Groups of Isometries of the Cuntz Algebras

Roberto Conti and Stefano Rossi

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Abstract. We provide a new interpretation of the group of Bogolubov automorphisms of the Cuntz algebras $\mathcal{O}_n$ and the group $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$ of all automorphisms preserving the UHF subalgebra $\mathcal{F}_n \subset \mathcal{O}_n$ as the isometry groups coming from two distinct spectral triples on $\mathcal{O}_n$.

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1 Introduction

The group $\mathrm{Iso}(M)$ of the isometries of a Riemannian manifold M is a natural invariant that provides useful information for investigating the properties of the manifold itself. For this reason it has been the focus of much work ever since its introduction. For instance, $\mathrm{Iso}(M)$ is known to be a compact Lie group if the manifold is compact, and moreover any compact Lie group arises as the group of the isometries of some compact Riemannian manifold. Here by isometry of a Riemannian manifold M we mean a bijective differentiable map that preserves the geodesic distance between the points of M. It is worth recalling that differentiability is actually not needed in the definition. Indeed, in light of a classical result by Myers and Steenrod, [MS39], any metric-space isometry is automatically a smooth function preserving the Riemannian metric also. Quite interestingly, questions to do with automatic regularity can also be posed in the much wider context provided by noncommutative geometry, and this is in part what the present work aims to do. Noncommutative geometry is a recent subject shaped by A. Connes that provides a rather elegant yet powerful way to fit classical differential geometry into the frame of operator algebras and quantum physics. One way it does so is through so-called spectral triples on C^*-algebras, [Con94]. In this paper we deal with two known spectral triples on the Cuntz algebras $\mathcal{O}_n$, which are among the most studied C^*-algebras. Now one good reason to set these problems right on the Cuntz algebras [Cun77] is their automorphisms have been investigated in great detail [CKS10, CS11, CHS12a, CHS12b, CHS15] Another is the Cuntz algebras often exhibit unexpected interplay with several different areas, and any novel emerging connection is often a source of inspiration for the papers that immediately afterwards ensue. Our attention, however, is here directed towards determining the resulting isometric isomorphisms. Problems of this sort have rarely been addressed before. A recent case in point, though, is the paper [LW17], where suitable spectral triples on twisted reduced C^*-algebras of discrete groups are looked at and the corresponding isometry groups are described completely. This is very much in line with the direction established by E. Park in [Eft95], where, to our knowledge, the notion of isometry with respect to a given spectral triple was defined for the first time. That notion should actually be interpreted as the non-commutative counterpart of the classical definition in which differentiability is assumed from the beginning. It is then natural to consider also the non-commutative counterpart of the classical definition in which differentiability is no longer required at the outset, so as to attempt a comparison between the two notions, which is made possible by using the so-called Connes distance between the states of our C^*-algebras. This is quite a legitimate question to ask, and yet it has not been formulated so far. In this paper, we attack the problem with a θ-summable spectral triple on $\mathcal{O}_n$ introduced and thoroughly studied in [GM18, GMR18]. What we actually do is give a complete description of the corresponding isometries with respect to the stronger notion. Moreover, the resulting group turns out to be compact as it consists precisely of the Bogolubov (i.e., quasi-free) automorphisms [Eva80], and it is thus isomorphic to the n-dimensional unitary group $U(n)$. Not unexpectedly, the isometries with respect to the weaker notion prove very hard to deal with, not least because explicit computations with Connes's distance are often very demanding. Even so, we do succeed in showing that the two notions are actually the same for the so-called modular spectral triple on $\mathcal{O}_n$, which semi-finite in a suitable sense [CNNR11]. The resulting isometry group, though, fails to be compact.

2 Preliminaries and notation

2.1 Some generalities on the Cuntz algebras

For an integer $n \geq 2$, $\mathcal{O}_n$ denotes the Cuntz algebra with n generating isometries, i.e. the universal C^*-algebra generated by n isometries S_i, $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^n S_i S_i^* = 1$. As is known, all endomorphisms of $\mathcal{O}_n$ can be obtained via the so-called Cuntz-Takesaki correspondence $\mathcal{U}(\mathcal{O}_n) \ni u \to \lambda_u \in \mathrm{End}(\mathcal{O}_n)$, where λ_u acts on the generating isometries as $\lambda_u(S_i) = uS_i$,

$i = 1, 2, \ldots n$. In addition, if $u, v \in \mathcal{U}(\mathcal{O}_n)$ are such that $\lambda_u = \lambda_v$, then $u = v$. Now the choice $u = z1$, $z \in \mathbb{T}$ yields an action of the compact group $\mathbb{T}$ on $\mathcal{O}_n$ through the so-called gauge automorphisms. The corresponding invariant subalgebra is known to be a UHF algebra, which is commonly denoted by $\mathcal{F}_n := \{x \in \mathcal{O}_n : \lambda_{z1}(x) = x, \text{ for every } z \in \mathbb{T}\}$. We then denote by τ the unique trace on $\mathcal{F}_n$, and by E the conditional expectation of $\mathcal{O}_n$ onto $\mathcal{F}_n$ obtained by integrating the gauge action of $\mathbb{T}$ on $\mathcal{O}_n$. The composition $\omega := \tau \circ E$ is known to be the unique KMS state of $\mathcal{O}_n$ (w.r.t. the rescaled gauge action). Henceforth, we simply denote by $(\pi, \mathcal{H}, \xi)$ the GNS triple corresponding to ω. A unitary $u \in \mathcal{F}_n$ is referred to as a gauge-invariant unitary. Now the KMS state is well known to be invariant under the action of any endomorphism λ_u coming from a gauge-invariant unitary u. In other terms, the following result holds true, see [Lon94, Lemma 2.1].

Lemma 2.1. *For any $u \in \mathcal{U}(\mathcal{F}_n)$, one has*

$$\omega \circ \lambda_u = \omega$$

In particular, any $\lambda_u \in \mathrm{Aut}(\mathcal{O}_n)$, with u a unitary in $\mathcal{F}_n$, is unitarily implemented on $\mathcal{H} = L^2(\mathcal{O}_n, \omega)$, namely there exists a unitary $U_u \in \mathcal{B}(\mathcal{H})$ such that $\pi(\lambda(x)) = U_u \pi(x) U_u^*$, for every $x \in \mathcal{O}_n$. The canonical choice is obviously given by $U_u \pi(x)\xi \doteq \pi(\lambda_u(x))\xi$, $x \in \mathcal{O}_n$. For brevity, when no confusion can arise, the underlying representation π is understood without explicit mention and thus simply dropped.

Finally, the Bogolubov automorphisms on $\mathcal{O}_n$ are those of the form λ_u, where u is a unitary in $\mathcal{F}_n^1 := \mathrm{span}\{S_i S_j^* : i, j = 1, 2, \ldots n\}$, namely a unitary of the form $u = \sum_{i,j=1}^n u_{i,j} S_i S_j^*$ with $(u_{i,j})$ being a unitary matrix in $M_n(\mathbb{C})$. With a very minor abuse of notation we identify our unitary u with the corresponding $n \times n$ matrix $(u_{i,j})$. Doing so, one immediately sees that for any two unitaries $u, v \in M_n(\mathbb{C})$, the composition $\lambda_u \circ \lambda_v$ is just given by λ_{uv}, where uv is the usual raw by column multiplication product between u and v. Phrased differently, the Bogolubov automorphisms provide a (faithful) representation of the unitary group $U(n)$ in $\mathrm{Aut}(\mathcal{O}_n)$, in which every non-trival λ_u is outer.

