Homotopy Groups of Highly Connected Poincaré Duality Complexes

Piotr Beben and Stephen Theriault

Received: April 19, 2019
Revised: January 28, 2022

Communicated by Mike Hill

Abstract. Methods are developed to relate the action of a principal fibration to relative Whitehead products in order to determine the homotopy type of certain spaces. The methods are applied to thoroughly analyze the homotopy type of the based loops on certain cell attachments. Key examples are $(n-1)$-connected Poincaré Duality complexes of dimension $2n$ or $2n+1$ with minor cohomological conditions.

2020 Mathematics Subject Classification: Primary 55P35, 57N65; Secondary 55Q15
Keywords and Phrases: Principal fibration, Whitehead product, loop space decomposition, Poincaré duality complex

1 Introduction

A long-standing problem in homotopy theory is to determine the effect on homotopy type from attaching a cell. Starting with a space X one considers a cofibration $S^m \xrightarrow{f} X \longrightarrow X \cup CS^m$ where, for a space A, CA is the reduced cone on A. The aim is to determine the homotopy type of $X \cup CS^m$, or the based loop space $\Omega(X \cup CS^m)$, from information on the homotopy type of X and the homotopy class of f. Rational homotopy theory has developed methods for dealing with certain kinds of cell attachments, called inert or lazy [FT, HaL, HeL]. These methods do not translate well to the p-local case, except for primes that are large relative to the dimension of X divided by its connectivity (see, for example, [A]), and therefore give limited information on the integral homotopy type.

If cell attachments are generalized to attaching a cone, and the context is specialized somewhat, then information can sometimes be obtained. For example, suppose that there is a fibration $F \longrightarrow E \longrightarrow B$. Then the map $E \longrightarrow B$ extends to a map $E \cup CF \longrightarrow B$. If G is the homotopy fibre of this new map, then Ganea [Ga] showed that $G \simeq F * \Omega B$, where the right side is the join of F and ΩB. Further, he showed that there is a homotopy equivalence $\Omega(E \cup CF) \simeq \Omega B \times \Omega(F * \Omega B)$. Note that this holds integrally. Ganea's result can be recovered as a special case of Mather's Cube Lemma [M], and the Cube Lemma has been used in a wide variety of contexts to identify the integral homotopy types of certain spaces.

We consider the following case related to the cell attachment problem and Ganea-type results. Suppose that there is a cofibration $A \xrightarrow{f} Y \longrightarrow Y \cup CA$; there is a map $Y \longrightarrow Z$ which induces a principal fibration $\Omega Z \longrightarrow E \xrightarrow{p} Y$; and the map $Y \longrightarrow Z$ extends to a map $Y \cup CA \longrightarrow Z$, inducing a principal fibration $\Omega Z \longrightarrow E' \xrightarrow{p'} Y \cup CA$. In Sections 2 and 3 we develop new techniques that relate the action of a principal fibration to relative Whitehead products in order to identify the homotopy type of E' and the homotopy class of p' in terms of the homotopy type of E and the homotopy classes of p and f. This requires certain hypotheses on the spaces and maps involved, but these are fulfilled in a wide variety of contexts.

The new methods are powerful and should have numerous applications. We use them to prove general results about certain cell attachments in Theorems 4.1 and 5.1. Philosophically, these cell attachments are integral versions of inert maps in rational homotopy theory. Key examples are $(n-1)$-connected $2n$-dimensional Poincaré Duality complexes M having $H^n(M;\mathbb{Z})) \cong \mathbb{Z}^d$ for $d \geq 2$ and $n \notin \{4, 8\}$, and $(n-1)$-connected $(2n+1)$-dimensional Poincaré Duality complexes M having $H^{n+1}(M;\mathbb{Z})) \cong \mathbb{Z}^d$ for $d \geq 1$. In Examples 4.2 and 4.4 we give an explicit homotopy decomposition of ΩM and identify the homotopy classes of the maps from the factors into ΩM. In Examples 5.2 and 5.3 we go further: if $\overline{M}$ is M with a point removed and $\overline{M} \xrightarrow{j} M$ is the inclusion, then we explicitly identify the homotopy fibre of j, the homotopy class of the map from the fibre into $\overline{M}$, and show that Ωj has a right homotopy inverse. Collectively, these results give a thorough picture of the homotopy theory associated to M. They subsume most of the results in [B, BB, BT, BW] and often go much further. For example, the statements about the map j were known only in the case when M is a simply-connected 4-manifold. Finally, in Section 6 we give an entirely new example that gives conditions for when the map $M \# N \longrightarrow M \vee N$ that collapses the collar in a connected sum of two Poincaré duality complexes has a right homotopy inverse after looping.

