta^*_{v+2i} = \sum_{J\subset I\subset\{1,\dots,m\},\mathrm{Card}\, I=\mathrm{Card}\, J+1=v+2i+2} \epsilon(J,I) i^*_{JI}.$$

See [32, Proposition 2.10] for details.

We will discuss the degeneracy of the weight spectral sequence. For the weight spectral sequence with $\mathbb{Q}_\ell$-coefficients, we have the following theorem:

Theorem 4.3. *The weight spectral sequence with $\mathbb{Q}_\ell$-coefficients degenerates at $E_2$ for every $\ell \neq \mathrm{char}(k)$.*

Proof. See [27, Theorem 0.1] or [22, Theorem 1.1 (1)]. □

For the weight spectral sequence with $\Lambda_\ell$-coefficients, where $\Lambda_\ell$ is either $\mathbb{F}_\ell$ or $\mathbb{Z}_\ell$, we can prove the following theorem, which relies on the Weil conjecture and Theorem 2.3.

Theorem 4.4.

*(i) Let $\Lambda_\ell$ be either $\mathbb{F}_\ell$ or $\mathbb{Z}_\ell$. For all but finitely many $\ell \neq \mathrm{char}(k)$, the weight spectral sequence with $\Lambda_\ell$-coefficients degenerates at $E_2$.*

*(ii) For all but finitely many $\ell \neq \mathrm{char}(k)$, the $\mathbb{Z}_\ell$-module $E^{v,w}_{r,\mathbb{Z}_\ell}$ is torsion-free and we have $E^{v,w}_{r,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\sim} E^{v,w}_{r,\mathbb{F}_\ell}$ for every $r \geq 1$ and all $v, w$.*

Proof. By Theorem 2.3, we see that $E^{v,w}_{1,\mathbb{Z}_\ell}$ is torsion-free and $E^{v,w}_{1,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\sim} E^{v,w}_{1,\mathbb{F}_\ell}$ for all but finitely many $\ell \neq \mathrm{char}(k)$.
If $\mathrm{char}(k) = 0$, by Remark 4.2 and the comparison of étale and singular cohomology for varieties over $\mathbb{C}$, it follows that the cokernel of the map $d^{v,w}_1 \colon E^{v,w}_{1,\mathbb{Z}_\ell} \to E^{v+1,w}_{1,\mathbb{Z}_\ell}$ is torsion-free for all but finitely many $\ell \neq \mathrm{char}(k)$ and all $v, w$. Thus Theorem 4.4 is a consequence of Theorem 4.3 and Lemma 2.8.
We assume that $p := \mathrm{char}(k) > 0$ for the rest of the proof. We claim that, for all but finitely many $\ell \neq p$, the weight spectral sequence with $\mathbb{F}_\ell$-coefficients degenerates at $E_2$. First, we assume that $k$ is finitely generated over $\mathbb{F}_p$. The family $\{E^{v,w}_{1,\mathbb{F}_\ell}\}_{\ell\neq p}$ of $G_k$-modules over $\mathbb{F}_\ell$ is of weight $w$ by Corollary 2.4. Since

$E_{r,\mathbb{F}_\ell}^{v,w}$ is a subquotient of $E_{1,\mathbb{F}_\ell}^{v,w}$ for $r \geq 2$, the family $\{E_{r,\mathbb{F}_\ell}^{v,w}\}_{\ell \neq p}$ is also of weight $w$. Since the map $d_r^{v,w} \colon E_{r,\Lambda_\ell}^{v,w} \to E_{r,\Lambda_\ell}^{v+r,w-r+1}$ is $G_k$-equivariant, it is a zero map for $r \geq 2$ and all but finitely many $\ell \neq p$ by (the proof of) Lemma 2.1. The general case can be deduced from the case where $k$ is finitely generated over $\mathbb{F}_p$ by using Néron's blowing up as in [22, Proposition 5.1].

We shall prove that, for all but finitely many $\ell \neq \mathrm{char}(k)$, the $\mathbb{Z}_\ell$-module $E_{2,\mathbb{Z}_\ell}^{v,w}$ is torsion-free and we have $E_{2,\mathbb{Z}_\ell}^{v,w} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\sim} E_{2,\mathbb{F}_\ell}^{v,w}$ for all $v, w$. We note that, with Theorem 4.3, this will imply that the weight spectral sequence with $\mathbb{Z}_\ell$-coefficients degenerates at $E_2$ for all but finitely many $\ell \neq p$ and that the assertion (ii) holds for $r \geq 3$. By the degeneracy of the weight spectral sequence with $\mathbb{F}_\ell$-coefficients, we have

$$\sum_{v,w} \dim_{\mathbb{F}_\ell} E_{2,\mathbb{F}_\ell}^{v,w} = \sum_i \dim_{\mathbb{F}_\ell} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$$

for all but finitely many $\ell \neq p$. By Theorem 4.3, we have

$$\sum_{v,w} \mathrm{rank}_{\mathbb{Z}_\ell} E_{2,\mathbb{Z}_\ell}^{v,w} = \sum_i \mathrm{rank}_{\mathbb{Z}_\ell} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell).$$

By Theorem 2.3 and Lemma 2.8, for all but finitely many $\ell \neq p$, we have

$$\sum_i \mathrm{rank}_{\mathbb{Z}_\ell} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell) = \sum_i \dim_{\mathbb{F}_\ell} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$$

and

$$\mathrm{rank}_{\mathbb{Z}_\ell} E_{2,\mathbb{Z}_\ell}^{v,w} \leq \dim_{\mathbb{F}_\ell} E_{2,\mathbb{F}_\ell}^{v,w}$$

for all $v, w$. It follows that, for all but finitely many $\ell \neq p$, we have

$$\mathrm{rank}_{\mathbb{Z}_\ell} E_{2,\mathbb{Z}_\ell}^{v,w} = \dim_{\mathbb{F}_\ell} E_{2,\mathbb{F}_\ell}^{v,w}$$

for all $v, w$. Now the assertion follows from Lemma 2.8.

The proof of Theorem 4.4 is complete. □

We shall discuss a relation between Conjecture 3.5 and the weight spectral sequence. Let $\sigma \in I_K$ be an element such that, for every $\ell \neq \mathrm{char}(k)$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator. Let $i \geq 0$ be an integer. The $i$-th iterate of $(1 \otimes t_\ell(\sigma))^i$ induces a homomorphism

$$(1 \otimes t_\ell(\sigma))^i \colon E_{2,\Lambda_\ell}^{-i,w+i} \to E_{2,\Lambda_\ell}^{i,w-i},$$

see Theorem 4.1 (ii). Then we have the following conjecture.

Conjecture 4.5. *Let $\mathfrak{X}$ be a proper strictly semi-stable scheme over* $\mathrm{Spec}\,\mathcal{O}_K$ *purely of relative dimension $d$. Let the notation be as above. We put $\Lambda_\ell = \mathbb{Q}_\ell$ (resp. $\Lambda_\ell = \mathbb{F}_\ell, \mathbb{Z}_\ell$). Let $w$ be an integer. Then for every $\ell \neq \mathrm{char}(k)$ (resp. all but finitely many $\ell \neq \mathrm{char}(k)$), the above morphism $(1 \otimes t_\ell(\sigma))^i \colon E_{2,\Lambda_\ell}^{-i,w+i} \to E_{2,\Lambda_\ell}^{i,w-i}$ is an isomorphism for every $i \geq 0$.*

Remark 4.6. Assume that $\mathrm{char}(k) = 0$. Then Conjecture 4.5 for $(\mathfrak{X}, \Lambda_\ell = \mathbb{Q}_\ell)$ is true; see [22, Theorem 1.1 (2)]. Therefore, by a similar argument as in the proof of Theorem 4.4, we see that Conjecture 4.5 also holds for $(\mathfrak{X}, \Lambda_\ell = \mathbb{F}_\ell)$ and $(\mathfrak{X}, \Lambda_\ell = \mathbb{Z}_\ell)$.

Lemma 4.7. *Conjecture 4.5 for $(\mathfrak{X}, w, \Lambda_\ell = \mathbb{F}_\ell)$ is equivalent to Conjecture 4.5 for $(\mathfrak{X}, w, \Lambda_\ell = \mathbb{Z}_\ell)$.*

Proof. By Theorem 4.4, it follows that, for all but finitely many $\ell \neq \mathrm{char}(k)$, the $\mathbb{Z}_\ell$-module $E_{2,\mathbb{Z}_\ell}^{v,w}$ is torsion-free and we have $E_{2,\mathbb{Z}_\ell}^{v,w} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \xrightarrow{\sim} E_{2,\mathbb{F}_\ell}^{v,w}$ for all $v, w$. Therefore the assertion follows from Nakayama's lemma. $\square$

In the rest of this subsection, we assume that $K$ is a non-archimedean local field.

Remark 4.8. It is well known that Conjecture 3.1 for $(X, w)$ is equivalent to Conjecture 4.5 for $(\mathfrak{X}, w, \Lambda_\ell = \mathbb{Q}_\ell)$; see [22, Proposition 2.5] for example.

Similarly to Remark 4.8, we have the following lemma.

Lemma 4.9. *Let $\mathfrak{X}$ be a proper strictly semi-stable scheme over $\mathrm{Spec}\, \mathcal{O}_K$ purely of relative dimension $d$ with generic fiber $X$ and let $w$ be an integer. Then Conjecture 3.5 for $(X, w)$ is equivalent to Conjecture 4.5 for $(\mathfrak{X}, w, \Lambda_\ell = \mathbb{F}_\ell)$.*

Proof. By Theorem 4.4, the weight spectral sequence with $\mathbb{F}_\ell$-coefficients degenerates at $E_2$ for all but finitely many $\ell \neq p$. Hence the claim follows from Lemma 2.2, Corollary 2.4, Theorem 4.1 (ii), and the definition of the monodromy filtration. $\square$

### 4.2 Standard reductions

In this subsection, let $K$ be a non-archimedean local field. We shall recall some standard techniques used to study the weight-monodromy conjecture, which are based on de Jong's alterations and the hard Lefschetz theorem.
We begin with the following easy lemma.

Lemma 4.10. *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. For a finite extension $L$ of $K$, Conjecture 3.5 for $(X, w)$ is equivalent to Conjecture 3.5 for $(X_L, w)$.*

Proof. This is a consequence of Lemma 2.7. $\square$

In the proof of Theorem 3.7, we will need to replace a proper smooth scheme $X$ over $K$ by a projective smooth alteration or a strictly semi-stable alteration of $X$ in some cases. The following lemma will be useful in such a context.

Lemma 4.11. *Let $F$ be a field. Let $X, Y$ be proper smooth connected schemes over $F$ of dimension $d$ and $f\colon X \to Y$ an alteration, i.e. a proper surjective generically finite morphism over $F$. Let $\Lambda_\ell = \mathbb{Q}_\ell$ (resp. $\Lambda_\ell = \mathbb{F}_\ell, \mathbb{Z}_\ell$). Let*

*$w \geq 0$ be an integer. Then the map $f^*\colon H^w_{\text{ét}}(Y_{\overline{F}}, \Lambda_\ell) \to H^w_{\text{ét}}(X_{\overline{F}}, \Lambda_\ell)$ maps $H^w_{\text{ét}}(Y_{\overline{F}}, \Lambda_\ell)$ isomorphically onto a direct summand of $H^w_{\text{ét}}(X_{\overline{F}}, \Lambda_\ell)$ as a $G_F$-module for every $\ell \neq \mathrm{char}(F)$ (resp. all but finitely many $\ell \neq \mathrm{char}(F)$). In particular, if $F = K$ and Conjecture 3.5 holds for $(X, w)$, then Conjecture 3.5 also holds for $(Y, w)$.*

Proof. The proof of this lemma is standard; see the proofs of [21, Lemma 6.1], [29, Théorème 6.2.2], and [36, Theorem 1.4]. We briefly recall the argument for the convenience of the reader.