2.2 The spectral triple

In what follows we will often identify $\mathcal{O}_n$ with $\pi(\mathcal{O}_n)$ to ease the notation. For $k \geq 1$, W_n^k denotes the set of words (multi-indices) μ in the alphabet $\{1, \ldots, n\}$ of length k, while W_n^0 only consists of the empty word $\emptyset$. Then we set $S_\mu := S_{\mu_1} \cdots S_{\mu_k}$ if $\mu = (\mu_1 \mu_2 \cdots \mu_k)$ and $S_\emptyset = 1$. For $\mu, \nu \in W_n = \cup_{k=0}^\infty W_n^k$, we define $e_{\emptyset,\emptyset} = \xi$ and, for $\mu, \nu \neq \emptyset, \emptyset$,

$$e_{\mu,\nu} = \begin{cases} n^{|\nu|/2} S_\mu S_\nu^* \xi & t(\mu) \neq t(\nu) \\ n^{|\nu|/2} \sqrt{\frac{n}{n-1}} \left(S_\mu S_\nu^* - \frac{1}{n} S_{\underline{\mu}} S_{\underline{\nu}}^* \right) \xi & t(\mu) = t(\nu) \neq \emptyset \end{cases}$$

where, if $|\mu| \geq 1$, $t(\mu)$ denotes the last entry of the multi-index μ and $t(\emptyset) = \emptyset$, and $\mu = \underline{\mu}t(\mu)$ (if $|\mu| = 1$, $\mu = t(\mu)$). Then the family $\{e_{\mu,\nu} : \mu, \nu \in W_n\}$ is a spanning set of unit vectors of $\mathcal{H}$.[1]

Following [GMR18], we consider a convenient decomposition of $\mathcal{H}$ into a direct sum. More precisely, we set $\mathcal{H}_{h,k} := \mathrm{span}\{e_{\mu,\nu} : |\mu| = h \,\text{and}\, |\nu| = k\}$ with $h, k \in \mathbb{Z}_{\geq 0}$. The Hilbert space $\mathcal{H}$ is then seen to decompose into $\bigoplus_{h,k\geq 0} \mathcal{H}_{h,k}$, *c.f.* [GMR18], where the finite-dimensional subspaces $\mathcal{H}_{h,k}$ are parametrized in a slightly different but equivalent way. We also need to consider the subspace $\mathcal{F} := \bigoplus_{h\geq 0} \mathcal{H}_{h,0}$.

We are now in a position to recall the definition of the Dirac operator D_κ on $\mathcal{H}$ as given in the above mentioned paper. This is the self-adjoint operator obtained as the closure of the diagonal operator w.r.t. the above decomposition acting by multiplication by h on $\mathcal{H}_{h,0}$ and by $-(k+|h-k|)$ on $\mathcal{H}_{h,k}$ with $k \neq 0$, and extended by linearity to the algebraic direct sum of the subspaces $\mathcal{H}_{h,k}$. In other terms, the action of D_κ on the vectors $e_{\mu,\nu}$ is given by

$$\begin{aligned} D_\kappa e_{\mu,\emptyset} &= |\mu| e_{\mu,\emptyset} \\ D_\kappa e_{\mu,\nu} &= -(|\nu| + |\,|\mu| - |\nu|\,|) e_{\mu,\nu}, \ \text{if}\, \nu \neq \emptyset. \end{aligned}$$

Note that the kernel of D_κ is the one-dimensional subspace spanned by ξ. Now $(\mathcal{O}_n, \mathcal{H}, D_\kappa)$ can be shown to be a θ-summable spectral triple, namely:

1. D_κ is an unbounded self-adjoint operator;
2. the set of elements $a \in A$ such that $a\,\mathrm{dom}(D_\kappa) \subset \mathrm{dom}(D_\kappa)$ and $[D_\kappa, a]$ is bounded on $\mathrm{dom}(D_\kappa)$ is (norm) dense in A;
3. $(1 + D_\kappa^2)^{-1}$ is a compact operator;
4. for any positive $t > 0$, $e^{-tD_\kappa^2}$ is a trace-class operator.

Note that the third condition is implied by the forth, since $(1 + D_\kappa^2)^{-1} = \int_0^\infty e^{-t(1+D_\kappa^2)} \mathrm{d}t$. The dense set above is actually a *-subalgebra and is usually referred to as the Lipschitz subalgebra, and shortly denoted as $\{a \in \mathcal{O}_n : \|[D_\kappa, a]\| < +\infty\}$; with a very minor abuse of notation, whenever a sits in the Lipschitz subalgebra, we continue to denote by $[D_\kappa, a]$ its unique bounded extension to $\mathcal{H}$. Moreover, the K-homology class of this spectral triple coincides with the generator $\widehat{[1]} \in K^1(\mathcal{O}_n) \simeq \mathbb{Z}/(n-1)\mathbb{Z}$, see [GM18, GMR18] As is known, associated with any general spectral triple $(A, \mathcal{H}, D)$ there is a pseudo-distance on $S(A)$, the state space of A, which is usually referred to as Connes' distance. This is defined as

$$d_D(\varphi, \varphi') \doteq \sup\left\{|\varphi(x) - \varphi'(x)| : x \in A \,\text{with}\, ||\,[x, D]\,|| \leq 1\right\}, \quad \varphi, \varphi' \in \mathcal{S}(A).$$

[1]The set $\{e_{\mu,\nu}\}$, though, does not form an orthonormal basis of $\mathcal{H}$; for instance, $e_{1,1}$ and $e_{2,2}$ are not orthogonal. As a matter of fact, examples can be given of finite subsets of $\{e_{\mu,\nu}\}$ which even fail to be linearly independent, for easy computations lead to the equality $\sum_{i=1}^n e_{i,i} = 0$.

In the above definition and throughout the paper it is tacitly understood that if one writes $\|[x, D]\| \leq 1$, x is assumed to lie in the Lipschitz subalgebra. See e.g. [CI06] for an extensive study of this distance in the case of some natural spectral triples on AF algebras, and [Mar16] for a broad coverage of many other cases.

Following [Eft95], we say that an automorphism $\alpha \in \mathrm{Aut}(A)$ is isometric with respect to the given spectral triple $(A, \mathcal{H}, D)$ if there exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\alpha(x) = UxU^*$ for any $x \in A$ and $[U, D] = 0$. Accordingly, we can also define $\mathrm{Iso}(A, \mathcal{H}, D) \doteq \{\alpha \in \mathrm{Aut}(A) : \alpha = \mathrm{ad}(U)$ for some $U \in \mathcal{U}(\mathcal{H})$ with $[D, U] = 0\}$. It is then not difficult to see that any automorphism $\alpha \in \mathrm{Iso}(A, \mathcal{H}, D)$ preserves Connes' distance between any pair of states, i.e. $d_D(\varphi, \varphi') = d_D(\varphi \circ \alpha, \varphi' \circ \alpha)$, for all $\varphi, \varphi' \in \mathcal{S}(A)$. In other words, if we define

$$\mathrm{ISO}(A, \mathcal{H}, D) \doteq \{\alpha \in \mathrm{Aut}(A) : d_D(\varphi, \varphi') = d_D(\varphi \circ \alpha, \varphi' \circ \alpha)\, \forall\, \varphi, \varphi' \in \mathcal{S}(A)\}.$$

then we have the group inclusions $\mathrm{Iso}(A, \mathcal{H}, D) \subset \mathrm{ISO}(A, \mathcal{H}, D) \subset \mathrm{Aut}(A)$.