Acknowledgement

The authors would like to thank the referee for a careful reading of the paper and numerous valuable comments.

2 Principal fibrations and relative Whitehead products

In this section we define relative Whitehead products and relate them to the action induced by a principal fibration. This will be an important technical tool used later in the paper. The material in this section is a development of that in [Gr, §3], which in turn was a development on [N, §6.10]. We give a full account in order to have to hand all the material needed for later.

In what follows, it should be pointed out that by a fibration we mean a strict fibration, not a fibration up to homotopy. All spaces are assumed to be path-connected and pointed, with the basepoint generically denoted by $*$, and to have the homotopy type of a finite type CW-complex. For a space X, let X^I be the space of (not necessarily pointed) continuous maps from the unit interval I to X. Let $PX = \{\omega \in X^I \mid \omega(0) = *\}$ be the *path space* of X. Let $ev_1 \colon PX \longrightarrow X$ be the evaluation map, defined by $ev_1(\omega) = \omega(1)$. The *loop space* ΩX is the subspace of PX with the property that $ev_1(\omega) = *$. It is well known that there is a fibration $\Omega X \longrightarrow PX \xrightarrow{ev_1} X$.

Let $\Omega Z \xrightarrow{i} E \xrightarrow{p} B$ be a principal fibration induced by a map $\varphi \colon B \longrightarrow Z$. Precisely, E and p are defined by the pullback

$$\begin{array}{ccc} E & \longrightarrow & PZ \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle ev_1} \\ B & \xrightarrow{\varphi} & Z. \end{array}$$

So $E = \{(b, \omega) \in B \times PZ \mid \omega(1) = b\}$ and p is the projection, $p(b, \omega) = b$.

This principal fibration has an action of the fibre on the total space,

$$a \colon \Omega Z \times E \longrightarrow E$$

defined by $a(\gamma, (b, \omega)) = (b, \omega \circ \gamma)$. One useful property this satisfies is the following. The definition of p as a projection implies that there is a strictly commutative diagram

$$\begin{array}{ccc} \Omega Z \times E & \xrightarrow{a} & E \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle p} \\ E & \xrightarrow{p} & B. \end{array} \tag{1}$$

Consider the map $B \vee E \xrightarrow{1 \vee p} B$. Since the composite $E \xrightarrow{p} B \xrightarrow{\varphi} Z$ is null homotopic, there is a homotopy commutative diagram

$$\begin{array}{ccc} B \vee E & \xrightarrow{1 \vee p} & B \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \varphi} \\ B & \xrightarrow{\varphi} & Z \end{array}$$

where p_1 is the pinch map onto the first wedge summand. We wish to have a model for the homotopy fibre of p_1.

In general, if $f\colon X \longrightarrow Y$ is a continuous map with Y connected, there is a standard way of turning f into a fibration. Define the space $\widetilde{X}$ as the pullback

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & Y^I \\ \downarrow & & \downarrow{\scriptstyle ev_1} \\ X & \xrightarrow{f} & Y \end{array}$$

where $ev_1(\omega) = \omega(1)$. Then $\widetilde{X} = \{(x,\omega) \in X \times Y^I \mid f(x) = \omega(1)\}$. Let $\widetilde{f}$ be the composite

$$\widetilde{f}\colon \widetilde{X} \longrightarrow Y^I \xrightarrow{ev_0} Y$$

where $ev_0(\omega) = \omega(0)$. Then $\widetilde{f}$ is a fibration. Define $\iota\colon X \longrightarrow \widetilde{X}$ by $\iota(x) = (x, c_{f(x)})$ where $c_{f(x)}$ is the constant path at $f(x)$. Then there is a strictly commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota} & \widetilde{X} \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \widetilde{f}} & \\ Y & & \end{array}$$