The second assertion follows from the first assertion. So we shall prove the first assertion. Let $f_*\colon H^w_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^w_{\text{ét}}(Y_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z})$ be the push-forward map, which can be defined as the dual of $f^*\colon H^{2d-w}_{\text{ét}}(Y_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z}(d)) \to H^{2d-w}_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z}(d))$ by using Poincaré duality. We claim that the composition

$$H^w_{\text{ét}}(Y_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z}) \xrightarrow{f^*} H^w_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z}) \xrightarrow{f_*} H^w_{\text{ét}}(Y_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z})$$

is the map $x \mapsto mx$, where $m$ is the degree of $f$. It follows that the same holds for $\mathbb{Z}_\ell$-coefficients by taking inverse limits, and then holds for $\mathbb{Q}_\ell$-coefficients by tensoring with $\mathbb{Q}_\ell$. The first assertion easily follows from these facts.

Let $\mathrm{Tr}_X$ denote the trace map $H^{2d}_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z}(d)) \to \mathbb{Z}/\ell^n\mathbb{Z}$, and similarly for $\mathrm{Tr}_Y$. In order to prove the claim, it suffices to prove that $\mathrm{Tr}_X \circ f^*$ coincides with $m\,\mathrm{Tr}_Y$ as a map from $H^{2d}_{\text{ét}}(Y_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z}(d))$ to $\mathbb{Z}/\ell^n\mathbb{Z}$. Let $V$ be a dense open subset of $Y$ such that $f^{-1}(V) \to V$ is a finite flat morphism. Since we have $H^{2d}_{\text{ét},c}(V_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z}(d)) \xrightarrow{\sim} H^{2d}_{\text{ét}}(Y_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z}(d))$ and $H^{2d}_{\text{ét},c}(f^{-1}(V)_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z}(d)) \xrightarrow{\sim} H^{2d}_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z}(d))$, the claim follows from [42, Exposé XVIII, Théorème 2.9]; see especially (Var 3) and (Var 4)(I). (We remark that $f^{-1}(V) \to V$ need not be étale here.) □

Let us record the following two results for future reference, which are based on de Jong's alterations and the hard Lefschetz theorem.

Lemma 4.12. *Let $X$ be a proper smooth geometrically connected scheme over $K$ of dimension $d$.*

*(i) (See [10, Theorem 6.5].) There exist a finite extension $L$ of $K$ and a proper strictly semi-stable connected scheme $\mathfrak{Z}$ over $\operatorname{Spec} \mathcal{O}_L$ purely of relative dimension $d$ such that the generic fiber $Z$ of $\mathfrak{Z}$ is an alteration of $X_L$.*

*(ii) If Conjecture 4.5 for $(\mathfrak{Z}, w, \Lambda_\ell = \mathbb{F}_\ell)$ holds, then Conjecture 3.5 for $(X, w)$ holds.*

Proof. (i) This is due to de Jong [10, Theorem 6.5].
(ii) Use Lemma 4.9, Lemma 4.10, and Lemma 4.11. □

Lemma 4.13. *Let $w \geq 0$ be an integer. We assume that $\mathrm{char}(K) = 0$. We assume further that, for every finite extension $L$ of $K$, Conjecture 3.5 for $(Z, w)$ holds for every proper smooth scheme $Z$ over $L$ with $\dim Z \leq w$. Then Conjecture 3.5 for $(X, w)$ holds for every proper smooth scheme $X$ over $K$.*

Proof. Let $X$ be a proper smooth scheme over $K$. We shall prove Conjecture 3.5 for $(X, w)$ under the assumptions. We may assume that $X$ is connected. By [10, Theorem 4.1], there exists a projective smooth connected scheme $Z$ over $K$ with an alteration $Z \to X$. In order to prove Conjecture 3.5 for $(X, w)$, it is enough to prove Conjecture 3.5 for $(Z, w)$ by Lemma 4.11. Thus we may assume that $X$ is projective.
Since the hard Lefschetz theorem with $\mathbb{Q}$-coefficients holds for singular cohomology of projective smooth varieties over $\mathbb{C}$, the hard Lefschetz theorem with $\mathbb{F}_\ell$-coefficients holds for étale cohomology of projective smooth varieties over $K$ for all but finitely many $\ell$ by Theorem 2.3. (See also Remark 4.14.) Therefore, by taking general hyperplane sections if necessary, we may assume that $\dim X \leq w$. Then Conjecture 3.5 holds for $(X, w)$ by assumption. □

Remark 4.14. In [14, Complément 6], Gabber announced the hard Lefschetz theorem with $\mathbb{Z}_\ell$-coefficients (for all but finitely many $\ell$) for étale cohomology of projective smooth varieties in positive characteristic. Using this fact, we can remove the assumption that $\mathrm{char}(K) = 0$ in Lemma 4.13. (We will not use this fact in this paper.)

We close this section with the following well known results.

Proposition 4.15. *Let $X$ be a proper smooth scheme over* $\operatorname{Spec} K$ *and $w$ an integer. Let* $\mathrm{Frob} \in G_K$ *be a lift of the geometric Frobenius element.*

*(i) There is a non-zero monic polynomial $P(T) \in \mathbb{Z}[T]$ such that, for all but finitely many $\ell \neq p$, we have $P(\mathrm{Frob}) = 0$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$.*

*(ii) For every $\ell \neq p$, there exists a unique increasing, separated, exhaustive filtration*

$$\{W_{i,\mathbb{Q}_\ell}\}_i$$

*on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ which is stable by the action of $G_K$ and satisfies the following property. For every $i$, there exists a Weil $q^{w+i}$-polynomial $P_i(T) \in \mathbb{Z}[T]$ such that $P_i(\mathrm{Frob}) = 0$ on the $i$-th graded piece $\mathrm{Gr}^W_{i,\mathbb{Q}_\ell} := W_{i,\mathbb{Q}_\ell}/W_{i-1,\mathbb{Q}_\ell}$. Moreover, we can take the polynomial $P_i(T) \in \mathbb{Z}[T]$ independent of $\ell \neq p$.*

*(iii) Assume that Conjecture 3.1 for $(X, w)$ is true. Then, for every $i$, there exists a Weil $q^{w+i}$-polynomial $P_i(T) \in \mathbb{Z}[T]$ such that, for every $\ell \neq p$, we have $P_i(\mathrm{Frob}) = 0$ on the $i$-th graded piece $\mathrm{Gr}^M_{i,\mathbb{Q}_\ell}$ of the monodromy filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$.*

Proof. We may assume that $X$ is geometrically connected. By [10, Theorem 6.5], there is a proper strictly semi-stable connected scheme $\mathfrak{Z}$ over $\operatorname{Spec} \mathcal{O}_L$ as in Lemma 4.12 (i). We may further assume that $K = L$. By Lemma 4.11, we see that $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is a direct summand of $H^w_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_\ell)$ as a $G_K$-representation for every $\ell \neq p$.

Let $\{F^i_{\mathbb{Q}_\ell}\}_i$ be the decreasing filtration on $H^w_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_\ell)$ arising from the weight spectral sequence. The filtration $\{F^i_{\mathbb{Q}_\ell}\}_i$ defines a decreasing filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, which is also denoted by $\{F^i_{\mathbb{Q}_\ell}\}_i$. Let $\{W_{i,\mathbb{Q}_\ell}\}_i$ be the increasing filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ defined by

$$W_{i,\mathbb{Q}_\ell} := F^{-i}_{\mathbb{Q}_\ell}.$$

Since the $i$-th graded piece $\mathrm{Gr}^W_{i,\mathbb{Q}_\ell} := W_{i,\mathbb{Q}_\ell}/W_{i-1,\mathbb{Q}_\ell}$ is a subquotient of $E^{-i,w+i}_{1,\mathbb{Q}_\ell}$, by the Weil conjecture, there exists a Weil $q^{w+i}$-polynomial $P_i(T) \in \mathbb{Z}[T]$ such that, for every $\ell \neq p$, we have $P_i(\mathrm{Frob}) = 0$ on $\mathrm{Gr}^W_{i,\mathbb{Q}_\ell}$. Thus the assertion (ii) follows.

The assertion (i) follows from (ii) and Theorem 2.3. If Conjecture 3.1 for $(X, w)$ is true, the filtration $\{W_{i,\mathbb{Q}_\ell}\}_i$ coincides with the monodromy filtration $\{M_{i,\mathbb{Q}_\ell}\}_i$. Therefore the assertion (iii) follows from (ii). □

Remark 4.16. We call the filtration $\{W_{i,\mathbb{Q}_\ell}\}_i$ in Proposition 4.15 the *weight filtration* on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. (The numbering used here differs from the one of [13, Proposition-définition (1.7.5)].)

Remark 4.17. Let $\mathrm{Frob} \in G_K$ be a lift of the geometric Frobenius element. Let $X$ be a proper smooth scheme over $K$. It is conjectured that the characteristic polynomial $P_{\mathrm{Frob},\ell}(T)$ of Frob acting on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is in $\mathbb{Z}[T]$ and independent of $\ell \neq p$. If $X$ is a surface or $K$ is of equal characteristic, this conjecture is true; see [28, Corollary 2.5] and [23, Theorem 1.4]. (See also [40, Theorem 3.3].) If this conjecture and Conjecture 3.1 for $(X, w)$ are true, then we can take $P_i(T)$ in Proposition 4.15 (iii) as the characteristic polynomial of Frob acting on $\mathrm{Gr}^M_{i,\mathbb{Q}_\ell}$.

## 5 Equal characteristic cases

In this section, we will prove Theorem 3.7 in the case (1). We will use the language of ultraproducts following [4]. We first recall some properties of ultraproducts which we need. For details, see [6, Appendix] for example. The notation used here is similar to that of [4].

### 5.1 Ultraproducts

Let $\mathcal{L}$ be an infinite set of prime numbers. We define

$$\underline{F} := \prod_{\ell \in \mathcal{L}} \overline{\mathbb{F}}_\ell,$$

where $\overline{\mathbb{F}}_\ell$ is an algebraic closure of $\mathbb{F}_\ell$. For a subset $S \subset \mathcal{L}$, let $e_S$ be the characteristic function of $\mathcal{L} \backslash S$, which we consider as an element of $\underline{F}$. Attaching to an ultrafilter $\mathfrak{u}$ on $\mathcal{L}$ a prime ideal

$$\mathfrak{m}_{\mathfrak{u}} := \langle e_S \mid S \in \mathfrak{u} \rangle \subset \underline{F}$$

defines a bijection from the set of ultrafilters on $\mathcal{L}$ to $\operatorname{Spec} \underline{F}$. Note that every prime ideal of $\underline{F}$ is a maximal ideal. We say that an ultrafilter $\mathfrak{u}$ on $\mathcal{L}$ is principal if it corresponds to a principal ideal. For a non-principal ultrafilter $\mathfrak{u}$, we define

$$\overline{\mathbb{Q}}_{\mathfrak{u}} := \underline{F}/\mathfrak{m}_{\mathfrak{u}}.$$

It is a field of characteristic 0 and is isomorphic to the field of complex numbers $\mathbb{C}$. The field $\overline{\mathbb{Q}}_{\mathfrak{u}}$ is called the *ultraproduct* of $\{\overline{\mathbb{F}}_\ell\}_{\ell\in\mathcal{L}}$ (with respect to the non-principal ultrafilter $\mathfrak{u}$). The ring homomorphism $\underline{F} \to \overline{\mathbb{Q}}_{\mathfrak{u}}$ is flat; see [6, Lemma in Section 4.1.4].

Remark 5.1. Let $\mathcal{L}' \subset \mathcal{L}$ be a subset such that $\mathcal{L}\backslash\mathcal{L}'$ is finite. The projection $\underline{F} \to \prod_{\ell\in\mathcal{L}'} \overline{\mathbb{F}}_\ell$ defines a bijection from the set of non-principal ultrafilters on $\mathcal{L}'$ to the set of non-principal ultrafilters on $\mathcal{L}$.

Let $\{M_\ell\}_{\ell\in\mathcal{L}}$ be a family of $\overline{\mathbb{F}}_\ell$-vector spaces. We define

$$\underline{M} := \prod_{\ell\in\mathcal{L}} M_\ell.$$

For the $\underline{F}$-module $\underline{M}$, the following assertions are equivalent.