We call $\mathrm{Iso}(A, \mathcal{H}, D)$ and $\mathrm{ISO}(A, \mathcal{H}, D)$ the *isometry group* and the *large isometry group* of the spectral triple, respectively. In general, one would expect the large isometry group to contain the isometry group strictly, and any result stating the equality between the two should be construed as a regularity result for the given spectral triple. Indeed, this is exactly what happens in the commutative case, where the groups defined above allow to recover the group of all isometries on the underlying Riemannian manifold. More precisely, if (M, g) is a compact oriented Riemannian manifold and $d + d^*$ is the de Rham operator on the Hilbert space $L^2(\Lambda^*(M))$ of complex forms on M, then $(C(M), L^2(\Lambda^*(M)), d + d^*)$ is a spectral triple whose Connes' distance gives back the geodesic distance between the points of M and $\mathrm{Iso}(C(M), L^2(\Lambda^*(M)), d+d^*)$ is (up to an isomorphism) nothing but the group of Riemannian isometries of (M, g) [Eft95, Theorem 1.2]. Furthermore, combining the automatic regularity results we have recalled (see also [Hel78, Theorem 11.1]), one can easily see that in the classical case all three groups do coincide, that is

$$\mathrm{Iso}(M) \simeq \mathrm{Iso}(C(M), L^2(\Lambda^*(M)), d + d^*) = \mathrm{ISO}(C(M), L^2(\Lambda^*(M)), d + d^*)$$

3 Bogolubov automorphisms as isometries of $(\mathcal{O}_n, \mathcal{H}, D_\kappa)$

This section provides the full characterization of the group $\mathrm{Iso}(\mathcal{O}_n, \mathcal{H}, D_\kappa)$. As announced, the group consists exactly of all Bogolubov automorphisms. We start by showing that any Bogolubov automorphism is isometric with respect to the spectral triple $(\mathcal{O}_n, \mathcal{H}, D_\kappa)$, which is done below. For the sake of self-containment we provide a detailed proof, even though this is basically an instance of the "frame independence" stated in [GMR18].

Proposition 3.1. *All Bogolubov automorphisms of* $\mathcal{O}_n$ *are in* $\mathrm{Iso}(\mathcal{O}_n, \mathcal{H}, D_\kappa)$.

Proof. Let $(u_{i,j})_{i,j=1,2,\dots,n}$ be a unitary matrix in $M_n(\mathbb{C})$ and let u be the corresponding unitary in $\mathcal{O}_n$, *i.e.* $u = \sum_{i,j=1}^n u_{i,j} S_i S_j^*$. What we have to prove is that $\lambda_u \in \mathrm{Aut}(\mathcal{O}_n)$ can be implemented by a unitary $U \in \mathcal{U}(\mathcal{H})$ such that $[U, D_\kappa] = 0$. We will show this is indeed the case if U is defined as $US_\mu S_\nu^*\xi \doteq \lambda_u(S_\mu S_\nu^*)\xi$, for every $\mu, \nu \in W_n$. In order to see that U commutes with D_κ, it will be sufficient to check that U leaves invariant all the finite-dimensional subspaces $\mathcal{H}_{h,k}$, $h, k \in \mathbb{Z}_{\geq 0}$.
To this end, since $\{e_{\mu,\nu} : |\mu| = h \,\text{and}\, |\nu| = k\}$ is a spanning set for $\mathcal{H}_{h,k}$, we need to make some computation to make sure that for any $e_{\mu,\nu}$, with $\mu, \nu \in W_n$, one has that $Ue_{\mu,\nu}$ is in fact a finite linear combination of vectors $e_{\mu',\nu'}$ with $|\mu'| = |\mu|$ and $|\nu'| = |\nu|$.
Let us set $\mu = (\mu_1, \mu_2, \dots \mu_r)$ and $\nu = (\nu_1, \nu_2, \dots, \nu_s)$, where r and s can be safely assumed to be greater than zero. There are two cases to deal with according to whether μ_r and ν_s differ or not.

If they do differ, then $e_{\mu,\nu}$ is equal to $S_\mu S_\nu^*\xi$ and $Ue_{\mu,\nu}$ is accordingly $\lambda_u(S_\mu S_\nu^*)\xi$, which can be computed as follows:

$$
\begin{aligned}
&\lambda_u(S_\mu S_\nu^*)\xi = uS_{\mu_1}uS_{\mu_2}\dots uS_{\mu_{r-1}}\left(uS_{\mu_r}S_{\nu_s}^*u^*\right)\dots S_{\nu_2}^*u^*S_{\nu_1}^*u^*\xi \\
&= uS_{\mu_1}uS_{\mu_2}\dots uS_{\mu_{r-1}}\big(\sum_{i,j} u_{i,j}S_iS_j^*S_{\mu_r}\big)S_{\nu_s}^*u^*\dots S_{\nu_2}^*u^*S_{\nu_1}^*u^*\xi \\
&= uS_{\mu_1}uS_{\mu_2}\dots uS_{\mu_{r-1}}\big(\sum_i u_{i,\mu_r}S_i\big)S_{\nu_s}^*u^*\dots S_{\nu_2}^*u^*S_{\nu_1}^*u^*\xi \\
&= uS_{\mu_1}uS_{\mu_2}\dots uS_{\mu_{r-1}}\big(\sum_i u_{i,\mu_r}S_i\big)\big(\sum_{i'} \overline{u_{i',\nu_s}}S_{i'}^*\big)\dots S_{\nu_2}^*u^*S_{\nu_1}^*u^*\xi \\
&= uS_{\mu_1}uS_{\mu_2}\dots uS_{\mu_{r-1}}\Big(\sum_{i,i':i\neq i'} u_{i,\mu_r}\overline{u_{i',\nu_s}}S_iS_{i'}^* \\
&\qquad +\sum_i u_{i,\mu_r}\overline{u_{i,\nu_s}}S_iS_i^*\Big)\dots S_{\nu_2}^*u^*S_{\nu_1}^*u^*\xi
\end{aligned}
$$

Now

$$
uS_{\mu_1}uS_{\mu_2}\dots uS_{\mu_{r-1}}\Big(\sum_{i,i':i\neq i'} u_{i,\mu_r}\overline{u_{i',\nu_s}}S_iS_{i'}^*\Big)S_{\nu_{s-1}}^*u^*\dots S_{\nu_2}^*u^*S_{\nu_1}^*u^*\xi
$$

is easily recognized to be a finite linear combination of vectors $e_{\mu',\nu'}$ with $|\mu'| = |\mu|$ and $|\nu'| = |\nu|$.