in which $\widetilde{f}$ is a fibration and ι is a homotopy equivalence. The *homotopy fibre* of f is the actual fibre of $\widetilde{f}$, which is the space $F_f = \{(x,\omega) \in \widetilde{X} \mid \omega(0) = *\}$, and this maps to X by the composite $F_f \longrightarrow \widetilde{X} \longrightarrow X$. Note that if this construction is applied to the inclusion of the basepoint $* \longrightarrow Y$ then $\widetilde{*}$ is the "reverse" path space on Y, $\widetilde{*} = \{\omega \in Y^I \mid \omega(1) = *\}$, and $F_f = \Omega Y$. If the construction is applied to the identity map $Y \longrightarrow Y$ then F_f is precisely the path space PY.

Apply this construction to the pinch map $B \vee E \xrightarrow{p_1} B$. The restriction of p_1 to B is the identity map on B. So the part of the fibre F_{p_1} corresponding to $B \subseteq B \vee E$ is PB and this maps to B by sending $\gamma \in PB$ to $\gamma(1) \in B$. The restriction of p_1 to E is the constant map to the basepoint, so the part of the fibre F_{p_1} corresponding to $E \subseteq B \vee E$ is $\Omega B \times E$, and this maps to E by projecting (γ, e) to e. The two parts of the fibre F_{p_1} that correspond to the basepoint $* \subseteq B \vee E$ match at ΩB. Thus $F_{p_1} = PB \cup_{\Omega B} \Omega B \times E$ and the map $PB \cup_{\Omega B} \Omega B \times E \longrightarrow B \vee E$ is given by sending $\gamma \in PB$ to $\gamma(1) \in B$ and projecting $(\gamma, e) \in \Omega B \times E$ to $e \in E$. The initial model for the homotopy fibre of p_1 is therefore $PB \cup_{\Omega B} \Omega B \times E$.

It is convenient to express this homotopy fibre in terms of the cone on ΩB, up to homotopy equivalence. For a space Y, the *reduced cone* on Y is defined by $CY = (Y \times I)/\sim$ where $(y,0) \sim *$ and $(*,t) \sim *$. Observe that Y includes into CY by sending y to $(y,1)$. Notice that CY is a lower cone, which we use instead of the more usual upper cone, as it makes several subsequent formulas easier to follow. It is well known that the map of pairs

$$\xi\colon (C\Omega B, \Omega B) \longrightarrow (PB, \Omega B)$$

defined by $\xi(\gamma,t)(s)=\gamma(st)$ is a homotopy equivalence.
Collecting all the information above, and noting that all the constructions involved are natural, we obtain the following.

LEMMA 2.1. *A model for the homotopy fibre of the pinch map* $B\vee E\xrightarrow{p_1}B$ *is* $C\Omega B\cup_{\Omega B}\Omega B\times E$, *and with respect to this model the map*

$$\psi\colon C\Omega B\cup_{\Omega B}\Omega B\times E\longrightarrow B\vee E$$

from the fibre is given by sending $(\gamma,t)\in C\Omega B$ *to* $\gamma(t)\in B$ *and projecting* $(\gamma,e)\in\Omega B\times E$ *to* $e\in E$. *Further, all of this is natural for maps* $B\vee E\longrightarrow B'\vee E'$. □

From this description we can immediately determine the following composition.

COROLLARY 2.2. *The composite* $C\Omega B\cup_{\Omega B}\Omega B\times E\xrightarrow{\psi}B\vee E\xrightarrow{1\vee p}B$ *is given by sending* $(\gamma,t)\in C\Omega B$ *to* $\gamma(t)\in B$ *and by sending* $(\gamma,e)\in\Omega B\times E$ *to* $p(e)$. □

Next, we wish to produce an alternative description of the composition in Corollary 2.2 which depends on the action of the principal fibration $\Omega Z\xrightarrow{i}E\xrightarrow{p}B$. The composite $\Omega B\xrightarrow{\Omega\varphi}\Omega Z\xrightarrow{i}E$ is null homotopic since it is the composition of two consecutive maps in a homotopy fibration sequence. An explicit null homotopy is as follows. Recall that

$$E=\{(b,\omega)\in B\times PZ\mid\omega(1)=\varphi(b)\}.$$

Define

$$H\colon\Omega B\times I\longrightarrow E$$

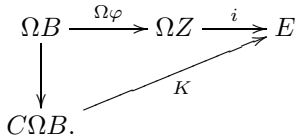
by $H(\gamma,t)=(\gamma(t),(\Omega\varphi)(\gamma_t))$ where $\gamma_t(s)=\gamma(st)$. Notice that γ_0 is the constant map and $\gamma_1=\gamma$. Observe that:

