Representation Stability
and Outer Automorphism Groups

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Abstract. In this paper we study families of representations of the outer automorphism groups indexed on a collection of finite groups $\mathcal{U}$. We encode this large amount of data into a convenient abelian category which generalizes the category of VI-modules appearing in the representation theory of the finite general linear groups. Inspired by work of Church–Ellenberg–Farb, we investigate for which choices of $\mathcal{U}$ the abelian category is locally noetherian and deduce analogues of central stability and representation stability results in this setting. Finally, we show that some invariants coming from rational global homotopy theory exhibit representation stability.

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1 Introduction

In this paper we develop a framework for studying families of representations of the outer automorphism groups. A common theme in representation theory is that there is a conceptual advantage in encoding this large amount of (possibly complicated) data into a single object, which lives in a convenient abelian category. Using purely algebraic techniques we will deduce strong constraints on naturally occurring families of representations of the outer automorphism groups. We will then provide a range of examples for our theory coming from rational global homotopy theory.

THE MAIN CHARACTER

Fix k a field of characteristic zero and let $\mathcal{G}$ denote the category of finite groups and conjugacy classes of surjective group homomorphisms. We are interested in the category $\mathcal{A} = [\mathcal{G}^{\mathrm{op}}, \mathrm{Vect}_k]$ of contravariant functors from $\mathcal{G}$ to the category of k-vector spaces. More generally, we will restrict our attention to a replete full subcategory $\mathcal{U} \leq \mathcal{G}$ and then consider the smaller category $\mathcal{AU} = [\mathcal{U}^{\mathrm{op}}, \mathrm{Vect}_k]$. Note that the endomorphism group of an object $G \in \mathcal{U}$ is the outer automorphism group $\mathcal{U}(G, G) = \mathrm{Out}(G)$. Therefore any object $X \in \mathcal{AU}$ gives rise to a collection of $\mathrm{Out}(G)$-representations $X(G)$ for $G \in \mathcal{U}$. The functoriality of X imposes further compatibility conditions on these representations. There are two main examples where all this data can be made very explicit.

EXAMPLE. Consider the category $\mathcal{C}[2^\infty]$ of cyclic 2-groups. An object $X \in \mathcal{AC}[2^\infty]$ gives rise to a consistent sequence of representations of cyclic 2-groups:

where the horizontal maps are induced by the canonical projections.

EXAMPLE. Fix a prime number p and consider the category $\mathcal{E}[p]$ of elementary abelian p-groups. An object $X \in \mathcal{AE}[p]$ gives rise to a consistent sequence of representations of the finite general linear groups:

where the horizontal maps are induced by the projection into the first coordinates.

As we have already seen in the previous examples, it will often be convenient to restrict attention to special subcategories $\mathcal{U}$ (always full and replete) for which certain phenomena stand out more clearly. For example:

- We might fix a prime p and restrict attention to p-groups.

- We might restrict attention to solvable, nilpotent or abelian groups.

- We might impose upper or lower bounds on the exponent, nilpotence class, order, or on the size of a minimal generating set.

- As special cases of the above, we might consider only cyclic groups, or only elementary abelian p-groups for some fixed prime p.

To ensure good homological properties, we will impose additional conditions on $\mathcal{U}$ such as:

- Closure under products: If $G, H \in \mathcal{U}$, then $G \times H \in \mathcal{U}$. If this holds, we say that $\mathcal{U}$ is *multiplicative*.

- Closure under passage to subgroups: If $G \in \mathcal{U}$ and $H \leq G$, then $H \in \mathcal{U}$.

- Downwards closure (i.e. closure under passage to quotients): If $G \in \mathcal{U}$ and $\mathcal{G}(G, H) \neq \emptyset$, then $H \in \mathcal{U}$.

- Upwards closure: If $H \in \mathcal{U}$ and $\mathcal{G}(G, H) \neq \emptyset$, then $G \in \mathcal{U}$.

We will see throughout this introduction that $\mathcal{A}\mathcal{U}$ has its best homological behaviour when $\mathcal{U}$ is *submultiplicative* (multiplicative and closed under passage to subgroups), or a *global family* (closed downwards and closed under passage to subgroups). We refer the reader to Section 3 for a detailed list of all the closure properties considered in this paper together with some examples.
Before presenting our results we put the abelian category $\mathcal{A}\mathcal{U}$ in the relevant context.

REPRESENTATIONS OF COMBINATORIAL CATEGORIES

The abelian category $\mathcal{A}\mathcal{U}$ is part of a larger family of categories appearing in representation theory and algebraic topology. Given a category $\mathcal{I}$ whose objects are finite sets (with possibly extra structure) and whose morphisms are functions (possibly respecting the extra structure), we can consider the associated diagram category $\mathcal{A}_{\mathcal{I}} = [\mathcal{I}, \text{Vect}_k]$. Some examples of interest include:

- Let FI be the category of finite sets and injections. The associated diagram category is the category of FI-modules which appears in [21] in the context of stable homotopy groups of symmetric spectra, and in [2,3] in relation to the representation theory of the symmetric groups.

- Let VI be the category of finite dimensional $\mathbb{F}_p$-vector spaces and injective linear maps. The associated diagram category is the category of VI-modules which appears in [6,14] in relation to the representation theory of the finite general linear groups. This category is equivalent by Pontryagin duality to the category $\mathcal{A}\mathcal{E}[p]$ mentioned earlier.

- Let VA be the category of finite dimensional $\mathbb{F}_p$-vector spaces and all linear maps. The associated diagram categories have been studied in relation to algebraic K-theory, rational cohomology, and the Steenrod algebra [12].

Despite the similarities with other abelian categories appearing in representation theory, there is a major difference between $\mathcal{A}\mathcal{U}$ and all these categories. We are no longer considering a one-parameter family of representations but rather

collections of representations which are indexed by a family of groups. This brings into play group-theoretic properties of the family $\mathcal{U}$ and so introduces a new level of complexity into the story which has so far not been explored.

Noetherian condition

The category $\mathcal{A}\mathcal{U}$ is a Grothendieck abelian category with generators given by the representable functors

$$e_G = k[\mathcal{U}(-, G)] \qquad G \in \mathcal{U}.$$

Many of the familiar notions from the theory of modules carry over to this setting. For example, there are notions of finitely generated and finitely presented objects, see Definition 11.1 for the details. We then say that the abelian category $\mathcal{A}\mathcal{U}$ is *locally noetherian* if all subobjects of e_G are finitely generated for all $G \in \mathcal{U}$. It is then a formal consequence of the definition that subobjects of finitely generated objects are again finitely generated, and that any finitely generated object is also finitely presented.

Work of Church–Ellenberg–Farb in the category of FI-modules showed that the noetherian condition plays a fundamental role when working with sequences of representations [2]. This key technical innovation allowed them to prove an asymptotic structure theorem for finitely generated FI-modules which gave an elegant explanation for the representation-theoretic patterns observed in earlier work [4]. Motivated by this, we investigate for which choices of $\mathcal{U}$ the category $\mathcal{A}\mathcal{U}$ is locally noetherian. The next result combines Proposition 13.3 and Theorem 13.15 in the body of the paper.

THEOREM A. *Let $\mathcal{U}$ be a subcategory of $\mathcal{G}$ and let p be a prime number.*

(a) *If $\mathcal{U}$ is a multiplicative global family of finite abelian p-groups, then $\mathcal{A}\mathcal{U}$ is locally noetherian. Such $\mathcal{U}$ have the form $\mathcal{Z}[p^n]$ for some $0 \leq n \leq \infty$, see Definition 3.3.*

(b) *If $\mathcal{U}$ is the global family of cyclic p-groups, then $\mathcal{A}\mathcal{U}$ is locally noetherian.*

If $\mathcal{U}$ contains the trivial group and infinitely many cyclic groups of prime order, then $\mathcal{A}\mathcal{U}$ is not locally noetherian. In particular, $\mathcal{A}$ is not locally noetherian.

There are several combinatorial criteria available in the literature to show that the category $\mathcal{A}\mathcal{U}$ is locally noetherian. We prove part (a) using the theory of Gröbner bases developed by Sam–Snowden [19], and part (b) using the criterion developed in [6]. Our result does not aim to give a complete classification of locally noetherian categories, as this would be costly and highly non-trivial, but rather aims to give a good range of examples and counterexamples to which our theory applies.

We then turn to study homological properties of our category of interest.

HOMOLOGICAL PROPERTIES

The levelwise tensor product of k-vector spaces gives $\mathcal{AU}$ a symmetric monoidal structure in which the unit object $\mathbb{1}$ is the constant functor with value k. For all $X, Y \in \mathcal{AU}$, there exists an internal hom object that we denote by $\underline{\mathrm{Hom}}(X, Y) \in \mathcal{AU}$.

We list a few interesting homological properties that our category enjoys.

(i) As is typical for diagram categories, the finitely generated projective objects are not strongly dualizable. In particular this means that the canonical map
$$e_G \otimes \underline{\mathrm{Hom}}(e_G, \mathbb{1}) \to \underline{\mathrm{Hom}}(e_G, e_G)$$
is in general not an isomorphism, see Remark 4.3. However, the finitely generated projective objects of $\mathcal{AU}$ still form a subcategory that is closed under tensor products and internal hom, see Proposition 4.11 and Theorem 4.18.

(ii) As is typical for diagram categories, any projective object is a retract of a direct sum of generators, see Lemma 8.2. However, under mild conditions on $\mathcal{U}$ (satisfied by $\mathcal{G}$) the projective objects of $\mathcal{AU}$ coincide with the torsion-free injective objects, see Proposition 15.1. In particular, the generators e_G are injectives.

(iii) Under mild conditions on $\mathcal{U}$ (which are satisfied by $\mathcal{G}$ itself), the only objects with a finite projective resolution are the projective ones, see Proposition 11.6.

(iv) The abelian category $\mathcal{AU}$ is semisimple if and only if $\mathcal{U}$ is a groupoid, see Proposition 6.3.

REPRESENTATION STABILITY

A common goal in the representation theory of categories is to give a uniform description of the representations encoded into an object $X \in \mathcal{AU}$. For example, one may prove that a finitely presented object can be recovered by finite amount of data via a "stabilization recipe". This phenomenon is called *central stability* and it was first introduced by Putman [17] for describing certain stability phenomena of the general linear groups. Since then, central stability has been shown to hold for various diagram categories such as FI-modules [3] and complemented categories [18]. In our framework this phenomenon can be formulated in the following way.

DEFINITION A. Let $\mathcal{U}$ be a subcategory of $\mathcal{G}$. We say that an object $X \in \mathcal{AU}$ satisfies *central stability* if there exists a natural number $n \in \mathbb{N}$ such that for all $G \in \mathcal{U}$, we have
$$X(G) = \varinjlim_{H \in N(G,n)} X(G/H)$$

where $N(G, n) = \{H \triangleleft G \mid |G/H| \le n\}$.

In Section 14 we give a slight generalization of the machinery described in [7] and show that any finitely presented objects satisfies central stability. This result illustrates the fact that the representations encoded in a finitely presented object need to satisfy strong compatibility conditions. It tells us that we can recover the value $X(G)$ from a finite amount of data, namely the poset $N(G, n)$ and the representations $X(G/H)$. We note that the poset $N(G, n)$ is always finite and often can be determined by purely combinatorial means. For instance, in the abelian p-group case its cardinality can be explicitly calculated using the Hall polynomials [1, 2.1.1].

Given an epimorphism $\alpha \colon B \to A$, we also investigate the behaviour of the structure maps $\alpha^* \colon X(A) \to X(B)$ for sufficiently large groups A and B. In this case however, we need to restrict to the locally noetherian case. Consider the following families of finite abelian p-groups:

$$\mathcal{F}[p^n] = \{\text{free } \mathbb{Z}/p^n\text{-modules}\} \quad \text{and} \quad \mathcal{C}[p^\infty] = \{\text{cyclic } p\text{-groups}\}.$$

The following is an adaptation in our setting of the injectivity and surjectivity conditions in the definition of representation stability due to Church–Farb [4, 1.1].

DEFINITION B. Let $\mathcal{U}$ be either $\mathcal{C}[p^\infty]$ or $\mathcal{F}[p^n]$ for some $n \ge 1$. Consider an object $X \in \mathcal{A}\mathcal{U}$.

- We say that X is *eventually torsion-free* if there exists $r_0 \in \mathbb{N}$ such that for every morphism $\alpha \colon B \to A$ with $|A| \ge r_0$, the induced map $\alpha^* \colon X(A) \to X(B)$ is injective.

- We say that X is *generated in finite degree* if there exists $r_0 \in \mathbb{N}$ such that the canonical map

$$X(A) \otimes k[\mathcal{U}(B, A)] \to X(B), \quad (x, \alpha) \mapsto \alpha^*(x)$$

  is surjective, for all $|B| \ge |A| \ge r_0$.

We are finally ready to state our second result, see Theorem 14.6 and Proposition 14.3 in the body of the paper.

THEOREM B. *Fix a prime number p. Let $\mathcal{Z}[p^\infty]$ be the family of finite abelian p-groups and consider a finitely generated object $X \in \mathcal{A}\mathcal{Z}[p^\infty]$. Then the restriction of X to $\mathcal{A}\mathcal{C}[p^\infty]$ and $\mathcal{A}\mathcal{F}[p^n]$, for $n \ge 1$, is generated in finite degree and eventually torsion-free. Moreover, X satisfies central stability.*

Since the family of cyclic p-groups is closed downwards, one can easily verify that the restriction of X to $\mathcal{A}\mathcal{C}[p^\infty]$ is again finitely generated. Therefore the first part of the previous result follows by combining our Theorem A with [6, Section 5]. On the other hand, it is not immediately clear that the restriction of X to $\mathcal{A}\mathcal{F}[n]$ is again finitely generated so an additional argument is required in this case.

GLOBAL HOMOTOPY THEORY

A good source of examples of finitely generated objects satisfying representation stability comes from global stable homotopy theory: the study of spectra with a uniform and compatible group action for all groups in a specific class. These are particular kinds of spectra that give rise to cohomology theories on G-spaces for all groups in the chosen class. The fact that all these individual cohomology theories come from a single object imposes extra compatibility conditions as the group varies. In this paper we will use the framework of global homotopy theory developed by Schwede [22]. His approach has the advantage of being very concrete as the category of global spectra is the usual category of orthogonal spectra but with a finer notion of equivalence, called global equivalence. As any orthogonal spectrum is a global spectrum, this approach comes with a good range of examples. For instance, there are global analogues of the sphere spectrum, cobordism spectra, K-theory spectra, Borel cohomology spectra and many others. It is a special feature of such a global spectrum X that the assignment $G \mapsto \pi_0(\Phi^G X) \otimes \mathbb{Q}$ defines an object $\underline{\Phi}_0(X) \in \mathcal{A}$, where we put $k = \mathbb{Q}$. The connection with global homotopy theory is even stronger as there is a triangulated equivalence

$$\Phi^{\mathcal{G}} \colon \operatorname{Sp}^{\mathbb{Q}}_{\mathcal{G}} \simeq_{\triangle} \mathcal{D}(\mathcal{A}) \tag{1.0.1}$$

between the homotopy category of rational $\mathcal{G}$-global spectra and the derived category of $\mathcal{A}$ [22, 4.5.29]. This equivalence is compatible with geometric fixed points in the sense that $\pi_*(\Phi^G X) = H_*(\Phi^{\mathcal{G}}(X))(G)$.

We obtain the following application to global homotopy theory which highlights the good behaviour of the geometric fixed points functor on the full subcategory of compact global spectra. Recall that an object X in a triangulated category $\mathcal{T}$ is said to be compact if the representable functor $\mathcal{T}(X, -)$ preserves arbitrary sums. The proof of the following result can be found in Section 14.

THEOREM C. *Let $\mathcal{Z}[p^\infty]$ be the family of finite abelian p-groups and let X be a rational $\mathcal{Z}[p^\infty]$-global spectrum. If X is compact, then for all $k \in \mathbb{Z}$ the geometric fixed points homotopy groups $\underline{\Phi}_k(X) \in \mathcal{AZ}[p^\infty]$ satisfy central stability and their restriction to $\mathcal{AC}[p^\infty]$ and $\mathcal{AF}[p^n]$, for $n \geq 1$, are generated in finite degree and eventually torsion-free.*

An interesting source of examples is given by the rational n-th symmetric product spectra.

EXAMPLE. For $n \geq 1$, we let Sp^n denote the orthogonal spectrum whose value at inner product space V is given by

$$\operatorname{Sp}^n(V) = (S^V)^{\times n}/\Sigma_n.$$

Its rationalization is a compact rational $\mathcal{Z}[p^\infty]$-global spectrum by [9, 2.10, 5.1]. Therefore its geometric fixed points satisfy central stability and their restriction are eventually torsion-free and generated in finite degree.

Related work

Our study of the representation theory and homological algebra of $\mathcal{AU}$
is inspired by earlier work in the categories of FI-modules [2, 3] and VI-
modules [8, 14]. Our Theorem A recovers the result that the category of
VI-modules is locally noetherian, which was proved independently by Sam–
Snowden [19, 8.3.3] and Gan–Li [6]. Versions of our representation stability
theorems were already known to hold for the category of FI-modules [3], VI-
modules [8] and complemented categories [18]. Finally our study of indecompos-
able injective objects recovers part of the classification of injective VI-modules
due to Nagpal [14].
Nonetheless, to the best of our knowledge the results of this introduction are
new and they generalize several known results to a wider class of examples of
interest.

2   Preliminaries

We start by introducing the main object of study of this paper, the abelian
category $\mathcal{A}$.

Definition 2.1.

- Let H be a group, and h an element of H. We write $c_h \colon H \to H$ for the
  inner automorphism $x \mapsto hxh^{-1}$.

- Let G be another group, and let $\varphi, \psi \colon G \to H$ be homomorphisms. We
  say that φ and ψ are *conjugate* if $\psi = c_h \circ \varphi$ for some $h \in H$. This is easily
  seen to be an equivalence relation that is compatible with composition.
  We write $[\varphi]$ for the conjugacy class of φ.

- We write $\mathcal{G}$ for the category whose objects are finite groups, and whose
  morphisms are conjugacy classes of surjective homomorphisms. We also
  write $\mathrm{Out}(G) = \mathcal{G}(G, G)$.

Lemma 2.2. *Let $\alpha \colon H \to G$ be a surjective group homomorphism between finite
groups. Then $[\alpha]$ is an epimorphism in $\mathcal{G}$.*

Proof. Consider two surjective group homomorphisms $\beta, \gamma \colon G \to K$, and sup-
pose that $[\beta\alpha] = [\gamma\alpha]$. This means that $c_k\beta\alpha = \gamma\alpha$ for some $k \in K$. Since α is
surjective we have $c_k\beta = \gamma$ which shows that $[\beta] = [\gamma]$.   □

DEFINITION 2.3. Fix a field k of characteristic zero, and set $\mathcal{A} = [\mathcal{G}^{\mathrm{op}}, \mathrm{Vect}_k]$. Given $G \in \mathcal{G}$, we write $\psi^G \colon \mathcal{A} \to \mathrm{Vect}_k$ for the evaluation functor $X \mapsto X(G)$.

DEFINITION 2.4. Let $\mathcal{U}$ be a subcategory of $\mathcal{G}$. Unless we explicitly say otherwise, such subcategories are assumed to be full and replete. (Replete means that any object of $\mathcal{G}$ isomorphic to an object of $\mathcal{U}$ is itself in $\mathcal{U}$.) We then put $\mathcal{A}\mathcal{U} = [\mathcal{U}^{\mathrm{op}}, \mathrm{Vect}_k]$.

REMARK 2.5. The category $\mathcal{A}\mathcal{U}$ is abelian and admits limits and colimits for all small diagrams. These (co)limits are computed pointwise, so they are preserved by the evaluation functors $\psi^G \colon \mathcal{A}\mathcal{U} \to \mathrm{Vect}_k$.

DEFINITION 2.6. Consider an object $X \in \mathcal{A}\mathcal{U}$.

- The *base* of X is defined by $\mathrm{base}(X) = \min\{|G| \mid X(G) \neq 0\} \in \mathbb{N}$. If X is zero, we set $\mathrm{base}(X) = \infty$.

- The *support* of X is defined by $\mathrm{supp}(X) = \{[G] \mid X(G) \neq 0\}$ where $[G]$ denotes the isomorphism class of the group G. We equipped the support with the partial order $[G] \gg [H]$ if and only if $\mathcal{U}(G, H) \neq \emptyset$.

DEFINITION 2.7. Consider a subcategory $\mathcal{U} \leq \mathcal{G}$. We define certain objects of $\mathcal{A}\mathcal{U}$ as follows. Most of them depend on an object $G \in \mathcal{U}$, and possibly also a module V over $k[\mathrm{Out}(G)]$.

- We define e_G by $e_G(T) = k[\mathcal{G}(T, G)]$. Yoneda's Lemma tells us that $\mathcal{A}\mathcal{U}(e_G, X) = X(G) = \psi^G(X)$.

- We define objects $e_{G,V}$ and $t_{G,V}$ by

$$e_{G,V}(T) = V \otimes_{k[\mathrm{Out}(G)]} k[\mathcal{G}(T, G)]$$

and

$$t_{G,V}(T) = \mathrm{Hom}_{k[\mathrm{Out}(G)]}(k[\mathcal{G}(G, T)], V).$$

- We put
$$c_G(T) = e_G(T)^{\mathrm{Out}(G)} = k[\mathcal{G}(T, G)/\mathrm{Out}(G)].$$

Note that the basis set $\mathcal{G}(T, G)/\mathrm{Out}(G)$ here can be identified with the set of normal subgroups $N \leq T$ such that $T/N \simeq G$. Alternatively, we can regard k as a k-linear representation of $\mathrm{Out}(G)$ with trivial action, and then $c_G = e_{G,k}$.

- The groups $e_{G,V}(G)$ and $t_{G,V}(G)$ are both canonically identified with V, and one can check that there is a unique morphism $\alpha \colon e_{G,V} \to t_{G,V}$ with $\alpha_G = 1$. We write $s_{G,V}$ for the image of this. If $T = G$ then $s_{G,V}(T) = V$. If $T \simeq G$ then $s_{G,V}(T)$ is canonically isomorphic to $e_{G,V}(T)$ or $t_{G,V}(T)$, but a choice of isomorphism $T \to G$ is needed to identify $s_{G,V}(T)$ with V. If $T \not\simeq G$ then $s_{G,V}(T) = 0$.

- Now let $\mathcal{C}$ be a subcategory of $\mathcal{U}$. Suppose that $\mathcal{C}$ is *convex*, which means that whenever $G \to H \to K$ are surjective homomorphisms with $G, K \in \mathcal{C}$ and $H \in \mathcal{U}$ we also have $H \in \mathcal{C}$. We then define the "characteristic function" $\chi_{\mathcal{C}} \in \mathcal{AU}$ by

$$\chi_{\mathcal{C}}(T) = \begin{cases} k & \text{if } T \in \mathcal{C} \\ 0 & \text{if } T \notin \mathcal{C}. \end{cases}$$

  (Convexity ensures that this can be made into a functor in an obvious way: the map $\chi_{\mathcal{C}}(T) \to \chi_C(T')$ is the identity if both groups are nonzero, and zero otherwise.)

If we need to specify the ambient category $\mathcal{U}$, we may write $e_G^{\mathcal{U}}$ rather than e_G, and so on.

REMARK 2.8. The abelian category $\mathcal{AU}$ is Grothendieck with generators given by e_G for all $G \in \mathcal{U}$. This means that filtered colimits are exact and that any $X \in \mathcal{A}$ admits an epimorphism $P \to X$ where P is a direct sum of generators.

LEMMA 2.9. *For $G \in \mathcal{U}$, we let $\mathcal{M}_G$ denote the category of $k[\mathrm{Out}(G)]$-modules. Then the evaluation functor*

$$\mathrm{ev}_G \colon \mathcal{AU} \to \mathcal{M}_G, \quad X \mapsto X(G)$$

has a left and right adjoint which are respectively given by $e_{G,\bullet}$ and $t_{G,\bullet}$. In particular, $e_{G,V}$ is projective and $t_{G,V}$ is injective.

Proof. The unit of the adjunction $\eta_V \colon V \to e_{G,V}(G) = V$ is the identity, and the counit is given by

$$\epsilon_X(T) \colon e_{G,X(G)}(T) \to X(T), \quad x \otimes [\alpha] \mapsto \alpha^*(x)$$

for all $T \in \mathcal{G}$. Similarly, the counit map $t_{G,V}(G) \to V$ is the identity, and the unit is given by

$$\eta_X(T) \colon X(T) \to t_{G,X(G)}(T), \quad x \mapsto ([\beta] \mapsto \beta^*(x))$$

for all $T \in \mathcal{G}$. We leave to the reader to check that these maps are natural and that they satisfy the triangular identities. The second part of the claim follows immediately from the fact that the evaluation functor is exact as colimits are computed pointwise. Here we are implicitly using that the field k has characteristic 0, so that all finitely generated $k[Out(G)]$-modules are projective. $\square$

REMARK 2.10. If $\mathcal{C}$ is a groupoid with finite hom sets, it is standard and easy that all objects in $[\mathcal{C}^{\mathrm{op}}, \mathrm{Vect}_k]$ are both projective and injective. (We will review these arguments in Section 7.) In some other cases where $\mathcal{C}$ is finite and an associated algebra is Frobenius, we find that the projectives and injectives are

the same, but that general objects do not have either property. For a typical small category, the projectives and injectives are unrelated. For many of the categories $\mathcal{U} \leq \mathcal{G}$ arising in this paper, we will show that the projectives in $\mathcal{A}\mathcal{U}$ are a strict subset of the injectives, which are a strict subset of the full subset of objects. We are not aware of any examples where this pattern has previously been observed; it has a number of interesting consequences.

3   Subcategories and their properties

Throughout this paper we will consider a wide range of subcategories $\mathcal{U} \leq \mathcal{G}$, and we will impose different conditions on $\mathcal{U}$ in different places. It is convenient to collect together the main examples and conditions here.

DEFINITION 3.1. Let $\mathcal{U}$ be a subcategory of $\mathcal{G}$ (assumed implicitly to be full and replete, as usual).

- We say that $\mathcal{U}$ is *subgroup-closed* if whenever $H \leq G \in \mathcal{U}$ we also have $H \in \mathcal{U}$.

- We say that $\mathcal{U}$ is *closed downwards* if whenever $G \to H$ is a surjective homomorphism with $G \in \mathcal{U}$, we also have $H \in \mathcal{U}$.

- We say that $\mathcal{U}$ is *closed upwards* if whenever $H \to K$ is a surjective homomorphism with $K \in \mathcal{U}$, we also have $H \in \mathcal{U}$.

- We say that $\mathcal{U}$ is *convex* if whenever $G \to H \to K$ are surjective homomorphisms with $G, K \in \mathcal{U}$, we also have $H \in \mathcal{U}$.

- We say that $\mathcal{U}$ is *multiplicative* if $1 \in \mathcal{U}$, and $G \times H \in \mathcal{U}$ whenever $G, H \in \mathcal{U}$. Equivalently, $\mathcal{U}$ should contain the product of any finite family of its objects, including the empty family.

- We say that $\mathcal{U}$ is *widely closed* if whenever $G \leftarrow H \to K$ are surjective homomorphisms with $G, H, K \in \mathcal{U}$, the image of the combined morphism $H \to G \times K$ is also in $\mathcal{U}$. (We will show that almost all of our examples have this property.)

- We say that $\mathcal{U}$ is *finite* if it has only finitely many isomorphism classes.

- We say that $\mathcal{U}$ is *groupoid* if all morphisms in $\mathcal{U}$ are isomorphisms.

- We say that $\mathcal{U}$ is *colimit-exact* if the functor $X \mapsto \varinjlim_{G \in \mathcal{U}^{\mathrm{op}}} X(G)$ is an exact functor $\mathcal{A}\mathcal{U} \to \mathrm{Vect}_k$. (We will show that almost all of our examples have this property.)

- We say that $\mathcal{U}$ is *submultiplicative* if it is multiplicative and subgroup-closed.

- We say that $\mathcal{U}$ is a *global family* if it is subgroup-closed and also closed downwards.

Remark 3.2.

- If $\mathcal{U}$ is closed upwards or downwards or is a groupoid, then it is convex.

- If $\mathcal{U}$ is submultiplicative then it is clearly widely closed.

- If $\mathcal{U}$ is convex, then it is also widely closed. Indeed, if $G \leftarrow H \rightarrow K$ are surjective homomorphisms with $G, H, K \in \mathcal{U}$ and L is the image of the resulting map $H \rightarrow G \times K$ then we have evident surjective homomorphisms $H \rightarrow L \rightarrow G$, showing that $L \in \mathcal{U}$.

- In particular, if $\mathcal{U}$ is closed upwards or downwards or is a groupoid, then it is widely closed.

Definition 3.3. We define subcategories of $\mathcal{G}$ as follows. Some of them depend on a prime number p and/or an integer $n \geq 1$.

- $\mathcal{Z}$ is the multiplicative global family of finite abelian groups.

- $\mathcal{C}$ is the global family of finite cyclic groups.

- $\mathcal{G}[p^\infty]$ is the multiplicative global family of finite p-groups.

- $\mathcal{Z}[p^\infty] = \mathcal{Z} \cap \mathcal{G}[p^\infty]$ is the multiplicative global family of finite abelian p-groups.

- $\mathcal{C}[p^\infty] = \mathcal{C} \cap \mathcal{G}[p^\infty]$ is the global family of finite cyclic p-groups.

- $\mathcal{G}[p^n]$ is the multiplicative global family of finite groups of exponent dividing p^n.