The second summand

$$
uS_{\mu_1}uS_{\mu_2}\dots uS_{\mu_{r-1}}\big(\sum_i u_{i,\mu_r}\overline{u_{i,\nu_s}}S_iS_i^*\big)S_{\nu_{s-1}}^*u^*\dots S_{\nu_2}^*u^*S_{\nu_1}^*u^*\xi
$$

can be more conveniently rewritten as

$$uS_{\mu_1}uS_{\mu_2}\dots uS_{\mu_{r-1}}\left(\sum_i u_{i,\mu_r}\overline{u_{i,\nu_s}}\left(S_iS_i^*-\frac{1}{n}I\right)\right)S^*_{\nu_{s-1}}u^*\dots S^*_{\nu_2}u^*S^*_{\nu_1}u^*\xi$$

thanks to the equality $\sum_i u_{i,\mu_r}\overline{u_{i,\nu_s}}=0$, which follows from the unitarity of $(u_{i,j})$. It is now straightforward to recognize in the sums above a finite linear combination of vectors $e_{\mu',\nu'}$ with $|\mu'|=|\mu|$ and $|\nu'|=|\nu|$ with $t(\mu')=t(\nu')$.

If μ_r and ν_s coincide, then $e_{\mu,\nu}=(S_\mu S_\nu^*-\frac{1}{n}S_{\underline{u}}S_{\underline{\nu}}^*)\xi$. Therefore, $Ue_{\mu,\nu}$ is now given by $\lambda_u(S_\mu S_\nu^*-\frac{1}{n}S_{\underline{u}}S_{\underline{\nu}}^*)\xi$. The computations can be carried out in much the same way as above by taking into account the orthogonality relation $\sum_i u_{i,\mu_r}\overline{u_{i,\nu_s}}=1$, which is crucial to get rid of the extra terms coming from $\frac{1}{n}S_{\underline{u}}S_{\underline{\nu}}^*\xi$. More precisely, we now have

$$\begin{aligned}
U_ue_{\mu,\nu}&=\lambda_u\big(S_\mu S_\nu^*-\frac{1}{n}S_{\underline{\mu}}S_{\underline{\nu}}^*\big)\xi\\
&=uS_{\mu_1}uS_{\mu_2}\dots uS_{\mu_{r-1}}\Big(\sum_{i,j=1}^n u_{i,j}S_iS_j^*S_{\mu_r}S^*_{\nu_s}u^*-\frac{1}{n}\Big)S^*_{\nu_{s-1}}u^*\dots S^*_{\nu_1}u^*\xi\\
&=uS_{\mu_1}uS_{\mu_2}\dots uS_{\mu_{r-1}}\Big(\sum_i u_{i,\mu_r}S_i\sum_j\overline{u_{j,\nu_s}}S_j^*-\frac{1}{n}\Big)S^*_{\nu_{s-1}}u^*\dots S^*_{\nu_1}u^*\xi\\
&=uS_{\mu_1}uS_{\mu_2}\dots uS_{\mu_{r-1}}\left(\sum_{i\neq j}u_{i,\mu_r}\overline{u_{j,\nu_s}}S_iS_j^*+\sum_i u_{i,\mu_r}\overline{u_{i,\nu_s}}\Big(S_iS_i^*-\frac{1}{n}\Big)\right)\\
&\quad\times S^*_{\nu_{s-1}}u^*\dots S^*_{\nu_1}u^*\xi
\end{aligned}$$

and the conclusion is exactly as above.

Finally, when the length of either μ or ν (or both) is zero, the claim can be checked without difficulties. □

We next prove that, conversely, any automorphism $\alpha\in\mathrm{Iso}(\mathcal{O}_n,\mathcal{H},D_\kappa)$ must be a Bogolubov automorphism. To this end, we need a preliminary result.

Lemma 3.2. *If $\lambda_u\in\mathrm{Aut}(\mathcal{O}_n)$ belongs to $\mathrm{Iso}(\mathcal{O}_n,\mathcal{H},D_\kappa)$, then the unitary u must lie in $\mathcal{F}_n$.*

Proof. All we need to check is that under our assumption the unitary u is forced to be invariant under the action of the gauge automorphisms. For brevity, from now on λ_u will be denoted by α. Let $U\in\mathcal{U}(\mathcal{H})$ be any unitary such that $U\pi(x)U^*=\pi(\alpha(x))$, for every $x\in\mathcal{O}_n$ and $UD_\kappa=D_\kappa U$. By applying the commutation relation on the GNS vector ξ we see that $D_\kappa U\xi=0$. Since the kernel of D_κ is the one-dimensional subspace generated by ξ, $U\xi$ must be a multiple of ξ. Therefore, without any loss of generality, we may safely assume that $U\xi = \xi$, which readily implies that $U\pi(x)\xi = \pi(\alpha(x))\xi$, for every $x \in \mathcal{O}_n$. But then for any $x \in \mathcal{O}_n$ one has $\omega(x) = (\pi(x)\xi, \xi) = (U\pi(x)\xi, U\xi) = (\pi(\alpha(x)\xi, \xi) = \omega(\alpha(x))$, that is $\omega \circ \alpha = \omega$. For the next step, we consider the unique normal extensions (still denoted in the same way) of ω and α to the von Neumann algebra $\mathcal{M} := \pi(\mathcal{O}_n)''$. Denoting by σ_t^ω and $\sigma_t^{\omega\circ\alpha}$, $t \in \mathbb{R}$, the modular automorphism groups of $\mathcal{M}$ with respect to ω and $\omega \circ \alpha$ respectively, a standard application of the KMS condition shows that $\sigma_t^{\omega\circ\alpha} = \alpha^{-1} \circ \sigma_t^\omega \circ \alpha$, for any real t, see *e.g.* Formula 5 on page 40 of [Str81]. But then σ_t^ω and $\alpha^{-1} \circ \sigma_t^\omega \circ \alpha$ must coincide as automorphisms of $\pi(\mathcal{O}_n)''$. In other terms, α and σ_t^ω commute for every $t \in \mathbb{R}$. In particular, their respective restrictions to $\pi(\mathcal{O}_n)$ still commute. Because the restriction of σ_t^ω's to $\pi(\mathcal{O}_n)$ are nothing but rescaled gauge automorphisms, that is $\sigma_\omega^t = \lambda_{n^{-it}1}$, $t \in \mathbb{R}$, (see e.g. [CP96, CNNR11]), we find that our given automorphisms $\alpha = \lambda_u$ commutes with the gauge automorphisms, that is $\lambda_u \circ \lambda_{z1} = \lambda_{z1} \circ \lambda_u$, for every $z \in \mathbb{T}$. But $\lambda_u \circ \lambda_{z1} = \lambda_{zu}$ and $\lambda_{z1} \circ \lambda_u = \lambda_{z\lambda_{z1}(u)}$, hence $u = \lambda_{z1}(u)$ for every $z \in \mathbb{T}$, which is exactly what we wanted to prove. □

We are finally in a position to prove the main result of the present section.