(i) $(\Omega\varphi)(\gamma_t)(1)=\varphi(\gamma_t(1))=\varphi(\gamma(t))$;

(ii) $H(\gamma,0)=(\gamma(0),(\Omega\varphi)(\gamma_0)=(*,*)$;

(iii) $H(\gamma,1)=(\gamma(1),(\Omega\varphi)(\gamma_1)=(*,(\Omega\varphi)(\gamma))=(i\circ\Omega\varphi)(\gamma)$.

Item (i) implies that $H(\gamma,t)\in E$ so H is well-defined, item (ii) implies that H_0 is the constant map and item (iii) implies that $H_1=i\circ\Omega\varphi$.
Recalling that $C\Omega B$ is a lower cone with ΩB including in by sending b to $(b,1)$, the homotopy H can be used to define a map $K\colon C\Omega B\longrightarrow E$ by $K(\gamma,t)=H(\gamma,t)$. Then there is a strictly commutative diagram



Consider the composite $C\Omega B \xrightarrow{K} E \xrightarrow{p} B$. From the description of E as $\{(b,\omega) \in B \times PZ \mid \omega(1) = \varphi(b)\}$ and p as the projection $p(b,\omega) = b$, we obtain

$$(p \circ K)(\gamma, t) = (p \circ H)(\gamma, t) = p((\gamma(t), (\Omega\varphi)(\gamma_t))) = \gamma(t). \tag{2}$$

That is, $p \circ K$ is the evaluation map.

We relate K to the action $\Omega Z \times E \xrightarrow{a} E$ for the principal fibration $\Omega Z \xrightarrow{i} E \xrightarrow{p} B$. Let θ be the composite

$$\theta\colon \Omega B \times E \xrightarrow{\Omega\varphi \times 1} \Omega Z \times E \xrightarrow{a} E.$$

Since the restriction of a to ΩZ is i, the restriction of θ to ΩB is $i \circ \Omega\varphi$. On the other hand, by definition of K in terms of the homotopy H and item (iii) above, the restriction of K to $\Omega B \subseteq C\Omega B$ is $i \circ \Omega\varphi$. Therefore there is a pushout map

$$\begin{array}{ccccc}
\Omega B & \longrightarrow & \Omega B \times E & & \\
\downarrow & & \downarrow & \searrow^{\theta} & \\
C\Omega B & \longrightarrow & C\Omega B \cup_{\Omega B} \Omega B \times E & \overset{\Gamma}{\dashrightarrow} & \\
 & \searrow_{K} & & & E
\end{array} \tag{3}$$

that defines Γ.

Lemma 2.3. *The composite $C\Omega B \cup_{\Omega B} \Omega B \times E \xrightarrow{\Gamma} E \xrightarrow{p} B$ is given by sending $(\gamma, t) \in C\Omega B$ to $\gamma(t) \in B$ and by sending $(\gamma, e) \in \Omega B \times E$ to $p(e)$.*

Proof. The restriction of Γ to $C\Omega B$ is K, so the restriction of $p \circ \Gamma$ to $C\Omega B$ is $p \circ K$, which by (2) is the evaluation map sending (γ, t) to $\gamma(t)$. The restriction of Γ to $\Omega B \times E$ is θ, so the restriction of $p \circ \Gamma$ to $\Omega B \times E$ is $p \circ \theta = p \circ a \circ (\Omega\varphi \times 1)$. Using (1), there is a strictly commutative diagram

$$\begin{array}{ccccc}
\Omega B \times E & \xrightarrow{\Omega\varphi \times 1} & \Omega Z \times E & \xrightarrow{a} & E \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle p} \\
E & = & E & \xrightarrow{p} & B
\end{array}$$

which shows that $p \circ \theta$ sends $(\gamma, e) \in \Omega B \times E$ to $p(e)$. □

Corollary 2.2 and Lemma 2.3 combine to give the following.