- $\mathcal{Z}[p^n] = \mathcal{Z} \cap \mathcal{G}[p^n]$ is the multiplicative global family of finite abelian groups of exponent dividing p^n, which is equivalent to the category of finitely generated modules over $\mathbb{Z}/p^n$.

- $\mathcal{C}[p^n] = \mathcal{C} \cap \mathcal{G}[p^n]$ is the global family of finite cyclic groups of exponent dividing p^n, which is equivalent to the category of cyclic modules over $\mathbb{Z}/p^n$.

- $\mathcal{F}[p^n]$ is the subcategory of groups isomorphic to $(\mathbb{Z}/p^n)^r$ for some $r \geq 0$, which is equivalent to the category of finitely generated free modules over $\mathbb{Z}/p^n$.

- $\mathcal{E}[p]$ is the multiplicative global family of elementary abelian p-groups, which is the same as $\mathcal{Z}[p]$ or $\mathcal{F}[p]$.

We also consider the following subcategories, primarily as a source of counter-examples:

- $\mathcal{W}_0$ is the subcategory of finite simple groups, which is a groupoid.

- $\mathcal{W}_1$ is the subcategory of (necessarily cyclic) groups of prime order, which is also a groupoid.

- $\mathcal{W}_2$ is the subcategory of finite 2-groups in which every square is a commutator. This is easily seen to be multiplicative and closed downwards. However, it contains the quaternion group Q_8 but not the cyclic group $C_4 < Q_8$, so it is not subgroup-closed.

- $\mathcal{W}_3$ is the subcategory of finite p-groups in which all elements of order p commute. This is clearly submultiplicative, but it is not closed downwards. Indeed, one can check that $\mathcal{W}_3$ contains the upper triangular group $UT_3(\mathbb{Z}/p^2)$ (provided that $p > 2$), but not the quotient group $UT_3(\mathbb{Z}/p)$. (We thank Yves de Cornulier, aka MathOverflow user YCor, for this example [11].)

Given a subcategory $\mathcal{U}$, we also define further subcategories as below, depending on an integer $n > 0$ or an object $N \in \mathcal{U}$:

- $\mathcal{U}_{\leq n} = \{G \in \mathcal{U} \mid |G| \leq n\}$. This is always finite. If $\mathcal{U}$ is subgroup-closed, closed downwards, convex, widely-closed or a groupoid then $\mathcal{U}_{\leq n}$ inherits the same property.

- $\mathcal{U}_{\geq n} = \{G \in \mathcal{U} \mid |G| \geq n\}$. If $\mathcal{U}$ is closed upwards, convex, widely closed, finite or a groupoid then $\mathcal{U}_{\geq n}$ inherits the same property.

- $\mathcal{U}_{=n} = \{G \in \mathcal{U} \mid |G| = n\} = \mathcal{U}_{\leq n} \cap \mathcal{U}_{\geq n}$. This is always a finite groupoid, and so is convex and widely closed.

- $\mathcal{U}_{\leq N} = \{G \in \mathcal{U} \mid \mathcal{G}(N, G) \neq \emptyset\}$. This is always finite. If $\mathcal{U}$ is closed downwards, convex, widely-closed or a groupoid then $\mathcal{U}_{\leq N}$ inherits the same property.

- $\mathcal{U}_{\geq N} = \{G \in \mathcal{U} \mid \mathcal{U}(G, N) \neq \emptyset\}$. If $\mathcal{U}$ is closed upwards, convex, widely closed, finite or a groupoid then $\mathcal{U}_{\geq N}$ inherits the same property.

- $\mathcal{U}_{\simeq N} = \{G \in \mathcal{U} \mid G \simeq N\} = \mathcal{U}_{\leq N} \cap \mathcal{U}_{\geq N}$. This is always a finite groupoid, and so is convex and widely closed.

EXAMPLE 3.4. Using Remark 3.2 we see that almost all of the specific subcategories listed above are widely closed. One exception is the subcategory $\mathcal{F}[p^n]$ for $n > 1$. We will identify this with the category of finitely generated free modules over $\mathbb{Z}/p^n$ and so use additive notation. We take $G = K = \mathbb{Z}/p^n$ and $H = (\mathbb{Z}/p^n)^2$, and we define maps $G \xleftarrow{\alpha} H \xrightarrow{\beta} K$ by $\alpha(i, j) = i$ and $\beta(i, j) = i + pj$. We find that the image of the combined map $H \to G \times K$ is isomorphic to $\mathbb{Z}/p^n \times \mathbb{Z}/p^{n-1}$ and so does not lie in $\mathcal{F}[p^n]$.

4   CLOSED MONOIDAL STRUCTURE

It is convenient to add a bit of structure on $\mathcal{A}$.

DEFINITION 4.1. We give $\mathcal{A}\mathcal{U}$ the symmetric monoidal structure given by $(X \otimes Y)(T) = X(T) \otimes Y(T)$. The unit object $\mathbb{1}$ is the constant functor with value k (so $\mathbb{1} = e_1$ provided that $1 \in \mathcal{U}$). We also put

$$\underline{\operatorname{Hom}}(X, Y)(T) = \mathcal{A}(e_T \otimes X, Y).$$

Standard arguments show that this defines an object of $\mathcal{A}\mathcal{U}$ with

$$\mathcal{A}\mathcal{U}(W, \underline{\operatorname{Hom}}(X, Y)) \simeq \mathcal{A}(W \otimes X, Y),$$

so $\mathcal{A}\mathcal{U}$ is a closed symmetric monoidal category. We write DX for $\underline{\operatorname{Hom}}(X, \mathbb{1})$, and call this the *dual* of X.

REMARK 4.2. Note that the tensor product is both left and right exact, so all objects are flat.

REMARK 4.3. We warn the reader that DX is not obtained from X by taking levelwise duals, so the canonical map $X \otimes DX \to \underline{\operatorname{Hom}}(X, X)$ is usually not an isomorphism. To demonstrate this consider the case $X = e_G$ for any non-trivial group G. If we evaluate at the trivial group, we find $e_G(1) \otimes De_G(1) = 0$ and $\underline{\operatorname{Hom}}(e_G, e_G)(1) = k[\operatorname{Out}(G)]$. Therefore the map is far from being an isomorphism.

For the rest of this section we study the effect of the tensor product and internal hom functor on the generators. The main results are Proposition 4.11 and Theorem 4.18 and they both rely on the following notion.

DEFINITION 4.4. Let $\mathcal{U}$ be a subcategory of $\mathcal{G}$. A *permuted family* of groups consists of a finite group Γ, a finite Γ-set A, a family of groups $G_a \in \mathcal{U}$ for each $a \in A$, and a system of isomorphisms $\gamma_* \colon G_a \to G_{\gamma(a)}$ (for $\gamma \in \Gamma$ and $a \in A$) satisfying the functoriality conditions $1_* = 1$ and $(\delta\gamma)_* = \delta_*\gamma_*$. The system of isomorphisms gives maps $\operatorname{stab}_\Gamma(a) \to \operatorname{Aut}(G_a)$ for each $a \in A$. We say that the family is *outer* if the image of this map contains the inner automorphism group $\operatorname{Inn}(G_a)$ for all a. Given a permuted family $\underline{G}$ which is outer, we define the set

$$\widetilde{B}(\underline{G})(T) = \{(a, \alpha) \mid a \in A, \ \alpha \in \operatorname{Epi}(T, G_a)\}.$$

The group Γ acts on $\widetilde{B}(\underline{G})(T)$ via the formula $\gamma \cdot (a, \alpha) = (\gamma(a), \gamma_* \circ \alpha)$. We define $B(\underline{G})(T) = \widetilde{B}(\underline{G})(T)/\Gamma$ and $F(\underline{G})(T) = k[B(\underline{G})(T)]$. This is contravariantly functorial in T, so $F(\underline{G}) \in \mathcal{A}\mathcal{U}$.

PROPOSITION 4.5. *For all $X \in \mathcal{A}\mathcal{U}$ there is a natural isomorphism*

$$\mathcal{A}\mathcal{U}(F(\underline{G}), X) = \left(\prod_{a \in A} X(G_a) \right)^\Gamma.$$

If we choose a subset $A_0 \subset A$ containing one element of each Γ-orbit, we get an isomorphism

$$F(\underline{G}) = \bigoplus_{a \in A_0} e_{G_a}^{\operatorname{stab}_\Gamma(a)}.$$

Thus, $F(\underline{G})$ is finitely projective (see Definition 11.1).

Proof. We can reduce to the case where A is a single orbit, say $A = \Gamma a \simeq \Gamma/\Delta$, where $\Delta = \operatorname{stab}_\Gamma(a)$. We can define $\phi \colon \operatorname{Epi}(T, G_a)/\Delta \to B(\underline{G})(T)$ by $\phi[\alpha] = [a, \alpha]$. If $[b, \beta] \in B(\underline{G})(T)$ then $b = \gamma(a)$ for some a. We can then put $\alpha = \gamma_*^{-1} \circ \beta \colon T \to G_a$ and we find that $[b, \beta] = \phi[\alpha]$. On the other hand, if $\phi(\alpha) = \phi(\alpha')$ then there exists $\gamma \in \Gamma$ with $(\gamma(a), \gamma_* \circ \alpha) = (a, \alpha')$ which means that $\gamma \in \Delta$ and $[\alpha] = [\alpha']$ in $\operatorname{Epi}(T, G_a)/\Delta$. It follows that ϕ is a natural bijection. Thus, if we let Φ denote the image of Δ in $\operatorname{Out}(G_a)$, we have $F(\underline{G}) \simeq e_{G_a}^\Phi$. Note that the inclusion $e_{G_a}^\Phi \leq e_{G_a}$ is split by the map $x \to |\Phi|^{-1} \sum_{\phi \in \Phi} \phi \cdot x$. It follows that $e_{G_a}^\Phi$ is projective. $\qquad\square$

DEFINITION 4.6. Let $(G_i)_{i \in I}$ be a finite family of groups in $\mathcal{U}$ with product $P = \prod_i G_i$.

- We say that a subgroup $W \leq P$ is *wide* if all the projections $\pi_i \colon W \to G_i$ are surjective.

- We say that a homomorphism $f \colon T \to P$ is *wide* if all the morphisms $\pi_i \circ f$ are surjective, or equivalently $f(T)$ is a wide subgroup of P.

For $G, H \in \mathcal{U}$, we let $\operatorname{Wide}(G, H)$ denote the set of wide subgroups of $G \times H$ which belong to $\mathcal{U}$. This set is covariantly functorial in G and H with respect to morphisms in $\mathcal{U}$. Given $\varphi \colon G' \to G$ in $\mathcal{U}$ and $W' \in \operatorname{Wide}(G', H)$, we put $\varphi_* W' = (\varphi \times \operatorname{id}_H)(W')$ which is wide in $G \times H$. This comes with a map $j_\varphi \colon W' \to \varphi_* W'$ which makes the following diagram

$$
\begin{array}{ccc}
G' \times H & \xrightarrow{\varphi \times \operatorname{id}} & G \times H \\
\uparrow & & \uparrow \\
W' & \xrightarrow{\ j_\varphi\ } & \varphi_* W'
\end{array}
$$

commute. The assignment $W' \mapsto \varphi_* W'$ defines a map $\varphi_* \colon \operatorname{Wide}(G', H) \to \operatorname{Wide}(G, H)$ between the set of wide subgroups. Similar functoriality holds for H as well.

EXAMPLE 4.7. Let G_1 and G_2 be finite groups.

(a) The full group $G_1 \times G_2$ is always wide. If G_1 and G_2 are nonisomorphic simple groups, then one can check (perhaps using Lemma 4.9 below) that this is the only example. Similarly, if $|G_1|$ and $|G_2|$ are coprime, then $G_1 \times G_2$ is the only wide subgroup.

(b) If $\alpha\colon G_1 \to G_2$ is a surjective homomorphism, then the graph

$$\mathrm{Gr}(\alpha) = \{(g, \alpha(g)) \mid g \in G_1\}$$

is always wide. If G_1 and G_2 are isomorphic simple groups, then one can check that every wide subgroup is of the form (a) or (b). Moreover, in (b) we see that α must be an isomorphism.

(c) Now let $\mathcal{U} \leq \mathcal{G}$ be a groupoid, and suppose that $G_1, G_2 \in \mathcal{U}$. If $W \leq G_1 \times G_2$ is wide and lies in $\mathcal{U}$, we see easily that W is the graph of an isomorphism $\alpha\colon G_1 \to G_2$.

(d) Now consider the case $\mathcal{U} = \mathcal{C}[p^\infty] = \{\text{cyclic } p\text{-groups}\}$. If $|G_1| \geq |G_2|$ then it is not hard to see that any cyclic wide subgroup of $G_1 \times G_2$ is the graph of a surjective homomorphism $\alpha\colon G_1 \to G_2$ as in (b). Similarly, if $|G_1| \leq |G_2|$ then any cyclic wide subgroup of $G_1 \times G_2$ is the graph of a surjective homomorphism $\beta\colon G_2 \to G_1$. Of course, if $|G_1| = |G_2|$ then any surjective homomorphism $\alpha\colon G_1 \to G_2$ is an isomorphism, and the graph of $\alpha\colon G_1 \to G_2$ is the same as the graph of $\alpha^{-1}\colon G_2 \to G_1$.

DEFINITION 4.8. Suppose we have finite groups G_1 and G_2, and normal subgroups $N_i \lhd G_i$, and an isomorphism $\alpha\colon G_1/N_1 \to G_2/N_2$. We can then put

$$H(N_1, \alpha, N_2) = \{(x_1, x_2) \in G_1 \times G_2 \mid \alpha(x_1 N_1) = x_2 N_2\} \leq G_1 \times G_2.$$

This is easily seen to be a wide subgroup.

LEMMA 4.9. *Every wide subgroup $K \leq G_1 \times G_2$ has the form $H(N_1, \alpha, N_2)$ for a unique triple (N_1, α, N_2) as above.*

Proof. Put

$$N_1 = \{n_1 \in G_1 \mid (n_1, 1) \in K\},$$

and similarly for N_2. If $n_1 \in N_1$ and $g_1 \in G_1$ then wideness gives $g_2 \in G_2$ such that $(g_1, g_2) \in K$. It follows that the element $(g_1 n_1 g_1^{-1}, 1) = (g_1, g_2)(n_1, 1)(g_1, g_2)^{-1}$ lies in K and that N_1 is normal. The same argument shows that N_2 is normal in G_2 too. This means that K is the preimage in $G_1 \times G_2$ of the subgroup $\bar{K} = K/(N_1 \times N_2) \leq (G/N_1) \times (G/N_2)$. We now find that the projections $\pi_i\colon \bar{K} \to G_i/N_i$ are both isomorphisms, so we can define $\alpha\colon \pi_2 \pi_1^{-1}\colon G/N_1 \to G/N_2$. It is now easy to see that $K = H(N_1, \alpha, N_2)$, as required. $\square$

DEFINITION 4.10. Given $G, H \in \mathcal{U}$, we let $\underline{W}(G, H)$ denote the tautological family indexed by $\mathrm{Wide}(G, H)$, so the group indexed by $U \in \mathrm{Wide}(G, H)$ is U itself. Then $G \times H$ acts on $\mathrm{Wide}(G, H)$ by conjugation. We use this to regard $\underline{W}(G, H)$ as a permuted family, and thus define a finitely projective object $F(\underline{W}(G, H)) \in \mathcal{AU}$.

We now consider tensor products of generators.

PROPOSITION 4.11. *Let $\mathcal{U}$ be a widely closed subcategory of $\mathcal{G}$, and suppose that $G, H \in \mathcal{U}$. Then $e_G \otimes e_H$ is naturally isomorphic to $F(\underline{W}(G,H))$ (and so is a finitely generated projective object of $\mathcal{AU}$).*

Proof. Consider another object $T \in \mathcal{U}$ and a pair $(\alpha, \beta) \in \mathrm{Epi}(T,G) \times \mathrm{Epi}(T,H)$. This gives a wide subgroup $U = \langle \alpha, \beta \rangle (T) \leq G \times H$, which lies in $\mathcal{U}$ because $\mathcal{U}$ is assumed to be widely closed. We can regard $\langle \alpha, \beta \rangle$ as a surjective homomorphism from T to U, so we have an element $\phi(\alpha, \beta) = (U, \langle \alpha, \beta \rangle) \in \widetilde{B}(\underline{W}(G,H))(T)$. This is easily seen to give a $(G \times H)$-equivariant natural bijection

$$\phi \colon \mathrm{Epi}(T,G) \times \mathrm{Epi}(T,H) \to \widetilde{B}(\underline{W}(G,H))(T).$$

It follows easily that we get an induced bijection $\mathcal{U}(T,G) \times \mathcal{U}(T,H) \to B(\underline{W}(G,H))(T)$ and an isomorphism $e_G \otimes e_H \to F(\underline{W}(G,H))$ as required. □

REMARK 4.12. If G and H are abelian, then $G \times H$ acts trivially on $\underline{W}$ and so $e_G \otimes e_H = \bigoplus_{U \in \mathrm{Wide}(G,H)} e_U$.

REMARK 4.13. It is not true that $e_G \otimes e_H$ is always a direct sum of objects of the form e_K. In particular, this fails when $G = H = D_8$. To see this, let N be the subgroup of G isomorphic to C_4, and put $W = \{(g,h) \in G \times H \mid gN = hN\}$. This is wide, and has index 2 in $G \times H$, so it is normal in $G \times H$. The group $G \times H$ acts by conjugation of the set $\mathrm{Wide}(G,H)$ and the stabilizer of the conjugacy class of W is the quotient $Q = (G \times H)/W$. Then the summand in the tensor product $e_G \otimes e_H$ corresponding to the conjugacy class of W is given by e_W^Q which is not of the form e_K.

DEFINITION 4.14. A *virtual homomorphism* from G to H is a pair $\alpha = (A, A')$ where $A' \lhd A \leq G \times H$ and A is wide and $A' \cap (1 \times H) = 1$ and $A/A' \in \mathcal{U}$. We write $\mathrm{VHom}(G,H)$ for the set of virtual homomorphisms. We then let $\underline{Q}(G,H)$ denote the collection of groups $Q_\alpha = A/A'$ indexed by all $\alpha = (A, A') \in \mathrm{VHom}(G,H)$. We call Q_α the *spread* of α. Note that $G \times H$ acts compatibly on $\mathrm{VHom}(G,H)$ and Q_α by conjugation. We use this to regard $\underline{Q}(G,H)$ as a permuted family, and thus to define a finitely projective object $\overline{F}(\underline{Q}(G,H)) \in \mathcal{AU}$.

EXAMPLE 4.15. Suppose that $\mathcal{U}$ contains the trivial group. For any surjective homomorphism $u \colon G \to H$, we can define

$$A = A' = \mathrm{graph}(u) = \{(g, u(g)) \mid g \in G\}.$$

This gives a virtual homomorphism with trivial spread. We claim that every virtual homomorphism with trivial spread arises in this way from a unique homomorphism. Indeed, let $\alpha = (A, A)$ be any such virtual homomorphism and consider the projection map $A \leq G \times H \to G$. The condition $A \cap (1 \times H) = 1$ ensures that every element $g \in G$ has a unique preimage $(g, u(g)) \in A$ under the projection. It is easy to check that the assignment $u \colon G \to H$ defines a surjective group homomorphism, and by construction $A = \mathrm{graph}(u)$.

EXAMPLE 4.16. Consider a virtual homomorphism $\alpha = (A, A') \in \mathrm{VHom}(1, G)$. The group A must be wide in $1 \times G$, which just means that $A = 1 \times G$. The group $A' \leq 1 \times G$ must satisfy $A' \cap (1 \times G) = 1$, which means that $A' = 1$. Thus, there is a unique virtual homomorphism $\alpha = (1 \times G, 1)$, whose spread is G.

EXAMPLE 4.17. Consider a virtual homomorphism $\alpha = (A, A') \in \mathrm{VHom}(G, 1)$. We find that A must be equal to $G \times 1$ (which we identify with G) and A' can be any normal subgroup of G such that $G/A' \in \mathcal{U}$.

THEOREM 4.18. *Let $\mathcal{U}$ be a multiplicative global family of finite groups. Fix groups $G, H \in \mathcal{U}$ and let $Q(G, H)$ be the permuted family of virtual homomorphisms from G to H. Then $\underline{\mathrm{Hom}}(e_G, e_H)$ is isomorphic to $F(\underline{Q}(G, H))$ (and so is a finitely generated projective object of $\mathcal{A}\mathcal{U}$).*

The general structure of the proof is as follows. We will fix G and H, and define finite sets $\mathcal{L}(T)$, $\mathcal{M}(T)$ and $\mathcal{N}(T)$ depending on a third object $T \in \mathcal{U}$. All of these will have actions of $G \times H$ by conjugation, and we will construct equivariant bijections between them. We will also construct isomorphisms $\underline{\mathrm{Hom}}(e_G, e_H)(T) \simeq k[\mathcal{N}(T)]^{G \times H}$ and $F(\underline{Q}(G, H))(T) \simeq k[\mathcal{M}(T)]^{G \times H}$. All of this is natural with respect to isomorphisms $T' \to T$, but unfortunately not with respect to arbitrary morphisms $T' \to T$ in $\mathcal{U}$. However, we will introduce filtrations of all the relevant objects and show that the failure of naturality involves terms that shift filtration. It will follow that the associated graded object for $\underline{\mathrm{Hom}}(e_G, e_H)$ is isomorphic to $F(\underline{Q}(G, H))$. As this object is projective, we see that the filtration splits, so $\underline{\mathrm{Hom}}(e_G, e_H)$ itself is isomorphic to $F(\underline{Q}(G, H))$, as claimed.

DEFINITION 4.19. Fix groups $G, H \in \mathcal{U}$. Let T be another group in $\mathcal{U}$.

(a) We define $\mathcal{L}(T)$ to be the set of wide subgroups $V \leq T \times G \times H$ such that $V \cap H = 1$. (Here we identify H with the subgroup $1 \times 1 \times H \leq T \times G \times H$, and we will make similar identifications in various places below.)

(b) We define $\mathcal{M}(T)$ to be the set of triples (A, A', θ) where $(A, A') \in \mathrm{VHom}(G, H)$ and $\theta \in \mathrm{Epi}(T, A/A')$.

(c) We define $\mathcal{N}(T)$ to be the set of pairs (W, λ) where W is a wide subgroup of $T \times G$, and $\lambda \in \mathrm{Epi}(W, H)$.

All of these sets have evident actions of $G \times H$ by conjugation.

DEFINITION 4.20. Given a surjective homomorphism $\varphi: T' \to T$, we define maps $\varphi^*: \mathcal{L}(T) \to \mathcal{L}(T')$, and similarly for $\mathcal{M}$ and $\mathcal{N}$, as follows:

(a) $\varphi^*(V) = (\varphi \times 1 \times 1)^{-1}(V) = \{(t', g, h) \in T' \times G \times H \mid (\varphi(t'), g, h) \in V\}$

(b) $\varphi^*(A, A', \theta) = (A, A', \theta\varphi)$

(c) $\varphi^*(W, \lambda) = ((\varphi \times 1)^{-1}(W), \lambda \circ (\varphi \times 1))$.

These constructions are clearly functorial.

CONSTRUCTION 4.21. We define a bijection $\mu\colon \mathcal{L}(T) \to \mathcal{M}(T)$ as follows. Given $V \in \mathcal{L}(T)$ we put $A = \pi_{G \times H}(V) \leq G \times H$ and $A' = \{(g,h) \in G \times H \mid (1,g,h) \in V\}$. As V is wide in $T \times G \times H$, we see that A is wide in $G \times H$. As $V \cap H = 1$, we see that $A' \cap H = 1$. This means that the pair (A, A') is an element of $\mathrm{VHom}(G, H)$. Next, for $t \in T$ we put

$$\theta(t) = \{(g,h) \in G \times H \mid (t,g,h) \in V\}.$$

This is a coset of A' in A, or in other word an element of A/A'. It is not hard to check that this gives a homomorphism $\theta\colon T \to A/A'$. From the definition of A we see that θ is surjective. We have thus defined an element $\mu(V) = (A, A', \theta) \in \mathcal{M}(T)$.

In the opposite direction, suppose we start with an element $(A, A', \theta) \in \mathcal{M}(T)$. We can then define

$$V = \{(t,g,h) \in T \times A \mid \theta(t) = (g,h).A'\}.$$

It is clear that $\pi_T(V) = T$ and $\pi_{G \times H}(V) = A$. As A is wide in $G \times H$, it follows that V is wide in $T \times G \times H$. Now suppose that $(1,1,h) \in V$, so $(1,h) \in A$ and the coset $(1,h).A'$ is the same as $\theta(1)$, or in other words $(1,h) \in A'$. It then follows from the definition of a virtual homomorphism that $h = 1$. This proves that $V \in \mathcal{L}(T)$. It is easy to check that this construction gives a map $\mathcal{M}(T) \to \mathcal{L}(T)$ that is inverse to μ. It is also straightforward to check that these bijections are natural with respect to the functoriality in Definition 4.20.

CONSTRUCTION 4.22. We define a bijection $\nu\colon \mathcal{L}(T) \to \mathcal{N}(T)$ as follows. Given $V \in \mathcal{L}(T)$ we define $W = \pi_{T \times G}(V) \leq T \times G$. As $V \in \mathcal{L}(T)$ we have $V \cap H = 1$, which means that the projection $\pi_{T \times G}\colon V \to W$ is an isomorphism. We define λ to be the composite

$$W \xrightarrow{\pi_{T \times G}^{-1}} V \xrightarrow{\pi_H} H.$$

As V is wide in $T \times G \times H$, we see that λ is surjective, so we have an element $\nu(V) = (W, \lambda) \in \mathcal{N}(T)$.

In the opposite direction, suppose we start with an element $(W, \lambda) \in \mathcal{N}(T)$. We then put

$$V = \{(t,g,h) \in W \times H \mid \lambda(t,g) = h\}.$$

As W is wide in $T \times G$ and $\lambda\colon W \to T$ is surjective, we see that V is wide in $T \times G \times H$. If $(1,1,h) \in V$ then we must have $h = \lambda(1,1) = 1$. This proves that $V \in \mathcal{L}(T)$. It is easy to check that this construction gives a map $\mathcal{N}(T) \to \mathcal{L}(T)$ that is inverse to ν. It is also easy to check that these bijections are natural with respect to the functoriality in Definition 4.20.

REMARK 4.23. It is straightforward to identify $\widetilde{B}(\underline{Q}(G,H))(T)$ with $\mathcal{M}(T)$, and so to identify $F(\underline{Q}(G,H))(T)$ with $k[\mathcal{M}(T)]^{G \times H}$.

Definition 4.24. For each element x in $\mathcal{L}(T)$, $\mathcal{M}(T)$ or $\mathcal{N}(T)$ we define a positive integer $\sigma(x)$ as follows.

(a) For $V \in \mathcal{L}(T)$ we put

$$V^{\#} = \{(t, g, h) \in V \mid (t, 1, 1),\ (1, g, h) \in V\},$$

and $\sigma(V) = |V|/|V^{\#}|$.

(b) For $(A, A', \theta) \in \mathcal{M}(T)$ we put $\sigma(A, A', \theta) = |A/A'|$.

(c) For $(W, \lambda) \in \mathcal{N}(T)$ we put

$$K(W, \lambda) = \{t \in T \mid (t, 1) \in W \text{ and } \lambda(t, 1) = 1\}$$

and $\sigma(W, \lambda) = |T|/|K(W, \lambda)|$.

We then put

$$
\begin{aligned}
F^n \mathcal{L}(T) &= \{x \in \mathcal{L}(T) \mid \sigma(x) \geq n\} \subseteq \mathcal{L}(T) \\
F^n k[\mathcal{L}(T)] &= k[F^n \mathcal{L}(T)] \leq k[\mathcal{L}(T)] \\
Q^n k[\mathcal{L}(T)] &= F^n k[\mathcal{L}(T)]/F^{n+1} k[\mathcal{L}(T)].
\end{aligned}
$$

Remark 4.25. For $(A, A', \theta) \in \mathcal{M}(T)$ it is clear that $\sigma(A, A', \theta) \leq |G||H|$. It follows that $\sigma(x) \leq |G||H|$ for $x \in \mathcal{L}(T)$ or $x \in \mathcal{N}(T)$ as well.

Lemma 4.26. *Suppose that the elements $V \in \mathcal{L}(T)$ and $(A, A', \theta) \in \mathcal{M}(T)$ and $(W, \lambda) \in \mathcal{N}(T)$ are related as in Constructions 4.21 and 4.22. Then $\sigma(V) = \sigma(A, A', \theta) = \sigma(W, \lambda)$. Thus, those constructions give bijections $F_n \mathcal{L}(T) \simeq F_n \mathcal{M}(T) \simeq F_n \mathcal{N}(T)$.*

Proof. As in Construction 4.21, we have a surjective projection $\pi \colon V \to A$, and it follows that $|A/A'| = |V|/|\pi^{-1}(A')|$. Moreover, we have $A' = \{(g, h) \mid (1, g, h) \in V\}$, and it follows easily that $\pi^{-1}(A') = V^{\#}$; this makes it clear that $\sigma(V) = \sigma(A, A', \theta)$. On the other hand, we also have a surjective projection $\pi' \colon V \to T$, and it follows that $|T|/|K(W, \lambda)| = |V|/|(\pi')^{-1}(K(W, \lambda))|$. Suppose we have $(t, g, h) \in V$ with $t \in K(W, \lambda)$. It then follows that $(t, 1, 1) \in V$, and thus that the product $(t, g, h).(t, 1, 1)^{-1} = (1, g, h)$ also lies in V, so $(t, g, h) \in V^{\#}$. This argument is reversible so we find that $(\pi')^{-1}(K(W, \lambda)) = V^{\#}$ and $\sigma(V) = \sigma(W, \lambda)$. $\qquad\square$

We now want to define an isomorphism

$$\zeta \colon k[\mathcal{N}(T)]^{G \times H} \to \underline{\mathrm{Hom}}(e_G, e_H)(T) = \mathcal{A}\mathcal{U}(e_T \otimes e_G, e_H).$$

One approach would be to split $e_T \otimes e_G$ as a sum over conjugacy classes of wide subgroups, but that involves choices which are awkward to control. We will therefore define ζ in a different way, and then use the splitting of $e_T \otimes e_G$ to verify that it is an isomorphism.