- $\underline{M}$ is a finitely generated $\underline{F}$-module.
- $\underline{M}$ is a finitely presented $\underline{F}$-module.
- $\sup_{\ell\in\mathcal{L}} \dim_{\overline{\mathbb{F}}_\ell} M_\ell < \infty$.

We put $M_{\mathfrak{u}} := \underline{M} \otimes_{\underline{F}} \overline{\mathbb{Q}}_{\mathfrak{u}}$ for a non-principal ultrafilter $\mathfrak{u}$. We will use a similar notation for a family $\{f_\ell\}_{\ell\in\mathcal{L}}$ of maps of $\overline{\mathbb{F}}_\ell$-vector spaces.

Lemma 5.2. *Let $\{M_\ell\}_{\ell\in\mathcal{L}}$ and $\{N_\ell\}_{\ell\in\mathcal{L}}$ be families of $\overline{\mathbb{F}}_\ell$-vector spaces. Assume that $\underline{M}$ and $\underline{N}$ are finitely generated $\underline{F}$-modules. Let $\{f_\ell\}_{\ell\in\mathcal{L}}$ be a family of maps $f_\ell\colon M_\ell \to N_\ell$ of $\overline{\mathbb{F}}_\ell$-vector spaces. Then the following assertions are equivalent.*

*(i) $f_{\mathfrak{u}}\colon M_{\mathfrak{u}} \to N_{\mathfrak{u}}$ is an isomorphism for every non-principal ultrafilter $\mathfrak{u}$.*

*(ii) $f_\ell\colon M_\ell \to N_\ell$ is an isomorphism for all but finitely many $\ell \in \mathcal{L}$.*

Proof. For a subset $S \subset \mathcal{L}$ which is contained in every non-principal ultrafilter, the complement $\mathcal{L}\backslash S$ is finite. Hence the lemma follows from [6, Lemma 4.3.3]. □

Let $K$ be a Henselian discrete valuation field. Assume that $p := \operatorname{char}(k) > 0$. Let $\mathcal{L}$ be the set of prime numbers $\ell \neq p$. Let $\mathfrak{X}$ be a proper strictly semi-stable scheme over $\mathcal{O}_K$ purely of relative dimension $d$. We retain the notation of Section 4.

Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathcal{L}$. Since the map $\underline{F} \to \overline{\mathbb{Q}}_{\mathfrak{u}}$ is flat, we have the following weight spectral sequence with $\overline{\mathbb{Q}}_{\mathfrak{u}}$-coefficients:

$$E_{1,\overline{\mathbb{Q}}_{\mathfrak{u}}}^{v,w} = \bigoplus_{i\geq\max(0,-v)} H^{w-2i}_{\text{ét}}(Y^{(v+2i)}_{\overline{k}}, \overline{\mathbb{Q}}_{\mathfrak{u}}(-i)) \Rightarrow H^{v+w}_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_{\mathfrak{u}}).$$

Here we define

$$H^w_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_{\mathfrak{u}}) := \Big(\prod_{\ell \neq p} H^w_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{F}}_\ell)\Big) \otimes_{\underline{F}} \overline{\mathbb{Q}}_{\mathfrak{u}},$$

and similarly for $H^{w-2i}_{\text{ét}}(Y^{(v+2i)}_{\overline{k}}, \overline{\mathbb{Q}}_{\mathfrak{u}}(-i))$. For an element $\sigma \in I_K$ such that, for every $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator, we have a monodromy operator

$$(1 \otimes t_{\mathfrak{u}}(\sigma))^i \colon E^{-i, w+i}_{2, \overline{\mathbb{Q}}_{\mathfrak{u}}} \to E^{i, w-i}_{2, \overline{\mathbb{Q}}_{\mathfrak{u}}}$$

for all $w, i \geq 0$ in a similar way as in Section 4.

Lemma 5.3. *Conjecture 4.5 for $(\mathfrak{X}, w, \Lambda_\ell = \mathbb{F}_\ell)$ is equivalent to the assertion that the morphism*

$$(1 \otimes t_{\mathfrak{u}}(\sigma))^i \colon E^{-i, w+i}_{2, \overline{\mathbb{Q}}_{\mathfrak{u}}} \to E^{i, w-i}_{2, \overline{\mathbb{Q}}_{\mathfrak{u}}}$$

*is an isomorphism for every non-principal ultrafilter $\mathfrak{u}$ on $\mathcal{L}$ and every $i \geq 0$.*

Proof. The $\underline{F}$-module $\prod_{\ell \neq p}(E^{v,w}_{2, \mathbb{F}_\ell} \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell)$ is finitely generated for all $v, w$ by Theorem 2.3. Hence the assertion follows from Lemma 5.2. □

Finally, we define an ultraproduct variant of the notion of weight. Let $F$ be a finitely generated field over $\mathbb{F}_p$ and let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathcal{L}$. Let $\{H_\ell\}_{\ell \in \mathcal{L}}$ be a family of finite dimensional $G_F$-representations over $\overline{\mathbb{F}}_\ell$ such that the $\underline{F}$-module $\underline{H}$ is finitely generated. Then $H_{\mathfrak{u}}$ is a finite dimensional representation of $G_F$ over $\overline{\mathbb{Q}}_{\mathfrak{u}}$. (We do not impose any continuity conditions here.) Let $w$ be an integer. Let $\iota \colon \overline{\mathbb{Q}}_{\mathfrak{u}} \cong \mathbb{C}$ be an isomorphism. We say that $H_{\mathfrak{u}}$ is *$\iota$-pure of weight $w$* if the following conditions are satisfied:

- There is an integral scheme $U$ of finite type over $\mathbb{F}_p$ with function field $F$ such that the family $\{H_\ell\}_{\ell \in \mathcal{L}}$ comes from a family $\{\mathcal{F}_\ell\}_{\ell \in \mathcal{L}}$ of locally constant constructible $\overline{\mathbb{F}}_\ell$-sheaves on $U$.
- Moreover, for every closed point $x \in U$ and for every eigenvalue $\alpha$ of $\mathrm{Frob}_x$ acting on $H_{\mathfrak{u}}$, we have $|\iota(\alpha)| = (q_x)^{w/2}$.

(See also Section 2.1.) We say that $H_{\mathfrak{u}}$ is *pure of weight $w$* if it is $\iota$-pure of weight $w$ for every $\iota \colon \overline{\mathbb{Q}}_{\mathfrak{u}} \cong \mathbb{C}$.

## 5.2 Proof of Theorem 3.7 in the case (1)

We shall prove Theorem 3.7 in the case (1). By Lemma 4.12, it suffices to prove the following theorem.

Theorem 5.4. *Let $K$ be a Henselian discrete valuation field of equal characteristic $p > 0$. Then Conjecture 4.5 for $\Lambda_\ell = \mathbb{F}_\ell$ is true.*

For this, we use the same strategy as in [22]. The only problem is that we cannot use Weil II [13] directly since it works with étale cohomology with $\mathbb{Q}_\ell$-coefficients. By using the ultraproduct variant of Weil II established by Cadoret [4], the same arguments as in [22] can be carried out.

Proof. The proof is divided into three steps. First, we reduce to the case where $\mathcal{O}_K$ is the Henselization of the local ring of the generic point of a smooth divisor in a smooth variety over $\mathbb{F}_p$. Second, we reduce to the case where $K$ is the function field of a smooth curve over $\mathbb{F}_p$. Finally, we apply the ultraproduct variant of Weil II [4]. We recall the arguments in more detail for the reader's convenience.

Let $\mathcal{L}$ be the set of prime numbers $\ell \neq p$. Let $\mathfrak{X}$ be a proper strictly semi-stable scheme over $\mathcal{O}_K$ purely of relative dimension $d$ with generic fiber $X$. We retain the notation of Section 5.1. By Lemma 5.3, it suffices to prove that the morphism $(1 \otimes t_{\mathfrak{u}}(\sigma))^i \colon E_{2,\overline{\mathbb{Q}}_{\mathfrak{u}}}^{-i,w+i} \to E_{2,\overline{\mathbb{Q}}_{\mathfrak{u}}}^{i,w-i}$ is an isomorphism for every non-principal ultrafilter $\mathfrak{u}$ on $\mathcal{L}$ and for all $w, i \geq 0$.

By using Néron's blowing up as in [22, Section 4] and by using an argument in the proof of [22, Lemma 3.2], we may assume that there exist a connected smooth scheme $\operatorname{Spec} A$ over $\mathbb{F}_p$ and an element $\varpi \in A$ satisfying the following properties:

- $D := \operatorname{Spec} A/(\varpi)$ is an irreducible divisor on $A$ which is smooth over $\mathbb{F}_p$ and $\mathcal{O}_K$ is the Henselization of the local ring of $\operatorname{Spec} A$ at the prime ideal $(\varpi) \subset A$.
- There is a proper scheme $\widetilde{\mathfrak{X}}$ over $\operatorname{Spec} A$ which is smooth over $\operatorname{Spec} A[1/\varpi]$ such that $\widetilde{\mathfrak{X}} \otimes_A \mathcal{O}_K \cong \mathfrak{X}$.

Let $f \colon \widetilde{\mathfrak{X}} \to \operatorname{Spec} A$ be the structure morphism. The function field of $D$ is the residue field $k$ of $K$, which is finitely generated over $\mathbb{F}_p$.

Let $w \geq 0$ be an integer. By the same construction as in Section 3.2, after removing finitely many $\ell \neq p$ from $\mathcal{L}$, we can construct the monodromy filtration $\{M_{i,\mathbb{F}_\ell}\}_i$ with $\mathbb{F}_\ell$-coefficients on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ for every $\ell \in \mathcal{L}$. We have $\sup_{\ell \in \mathcal{L}} \dim_{\mathbb{F}_\ell} \operatorname{Gr}^M_{i,\mathbb{F}_\ell} < \infty$, where $\operatorname{Gr}^M_{i,\mathbb{F}_\ell} := M_{i,\mathbb{F}_\ell}/M_{i-1,\mathbb{F}_\ell}$ is the $i$-th graded piece. Let $\mathfrak{u}$ be a non-principal ultrafilter on $\mathcal{L}$. By an analogue of Lemma 4.9, it suffices to prove that the $G_k$-representation over $\overline{\mathbb{Q}}_{\mathfrak{u}}$

$$(\prod_{\ell \in \mathcal{L}} \operatorname{Gr}^M_{i,\mathbb{F}_\ell} \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell) \otimes_{\underline{F}} \overline{\mathbb{Q}}_{\mathfrak{u}}$$

is pure of weight $w + i$ for every $i$.

By applying a construction given in [13, Variante (1.7.8)] to the higher direct image sheaf $R^w f_* \mathbb{F}_\ell$ and by using a similar construction as in Section 3.2, we get a locally constant constructible $\overline{\mathbb{F}}_\ell$-sheaf $\mathcal{F}_\ell[D]$ on $D$ with a filtration $\{\mathcal{M}_{i,\ell}\}_i$ (after removing finitely many $\ell \neq p$ from $\mathcal{L}$). For every $i$, the stalk of the quotient

$$\operatorname{Gr}^{\mathcal{M}}_{i,\ell} := \mathcal{M}_{i,\ell}/\mathcal{M}_{i-1,\ell}$$

at the geometric generic point of $D$ is isomorphic to $\operatorname{Gr}^M_{i,\mathbb{F}_\ell} \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell$ as a $G_k$-representation for every $\ell \in \mathcal{L}$.