Proposition 2.4. *There is a strictly commutative diagram*

$$\begin{array}{ccc}
C\Omega B \cup_{\Omega B} \Omega B \times E & \xrightarrow{\Gamma} & E \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle p} \\
B \vee E & \xrightarrow{1 \vee p} & B.
\end{array}$$

□

Proposition 2.4 is a key technical result. It relates the homotopy fibre of the pinch map from $B \vee E$ to B to the action induced by the principal fibration p. Its importance will be seen in how it is used to relate certain Whitehead products on B to the principal action.

Let G be an H-group, which is a homotopy associative H-space with a homotopy inverse. For example, any loop space is an H-group. By [S], the multiplication m on G is homotopic to one in which the basepoint is a strict unit, that is, one for which $m(x, *) = x = m(*, x)$ for any $x \in G$ (this is true for any H-space, not just H-groups). From here on it will be assumed that the multiplication on G has a strict unit. If there are maps $f\colon A \longrightarrow G$ and $g\colon B \longrightarrow G$ the commutator $c(f, g)\colon A \times B \longrightarrow G$ of f and g is defined pointwise by $c(f, g)(a, b) = f(a)g(b)f(a)^{-1}g(b)^{-1}$. Since the multiplication on G has a strict unit, the restriction of $c(f, g)$ to $G \vee G$ is null homotopic, so $c(f, g)$ extends to a map

$$\langle f, g\rangle\colon A \wedge B \longrightarrow G$$

called the *Samelson product* of f and g. As the connecting map for the homotopy cofibration $A \vee B \longrightarrow A \times B \longrightarrow A \wedge B$ is null homotopic (since $\Sigma A \vee \Sigma B$ retracts off $\Sigma(A \times B)$), the homotopy class of $\langle f, g\rangle$ is determined uniquely by that of $c(f, g)$.

In general, if X and Y are path-connected spaces then there are homeomorphisms $\Sigma(X \wedge Y) \cong \Sigma X \wedge Y \cong X \wedge \Sigma Y$. In what follows the version used will depend only on aesthetics. Carrying on from the previous paragraph, if $G = \Omega Z$ then f, g have adjoints $f'\colon \Sigma A \longrightarrow Z$ and $g'\colon \Sigma B \longrightarrow Z$ respectively. The *Whitehead product* of f' and g' is the map $[f', g']\colon \Sigma A \wedge B \longrightarrow Z$ obtained by taking the adjoint of the Samelson product $\langle f, g\rangle$.

Let X and Y be path-connected spaces. Let $i_1\colon \Sigma X \longrightarrow \Sigma X \vee \Sigma Y$ and $i_2\colon \Sigma Y \longrightarrow \Sigma X \vee \Sigma Y$ be the inclusion of the left and right wedge summands respectively. Let

$$W\colon \Sigma X \wedge Y \longrightarrow \Sigma X \vee \Sigma Y$$

be the Whitehead product $W = [i_1, i_2]$. It is well known that there is a homotopy cofibration $\Sigma X \wedge Y \xrightarrow{W} \Sigma X \vee \Sigma Y \longrightarrow \Sigma X \times \Sigma Y$ where the right map is the inclusion of the wedge into the product. Observe that the pinch map $p_1\colon \Sigma X \vee \Sigma Y \longrightarrow \Sigma X$ factors as the composite $\Sigma X \vee \Sigma Y \longrightarrow \Sigma X \times \Sigma Y \xrightarrow{\pi_1} \Sigma X$ where π_1 is the projection onto the first factor. Thus $p_1 \circ W$ is null homotopic, implying that W lifts to the homotopy fibre of p_1. Using the model for the homotopy fibre of p_1 already established, we obtain a lift

$$\begin{array}{ccc} & & C\Omega\Sigma X \cup_{\Omega\Sigma X} \Omega\Sigma X \times \Sigma Y \\ & \overset{\lambda}{\nearrow} & \downarrow \psi \\ \Sigma X \wedge Y & \xrightarrow{W} & \Sigma X \vee \Sigma Y \end{array} \tag{4}$$