Construction 4.27. Fix an element $(W, \lambda) \in \mathcal{N}(T)$. Now consider an object $P \in \mathcal{U}$ and a pair of surjective homomorphisms $\alpha \colon P \to T$ and $\beta \colon P \to G$, giving an element $[\alpha] \otimes [\beta] \in (e_T \otimes e_G)(P)$ and a wide subgroup $\langle \alpha, \beta \rangle (P) \leq T \times G$. If there exists an element $(t, g) \in T \times G$ such that $c_{(t,g)}(\langle \alpha, \beta \rangle (P)) = W$, then we can form the composite

$$ P \xrightarrow{\langle c_t \alpha, c_g \beta \rangle} W \xrightarrow{\lambda} H. $$

This is a surjective homomorphism. Its conjugacy class depends only on the conjugacy classes of α and β, and not on the choice of (t, g). Moreover, everything that we have done is natural for morphisms $P' \to P$ in $\mathcal{U}$. We can thus define an element $\zeta_0(W, \lambda) \in \mathcal{AU}(e_T \otimes e_G, e_H)$ by $\zeta_0(W, \lambda)([\alpha] \otimes [\beta]) = \lambda \circ \langle c_t \alpha, c_g \beta \rangle$ in the case discussed above, and $\zeta_0(W, \lambda)([\alpha] \otimes [\beta]) = 0$ in the case where $\langle \alpha, \beta \rangle (P)$ is not conjugate to W. It is easy to see that if (W_0, λ_0) and (W_1, λ_1) lie in the same $(G \times H)$-orbit of $\mathcal{N}(T)$, then $\zeta_0(W_0, \lambda_0) = \zeta_0(W_1, \lambda_1)$. We now extend linearly to get a map $k[\mathcal{N}(T)] \to \underline{\mathrm{Hom}}(e_G, e_H)(T)$, and restrict to get a map $\zeta \colon k[\mathcal{N}(T)]^{G \times H} \to \underline{\mathrm{Hom}}(e_G, e_H)(T)$.

We can now choose a list of wide subgroups $W_1, \ldots, W_r \leq T \times G$ containing precisely one representative of each conjugacy class, and let Δ_i be the normaliser of W_i in $T \times G$. We have seen that this gives a decomposition $e_T \otimes e_G = \bigoplus_i e_{W_i}^{\Delta_i}$, and thus an isomorphism

$$ (e_T \otimes e_G)(H) = \bigoplus_k k[\mathrm{Epi}(W_i, H)/\Delta_i]. $$

(Note here that $\Delta_i \geq W_i$ so the conjugation action of Δ_i on $\mathrm{Epi}(W_i, H)$ encompasses the action of inner automorphisms.) From this it is not hard to see that ζ is an isomorphism.

Definition 4.28. We put $F^n \underline{\mathrm{Hom}}(e_G, e_H)(T) = \zeta(F^n k[\mathcal{N}(T)])$, and

$$ Q^n \underline{\mathrm{Hom}}(e_G, e_H)(T) = \frac{F^n \underline{\mathrm{Hom}}(e_G, e_H)(T)}{F^{n+1} \underline{\mathrm{Hom}}(e_G, e_H)(T)}. $$

Now consider a surjective homomorphism $\varphi \colon T' \to T$. This gives a map $\varphi^* \colon \mathcal{M}(T) \to \mathcal{M}(T')$ given by $\varphi^*(A, A', \theta) = (A, A', \theta \varphi)$, and this is straightforwardly compatible with our identification $F(\underline{Q}(G, H))(T) \simeq k[\mathcal{M}(T)]^{G \times H}$. However, the situation with $\mathcal{N}(T)$ and $\underline{\mathrm{Hom}}(e_G, e_H)(T)$ is more complicated.

Definition 4.29. Consider an element $(W, \lambda) \in \mathcal{N}(T)$, and a surjective homomorphism $\varphi \colon T' \to T$. Let $E(\varphi, (W, \lambda))$ be the set of pairs $(W', \lambda') \in \mathcal{N}(T')$ such that $(\varphi \times 1)(W') = W$ and λ' is the same as the composite

$$ W' \xrightarrow{\varphi \times 1} W \to H. $$

It is easy to see that the element $\varphi^*(W, \lambda) = ((\varphi \times 1)^{-1}(W), \lambda \circ (\varphi \times 1))$ is an element of $E(\varphi, (W, \lambda))$.

LEMMA 4.30. *Suppose that $(W', \lambda') \in E(\varphi, (W, \lambda))$. Then φ restricts to give a surjective homomorphism $K(W', \lambda') \to K(W, \lambda)$. It follows that $\sigma(W', \lambda') \geq \sigma(W, \lambda)$, with equality iff $(W', \lambda') = \varphi^*(W, \lambda)$.*

Proof. Suppose that $t' \in K(W', \lambda')$, so that $(t', 1) \in W'$ and $\lambda'(t', 1) = 1$. As $(\varphi \times 1)(W') = W$, we see that $(\varphi(t'), 1) \in W$. As $\lambda \circ (\varphi \times 1) = \lambda'$, we see that $\lambda(t, 1) = 1$. This shows that $t \in K(W, \lambda)$.

Conversely, suppose that $t \in K(W, \lambda)$. This means that $(t, 1) \in W = (\varphi \times 1)(W')$, so there exists $(t', g) \in W'$ with $(\varphi(t'), g) = (t, 1)$. In other words, there exists $t' \in T'$ such that $(t', 1) \in W'$ and $\varphi(t') = t$. Using the relation $\lambda \circ (\varphi \times 1) = \lambda'$ again, we also see that $\lambda'(t', 1) = 1$, so $t' \in K(W', \lambda')$.

We now see that

$$|K(W', \lambda')| = |K(W, \lambda)||\ker(\varphi) \cap K(W', \lambda')| \leq |K(W, \lambda)|.\frac{|T'|}{|T|}.$$

Rearranging this gives

$$\sigma(W', \lambda') = \frac{|T'|}{|K(W', \lambda')|} \geq \frac{|T|}{|K(W, \lambda)|} = \sigma(W, \lambda).$$

We have equality iff $\ker(\varphi) \leq K(W', \lambda')$. Because λ' factors through $\varphi \times 1$, we see that the second condition in the definition of $K(W', \lambda')$ is automatic, so we have equality iff $\ker(\varphi) \times 1 \leq W'$. This clearly holds if $W' = (\varphi \times 1)^{-1}(W)$. Conversely, suppose that $\ker(\varphi) \times 1 \leq W'$. We are given that $(\varphi \times 1)(W') = W$, so $W' \leq (\varphi \times 1)^{-1}(W)$. In the other direction, suppose that $(t', g) \in (\varphi \times 1)^{-1}(W)$, so $(\varphi(t'), g) \in W$. As $W = (\varphi \times 1)(W')$, we can choose $(t'_0, g_0) \in W'$ with $(\varphi \times 1)(t'_0, g_0) = (t', g)$. In other words, we can find $t'_0 \in T'$ such that $\varphi(t'_0) = \varphi(t')$ and $(t'_0, g) \in W'$. We now have $t' = t'_0 t'_1$ for some $t'_1 \in \ker(\varphi)$, so $(t'_1, 1) \in W'$ by assumption. It follows that the product $(t', g) = (t'_0, g)(t'_1, 1)$ also lies in W' as required. $\square$

PROPOSITION 4.31. *The subspaces $F^n \underline{\mathrm{Hom}}(e_G, e_H)(T)$ form a subobject of $\underline{\mathrm{Hom}}(e_G, e_H)$ in $\mathcal{AU}$, so the quotient $Q^n \underline{\mathrm{Hom}}(e_G, e_H)$ can also be regarded as an object of $\mathcal{AU}$. Moreover, the sum $Q^* \underline{\mathrm{Hom}}(e_G, e_H) = \bigoplus_n Q^n \underline{\mathrm{Hom}}(e_G, e_H)$ is naturally isomorphic to $F(\underline{Q}(G, H))$.*

Proof. Consider an element $m \in F^n \underline{\mathrm{Hom}}(e_G, e_H)(T)$ and a surjective homomorphism $\varphi \colon T' \to T$. We can regard m as a morphism $e_T \otimes e_G \to e_H$. Now suppose we have a surjective homomorphism $\varphi \colon T' \to T$. Now $\varphi^* m$ corresponds to the composite $m \circ (e_\varphi \otimes 1) \colon e_{T'} \otimes e_G \to e_H$. Consider a wide subgroup $W' \leq T' \times G$, and the resulting map $j' \colon e_{W'} \to e_{T'} \otimes e_G$. Put $W = (\varphi \times 1)(W')$, which is wide in $T \times G$, and let j be the resulting map $e_W \to e_T \otimes e_G$. The composite $mj \colon e_W \to e_H$ can be expressed as a k-linear combination of morphisms $\lambda \in \mathrm{Epi}(W, H)$. The condition $m \in F^n \underline{\mathrm{Hom}}(e_G, e_H)(T)$ means that for all λ appearing here, we have $\sigma(W, \lambda) \geq n$. It follows that $\varphi^*(m)j'$ can be expressed as a k-linear combination of the corresponding morphisms

$\lambda' = \lambda \circ (\varphi \times 1) \colon W' \to H$. Lemma 4.30 tells us that the resulting pairs satisfy $\sigma(W', \lambda') \geq n$. It follows that $F^n\underline{\mathrm{Hom}}(e_G, e_H)$ is a subobject, as claimed. Moreover, the edge case in Lemma 4.30 tells us that in the associated graded, we see only terms of the form $\varphi^*(W, \lambda)$. This means that the associated graded is isomorphic in $\mathcal{AU}$ to $k[\mathcal{N}]$ or $k[\mathcal{L}]$ or $k[\mathcal{M}]$ or $F(\underline{Q}(G, H))$, as claimed. $\qquad\square$

Proof of Theorem 4.18. The subobjects $F^n\underline{\mathrm{Hom}}(e_G, e_H)$ form a finite-length filtration of $\underline{\mathrm{Hom}}(e_G, e_H)$ with finitely projective quotients, so the filtration must split. The claim follows easily from this. $\qquad\square$

REMARK 4.32. We do not know whether there is a splitting of the filtration that is natural in G and H as well as T. There may be some interesting group theory and combinatorics involved here.

5 FUNCTORS FOR SUBCATEGORIES

In this section we study the formalism that relates the abelian category $\mathcal{A}$ to its smaller subcategories $\mathcal{AU}$.

DEFINITION 5.1. Let $\mathcal{U}$ and $\mathcal{V}$ be full and replete subcategories of $\mathcal{G}$, with $\mathcal{U} \subseteq \mathcal{V}$. The inclusion $i = i_{\mathcal{UV}} \colon \mathcal{U} \to \mathcal{V}$ gives a pullback functor $i^* \colon \mathcal{AV} \to \mathcal{AU}$. We write $i_!$ and i_* for the left and right adjoints of i^* (so $i_!, i_* \colon \mathcal{AU} \to \mathcal{AV}$). These are given by the usual Kan formulae (in their contravariant versions):

(a) $(i_!X)(G)$ is the colimit over the comma category $(G \downarrow \mathcal{U})$ of the functor sending each object $(G \xrightarrow{u} iH)$ to $X(H)$.

(b) $(i_*X)(G)$ is the limit over the comma category $(\mathcal{U} \downarrow G)$ of the functor sending each object $(iK \xrightarrow{v} G)$ to $X(K)$.

REMARK 5.2. The above definition covers most of the inclusion functors that we need to consider, with one class of exceptions, as follows. Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$. We then let $\mathcal{U}^\times$ be the category with the same objects, but with only group isomorphisms as the morphisms, and we let $l \colon \mathcal{U}^\times \to \mathcal{U}$ be the inclusion. Then $\mathcal{U}^\times$ is not a full subcategory of $\mathcal{U}$, so definition 5.1 does not officially apply. Nonetheless, we still have functors l^*, $l_!$ and l_*, whose behaviour is slightly different from what we see in the main case. Details will be given later.

LEMMA 5.3. *Let $i \colon \mathcal{U} \to \mathcal{V}$ be an inclusion of replete full subcategories of $\mathcal{G}$.*

(a) *The (co)unit maps $i^*i_*(X) \to X \to i^*i_!(X)$ are isomorphisms, for all $X \in \mathcal{AU}$. Thus, the functors $i_!$ and i_* are full and faithful embeddings.*

(b) *The essential image of $i_!$ is $\{Y \in \mathcal{AV} \mid \epsilon_Y \colon i_!i^*(Y) \to Y \text{ is iso}\}$.*
The essential image of i_* is $\{Y \in \mathcal{AV} \mid \eta_Y \colon Y \to i_*i^*(Y) \text{ is iso}\}$.

(c) *There are natural isomorphisms $i^*(\mathbb{1}) = \mathbb{1}$ and $i^*(X \otimes Y) = i^*(X) \otimes i^*(Y)$ giving a strong monoidal structure on i^*. However, the corresponding map $i^*\underline{\mathrm{Hom}}(X,Y) \to \underline{\mathrm{Hom}}(i^*X, i^*Y)$ is typically not an isomorphism.*

(d) *There are natural maps $i_!\mathbb{1} \to \mathbb{1} \to i_*\mathbb{1}$ and $i_!(X \otimes Y) \to i_!(X) \otimes i_!(Y)$ and $i_*(X) \otimes i_*(Y) \to i_*(X \otimes Y)$ giving (op)lax monoidal structures.*

(e) *In all cases $i_!$ preserves all colimits and i_* preserves all limits and i^* preserves both limits and colimits. Also $i_!$ preserves projective objects and i_* preserves injective objects. Both i_* and $i_!$ preserve indecomposable objects.*

(f) *If $\mathcal{U}$ is closed upwards in $\mathcal{V}$, then $i_!$ is extension by zero and so preserves all limits, colimits and tensors (but not the unit).*

(g) *If $\mathcal{U}$ is closed downwards in $\mathcal{V}$ then i_* is extension by zero and so preserves all limits, colimits and tensors (but not the unit).*

(h) *If $\mathcal{U}$ is submultiplicative, then $i_!$ preserves the unit and all tensors; in other words, is strongly monoidal.*

(i) *If i has a left adjoint $q \colon \mathcal{V} \to \mathcal{U}$ then $i_! = q^*$ (and so $i_!$ preserves all (co)limits).*

(j) *Suppose that $G \in \mathcal{U}$ and $\mathcal{C} \leq \mathcal{U}$ is convex. Then, for the objects defined in Definition 2.7 we have*

$$i^*(e_{G,V}) = e_{G,V} \qquad i^*(t_{G,V}) = t_{G,V} \qquad i^*(s_{G,V}) = s_{G,V}$$
$$i^*(\chi_\mathcal{C}) = \chi_\mathcal{C} \qquad i_!(e_{G,V}) = e_{G,V} \qquad i_*(t_{G,V}) = t_{G,V}.$$

If $\mathcal{U}$ is closed upwards, we also have

$$i_!(\chi_\mathcal{C}) = \chi_\mathcal{C} \quad i_!(s_{G,V}) = s_{G,V} \quad i_!(t_{G,V}) = \chi_\mathcal{U} \otimes t_{G,V}.$$

On the other hand, if $\mathcal{U}$ is closed downwards, we also have

$$i_*(e_{G,S}) = \chi_\mathcal{U} \otimes e_{G,S} \quad i_!(s_{G,V}) = s_{G,V} \quad i_*(\chi_\mathcal{C}) = \chi_\mathcal{C}.$$

Proof. Almost all of this is standard, but we recall proofs for ease of reference. If $G \in \mathcal{U}$ then $(G \xrightarrow{1} G)$ is terminal in the comma category $\mathcal{U} \downarrow G$, so the Kan formula reduces to $(i_!X)(G) = X(G)$. Using this, we see that the unit map $X \to i^*i_!(X)$ is an isomorphism for all X. It follows that the map $i_! \colon A\mathcal{U}(X,Y) \to A\mathcal{V}(i_!X, i_!Y)$ is an isomorphism, with inverse essentially given by i^*, so $i_!$ is a full and faithful embedding. A dual argument shows that the counit map $i^*i_*(X) \to X$ is also an isomorphism, and that i_* is also a full and faithful embedding. This proves claim (a).

Now put

$$\mathcal{B} = \{Y \in \mathcal{AV} \mid \epsilon_Y \colon i_! i^*(Y) \to Y \text{ is iso}\}$$
$$\mathcal{C} = \{Y \in \mathcal{AV} \mid \eta_Y \colon Y \to i_* i^*(Y) \text{ is iso}\}.$$

If $Y \in \mathcal{B}$ then $Y \simeq i_! i^*(Y)$ so Y is in the essential image of $i_!$. Conversely, for $X \in \mathcal{AU}$ we have seen that the unit $X \to i^* i_!(X)$ is an isomorphism, so the same is true of the map $i_!(X) \to i_! i^* i_!(X)$. By the triangular identities for the $(i_!, i^*)$-adjunction, it follows that the counit $i_! i^* i_!(X) \to i_!(X)$ is also an isomorphism, so $i_!(X) \in \mathcal{B}$, so any object isomorphic to $i_!(X)$ also lies in $\mathcal{B}$. This proves that $\mathcal{B}$ is the essential image of $i_!$, and a dual argument shows that $\mathcal{C}$ is the essential image of i_*. This proves claim (b).

We now consider claim (c). For all $G \in \mathcal{U}$, we have $(i^* \mathbb{1})(G) = k = \mathbb{1}(G)$ and $i^*(X \otimes Y)(G) = X(G) \otimes Y(G) = (i^* X \otimes i^* Y)(G)$ which proves that i^* is strongly monoidal as claimed. For the negative part of (c), consider the case where $\mathcal{U} = \{1\}$. For any G we have $(i^* \underline{\mathrm{Hom}}(e_G, e_G))(1) = \mathcal{AV}(e_G, e_G) = k[\mathrm{Out}(G)] \neq 0$, but if G is nontrivial, then $i^*(e_G) = 0$ and so $\underline{\mathrm{Hom}}(i^*(e_G), i^*(eG))(1) = 0$. This shows that the natural map $i^* \underline{\mathrm{Hom}}(X, Y) \to \underline{\mathrm{Hom}}(i^*(X), i^*(Y))$ (adjoint to the evaluation map) is not always an isomorphism.

From claim (c) we get a natural isomorphism

$$i^*(i_*(X) \otimes i_*(Y)) \simeq i^* i_*(X) \otimes i^* i_*(Y) \xrightarrow{\epsilon_X \otimes \epsilon_Y} X \otimes Y,$$

and using the (i^*, i_*)-adjunction we get a natural map $i_*(X) \otimes i_*(Y) \to i_*(X \otimes Y)$. A standard argument shows that this makes i_* into a lax monoidal functor. By a dual construction, we get a natural map $i_!(X \otimes Y) \to i_!(X) \otimes i_!(Y)$ making $i_!$ into an oplax monoidal functor. This proves (d).

Most of claim (e) is formal and follows from the properties of adjunctions. If P is indecomposable, we see that the only idempotent elements in $\mathrm{End}(P)$ are 0 and 1, and that $0 \neq 1$. As $i_!$ is full and faithful, we see that $\mathrm{End}(i_! P)$ is isomorphic to $\mathrm{End}(P)$, and so has the same idempotent structure. A similar proof works for i_* too.

Now consider claim (f). Suppose that $\mathcal{U}$ is closed upwards in $\mathcal{V}$, that $X \in \mathcal{AU}$, and that $G \in \mathcal{V}$. If $G \in \mathcal{U}$ then $i_!(X)(G) = X(G)$ by claim (a). If $G \notin \mathcal{U}$ then the upward closure assuption implies that $G \downarrow \mathcal{U} = \emptyset$, so the Kan formula reduces to $i_!(X)(G) = 0$. In other words, $i_!$ is extension by zero, and the rest of claim (f) follows immediately. A dual argument proves (g).

Now suppose instead that $\mathcal{U}$ is submultiplicative, as in (h), so in particular $1 \in \mathcal{U}$. We claim that the natural map $i_!(X \otimes Y) \to i_!(X) \otimes i_!(Y)$ is an isomorphism. As $i_!$ and the tensor product preserve colimits, we can reduce to the case where $X = e_G$ and $Y = e_H$ for some $G, H \in \mathcal{U}$. Recall that $i_!(e_G^{\mathcal{U}}) = e_G^{\mathcal{V}}$ (or more briefly $i_!(e_G) = e_G$), and similarly for H. Using Proposition 4.11 we see that $i_!(e_G) \otimes i_!(e_H)$ can be expressed in terms of the wide subgroups $W \leq G \times H$ such that $W \in \mathcal{V}$, whereas $i_!(e_G \otimes e_H)$ is similar but involves only groups W that lie in $\mathcal{U}$. However, the submultiplicativity condition ensures that any wide

subgroup $W \leq G \times H$ lies in $\mathcal{U}$, so we see that $i_!(e_G) \otimes i_!(e_H) = i_!(e_G \otimes e_H)$. We also have $i_!(\mathbb{1}) = i_!(e_1) = e_1 = \mathbb{1}$. This shows that $i_!$ is strongly monoidal. Now suppose that i has a left adjoint q as in (i). Then the comma category $T \downarrow \mathcal{U}$ is equivalent to $qT \downarrow \mathcal{U}$ which has a terminal object $(qT \xrightarrow{1} qT)$ giving $Y(T) = X(qT)$. It follows that q^* and $i_!$ are naturally isomorphic as claimed. In claim (j), all the statements about i^* are straightforward. For any $X \in \mathcal{AV}$ we have

$$\mathcal{AV}(i_!(e_{G,V}), X) = \mathcal{AU}(e_{G,V}, i^*(X)) = \mathcal{M}_G(V, X(G)) = \mathcal{AV}(e_{G,V}, X)$$

where we used Lemma 2.9. It follows by the Yoneda Lemma that $i_!(e_{G,V}) = e_{G,V}$, and a similar proof gives that $i_*(t_{G,V}) = t_{G,V}$. The remaining claims in (j) follows from (f) and (g) as the functor $i_!$ and i_* are extension by zero. $\qquad\square$

REMARK 5.4. Part (f) of the lemma gives conditions under which $i_!$ preserves tensor products. However, this does not always hold if we remove those conditions, as shown by the following counterexample. Take $\mathcal{U} = \mathcal{C}[2^\infty]$ (the family of cyclic 2-groups). Note that the only wide subgroups of $C_4 \times C_2$ are the whole group $C_4 \times C_2$ and the graph subgroup $\mathrm{Gr}(\pi) \simeq C_4$ of the canonical projection $\pi : C_4 \to C_2$. Using Proposition 4.11, we see that $e_{C_2} \otimes e_{C_4} \simeq e_{C_4 \times C_2} \oplus e_{\mathrm{Gr}(\pi)}$ in $\mathcal{A}$ but $e_{C_2}^{\mathcal{U}} \otimes e_{C_4}^{\mathcal{U}} \simeq e_{\mathrm{Gr}(\pi)}^{\mathcal{U}}$ in $\mathcal{AU}$. Thus, the canonical map

$$e_{\mathrm{Gr}(\pi)} = i_!(e_{\mathrm{Gr}(\pi)}^{\mathcal{U}}) \simeq i_!(e_{C_2}^{\mathcal{U}} \otimes e_{C_4}^{\mathcal{U}}) \to i_!(e_{C_2}^{\mathcal{U}}) \otimes i_!(e_{C_4}^{\mathcal{U}}) = e_{C_4 \times C_2}$$

is not an isomorphism in $\mathcal{A}$.

LEMMA 5.5. *Let $\mathcal{V}$ be a replete full subcategory of $\mathcal{G}$. Let $\mathcal{U}$ and $\mathcal{W}$ be two replete full subcategories of $\mathcal{V}$ that are complements of each other, with inclusions $i : \mathcal{U} \to \mathcal{V}$ and $j : \mathcal{W} \to \mathcal{V}$. Suppose that $\mathcal{U}$ is closed upwards, or equivalently, that $\mathcal{W}$ is closed downwards. Then:*

(a) *The functor $i_! : \mathcal{AU} \to \mathcal{AV}$ admits a left adjoint $i^! : \mathcal{AV} \to \mathcal{AU}$ given by $i^!(Y) = i^*(\mathrm{cok}(j_! j^*(Y) \to Y))$.*

(b) *The functor $j_* : \mathcal{AW} \to \mathcal{AV}$ admits a right adjoint $j^\sharp : \mathcal{AV} \to \mathcal{AW}$ given by $j^\sharp(X) = i^*(\ker(X \to j_* j^*X))$.*

Proof. We will only prove (a) as the argument for (b) is similar. Consider a morphism $u : Y \to i_!(X)$. This fits in a naturality square as follows:

$$
\begin{array}{ccc}
j_! j^*(Y) & \xrightarrow{\ \epsilon_Y\ } & Y \\
{\scriptstyle j_! j^*(u)}\big\downarrow & & \big\downarrow{\scriptstyle u} \\
j_! j^* i_!(X) & \xrightarrow[\ \epsilon_{i_!(X)}\ ]{} & i_!(X).
\end{array}
$$

Lemma 5.3(f) tells us that $i_!$ is extension by zero, so $j^* i_! = 0$, so the bottom left corner of the square is zero, so there is a unique morphism $\overline{u} : \mathrm{cok}(\epsilon_Y) \to i_!(X)$ induced by u. We can now compose $i^*(\overline{u})$ with the inverse of the unit map

$X \to i^* i_!(X)$ to get a morphism $u^\#\colon i^!(Y) \to X$. We leave it to the reader to check that this construction gives the required bijection $\mathcal{AV}(Y, i_!(X)) \simeq \mathcal{AU}(i^!(Y), X)$. $\qquad\square$

We can use the formalism of change of subcategory to construct functorial projective and injective resolutions.

CONSTRUCTION 5.6. As in Remark 5.2, we let $\mathcal{U}^\times$ denote the subcategory with the same objects as $\mathcal{U}$ but only isomorphisms as morphisms. Let $l\colon \mathcal{U}^\times \to \mathcal{U}$ be the inclusion, and consider the functors $l_!, l_*\colon \mathcal{AU}^\times \to \mathcal{AU}$. If we choose a skeleton $\mathcal{U}' \subset \mathcal{U}$, it is not hard to check that

$$l_!(W) = \bigoplus_{G \in \mathcal{U}'} e_{G, W(G)}$$

$$l_*(W) = \prod_{G \in \mathcal{U}'} t_{G, W(G)}$$

for all $W \in \mathcal{AU}^\times$. It follows that $l_!(W)$ is always projective and $l_*(W)$ is always injective. Moreover, we see that the counit $l_! l^*(X) \to X$ is always an epimorphism for all $X \in \mathcal{AU}$, and the unit $X \to l_* l^*(X)$ is always a monomorphism. We now set

$$P_0 = l_! l^*(X) \quad P_1 = l_! l^*(\ker(P_0 \to X)) \quad P_{i+2} = l_! l^*(\ker(P_{i+1} \to P_i)) \ \forall i \ge 0$$
$$I_0 = l_* l^*(X) \quad I_1 = l_* l^*(\mathrm{cok}(X \to I_0)) \quad I_{i+2} = l_* l^*(\mathrm{cok}(I_i \to I_{i+1})) \ \forall i \ge 0$$

Then $P_\bullet \to X$ and $X \to I_\bullet$ define functorial projective and injective resolutions of X, respectively.

Recall the base of an object

REMARK 5.7. Recall from Definition 2.6 that

$$\mathrm{base}(X) = \min\{|G| \mid X(G) \ne 0\}.$$

If $|G| < \mathrm{base}(X)$ then we find that $(l_! l^*(X))(G) = 0$, and if $|G| = \mathrm{base}(X)$ we find that the counit map $(l_! l^*(X))(G) \to X(G)$ is an isomorphism. Using this, we see that $\mathrm{base}(P_k) \ge \mathrm{base}(X) + k$ for all $k \ge 0$. Thus, our canonical projective resolution is convergent in a convenient sense.

We now give other useful constructions and examples that we will use later on.