Let $x \in D$ be a closed point. We can find a connected smooth curve $C \subset \operatorname{Spec} A$ over $\mathbb{F}_p$ such that $C \cap D = \{x\}$ and the image of $\varpi \in A$ in $\mathcal{O}_{C,x}$ is a uniformizer. Let $L$ be the field of fractions of the completion $\widehat{\mathcal{O}}_{C,x}$ of $\mathcal{O}_{C,x}$. We write $Z := \widetilde{\mathfrak{X}} \otimes_A L$. By the construction, for all but finitely many $\ell \in \mathcal{L}$, the stalk $(\mathrm{Gr}^{\mathcal{M}}_{i,\ell})_{\overline{x}}$ is isomorphic to the base change of the $i$-th graded piece of the monodromy filtration with $\mathbb{F}_\ell$-coefficients on $H^w_{\text{ét}}(Z_{\overline{L}}, \mathbb{F}_\ell)$ as a $G_{\kappa(x)}$-representation. Thus we see that $(\prod_{\ell \in \mathcal{L}} (\mathrm{Gr}^{\mathcal{M}}_{i,\ell})_{\overline{x}}) \otimes_{\underline{F}} \overline{\mathbb{Q}}_{\mathfrak{u}}$ is pure of weight $w+i$ by [4, Corollary 5.3.2.4] together with Corollary 2.4 and [4, Lemma in 11.3]. This fact completes the proof of Theorem 5.4. $\square$

## 6 The case of set-theoretic complete intersections in projective smooth toric varieties

In this section, we will prove Theorem 3.7 in the case (5). Let $K$ be a non-archimedean local field. We assume that $\mathrm{char}(K) = 0$.

### 6.1 A uniform variant of a theorem of Huber

We will recall a result from [19] which will be used in the proof of Theorem 3.7 in the case (5). In this section, we will freely use the theory of adic spaces developed by Huber. The theory of étale cohomology for adic spaces was developed in [17].

Let $\mathbb{C}_p$ be the completion of $\overline{K}$, which is a complete non-archimedean field (in the sense of [17, Definition 1.1.3]). Let $L \subset \mathbb{C}_p$ be a subfield such that it is also a complete non-archimedean field with the induced topology. Let $\mathcal{O}_L$ be the ring of integers of $L$. For a scheme $X$ of finite type over $L$, the adic space associated with $X$ is denoted by

$$X^{\mathrm{ad}} := X \times_{\operatorname{Spec} L} \operatorname{Spa}(L, \mathcal{O}_L);$$

see [16, Proposition 3.8]. For an adic space $Y$ locally of finite type over $\operatorname{Spa}(L, \mathcal{O}_L)$, we denote by

$$Y_{\mathbb{C}_p} := Y \times_{\operatorname{Spa}(L,\mathcal{O}_L)} \operatorname{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$$

the base change of $Y$ to $\operatorname{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$.

Let us recall the following theorem due to Huber:

- Let $Y$ be a proper scheme over $L$ and $X \hookrightarrow Y$ a closed immersion. We have a closed immersion $X^{\mathrm{ad}} \hookrightarrow Y^{\mathrm{ad}}$ of adic spaces over $\operatorname{Spa}(L, \mathcal{O}_L)$. We fix a prime number $\ell \neq p$. Then, there is an open subset $V$ of $Y^{\mathrm{ad}}$ containing $X^{\mathrm{ad}}$ such that the pull-back map

  $$H^w_{\text{ét}}(V_{\mathbb{C}_p}, \mathbb{F}_\ell) \to H^w_{\text{ét}}(X^{\mathrm{ad}}_{\mathbb{C}_p}, \mathbb{F}_\ell)$$

  of étale cohomology groups is an isomorphism for every $w$.

(See [18, Theorem 3.6] for a more general result.) Scholze used this theorem in his proof of the weight-monodromy conjecture in the case (5).

In our case, we need the following uniform variant of Huber's theorem:

Theorem 6.1 ([19, Corollary 4.11]). *Let $Y$ be a proper scheme over $L$ and $X \hookrightarrow Y$ a closed immersion. We have a closed immersion $X^{\mathrm{ad}} \hookrightarrow Y^{\mathrm{ad}}$ of adic spaces over* $\mathrm{Spa}(L, \mathcal{O}_L)$*. Then, there is an open subset $V$ of $Y^{\mathrm{ad}}$ containing $X^{\mathrm{ad}}$ such that, for every prime number $\ell \neq p$, the pull-back map*

$$H^w_{\text{ét}}(V_{\mathbb{C}_p}, \mathbb{F}_\ell) \to H^w_{\text{ét}}(X^{\mathrm{ad}}_{\mathbb{C}_p}, \mathbb{F}_\ell)$$

*is an isomorphism for every $w$.*

Proof. See [19, Corollary 4.11]. □

### 6.2 Proof of Theorem 3.7 in the case (5)

The proof is the same as that of [33, Theorem 9.6], except that we use Theorem 6.1 instead of Huber's theorem. We shall give a sketch here. We will use the terminology in [33].

Let $X$ be a geometrically connected projective smooth scheme over $K$ which is a set-theoretic complete intersection in a projective smooth toric variety $Y_{\Sigma,K}$ over $K$ associated with a fan $\Sigma$. After replacing $K$ by a finite extension, we may assume that the action of $I_K$ on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ is unipotent and factors through $t_\ell$ for every $w$ and for every $\ell \neq p$.

Let $\varpi$ be a uniformizer of $K$. We fix a system $\{\varpi^{1/p^n}\}_{n\geq 0} \subset \overline{K}$ of $p^n$-th roots of $\varpi$. Let $L$ be the completion of $\bigcup_{n\geq 0} K(\varpi^{1/p^n})$, which is a perfectoid field. Let $G_L = \mathrm{Aut}(\overline{L}/L)$ be the absolute Galois group of $L$, where $\overline{L}$ is the algebraic closure of $L$ in $\mathbb{C}_p$. Then we have a surjection $G_L \to G_k$. Thus there exists a lift $\mathrm{Frob} \in G_L$ of the geometric Frobenius element. Let $I_L$ be the kernel of the map $G_L \to G_k$. We have $I_L \subset I_K$. Since $\bigcup_{n\geq 0} K(\varpi^{1/p^n})$ is a pro-$p$ extension of $K$, there exists an element $\sigma \in I_L$ such that, for every $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator. In other words, there exists an element $\sigma \in I_L$ such that it defines the monodromy filtration with $\mathbb{F}_\ell$-coefficients on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)$ for all but finitely many $\ell \neq p$. Therefore, it suffices to prove a natural analogue of Theorem 3.7 for the family $\{H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell)\}_{\ell\neq p}$ of $G_L$-representations. Moreover, in order to prove this, we can replace $L$ by a finite extension if necessary.

Let $L^\flat$ be the tilt of $L$. We have an identification $G_L = G_{L^\flat}$. The choice of the system $\{\varpi^{1/p^n}\}_{n\geq 0} \subset \overline{K}$ gives an identification between $L^\flat$ and the completion of the perfection of the field of formal Laurent series $k((t))$ over $k$.

Let $Y_{\Sigma,L}$ be the toric variety over $L$ associated with the fan $\Sigma$ and let $Y^{\mathrm{ad}}_{\Sigma,L}$ be the adic space associated with $Y_{\Sigma,L}$. We define $Y^{\mathrm{ad}}_{\Sigma,L^\flat}$ similarly. By [33, Theorem 8.5 (iii)], we have a projection

$$\pi \colon Y^{\mathrm{ad}}_{\Sigma,L^\flat} \to Y^{\mathrm{ad}}_{\Sigma,L}$$

of topological spaces. By Theorem 6.1, there exists an open subset $V$ of $Y^{\mathrm{ad}}_{\Sigma,L}$ containing $X^{\mathrm{ad}}_L$ such that, for every prime number $\ell \neq p$, the pull-back map

$$H^w_{\text{ét}}(V_{\mathbb{C}_p}, \mathbb{F}_\ell) \to H^w_{\text{ét}}(X^{\mathrm{ad}}_{\mathbb{C}_p}, \mathbb{F}_\ell)$$

is an isomorphism for every $w$. By [33, Corollary 8.8], there exists a closed subscheme $Z$ of $Y_{\Sigma,L^\flat}$, which is defined over a global field (i.e. the function field of a smooth connected curve over $k$), such that $Z^{\mathrm{ad}}$ is contained in $\pi^{-1}(V)$ and $\dim Z = \dim X$. We may assume that $Z$ is irreducible. By [10, Theorem 4.1], there exists an alteration $Z' \to Z$, which is defined over a global field, such that $Z'$ is projective and smooth over $L^\flat$. (We note that $L^\flat$ is a perfect field.) We may assume further that $Z'$ and $Z$ are geometrically irreducible.

We have the following composition for every $\ell \neq p$ and every $w$:

$$\begin{aligned} H^w_{\text{ét}}(X^{\mathrm{ad}}_{\mathbb{C}_p}, \mathbb{F}_\ell) \xrightarrow{\sim} H^w_{\text{ét}}(V_{\mathbb{C}_p}, \mathbb{F}_\ell) &\to H^w_{\text{ét}}(\pi^{-1}(V)_{\mathbb{C}_p^\flat}, \mathbb{F}_\ell) \\ &\to H^w_{\text{ét}}(Z^{\mathrm{ad}}_{\mathbb{C}_p^\flat}, \mathbb{F}_\ell) \to H^w_{\text{ét}}((Z'_{\mathbb{C}_p^\flat})^{\mathrm{ad}}, \mathbb{F}_\ell), \end{aligned}$$

where the first map is the inverse map of the pull-back map, the second map is induced by [33, Theorem 8.5 (v)], and the last two maps are the pull-back maps. By using a comparison theorem of Huber [17, Theorem 3.8.1], we obtain a map

$$H^w_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{F}_\ell) \to H^w_{\text{ét}}(Z'_{\mathbb{C}_p^\flat}, \mathbb{F}_\ell)$$

for every $\ell \neq p$ and every $w$. This map is compatible with the actions of $G := G_L = G_{L^\flat}$ on both sides and compatible with the cup products.

For $w = 2\dim X$, by the same argument as in the proof of [33, Lemma 9.9], we conclude that the above map is an isomorphism for all but finitely many $\ell \neq p$ from the fact that the image of the $(\dim X)$-th power of the Chern class of an ample line bundle on $Y_{\Sigma,\mathbb{C}_p^\flat}$ under the map

$$H^{2\dim X}_{\text{ét}}(Y_{\Sigma,\mathbb{C}_p^\flat}, \mathbb{F}_\ell) \to H^{2\dim X}_{\text{ét}}(Z'_{\mathbb{C}_p^\flat}, \mathbb{F}_\ell)$$

is not zero for all but finitely many $\ell \neq p$. By Poincaré duality, it follows that $H^w_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{F}_\ell)$ is a direct summand of $H^w_{\text{ét}}(Z'_{\mathbb{C}_p^\flat}, \mathbb{F}_\ell)$ as a $G$-representation for every $w$ and for all but finitely many $\ell \neq p$. Since $Z'$ is defined over a global field, a natural analogue of Theorem 3.7 holds for the family $\{H^w_{\text{ét}}(Z'_{\mathbb{C}_p^\flat}, \mathbb{F}_\ell)\}_{\ell\neq p}$ of $G$-representations by the case (1). This fact completes the proof of Theorem 3.7 in the case (5).

## 7 The case of abelian varieties

We shall prove Theorem 3.7 in the case (2). We use the same notation as in Section 3. Let $A$ be an abelian variety over $K$. Let $\mathscr{A}$ be the Néron model of $A$. After replacing $K$ by a finite extension, we may assume that $A$ has semi-abelian reduction, i.e. the identity component $\mathscr{A}^0_s$ of the special fiber $\mathscr{A}_s$ of $\mathscr{A}$ is a semi-abelian variety over $k$. In this case, the action of $I_K$ on the $\ell$-adic Tate module $T_\ell A_{\overline{K}}$ of $A$ is unipotent and factors through $t_\ell\colon I_K \to \mathbb{Z}_\ell(1)$ for every $\ell \neq p$. Let $\sigma \in I_K$ be an element such that, for every $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator.