for some map λ. In the homotopy fibration sequence

$$\Omega(\Sigma X \vee \Sigma Y) \xrightarrow{\Omega p_1} \Omega\Sigma X \xrightarrow{\partial} C\Omega\Sigma X \cup_{\Omega\Sigma X} \Omega\Sigma X \times \Sigma Y,$$

where ∂ is the fibration connecting map, the map Ωp_1 has a right homotopy inverse, implying that ∂ is null homotopic. Thus the homotopy class of the lift λ is uniquely determined by the homotopy class of W. The naturality of W therefore implies the naturality of the homotopy class of the lift λ.
We develop this in the context of the wedge $B \vee E$ used previously. Suppose that there are maps $f\colon \Sigma X \longrightarrow B$ and $g\colon \Sigma Y \longrightarrow E$. Consider the diagram

$$\begin{array}{ccccccc} & & C\Omega\Sigma X \cup_{\Omega\Sigma X} \Omega\Sigma X \times \Sigma Y & \xrightarrow{\Theta} & C\Omega B \cup_{\Omega B} \Omega B \times E & \xrightarrow{\Gamma} & E \\ & \overset{\lambda}{\nearrow} & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle p} \\ \Sigma X \wedge Y & \xrightarrow{W} & \Sigma X \vee \Sigma Y & \xrightarrow{f\vee g} & B \vee E & \xrightarrow{1\vee p} & B \end{array} \tag{5}$$

where $\Theta = C\Omega f \cup_{\Omega f} (\Omega f \times g)$. The left triangle commutes by (4). The middle square strictly commutes by the naturality of Lemma 2.1. The right square strictly commutes by Proposition 2.4. Observe that the naturality of the Whitehead product $W = [i_1, i_2]$ implies that the composite along the bottom row of (5) is homotopic to the Whitehead product $[f, p \circ g]$. Thus $\Gamma \circ \Theta \circ \lambda$ is a lift of $[f, p \circ g]$ through p.

Definition 2.5. *The composite* $[f, g]_r = \Gamma \circ \Theta \circ \lambda$ *is the* relative Whitehead product *of the maps* f *and* g.

Remark 2.6. *The relative Whitehead product generalizes the usual Whitehead product. If the underlying homotopy fibration* $E \xrightarrow{p} B \longrightarrow Z$ *has* $Z = *$ *then* $E = B$ *and* p *is the identity map, so (5) implies that* $[f, g]_r$ *is homotopic to the composite along the lower row of the diagram, which with* $p = 1$ *is the usual Whitehead product* $[f, g]$.

Remark 2.7. *The naturality of the construction of* Γ *and* Θ*, and the naturality of the homotopy class of* λ*, implies that the relative Whitehead product is natural, up to homotopy, for maps of principal fibrations*

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \longrightarrow & B' \end{array}$$

and maps

$$\begin{array}{ccc} \Sigma X & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \Sigma a} & & \downarrow \\ \Sigma X' & \xrightarrow{f'} & B' \end{array} \qquad\qquad \begin{array}{ccc} \Sigma Y & \xrightarrow{g} & B \\ \downarrow{\scriptstyle \Sigma b} & & \downarrow \\ \Sigma Y' & \xrightarrow{g'} & B'. \end{array}$$

It will be useful in what follows to now introduce some homotopies. Let X and Y be path-connected pointed spaces. Let $X \ltimes Y$ be the *left half-smash* of X and Y, defined as the quotient space $(X \times Y)/\sim$ where $(x, *) \sim *$. Let

$q\colon X \times Y \longrightarrow X \ltimes Y$ be the quotient map. Observe that there is a pushout diagram

$$\begin{array}{ccccc} X & \longrightarrow & X \times Y & \xrightarrow{q} & X \ltimes Y \\ \downarrow & & \downarrow & & \| \\ CX & \longrightarrow & CX \cup_X X \times Y & \xrightarrow{e} & X \ltimes Y \end{array} \tag{6}$$

where e collapes the cone to a point. Since CX is (naturally) contractible, e is a natural homotopy equivalence. In our context, the space $C\Omega B \cup_{\Omega B} \Omega B \times E$ that is the homotopy fibre of the pinch map $B \vee E \xrightarrow{p_1} B$ is naturally homotopy equivalent to $\Omega B \ltimes E$.
Let