CONSTRUCTION 5.8. Let $\mathcal{V} \le \mathcal{G}$ be a subcategory with only finitely many isomorphism classes. Let $\mathcal{V}^\star$ be the submultiplicative closure of $\mathcal{V}$, so $G \in \mathcal{V}^*$ iff G is isomorphic to a subgroup of $\prod_{i=1}^n H_i$ for some family of groups $H_i \in \mathcal{V}$. For a finitely generated group F we put

$$\mathcal{K}(F; \mathcal{V}) = \{N \lhd F \mid F/N \in \mathcal{V}\}.$$

We can choose a finite list of groups containing one representative of each isomorphism class in $\mathcal{V}$, then each group in $\mathcal{K}(F;\mathcal{V})$ will occur as the kernel of one of the finitely many surjective homomorphisms from F to one of these groups. It follows that $\mathcal{K}(F;\mathcal{V})$ is a finite collection of normal subgroups of finite index. We define $\mathcal{N}(F;\mathcal{V})$ to be the intersection of all the subgroups in $\mathcal{K}(F;\mathcal{V})$, then we put $q_{\mathcal{V}}(F) = F/\mathcal{N}(F;\mathcal{V})$. This is isomorphic to the image of the natural map

$$F \to \prod_{N \in \mathcal{K}(F;\mathcal{V})} F/N,$$

so the submultiplicativity condition ensures that $q_{\mathcal{V}}(F) \in \mathcal{V}^{\star}$. It is straightforward to check that any surjective homomorphism $\phi\colon F_0 \to F_1$ has $\phi(\mathcal{N}(F_0;\mathcal{V})) \leq \mathcal{N}(F_1;\mathcal{V})$ and so induces a homomorphism $q_{\mathcal{V}}(F_0) \to q_{\mathcal{V}}(F_1)$. This makes $q_{\mathcal{V}}$ into a functor on the category of finitely generated groups and surjective homomorphisms. If we restrict to finite groups, then the functor $q_{\mathcal{V}}$ is the left adjoint to the inclusion $i\colon \mathcal{V}^{\star} \to \mathcal{G}$. We therefore have $i_{!} = q_{\mathcal{V}}^*$ by Lemma 5.3(i).

EXAMPLE 5.9. Let $\mathcal{U}$ be a submultiplicative subcategory of $\mathcal{G}$ and fix an integer $n \geq 1$. If we take $\mathcal{V} = \mathcal{U}_{\leq n}$ then $\mathcal{V}^{\star} \subseteq \mathcal{U}$. In this case we will use the abbreviated notation $q_{\leq n}(F), \mathcal{K}_{\leq n}(F)$ and $\mathcal{N}_{\leq n}(F)$.

EXAMPLE 5.10. Let $\mathcal{U} \leq \mathcal{G}$ be a submultiplicative subcategory and fix an integer $n \geq 1$. For any finite set X of cardinality n, let FX be the free group on X. Then we put $TX = q_{\leq n}(FX) \in \mathcal{U}_{\leq n}^{\star} \subseteq \mathcal{U}$. This is finite and functorial for bijections of X. If G is any group in $\bar{\mathcal{U}}_{\leq n}$, then we can choose a surjective map $X \to G$, and extend it to a surjective homomorphism $FX \to G$. The kernel of this homomorphism is in $\mathcal{K}_{\leq n}(FX)$ and so contains $\mathcal{N}_{\leq n}(FX)$, so we get an induced surjective homomorphism $TX \to G$. In particular, we can take $X = G$ and use the identity map to get a canonical epimorphism $\epsilon\colon TG \to G$.

We now consider a natural filtration on objects of $\mathcal{AU}$ which will be useful later on.

CONSTRUCTION 5.11. Consider an object $X \in \mathcal{AU}$. For $n \geq 0$, we let $L_{\leq n}X$ denote the image of the counit map $i_{!}^{\leq n}i_{\leq n}^{*}X \to X$. By construction, $L_{\leq n}X$ is the smallest subobject of X containing $X(H)$ for all $H \in \mathcal{U}_{\leq n}$ This gives a filtration

$$0 = L_{\leq 0}X \leq L_{\leq 1}X \leq \cdots \leq L_{\leq n}X \leq L_{\leq n+1}X \leq \cdots \leq X$$

with subquotients denoted by $L_n X$. Consider a map $f\colon X \to Y$ and an element $x \in (L_{\leq n}X)(G)$. We can write $x = \sum_{i=1}^{s} \alpha_i^*(x_i)$ where $x_i \in X(H_i)$ with $|H_i| \leq n$ and $\alpha_i \in \mathcal{U}(G, H_i)$. Note that

$$f(x) = \sum_i f\alpha_i^*(x) = \sum_i \alpha_i^* f(x) \in (L_{\leq n}Y)(G),$$

so the filtration is natural in X. Therefore we also have induced maps $L_{\leq n}f\colon L_{\leq n}X \to L_{\leq n}Y$ and $L_n f\colon L_n X \to L_n Y$ for all n.

EXAMPLE 5.12. For all $G \in \mathcal{U}$, we have

$$L_{\leq n}(e_G) = \begin{cases} 0 & \text{if } n < |G| \\ e_G & \text{if } n \geq |G|. \end{cases}$$

From this we see that $L_n(e_G) = e_G$ if $|G| = n$, and $L_n(e_G) = 0$ otherwise.

CONSTRUCTION 5.13. Consider an object $X \in \mathcal{AU}$. We define

$$(QX)(G) = X(G) / \sum_{1 \neq N \triangleleft G} \pi^* X(G/N)$$

where $\pi \colon G \to G/N$ denotes the projection. Equivalently, if $|G| = n$ then this is

$$(QX)(G) = \mathrm{cok}(i_!^{<n} i_{<n}^* X \to X)(G) = X(G) / X_{<n}(G).$$

We refer to QX as the object of indecomposables of X. This is functorial for isomorphisms of G, so we can regard Q as a functor $\mathcal{AU} \to \mathcal{AU}^\times$. In fact, it is not hard to see that Q is left adjoint to the functor $l_! \colon \mathcal{AU}^\times \to \mathcal{AU}$, and that the counit map $Ql_!(W) \to W$ is an isomorphism for all $W \in \mathcal{AU}^\times$. (Indeed, it is sufficient to check this in the case $W = e_{G,V}$.)

LEMMA 5.14. *If $f \colon X \to Y$ in $\mathcal{AU}$ and Qf is an epimorphism, then f is an epimorphism.*

Proof. We will show by induction on $n = |G|$ that $f(G) \colon X(G) \to Y(G)$ is surjective. If $n = 1$, then $f(G) = (Qf)(G)$ is surjective by assumption. Now suppose that $n > 1$ and consider the diagram

$$\begin{array}{ccccccc} (i_!^{<n} i_{<n}^* X)(G) & \longrightarrow & X(G) & \longrightarrow & (QX)(G) & \longrightarrow & 0 \\ \downarrow{\scriptstyle (i_!^{<n} i_{<n}^* f)(G)} & & \downarrow{\scriptstyle f(G)} & & \downarrow{\scriptstyle (Qf)(G)} & & \\ (i_!^{<n} i_{<n}^* Y)(G) & \longrightarrow & Y(G) & \longrightarrow & (QY)(G) & \longrightarrow & 0. \end{array}$$

By induction we know that $i_{<n}^* f$ is an epimorphism and it follows that the left vertical map in an epimorphism. As $(Qf)(G)$ is an epimorphism too, the claim follows from the four lemma. $\square$

REMARK 5.15. We can now use Q to build minimal projective resolutions. Consider an object $X \in \mathcal{AU}$. Then QX is a quotient of $l^* X$ in the semisimple category $\mathcal{AU}^\times$, so we can choose a section $QX \to l^* X$. By passing to the adjoint, we get an morphism $e \colon P_0' \to X$, where $P_0' = l_! QX$. We find that Qe is an isomorphism, so e is an epimorphism. We can iterate this in the same way as in Construction 5.6 to get a projective resolution $P_\bullet'$ which is minimal in the sense that the differential $d_k \colon P_k' \to P_{k-1}'$ has $Q(d_k) = 0$ for all k. As is familiar for minimal resolutions in other contexts, it follows that $P_\bullet'$ is a summand in any other projective resolution of X.

6 Simple objects

In this section we classify the simple objects and show that $\mathcal{AU}$ is semisimple if and only if $\mathcal{U}$ is a groupoid.

DEFINITION 6.1. Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$.

- An object $X \in \mathcal{AU}$ is *simple* if the only subobjects are 0 and X.

- An object $X \in \mathcal{AU}$ is *semisimple* if it is a sum of simple objects.

- The abelian category $\mathcal{AU}$ is *semisimple* if every object is semisimple.

- The abelian category $\mathcal{AU}$ is *split* if every short exact sequence in $\mathcal{AU}$ splits. Equivalently, every object of $\mathcal{AU}$ is both injective and projective

We immediately get the following result.

LEMMA 6.2. *An object $X \in \mathcal{AU}$ is simple if and only if it is isomorphic to $s_{G,V}$ for some G and some irreducible $k[\mathrm{Out}(G)]$-module V.*

Proof. Consider a simple object $X \in \mathcal{AU}$. Choose G of minimal order so that $X(G) \neq 0$. It is standard that the category of $k[\mathrm{Out}(G)]$-modules is semisimple, so we can choose a simple quotient V of $X(G)$ in this category. The projection $X(G) \to V$ is adjoint to a morphism $X \to t_{G,V}$ in $\mathcal{AU}$. As $X(H) = 0$ when $|H| < |G|$, we see that this factors through the subobject $s_{G,V} \leq t_{G,V}$. The morphism $X \to s_{G,V}$ is then an epimorphism whose kernel is a proper subobject, and so must be zero by simplicity. Thus $X \simeq s_{G,V}$ as required. $\qquad\square$

We are now ready to study when our abelian category is semisimple.

PROPOSITION 6.3. *The following are equivalent:*

(a) $\mathcal{U}$ is a groupoid;

(b) the abelian category $\mathcal{AU}$ is split;

(c) the abelian category $\mathcal{AU}$ is semisimple.

Proof. The fact that (b) and (c) are equivalent is well-known and proved for instance in [23, V.6.7]. It is also standard that (a) implies (b); the argument will be recalled as Proposition 7.3(a) below. Thus, we need only prove that (b) implies (a), or the contrapositive of that. Suppose that $\mathcal{U}$ is not a groupoid so there exists an epimorphism $\varphi \colon G \to H$ which is not an isomorphism. Consider the canonical epimorphism $\pi \colon e_{H,k} \to s_{H,k}$. The map $\varphi^* \colon e_{H,k}(H) \to e_{H,k}(G)$ is easily seen to be injective. The map $\varphi^* \colon s_{H,k}(H) \to s_{H,k}(G)$ is of the form $k \to 0$ and so is not injective. It follows that $s_{H,k}$ cannot be a retract of $e_{H,k}$, so π cannot split. Thus, $\mathcal{AU}$ is not a split abelian category. $\qquad\square$

7 Finite groupoids

In this section we study the abelian category $\mathcal{AU}$ in the special case that $\mathcal{U} \leq \mathcal{G}$ is a finite groupoid. For example we could take $\mathcal{U} = \{G \in \mathcal{G} \mid |G| = n\}$.

LEMMA 7.1. *Suppose we choose a list of groups $G_1, \ldots, G_r$ containing precisely one representative of each isomorphism class of groups in $\mathcal{U}$, so $\mathcal{G}(G_i, G_j) = \emptyset$ for $i \neq j$. Let $\mathcal{M}_i$ be the category of modules for the group ring $k[\mathrm{Out}(G_i)]$ and put $\mathcal{M} = \prod_{i=1}^r \mathcal{M}_i$. Then the functor $X \mapsto (X(G_i))_{i=1}^r$ gives an equivalence of categories $\mathcal{AU} \to \mathcal{M}$.*

Proof. The inverse functor is given by $(V_i)_{i=1}^r \mapsto \bigoplus_{i=1}^r e_{G_i, V_i}$. $\qquad\square$

REMARK 7.2. Suppose that $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{G}$ with $\mathcal{U}$ a finite groupoid and $\mathcal{V}$ a subcategory, and let $i \colon \mathcal{U} \to \mathcal{V}$ denote the inclusion. After choosing a list of groups $G_1, \ldots, G_r \in \mathcal{U}$ as in Lemma 7.1, we have identifications

$$i_! = \bigoplus_{i=1}^r e_{G_i, \bullet} \quad \text{and} \quad i_* = \bigoplus_{i=1}^r t_{G_i, \bullet}.$$

PROPOSITION 7.3. *Suppose that $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{G}$ with $\mathcal{U}$ a finite groupoid and $\mathcal{V}$ a subcategory, and let $i \colon \mathcal{U} \to \mathcal{V}$ denote the inclusion.*

(a) *All monomorphisms and epimorphisms in $\mathcal{AU}$ are split.*

(b) *All objects in $\mathcal{AU}$ are both injective and projective.*

(c) *All objects in the image of $i_!$ are projective, and all objects in the image of i_* are injective.*

(d) *The functor $i_!$ preserves all limits and colimits, as does the functor i_*.*

Proof. We identify $\mathcal{AU}$ with $\mathcal{M}$ as in Lemma above. Maschke's Theorem shows that (a) and (b) hold in $\mathcal{M}_i$, and it follows that they also hold in $\mathcal{M}$ and $\mathcal{AU}$. If $X \in \mathcal{AU}$ then the functor $\mathcal{AV}(i_!(X), -)$ is isomorphic to $\mathcal{AU}(X, i^*(-))$. Here i^* and $\mathcal{AU}(X, -)$ preserve epimorphisms, so $i_!(X)$ is projective. Similarly, we see that $i_*(X)$ is injective, which proves (c).

We next claim that i_* preserves all limits and colimits. As it is a right adjoint it is enough to show that it preserves all colimits. By Remark 7.2, it is enough to show that the functor $t_{G_k, \bullet}$ preserves colimits for all $1 \leq k \leq r$. Choose $f_1, \ldots, f_s \in \mathcal{G}(G_k, G)$, containing precisely one element from each $\mathrm{Out}(G_k)$-orbit. Let $\Delta_s \leq \mathrm{Out}(G_k)$ be the stabiliser of f_s. We find that

$$t_{G_k, V}(G) = \mathrm{Hom}_{k[\mathrm{Out}(G_k)]}(k[\mathcal{G}(G_k, G)], V) = \prod_s V^{\Delta_s},$$

and this is easily seen to preserve all colimits as required. A similar argument shows that $i_!$ preserves all limits and colimits. As before, it is enough to show that the functor $e_{G_k, \bullet}$ preserves all limits. We find that

$$e_{G_k, V}(G) = k[\mathcal{G}(G, G_k)] \otimes_{k[\mathrm{Out}(G_k)]} V = \bigoplus_s V_{\Delta_s}$$

and this is easily seen to preserve all limits.   □

The following results are standard.

Proposition 7.4.

  (a) *The simple objects of $\mathcal{AU}$ are the same as the indecomposable objects, and these are precisely the objects $e_{G,V}$ for some $G \in \mathcal{U}$ and irreducible $k[\mathrm{Out}(G)]$-module V.*

  (b) *Every nonzero morphism to a simple object is a split epimorphism, and every nonzero morphism from a simple object is a split monomorphism.*

  (c) *If S and S' are non-isomorphic simple objects in $\mathcal{AU}$, then $\mathcal{AU}(S,S') = 0$.*

  (d) *If S is a simple object in $\mathcal{AU}$, then $\mathrm{End}(S)$ is a division algebra of finite dimension over k.*

  (e) *The category $\mathcal{AU}$ has finitely many isomorphism classes of simple objects.*

  (f) *Suppose that the list $S_1, \ldots, S_s$ contains precisely one simple object from each isomorphism class, and put $D_j = \mathrm{End}(S_j)$. Let $\mathcal{N}_j$ be the category of right modules over D_j, and put $\mathcal{N} = \prod_j \mathcal{N}_j$. Define functors*

$$\mathcal{AU} \xrightarrow{\phi} \mathcal{N} \xrightarrow{\psi} \mathcal{AU}$$

  by $\phi(X)_j = \mathcal{AU}(S_j, X)$ and $\psi(N) = \bigoplus_j N_j \otimes_{D_j} S_j$. Then ϕ and ψ are inverse to each other, and so are equivalences.

Proof. The first part of (a) is clear from the fact that all monomorphisms are split. As any morphism in $\mathcal{U}$ is an isomorphism we see that $e_{G,V} = s_{G,V}$ and this is simple when V is irreducible, see Lemma 6.2. For (b), suppose that $\alpha \colon X \to S$ is nonzero, where S is simple. Then $\mathrm{image}(\alpha)$ is a nonzero subobject of S, so it must be all of S, so α is an epimorphism, and all epimorphisms are split. This gives half of (b), and the other half is similar. Now suppose that $\alpha \colon S \to S'$, where both S and S' are simple. If $\alpha \neq 0$ then (b) tell us that α is both a split monomorphism and a split epimorphism, so it is an isomorphism. The contrapositive gives claim (c), and the special case $S' = S$, gives most of (d), apart from the finite-dimensionality statement. For that, we choose a list of groups G_i as in Lemma 7.1, and put $U = \bigoplus_i e_{G_i}$ which is a generator for $\mathcal{AU}$. We can decompose U as a finite direct sum of indecomposables, say $U = \bigoplus_{j=1}^s S_j^{d_j}$ with $0 < d_j < \infty$ and $S_j \not\simeq S_k$ for $j \neq k$. If S is simple, there is an nonzero map $U \to S$ and so a nonzero map $S_j \to S$ for some j, that has to be an isomorphism from (b). This proves (e). We also note that S is a summand in U, so $\mathrm{End}(S)$ is a summand in $\mathrm{End}(U)$ and hence it has finite dimension over k, completing the proof of (d).

Now define ϕ and ψ as in (f). Put $T_m = \psi(S_m) \in \mathcal{N}$, so $(T_m)_m = D_m$ and $(T_m)_j = 0$ for $j \neq m$. Define

$$\eta_N \colon N \to \phi\psi(N) = \mathcal{AU}(S_j, \bigoplus_k N_k \otimes_{D_k} S_k)$$

$$\epsilon_X \colon \psi\phi(X) = \bigoplus_j \mathcal{AU}(S_j, X) \otimes_{D_j} S_j \to X$$

as follows. First, any $n \in N_j$ gives a map $D_j \to N_j$ and thus a map

$$S_j = S_j \otimes_{D_j} D_j \to S_j \otimes_{D_j} N_j \leq \bigoplus_k N_k \otimes_{D_k} S_k$$

we take this to be the j-th component of η_N. Similarly, there is an evaluation morphism $\mathcal{AU}(S_j, X) \otimes S_j \to X$, which is easily seen to factor through $\mathcal{AU}(S_j, X) \otimes_{D_j} S_j$. We combine these maps to give ϵ_X.

We claim that ϵ_X is an isomorphism. Indeed, we know that the object U is a generator for $\mathcal{AU}$, so the objects S_j form a generating family. As all epimorphisms in $\mathcal{AU}$ split, we see that every object is a retract of a direct sum of objects of the form S_m. We also see that both ϕ and ψ preserve all direct sums. It will therefore suffice to check that ϵ_{S_m} is an isomorphism, and this follows easily from our description of $T_m = \psi(S_m)$.

Because every module over a division algebra is free, we also see that every object of $\mathcal{N}$ is a direct sum of objects of the form T_m. It is easy to see that η_{T_m} is an isomorphism, and it follows that η_N is an isomorphism for all N. $\qquad\square$

8 PROJECTIVES

In this section we study and classify the projective objects of $\mathcal{AU}$ for a replete full subcategory $\mathcal{U}$ of $\mathcal{G}$.

LEMMA 8.1. *Consider an object P in $\mathcal{AU}$. Then the following are equivalent:*

(a) *P is projective in $\mathcal{AU}$.*

(b) *P is isomorphic to a retract of a direct sum of objects of the form e_G with $G \in \mathcal{U}$.*

Proof. First suppose that (a) holds. Let $\mathcal{U}_0$ be a countable collection of objects of $\mathcal{U}$ that contains at least one representative of every isomorphism class. Put

$$FP = \bigoplus_{G \in \mathcal{U}_0} \bigoplus_{x \in P(G)} e_G \in \mathcal{AU}.$$

Each pair (G, x) defines a morphism $\epsilon_{(G,x)} \colon e_G \to P$ by the Yoneda Lemma. By combining these for all pairs (G, x), we get a morphism $\epsilon \colon FP \to P$ which is an epimorphism by construction. As P is assumed to be projective this

epimorphism must split, so P is a retract of FP, so (b) holds. Conversely, suppose that P is as in (b) and note that e_G is projective since $\mathcal{AU}(e_G, -)$ is exact by the Yoneda Lemma. Sums and retracts of projective objects is again projective so (a) follows. $\square$

LEMMA 8.2. *Let* $i: \mathcal{U} \to \mathcal{V}$ *be an inclusion of replete full subcategories of* $\mathcal{G}$, *and let* P *be an object of* $\mathcal{AU}$. *Then* P *is projective in* $\mathcal{AU}$ *iff* $i_!(P)$ *is projective in* $\mathcal{AV}$.

Proof. First, if P is projective then the functor $\mathcal{AV}(i_!(P), -)$ is isomorphic to the composite of the exact functors $i^*: \mathcal{AV} \to \mathcal{AU}$ and $\mathcal{AU}(P, -)$, so it is exact, so $i_!(P)$ is projective.
Conversely, suppose that $i_!(P)$ is projective. We can certainly choose a projective object $Q \in \mathcal{AU}$ and an epimorphism $u: Q \to P$. As $i_!$ is a left adjoint, it preserves colimits and epimorphisms, so $i_!(u): i_!(Q) \to i_!(P)$ is an epimorphism. As $i_!(P)$ is assumed projective, we can choose $v: i_!(P) \to i_!(Q)$ with $i_!(u) \circ v = 1$. We now apply i^* to this, recalling that $i^* i_! \simeq 1$; we find that $i^*(v)$ gives a section for u, so u is a split epimorphism, so P is projective. $\square$

PROPOSITION 8.3. *Let* $i: \mathcal{U} \to \mathcal{V}$ *be an inclusion of replete full subcategories of* $\mathcal{G}$, *and let* Q *be an object of* $\mathcal{AV}$. *Then the following are equivalent:*

(a) $Q \simeq i_!(P)$ *for some projective object* $P \in \mathcal{AU}$.

(b) Q *is a retract of* $i_!(P)$ *for some projective object* $P \in \mathcal{AU}$.

(c) Q *is a retract of some direct sum of objects* e_G, *with* $G \in \mathcal{U}$.

(d) Q *is projective, and the counit map* $i_! i^* Q \to Q$ *is an isomorphism.*

Moreover, if these conditions hold then $i^*(Q)$ *is projective in* $\mathcal{AU}$.

Proof. From what we have seen already it is clear that $(a) \Rightarrow (b) \Leftrightarrow (c)$ and that $(a) \Rightarrow (d)$. Now suppose that (b) holds, so there is a projective object $P \in \mathcal{AU}$ and an idempotent $e: i_! P \to i_! P$ with $Q = e.(i_! P) = \operatorname{cok}(1 - e)$. As $i_!$ is full and faithful, there is an idempotent $f: P \to P$ with $i_!(f) = e$. As $i_!$ preserves cokernels, it follows that $Q = i_!(f.P)$, and of course $f.P$ is projective, so (a) holds. Also, if $Q \simeq i_! P$ as in (a) holds then $i^* Q$ is isomorphic to P and so is projective.
Now all that is left is to prove that $(d) \Rightarrow (b)$. Suppose that Q is projective, and that the counit map $i_! i^* Q \to Q$ is an isomorphism. Choose a projective $P \in \mathcal{AU}$ and an epimorphism $f: P \to i^* Q$. As $i_!$ preserves epimorphisms, we see that $i_!(f): i_! P \to i_! i^* Q \simeq Q$ is an epimorphism, but Q is projective, so Q is a retract of $i_! P$ as required. $\square$

Recall the functors $L_{\leq n}$ and L_n from Construction 5.11. Recall also that we put $\mathcal{U}_n = \{G \in \mathcal{U} \mid |\overline{G}| = n\}$ (which is a groupoid), and we write i_n for the inclusion $\mathcal{U}_n \to \mathcal{U}$.

PROPOSITION 8.4. *If P is projective in $\mathcal{AU}$, then the filtration $L_{\leq *}P$ can be split, so there is an unnatural isomorphism $P \simeq \bigoplus_n L_n P$, and the filtration quotients $L_n P$ are themselves projective. Furthermore, $i_n^*(L_n P)$ is projective in $\mathcal{AU}_n$ and $(i_n)_!(i_n^* L_n P) = L_n P$.*

Proof. We have seen that P can be written as a retract of a direct sum of generators. In more detail, we can choose an object $Q = \bigoplus_\alpha e_{G_\alpha}$ and an idempotent $u \colon Q \to Q$ such that $P \simeq u.Q$, so without loss of generality $P = u.Q$. Let Q_n be the sum of the terms e_{G_α} for which $|G_\alpha| = n$, so that $Q = \bigoplus_n Q_n$. We can then decompose u as a sum of morphisms $u_{nm} \colon Q_m \to Q_n$. When $m < n$ we have $\mathcal{AU}(Q_m, Q_n) = 0$ and so $u_{nm} = 0$. Given this, the relation $u^2 = u$ implies that $u_{nn}^2 = u_{nn}$. The object $P_n' = u_{nn}.Q_n$ is therefore projective. Put $P' = \bigoplus_n P_n'$ and let $f \colon P' \to P$ be the composite $P' \xrightarrow{\text{inc}} Q \xrightarrow{u} u.Q = P$. We claim that this is an isomorphism. By passing to the colimit, it will suffice to show that $L_{\leq n}(f)$ is an isomorphism for all n. By an evident reduction, it will suffice to show that $L_n(f)$ is an isomorphism for all n. As L_n is an additive functor we have $L_n(P) = L_n(u).L_n(Q) = L_n(u).Q_n = u_{nn}.Q_n = P_n'$, as required. All remaining claims are now easy. $\square$

COROLLARY 8.5. *Suppose we choose a complete system of simple objects in $\mathcal{AU}_n$ for all n, giving a sequence $(e_{G_i, S_i} \mid G_i \in \mathcal{U}_n)_n$ of indecomposable projectives in $\mathcal{AU}$. Then every projective object is a direct sum of objects of the form e_{G_i, S_i}. In particular, every indecomposable projective is isomorphic to some e_{G_i, S_i}.*

Proof. Because $\mathcal{AU}_n$ is semisimple, we see that $i_n^*(L_n P)$ splits in the indicated way. As $L_n P \simeq (i_n)_!(i_n^*(L_n P))$, we see that $L_n P$ also splits, as does $\bigoplus_n L_n P$, which is isomorphic to P. $\square$

PROPOSITION 8.6. *Any projective object P can be written as $P \simeq \prod_n L_n P$. Furthermore, products of projective objects are projective.*

Proof. By Proposition 8.4 we can write $P = \bigoplus_n L_n P$. Now note that for a fixed $G \in \mathcal{U}$, there are only finitely many indices n such that $P_n(G)$ is nonzero, so the inclusion $\bigoplus_n L_n P \to \prod_n L_n P$ is an isomorphism.
For the second claim, let (P_α) be a family of projectives, and put $P = \prod_\alpha P_\alpha$. We can write $P_\alpha = \prod_k L_k P_\alpha$ as above, so $P = \prod_k Q_k$ where $Q_k = \prod_\alpha L_k P_\alpha$. We know from Proposition 7.3 that $(i_k)_!$ preserves products, so Q_k is in the image of $(i_k)_!$. It follows that Q_k is projective and also that $P = \prod_k Q_k$ is the same as $\bigoplus_k Q_k$, so P is projective. $\square$

PROPOSITION 8.7. *Let $\mathcal{U}$ be a widely closed subcategory of $\mathcal{G}$. Then the full subcategory of projective objects is closed under tensor products. If $\mathcal{U}$ is a multiplicative global family, then the full subcategory of projective objects is also closed under the internal homs.*

Proof. Consider projective objects $P, Q \in \mathcal{AU}$. We can write P as a retract of a direct sum of terms e_G. The functor $(-) \otimes Q$ sends sums to sums, and

the functor $\underline{\mathrm{Hom}}(-, Q)$ sends sums to products, and both sums and products of projectives are projective. We can therefore reduce to the case $P = e_G$. Next, we can split Q as a direct sum or product of terms $L_n Q$. The functor $e_G \otimes (-)$ preserves sums, and the functor $\underline{\mathrm{Hom}}(e_G, -)$ preserves products, so we can reduce to the case where $Q = L_n Q$, or equivalently $Q = (i_n)_!(M)$ for some $M \in \mathcal{AU}_n$. We can now write M as a retract of a sum of terms e_{H_α} with $|H_\alpha| = n$. We know from Proposition 4.11 that $e_G \otimes e_{H_\alpha}$ is projective, and it follows easily that $e_G \otimes Q$ is projective as claimed.

It also follows from Proposition 4.11, together with the formula $\underline{\mathrm{Hom}}(e_G, Z)(H) = \mathcal{AU}(e_G \otimes e_H, Z)$, that the functor $\underline{\mathrm{Hom}}(e_G, -)$ preserves sums. If $\mathcal{U}$ is a multiplicative global family, then Theorem 4.18 tells us that $\underline{\mathrm{Hom}}(e_G, e_{H_\alpha})$ is projective. From these two facts it follows that $\underline{\mathrm{Hom}}(e_G, Q)$ is also projective, which finishes the proof. $\square$

9   Colimit-exactness

Let $\mathcal{U}$ be a subcategory of $\mathcal{G}$. In this section we will write L for the colimit functor $X \mapsto \varinjlim_{G \in \mathcal{U}^{\mathrm{op}}} X(G)$ from $\mathcal{AU}$ to Vect_k. Recall that $\mathcal{U}$ is said to be *colimit-exact* if L is an exact functor. We will show that most of our examples have this property. First, however, we give an equivalent condition.