Since the quotient $\mathscr{A}_s/\mathscr{A}_s^0$ is a finite étale group scheme over $k$, for all but finitely many $\ell \neq p$, we have

$$A[\ell^n](\overline{K})^{I_K} = \mathscr{A}_s^0[\ell^n](\overline{k})$$

for every $n \geq 1$ by the Néron mapping property and [3, Section 7.3, Proposition 3]. It follows that

$$(T_\ell A_{\overline{K}})^{I_K} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell = (T_\ell A_{\overline{K}} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell)^{I_K}$$

for all but finitely many $\ell \neq p$. For such $\ell \neq p$, the cokernel of $\sigma - 1$ acting on $T_\ell A_{\overline{K}}$ is torsion-free by Lemma 2.8. Note that we have $(\sigma-1)^2 = 0$ on $T_\ell A_{\overline{K}}$; see the proof of [44, Exposé I, Théorème 6.1] for instance. Therefore we see that Conjecture 3.5 for $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell)$ is true by Theorem 3.2 and Proposition 3.10.
Let $w$ be an integer. We can define the monodromy filtration on $\otimes^w H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell)$ for all but finitely many $\ell \neq p$; see Section 3.2. The assertion of Conjecture 3.5 also holds for $\otimes^w H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell)$ by the formula in [13, Proposition (1.6.9)(i)]. (Although the base field is of characteristic 0 in *loc. cit.*, the same formula holds with $\mathbb{F}_\ell$-coefficients for all but finitely many $\ell \neq p$.) Since $H^w_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell) = \wedge^w H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell)$ is a direct summand of $\otimes^w H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell)$ as a $G_K$-representation for all but finitely many $\ell \neq p$, it follows that Conjecture 3.5 holds for $H^w_{\text{ét}}(A_{\overline{K}}, \mathbb{F}_\ell)$.
The proof of Theorem 3.7 in the case (2) is complete.

## 8 The cases of surfaces

We shall prove Theorem 3.7 in the case (3). We retain the notation of Section 3. By Poincaré duality, it is enough to prove the case where $w \leq 2$. By the theory of Picard varieties, the case where $w = 1$ (and the case where $\dim X = 1$) follows from the case (2). We may assume that $w = 2$. Since we have already proved Theorem 3.7 in the case (1), we may assume that $\mathrm{char}(K) = 0$. By Lemma 4.7, Lemma 4.12, and Lemma 4.13, it suffices to prove that, for a proper strictly semi-stable scheme $\mathfrak{X}$ over $\mathcal{O}_K$ purely of relative dimension 2, Conjecture 4.5 for $(\mathfrak{X}, w = 2, \Lambda_\ell = \mathbb{Z}_\ell)$ is true.
We use the same notation as in Section 4.1. We fix an element $\sigma \in I_K$ such that, for every $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator. Using the generator $t_\ell(\sigma)$, we identify $\mathbb{Z}_\ell(i)$ with $\mathbb{Z}_\ell$. We shall prove that the map $(1 \otimes t_\ell(\sigma))^2 \colon E_{2,\mathbb{Z}_\ell}^{-2,4} \to E_{2,\mathbb{Z}_\ell}^{2,0}$ is an isomorphism for all but finitely many $\ell \neq p$. This map is identified with the map

$$\begin{aligned} &\mathrm{Ker}(d_1^{-2,4} \colon H^0(Y^{(2)}, \mathbb{Z}_\ell) \to H^2(Y^{(1)}, \mathbb{Z}_\ell)) \\ &\qquad \to \mathrm{Coker}(d_1^{1,0} \colon H^0(Y^{(1)}, \mathbb{Z}_\ell) \to H^0(Y^{(2)}, \mathbb{Z}_\ell)) \end{aligned}$$

induced by the identity map on $H^0(Y^{(2)}, \mathbb{Z}_\ell)$. Here we put $H^i(Y^{(j)}, \mathbb{Z}_\ell) := H^i_{\text{ét}}(Y^{(j)}_{\overline{k}}, \mathbb{Z}_\ell)$ for simplicity. The map $d_1^{-2,4}$ is a linear combination of Gysin maps and the map $d_1^{1,0}$ is a linear combination of restriction maps; see Remark 4.2. Since $\dim Y^{(1)} = 1$ and $\dim Y^{(2)} = 0$ (if they are not empty), each cohomology group is the base change of a finitely generated $\mathbb{Z}$-module and the above morphism is defined over $\mathbb{Z}$. These $\mathbb{Z}$-structures are independent of $\ell \neq p$. Hence $E_{2,\mathbb{Z}_\ell}^{-2,4}$, $E_{2,\mathbb{Z}_\ell}^{2,0}$, and the cokernel of the map $E_{2,\mathbb{Z}_\ell}^{-2,4} \to E_{2,\mathbb{Z}_\ell}^{2,0}$ are torsion-free for all but finitely many $\ell \neq p$. Therefore the assertion follows from the fact that the map $E_{2,\mathbb{Q}_\ell}^{-2,4} \to E_{2,\mathbb{Q}_\ell}^{2,0}$ is an isomorphism for every $\ell \neq p$; see Theorem 3.2 and Remark 4.8.

To prove that the map $1 \otimes t_\ell(\sigma)\colon E_{2,\mathbb{Z}_\ell}^{-1,3} \to E_{2,\mathbb{Z}_\ell}^{1,1}$ is an isomorphism for all but finitely many $\ell \neq p$, it suffices to prove that the restriction of the canonical pairing on $H^1(Y^{(1)}, \mathbb{Z}_\ell)$ to the image of the boundary map

$$d_1^{0,1}\colon E_{1,\mathbb{Z}_\ell}^{0,1} = H^1(Y^{(0)}, \mathbb{Z}_\ell) \to E_{1,\mathbb{Z}_\ell}^{1,1} = H^1(Y^{(1)}, \mathbb{Z}_\ell)$$

is perfect for all but finitely many $\ell \neq p$. For every $i$, let $\mathrm{Pic}^0_{D_i}$ be the Picard variety of $D_i$, i.e. the underlying reduced subscheme of the identity component of the Picard scheme associated with $D_i$. Similarly, let $\mathrm{Pic}^0_{D_i \cap D_j}$ be the Picard variety of $D_i \cap D_j$ for every $i < j$. Since $D_i$ and $D_i \cap D_j$ are proper smooth schemes, the group schemes $\mathrm{Pic}^0_{D_i}$ and $\mathrm{Pic}^0_{D_i \cap D_j}$ are abelian varieties. The Kummer sequence gives isomorphisms $H^1(D_i, \mathbb{Z}_\ell) \cong T_\ell(\mathrm{Pic}^0_{D_i})_{\overline{k}}$ and $H^1(D_i \cap D_j, \mathbb{Z}_\ell) \cong T_\ell(\mathrm{Pic}^0_{D_i \cap D_j})_{\overline{k}}$. (Recall that we have fixed the isomorphism $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$.) By Remark 4.2, under these isomorphisms, the map $d_1^{0,1}$ can be identified with the homomorphism of Tate modules induced by a linear combination of pull-back maps

$$\rho\colon \times_i \mathrm{Pic}^0_{D_i} \to \times_{i<j} \mathrm{Pic}^0_{D_i \cap D_j}.$$

We write $A := \times_{i<j} \mathrm{Pic}^0_{D_i \cap D_j}$. Let $B \subset A$ be the image of $\rho$. By the Poincaré complete reducibility theorem, the image of $d_1^{0,1}$ coincides with $T_\ell B_{\overline{k}}$ for all but finitely many $\ell \neq p$. The canonical pairing on $H^1(Y^{(1)}, \mathbb{Z}_\ell)$ is equal to the pairing on $T_\ell A_{\overline{k}}$ induced by a principal polarization on $A_{\overline{k}}$. The restriction of the pairing on $T_\ell A_{\overline{k}}$ to $T_\ell B_{\overline{k}}$ is induced by a polarization on $B_{\overline{k}}$, which is perfect for all but finitely many $\ell \neq p$. This fact proves our assertion.

The proof of Theorem 3.7 in the case (3) is complete.

Remark 8.1. The proof given above (or the proof of Conjecture 3.1 for surfaces given in [30]) cannot be generalized directly to higher dimensional cases. For example, if $\mathfrak{X}$ is a proper strictly semi-stable scheme over $\mathcal{O}_K$ purely of relative dimension 3, then in order to prove Conjecture 4.5 for $(\mathfrak{X}, w = 3)$, we have to investigate the linear combination $\delta_0^*\colon H^2(Y^{(0)}, \mathbb{Z}_\ell) \to H^2(Y^{(1)}, \mathbb{Z}_\ell)$ of restriction maps defined in Remark 4.2. However, not much is known about this map in general.

# 9 The cases of varieties uniformized by Drinfeld upper half spaces

## 9.1 The $\ell$-independence of the weight-monodromy conjecture in certain cases

In this subsection, we make some preparations for the proof of Theorem 3.7 in the case (4). Let $k$ be a finite field of characteristic $p$. Let $Y$ be a projective smooth scheme over $k$. Let $\ell \neq p$ be a prime number. The cycle map for codimension $w$ cycles is denoted by

$$\mathrm{cl}_\ell^w \colon Z^w(Y) \to H^{2w}_{\text{ét}}(Y_{\overline{k}}, \mathbb{Z}_\ell(w)),$$

where $Z^w(Y)$ is the group of algebraic cycles of codimension $w$ on $Y$. We denote by $N^w(Y) := Z^w(Y)/\sim_{\text{num}}$ the group of algebraic cycles of codimension $w$ on $Y$ modulo numerical equivalence. It is known that $N^w(Y)$ is a finitely generated $\mathbb{Z}$-module [43, Exposé XIII, Proposition 5.2].

Assumption 9.1 (Assumption $(*)$). We say that $Y$ *satisfies the assumption* $(*)$ if, for every $\ell \neq p$, we have $H^w_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_\ell) = 0$ for every odd integer $w$ and the $\mathbb{Q}_\ell$-vector space $H^{2w}_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_\ell(w))$ is spanned by the image of $\mathrm{cl}_\ell^w$ for every $w \geq 0$.

Lemma 9.2. *Let $Y$ be a projective smooth scheme over $k$. Assume that $Y$ satisfies the assumption $(*)$.*

*(i) The cycle map $\mathrm{cl}_\ell^w$ induces an isomorphism*

$$N^w(Y) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{\sim} H^{2w}_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_\ell(w))$$

*for every $\ell \neq p$ and $w \geq 0$.*

*(ii) For all but finitely many $\ell \neq p$, we have $H^w_{\text{ét}}(Y_{\overline{k}}, \mathbb{Z}_\ell) = 0$ for every odd integer $w$ and the cycle map $\mathrm{cl}_\ell^w$ induces an isomorphism*

$$N^w(Y) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} H^{2w}_{\text{ét}}(Y_{\overline{k}}, \mathbb{Z}_\ell(w))$$

*for every $w \geq 0$.*

Proof. See [21, Lemma 2.1] for the assertion (i). The first part of (ii) follows from (i) and Theorem 2.3. The second part of (ii) can be proved by using the same argument as in [21, Lemma 2.1] together with Theorem 2.3. □

Let $K$ be a non-archimedean local field with residue field $k$. Let $\mathfrak{X}$ be a projective strictly semi-stable scheme over $\mathcal{O}_K$ purely of relative dimension $d$. We use the same notation as in Section 4. So $D_1, \ldots, D_m$ are the irreducible components of the special fiber $Y$ of $\mathfrak{X}$ and for every non-empty subset $I \subset \{1, \ldots, m\}$, we define $D_I := \cap_{i \in I} D_i$. We will consider the weight spectral sequences arising from $\mathfrak{X}$. We fix an element $\sigma \in I_K$ such that, for every $\ell \neq p$, the image $t_\ell(\sigma) \in \mathbb{Z}_\ell(1)$ is a generator.