$$\epsilon\colon X \ltimes Y \longrightarrow CX \cup_X X \times Y \tag{7}$$

be a natural right homotopy inverse of e. Considering the spaces and maps in (5), let $\overline{\Gamma}$ be the composite

$$\overline{\Gamma}\colon \Omega B \ltimes E \xrightarrow{\epsilon} C\Omega B \cup_{\Omega B} \Omega B \times E \xrightarrow{\Gamma} E$$

and let $\overline{\lambda}$ be the composite

$$\overline{\lambda}\colon \Sigma X \wedge Y \xrightarrow{\lambda} C\Omega\Sigma X \cup_{\Omega\Sigma X} \Omega\Sigma X \times \Sigma Y \xrightarrow{e} \Omega\Sigma X \ltimes \Sigma Y.$$

Then the definitions of $\overline{\Gamma}$, $\overline{\lambda}$ and Θ, and the naturality of ϵ, imply that there is a homotopy commutative diagram

$$\begin{array}{ccccccc} \Sigma X \wedge Y & \xrightarrow{\overline{\lambda}} & \Omega\Sigma X \ltimes \Sigma Y & \xrightarrow{\Omega f \ltimes g} & \Omega B \ltimes E & \xrightarrow{\overline{\Gamma}} & E \\ \| & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} & & \| \\ \Sigma X \wedge Y & \xrightarrow{\lambda} & C\Omega\Sigma X \cup_{\Omega\Sigma X} \Omega\Sigma X \times \Sigma Y & \xrightarrow{\Theta} & C\Omega B \cup_{\Omega B} \Omega B \times E & \xrightarrow{\Gamma} & E. \end{array} \tag{8}$$

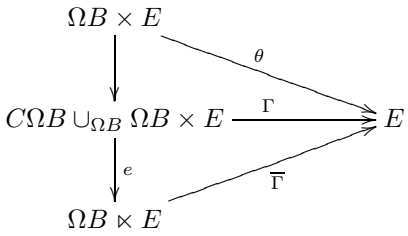
Combining (5) and (8) and the definition of the relative Whitehead product $[f,g]_r$ as $\Gamma \circ \Theta \circ \lambda$, we immediately obtain the following.

LEMMA 2.8. *There is a homotopy commutative diagram*

$$\begin{array}{ccccccc} & & \Omega\Sigma X \ltimes \Sigma Y & \xrightarrow{\Omega f \ltimes g} & \Omega B \ltimes E & \xrightarrow{\overline{\Gamma}} & E \\ & \overset{\overline{\lambda}}{\nearrow} & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ \Sigma X \wedge Y & \xrightarrow{W} & \Sigma X \vee \Sigma Y & \xrightarrow{f \vee g} & B \vee E & \xrightarrow{1 \vee p} & B \end{array}$$

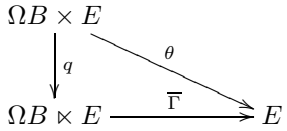
and the relative Whitehead product $\Sigma X \wedge Y \xrightarrow{[f,g]_r} E$ is homotopic to the composite $\overline{\Gamma} \circ (\Omega f \ltimes g) \circ \overline{\lambda}$. □

Note that $\overline{\Gamma} \circ (\Omega f \ltimes g) \circ \overline{\lambda}$ satisfies the same naturality properties as $\Gamma \circ \Theta \circ \lambda$ stated in Remark 2.13.
Consider the diagram



where θ is the composite $\Omega B \times E \xrightarrow{\Omega\varphi \times 1} \Omega Z \times E \xrightarrow{a} E$ and a is the action from the principal fibration $\Omega Z \longrightarrow E \longrightarrow B$. The upper triangle commutes by (3). The lower triangle homotopy commutes since, by definition, $\overline{\Gamma} = \Gamma \circ \epsilon$, and as $\epsilon \circ e$ is homotopic to the identity map on $\Omega B \ltimes E$, we obtain $\overline{\Gamma} \circ e \simeq \Gamma$. Also, by (6), the left vertical composite is the quotient map q. This establishes the following proposition, which encapsulates the connection between the action of a principal fibration and the relative Whitehead product.