Proposition 9.1. *There is a natural isomorphism $\mathcal{AU}(X, \mathbb{1}) \simeq \mathrm{Vect}_k(LX, k)$. Thus, the object $\mathbb{1} \in \mathcal{AU}$ is injective if and only if $\mathcal{U}$ is colimit-exact. If so, then all objects of the form $DX = \underline{\mathrm{Hom}}(X, \mathbb{1})$ are also injective.*

Proof. The natural isomorphism $\mathcal{AU}(X, \mathbb{1}) \simeq \mathrm{Vect}_k(LX, k)$ is clear. The functor $V \mapsto V^* = \mathrm{Vect}_k(-, k)$ is certainly exact, so if L is exact then $\mathcal{AU}(-, \mathbb{1})$ is exact, so $\mathbb{1}$ is injective. Conversely, suppose that $\mathbb{1}$ is injective. For any short exact sequence $X \to Y \to Z$ in $\mathcal{AU}$, we deduce that the resulting sequence $(LZ)^* \to (LY)^* \to (LZ)^*$ is also short exact, and then linear algebra shows that $LX \to LY \to LZ$ is short exact as well. This proves that L is exact. Also, there is a natural isomorphism $\mathcal{AU}(X, DW) = \mathcal{AU}(W \otimes X, \mathbb{1})$. The functors $W \otimes (-)$ and $\mathcal{AU}(-, \mathbb{1})$ are exact, and it follows that DW is injective as claimed. $\square$

Lemma 9.2.

(a) *For any $G \in \mathcal{U}$ we have $L e_G = k$. In particular, if $\mathbb{1} \in \mathcal{U}$ then $L\mathbb{1} = L e_1 = k$.*

(b) *For any $G \in \mathcal{U}$ and any $k[\mathrm{Out}(G)]$-module V we have $L(e_{G,V}) = V_{\mathrm{Out}(G)}$ (the module of coinvariants).*

(c) *Unless G is maximal in $\mathcal{U}$ we also have $L(t_{G,V}) = L(s_{G,V}) = 0$.*

Proof. For $T \in \mathrm{Vect}_k$ we have

$$\mathrm{Vect}_k(L(e_{G,V}), T) = \mathcal{AU}(e_{G,V}, T \otimes \mathbb{1})$$
$$= \mathrm{Mod}_{k[\mathrm{Out}(G)]}(V, T) = \mathrm{Vect}_k(V_{\mathrm{Out}(G)}, T).$$

By the Yoneda Lemma, we therefore have $L(e_{G,V}) = V_{\mathrm{Out}(G)}$. Taking $V = k[\mathrm{Out}(G)]$ gives $L(e_G) = k$.

Now consider the object $L(t_{G,V})$. This is the colimit over $H \in \mathcal{U}$ of the groups $t_{G,V}(H) = \mathrm{Map}_{\mathrm{Out}(G)}(\mathcal{U}(G, H), V)$. If there are no morphisms $G \to H$, then $t_{G,V}(H) = 0$. If there is a morphism $\alpha \colon G \to H$, then by definition the limit map $t_{G,V}(H) \to L(t_{G,V})$ factors through α^*. This makes it clear that the map $t_{G,V}(G) \to L(t_{G,V})$ is surjective. Now suppose that G is not maximal in $\mathcal{U}$, so we can choose $\beta \colon K \to G$ in $\mathcal{U}$ that is not an isomorphism. The map $t_{G,V}(G) \to L(t_{G,V})$ will then factor through β^*, but the codomain of β^* is zero, so $L(t_{G,V}) = 0$. A simpler version of the same argument also gives $L(s_{G,V}) = 0$. $\square$

REMARK 9.3. For $X, Y \in \mathcal{AU}$ there are natural unit maps $X \to (LX) \otimes \mathbb{1}$ and $Y \to (LY) \otimes \mathbb{1}$. We can tensor these together and take adjoints to get a map $L(X \otimes Y) \to (LX) \otimes (LY)$. This gives an oplax monoidal structure on L. However, the map $L(X \otimes Y) \to (LX) \otimes (LY)$ is rarely an isomorphism. For example, we have $Le_G \otimes Le_H = k$ but Proposition 4.11 shows that $L(e_G \otimes e_H)$ is freely generated by the set $\mathrm{Wide}(G, H)/\text{conjugacy}$.

We now start to prove that various categories are colimit-exact. Our first example is easy:

PROPOSITION 9.4. *If $\mathcal{U} \leq \mathcal{G}$ is a groupoid, then it is colimit-exact.*

Proof. Choose a family $(G_i)_{i \in I}$ containing precisely one representative of each isomorphism class in $\mathcal{U}$. If $X \in \mathcal{AU}$ then the group $\mathrm{Out}(G_i)$ acts on $X(G_i)$, and we write $X(G_i)_{\mathrm{Out}(G_i)}$ for the module of coinvariants. As we work over a field of characteristic zero and $\mathrm{Out}(G_i)$ is finite, this is an exact functor of X. It is easy to identify $\varinjlim X$ with $\bigoplus_i X(G_i)_{\mathrm{Out}(G_i)}$, and this makes it clear that the colimit functor is exact as well. $\square$

For other examples we will use the following notion:

DEFINITION 9.5. A *colimit tower* for $\mathcal{U}$ is a diagram

$$G_0 \xleftarrow{\epsilon_0} G_1 \xleftarrow{\epsilon_1} G_2 \leftarrow \cdots$$

in $\mathcal{U}$ such that

(a) For every $H \in \mathcal{U}$ there is a pair (i, α) with $i \in \mathbb{N}$ and $\alpha \in \mathcal{U}(G_i, H)$.

(b) For every diagram $G_i \xrightarrow{\alpha} H \xleftarrow{\beta} G_i$ in $\mathcal{U}$ there exists $\gamma \in \mathcal{U}(G_{i+1}, G_{i+1})$ making the following diagram commute:

$$
\begin{array}{ccc}
G_{i+1} & \xrightarrow{\ \gamma\ } & G_{i+1} \\
{\scriptstyle \epsilon_i}\downarrow & & \epsilon_i\downarrow \\
G_i & \xrightarrow[\alpha]{} H \xleftarrow[\beta]{} & G_i.
\end{array}
$$

(c) For every diagram $G_{i+1} \xrightarrow{\alpha} H \xleftarrow{\phi} K$ in $\mathcal{U}$ with $\mathcal{U}(G_i, K) \neq \emptyset$, there exists $\beta \in \mathcal{U}(G_{i+1}, K)$ such that $\phi \circ \beta = \alpha$.

CONSTRUCTION 9.6. Suppose we have a colimit tower as above. For any $X \in \mathcal{AU}$ we define $\Lambda_i X$ to be the group of coinvariants $X(G_i)_{\mathrm{Out}(G_i)}$, and we let $\rho_i \colon X(G_i) \to \Lambda_i X$ be the obvious reduction map. By taking $H = G_i$ and $\beta = 1$ in condition (b), we see that every automorphism of G_i can be covered by an automorphism of G_{i+1}. It follows that there is a unique map $\Lambda_i X \to \Lambda_{i+1} X$ making the following diagram commute:

$$
\begin{array}{ccc}
X(G_i) & \xrightarrow{\ \epsilon_i^*\ } & X(G_{i+1}) \\
{\scriptstyle \rho_i}\downarrow & & \downarrow{\scriptstyle \rho_{i+1}} \\
\Lambda_i X & \longrightarrow & \Lambda_{i+1} X
\end{array}
$$

We will just write ϵ_i^* for this map. We define $\Lambda_\infty X$ to be the colimit of the sequence

$$
\Lambda_0 X \xrightarrow{\epsilon_0^*} \Lambda_1 X \xrightarrow{\epsilon_1^*} \Lambda_2 X \xrightarrow{\epsilon_2^*} \cdots,
$$

and we let σ_i denote the canonical map $\Lambda_i X \to \Lambda_\infty X$. As we are working over a field of characteristic zero and $\mathrm{Out}(G_n)$ is finite, we see that Λ_n is an exact functor. As sequential colimits are exact, we see that $\Lambda_\infty \colon \mathcal{AU} \to \mathrm{Vect}_k$ is also an exact functor.

PROPOSITION 9.7. *For any colimit tower, there is a natural isomorphism* $\Lambda_\infty X \to LX$. *Thus, if $\mathcal{U}$ has a colimit tower, then it is colimit-exact.*

Proof. Let $\theta_H \colon X(H) \to LX$ be the canonical morphism. It is formal that there is a unique map $\phi \colon \Lambda_\infty X \to LX$ with $\phi \sigma_i \rho_i = \theta_{G_i}$ for all i. In the opposite direction, suppose we have $H \in \mathcal{U}$. We can choose (i, α) as in condition (a) and define

$$
\psi_{H,i,\alpha} = \sigma_i \rho_i \alpha^* \colon X(H) \to \Lambda_\infty X.
$$

Using the obvious cone properties we see that this is the same as $\psi_{H,i+1,\alpha\epsilon_i}$, or as $\psi_{H,i,\alpha\mu}$ for any $\mu \in \mathrm{Out}(G_i)$. By using these rules together with condition (b), we see that $\psi_{H,i,\alpha}$ is independent of the choice of (i, α), so we can just denote it by ψ_H. It is now easy to see that for any $\zeta \colon H \to K$ we have $\psi_H \zeta^* = \psi_K \colon X(K) \to \Lambda_\infty X$. This means that there is a unique $\psi \colon LX \to \Lambda_\infty X$ with $\psi \theta_H = \psi_H$ for all H. This is clearly inverse to ϕ. $\qquad\square$

REMARK 9.8. So far we only used conditions (a) and (b) in the definition of colimit tower. Condition (c) will play an important role in Section 12.

EXAMPLE 9.9. Let $\mathcal{C}$ be the category of cyclic groups. The morphisms can be described as follows:

(a) If $|G| = n$ then the group $\mathrm{Aut}(G) = \mathrm{Out}(G)$ is canonically identified with $(\mathbb{Z}/n)^{\times}$.

(b) If $|H|$ divides $|G|$ then $\mathcal{C}(G, H)$ is a torsor for $\mathrm{Aut}(H)$. Moreover, for any $\alpha \colon G \to H$ and $\phi \in \mathrm{Aut}(H) = (\mathbb{Z}/|H|)^{\times}$ there exists $\psi \in \mathrm{Aut}(G) = (\mathbb{Z}/|G|)^{\times}$ that reduces to ϕ, and any such ψ satisfies $\alpha\psi = \phi\alpha$.

(c) On the other hand, if $|H|$ does not divide $|G|$ then $\mathcal{C}(G, H) = \emptyset$.

From these observations it follows easily that the groups $G_n = \mathbb{Z}/n!$ form a colimit tower, and so $\mathcal{C}$ is colimit-exact. Similarly, the groups $\mathbb{Z}/p^n$ form a colimit tower in the category $\mathcal{C}[p^{\infty}]$ of cyclic p-groups, so $\mathcal{C}[p^{\infty}]$ is also colimit exact. For a more degenerate example, we can fix a positive integer d and consider the category $\mathcal{C}[d]$ of cyclic groups of order dividing d. We find that the constant sequence with value $\mathbb{Z}/d$ is a colimit tower for $\mathcal{C}[d]$, so this category is again colimit-exact.

Recall the category $\mathcal{Z}[p^r]$ of finitely generated $\mathbb{Z}/p^r$-modules and its subcategory $\mathcal{F}[p^r]$ of free $\mathbb{Z}/p^r$-modules.

LEMMA 9.10. *Consider a diagram of epimorphisms*

$$
\begin{array}{ccc}
A & \xrightarrow{\quad\gamma\quad} & B \\
 & {}_{\alpha}\searrow \quad \swarrow{}_{\beta} & \\
 & C &
\end{array}
$$

with $A \in \mathcal{F}[p^r], B \in \mathcal{Z}[p^r]$ and $\mathrm{rk}(A) \geq \mathrm{rk}(B)$. Then the dotted arrow can be filled by another epimorphism.

Proof. Put $\mathrm{rk}(A) = n$, $\mathrm{rk}(B) = m$ and $\mathrm{rk}(C) = l$ so that $n \geq m \geq l$. Choose elements $c_1, \ldots, c_l \in C$ that project to a basis of C/pC over $\mathbb{Z}/p$ (so they form a minimal generating set for C). Choose elements $b_1, \ldots, b_l \in B$ with $\beta(b_i) = c_i$. The images of $b_1, \ldots, b_l$ in B/pB will then be linearly independent. Choose additional elements $b_{l+1}, \ldots, b_m \in B$ so that $b_1, \ldots, b_m$ gives a basis for B/pB. After adding multiples of $b_1, \ldots, b_l$ to $b_{l+1}, \ldots, b_m$ if necessary, we can ensure that $\beta(b_i) = 0$ for $i > l$. In the same way, we can find elements $a_1, \ldots, a_n \in A$ such that $\alpha(a_i) = c_i$ for $i \leq l$, and $\alpha(a_i) = 0$ for $i > l$, and $a_1, \ldots, a_n$ gives a basis for A/pA over $\mathbb{Z}/p$. As A is free of rank n over $\mathbb{Z}/p^r$, it follows that the same elements give a basis over $\mathbb{Z}/p^r$. Thus, there is a unique morphism $\gamma \colon A \to B$ with

$$
\gamma(a_i) = \begin{cases} b_i & \text{if} \quad 0 \leq i \leq m \\ 0 & \text{if} \quad m < i \leq n. \end{cases}
$$

As all the generators b_i lie in the image of γ, we see that γ is surjective. It also satisfies $\beta\gamma = \alpha$ by construction. $\qquad\square$

EXAMPLE 9.11. Consider the category $\mathcal{Z}[p^r]$ of finite abelian p-groups of exponent dividing p^r, which is equivalent to the category of finitely generated $\mathbb{Z}/p^r$-modules and linear maps. Using Lemma 9.10 one sees that the groups $(\mathbb{Z}/p^r)^n$ form a colimit tower. It follows that $\mathcal{Z}[p^r]$ is colimit-exact. As these groups lie in the subcategory $\mathcal{F}[p^r] \leq \mathcal{Z}[p^r]$, it is clear that they form a colimit tower for that subcategory as well.

Most of the rest of this section is devoted to the proof of the following result:

THEOREM 9.12. *If $\mathcal{U} \leq \mathcal{G}$ is submultiplicative then it is colimit-exact.*

We will prove this by giving a less explicit but much more general construction of colimit towers. For this, we will need a bit of preparation. Recall the functor T from Example 5.10.

LEMMA 9.13. *Let X be a finite set and consider a diagram of epimorphisms between groups in $\mathcal{U}$*

$$
\begin{array}{ccc}
 & & G \\
 & \overset{\mu}{\nearrow} & \downarrow \alpha \\
TX & \overset{\lambda}{\twoheadrightarrow} & H
\end{array}
$$

in which $|G| \leq |X|$. Then the dotted arrow can be filled in by another epimorphism.

Proof. Put $L = \ker(\alpha)$, so $|L||H| = |G| \leq |X|$. Let $i\colon X \to TX$ be the canonical inclusion, and put $X_h = (\lambda i)^{-1}\{h\} \subseteq X$ for each $h \in H$. We then have $\sum_h |X_h| = |X| \geq |H||L|$, so we can choose h_0 with $|X_{h_0}| \geq |L|$. Let $\mu_h\colon X_h \to \alpha^{-1}\{h\}$ be chosen arbitrarily, except that we choose μ_{h_0} to be surjective. By combining these maps, we get $\mu'\colon X \to G$ such that $\alpha\mu' = \lambda i$. By the defining properties of TX, we see that there is a unique homomorphism $\mu\colon TX \to G$ with $\mu i = \mu'$. This satisfies $\alpha\mu i = \lambda i$ and $i(X)$ generates TX so $\alpha\mu = \lambda$. Now note that the restriction of α to the image of μ is an epimorphism since $\alpha\mu$ is surjective. Also, the image of μ contains L as μ_{h_0} is surjective. It follows that $|\mathrm{Im}(\mu)| = |L||H| = |G|$ so μ is surjective as required. $\qquad\square$

LEMMA 9.14. *If $G \neq 1$ then $\epsilon\colon TG \to G$ is not injective, so $|TG| \geq 2|G|$.*

Proof. Choose any nontrivial $g \in G$ and let $\tau\colon G \to G$ be the transposition that exchanges 1 and g. Let e_1 and e_g denote the corresponding generators of FG or TG. The map τ induces an automorphism α of TG which exchanges e_1 and e_g. The homomorphism $\epsilon\alpha$ sends e_1 to $g \neq 1$, so $e_1 \notin N$, so e_1 gives a nontrivial element of TG. However, this lies in the kernel of ϵ, so ϵ is not injective, and $|TG| = |G||\ker(\epsilon)| \geq 2|G|$. $\qquad\square$

REMARK 9.15. This lower bound is pitifully weak; in practice TG is enormously larger than G.

LEMMA 9.16. *Suppose that $\alpha, \beta: G \to H$ are surjective homomorphisms in $\mathcal{U}$. Then there is an automorphism γ of TG making the following diagram commute:*

$$
\begin{array}{ccc}
TG & \xrightarrow{\quad \gamma \quad} & TG \\
{\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle \epsilon} \\
G & \xrightarrow{\quad \alpha \quad} H \xleftarrow{\quad \beta \quad} & G.
\end{array}
$$

Proof. Put $m = |G|/|H| = |\ker(\alpha)| = |\ker(\beta)|$. For each $h \in H$ we have $|\alpha^{-1}\{h\}| = m = |\beta^{-1}\{h\}|$, so we can choose a bijection $\alpha^{-1}\{h\} \to \beta^{-1}\{h\}$. By combining these choices, we obtain a bijection $\sigma: G \to G$ such that $\beta\sigma = \alpha$. This gives an automorphism $\gamma = T\sigma$ of TG. We claim that $\beta\epsilon\gamma = \alpha\epsilon: TG \to H$. It will suffices to check this on the generating set $G \subset TG$, and that reduces to the relation $\beta\sigma = \alpha$, which holds by construction. $\qquad\square$

Proof of Theorem 9.12. The claim is clear if $\mathcal{U} = \{1\}$. Suppose instead that $\mathcal{U}$ contains a nontrivial group G_0. Put $G_n = T^n G_0$, so we have a tower

$$
G_0 \xleftarrow{\epsilon} G_1 \xleftarrow{\epsilon} G_2 \xleftarrow{\epsilon} \cdots .
$$

We claim that this is a colimit tower for $\mathcal{U}$. Using Remark 9.15 we see that $|G_n| \to \infty$ as $n \to \infty$. For fixed $H \in \mathcal{U}$ we can therefore choose a surjective function $G_n \to H$ for some n, and this will induce a surjective homomorphism $G_{n+1} \to H$ giving condition (a) of the colimit tower. Condition (b) holds by Lemma 9.16 and (c) follows from Lemma 9.13, so $\mathcal{U}$ is colimit-exact. $\qquad\square$

PROPOSITION 9.17. *Suppose that $\mathcal{V} \subseteq \mathcal{U} \subseteq \mathcal{G}$, that $\mathcal{U}$ is colimit-exact and that $\mathcal{V}$ is closed upwards in $\mathcal{U}$. Then $\mathcal{V}$ is also colimit-exact.*

Proof. Let $i: \mathcal{V} \to \mathcal{U}$ be the inclusion, and let c be the functor $\mathcal{U} \to 1$. Note that $i_!: \mathcal{A}\mathcal{V} \to \mathcal{A}\mathcal{U}$ is just extension by zero, as proved in Lemma 5.3, and so is exact. We are given that the functor $L_{\mathcal{U}} = c_!$ is exact, so the composite $L_{\mathcal{V}} = (ci)_! = c_! i_!$ is exact as well. $\qquad\square$

We conclude with a counterexample.

EXAMPLE 9.18. *The category $\mathcal{G}_{\leq 3}$ is not colimit-exact.*

Proof. The subcategory $\mathcal{U}' = \{1, C_2, C_3\}$ is a skeleton of $\mathcal{G}_{\leq 3}$, which makes it easy to calculate colimits. Let $X < \mathbb{1}$ be given by $X(G) = 0$ when $|G| = 1$ and $X(G) = \mathbb{1}(G) = k$ when $|G| > 1$. We find that $LX = k^2$ but $L\mathbb{1} = k$, so L does not send the monomorphism $X \to \mathbb{1}$ to a monomorphism, so L is not exact. $\qquad\square$

10   Complete subcategories

In this section we introduce a well-behaved type of subcategory and present some examples.

DEFINITION 10.1. Let $\mathcal{U}$ be a subcategory of $\mathcal{G}$.

(a) For $T \in \mathcal{G}$, we denote by $\delta(T)$ the minimum possible size of a generating set for T.

(b) For $m \in \mathbb{N}$, we put $\mathcal{R}_m = \{T \in \mathcal{U} \mid \delta(T) \geq m\}$.

(c) We say that $\mathcal{U}$ is *expansive* if for all $G \in \mathcal{U}$ and $m \in \mathbb{N}$ we have $\mathcal{R}_m \downarrow G \neq \emptyset$.

(d) Let $\mathcal{U}$ be expansive. For $X \in \mathcal{A}\mathcal{U}$ and $n > 0$ we put

$$\omega_n^{\mathcal{U}}(X) = \limsup_{m \to \infty} \{\dim(X(T))/n^{\delta(T)} \mid T \in \mathcal{R}_m\} \in [0, \infty].$$

and

$$\mathcal{W}(\mathcal{U})_n = \{X \in \mathcal{A}\mathcal{U} \mid \omega_n^{\mathcal{U}}(X) < \infty\}.$$

It is easy to see that if $\omega_n^{\mathcal{U}}(X) > 0$ then $\omega_m^{\mathcal{U}}(X) = \infty$ for $m < n$. Similarly, if $\omega_n^{\mathcal{U}}(X) < \infty$ then $\omega_m^{\mathcal{U}}(X) = 0$ for $m > n$. Thus, there is at most one n such that $0 < \omega_n^{\mathcal{U}}(X) < \infty$. If such an n exists, we call it the *order* of X.

REMARK 10.2. We will often drop the superscript and just write $\omega_n(X)$.

Using the properties of the limsup we obtain the following result.

LEMMA 10.3. *For any short exact sequence* $X \to Y \to Z$ *in* $\mathcal{A}\mathcal{U}$ *we have*

$$\max(\omega_n(X), \omega_n(Z)) \leq \omega_n(Y) \leq \omega_n(X) + \omega_n(Z).$$

In particular, for any X and Z we have

$$\max(\omega_n(X), \omega_n(Z)) \leq \omega_n(X \oplus Z) \leq \omega_n(X) + \omega_n(Z). \quad \square$$

COROLLARY 10.4. *The category* $\mathcal{W}(\mathcal{U})_n$ *is closed under finite direct sums, subobjects, quotients, extensions and retracts. It also contains* e_G *for all* $G \in \mathcal{U}_{\leq n}$.

Proof. The closure properties easily follow from Lemma 10.3. For the second claim, note that if $A \subset T$ is a generating set for $T \in \mathcal{U}$, then the restriction map $\mathrm{Hom}(T, G) \to \mathrm{Map}(A, G)$ is injective, so $|\mathrm{Hom}(T, G)| \leq |G|^{|A|}$. It follows that

$$|\mathcal{U}(T, G)| = |\mathrm{Epi}(T, G)|/|\mathrm{Inn}(G)| \leq |\mathrm{Hom}(T, G)|/|\mathrm{Inn}(G)|$$
$$\leq |G|^{\delta(T)}/|\mathrm{Inn}(G)| = |G|^{\delta(T)-1}|ZG|.$$

From this it is easy to see that $\omega_n(e_G) \leq |\mathrm{Inn}(G)|^{-1}$ if $|G| = n$, and $\omega_n(e_G) = 0$ if $|G| < n$. $\square$

We are now ready to introduce an important family of subcategories.

DEFINITION 10.5. A subcategory $\mathcal{U}$ of $\mathcal{G}$ is *complete* if the following conditions are satisfied:

- $\mathcal{U}$ is expansive, i.e., for all $G \in \mathcal{U}$ and $n > 0$ there exists $T \in \mathcal{U}$ with $\delta(T) \geq n$ and $\mathcal{U}(T, G) \neq \emptyset$;

- For all $n > 0$ and $G \in \mathcal{U}_n$, we have $0 < \omega_n^{\mathcal{U}}(e_G) < \infty$. In other words, e_G has order exactly $|G|$.

EXAMPLE 10.6. Recall that we always have $\omega_n(e_G) \leq |\operatorname{Inn}(G)|^{-1}$ if $|G| = n$.

- The category $\mathcal{C}[p^\infty]$ of cyclic p-groups is not complete, as it is not expansive.

- The category $\mathcal{E}[p]$ of elementary abelian p-groups is complete. Indeed we have

$$
\begin{aligned}
\omega_{p^n}(e_{C_p^n}) &= \lim_{m \to \infty} \frac{|\operatorname{Epi}(C_p^m, C_p^n)|}{p^{nm}} \\
&= \lim_{m \to \infty} \frac{(p^m - 1)(p^m - p) \cdots (p^m - p^{n-1})}{p^{nm}} = 1.
\end{aligned}
$$

Let us produce more examples of complete subcategories.

PROPOSITION 10.7. *If $\mathcal{U} \leq \mathcal{G}$ is nontrivial and submultiplicative, then it is complete.*

Proof. As $\mathcal{U}$ is nontrivial and subgroup-closed, it must contain C_p for some p. Then for $G \in \mathcal{U}$ we have $G \times C_p^n \in \mathcal{U}$ with $\delta(G \times C_p^n) \geq n$, showing that $\mathcal{U}$ is expansive. We now need to show that $\omega_{|G|}(e_G) > 0$ for all $G \in \mathcal{U}$. Without loss of generality we can assume that $G \neq 1$. For X_m a set with m elements, consider the group $T X_m \in \mathcal{U}$ as defined in Example 5.10. By definition, there is a natural bijection $\operatorname{Hom}(T X_m, G) = \operatorname{Hom}(F X_m, G) \simeq G^m$ for all the groups $G \in \mathcal{U}_{\leq m}$. Since by [16, Theorem 1] we have

$$
\lim_{m \to \infty} |\operatorname{Epi}(F X_m, G)|/|G|^m = 1
$$

we deduce that

$$
\lim_{m \to \infty} |\operatorname{Epi}(T X_m, G)|/|G|^m = 1.
$$

It only remains to notice that $\delta(T X_m) \leq m$ so

$$
\omega_{|G|}(e_G) \geq \lim_{m \to \infty} \frac{|\mathcal{U}(T X_m, G)|}{|G|^m} = \lim_{m \to \infty} \frac{|\operatorname{Epi}(T X_m, G)|}{|\operatorname{Inn}(G)||G|^m} = \frac{1}{|\operatorname{Inn}(G)|} > 0.
$$

$\square$

The completeness assumption give us information on the growth of the indecomposable projectives.

LEMMA 10.8. *Let $\mathcal{U}$ be a complete subcategory of $\mathcal{G}$. For $G \in \mathcal{U}$ and V an Out(G)-representation, we have $0 < \omega_{|G|}(e_{G,V}) < \infty$.*

Proof. We show that $\dim(e_{G,V}(T)) = \dim(V)|\operatorname{Out}(G)|^{-1}|\dim(e_G(T))|$, and so the claim follows by completeness. It is easy to see that $\operatorname{Out}(G)$ acts freely on $\mathcal{U}(T, G)$. Choose a subset $M \subset \mathcal{U}(T, G)$ containing one representative of every orbit, so that $|M| = |\operatorname{Out}(G)|^{-1}|\mathcal{U}(T, G)|$. We also see that M is a basis for $e_G(T)$ as a module over the ring $R = k[\operatorname{Out}(G)]$, so

$$e_{G,V}(T) = V \otimes_R e_G(T) \simeq V^{|M|}.$$

This gives

$$\dim(e_{G,V}(T)) = \dim(V)|M| = \dim(V)|\operatorname{Out}(G)|^{-1}\dim(e_G(T))$$

as claimed. □

PROPOSITION 10.9. *Let $\mathcal{U}$ be complete subcategory of $\mathcal{G}$. Then any monomorphism between projective objects of $\mathcal{A}\mathcal{U}$ is split.*

Proof. Let $u \colon P \to Q$ be a monomorphism between projective objects. By Proposition 8.4, we can write $P = \bigoplus_n P_n$ and $Q = \bigoplus_n Q_n$, where P_n and Q_n are in the image of $(i_n)_! \colon \mathcal{A}\mathcal{U}_n \to \mathcal{A}\mathcal{U}$, so $\mathcal{A}\mathcal{U}(P_n, Q_m) = 0$ when $n < m$. We put $P_{\leq m} = \bigoplus_{k \leq m} P_k = L_{\leq m}P$, and similarly for Q. It is then clear that u restricts to give a monomorphism $u_{\leq m} \colon P_{\leq m} \to Q_{\leq m}$. We will prove by induction on m that $u_{\leq m}$ splits. The claim is trivial if $m = 0$. Let $m > 0$ and let $s_{<m} \colon Q_{<m} \to P_{<m}$ be a splitting of $u_{<m} \colon P_{<m} \to Q_{<m}$. Now let K_m be the kernel of the map $u_m \colon P_m \to Q_m$. As all monomorphisms in $\mathcal{A}\mathcal{U}_m$ are split, we see that K_m is a retract of P_m. As $u_m(K_m) = 0$ and $u_{\leq m}$ is a monomorphism, we see that $u_{\leq m}$ induces a monomorphism from K_m to $Q_{<m}$. However, by completeness the order of $Q_{<m}$ is at most $m - 1$, whereas if K_m is nonzero, it must have order m. It follows that K_m must actually be zero, so u_m is a monomorphism in $\mathcal{A}\mathcal{U}_m$, so there is a splitting $v \colon Q_m \to P_m$. Let $s_{\leq m} \colon Q_{\leq m} \to P_{\leq m}$ be given by $s_{<m}$ on $Q_{<m}$, and by v on Q_m. Then $s_{\leq m}u_{\leq m}$ is the identity of $P_{<m}$, and it is the identity modulo $P_{<m}$ on P_m, so it is an automorphism of $P_{\leq m}$. It follows that $(s_{\leq m}u_{\leq m})^{-1} \circ s_{\leq m}$ is a splitting of $u_{\leq m}$, as required. By construction, the sections $s_{\leq m}$ assemble into a map $s \colon Q \to P$ satisfying $s \circ u = \operatorname{id}_P$, so u splits. □

11　FINITENESS CONDITIONS

We introduce various finiteness conditions on objects of $\mathcal{A}$ and prove some implications amongst them. We refer the reader to Remarks 11.10 and 13.2 for a summary.