Proposition 9.3. *Let the notation be as above. Assume that for every non-empty subset $I \subset \{1, \dots, m\}$, the intersection $D_I$ satisfies the assumption $(*)$. We assume further that, for some prime number $\ell' \neq p$, the map*

$$(1 \otimes t_{\ell'}(\sigma))^i \colon E_{2,\mathbb{Q}_{\ell'}}^{-i,w+i} \to E_{2,\mathbb{Q}_{\ell'}}^{i,w-i}$$

*is an isomorphism for all $w, i \geq 0$. Then Conjecture 4.5 for $\mathfrak{X}$ is true.*

Proof. Using the generator $t_\ell(\sigma)$, we identify $\mathbb{Z}_\ell(i)$ with $\mathbb{Z}_\ell$. Let $\Lambda_\ell$ be $\mathbb{Q}_\ell$ (resp. $\mathbb{Z}_\ell$). The map $d_1^{v,w} \colon E_{1,\Lambda_\ell}^{v,w} \to E_{1,\Lambda_\ell}^{v+1,w}$ is a linear combination of Gysin maps and restriction maps, whose coefficients are in $\mathbb{Z}$ and independent of $\ell \neq p$; see Remark 4.2. By Lemma 9.2, for every $\ell \neq p$ (resp. all but finitely many $\ell \neq p$), this map is the base change of a homomorphism of finitely generated $\mathbb{Z}$-modules which is independent of $\ell \neq p$. Moreover, the same holds for the map $(1 \otimes t_\ell(\sigma))^i \colon E_{2,\Lambda_\ell}^{-i,w+i} \to E_{2,\Lambda_\ell}^{i,w-i}$. Conjecture 4.5 for $\mathfrak{X}$ follows from this fact. $\square$

### 9.2 Proof of Theorem 3.7 in the case (4)

We shall explain the precise statement. Let $K$ be a non-archimedean local field of characteristic 0. Let $\Omega_K^d$ be the Drinfeld upper half space over $K$ of dimension $d$. It is a rigid analytic variety over $K$. Let $\Gamma \subset \mathrm{PGL}_{d+1}(K)$ be a discrete cocompact torsion-free subgroup. It is known that the quotient $\Gamma \backslash \Omega_K^d$ is the rigid analytic variety associated with a projective smooth scheme $X$ over $K$. In this case, we say that $X$ is uniformized by a Drinfeld upper half space. We shall prove Conjecture 3.5 for $X$.
Let $\widehat{\Omega}_K^d$ be the formal model of $\Omega_K^d$ considered in [26], which is a flat formal scheme locally of finite type over $\operatorname{Spf} \mathcal{O}_K$. We can take the quotient $\Gamma \backslash \widehat{\Omega}_K^d$. There is a flat projective scheme $\mathfrak{X}$ over $\operatorname{Spec} \mathcal{O}_K$ whose $\varpi$-adic completion is isomorphic to $\Gamma \backslash \widehat{\Omega}_K^d$. Here $\varpi$ is a uniformizer of $K$. The generic fiber of $\mathfrak{X}$ is isomorphic to $X$. Let $D_1, D_2, \dots, D_m$ be the irreducible components of the special fiber of $\mathfrak{X}$. As in the proof of [21, Theorem 1.1], after replacing $\Gamma$ by a finite index subgroup, we may assume that $\mathfrak{X}$ is a projective strictly semi-stable scheme over $\mathcal{O}_K$ purely of relative dimension $d$ and, for every non-empty subset $I \subset \{1, 2, \dots, m\}$, the intersection $D_I := \cap_{i \in I} D_i$ satisfies the assumption $(*)$. Since the weight-monodromy conjecture for $X$ is true, we see that Conjecture 3.5 for $X$ is true by Lemma 4.9 and Proposition 9.3.

## 10 Applications to Brauer groups and Chow groups of codimension two cycles

In this section, let $K$ be a non-archimedean local field.

### 10.1 Brauer groups

First, we recall well known results on the Chern class maps for divisors. Let $Z$ be a proper smooth scheme over a field $F$. Let $\mathrm{NS}(Z_{\overline{F}})$ be the Néron-Severi group of $Z_{\overline{F}}$, which is a finitely generated $\mathbb{Z}$-module. The absolute Galois group $G_F$ of $F$ acts on $\mathrm{NS}(Z_{\overline{F}})$ via the isomorphism $\mathrm{Aut}(\overline{F}/F) \xrightarrow{\sim} G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$. Let $\Lambda_\ell$ be either $\mathbb{Q}_\ell$ or $\mathbb{Z}_\ell$. We put $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} := \mathrm{NS}(Z_{\overline{F}}) \otimes_{\mathbb{Z}} \Lambda_\ell$. The Chern class map with $\Lambda_\ell$-coefficients gives an injection

$$\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \hookrightarrow H^2_{\text{ét}}(Z_{\overline{F}}, \Lambda_\ell(1))$$

for every $\ell \neq \mathrm{char}(F)$.

Lemma 10.1. *Let the notation be as above. Let $\Lambda_\ell = \mathbb{Q}_\ell$ (resp. $\Lambda_\ell = \mathbb{Z}_\ell$). Then the injection $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \hookrightarrow H^2_{\text{ét}}(Z_{\overline{F}}, \Lambda_\ell(1))$ maps $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell}$ onto a direct summand of $H^2_{\text{ét}}(Z_{\overline{F}}, \Lambda_\ell(1))$ as a $G_F$-module for every $\ell \neq \mathrm{char}(F)$ (resp. all but finitely many $\ell \neq \mathrm{char}(F)$).*

Proof. We may assume that $Z$ is connected. We first assume that $Z$ is projective. Let $d := \dim Z$. If $d = 1$, then $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \to H^2_{\text{ét}}(Z_{\overline{F}}, \Lambda_\ell(1))$ is an isomorphism for every $\ell \neq \mathrm{char}(F)$ and the assertion is trivial. So we assume that $d \geq 2$. Let $D$ be an ample divisor on $Z$. The cohomology class of $D$ in $H^2_{\text{ét}}(Z_{\overline{F}}, \Lambda_\ell(1))$ is also denoted by $D$. Let $D^{d-2} \in H^{2d-4}_{\text{ét}}(Z_{\overline{F}}, \Lambda_\ell(d-2))$ be the $(d-2)$-times self-intersection of $D$ with respect to the cup product. We have the following $G_F$-equivariant map:

$$\begin{aligned} f_D \colon H^2_{\text{ét}}(Z_{\overline{F}}, \Lambda_\ell(1)) &\to \mathrm{Hom}_{\Lambda_\ell}(\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell}, \Lambda_\ell) \\ x &\mapsto (y \mapsto \mathrm{tr}(D^{d-2} \cup x \cup y)), \end{aligned}$$

where $D^{d-2} \cup x \cup y \in H^{2d}_{\text{ét}}(Z_{\overline{F}}, \Lambda_\ell(d))$ is the cup product of the triple $(D^{d-2}, x, y)$, and $\mathrm{tr} \colon H^{2d}_{\text{ét}}(Z_{\overline{F}}, \Lambda_\ell(d)) \to \Lambda_\ell$ is the trace map. By [43, Exposé XIII, Théorème 4.6], we see that for every $\ell \neq \mathrm{char}(F)$ (resp. all but finitely many $\ell \neq \mathrm{char}(F)$), the restriction of the map $f_D$ to $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell}$ is an isomorphism, and hence $f_D$ gives a $G_F$-equivariant splitting of $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \hookrightarrow H^2_{\text{ét}}(Z_{\overline{F}}, \Lambda_\ell(1))$. This fact proves our claim.

The general case can be reduced to the case where $Z$ is projective as follows. We may assume that $F$ is perfect after replacing $F$ by the perfect closure of it. By [10, Theorem 4.1], there exists an alteration $Z' \to Z$ such that $Z'$ is a projective smooth connected scheme over $F$. Since we have already proved the assertion for $Z'$, it suffices to prove the claim that the pull-back map $\mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \to \mathrm{NS}(Z'_{\overline{F}})_{\Lambda_\ell}$ gives a decomposition $\mathrm{NS}(Z'_{\overline{F}})_{\Lambda_\ell} \cong \mathrm{NS}(Z_{\overline{F}})_{\Lambda_\ell} \oplus N_\ell$ as a $G_F$-module for every $\ell \neq \mathrm{char}(F)$ (resp. all but finitely many $\ell \neq \mathrm{char}(F)$). The pull-back map $\mathrm{NS}(Z_{\overline{F}})_{\mathbb{Q}} \to \mathrm{NS}(Z'_{\overline{F}})_{\mathbb{Q}}$ is a $G_F$-equivariant injection. Since both $\mathrm{NS}(Z_{\overline{F}})$ and $\mathrm{NS}(Z'_{\overline{F}})$ are finitely generated $\mathbb{Z}$-modules and the action of $G_F$ on $\mathrm{NS}(Z'_{\overline{F}})$ factors through a finite quotient of $G_F$, the claim follows. □

For a scheme $Z$, let $\mathrm{Br}(Z) := H^2_{\text{ét}}(Z, \mathbb{G}_m)$ be the cohomological Brauer group. Recall that $\mathrm{Br}(Z)$ is a torsion abelian group if $Z$ is a Noetherian regular scheme; see [15, Corollaire 1.8]. For an integer $n$, let $\mathrm{Br}(Z)[n]$ be the set of elements killed by $n$. Let $\mathrm{Br}(Z)[p']$ be the prime-to-$p$ torsion part, i.e. the set of elements

$x \in \mathrm{Br}(Z)$ such that we have $nx = 0$ for some non-zero integer $n$ which is not divisible by $p$.

Let $X$ be a proper smooth scheme over the non-archimedean local field $K$. Let $\ell \neq \mathrm{char}(K)$ be a prime number. Let

$$\mathrm{ch}_{\mathbb{Q}_\ell} \colon \mathrm{Pic}(X)_{\mathbb{Q}_\ell} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))$$

be the $\ell$-adic Chern class map and let

$$\mathrm{ch}_{\mathbb{F}_\ell} \colon \mathrm{Pic}(X) \to H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell(1))$$

be the $\ell$-torsion Chern class map. They induce homomorphisms

$$\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell} \colon \mathrm{Pic}(X)_{\mathbb{Q}_\ell} \to H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{G_K}$$

and

$$\widetilde{\mathrm{ch}}_{\mathbb{F}_\ell} \colon \mathrm{Pic}(X) \to H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell(1))^{G_K}.$$

We will also call $\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell}$ (resp. $\widetilde{\mathrm{ch}}_{\mathbb{F}_\ell}$) the $\ell$-adic (resp. $\ell$-torsion) Chern class map. We shall study the relation between the Chern class maps and the $G_K$-fixed part of the cohomological Brauer group $\mathrm{Br}(X_{\overline{K}})$ of $X_{\overline{K}}$. (Here $G_K$ acts on $\mathrm{Br}(X_{\overline{K}})$ via $\mathrm{Aut}(\overline{K}/K) \xrightarrow{\sim} G_K$.)

Theorem 10.2. *Let $X$ be a proper smooth scheme over $K$. Assume that the $\ell$-adic Chern class map $\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell}$ is surjective for all but finitely many $\ell \neq p$. Then the following assertions hold:*

*(i) The $\ell$-torsion Chern class map $\widetilde{\mathrm{ch}}_{\mathbb{F}_\ell}$ is surjective for all but finitely many $\ell \neq p$.*

*(ii) The $G_K$-fixed part $\mathrm{Br}(X_{\overline{K}})[\ell]^{G_K}$ is zero for all but finitely many $\ell \neq p$.*

Proof. (i) By Lemma 10.1, there is a decomposition

$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(1)) = \mathrm{NS}(X_{\overline{K}})_{\mathbb{Z}_\ell} \oplus M_\ell$$

as a $G_K$-module for all but finitely many $\ell \neq p$. By the assumption, we have $M_\ell[1/\ell]^{G_K} = 0$ for all but finitely many $\ell \neq p$. It follows that, for all but finitely many $\ell \neq p$, every eigenvalue of a lift $\mathrm{Frob} \in G_K$ of the geometric Frobenius element acting on $M_\ell[1/\ell]^{I_K}$ is different from 1.