Theorem 3.3. *The group* $\mathrm{Iso}(\mathcal{O}_n, \mathcal{H}, D_\kappa)$ *reduces to the Bogolubov automorphisms, i.e.*

$$\mathrm{Iso}(\mathcal{O}_n, \mathcal{H}, D_\kappa) = \{\lambda_u : u \in \mathcal{U}(\mathcal{F}_n^1)\}$$

Proof. Thanks to Proposition 3.1 we need only prove the inclusion $\mathrm{Iso}(\mathcal{O}_n, \mathcal{H}, D_\kappa) \subset \{\lambda_u : u \in \mathcal{U}(\mathcal{F}_n^1)\}$. If α sits in $\mathrm{Iso}(\mathcal{O}_n, \mathcal{H}, D_\kappa)$, Lemma 3.2 tells us that α is actually equal to λ_u, where u is a suitable unitary in $\mathcal{F}_n$. All is left to show, therefore, is that u lies in fact in $\mathcal{F}_n^1$. The equality $D_\kappa U = UD_\kappa$, applied on the vectors of the form $\pi_\omega(S_i)\xi$, gives $D_k\pi_\omega(\lambda_u(S_i))\xi = \pi_\omega(\lambda_u(S_i))\xi$, for every $i = 1, 2, \dots, n$. We now claim that this equality can only be fulfilled if $\pi_\omega(\lambda_u(S_i))\xi$ lies in $\mathrm{span}\{e_{\mu,\nu} : \mu, \nu \in W_n \text{ such that } |\mu| = 1, |\nu| = 0\}$. In other words, λ_u preserves the space spanned by $S_1, \dots, S_n$ in $\mathcal{O}_n$ and as such is a Bogolubov automorphism. □

Note that, except for the identity, the isometries in $\mathrm{Iso}(\mathcal{O}_n, \mathcal{H}, D_\kappa)$ are all outer automorphisms.

4 Modular isometry group

In this section the focus is on the so-called modular spectral triple, whose Dirac operator D_ω acting on $\mathcal{H}_\omega$ is simply given by the logarithm of the modular operator Δ_ω, i.e. $D_\omega \doteq \log \Delta_\omega$, cf. [CNNR11]. Before going any further, we ought to point out that this triple, strictly speaking, is not a spectral triple in the sense of Connes in that the resolvent of D_ω fails to be a compact operator on the Hilbert space $\mathcal{H}_\omega$, but rather a semi-finite spectral triple. Even so, the pseudo-distance between states as defined above continues to make perfect sense. Recall that if $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$ denotes the group of all automorphisms of $\mathcal{O}_n$ leaving $\mathcal{F}_n$ globally invariant, we have (cf. [CRS10])

$$\begin{aligned}\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) &= \{\lambda_u \in \mathrm{Aut}(\mathcal{O}_n) \mid u \in \mathcal{U}(\mathcal{F}_n)\} \\ &= \{\alpha \in \mathrm{Aut}(\mathcal{O}_n) \mid \alpha \circ \lambda_{z1} = \lambda_{z1} \circ \alpha,\ z \in \mathbb{T}\}\ .\end{aligned}$$

Proposition 4.1. *There holds the group inclusion*

$$\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) \subset \mathrm{Iso}(\mathcal{O}_n, \mathcal{H}, D_\omega).$$

Proof. By Lemma 2.1, any $\lambda_u \in \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$ preserves the KMS state ω, and thus is unitarily implemented on $\mathcal{H}$ by U_u, i.e. $\pi_\omega(\lambda_u(x)) = U_u \pi_\omega(x) U_u^*$, for every $x \in \mathcal{O}_n$. The proof is then completed by showing that U_u and the Dirac operator D_ω commute, that is $U_u \mathcal{D}(D_\omega) \subset \mathcal{D}(D_\omega)$ and $U_u D_\omega \xi = D_\omega U_u \xi$, for any $\xi \in \mathcal{D}(D_\omega)$. As is known, this amounts to proving that $U_u \Delta_\omega^{it} = \Delta_\omega^{it} U_u$ for every $t \in \mathbb{R}$, which follows by easy computations taking into account that $\lambda_u \circ \alpha_t = \alpha_t \circ \lambda_u$ and $\Delta_\omega^{it} \pi_\omega(x) \Delta_\omega^{-it} = \pi_\omega(\alpha_{-t \log n}(x))$, for any $t \in \mathbb{R}$, where $\{\alpha_t : t \in \mathbb{R}\}$ are gauge automorphisms of $\mathcal{O}_n$. □

Remark 4.2. It is worth noting that the sole possibility for an inner automorphism α of $\mathcal{O}_n$ to be in $\mathrm{Iso}(D_\omega)$ is that $\alpha = \mathrm{ad(u)}$ for some $u \in \mathcal{U}(\mathcal{F}_n)$. Indeed, if α is also implemented by a certain $V \in \mathcal{U}(\mathcal{H})$ such that $V D_\omega = D_\omega V$, then V commutes with Δ_ω^{it} as well, for any real number t. But then $\mathrm{ad(u)} \circ \alpha_{\mathrm{t}} = \alpha_{\mathrm{t}} \circ \mathrm{ad(u)}$, $t \in \mathbb{R}$, which is only possible when $u \in \mathcal{F}_n$.

Connes' distance with respect to D_ω is always infinite between two states whose restrictions to $\mathcal{F}_n$ do not coincide. This is the content of the following result, for which we first need to prove a lemma that has an interest in its own. We first set some notation. We denote by $\mathcal{O}_n^{(k)} \subset \mathcal{O}_n$ the so-called spectral eigenspaces for the action of the gauge automorphisms, that is $\mathcal{O}_n^{(k)} := \{x \in \mathcal{O}_n : \alpha_t(x) = e^{ikt}x, \text{ for any}\, t \in \mathbb{R}\}$. As can be easily seen, $\mathcal{O}_n^{(k)}$ is the norm closure of $\mathrm{span}\{S_\alpha S_\beta^* : |\alpha| - |\beta| = k\}$, which we also denote by ${}^{\mathrm{alg}}\mathcal{O}_n^{(k)}$.

Lemma 4.3. *If x is in ${}^{\mathrm{alg}}\mathcal{O}_n^{(k)}$, $k \in \mathbb{Z}$, then x sits in the domain of the derivation induced by D_ω and $[D_\omega, x] = -ik\log(n)x$. In particular, $\|[D_\omega, x]\| = |k|\log n \|x\|$.*

Proof. It is enough to recall that for such an x the equality $\Delta_\omega^{it} x \Delta_\omega^{-it} = e^{-itk\log n}x$ holds. Notably, the function $\mathbb{R} \ni t \to \Delta_\omega^{it} x \Delta_\omega^{-it} \in \mathcal{O}_n$ is differentiable and so the commutator $[D_\omega, x]$ is a bounded operator given by $\frac{d}{dt}(\Delta_\omega^{it} x \Delta_\omega^{-it})|_{t=0} = -ik\log(n)x$. □

Proposition 4.4. *If $\varphi, \varphi' \in \mathcal{S}(\mathcal{O}_n)$ are two states such that $\varphi \upharpoonright_{\mathcal{F}_n} \neq \varphi' \upharpoonright_{\mathcal{F}_n}$, then $d_{D_\omega}(\varphi, \varphi') = +\infty$.*

Proof. By hypothesis, there exists x in ${}^{\mathrm{alg}}\mathcal{O}_n^0$ such that $\varphi(x) \neq \varphi'(x)$. Thanks to Lemma 4.3 we have $\|[D_\omega, x]\| = 0$. But then we have

$$\begin{aligned} d_{D_\omega}(\varphi, \varphi') &\doteq \sup\{|\varphi(y) - \varphi'(y)| \ : \ y \in \mathcal{O}_n \text{ s.t. } \|[D_\omega, y]\| \leq 1\} \\ &\geq \sup_{\lambda \in \mathbb{R}} |\lambda|\, |\varphi(x) - \varphi'(x)| = +\infty\,, \end{aligned}$$

which ends the proof. □

Therefore, a natural question is whether there exist two states of $\mathcal{O}_n$ which are at a finite Connes' distance from each other. The following theorem fully settles the question, showing that the only possibility for two states of $\mathcal{O}_n$ to be at an infinite distance is precisely when their restrictions to $\mathcal{F}_n$ differ from each other.