DEFINITION 11.1. Consider a subcategory $\mathcal{U} \leq \mathcal{G}$ and an object $X \in \mathcal{A}\mathcal{U}$.

(a) We say that X has *finite type* if $\dim(X(G)) < \infty$ for all $G \in \mathcal{U}$.

(b) We say that X is *finitely projective* if it can be expressed as the direct sum of a finite family of indecomposable projectives.

(c) We say that X is *finitely generated* if there is an epimorphism $P_0 \to X$, for some finitely projective object P_0 (or equivalently, for some object P_0 of the form $\bigoplus_{i=1}^n e_{G_i}$).

(d) We say that X is *finitely presented* if there is a right exact sequence $P_1 \to P_0 \to X$, where P_0 and P_1 are finitely projective.

(e) We say that X is *finitely resolved* if there is a resolution $P_* \to X$, where each P_i is finitely projective.

(f) We say that X is *perfect* if there is a resolution $P_* \to X$, where P_i is finitely projective for all i, and $P_i = 0$ for $i \gg 0$.

(g) We say that X has *finite order* if there exists $n > 0$ such that $\omega_n(X) < \infty$. (This is only meaningful in the case where $\mathcal{U}$ is expansive.)

LEMMA 11.2. *Let $i\colon \mathcal{U} \to \mathcal{V}$ be the inclusion of a subcategory.*

(a) *The functor i^* always preserves objects of finite type. If $\mathcal{U}$ is closed downwards, then i^* preserves all finiteness conditions from Definition 11.1 excluding that of finite order.*

(b) *The functor $i_!$ always preserves finitely presented and finitely generated objects. If $\mathcal{U}$ is closed upwards (and therefore expansive), then $i_!$ preserves all finiteness conditions.*

(c) *If $\mathcal{U}$ is closed downwards, then i_* preserves objects of finite type.*

Proof. Clearly, i^* preserves objects of finite type. If $\mathcal{U}$ is closed downwards, then $i^*(e_G)$ is either e_G (if $G \in \mathcal{U}$) or 0 (if $G \notin \mathcal{U}$). It follows that i^* preserves (finitely) projective objects. Since i^* is also exact by Lemma 5.3(e), it follows that i^* preserves conditions (a) to (f) in Definition 11.1.

By Lemma 5.3(e) and (i), the functor $i_!$ preserves colimits and preserves (finitely) projective objects. It follows that $i_!$ preserves finitely presented and finitely generated objects. If $\mathcal{U}$ is closed upwards, then $i_!$ is extension by zero by Lemma 5.3(f) so it preserves objects of finite type and finite order (if $\mathcal{U}$ expansive). It is also exact so it preserves all the other finiteness conditions. Finally, part (c) follows from Lemma 5.3(g) as i_* is extension by zero. $\square$

REMARK 11.3. We have seen that the restriction functor i^* preserves projectives if $\mathcal{U}$ is closed downwards. This is no longer true if we relax the conditions on $\mathcal{U}$ as the following example shows. Choose a group $G \in \mathcal{G}$, and consider

$$\mathcal{U} = \{H \in \mathcal{G} \mid \mathcal{U}(H,G) \neq \emptyset, \ \mathcal{U}(G,H) = \emptyset\} = \{H \in \mathcal{G}_{\geq G} \mid H \not\simeq G\}.$$

Note that $\mathcal{U}$ is complete as it is closed upwards in $\mathcal{G}$. Let $i\colon \mathcal{U} \to G$ denote the inclusion functor. We claim that i^*e_G is not projective in $\mathcal{U}$. Suppose that i^*e_G was projective, so we could write $i^*e_G = \bigoplus_i e_{H_i, V_i}$ for some groups $H_i \in \mathcal{U}$. Note that we must have $|H_i| > |G|$ for all i. If we calculate the order of these objects we see that $\omega^{\mathcal{U}}_{|G|}(i^*e_G) = \omega^{\mathcal{G}}_{|G|}(e_G)$ and so i^*e_G has order $|G|$ by completeness of $\mathcal{G}$. On the other hand, for $n = \max_i |H_i|$ we have $0 < \omega^{\mathcal{U}}_n(\bigoplus_i e_{H_i, V_i}) < \infty$ so this has order n. We have found a contradiction since $n > |G|$ so i^*e_G cannot be projective.

LEMMA 11.4. *Consider the inclusion* $i_n\colon \mathcal{F}[p^n] \to \mathcal{Z}[p^\infty]$ *for some* $n \geq 1$. *Then the restriction functor* $i_n^*\colon \mathcal{A}\mathcal{Z}[p^\infty] \to \mathcal{A}\mathcal{F}[p^n]$ *preserves finitely generated objects.*

Proof. Consider a finitely generated object $X \in \mathcal{A}\mathcal{Z}[p^\infty]$ and choose an epimorphism $\varphi\colon \bigoplus_{i=1}^s e_{A_i} \to X$. Since i_n^* preserves epimorphisms by Lemma 5.3(e), it will suffice to prove the following claim: if $A \in \mathcal{Z}[p^\infty]$, then $i_n^*e_A \in \mathcal{A}\mathcal{F}[p^n]$ is finitely generated. Let $F \in \mathcal{F}[p^n]$ be minimal such that $\mathcal{Z}[p^\infty](F, A) \neq \emptyset$. A choice of an epimorphism $\varphi\colon F \to A$, then gives a morphism $e_\varphi\colon e_F \to i_n^*e_A$ and we claim this is an epimorphism. In other words, we ought to show that for any epimorphism $\psi\colon F' \to A$ with $F' \in \mathcal{F}[p^n]$, there exists an epimorphism $\zeta\colon F' \to F$ making the following diagram commute:

$$
\begin{array}{ccc}
F' & \xrightarrow{\ \zeta\ } & F \\
& \searrow{\scriptstyle \psi} & \downarrow{\scriptstyle \varphi} \\
& & A.
\end{array}
$$

This is the content of Lemma 9.10. $\square$

It is useful to have a criterion to detect objects which are not finitely generated. Recall the notion of support from Definition 2.6.

LEMMA 11.5. *If X is finitely generated, then* $\min(\mathrm{supp}(X))$ *is finite.*

Proof. If X is finitely generated, we can find an epimorphism $\varphi\colon \bigoplus_{i=1}^r e_{G_i} \to X$. Without loss of generality we can assume that each component $e_{G_i} \to X$ is nonzero so that $X(G_i) \neq 0$ for all i. We claim that $\min(\mathrm{supp}(X)) \subseteq \{[G_1], \dots, [G_r]\}$ which will prove the lemma. If $[H] \in \min(\mathrm{supp}(X))$, then $X(H) \neq 0$, so $\bigoplus_i e_{G_i}(H) \neq 0$, so we can choose an index i with $e_{G_i}(H) \neq 0$, so we can choose a morphism $\alpha\colon H \to G_i$ in $\mathcal{U}$. Now both $[H]$ and $[G_i]$ lie in $\mathrm{supp}(X)$, and $[H]$ is assumed to be minimal, so α must be an isomorphism, so $[H] = [G_i]$ as required. $\square$

PROPOSITION 11.6. *Let $\mathcal{U}$ be a complete subcategory of $\mathcal{G}$. Then any object of $\mathcal{A}\mathcal{U}$ with a finite projective resolution is projective. In particular any perfect object is finitely projective.*

Proof. Let $P_* \to X$ be a projective resolution and suppose that $P_i = 0$ for all $i > n$. If $n > 0$ it follows that the differential $d_n \colon P_n \to P_{n-1}$ must be a monomorphism, so Proposition 10.9 tells us that it is split. Now let Q_* be the same as P_* except that $Q_n = 0$ and $Q_{n-1} = \text{cok}(d_n)$. We find that this is again a projective resolution of X. By repeating this construction, we eventually obtain a projective resolution of length one, showing that X itself is projective. $\square$

REMARK 11.7. The Proposition above is not true if we drop the completeness condition. For example let $\mathcal{C}[p^\infty]$ be the subcategory of cyclic p-groups. Then there is a short exact sequence $0 \to c_{C_{p^2}} \to c_{C_p} \to t_{C_p,k} \to 0$ which shows that $t_{C_p,k}$ is perfect. On the other hand, we have $t_{C_p,k}(C_{p^r}) = 0$ for all $r > 1$, and it follows easily from this that $t_{C_p,k}$ is not projective.

PROPOSITION 11.8. *Let $\mathcal{U}$ be a complete subcategory of $\mathcal{G}$. Then any finitely projective object in $\mathcal{AU}$ has finite order.*

Proof. The zero object has by definition finite order. For $r \geq 1$, we have

$$0 < \omega_n \left(\bigoplus_{i=1}^r e_{G_i, S_i} \right) < \infty \qquad \text{if } n = \max_i(|G_i|)$$

by Lemma 10.8. $\square$

LEMMA 11.9. *Let $\mathcal{U} \leq \mathcal{G}$ be finite (meaning that it has only finitely many isomorphism classes). Then the following are equivalent for an object $X \in \mathcal{AU}$:*

(a) *X has finite type;*

(b) *X is finitely generated;*

(c) *X is perfect.*

Proof. Recall the explicit projective resolution from Construction 5.6. We have $P_0 = l_! l^*(X) = \bigoplus_{G \in \mathcal{U}'} e_{G, X(G)}$. If X has finite type, then P_0 is a finitely generated projective object since $\mathcal{U}$ is finite. This gives (a) $\Rightarrow$ (b). Clearly, (b) $\Rightarrow$ (a) so (a) is equivalent to (b).
Now suppose that X is finitely generated (and hence of finite type) and consider the canonical projective resolution $P_\bullet \to X$. The explicit formula for P_i tells us that P_i has finite type, and it follows from the previous paragraph that P_i is finitely generated too. To prove (b) $\Rightarrow$ (c), we need to show that $P_n = 0$ for $n \gg 0$. Recall from Remark 5.7 that $\text{base}(P_n) \geq \text{base}(X) + n$. Now note that any object in $\mathcal{AU}$ with sufficiently large base is zero as $\mathcal{U}$ is finite. Hence $P_n = 0$ for $n \gg 0$ as required. The final implication (c) $\Rightarrow$ (a) is clear. $\square$

REMARK 11.10. So far we have the following implications:

$$\text{fin. res.} \implies \text{fin. pres.} \implies \text{fin. gen.} \implies \text{fin. type}$$

$$\Big\Uparrow$$

$$\text{perfect} \xleftrightarrow{\ complete\ } \text{fin. proj.} \xRightarrow{\ complete\ } \text{fin. order}$$

12 Torsion and torsion-free objects

In this section we introduce the notions of torsion, absolutely torsion and torsion-free object. We study their formal properties and give some examples.

DEFINITION 12.1. Consider an object X of $\mathcal{AU}$.

- We say that $x \in X(G)$ is *torsion* if there exists $H \in \mathcal{U}$ and $f \in \mathcal{U}(H, G)$ such that $f^*(x) = 0$.

- We say that $x \in X(G)$ is *absolutely torsion* if there exists $m \in \mathbb{N}$ such that for all $f \in \mathcal{U}(H, G)$ with $|H| \geq m$ we have $f^*(x) = 0$.

- We say that X is *torsion* (resp., *absolutely torsion*) if it consists entirely of torsion (resp., absolutely torsion) elements.

- We say that X is *torsion-free* if it does not contain any nonzero torsion element. Equivalently, X is torsion-free if and only if all the maps $\alpha^* \colon X(G) \to X(H)$ are injective.

- We write $\mathrm{tors}(X)(G)$ for the subset of torsion elements in $X(G)$.

HYPOTHESIS 12.2. Throughout we will assume that $\mathcal{U} \leq \mathcal{G}$ has a colimit tower as in Definition 9.5. This is not a very restrictive assumption as we have shown in Section 9 that most natural examples satisfy this.

LEMMA 12.3. *For an element $x \in X(H)$, the following are equivalent:*

(a) *x is torsion.*

(b) *There exists $\alpha \in \mathcal{U}(G_n, H)$ for some n such that $\alpha^*(x) = 0$ in $X(G_n)$.*

(c) *There exists n_0 such that for all $n \geq n_0$ and all $\alpha \in \mathcal{U}(G_n, H)$ we have $\alpha^*(x) = 0$ in $X(G_n)$.*

Proof. By condition (a) of the colimit tower, we see that $\mathcal{U}(G_n, H) \neq \emptyset$ for large n. It follows that $(c) \Rightarrow (b) \Rightarrow (a)$. Now suppose that (a) holds, so we can choose $\beta \in \mathcal{U}(G, H)$ for some G with $\beta^*(x) = 0$. Now let n_0 be least such that $\mathcal{U}(G_{n_0-1}, G) \neq \emptyset$. Suppose that $n \geq n_0$, so $\mathcal{U}(G_{n-1}, G) \neq \emptyset$. If $\alpha \in \mathcal{U}(G_n, H)$, then condition (c) of the colimit tower gives us a morphism $\gamma \in \mathcal{U}(G_n, G)$ with $\alpha = \beta \circ \gamma$, and it follows that $\alpha^*(x) = 0$. Thus, part (c) holds. □

Recall the colimit functor $L \colon \mathcal{AU} \to \mathrm{Vect}_k$ from Section 9.

LEMMA 12.4. *Consider an object $X \in \mathcal{AU}$ and an element $x \in X(G)$. Then x is torsion if and only if the element $1_G \otimes x \in (e_G \otimes X)(G)$ maps to zero in $L(e_G \otimes X)$.*

Proof. Suppose that x is torsion, so we can choose $\alpha\colon G' \to G$ with $\alpha^*(x) = 0$. This means that $\alpha^*(1_G \otimes x) = \alpha \otimes \alpha^*(x) = 0$. The description $L(e_G \otimes X) = \varinjlim_H (e_G \otimes X)(H)$ shows that $1_G \otimes x$ is sent to zero in $L(e_G \otimes X)$.

For the converse, suppose we have an integer n and a morphism $\alpha \in \mathcal{U}(G_n, G)$. Put $\Gamma = \mathrm{Out}(G_n)$ and $\Delta = \{\delta \mid \alpha\delta = \alpha\}$. Define a map

$$\xi\colon (e_G \otimes X)(G_n) \to X(G_n), \quad \xi(\pi \otimes m) = \sum\{\gamma^*m \mid \gamma \in \Gamma, \ \pi\gamma = \alpha\}$$

One checks that $\xi\theta^* = \xi$ for all $\theta \in \Gamma$, so there is an induced map from the coinvariants $\overline{\xi}\colon (e_G \otimes X)(G_n)_\Gamma \to X(G_n)$. One also checks that $\xi(\alpha^*(1_G \otimes x)) = |\Delta|\alpha^*(x)$ for all $x \in X(G)$. The condition that $\alpha^*(1_G \otimes x)$ maps to zero in $L(e_G \otimes X)$ is equivalent to $\alpha^*(1_G \otimes x)$ mapping to zero in $(e_G \otimes X)(G_n)_\Gamma$ for some $n \geq 0$ by Proposition 9.7. It then follows that $0 = \xi(0) = \xi(\alpha^*(1_G \otimes x)) = |\Delta|\alpha^*(x)$ so x is torsion. $\square$

The next result will be a crucial ingredient to proving Proposition 12.16 later on in this section.

COROLLARY 12.5. *If $x \in X(G)$ is not torsion, then there is a morphism $u\colon e_G \otimes X \to \mathbb{1}$ such that $u(1_G \otimes x) = 1$.*

Proof. As the image of $1_G \otimes x$ is nonzero in $L(e_G \otimes X)$, we can choose a k-linear map $u_0\colon L(e_G \otimes X) \to k$ sending this image to 1. Then u_0 is adjoint to a morphism $u\colon e_G \otimes X \to \mathbb{1}$ as claimed. $\square$

LEMMA 12.6. *For any finite dimensional subspace $V \leq \mathrm{tors}(X)(G)$, there is a map $\alpha\colon H \to G$ in $\mathcal{U}$ with $\alpha^*(V) = 0$. Moreover, $\mathrm{tors}(X)$ defines a subobject of X in $\mathcal{AU}$ which is the largest torsion subobject of X. The assignment tors is functorial in X so we have a functor $\mathrm{tors}\colon \mathcal{AU} \to \mathcal{AU}$.*

Proof. Suppose we have torsion elements $x_1, \ldots, x_s \in \mathrm{tors}(X)(G)$. By Lemma 12.3(c), we can choose n large so that $\alpha^*(x_i) = 0$ for all $\alpha \in \mathcal{U}(G_n, G)$ and all $1 \leq i \leq s$. Thus, if V is the span of $\{x_1, \ldots, x_n\}$, we have $\alpha^*(V) = 0$, so $V \leq \mathrm{tors}(X)(G)$. This proves in particular that $\mathrm{tors}(X)(G)$ is a vector subspace of $X(G)$.

Now suppose we have $\alpha^*(x) = 0$, and we also have another morphism $\beta\colon G' \to G$ in $\mathcal{U}$. By condition (b) of the colimit tower, we can fill the dotted arrow in the diagram

$$\begin{array}{ccc} & & G' \\ & {\gamma}\nearrow\raisebox{1ex}{$\cdot$} & \downarrow{\beta} \\ H & \xrightarrow{\ \alpha\ } & G. \end{array}$$

We have $\gamma^*\beta^*(x) = \alpha^*(x) = 0$, so $\beta^*(x)$ is a torsion element. This shows that $\mathrm{tors}(X)$ is a subobject of X. All remaining claims are now clear. $\square$

The following example illustrates the fact that many things can go wrong if we do not assume that $\mathcal{U}$ has a colimit tower.

EXAMPLE 12.7. Consider the following object of $\mathcal{AG}_{\leq 3}$

$$X = (k_x \xleftarrow{\mathrm{pr}_x} k_x \oplus k_y \xrightarrow{\mathrm{pr}_y} k_y).$$

Note that $x, y \in X(1)$ are torsion since $\mathrm{pr}_x(y) = 0 = \mathrm{pr}_y(x)$. On the other hand, $x + y \in X(1)$ is not torsion since $\mathrm{pr}_x(x + y) = x$ and $\mathrm{pr}_y(x + y) = y$. In particular $\mathrm{tors}(X)(1)$ is not a vector subspace of $X(1)$.

REMARK 12.8. The sum of two torsion-free subobjects need not be torsion-free. To see this, consider a torsion-free object Y, a nonzero torsion object Z and an epimorphism $f \colon Y \to Z$. In $Y \oplus Z$ we have a copy of Y, and the graph of f is another subobject $Y' \leq Y \oplus Z$ which is also isomorphic to Y and so is torsion-free. However, $Y + Y'$ is all $Y \oplus Z$ and so is not torsion-free.

LEMMA 12.9. *For any object X of $\mathcal{AU}$, the quotient $X/\mathrm{tors}(X)$ is torsion-free.*

Proof. Consider an element $\overline{x} \in (X/\mathrm{tors}(X))(G)$, so $\overline{x}$ is represented by some element $x \in X(G)$. If $\overline{x}$ is a torsion element, then we have $\alpha^*(\overline{x}) = 0$ for some $\alpha \in \mathcal{U}(H, G)$, or equivalently $\alpha^*(x) \in \mathrm{tors}(X)(H)$. This means that there exists $\beta \in \mathcal{U}(K, H)$ with $(\alpha\beta)^*(x) = \beta^*(\alpha^*(x)) = 0$. Thus x is a torsion element and $\overline{x} = 0$ as required. $\square$

Recall the objects $e_{G,V}$ and $t_{G,V}$ from Definition 2.7.

LEMMA 12.10. *For all $G \in \mathcal{U}$, we have that $e_{G,V}$ is torsion-free and $t_{G,V}$ is absolutely torsion. Thus, any projective object is torsion-free.*

Proof. It is clear that $t_{G,V}$ is absolutely torsion as $t_{G,V}(K)$ is zero as soon as $|K| > |G|$. It is enough to show that e_G is torsion-free as $e_{G,V}$ is a retract of a direct sum of e_G's. Thus, we need to show that for any epimorphism $\varphi \colon H \to K$ the linear map $\varphi^* \colon k[\mathcal{U}(K, G)] \to k[\mathcal{U}(H, G)]$ is injective. This is equivalent to proving that the map $\varphi^* \colon \mathcal{U}(K, G) \to \mathcal{U}(H, G)$ is injective, or in other words that φ is an epimorphism in the category $\mathcal{U}$. This is the content of Lemma 2.2. $\square$

We write $\mathcal{AU}_t$ and $\mathcal{AU}_f$ for the subcategories of torsion and torsion-free objects.

LEMMA 12.11.

(a) *For an object $X \in \mathcal{AU}$, we have $X \in \mathcal{AU}_t$ if and only if $\mathcal{AU}(X, Y) = 0$ for all $Y \in \mathcal{AU}_f$.*

(b) *For an object $Y \in \mathcal{AU}$, we have $Y \in \mathcal{AU}_f$ if and only if $\mathcal{AU}(X, Y) = 0$ for all $X \in \mathcal{AU}_t$.*

Proof. If $f \colon X \to Y$ then $f(\mathrm{tors}(X)) \leq \mathrm{tors}(Y)$. If $X \in \mathcal{AU}_t$ and $Y \in \mathcal{AU}_f$ then $\mathrm{tors}(X) = X$ and $\mathrm{tors}(Y) = 0$ so this becomes $f(X) = 0$ and $f = 0$. Thus, for $X \in \mathcal{AU}_t$ and $Y \in \mathcal{AU}_f$ we have $\mathcal{A}(X, Y) = 0$.

Now suppose that X is such that $\mathcal{AU}(X, Y) = 0$ for all $Y \in \mathcal{AU}_t$. In particular, this means that the projection $X \to X/\operatorname{tors}(X)$ is zero, so $\operatorname{tors}(X) = X$ and $X \in \mathcal{AU}_t$.

Suppose instead that Y is such that $\mathcal{AU}(X, Y) = 0$ for all $X \in \mathcal{AU}_t$. In particular, this means that the inclusion $\operatorname{tors}(Y) \to Y$ is zero, so $\operatorname{tors}(Y) = 0$ and $Y \in \mathcal{AU}_f$. $\qquad\square$

LEMMA 12.12. *Consider objects $X \in \mathcal{AU}_t$ and $Y \in \mathcal{AU}_f$. Then for all $Z \in \mathcal{AU}$, we have*

(a) $X \otimes Z$ *is torsion;*

(b) $\underline{\operatorname{Hom}}(X \otimes Z, Y) = 0$.

Proof. Any element of $(X \otimes Z)(G)$ can be written as a finite linear combination of homogeneous terms $x_i \otimes z_i$. For each of such term we can find $\alpha_i \colon H_i \to G$ such that $\alpha_i^*(x_i) = 0$. Thus we have $\alpha^*(x \otimes z) = \alpha^*(x) \otimes \alpha^*(z) = 0$. As a finite linear combination of torsion elements is again torsion by Lemma 12.6, we deduce that $X \otimes Z$ is torsion. For all $G \in \mathcal{U}$, we have

$$\underline{\operatorname{Hom}}(X \otimes Z, Y)(G) = \mathcal{AU}(e_G \otimes X \otimes Z, Y) = 0$$

by part (a) and Lemma 12.11. $\qquad\square$

LEMMA 12.13. *The subcategory $\mathcal{AU}_t$ is localizing that is, it is closed under arbitrary sums, subobjects, extensions and quotients.*

Proof. Consider an exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in which X and Z are torsion objects. Consider an element $y \in Y(G)$. As Z is a torsion object, we can choose $\alpha \colon H \to G$ with $\alpha^*(p(y)) = 0$. This means that $p(\alpha^*(y)) = 0$, so $\alpha^*(y) = i(x)$ for some $x \in X(H)$. As X is a torsion object, we can choose $\beta \colon K \to H$ with $\beta^*(x) = 0$, and it follows that

$$(\alpha\beta)^*(y) = \beta^* i(x) = i(\beta^*(x)) = i(0) = 0.$$

This shows that Y is also a torsion object so $\mathcal{AU}_t$ is closed under extensions. Now let X be a sum of torsion objects X_i and consider an element $x \in X(G)$. By definition, we can write $x = x_{i_1} + \ldots + x_{i_n}$ for torsion elements $x_{i_k} \in X_{i_k}(G)$. By Lemma 12.3(c), we can find large n such that $\alpha^*(x_{i_k}) = 0$ for all i_k and $\alpha \in \mathcal{U}(G_n, H_{i_k})$. Thus, we have

$$\alpha^*(x) = \alpha_{i_1}^*(x_{i_1}) + \ldots \alpha_{i_n}^*(x_{i_n}) = 0$$

so x is torsion. This shows that $\mathcal{AU}_t$ is closed under arbitrary sums. All the other claims are clear. $\qquad\square$

LEMMA 12.14. *The subcategory $\mathcal{AU}_f$ is closed under subobjects, extensions, arbitrary sums and arbitrary products.*

Proof. From Lemma 12.11(b) it is clear that $\mathcal{AU}_f$ is closed under products and subobjects. As products and sums are computed objectwise, we see that every sum injects in the corresponding product, so $\mathcal{AU}_f$ is also closed under coproducts. Now consider a short exact sequence as follows, in which X and Z are torsion-free

$$X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z.$$

If T is a torsion object, this gives a left exact sequence

$$0 = \mathcal{AU}(T, X) \overset{f_*}{\rightarrowtail} \mathcal{AU}(T, Y) \overset{g_*}{\longrightarrow} \mathcal{AU}(T, Z) = 0,$$

proving that $\mathcal{AU}(T, Y) = 0$. It follows that $\mathcal{AU}_f$ is also closed under extensions. $\square$

We will now give another characterization of torsion-free objects under some mild conditions on $\mathcal{U}$. But first we need a little bit of preparation.

CONSTRUCTION 12.15. Recall the inclusion functor $l \colon \mathcal{U}^{\times} \to \mathcal{U}$ and the functor $l_! l^*$ from Construction 5.6. For any object $X \in \mathcal{AU}$, we set $SX = D(l_! l^*(DX))$ which is injective by Proposition 9.1. Adjoint to the counit map $l_! l^*(DX) \to DX$, we have a map $X \otimes l_! l^*(DX) \to \mathbb{1}$ which is itself adjoint to a map $\xi \colon X \to SX$. If we fix a skeleton $\mathcal{U}'$ for $\mathcal{U}$, we have the explicit formula

$$SX = \prod_{G \in \mathcal{U}'} De_{G, DX(G)},$$

and the map ξ has G-component which is adjoint to the evaluation map $\mathrm{ev} \colon X \otimes e_{G, DX(G)} \to \mathbb{1}$. More explicitly, we have

$$\mathrm{ev} \colon X(H) \otimes e_G(H) \otimes_{\mathrm{Out}(G)} \mathcal{AU}(e_G \otimes X, \mathbb{1}) \to k, \quad x \otimes [\alpha] \otimes f \mapsto f([\alpha] \otimes x)$$

for all $H \in \mathcal{U}$.

PROPOSITION 12.16. *Suppose that $\mathcal{U}$ is a multiplicative global family and consider an object $X \in \mathcal{AU}$. Then SX is projective and we have an exact sequence*

$$0 \to \mathrm{tors}(X) \to X \overset{\xi}{\to} SX.$$

In particular, X is torsion-free if and only if it can be embedded in a projective object.

Proof. We first show that SX is projective. By Proposition 8.6, it is enough to show that $De_{G, DX(G)}$ is projective. Note that $\mathrm{Out}(G)$ acts freely on $DX(G) = \mathcal{AU}(e_G \otimes X, \mathbb{1})$. Choose $u_1, \ldots, u_r \in DX(G)$ containing precisely one element from each $\mathrm{Out}(G)$-orbit so that $DX(G) = \bigoplus_{i=1}^r k[\mathrm{Out}(G)]$ and hence $e_{G, DX(G)} = \bigoplus_{i=1}^r e_G$. Therefore we have reduced the problem to showing that De_G is projective. This now follows from Theorem 4.18.

If SX is projective, then it is also torsion-free by Lemma 12.10, so we get $\text{tors}(X) \subseteq \ker(\xi)$. If $x \in X(G)$ is not torsion, then by Corollary 12.5 we can find $u \in DX(G)$ such that $u(1_G \otimes x) \neq 0$. In particular, we have $\text{ev}(1_G \otimes x \otimes u) \neq 0$ and hence $\ker(\xi) \subseteq \text{tors}(X)$. This shows that the sequence is exact. $\square$

Let X be a finitely generated torsion object. It is tempting to conclude that $X(G)$ should be zero when G is sufficiently large, in some sense. However, the following example shows that this is not correct.

EXAMPLE 12.17. Let $\theta \colon P \to Q$ be a non-split epimorphism between groups in $\mathcal{U}$. This gives a map $\theta_* \colon e_P \to e_Q$, and we define X to be the cokernel (so X is finitely presented). The obvious generator $x \in X(Q)$ satisfies $\theta^*(x) = 0$ by construction, so x is torsion. As x generates X, it follows that X is a torsion object. Note that $X(G)$ is the quotient of $k[\mathcal{U}(G, Q)]$ in which we kill every basis element $[\alpha]$ for which the homomorphism $\alpha \colon G \to Q$ can be lifted to P. Note that no split epimorphism $\alpha \colon G \to Q$ can be lifted to P, because that would give rise to a splitting of θ. In particular, if H admits a split epimorphism to Q, then $X(H) \neq 0$. Thus, we have $X(H \times Q) \neq 0$ for all $H \in \mathcal{U}$.