By Proposition 4.15 (i), there exists a non-zero monic polynomial $P(T) \in \mathbb{Z}[1/p][T]$ such that, for all but finitely many $\ell \neq p$, we have $P(\mathrm{Frob}) = 0$ on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(1))$. We write $P(T)$ in the form $(T-1)^m Q(T)$ for some non-negative integer $m$ and $Q(T) \in \mathbb{Z}[1/p][T]$ with $Q(1) \neq 0$. Then $Q(\mathrm{Frob}) = 0$ on $M_\ell[1/\ell]^{I_K}$, and hence $Q(\mathrm{Frob}) = 0$ on $M_\ell^{I_K}$ for all but finitely many $\ell \neq p$. By Corollary 3.11, we have $Q(\mathrm{Frob}) = 0$ on $(M_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell)^{I_K} = 0$ for all but finitely many $\ell \neq p$. Since $Q(T)$ and $T-1$ are relatively prime in $\mathbb{Q}[T]$, we have $(M_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell)^{G_K} = 0$ for all but finitely many $\ell \neq p$ by Lemma 2.2. Now, the

assertion follows from the fact that the natural map $\mathrm{Pic}(X) \to (\mathrm{NS}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{F}_\ell)^{G_K}$ is surjective for all but finitely many $\ell \neq p$.
(ii) The Kummer sequence gives a short exact sequence

$$0 \to \mathrm{NS}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{F}_\ell \to H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell(1)) \to \mathrm{Br}(X_{\overline{K}})[\ell] \to 0$$

for every $\ell \neq \mathrm{char}(K)$. Thus, by Lemma 10.1, there is a decomposition

$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_\ell(1)) \cong (\mathrm{NS}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{F}_\ell) \oplus \mathrm{Br}(X_{\overline{K}})[\ell]$$

as a $G_K$-module for all but finitely many $\ell \neq p$. Thus the assertion follows from (i). □

Corollary 10.3. *Assume that* $\mathrm{char}(K) = 0$ *(resp.* $\mathrm{char}(K) = p$*). Let $X$ be a proper smooth scheme over $K$. Assume that the $\ell$-adic Chern class map* $\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell}$ *is surjective for every* $\ell \neq \mathrm{char}(K)$*. Then* $\mathrm{Br}(X_{\overline{K}})^{G_K}$ *(resp.* $\mathrm{Br}(X_{\overline{K}})[p']^{G_K}$*) is finite.*

Proof. This follows from Theorem 10.2 and the fact that the union $\cup_n \mathrm{Br}(X_{\overline{K}})[\ell^n]^{G_K}$ is finite for every $\ell \neq \mathrm{char}(K)$ under the assumptions; see the proof of [6, Corollary 1.5]. We shall give a proof of this fact for the convenience of the reader.
We put

$$T_\ell \mathrm{Br}(X_{\overline{K}}) := \varprojlim_n \mathrm{Br}(X_{\overline{K}})[\ell^n]$$

and $V_\ell \mathrm{Br}(X_{\overline{K}}) := T_\ell \mathrm{Br}(X_{\overline{K}}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. As in the proof of Theorem 10.2, the Kummer sequence and Lemma 10.1 give a decomposition

$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1)) \cong \mathrm{NS}(X_{\overline{K}})_{\mathbb{Q}_\ell} \oplus V_\ell \mathrm{Br}(X_{\overline{K}})$$

as a $G_K$-module for every $\ell \neq \mathrm{char}(K)$. By the assumption, we have $(V_\ell \mathrm{Br}(X_{\overline{K}}))^{G_K} = 0$. Since $T_\ell \mathrm{Br}(X_{\overline{K}})$ is torsion-free, we have $(T_\ell \mathrm{Br}(X_{\overline{K}}))^{G_K} = 0$ for every $\ell \neq \mathrm{char}(K)$. It follows that $\cup_n \mathrm{Br}(X_{\overline{K}})[\ell^n]^{G_K}$ is finite for every $\ell \neq \mathrm{char}(K)$. □

Here we give an example of a projective smooth scheme over $K$ for which $\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell}$ is surjective for every $\ell \neq \mathrm{char}(K)$.

Corollary 10.4. *Let $X$ be a projective smooth scheme over $K$ which is uniformized by a Drinfeld upper half space.*

*(i) The $\ell$-adic Chern class map* $\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell}$ *is surjective for every* $\ell \neq \mathrm{char}(K)$.

*(ii) The $G_K$-fixed part* $\mathrm{Br}(X_{\overline{K}})^{G_K}$ *(resp.* $\mathrm{Br}(X_{\overline{K}})[p']^{G_K}$*) is finite if* $\mathrm{char}(K) = 0$ *(resp.* $\mathrm{char}(K) = p$*).*

Proof. (i) If $d := \dim X \neq 2$, then $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))$ is one-dimensional for every $\ell \neq \mathrm{char}(K)$. If $d = 2$, then the $G_K$-fixed part $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{G_K}$ is one-dimensional for every $\ell \neq \mathrm{char}(K)$ by [21, Lemma 7.1]. Therefore, for any $d \geq 1$, the $\ell$-adic Chern class map $\widetilde{\mathrm{ch}}_{\mathbb{Q}_\ell}$ is surjective for every $\ell \neq \mathrm{char}(K)$.
(ii) The assertion follows from (i) and Corollary 10.3. □

Remark 10.5. Let $F$ be a field which is finitely generated over its prime subfield.

(i) Assume that $\mathrm{char}(F) = p > 0$. Let $Z$ be a projective smooth variety over $F$. Cadoret-Hui-Tamagawa proved that the Tate conjecture for divisors on $Z$ implies the finiteness of $\mathrm{Br}(Z_{\overline{F}})[p']^{G_F}$; see [6, Corollary 1.5]. (If $F$ is finite, this result was proved by Tate; see also the references given in [38, Section 4].)

(ii) Assume that $\mathrm{char}(F) = 0$. Let $Z$ be an abelian variety or a K3 surface over $F$. By using the Tate conjecture for divisors on $Z$ and its torsion analogue, Skorobogatov-Zarhin proved that $\mathrm{Br}(Z_{\overline{F}})^{G_F}$ is finite; see [35] for details.

Remark 10.6. Let $X$ be a proper smooth scheme over $K$. Assume that $\mathrm{char}(K) = p$ or $\dim X = 2$. If the $\ell'$-adic Chern class map $\widetilde{\mathrm{ch}}_{\mathbb{Q}_{\ell'}}$ is surjective for some $\ell' \neq p$, then the same holds for every prime number $\ell \neq \mathrm{char}(K)$. For $\ell \neq p$, this fact can be proved by using Lemma 10.1 and the $\ell$-independence conjecture stated in Remark 4.17 (it is a theorem under the assumptions). If $\mathrm{char}(K) = 0$, $\dim X = 2$, and $\ell = p$, we use a $p$-adic analogue of the $\ell$-independence conjecture for the Weil-Deligne representation associated with $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$; see [28, Theorem 3.1].

### 10.2 Chow groups of codimension two cycles

In this subsection, following the strategy of Colliot-Thélène and Raskind [7], we show some finiteness properties of the Chow group of codimension two cycles on a proper smooth scheme over $K$.

First, we briefly recall a $p$-adic analogue of the weight-monodromy conjecture. Assume that $\mathrm{char}(K) = 0$. Let $W_K$ be the Weil group of $K$. Let $X$ be a proper smooth scheme over $K$. Let

$$\mathrm{WD}(H^w_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_p))$$

be the Weil-Deligne representation of $W_K$ over $\overline{\mathbb{Q}}_p$ associated with $H^w_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_p)$; see [39, p.469]. We say that the $p$-adic analogue of the weight-monodromy conjecture holds for $(X, w)$ if $\mathrm{WD}(H^w_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_p))$ is pure of weight $w$ in the sense of [39, p.471].

Assume that there exists a proper strictly semi-stable scheme $\mathfrak{X}$ over $\mathcal{O}_K$ purely of relative dimension $d$ whose generic fiber is isomorphic to $X$. Let $Y$ be the special fiber of $\mathfrak{X}$. Then, by the semi-stable comparison isomorphism [41, Theorem 0.2], the $p$-adic analogue of the weight-monodromy conjecture holds for $(X, w)$ if and only if the assertion of [25, Conjecture 3.27] holds for the logarithmic crystalline cohomology group $H^w_{\log\,\mathrm{cris}}(Y/W(k))[1/p]$, where we endow $Y$ with the canonical log structure arising from the strictly semi-stable scheme $\mathfrak{X}$. (Here $W(k)$ is the ring of Witt vectors of $k$.)

The following results are analogues of [7, Theorem 1.5 and Theorem 1.5.1].

Proposition 10.7. *Let $X$ be a proper smooth scheme over $K$ and $w$ an integer. Let $i$ be an integer with $w < 2i$.*

*(i) Assume that Conjecture 3.1 holds for $(X, w)$. Then $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))^{G_K}$ is finite for every $\ell \neq p$. Assume further that Conjecture 3.5 for $(X, w)$ is true. Then we have $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))^{G_K} = 0$ for all but finitely many $\ell \neq p$.*

*(ii) If $\mathrm{char}\,(K) = 0$ and the $p$-adic analogue of the weight-monodromy conjecture is true for $(X, w)$, then $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(i))^{G_K}$ is finite.*

Proof. For every $\ell \neq \mathrm{char}(K)$, we have the following exact sequence of $G_K$-modules:

$$\begin{aligned} H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(i)) \to H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(i)) \xrightarrow{f_\ell} H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \\ \to H^{w+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(i))_{\text{tor}} \to 0. \end{aligned}$$

Here $H^{w+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(i))_{\text{tor}}$ is the torsion part of $H^{w+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(i))$. Let $H_\ell$ denote the free part of $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(i))$. We will use the continuous cohomology group $H^j(G_K, H_\ell)$ defined in [37, Section 2]. It is a finitely generated $\mathbb{Z}_\ell$-module for every $\ell \neq \mathrm{char}(K)$.

(i) We assume that Conjecture 3.1 holds for $(X, w)$. Since $w < 2i$, it follows that $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(i))^{G_K} = 0$ for every $\ell \neq p$. To show that $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))^{G_K}$ is finite for every $\ell \neq p$, it suffices to show that $(\mathrm{Im}\, f_\ell)^{G_K}$ is finite for every $\ell \neq p$. For every $\ell \neq p$, since $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(i))^{G_K} = 0$, we see that $(\mathrm{Im}\, f_\ell)^{G_K}$ is isomorphic to the torsion part of $H^1(G_K, H_\ell)$ by [37, Proposition (2.3)]. Hence $(\mathrm{Im}\, f_\ell)^{G_K}$ is finite.

Assume further that Conjecture 3.5 for $(X, w)$ is true. Since $H^{w+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(i))_{\text{tor}} = 0$ for all but finitely many $\ell \neq p$ by Theorem 2.3, we have $\mathrm{Im}\, f_\ell = H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ for all but finitely many $\ell \neq p$. Thus, to show that $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))^{G_K} = 0$ for all but finitely many $\ell \neq p$, it suffices to prove that the $\mathbb{Z}_\ell$-module $H^1(G_K, H_\ell)$ is torsion-free for all but finitely many $\ell \neq p$. We have the following exact sequence:

$$0 \to H^1(G_k, H_\ell^{I_K}) \to H^1(G_K, H_\ell) \to H^1(I_K, H_\ell).$$

Let $\mathrm{Frob} \in G_K$ be a lift of the geometric Frobenius element. We have

$$H^1(G_k, H_\ell^{I_K}) = \mathrm{Coker}(\mathrm{Frob} - 1 \colon H_\ell^{I_K} \to H_\ell^{I_K})$$

and

$$H^1(I_K, H_\ell) = (H_\ell)_{I_K} \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(-1).$$

We have $H_\ell^{I_K} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \subset M_{0,\mathbb{Q}_\ell} \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(i)$, where $M_{0,\mathbb{Q}_\ell}$ is the 0-th part of the monodromy filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. By Proposition 4.15 (iii), there exists a non-zero monic polynomial $P(T) \in \mathbb{Z}[1/p][T]$ such that every root of $P(T)$ has complex absolute values $q^{(w+j)/2}$ with $j \leq -2i$ and, for every $\ell \neq p$, we

have $P(\mathrm{Frob}) = 0$ on $H_\ell^{I_K} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Thus we also have $P(\mathrm{Frob}) = 0$ on $H_\ell^{I_K}$ for every $\ell \neq p$. Since $w < 2i$, the polynomials $P(T)$ and $T-1$ are relatively prime in $\mathbb{Q}[T]$. Thus, we have $H^1(G_k, H_\ell^{I_K}) = 0$ for all but finitely many $\ell \neq p$ by Lemma 2.2. Now, it remains to prove that the $\mathbb{Z}_\ell$-module $H^1(I_K, H_\ell)$ is torsion-free for all but finitely many $\ell \neq p$. This follows from Proposition 3.10.
(ii) If $\mathrm{char}\,(K) = 0$ and the $p$-adic analogue of the weight-monodromy conjecture holds for $(X, w)$, then we have $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(i))^{G_K} = 0$ if $w < 2i$. Then the same argument as above shows that $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(i))^{G_K}$ is finite.
The proof of Proposition 10.7 is complete. □

Let $X$ be a proper smooth scheme over $K$. The Chow group of codimension two cycles on $X_{\overline{K}}$ is denoted by $\mathrm{CH}^2(X_{\overline{K}})$. By combining Proposition 10.7 and [7, Proposition 3.1], we have the following results on the torsion part of $\mathrm{CH}^2(X_{\overline{K}})$, which are local analogues of [7, Theorem 3.3 and Theorem 3.4].