Before doing this, we need to spell out the way the commutator with the Dirac operator D_ω is understood. To this aim, we consider $\mathcal{M}$, the von Neumann algebra generated by $\mathcal{O}_n$ in the GNS representation of the KMS state ω, acted upon by the associated modular group of automorphisms $\mathrm{ad}(\Delta_\omega^{it})$. Now the closed densely defined derivation, δ, on $\mathcal{M}$ associated with this one-parameter group of automorphisms (see *e.g.* [BR87]) is nothing but the commutator with the Dirac operator. As a consequence of a minor variation of [BR76, Theorem 4], the domain of δ intersected with $\mathcal{O}_n$ coincides with the *-subalgebra $\mathcal{A}$ of the Lipschitz elements. In particular, we find that the commutator $[D_\omega, x]$ lies in the von Neumann algebra $\mathcal{M}$ whenever x is a Lipschitz element of $\mathcal{O}_n$.

Note that if $x \in \mathcal{O}_n^{(k)}$, then there exists a sequence $\{x_l : l \in \mathbb{N}\} \subset {}^{\mathrm{alg}}\mathcal{O}_n^{(k)}$ such that $\|x - x_l\| \to 0$ for $l \to \infty$. But then $[D_\omega, x_l] = -ik\log(n)x_l$ converges to $-ik\log(n)x$ for $l \to \infty$. Therefore, since the derivation $\delta(\cdot) = [D_\omega, \cdot]$ is closed, the commutator $[D_\omega, x]$ exists as densely defined (on the domain of D_ω) bounded operator and is given by $[D_\omega, x] = -ik\log(n)x$.

THEOREM 4.5. *If $\varphi, \varphi' \in \mathcal{S}(\mathcal{O}_n)$ are two states whose restrictions to $\mathcal{F}_n$ coincide, then their Connes' distance is finite and moreover $d_{D_\omega}(\varphi, \varphi') \leq \frac{2\pi}{\sqrt{3}}$.*

Proof. Let E be the canonical expectation of $\mathcal{O}_n$ onto $\mathcal{F}_n$. For any $y \in \mathcal{O}_n$ we have the inequality $\|E(y^*y)\| \leq \|y\|^2$ since E is contractive. We now want to apply this inequality to $y \doteq [D_\omega, x]$, with $x \in \mathcal{O}_n$ given by a finite sum of the type $\sum_{0<|k|\leq N} x_k$, where each x_k sits in $\mathcal{O}_n^k$, i.e. $\alpha_\theta(x_k) = e^{ik\theta}x_k$. As $[D_\omega, x] = \sum_{0<|k|\leq N} kx_k$, we find $\|\sum_{0<|k|\leq N} k^2 x_k^* x_k\| = \|E(y^*y)\| \leq \|[D_\omega, x]\|^2$. In particular, for any x of the form above we see that $\|[D_\omega, x]\| \leq 1$ implies $\|\sum_{0<|k|\leq N} k^2 x_k^* x_k\| \leq 1$. Now a standard application of the Cauchy-Schwarz inequality to the free Hilbert module of any finite rank obtained by taking the direct sum of $\mathcal{O}_n$ with itself as many times as needed yields the inequality

$$\begin{aligned} \|x\| = \Big\| \sum_{0<|k|\leq N} x_k \Big\| = \Big\| \sum_{0<k\leq N} \frac{1}{k} k x_k \Big\| \leq \Big(\sum_{0<|k|\leq N} \frac{1}{k^2} \Big)^{\frac{1}{2}} \Big\| \sum_{0<|k|\leq N} k^2 x_k^* x_k \Big\|^{\frac{1}{2}} \\ \leq \frac{\pi}{\sqrt{3}} \end{aligned}$$

which holds true as long as $\|[D_\omega, x]\| \leq 1$ (a similar estimate appears in the arXiv version of [AC04]). Now we claim that for any Lipschitz $x \in \mathcal{O}_n$ such that $E(x) = 0$ and $\|[D_\omega, x]\| \leq 1$ one still has $\|x\| \leq \frac{\pi}{\sqrt{3}}$. In order to prove this, it is enough to reason as above taking into account that with little effort one can also see that:

- any $x \in \mathcal{O}_n$ with $E(x) = 0$ is the norm limit of the sequence

$$F_N(x) = \sum_{0<|k|\leq N} \Big(1 - \frac{|k|}{N}\Big) x_k$$

  by virtue of a well-known version of Fejér's theorem;

- for any Lipschitz element $x \in \mathcal{O}_n$ one has $[D_\omega, x]_k = k x_k$, $k \in \mathbb{Z}$, if $x_k \in \mathcal{O}_n^{(k)}$ is the k-th spectral component of x, and moreover

$$\|\sum_{k\in\mathbb{Z}} k^2 x_k^* x_k\|^{\frac{1}{2}} \leq \|[D_\omega, x]\|$$

We are now ready to prove the theorem. Let $\varphi, \varphi' \in \mathcal{S}(\mathcal{O}_n)$ as in the statement. Since their restrictions to $\mathcal{F}_n$ coincide, we have $|\varphi(y) - \varphi'(y)| = |\varphi(y - E(y)) - \varphi'(y - E(y))|$, which means the distance $d_{D_\omega}(\varphi, \varphi')$ can also be obtained as the supremum of the set

$$\{|\varphi(x) - \varphi'(x)| : x \in \mathcal{O}_n \text{ is Lipschitz with } E(x) = 0 \text{ and } \|[D_\omega, x]\| \leq 1\}.$$

The conclusion is now immediate. □

To complete the picture, one might also want to explicitly compute the modular distance in a number of relevant cases.

Theorem 4.5 is instrumental in proving the following full characterization of the group of the isometric automorphisms of the Cuntz algebra $\mathcal{O}_n$ with respect to the modular spectral triple.