It is true, however, that if X is a finitely generated torsion object, and G is both sufficiently large and sufficiently free, then $X(G) = 0$. We now proceed to make a precise version of this statement.

DEFINITION 12.18. We say that an object $X \in \mathcal{A}\mathcal{U}$ is G_*-null if $X(G_n) = 0$ for large n.

LEMMA 12.19. *If X is G_*-null, then it is torsion. The converse holds if X is finitely generated.*

Proof. First suppose that X is G_*-null. Consider an element $x \in X(H)$. Choose n large enough that $X(G_n) = 0$ and $\mathcal{U}(G_n, H) \neq \emptyset$. Then for $\alpha \in \mathcal{U}(G_n, H)$ we have $\alpha^*(X) = 0$, as required.

Conversely, suppose that X is finitely generated, with generators $x_i \in X(H_i)$ for $i = 1, \ldots, d$ say. By Lemma 12.3 we can choose n_i such that $\alpha^*(x_i) = 0$ in $X(G_m)$ for all $m \geq n_i$, and all $\alpha \in \mathcal{U}(G_m, H_i)$. Put $n = \max(n_1, \ldots, n_d)$; then we find that $X(G_n) = 0$ for all $n \geq m$. $\square$

We finish this section by giving some examples of torsion objects.

EXAMPLE 12.20. Let G be cyclic of order p, so $\text{Aut}(G)$ is cyclic of order $p-1$, and let $\psi \in \text{Aut}(G)$ be a generator. Let X be the cokernel of $\psi_* - 1 \colon e_G \to e_G$. By definition $X(H)$ is the quotient of $k[\mathcal{U}(H, G)]$ by the subspace generated by the elements $[\psi\alpha] - [\alpha]$ for all α. As ψ is a generator of $\text{Aut}(G)$, we can identify X with c_G from Definition 2.7. In particular, X is projective and torsion-free. This illustrates the fact that we can introduce quite a lot of relations without creating torsion.

EXAMPLE 12.21. Take $\mathcal{U} = \mathcal{Z}[p^\infty]$ and let C be cyclic of order p. Let $\lambda, \rho \colon C^2 \to C$ be the two projections, and let X be the cokernel of $\lambda_* - \rho_* \colon e_{C^2} \to e_C$. This means that $X(G) = k[T(G)]$, where TG is the coequaliser of the maps $\lambda_*, \rho_* \colon k[\mathcal{U}(G, C^2)] \to k[\mathcal{U}(G, C)]$. Let $Q(G)$ be the Frattini quotient of G, so $Q(G) \simeq C^{d(G)}$ for some $d(G) \geq 0$. If $d(G) = 0$ then $G = 1$ and $T(G) = \emptyset$ and $X(G) = 0$. If $d(G) = 1$ then G is cyclic and $\mathcal{U}(G, C^2) = \emptyset$ so $T(G) = \mathcal{U}(G, C) = \mathcal{U}(Q(G), C)$ (which is a set of size $p - 1$) so $X(G) \simeq k^{p-1}$. Now suppose that $d(G) \geq 2$. If α and β are epimorphisms from G to C with different kernels then the combined map $\phi = (\alpha, \beta) \colon G \to C^2$ is again surjective with $\lambda\phi = \alpha$ and $\rho\phi = \beta$ so $[\alpha] = [\beta]$ in $T(G)$. Even if α and β have the same kernel, we can choose a third epimorphism $\gamma \colon G \to C$ with different kernel (because of the fact that $d(G) \geq 2$); we then have $[\alpha] = [\gamma] = [\beta]$. From this we see that $T(G)$ is a singleton and so $X(G) = k$. To summarize

$$X(G) = k[T(G)] = \begin{cases} 0 & \text{if } d(G) = 0 \\ \mathcal{U}(G, C) \simeq k^{p-1} & \text{if } d(G) = 1 \\ k & \text{if } d(G) \geq 2. \end{cases}$$

From our discussion we also see that

$$\operatorname{tors}(X)(G) \simeq \begin{cases} k^{p-2} & \text{if } G \text{ is nontrivial and cyclic} \\ 0 & \text{otherwise} \end{cases}$$

$$(X/\operatorname{tors}(X))(G) \simeq \begin{cases} 0 & \text{if } G = 1 \\ k & \text{if } G \neq 1. \end{cases}$$

EXAMPLE 12.22. Take $\mathcal{U} = \mathcal{Z}[2^\infty]$. There are then three morphisms $\lambda, \rho, \sigma \in \mathcal{U}(C^2, C)$, and we define X to be the cokernel of $\lambda_* + \rho_* + \sigma_* \colon e_{C^2} \to e_C$. We claim that X is a torsion object. To see this, we put $u = \lambda + \rho + \sigma \in e_C(C^2)$ so that $X(G)$ is the quotient of $k[\mathcal{U}(G, C)]$ by all elements of the form $\phi^*(r)$ as ϕ runs over $\mathcal{U}(G, C^2)$. If $d(G) = 1$ then $\mathcal{U}(G, C)$ is a singleton and $\mathcal{U}(G, C^2) = \emptyset$ and $X(G) = k$. If $d(G) = 2$ then $k[\mathcal{U}(G, C)]$ has three elements, say α, β, γ, and

$$X(G) = k\{\alpha, \beta, \gamma\}/(\alpha + \beta + \gamma) \simeq k^2.$$

Now consider $X(C^3)$. This is spanned by the seven nonzero homomorphisms $C^3 \to C$. There are seven subgroups of order 4 in $\operatorname{Hom}(C^3, C) \simeq C^3$:

$$A_1 = \{0, e_1^*, e_2^*, (e_1 + e_2)^*\}$$
$$A_2 = \{0, e_1^*, e_3^*, (e_1 + e_3)^*\}$$
$$A_3 = \{0, e_2^*, e_3^*, (e_2 + e_3)^*\}$$
$$A_4 = \{0, e_3^*, (e_1 + e_2)^*, (e_1 + e_2 + e_3)^*\}$$
$$A_5 = \{0, e_2^*, (e_1 + e_3)^*, (e_1 + e_2 + e_3)^*\}$$
$$A_6 = \{0, e_1^*, (e_2 + e_3)^*, (e_1 + e_2 + e_3)^*\}$$
$$A_7 = \{0, (e_1 + e_2)^*, (e_2 + e_3)^*, (e_1 + e_3)^*\}$$

where e_1^*, e_2^* and e_3^* denote the canonical generators. For each of these A_i we have a relation, saying that the sum of the three nonzero homomorphisms in that subgroup is zero. For example, the relation attached to A_1 tells us that $e_1^* + e_2^* + (e_1 + e_2)^* = 0$. Let u be the sum of all these relations, and let v_α be the sum of the subset that involve a particular morphism α. A calculation shows that $(3v_\alpha - u)/6 = \alpha$. It follows that the resulting quotient $X(C^3)$ is zero. If $d(G) \geq 3$ then any $\alpha \in \mathcal{U}(G, C)$ can be factored through C^3, and it follows from this that $X(G) = 0$. Thus X is a torsion object as claimed.
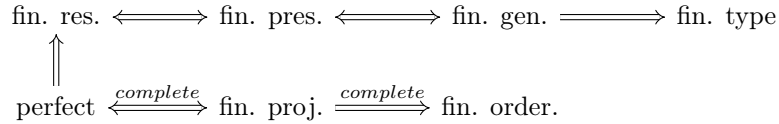
13 NOETHERIAN ABELIAN CATEGORIES

The goal of this section is to study when the category $\mathcal{A}\mathcal{U}$ is locally noetherian.

DEFINITION 13.1. Let $\mathcal{U}$ be a subcategory of $\mathcal{G}$.

- An object $X \in \mathcal{A}\mathcal{U}$ is *noetherian* if every subobject of X is finitely generated.

- The category $\mathcal{A}\mathcal{U}$ is *locally noetherian* if e_G is noetherian for all $G \in \mathcal{U}$.

REMARK 13.2. Suppose that $\mathcal{U}$ is locally noetherian. After adding the obvious consequences of the noetherian property to Remark 11.10 we get the following diagram of finiteness conditions for objects in $\mathcal{A}\mathcal{U}$:

$$\text{fin. res.} \iff \text{fin. pres.} \iff \text{fin. gen.} \implies \text{fin. type}$$
$$\Uparrow$$
$$\text{perfect} \xleftrightarrow{complete} \text{fin. proj.} \xrightarrow{complete} \text{fin. order.}$$

It is not difficult to find subcategories of $\mathcal{G}$ for which $\mathcal{A}\mathcal{U}$ is not locally noetherian.

PROPOSITION 13.3. *Let $\mathcal{U}$ be a full subcategory containing the trivial group and infinitely many cyclic groups of prime order. Then $\mathcal{A}\mathcal{U}$ is not locally noetherian.*

Proof. Let $\chi^+ \in \mathcal{A}\mathcal{U}$ be the subobject of $\mathbb{1}$ given by

$$\chi^+(T) = \begin{cases} 0 & \text{if } |T| = 1 \\ k & \text{if } |T| > 1. \end{cases}$$

Note that $\min(\text{supp}(\chi^+))$ contains the isomorphism classes of all cyclic groups of prime orders. Apply Lemma 11.5 to see that χ^+ cannot be finitely generated. $\square$

The rest of this section will be devoted to prove the following theorem.

THEOREM 13.4. *Fix a prime number p and a positive integer n. The abelian category $\mathcal{A}\mathcal{U}$ is locally noetherian for the following choices of subcategories $\mathcal{U}$:*

(a) $\mathcal{F}[p^n] = \{\text{free } \mathbb{Z}/p^n\text{-modules}\}$.

(b) $\mathcal{C}[p^\infty] = \{\text{cyclic } p\text{-groups}\}$.

(c) $\mathcal{Z}[p^n] = \{\text{fin. gen. } \mathbb{Z}/p^n\text{-modules}\}$.

(d) $\mathcal{Z}[p^\infty] = \{\text{finite abelian } p\text{-groups}\}$.

Proof. Sam and Snowden proved part (a) [20, 8.3.1]. The proofs of part (b),(c),(d) will be given in the next subsections. □

13.1 Part (b)

We start by introducing the criterion for noetherianity developed in [6] which applies to a special type of subcategories.

DEFINITION 13.5 ([6, 2.2]). Let $\mathcal{U}$ be a subcategory of $\mathcal{G}$ and fix a skeleton $\mathcal{U}'$ for $\mathcal{U}$. If $G, H \in \mathcal{U}$ we write $G \gg H$ to mean that $\mathcal{U}(G, H) \neq \emptyset$. We say that $\mathcal{U}$ has *type* A_∞ if there exists an isomorphism of posets $(\mathcal{U}', \gg) \simeq (\mathbb{N}, \geq)$.

EXAMPLE 13.6. The subcategory $\mathcal{C}$ of all cyclic groups is not of type A_∞ as there are no epimorphisms $C_3 \to C_2$ or $C_2 \to C_3$. However if we fix a prime number p, then the subcategory $\mathcal{C}[p^\infty]$ of cyclic p-groups has type A_∞. Recall that $\mathcal{F}[p^n]$ is (equivalent to) the category of finitely generated free modules over $\mathbb{Z}/p^n$; this also has type A_∞. The same is true of the category $\mathcal{E}[p]$ of elementary abelian p-groups (because it is the same as $\mathcal{F}[p]$).

For compatibility with our work, we reformulate [6, 3.1] for contravariant diagrams.

DEFINITION 13.7. We say that the category $\mathcal{U}$ has the *transitivity property* if the action of $\text{Out}(G)$ on $\mathcal{U}(G, H)$ is transitive whenever $G \gg H$.

DEFINITION 13.8. Suppose that $\mathcal{U}$ has the transitivity property. For any pair (G, H) with $G \gg H$ we let $\text{Out}(G)$ act diagonally on $\mathcal{U}(G, H)^2$ and put $\mathcal{U}_2(G, H) = \mathcal{U}(G, H)^2 / \text{Out}(G)$.

LEMMA 13.9. *Suppose we fix $\alpha \in \mathcal{U}(G, H)$ and put $\Phi(\alpha) = \{\phi \in \text{Out}(G) \mid \alpha\phi = \alpha\}$. Then there is a natural bijection $\zeta \colon \mathcal{U}(G, H)/\Phi(\alpha) \to \mathcal{U}_2(G, H)$.*

Proof. We have a map $\mathcal{U}(G, H) \to \mathcal{U}(G, H)^2$ given by $\gamma \mapsto (\alpha, \gamma)$, and this induces a map
$$\zeta \colon \mathcal{U}(G, H)/\Phi(\alpha) \to \mathcal{U}_2(G, H).$$
If $(\beta, \gamma) \in \mathcal{U}(G, H)^2$ then the transitivity property gives $\theta \in \text{Out}(G)$ with $\beta\theta = \alpha$ and it follows that $[\beta, \gamma] = [\beta\theta, \gamma\theta] = \zeta(\gamma\theta)$ in $\mathcal{U}_2(G, H)$. This shows that ζ is surjective.
On the other hand, if $\zeta[\beta_0] = \zeta[\beta_1]$ then there exists $\phi \in \mathcal{U}(G)$ with $(\alpha\phi, \beta_0\phi) = (\alpha, \beta_1)$. This means that $\alpha\phi = \alpha$ (so $\phi \in \Phi(\alpha)$) and $\beta_0\phi = \beta_1$ (so $[\beta_0] = [\beta_1]$ in $\mathcal{U}(G, H)/\Phi(\alpha)$). This shows that ζ is also injective. □

LEMMA 13.10. *Suppose that $G' \gg G$ and $u \in \mathcal{U}_2(G, H)$, so $u \subseteq \mathcal{U}(G, H)^2$. Put*

$$\lambda(u) = \lambda_{GG'}(u) = \{(\alpha\phi, \beta\phi) \mid (\alpha, \beta) \in u, \ \phi \in \mathcal{U}(G', G)\} \subseteq \mathcal{U}(G', H)^2.$$

Then $\lambda(u)$ is a $\mathrm{Out}(G')$-orbit, or in other words an element of $\mathcal{U}_2(G', H)$. The map λ can also be characterised by $\lambda[\alpha, \beta] = [\alpha\phi, \beta\phi]$ for any $\phi \in \mathcal{U}(G', G)$.

Proof. A typical element of $\lambda(u)$ has the form $x = (\alpha\phi, \beta\phi)$ with $(\alpha, \beta) \in u$ and $\phi \in \mathcal{U}(G, H)$. If $\theta \in \mathrm{Out}(G)$ then the map $\phi' = \phi\theta$ also lies in $\mathcal{U}(G, H)$ and $\theta^* x = (\alpha\phi', \beta\phi')$; this shows that $\lambda(u)$ is preserved by $\mathrm{Out}(G)$.

Now suppose we fix an element $x = (\alpha, \beta) \in u$ and a map $\phi \in \mathcal{U}(G, H)$ and put $x' = (\alpha\phi, \beta\phi) \in \lambda(u)$. Any element of u has the form $(\alpha\zeta, \beta\zeta)$ for some $\zeta \in \mathrm{Out}(G)$. Thus, any element $y \in \lambda(u)$ has the form $y = (\alpha\zeta\psi, \beta\zeta\psi)$ for some $\zeta \in \mathrm{Out}(G)$ and $\psi \in \mathcal{U}(G', G)$. By the transitivity property we can find $\xi \in \mathrm{Out}(G')$ with $\zeta\psi = \phi\xi$, so $y = (\alpha\phi\xi, \beta\phi\xi) = \xi^*(x')$. It follows that $\lambda[x] = [x']$, so in particular $\lambda[x]$ is an orbit as claimed. $\square$

DEFINITION 13.11. We say that $\mathcal{U}$ has the *bijectivity property* if for all H there exists $G \gg H$ such that for all $G' \gg G$ the map

$$\lambda\colon \mathcal{U}_2(G, H) \to \mathcal{U}_2(G', H)$$

is bijective.

REMARK 13.12. Our bijectivity property is not visibly the same as that of [6, 3.2]. However, Lemma 13.9 shows that they are equivalent (and we consider that our version is more transparent).

We are finally ready to state the criterion.

THEOREM 13.13 ([6, 3.7]). *Let $\mathcal{U}$ be a subcategory of $\mathcal{G}$ of type A_∞. Suppose that $\mathcal{U}$ satisfies the transitivity and bijectivity properties. Then $\mathcal{A}\mathcal{U}$ is locally noetherian.*

We now apply the criterion to our case of interest.

THEOREM 13.14. *Fix a prime number p and let $\mathcal{C}[p^\infty]$ be the family of cyclic p-groups. Then the category $\mathcal{A}\mathcal{C}[p^\infty]$ is locally noetherian.*

Proof. We have already seen that $\mathcal{C}[p^\infty]$ has type A_∞ so it is enough to check that it satisfies the transitivity and bijectivity property. Recall the discussion on the morphisms of $\mathcal{C}[p^\infty]$ from Example 9.9.

Consider cyclic groups G and H and suppose that $|H|$ divides $|G|$ so that $\mathcal{U}(G, H) \neq \emptyset$. We know that for any $\alpha \in \mathcal{U}(G, H)$ and $\phi \in \mathrm{Aut}(H)$ there exists $\psi \in \mathrm{Aut}(G)$ such that $\alpha\psi = \phi\alpha$. Combining this with the fact that $\mathcal{U}(G, H)$ is a torsor for $\mathrm{Aut}(H)$, we find that $\mathcal{U}(G, H)$ is a single orbit for $\mathrm{Aut}(G)$. Thus $\mathcal{C}[p^\infty]$ satisfies the transitivity condition.

If $(\alpha, \beta) \in \mathcal{U}(G, H)^2$ then there is a unique element $\phi \in \mathrm{Aut}(H)$ with $\beta = \phi \circ \alpha$. This is unchanged if we compose α and β with any surjective homomorphism $\epsilon\colon G' \to G$. It follows that the rule $[\alpha, \beta] \mapsto \phi$ gives a well-defined bijections $\xi = \xi_{GH}\colon \mathcal{U}_2(G, H) \to \mathrm{Aut}(H)$. This also satisfies $\xi_{G'H}\lambda = \xi_{GH}$, so all the maps λ are bijective, and so $\mathcal{C}[p^\infty]$ satisfies the bijectivity condition. $\square$

13.2 Part (c) and (d)

The rest of this section will be devoted to proving the following result.

Theorem 13.15. *Fix a prime number p. Recall that $\mathcal{Z}[p^\infty]$ is the category of finite abelian p-groups, and that $\mathcal{Z}[p^n]$ is the subcategory where the exponent divides p^n. Then the categories $\mathcal{AZ}[p^\infty]$ and $\mathcal{AZ}[p^n]$ are locally noetherian.*

We will apply a different criterion due to Sam and Snowden that we shall now recall [20]. The basic outline is as follows. One way to prove that polynomial rings are noetherian is to use the technology of Gröbner bases. If $\mathcal{C}$ is a category satisfying appropriate combinatorial and order-theoretic conditions, we can use similar techniques to prove that $[\mathcal{C}, \text{Vect}_k]$ is locally noetherian. If $\mathcal{U} \leq \mathcal{G}$ and we have a functor $\mathcal{C} \to \mathcal{U}^{\text{op}}$ with appropriate finiteness properties, we can then deduce that $\mathcal{AU}$ is locally noetherian. In the case $\mathcal{U} = \mathcal{Z}[p^\infty]$ we will take $\mathcal{C}$ to be something like the category of finite abelian p-groups with a specified presentation, although the precise details are somewhat complex.

Remark 13.16. Some of the definitions and constructions below can be done for preordered sets or for small categories. We regard a preordered set P as a small category with one morphism $a \to b$ whenever $a \leq b$, and no morphisms $a \to b$ if $a \not\leq b$. We regard a small category $\mathcal{C}$ as a preordered set by declaring that $a \leq b$ if and only if $\mathcal{C}(a, b) \neq \emptyset$.

The first combinatorial condition that we need to use is as follows:

Definition 13.17. Let $\mathcal{C}$ be a small category.

- A *sequence* in $\mathcal{C}$ means a map $u \colon \mathbb{N} \to \text{obj}(\mathcal{C})$ where $\mathbb{N}$ is viewed as a discrete category.

- A *subsequence* of u is a map of the form $u \circ f$, where $f \colon \mathbb{N} \to \mathbb{N}$ is strictly increasing.

- We say that u is *good* if there exists $i < j$ such that $u(i) \leq u(j)$ (meaning that $\mathcal{C}(u(i), u(j)) \neq \emptyset$, as in Remark 13.16).

- We say that u is *very good* if $u(i) \leq u(j)$ for all $i \leq j$.

- We say that $\mathcal{C}$ is *well-quasi-ordered* (or *wqo*) if every sequence in $\mathcal{C}$ is good.

- We say that $\mathcal{C}$ is *cowqo* if $\mathcal{C}^{\text{op}}$ is wqo.

- We say that $\mathcal{C}$ is *slice-wqo* if the slice category $X \downarrow \mathcal{C}$ is wqo for all objects X.

Remark 13.18. It is clear that the definition of wqo is compatible with the identifications in Remark 13.16.

Remark 13.19. If $\mathcal{C}$ is finite then any sequence $u\colon \mathbb{N} \to \mathrm{obj}(\mathcal{C})$ is non-injective and therefore good.

Remark 13.20. Now let P be a well-ordered set. For any sequence $u\colon \mathbb{N} \to P$, the set $u(\mathbb{N})$ must have a smallest element, say $u(k)$, and then we have $u(k) \leq u(k+1)$, showing that u is good. It follows that P is wqo.

The following lemma is a basic ingredient.

Lemma 13.21. *Suppose that $\mathcal{C}$ is wqo. Then any sequence in $\mathcal{C}$ has a very good subsequence.*

Proof. Given any sequence $u\colon \mathbb{N} \to \mathrm{obj}(\mathcal{C})$ and $i \in \mathbb{N}$, put

$$I(u,i) = \{j > i \mid u(i) \leq u(j)\}.$$

Then put $J(u) = \{i \mid |I(u,i)| = \infty\}$. Suppose that $J(u)$ is empty, so $I(u,i)$ is finite for all i. Define $f\colon \mathbb{N} \to \mathbb{N}$ recursively by $f(0) = 0$ and

$$f(i+1) = \min\{j \mid j > f(i) \text{ and } j > k \text{ for all } k \in I(u, f(i))\}.$$

It is then not hard to see that $u \circ f$ is bad, contradicting the assumption that $\mathcal{C}$ is wqo. It follows that $J(u)$ must actually be nonempty. Put $j(u) = \min(J(u))$, so $I(u, j(u))$ is infinite. Put $T(u) = u \circ f$, where $f\colon \mathbb{N} \to \mathbb{N}$ is the unique strictly increasing map with image $I(u, j(u))$. Now define $R(u)\colon \mathbb{N} \to \mathrm{obj}(\mathcal{C})$ recursively by $R(u)(0) = u(j(0))$ and $R(u)(i+1) = R(T(u))(i)$. We find that $R(u)$ is a very good subsequence of u. $\qquad\square$

Definition 13.22. Let $\mathcal{C}$ be a small category.

- We say that $\mathcal{C}$ is *rigid* if every endomorphism is an identity.

- A *hom-ordering* on $\mathcal{C}$ consists of a system of well-orderings of the hom sets $\mathcal{C}(X,Y)$ such that for all $\alpha\colon Y \to Z$, the induced map $\alpha_*\colon \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$ is monotone.

Definition 13.23. Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be essentially small.

- We say that $\mathcal{C}$ is *Gröbner* if it is rigid, slice-wqo and it admits a hom-ordering.

- We say that $\mathcal{D}$ is *quasi-Gröbner* if there is a Gröbner category $\mathcal{C}$ and an essentially surjective functor $M\colon \mathcal{C} \to \mathcal{D}$ such that each comma category $(x \downarrow M)$ has a finite weakly initial set. In more detail, the condition is as follows: for each $x \in \mathcal{D}$ there must exist a finite list of objects $y_1, \ldots, y_n \in \mathcal{C}$ and morphisms $f_i\colon x \to M(y_i)$, such that for any $y \in \mathcal{C}$ and any $f\colon x \to M(y)$ there exists i and $g\colon y_i \to y$ with $f = M(g) \circ f_i$. This is known as *Condition (F)*.

We are finally ready to state the criterion.

THEOREM 13.24. [20, 4.3.2] *Let $\mathcal{D}$ be a quasi-Gröbner category. Then the category $[\mathcal{D}, \mathrm{Vect}_k]$ is locally noetherian.*

REMARK 13.25. Here and elsewhere we have used terminology and notation that seems clear to us and compatible with the rest of our work, but which differs from that in [20] and related references. In particular, our "rigid" (as in Definition 13.22) is their "direct", and our "wqo" is their "noetherian". Our "hom-ordering" is their condition (G1), and our "slice-wqo" condition is their (G2).

Before proving Theorem 13.15 we need to introduce more notation and prove some technical results.

WELL-QUASI ORDERS

REMARK 13.26. To deal with some set-theoretic issues, we let $\mathcal{X}$ denote the set of hereditarily finite sets, so $\mathcal{X}$ is countable and closed under taking subsets, products and quotients, and contains sets of all finite orders. When we discuss categories of finite sets with extra structure, we will implicitly assume that the underlying sets are in $\mathcal{X}$, so that the category will be small.

DEFINITION 13.27. Let $\mathcal{C}$ and $\mathcal{D}$ be preordered sets, and let $f \colon \mathcal{C} \to \mathcal{D}$ be a function.

(a) We say that f is *monotone* if $p \leq p'$ implies $f(p) \leq f(p')$.

(b) We say that f is *comonotone* if $f(p) \leq f(p')$ implies $p \leq p'$.

REMARK 13.28. Here $\mathcal{C}$ is and $\mathcal{D}$ might be small categories, regarded as preordered sets as in Remark 13.16. In that case, any functor $f \colon \mathcal{C} \to \mathcal{D}$ gives a monotone map.

PROPOSITION 13.29. *If $f \colon \mathcal{C} \to \mathcal{D}$ is comonotone and $\mathcal{D}$ is wqo then $\mathcal{C}$ is wqo.*

Proof. If $u \colon \mathbb{N} \to \mathcal{C}$ is a sequence, then $f \circ u$ must be good, so there exists $i \leq j$ with $fu(i) \leq fu(j)$, but that implies $u(i) \leq u(j)$ by the comonotone property. □

PROPOSITION 13.30. *Any finite product of wqo preordered sets is again wqo.*

Proof. It suffices to show that if P and Q are wqo, then so is $P \times Q$. Let $u \colon \mathbb{N} \to P \times Q$ be a sequence. As P is wqo, we can find a subsequence v such that $\pi_P \circ v$ is nondecreasing. As Q is wqo, we can then find a subsequence w of v such that $\pi_Q \circ w$ is nondecreasing. Now w is nondecreasing subsequence of u. □

We now recall the Nash-Williams theory of minimal bad sequences [15].

DEFINITION 13.31. Let P be a preordered set. We say that a finite list $u \in P^n$ is *bad* if there is no pair (i,j) with $0 \leq i < j < n$ and $u(i) \leq u(j)$. We say that such a finite list u is *very bad* if there is an infinite bad sequence extending it. If so, the set

$$E(u) = \{u' \in P \mid (u(0), \dots, u(n-1), u') \text{ is } \quad \text{very} \quad \text{bad}\}$$

is nonempty. Now suppose we have a well-ordered set W and a function $\lambda \colon P \to W$. Put

$$EM(u) = \{u' \in E(u) \mid \lambda(u') = \min(\lambda(E(u)))\} \neq \emptyset.$$

We say that a very bad list $u \in P^n$ is λ-*minimal* if for all $k < n$ we have $u(k) \in EM(u_{<k})$. We say that a bad sequence u is λ-*minimal* if every initial segment $u_{<k}$ is λ-minimal.

LEMMA 13.32. *If P is not wqo, then it has a λ-minimal bad sequence.*

Proof. Start with the empty sequence, which is very bad by the assumption that P is not wqo. Then choose recursively $u(k) \in EM(u_{<k})$ for all $k \geq 0$. $\square$

The following result abstracts the logic used for various wqo proofs in the literature.

PROPOSITION 13.33. *Let P and λ be as above. Let P_0 be a subset of P, and let $\chi \colon P_0 \to P$ be a map such that*

(a) *For all $x \in P_0$ we have $\chi(x) \leq x$ and $\lambda(\chi(x)) < \lambda(x)$.*

(b) *Every bad sequence $u \colon \mathbb{N} \to P$ has a subsequence v contained in P_0 with the following property: if $i < j$ with $\chi(v(i)) \leq \chi(v(j))$, then $v(i) \leq v(j)$.*

Then P is wqo.

Proof. Suppose not, so there exists a minimal bad sequence u. Let v be a subsequence as in (b), so $v(n) = u(f(n))$ for some strictly increasing map $f \colon \mathbb{N} \to \mathbb{N}$. Define $w(n) = u(n)$ for $n < f(0)$ and $w(f(0) + k) = \chi(v(k))$. We claim that w is bad. If not, we have $i < j$ with $w(i) \leq w(j)$. If $j < f(0)$ this gives $u(i) \leq u(j)$, contradicting the badness of u. Suppose instead that $i < f(0) \leq j$, so $w(i) = u(i)$ and $w(j) = \chi(v(j')) = \chi(u(j''))$ for some $j' \geq 0$ and $j'' \geq f(0)$. We now have $u(i) \leq \chi(u(j'')) \leq u(j'')$, again contradicting the badness of u. This just leaves the possibility that $f(0) \leq i < j$, so $w(i) = \chi(v(i')) = \chi(u(i''))$ and $w(j) = \chi(v(j')) = \chi(u(j''))$ for some i', j', i'', j'' with $i' < j'$ and $i'' < j''$. We now have $\chi(v(i')) \leq \chi(v(j'))$ so $v(i') \leq v(j')$ b y condition (b), so $u(i'') \leq u(j'')$, yet again contradicting the badness of u. It follows that w must be bad after all. However, this contradicts the λ-minimality of $u(f(0))$ in $E(u_{<f(0)})$. $\square$

DEFINITION 13.34. Let $\mathcal{C}$ be a wqo category. We define $\mathcal{SC}$ to be the category of pairs (X, p), where X is a finite, totally ordered set, and $p\colon X \to \mathcal{C}$. A morphism from (X, p) to (Y, q) consists of a strictly monotone map $\overline{\phi}\colon X \to Y$ together with a family of morphisms $\phi_x\colon p(x) \to q(\overline{\phi}(x))$ for each $x \in X$. These are composed in the obvious way. We put $\lambda(X, p) = |X|$.