Corollary 10.8. *Let $X$ be a proper smooth scheme over $K$.*

(i) *Assume that Conjecture 3.1 and Conjecture 3.5 hold for $(X, w = 3)$. The prime-to-p torsion part of $\mathrm{CH}^2(X_{\overline{K}})^{G_K}$ is finite.*

(ii) *Assume that $\mathrm{char}\,(K) = 0$ and the p-adic analogue of the weight-monodromy conjecture holds for $(X, w = 3)$. Then $\cup_n \mathrm{CH}^2(X_{\overline{K}})[p^n]^{G_K}$ is finite.*

Proof. By [7, Proposition 3.1], there is a $G_K$-equivariant injection

$$\cup_n \mathrm{CH}^2(X_{\overline{K}})[\ell^n] \hookrightarrow H^3_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$$

for every $\ell \neq \mathrm{char}(K)$. Thus the assertions follow from Proposition 10.7. □

Corollary 10.9.

(i) *If $(X, w = 3)$ satisfies one of the conditions (1)–(5) in Theorem 3.2, then the prime-to-p torsion part of $\mathrm{CH}^2(X_{\overline{K}})^{G_K}$ is finite.*

(ii) *Assume that $\mathrm{char}\,(K) = 0$ and $(X, w = 3)$ satisfies one of the conditions (2)–(4) in Theorem 3.2. Then the torsion part of $\mathrm{CH}^2(X_{\overline{K}})^{G_K}$ is finite.*

Proof. (i) Use Theorem 3.2, Theorem 3.7, and Corollary 10.8 (i).
(ii) Under the assumptions, the $p$-adic analogue of the weight-monodromy conjecture holds for $(X, w = 3)$. Indeed, if $X$ is an abelian variety over $K$, then this is well known; since we have $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) = \wedge^w H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, it suffices to prove the $p$-adic analogue of the weight-monodromy conjecture for $(Z, w = 1)$ for every proper smooth scheme $Z$ over $K$, and this follows from the hard Lefschetz theorem and [25, Théorème 5.3]. If $X$ is a proper smooth surface over $K$, by, this follows from what we have just seen. If $X$ is uniformized by a Drinfeld upper half space, this follows from [21, Theorem 6.3]. (See also [25, 3.33].)
Therefore, the assertion follows from (i) and Corollary 10.8 (ii). □

Remark 10.10.

(i) If $X$ has good reduction over $\mathcal{O}_K$, the finiteness of the prime-to-$p$ torsion part of $\mathrm{CH}^2(X_{\overline{K}})^{G_K}$ was known; see the proof of [7, Theorem 3.4].

(ii) We assume that $\mathrm{char}(K) = 0$. If $\dim X = 2$ or $H^3_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) = 0$ for some (and hence every) $\ell$, then the finiteness of the torsion part of $\mathrm{CH}^2(X_{\overline{K}})^{G_K}$ is known; see [8, Section 4] and the proof of [31, Theorem 4.1]. When $\dim X = 2$, it is a consequence of Roitman's theorem; see [7, Remark 3.5] for details.

Acknowledgements

The author would like to thank his thesis adviser Tetsushi Ito for several invaluable discussions and pointing out a mistake in an earlier version of this paper. The author also would like to thank Teruhisa Koshikawa and Kanetomo Sato for helpful comments. Finally, the author would like to thank the referee for remarks and comments which improve the paper significantly. The work of the author was supported by JSPS Research Fellowships for Young Scientists KAKENHI Grant Number 18J22191 and the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 851146).

References

[1] Bădescu, L., Valla, G., *Grothendieck-Lefschetz theory, set-theoretic complete intersections and rational normal scrolls*, J. Algebra 324 (2010), no. 7, 1636–1655.

[2] Berthelot, P., *Altérations de variétés algébriques (d'après A. J. de Jong)*, Séminaire Bourbaki, Astérisque no. 241 (1997), 273–311.

[3] Bosch, S., Lütkebohmert, W., Raynaud, M., *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 21. Springer-Verlag, Berlin, 1990.

[4] Cadoret, A., *Ultraproduct Weil II for curves and integral models in compatible families of $\ell$-adic local systems*, preprint, 2019, https://webusers.imj-prg.fr/~anna.cadoret/Weil2Ultra.pdf.

[5] Cadoret, A., Hui, C.-Y., Tamagawa, A., *Geometric monodromy–semisimplicity and maximality*, Ann. of Math. (2) 186 (2017), no. 1, 205–236.

[6] Cadoret, A., Hui, C.-Y., Tamagawa, A., *$\mathbb{Q}_\ell$-versus $\mathbb{F}_\ell$-coefficients in the Grothendieck-Serre/Tate conjectures*, preprint, 2017, https://webusers.imj-prg.fr/~anna.cadoret/GST.pdf.

[7] Colliot-Thélène, J.-L., Raskind, W., *$\mathscr{K}_2$-cohomology and the second Chow group*, Math. Ann. 270 (1985), no. 2, 165–199.

[8] Colliot-Thélène, J.-L., Raskind, W., *Groupe de Chow de codimension deux des variétés définies sur un corps de nombres: un théorème de finitude pour la torsion*, Invent. Math. 105 (1991), no. 2, 221–245.

[9] Dat, J.-F., *Espaces symétriques de Drinfeld et correspondance de Langlands locale*, Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 1, 1–74.

[10] de Jong, A. J., *Smoothness, semi-stability and alterations*, Publ. Math. Inst. Hautes Études Sci. 83 (1996), 51–93.

[11] Deligne, P., *Théorie de Hodge. I*, Actes du Congrès International des Mathématiciens (Nice, 1970), tome 1, 425–430.

[12] Deligne, P., *La conjecture de Weil. I*, Publ. Math. Inst. Hautes Études Sci. 43 (1974), 273–307.

[13] Deligne, P., *La conjecture de Weil. II*, Publ. Math. Inst. Hautes Études Sci. 52 (1980), 137–252.

[14] Gabber, O., *Sur la torsion dans la cohomologie l-adique d'une variété*, C. R. Acad. Sci. Paris Sér. I Math. 297 (1983), no. 3, 179–182.

[15] Grothendieck, A., *Le groupe de Brauer. II. Théorie cohomologique*, Dix exposés sur la cohomologie des schémas, 67–87, Adv. Stud. Pure Math., 3, North-Holland, Amsterdam, 1968.

[16] Huber, R., *A generalization of formal schemes and rigid analytic varieties*, Math. Z. 217 (1994), no. 4, 513–551.

[17] Huber, R., *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30. Vieweg, Braunschweig, 1996.

[18] Huber, R., *A finiteness result for direct image sheaves on the étale site of rigid analytic varieties*, J. Algebraic Geom. 7 (1998), no. 2, 359–403.

[19] Ito, K., *Uniform local constancy of étale cohomology of rigid analytic varieties*, preprint, 2020, https://arxiv.org/abs/2008.07794.

[20] Ito, T., *Weight-monodromy conjecture for certain threefolds in mixed characteristic*, Int. Math. Res. Not. 2004, no. 2, 69–87.

[21] Ito, T., *Weight-monodromy conjecture for p-adically uniformized varieties*, Invent. Math. 159 (2005), no. 3, 607–656.

[22] Ito, T., *Weight-monodromy conjecture over equal characteristic local fields*, Amer. J. Math. 127 (2005), no. 3, 647–658.

[23] Lu, Q., Zheng, W., *Compatible systems and ramification*, Compos. Math. 155 (2019), no. 12, 2334–2353.

[24] Mieda, Y., *Note on weight-monodromy conjecture for p-adically uniformized varieties*, Proc. Amer. Math. Soc. 147 (2019), no. 5, 1911–1920.

[25] Mokrane, A., *La suite spectrale des poids en cohomologie de Hyodo-Kato*, Duke Math. J. 72 (1993), no. 2, 301–337.

[26] Mustafin, G. A., *Non-Archimedean uniformization*, Math. USSR-Sb. 34 (1978), no. 2, 187–214.

[27] Nakayama, C., *Degeneration of l-adic weight spectral sequences*, Amer. J. Math. 122 (2000), no. 4, 721–733.

[28] Ochiai, T., *l-independence of the trace of monodromy*, Math. Ann. 315 (1999), no. 2, 321–340.

[29] Orgogozo, F., *Constructibilité et modération uniformes en cohomologie étale*, Compos. Math. 155 (2019), no. 4, 711–757.

[30] Rapoport, M., Zink, Th., *Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, Invent. Math. 68 (1982), no. 1, 21–101.

[31] Saito, S., *On the cycle map for torsion algebraic cycles of codimension two*, Invent. Math. 106 (1991), no. 3, 443–460.

[32] Saito, T., *Weight spectral sequences and independence of $\ell$*, J. Inst. Math. Jussieu 2 (2003), no. 4, 583–634.

[33] Scholze, P., *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. 116 (2012), 245–313.

[34] Scholze, P., *Perfectoid spaces and their applications*, Proceedings of the International Congress of Mathematicians, Seoul 2014, vol. II, 461–486, Kyung Moon Sa, Seoul, 2014.

[35] Skorobogatov, A., Zarhin, Yu. G., *A finiteness theorem for the Brauer group of abelian varieties and K3 surfaces*, J. Algebraic Geom. 17 (2008), no. 3, 481–502.

[36] Suh, J., *Symmetry and parity in Frobenius action on cohomology*, Compos. Math. 148 (2012), no. 1, 295–303.

[37] Tate, J., *Relations between $K_2$ and Galois cohomology*, Invent. Math. 36 (1976), 257–274.

[38] Tate, J., *Conjectures on algebraic cycles in $\ell$-adic cohomology*, Motives (Seattle, WA, 1991), 71–83, Proc. Sympos. Pure Math., 55, part 1, Amer. Math. Soc., Providence, RI, 1994.

[39] Taylor, R., Yoshida, T., *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. 20 (2007), no. 2, 467–493.

[40] Terasoma, T., *Monodromy weight filtration is independent of l*, preprint, 1998, unpublished, https://arxiv.org/abs/math/9802051.

[41] Tsuji, T., *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. 137 (1999), no. 2, 233–411.

[42] *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Mathematics, 305. Springer-Verlag, Berlin-New York, 1973.

[43] *Théorie des intersections et théorème de Riemann-Roch*, Lecture Notes in Mathematics, 225. Springer-Verlag, Berlin-New York, 1971.

[44] *Groupes de monodromie en géométrie algébrique. I*, Lecture Notes in Mathematics, 288. Springer-Verlag, Berlin-New York, 1972.

Kazuhiro Ito
Kavli Institute for the Physics and Mathematics
of the Universe (WPI)
University of Tokyo
5-1-5 Kashiwanoha
Kashiwa, Chiba, 277-8583
Japan
kazuhiro.ito@ipmu.jp