Theorem 4.6. *The chain of equalities* $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_\omega) = \mathrm{Iso}(\mathcal{O}_n, \mathcal{H}, D_\omega) = \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$ *holds.*

Proof. Since the inclusions $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) \subset \mathrm{Iso}(\mathcal{O}_n, \mathcal{H}, D_\omega) \subset \mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_\omega)$ have already been established, we need only prove the inclusion $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_\omega) \subset \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$. To this end, let Φ be an automorphism in $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_\omega)$. All we have to show is $\Phi(\mathcal{F}_n) \subset \mathcal{F}_n$, for the same argument will then apply to Φ^{-1}. Suppose, on the contrary, that $\Phi(\mathcal{F}_n)$ is not contained in $\mathcal{F}_n$. Then there exists $x \in \mathcal{F}_n$ such that $\Phi(x)$ does not belong to $\mathcal{F}_n$. In particular, we find that $\Phi(x) - E(\Phi(x))$ is different from zero, where E is the canonical conditional expectation of $\mathcal{O}_n$ onto $\mathcal{F}_n$. Since the states of any C^*-algebra separate the elements of the C^*-algebra itself, there must exist $\varphi \in \mathcal{S}(\mathcal{O}_n)$ such that $\varphi(\Phi(x)) \neq \varphi(E(\Phi(x)))$. Let us now denote by

$\varphi' \in \mathcal{S}(\mathcal{O}_n)$ the state obtained by compounding φ with E, namely $\varphi' := \varphi \circ E$. By construction φ and φ' coincide on $\mathcal{F}_n$. By Theorem 4.5 φ and φ' are then at a finite Connes' distance from each other, i.e. $d_{D_\omega}(\varphi, \varphi') < \infty$. On the other hand, because $\varphi \circ \Phi$ and $\varphi' \circ \Phi$ fail to coincide on $\mathcal{F}_n$ by construction, the distance between $\varphi \circ \Phi$ and $\varphi' \circ \Phi$ is infinite by Proposition 4.4. This means Φ cannot preserve Connes' distance, which means Φ is not in $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_\omega)$. □

Now a few results concerning the homogeneity of the action of automorphisms on the pure states of $\mathcal{O}_n$ are known. One example worth mentioning is certainly a result due to Bratteli and Kishimoto, see [BK00], that any two inequivalent gauge-invariant pure states $\omega_1, \omega_2 \in \mathcal{S}(\mathcal{O}_n)$ can be transformed into each other, i.e. $\omega_2 = \omega_1 \circ \alpha$, through an automorphism $\alpha \in \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$. In light of the interpretation of $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$ as the isometries of $\mathcal{O}_n$ with respect to the modular spectral triple, we can restate this result saying that the modular isometries of $\mathcal{O}_n$ act transitively on gauge-invariant pure states. Moreover, this seems to indicate that there might be room for some speculation on the differential geometric features of the Cuntz algebras, as is done for instance in [Joa19], for not always a Riemannian manifold is acted upon transitively by its isometry group.
There is another consequence of the homogeneity result recalled above that we would like to discuss, which might provide a valid test for probing the presently unknown relative size of $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_\kappa)$. Indeed, should $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_\kappa)$ turn out to contain $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$, then all gauge-invariant pure states would be at the same distance from the KMS state ω.

Proposition 4.7. *Suppose that* $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_\kappa) \supset \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$. *If* $\omega = \tau \circ E$ *is the KMS state, then we have*

$$d_{D_\kappa}(\omega, \omega_1) = d_{D_\kappa}(\omega, \omega_2)$$

for all ω_1, ω_2 *gauge-invariant pure states in* $\mathcal{S}(\mathcal{O}_n)$

Proof. We first deal with the case when ω_1 and ω_2 are inequivalent. Thanks to the result of Bratteli and Kishimoto, there exists an automorphism $\alpha \in \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) = \{\beta \in \mathrm{Aut}(\mathcal{O}_n) : \beta \circ \alpha_t = \alpha_t \circ \beta,\, t \in \mathbb{R}\}$ such that $\omega_2 = \omega_1 \circ \alpha$, where $\alpha_t = \lambda_{e^{it}1}$, $t \in \mathbb{R}$, are the gauge automorphisms. But then $d_{D_\kappa}(\omega, \omega_2) = d_{D_\kappa}(\omega \circ \alpha, \omega_1 \circ \alpha) = d_{D_\kappa}(\omega, \omega_1)$, as under our hypothesis α sits in $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_\kappa)$ and ω is α-invariant.
If ω_1 and ω_2 are equivalent, we can always find a third gauge-invariant pure state ω' which is inequivalent to ω_1 and therefore to ω_2 as well. But then from the previous part of the proof we see that $d_{D_\kappa}(\omega, \omega_1) = d_{D_\kappa}(\omega, \omega') = d_{D_\kappa}(\omega, \omega_2)$, and the proof is thus complete. □

Of course, the conclusion above, in particular, holds for the modular spectral triple, where $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_\omega)$ is just $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$. However, this can easily be checked directly thanks to Proposition 4.4.

It would be interesting to prove similar results to those established in the previous sections for many other related C^*-algebras as well, e.g. the 2-adic ring C^*-algebra $\mathcal{Q}_2$ and its relatives, whose endomorphisms and automorphisms of late have been given much attention (see e.g. [ACR18a, ACR18b, ACR20, ACRS20])).

5 On the Connes distance between particular states

As an outlook for the foreseeable future, we gather in this section some results to do with the Connes distance induced by our spectral triple $(\mathcal{O}_n, \mathcal{H}, D_\kappa)$, which we believe are likely to play a role in describing $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_\kappa)$ completely. First, it is worth pointing out that quite a large set of vector states exist whose Connes' distance from the KMS state is finite. This will result as an application of the following.

Lemma 5.1. *Let $\omega \in \mathcal{S}(\mathcal{O}_n)$ be the KMS state considered above. If $\varphi \in \mathcal{S}(\mathcal{O}_n)$ is dominated by ω, that is $\varphi(x^*x) \leq M\omega(x^*x)$ for some $M \geq 1$ for any $x \in \mathcal{O}_n$, then*

$$d_{D_\kappa}(\omega, \varphi) \leq M^{\frac{1}{2}}$$

Proof. By definition, the distance $d_{D_\kappa}(\omega, \varphi)$ is obtained as the sup of the numerical set $\{|\omega(x) - \varphi(x)| : x \in \mathcal{O}_n, \text{with } \|[x, D_\kappa]\| \leq 1\}$. Since the two commutators $[D_\kappa, x]$ and $[D_\kappa, x - \omega(x)1]$ coincide, there is no loss of generality if we further assume that $\omega(x) = 0$. Therefore, the distance $d_{D_\kappa}(\omega, \varphi)$ is also given by the sup of the set

$$\{|\varphi(x)| : x \in \mathcal{O}_n, \omega(x) = 0 \,\text{and}\, \|[D_\kappa, x]\| \leq 1\}$$

Now we have the chain of inequalities $\|[D_\kappa, x]\| \geq \|[D_\kappa, x]\xi\| = \|D_\kappa x\xi\| \geq \|x\xi\|$, where the last one is due to the assumption that ω vanishes on x, which says $x\xi$ is a direct sum of eigenvectors of D_κ associated with eigenvalues whose absolute value is greater than or equal to 1. But then, for any such x, we must have $|\varphi(x)| \leq \varphi(x^*x)^{\frac{1}{2}} \leq M^{\frac{1}{2}}\omega(x^*x)^{\frac{1}{2}} = M^{\frac{1}{2}}\|x\xi\| \leq M^{\frac{1}{2}}$. ☐

Of course, Lemma 5.1 actually implies that $d_{D_\kappa}(\varphi, \varphi') < \infty$ for all pairs of states φ, φ' of $\mathcal{O}_n$ dominated by ω. Now the above lemma applies in particular to states of the form $\varphi_{x'}(x) := (x'x\xi, \xi)$, with $x' \in \pi(\mathcal{O}_n)'$ with $x' \geq 0$ and $(x'\xi, \xi) = 1$, which is a fairly rich set of states in that the subspace $\{x'\xi : x' \in \pi(\mathcal{O}_n)'\}$ is dense in $\mathcal{H}$. However, proving that Connes's distance between any two vector spaces associated with π is still finite appears to be no easy task. The information obtained above, though, is already enough to provide a simple, if indirect, proof that no non-scalar operator in $\mathcal{O}_n$ can commute with the Dirac operator D_κ.