REMARK 13.35. If $\mathcal{C}$ is just a preordered set, then a morphism from (X, p) to (Y, q) is just a strictly monotone map $\overline{\phi}\colon X \to Y$ such that $p(x) \leq q(\overline{\phi}(x))$ for all x.

The following result is standard (although typically formulated a little differently). We give the proof to illustrate the use of Proposition 13.33.

PROPOSITION 13.36 (Higman's Lemma). $\mathcal{SC}$ *is wqo.*

Proof. For (X, p) with $X \neq \emptyset$ we define $x_0 = \min(X)$ and $\epsilon(X, p) = p(x_0) \in \mathcal{C}$ and $\chi(X, p) = (X', p')$, where $X' = X \setminus \{x_0\}$ and $p' = p|_{X'}$. This clearly satisfies condition (a) of Proposition 13.33. If $u\colon \mathbb{N} \to \mathcal{SC}$ is bad then $u(n)$ can never be empty (otherwise we would have $u(n) \leq u(n+1)$), so we have a sequence $u_1 = \epsilon \circ u\colon \mathbb{N} \to \mathcal{C}$. As $\mathcal{C}$ is wqo, we can choose a strictly increasing map $f\colon \mathbb{N} \to \mathbb{N}$ such that $u_1 \circ f\colon \mathbb{N} \to P$ is very good. Now put $v = u \circ f$. If $i < j$ and $\chi(v(i)) \leq \chi(v(j))$ then we also have $\epsilon(v(i)) \leq \epsilon(v(j))$ and it follows easily that $v(i) \leq v(j)$. Using Proposition 13.33 we can now see that $\mathcal{SC}$ is wqo. $\square$

DEFINITION 13.37. Let X and Y be nonempty finite totally ordered sets. Let $\phi\colon X \to Y$ be a surjective map, which need not preserve the order. We define an $\phi^\dagger\colon Y \to X$ by $\phi^\dagger(y) = \min(\phi^{-1}\{y\})$. We say that ϕ is †-*monotone* if $\phi^\dagger$ is monotone.

LEMMA 13.38. *For any ϕ we have $\phi\phi^\dagger(y) = y$ for all $y \in Y$, and $\phi^\dagger\phi(x) \leq x$ for all $x \in X$. If ϕ is †-monotone then we have $\phi(x) < y$ whenever $x < \phi^\dagger(y)$. In particular, if x_0 and y_0 are the smallest elements of X and Y, then $\phi(x_0) = y_0$ and $\phi^\dagger(y_0) = x_0$.*

Proof. It is clear by definition that $\phi\phi^\dagger(y) = y$. Next, if $x \in X$ then x is a preimage of $\phi(x)$, whereas $\phi^\dagger\phi(x)$ is the smallest preimage, so $\phi^\dagger\phi(x) \leq x$. Now suppose that ϕ is †-monotone. If $y \leq \phi(x)$ then $\phi^\dagger(y) \leq \phi^\dagger\phi(x) \leq x$. By the contrapositive, if $x < \phi^\dagger(y)$ we must have $\phi(x) < y$, as claimed. We now claim that $x_0 = \phi^\dagger(y_0)$. Indeed, if not then $x_0 < \phi^\dagger(y_0)$ so $\phi(x_0) < y_0$, contradicting the definition of y_0. We must therefore have $x_0 = \phi^\dagger(y_0)$ after all, and it follows that $\phi(x_0) = \phi\phi^\dagger(y_0) = y_0$. $\square$

COROLLARY 13.39. *Suppose we have †-monotone maps*

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z.$$

Then $(\psi\phi)^\dagger = \phi^\dagger\psi^\dagger$, and so $\psi\phi$ is also †-monotone.

Proof. Given $z \in Z$ put $y = \psi^\dagger(z)$ and $x = \phi^\dagger(y) = \phi^\dagger\psi^\dagger(z)$. Using the Lemma we get $\psi\phi(x) = z$. We also see that if $x' < x = \phi^\dagger(y)$ then $\phi(x') < y = \psi^\dagger(z)$ and thus $\psi(\phi(x')) < z$. This means that x has the defining property of $(\psi\phi)^\dagger(z)$. We therefore have $(\psi\phi)^\dagger = \phi^\dagger\psi^\dagger$. This is the composite of two increasing maps, so it is again increasing, so $\psi\phi$ is $\dagger$-monotone. $\square$

DEFINITION 13.40. We define a category $\mathcal{L}_\dagger$ as follows. The objects are finite nonempty sets X equipped with a map $e_X \colon X \to \mathbb{N}$, together with a total order on X. The morphisms from X to Y are $\dagger$-monotone surjective maps $\phi \colon X \to Y$ such that $e_Y(\phi(x)) \leq e_X(x)$ for all $x \in X$.

DEFINITION 13.41. We define $\alpha, \beta \colon \mathcal{L}_\dagger \to \mathbb{N}$ by $\alpha(X) = e_X(\min(X))$ and $\beta(X) = \min(e_X(X))$. Next, for $x \in X \setminus \{\min(X)\}$ we define

$$e'_X(x) = \min\{e_X(x') \mid x' < x\} \in \mathbb{N},$$

and $e^*_X(x) = (e_X(x), e'_X(x)) \in \mathbb{N}^2$. The set $X \setminus \{\min(X)\}$ together with the map e^*_X define an object $\gamma(X) \in \mathcal{S}(\mathbb{N}^2)$.

PROPOSITION 13.42. *The map* $(\alpha, \beta, \gamma) \colon \mathcal{L}_\dagger^{\mathrm{op}} \to \mathbb{N}^2 \times \mathcal{S}(\mathbb{N}^2)$ *is comonotone, so* $\mathcal{L}_\dagger$ *is cowqo.*

Proof. Suppose that $\alpha(X) \leq \alpha(Y)$ and $\beta(X) \leq \beta(Y)$ and $\gamma(X) \leq \gamma(Y)$; we need to construct a morphism from Y to X. As $\beta(X) \leq \beta(Y)$, we can choose a strictly increasing map $\psi \colon X \setminus \{\min(X)\} \to Y \setminus \{\min(Y)\}$ with $e_X(x) \leq e_Y(\psi(x))$ and $e'_X(x) \leq e'_X(\psi(x))$ for all x. We extend ψ over all of X by putting $\psi(\min(X)) = \min(Y)$, and note that the relation $e_X(x) \leq e_Y(\psi(x))$ remains true. We define $\phi \colon \psi(X) \to X$ by $\phi(\psi(x)) = x$. Now consider an element $y \in Y \setminus \psi(X)$, so $y \neq \min(Y)$. If $y > \max(\psi(X))$ we choose x with $e_X(x) = \beta(X)$ and define $\phi(y) = x$, noting that $e_Y(y) \geq \beta(Y) \geq \beta(X) = e_X(x)$. Otherwise, we let x' be least such that $\psi(x') > y$, then choose $x < x'$ with $e_X(x) = e'_X(x')$. This gives

$$e_Y(y) \geq e'_Y(\psi(x')) \geq e'_X(x') = e_X(x),$$

and we define $\phi(y) = x$. We now have a surjective map $\phi \colon Y \to X$ with $e_Y(y) \geq e_X(\phi(y))$ for all y. We also have $\phi(\psi(x)) = x$, and $\phi(y) < x$ whenever $y < \psi(x)$, so that $\psi = \phi^\dagger$. This means that ϕ is a morphism in $\mathcal{L}^\dagger$, as required. $\square$

COROLLARY 13.43. $\mathcal{L}_\dagger$ *is slice-cowqo*

Proof. The construction $(X \xleftarrow{p} U) \mapsto (p^{-1}\{x\})_{x \in X}$ gives a full and faithful embedding $\mathcal{L}_\dagger \downarrow X \to \prod_{x \in X} \mathcal{L}_\dagger$. Finally apply Proposition 13.30. $\square$

HOM-ORDERINGS

REMARK 13.44. In Definition 13.22 we defined the notion of a hom-ordering on $\mathcal{C}$. We can spell out the dual notion as follows: a hom-ordering of $\mathcal{C}^{op}$ consists of a system of well-orderings of the hom sets $\mathcal{C}(X, Y)$ such that for all $\beta \colon W \to X$, the induced map $\beta^* \colon \mathcal{C}(X, Y) \to \mathcal{C}(W, Y)$ is monotone.

Remark 13.45. If $F\colon \mathcal{C} \to \mathcal{D}$ is a faithful functor and we have a hom-ordering on $\mathcal{D}$ then we can define a hom-ordering on $\mathcal{C}$ by declaring that $\phi \leq \psi$ if and only if $F\phi \leq F\psi$.

Definition 13.46. Let $\mathcal{F}_\dagger$ be the category of finite totally ordered sets and $\dagger$-monotone surjections. We order $\mathcal{F}_\dagger(X,Y)$ lexicographically, so $\phi < \psi$ if and only if there exists $x_0 \in X$ with $\phi(x_0) < \psi(x_0)$ and $\phi(x) = \psi(x)$ for all $x < x_0$.

Proposition 13.47. *This gives a hom-ordering on $\mathcal{F}_\dagger^{\mathrm{op}}$.*

Proof. It is standard and easy that the above rule gives a total order on the finite set of surjections from X to Y. Now suppose we have $\theta\colon W \to X$ and $\phi, \psi\colon X \to Y$ with $\phi \leq \psi$; we must show that $\phi\theta \leq \psi\theta$. By assumption there exists $x_0 \in X$ with $\phi(x_0) < \psi(x_0)$ and $\phi(x) = \psi(x)$ for all $x < x_0$. Put $w_0 = \theta^\dagger(x_0) = \min(\theta^{-1}\{x_0\})$. Then $(\phi\theta)(w_0) = \phi(x_0) < \psi(x_0) = (\psi\theta)(w_0)$. On the other hand, if $w < w_0$ then Lemma 13.38 tells us that $\theta(w) < x_0$ and so $(\phi\theta)(w) = (\psi\theta)(w)$. $\square$

Corollary 13.48. *The faithful forgetful functor $\mathcal{L}_\dagger^{\mathrm{op}} \to \mathcal{F}_\dagger^{\mathrm{op}}$ gives a hom-ordering to $\mathcal{L}_\dagger^{\mathrm{op}}$.* $\square$

Proof of Theorem 13.15

For the duration of this proof we put

$$\mathcal{P} = \mathcal{Z}[p^\infty] = \{\text{finite abelian } p\text{-groups}\}$$

and $C[k] = \mathbb{Z}/p^k \in \mathcal{P}$. If $k \geq m$, we write π for the standard surjective homomorphism $C[k] \to C[m]$. For $A \in \mathcal{P}$ and $a \in A$, we let η_a be the natural number such that a has order p^{η_a}

By combining Corollaries 13.43 and 13.48, we see that $\mathcal{L}_\dagger^{\mathrm{op}}$ is Gröbner.

We define an essentially surjective functor $M\colon \mathcal{L}_\dagger^{\mathrm{op}} \to \mathcal{P}^{\mathrm{op}}$ as follows. For an object $X \in \mathcal{L}_\dagger$, we set $MX = \prod_{x \in X} C[e_X(x)]$. Given a morphism $\phi\colon X \to Y$ in $\mathcal{L}_\dagger$, we define $\phi_*\colon MX \to MY$ by

$$(\phi_* m)_y = \prod_{\phi(x)=y} \pi(m_x).$$

Let us introduce some terminology before proceeding with the proof. A *framing* of $A \in \mathcal{P}$ is a surjective homomorphism $MX \to A$ for some $X \in \mathcal{L}_\dagger$. This corresponds to a map $\alpha_0\colon X \to A$ such that $\eta(\alpha_0(x)) \leq e_X(x)$ for all x, and $\alpha_0(X)$ generates A. We say that the framing is *tautological* if X is a subset of A and α_0 is just the inclusion and

$$e_X(x) = \max\{\eta(w) \mid w \in X,\ w \leq x\}.$$

It is clear from the definition that there are only finitely many tautological framings. Unravelling the definitions, we see that M satisfies condition (F) if any framing $\alpha_0 \colon X \to A$ factors as

$$X \to \overline{X} \to A$$

where the first arrow is in $\mathcal{L}_\dagger$ and the second one is a tautological framing. So if $\alpha \colon X \to A$ is an arbitrary framing, we define $\overline{X} = \alpha_0(X) \subset A$ and $e_{\overline{X}} = \eta|_{\overline{X}}$ and set $\overline{\alpha}_0 \colon \overline{X} \to A$ to be the inclusion. We also define $\alpha_0^\dagger \colon A \to X$ by $\alpha_0^\dagger(a) = \min(\alpha_0^{-1}(a))$ and order $\overline{X}$ by declaring that $a < b$ iff $\alpha_0^\dagger(a) < \alpha_0^\dagger(b)$. This makes $\overline{\alpha}_0$ into a tautological framing and gives the required factorization. Therefore $\mathcal{P}^{\mathrm{op}}$ is quasi-Gröbner and so part (d) holds.

For part (c), we put

$$\Omega = \{\eta_a \mid A \in \mathcal{U}, \; a \in A\} \subset \mathbb{N}.$$

Define $\mathcal{L}_\dagger^{\mathcal{U}}$ to be the full subcategory of $\mathcal{L}_\dagger$ consisting of objects X with image$(e_X) \subset \Omega$. This is still Gröbner by [20, 4.4.2]. It is now easy to check that the functor $M \colon (\mathcal{L}_\dagger^{\mathcal{U}})^{\mathrm{op}} \to \mathcal{U}^{\mathrm{op}}$ defined as above is essentially surjective and satisfies property (F). Thus $\mathcal{U}^{\mathrm{op}}$ is quasi-Gröbner and $\mathcal{A}\mathcal{U}$ is locally noetherian.

14   REPRESENTATION STABILITY

In this section we show that any finitely presented object can be recovered by a finite amount of data via a stabilization recipe. This phenomenon is called central stability and it was first introduced by Putman [17]. We also show that under the noetherian assumption, any finitely generated object satisfies the analogue of the injectivity and surjectivity conditions in the definition of representation stability due to Church–Farb [4, 1.1].

DEFINITION 14.1. Let $\mathcal{U}$ be a subcategory of $\mathcal{G}$. For $X \in \mathcal{A}\mathcal{U}$, we put

$$\tau_n(X) = i_!^{\leq n} i_{\leq n}^* (X) \in \mathcal{A}\mathcal{U},$$

and note that there is a counit map $\tau_n(X) \to X$. We also define natural maps $\tau_n(X) \to \tau_{n+1}(X)$ as follows. Let j denote the inclusion $\mathcal{U}_{\leq n} \to \mathcal{U}_{\leq(n+1)}$, so we have a counit map $j_! j^*(Y) \to Y$ for all $Y \in \mathcal{A}\mathcal{U}_{\leq(n+1)}$. Taking $Y = i_{\leq(n+1)}^*(X)$ for some $X \in \mathcal{A}\mathcal{U}$, we get a map $j_! i_{\leq(n+1)}^* X \to i_{\leq(n+1)}^* X$. Applying the functor $i_!^{\leq(n+1)}$ to this gives the required map $\tau_n(X) \to \tau_{n+1}(X)$.

We list a few important properties of the truncation functor.

PROPOSITION 14.2. *Consider an object $X \in \mathcal{A}\mathcal{U}$.*

(a) *Then X is the colimit of the objects $\tau_n(X)$.*

(b) *We have $\tau_n(e_G) = e_G$ if $G \in \mathcal{U}_{\leq n}$ and $\tau_n(e_G) = 0$ otherwise.*

(c) For all $G \in \mathcal{U}$ and $n \geq 0$, we have

$$\tau_n(X)(G) = \varinjlim_{H \in N(G,n)} X(G/H)$$

where $N(G,n) = \{H \lhd G \mid |G/H| \leq n\}$.

Proof. For part (a) it is enough to notice that $\tau_n(X)(G) = X(G)$ for $|G| \leq n$. Part (b) follows from Lemma 5.3(i). Using the formula for Kan extensions, we see that $\tau_n(X)(G)$ can be written as a colimit over the comma category $(G \downarrow \mathcal{U}_{\leq n})$. Suppose we have objects $(G \xrightarrow{\alpha} A)$ and $(G \xrightarrow{\beta} B)$ in the comma category so $A, B \in \mathcal{U}_{\leq n}$. As α and β are surjective, we find that there is a unique morphism from α to β if $\ker(\alpha) \leq \ker(\beta)$, and no morphisms otherwise. This shows that the comma category is equivalent to the poset $N(G,n)$ so part (c) follows. $\qquad\square$

The following is a characterization of finitely generated and finite presented objects.

PROPOSITION 14.3. *Consider an object $X \in \mathcal{AU}$.*

(a) X is finitely generated if and only if X has finite type and there exists $N \in \mathbb{N}$ such that the canonical map $\tau_n(X) \to X$ is an epimorphism for all $n \geq N$.

(b) X is finitely presented if and only if X has finite type and there exists $N \in \mathbb{N}$ such that the canonical map $\tau_n(X) \to X$ is an isomorphism for all $n \geq N$.

Proof. For part (a), assume that the map $\tau_n(X) \to X$ is an epimorphism for all $n \geq N$. Note that we can construct an epimorphism

$$\bigoplus_{G \in \mathcal{U}_{\leq n}} \dim(X(G)) \, e_G \to i^*_{\leq n}(X)$$

as X has finite type. We apply $i^{\leq n}_!$ to get an epimorphism

$$\bigoplus_{G \in \mathcal{U}_{\leq n}} \dim(X(G)) \, e_G \to \tau_n(X)$$

since $i^{\leq n}_!$ preserves all colimits by Lemma 5.3(f). Post-composition with $\tau_n(X) \to X$ gives the desired epimorphism. Conversely, assume that X is finitely generated so that we have a short exact sequence $0 \to K \to P \to X \to 0$ with P finitely projective. Note that by Proposition 14.2(b), there must exist $N \in \mathbb{N}$ such that $\tau_n(P) \simeq P$ for all $n \geq N$. The commutativity of the diagram

$$\begin{array}{ccccc} P & \longrightarrow & X & \longrightarrow & 0 \\ \simeq\big\uparrow & & \big\uparrow & & \\ \tau_n(P) & \longrightarrow & \tau_n(X) & & \end{array}$$

implies that the map $\tau_n(X) \to X$ is an epimorphism for all $n \geq N$.

For part (b), assume that X is finitely presented. Then there exists a short exact sequence $0 \to K \to P \to X \to 0$ with P finitely projective and K finitely generated. By Part (a), it is enough to show that the canonical map $\tau_n(X) \to X$ is eventually monic. Note that for large n, we have a diagram

$$
\begin{array}{ccccc}
\ker(i_K^n) & & 0 & & \ker(i_X^n) \\
\downarrow & & \downarrow & & \downarrow \\
\tau_n(K) & \longrightarrow & \tau_n(P) & \longrightarrow & \tau_n(X) \longrightarrow 0 \\
\downarrow{\scriptstyle i_K^n} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle i_X^n} \\
0 \longrightarrow K & \longrightarrow & P & \longrightarrow & X \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
\cok(i_K^n) & & 0 & & \cok(i_X^n)
\end{array}
$$

where the bottom row is exact and the top is only right exact. By assumption both K and X are finitely generated, so the maps i_K^n and i_X^n are epimorphisms by part (a). Thus, the Snake Lemma tell us that $\ker(i_X^n) = 0$. Conversely, assume that the natural map is an isomorphism. By part (a), X is finitely generated so we have a short exact sequence $0 \to K \to P \to X \to 0$ with P finitely projective. By applying the Snake Lemma to the diagram above, we see that $\cok(i_K^n) = 0$ for large n, so K is finitely generated and X is finitely presented. $\qquad\square$

We note that by combining Propositions 14.2 and 14.3 we obtain that any finitely presented object satisfies central stability as mentioned in the introduction.

REMARK 14.4. Recall the functor $q_{\leq n}$ from Example 5.9. We have seen that $q_{\leq n}$ is left adjoint to the inclusion $\mathcal{U}_{\leq n}^\star \to \mathcal{G}$. If $\mathcal{U}$ is closed downwards, then $q_{\leq n}$ is also the left adjoint to the inclusion $\mathcal{U}_{\leq n} \to \mathcal{U}$.

PROPOSITION 14.5. *Let $\mathcal{U}$ be multiplicative and closed under passage to subgroups, and consider a finitely presented object $X \in \mathcal{A}\mathcal{U}$. Then there exists $n \in \mathbb{N}$ such that $X(G) = X(q_{\leq n}G)$ for all $G \in \mathcal{U}$.*

Proof. Choose a finite presentation

$$
\bigoplus_{i=1}^{r} e_{G_i} \xrightarrow{f} \bigoplus_{j=1}^{s} e_{H_j} \to X \to 0.
$$

Choose n large enough so that $G_i, H_j \in \mathcal{U}_{\leq n}^*$ for all i and j. Let Y be cokernel of f in $\mathcal{A}\mathcal{U}_{\leq n}^*$. We claim that $X = q_{\leq n}^*(\bar{Y})$. As the functor q_n^* preserves all colimits it is enough to show that $q_{\leq n}^* e_G = e_G$ for all $G \in \mathcal{U}_{\leq n}^*$. Using that $q_{\leq n}$ is left adjoint to the inclusion $\mathcal{U}_{\leq n}^* \to \mathcal{U}$ we see that

$$
(q_{\leq n}^* e_G)(H) = k[\mathcal{U}(q_{\leq n}H, G)] = k[\mathcal{U}(H, G)] = e_G(H)
$$

which concludes the proof.   □

We now restrict to the locally noetherian case. Recall the definition of eventually torsion-free and generated in finite degree object from the introduction, see Definition B.

THEOREM 14.6. *Let* $X \in \mathcal{AZ}[p^\infty]$ *be a finitely generated object. Then the restriction of* X *to* $\mathcal{AC}[p^\infty]$ *and* $\mathcal{AF}[p^n]$, *for all* $n \geq 1$, *is generated in finite degree and eventually torsion-free.*

Proof. Firstly we note that the restriction of X to $\mathcal{AC}[p^\infty]$ and $\mathcal{AF}[p^n]$ is again finitely generated by Lemmas 11.2 and 11.4. Note also that $\mathcal{C}[p^\infty]$ and $\mathcal{F}[p^n]$ satisfy the transitivity property, see Definition 13.7. For the family of cyclic p-groups, we have proved this in the proof of Theorem 13.14. For the families $\mathcal{F}[p^n]$ this is a special case of Lemma 9.10. Since the abelian categories $\mathcal{AC}[p^\infty]$ and $\mathcal{AF}[p^n]$ are locally noetherian by Theorem 13.4, we can apply [6, 5.1, 5.2] and deduce that the restriction is generated in finite degree and eventually torsion-free.   □

We conclude this section by proving Theorem C from the introduction.

Proof of Theorem C. First of all note that the equivalence (1.0.1) in the introduction descends to an equivalence between the full subcategories of compact objects $(\mathrm{Sp}_{\mathcal{U}}^{\mathbb{Q}})^\omega \simeq \mathcal{D}(\mathcal{AU})^\omega$ for any family $\mathcal{U} \leq \mathcal{G}$. We can apply [10, 2.3.12] to deduce that
$$\mathcal{D}(\mathcal{AZ}[p^\infty])^\omega = \mathrm{thick}(e_G \mid G \in \mathcal{Z}[p^\infty])$$
where the right hand side denotes the smallest thick (=closed under retracts) triangulated subcategory containing the generators e_G for $G \in \mathcal{Z}[p^\infty]$.
Consider the full subcategory
$$\mathcal{T} = \{X \mid H_*(X) \text{ is finitely generated}\} \subset \mathcal{D}(\mathcal{AZ}[p^\infty])^\omega.$$

Since $\mathcal{AZ}[p^\infty]$ is locally noetherian one easily checks that $\mathcal{T}$ is a thick triangulated subcategory. Clearly $e_G \in \mathcal{T}$ for all $G \in \mathcal{Z}[p^\infty]$ so by the discussion in the previous paragraph we see that any compact object lies in $\mathcal{T}$. Finally apply Theorem B.   □

15   Injectives

We now turn to study the injective objects of $\mathcal{AU}$. Unlike in the projective case, a complete classification of the indecomposable injective objects seems at the moment far out of reach. The main difficulty arises from the fact that any projective object is necessarily torsion-free whereas an injective object can be torsion, absolutely torsion or torsion-free.
Recall that if $\mathcal{U}$ has a colimit tower then the dual of any object is injective by Proposition 9.1. Let us produce more examples of injective objects.

PROPOSITION 15.1. *Let $\mathcal{U}$ be a multiplicative global family. Then the torsion-free injective objects coincide with the projective objects.*

Proof. Suppose that $\mathcal{U}$ is a multiplicative global family and consider a projective object P. We will show that P is injective giving one of the implications in the proposition. We can write $P = \prod_n P_n$ by Proposition 8.6, so it will suffice to show that P_n is injective. We have $P_n = (i_n)_!(i_n^* P_n)$ and $i_n^* P_n$ is projective in $\mathcal{A}\mathcal{U}_n$. We can write $i_n^* P_n$ as a retract of an object $Q = \bigoplus_t e_{G_t}$ with $G_t \in \mathcal{U}_n$. This embeds in the product $R = \prod_t e_{G_t}$, and all monomorphisms in $\mathcal{A}\mathcal{U}_n$ are split, so $i_n^* P_n$ is a retract of R. We know that $(i_n)_!$ preserves products by Proposition 7.3, so $P_n = (i_n)_!(i_n^* P_n)$ is a retract of $\prod_t (i_n)_!(e_{G_t}) = \prod_t e_{G_t}$. Therefore, it is enough to show that e_{G_t} is injective. This now follows from the fact that De_{G_t} is injective and that e_{G_t} is a summand of De_{G_t} by Theorem 4.18. Therefore P is injective as claimed. Conversely, let I be a torsion-free injective. By Proposition 12.16, we can embed I into a projective object SI. Since I is injective, the inclusion $I \to SI$ splits showing that I is projective as required. $\square$

REMARK 15.2. Let $\mathcal{C}[2^\infty]$ be the family of cyclic 2-groups. Then we have a short exact sequence
$$0 \to e_{C_2} \to \mathbb{1} \to t_{1,k} \to 0$$
that cannot split as $\mathbb{1}$ is torsion-free and $t_{1,k}$ is torsion. Hence e_{C_2} is not injective in $\mathcal{A}\mathcal{C}[2^\infty]$.

The following structural result, classically due to Matlis [13], suggests that we can restrict our attention to indecomposable injectives.

THEOREM 15.3 ([5, Chaper IV]). *Any injective object in a locally noetherian abelian category is a sum of indecomposable injectives.*

LEMMA 15.4. *Let $\mathcal{U}$ be multiplicative global family of $\mathcal{V}$.*

(a) *For any $G \in \mathcal{V}$ and V irreducible $\mathrm{Out}(G)$-representation, the object $t_{G,V}$ is indecomposable and injective in $\mathcal{A}\mathcal{V}$. Furthermore, $t_{G,V}$ is the injective envelope of $s_{G,V}$.*

(b) *For any $G \in \mathcal{U}$ and V irreducible $\mathrm{Out}(G)$-representation, the object $\chi_{\mathcal{U}} \otimes e_{G,V}$ is indecomposable and injective in $\mathcal{A}\mathcal{V}$.*

Proof. We have seen that $t_{G,V}$ is injective and it is indecomposable by Lemma 5.3(e). If $\mathcal{U}$ is a multiplicative global family, then $e_{G,V}$ is injective and so combining part (e) and (i) of Lemma 5.3 we see that $i_*(e_{G,V}) = \chi_{\mathcal{U}} \otimes e_{G,V}$ is an indecomposable injective. Finally note that there is a canonical monomorphism $s_{G,V} \to t_{G,V}$, so the injective hull of $s_{G,V}$ is a direct summand of $t_{G,V}$ so the claim follows by indecomposability $\square$

The next result classifies the indecomposable injective objects which are absolutely torsion.

Lemma 15.5. *Let $\mathcal{U}$ be a subcategory of $\mathcal{G}$ and let $I \in \mathcal{A}\mathcal{U}$ be injective. Then I is a retract of a product of objects $t_{G,V}$ with $G \in \mathcal{U}$. If in addition I is absolutely torsion, then it is a retract of a sum of objects $t_{G,V}$ with $G \in \mathcal{U}$.*

Proof. By Construction 5.6, we have a monomorphism

$$\text{env}\colon I \to \prod_{G \in \mathcal{U}'} t_{G,I(G)} = l_* l^*(I)$$

By injectivity of I, the map env splits and so I is a retract of $l_* l^*(I)$. If in addition I is absolutely torsion, then the image of any element of I under env is nonzero only for finitely many $G \in \mathcal{U}'$, so the morphism env factors through the direct sum. $\square$

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