Proposition 5.2. *If $x \in \mathcal{O}_n$ commutes with the Dirac operator D_κ, i.e. $[x, D_\kappa] = 0$, then $x = \lambda 1$ for some $\lambda \in \mathbb{C}$.*

Proof. Let $x \in \mathcal{O}_n$ be such that $[x, D_\kappa] = 0$. For any pair of states φ, φ' at a finite Connes' distance $d_{D_\kappa}(\varphi, \varphi') < \infty$, we must have $\varphi(x) = \varphi'(x)$. For, otherwise the supremum of the set

$$\{|\varphi(y) - \varphi'(y)| : y \in \mathcal{O}_n \text{ with } \|[y, D_\kappa]\| \leq 1\}$$

would be clearly infinite. Since all vector states $\omega_{x'\xi}$, with $x' \in \pi_\omega(\mathcal{O}_n)'$, are at a finite distance from each other, we then find $(x\eta_1, \eta_1) = (x\eta_2, \eta_2)$ for any normalized $\eta_1, \eta_2 \in \mathcal{H}_\omega$ due to cyclity of ξ, which implies $x = \lambda 1$, with $\lambda = (x\xi, \xi)$. This is seen as follows. We decompose x into the sum of its real and imaginary parts, namely $x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i} = y_1 + iy_2$, and observe that $(y_i\eta_1, \eta_1) = (y_i\eta_2, \eta_2)$ for any normalized $\eta_1, \eta_2 \in \mathcal{H}_\omega$, $i = 1, 2$. But if $y = y^*$ then the equality $(y\eta_1, \eta_1) = (y\eta_2, \eta_2)$ becomes $((y - \lambda 1)\eta, \eta) = 0$ for any $\eta \in \mathcal{H}_\omega$, where $\lambda := (y\xi, \xi) \in \mathbb{R}$. But because $y - \lambda 1$ is self-adjoint, we have $y - \lambda 1 = 0$. □

Connes' distance also proves to be finite between vector states associated e.g. with the eigenvectors $e_{\mu,\nu}$, with $\mu, \nu \in W_n$, of the Dirac operator D_κ. If $x_{\mu,\nu}$ is the unique element in $\mathcal{O}_n$ such that $e_{\mu,\nu} = x_{\mu,\nu}\xi$, we have the following estimate, where $\omega_{\mu,\nu}$ is the vector state associated with $w_{\mu,\nu}$, *i.e.* $\omega_{\mu,\nu}(x) := (xe_{\mu,\nu}, e_{\mu,\nu})$, $x \in \mathcal{O}_n$.

PROPOSITION 5.3. *For any μ, ν and μ', ν' in W_n, Connes' distance between $\omega_{\mu,\nu}$ and $\omega_{\mu',\nu'}$ is finite. More precisely, the following upper bound holds:*

$$d_{D_\kappa}(\omega_{\mu,\nu}, \omega_{\mu',\nu'}) \leq \|\big(n^{|\mu|-|\nu|}x_{\mu,\nu}x^*_{\mu,\nu} - n^{|\mu'|-|\nu'|}x_{\mu',\nu'}x^*_{\mu',\nu'}\big)\xi\|$$

Proof. After making some computations, the difference $\omega_{e_{\mu,\nu}}(x) - \omega_{e_{\mu',\nu'}}(x)$, $x \in \mathcal{O}_n$, is easily seen to coincide with the scalar product $(x^*\xi, (x_{\mu,\nu}\sigma_{-i}(x^*_{\mu,\nu}) - x_{\mu',\nu'}\sigma_{-i}(x^*_{\mu',\nu'}))\xi)$ where we have used the KMS condition and the equality $\sigma_t = \lambda_{n^{-it}1}$, $t \in \mathbb{R}$. Therefore, the absolute value of the difference above can be estimated in the following way

$$|\omega_{e_{\mu,\nu}}(x) - \omega_{e_{\mu',\nu'}}(x)| \leq \|x^*\xi\|\|(x_{\mu,\nu}\sigma_{-i}(x^*_{\mu,\nu}) - x_{\mu',\nu'}\sigma_{-i}(x^*_{\mu',\nu'}))\xi\|$$

Since there is no loss of generality if we also assume $\omega(x) = 0$ (i.e. $x\xi \perp \xi$), we have $\|x^*\xi\| \leq \|[D, x^*]\xi\|$, so the above inequality can also be rewritten as

$$|\omega_{e_{\mu,\nu}}(x) - \omega_{e_{\mu',\nu'}}(x)| \leq \|[D, x^*]\xi\|\|(x_{\mu,\nu}\sigma_{-i}(x^*_{\mu,\nu}) - x_{\mu',\nu'}\sigma_{-i}(x^*_{\mu',\nu'}))\xi\|$$

which gives the sought inequality since $\sigma_{-i}(S_\alpha S^*_\beta)$ is easily seen to be $n^{|\alpha|-|\beta|}S_\alpha S^*_\beta$. □

Finally, we would like to end the present section by discussing a general property enjoyed by Connes's distance along with some consequences that might be relevant to our case: Connes' distance is a convex function in one of the two variables. More precisely, we have the following.

Lemma 5.4. *For any $\omega, \omega_1, \omega_2 \in \mathcal{S}(\mathcal{O}_n)$ one has*

$$d_{D_\kappa}(\omega, \alpha\omega_1 + \beta\omega_2) \leq \alpha d_{D_\kappa}(\omega, \omega_1) + \beta d_{D_\kappa}(\omega, \omega_2)$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Proof. A straightforward application of the definition:

$$d_{D_\kappa}(\omega, \alpha\omega_1+\beta\omega_2) = \sup\{|\omega(x)-\alpha\omega_1(x)-\beta\omega_2(x)| : x \in \mathcal{O}_n \text{ with } \|[x, D_\kappa]\| \leq 1\}$$

But $|\omega(x) - \alpha\omega_1(x) - \beta\omega_2(x)| \leq \alpha|\omega(x) - \omega_1(x)| + \beta|\omega(x) - \omega_2(x)|$, whence the conclusion as the supremum of a sum is less than or equal to the sum of the corresponding suprema. □

In particular, if we apply the above inequality to ω and $\alpha\omega + \beta\omega'$, we see that $d_{D_\kappa}(\omega, \alpha\omega + \beta\omega') \leq \beta d_{D_\kappa}(\omega, \omega')$. Notably, if ω is a state for which there exists another state ω' such that $0 < d_{D_\kappa}(\omega, \omega') < \infty$, then the sequence of states $\omega_n := \frac{n-1}{n}\omega + \frac{1}{n}\omega'$ converges to ω with respect to d_{D_κ}, i.e. $0 < d_{D_\kappa}(\omega, \omega_n) \leq \frac{1}{n}$. This also says that the distance must al least take countably many distinct values.

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Roberto Conti
Dipartimento di Scienze di Base
e Applicate per l'Ingegneria
Sapienza Università di Roma
Via A. Scarpa 16
00161 Roma
Italy
roberto.conti@sbai.uniroma1.it

Stefano Rossi
Dipartimento di Matematica
Università degli Studi di Bari
Aldo Moro
Via Edoardo Orabona 4
70125 Bari
Italy
stefano.rossi@uniba.it