Proposition 2.9. *Let $\Omega Z \longrightarrow E \longrightarrow B$ be a principal fibration induced by a map $B \xrightarrow{\varphi} Z$. Let θ be the composite $\Omega B \times E \xrightarrow{\Omega\varphi \times 1} \Omega Z \times E \xrightarrow{a} E$ where a is the action associated to the principal fibration. Then there is a homotopy commutative diagram*



which is natural for maps of principal fibrations.

Next, we aim to better identify the maps involved in the homotopy commutative diagram in Lemma 2.8. This requires two preliminary lemmas.

Lemma 2.10. *Let Q and R be pointed spaces. Then there is a homotopy equivalence $Q \ltimes \Sigma R \simeq (\Sigma Q \wedge R) \vee \Sigma R$. Further, this decomposition is natural for maps $Q \longrightarrow Q'$ and $R \longrightarrow R'$.*

Proof. In general, if T is a pointed space then there is a homeomorphism $Q \ltimes T \cong (Q_+) \wedge T$, where Q_+ is the union of Q and a disjoint basepoint. Note that $\Sigma(Q_+) \simeq \Sigma Q \vee S^1$. Therefore, if $T = \Sigma R$ then there are homotopy equivalences

$$Q \ltimes \Sigma R \cong (Q_+) \wedge \Sigma R \simeq \Sigma(Q_+) \wedge R \simeq (\Sigma Q \vee S^1) \wedge R \simeq (\Sigma Q \wedge R) \vee \Sigma R.$$

The naturality of the decomposition follows from the naturality of each homotopy equivalence. □

In general, if X and Y are path-connected spaces and $j_1\colon X \longrightarrow X \vee Y$ and $j_2\colon Y \longrightarrow X \vee Y$ are the inclusions, then Ganea [Ga] showed that there is a homotopy fibration

$$\Sigma\Omega X \wedge \Omega Y \xrightarrow{[ev_1, ev_2]} X \vee Y \longrightarrow X \times Y$$

where the right map is the inclusion of the wedge into the product and $[ev_1, ev_2]$ is the Whitehead product of the composites $ev_1\colon \Sigma\Omega X \xrightarrow{ev} X \xrightarrow{j_1} X \vee Y$ and $ev_2\colon \Sigma\Omega Y \xrightarrow{ev} Y \xrightarrow{i_2} X \vee Y$. Further, this fibration splits after looping, that is, $\Omega[ev_1, ev_2]$ has a left homotopy inverse. We will use Ganea's result to prove the following lemma.

LEMMA 2.11. *Let X and Y be path-connected spaces. Then there is a homotopy equivalence $e\colon (\Sigma\Omega X \wedge Y) \vee \Sigma Y \longrightarrow \Omega X \ltimes \Sigma Y$ that satisfies a homotopy commutative diagram*

$$\begin{array}{ccc} (\Sigma\Omega X \wedge Y) \vee \Sigma Y & \xrightarrow{\quad e \quad} & \Omega X \ltimes \Sigma Y \\ & {\scriptstyle [ev_1, j_2]+j_2} \searrow \quad \swarrow & \\ & X \vee \Sigma Y. & \end{array}$$

Further, this homotopy equivalence is natural for maps $X \longrightarrow X'$ and $Y \longrightarrow Y'$.

Proof. Since the pinch map $X \vee \Sigma Y \xrightarrow{p_1} X$ factors as the composite $X \vee \Sigma Y \longrightarrow X \times \Sigma Y \xrightarrow{\pi_1} X$ where π_1 is the projection, using Ganea's result we obtain a homotopy fibration diagram

$$\begin{array}{ccccc} \Sigma\Omega X \wedge \Omega\Sigma Y & \xrightarrow{a} & \Omega X \ltimes \Sigma Y & \xrightarrow{b} & \Sigma Y \\ \| & & \downarrow & & \downarrow{\scriptstyle i_2} \\ \Sigma\Omega X \wedge \Omega\Sigma Y & \xrightarrow{[ev_1, ev_2]} & X \vee \Sigma Y & \longrightarrow & X \times \Sigma Y \\ & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_1} \\ & & X & = & X \end{array} \tag{9}$$

where i_2 is the inclusion of the second factor and a and b are induced by the fibration diagram. Let $\eta\colon Y \longrightarrow \Omega\Sigma Y$ be the suspension map, which is the adjoint of the identity map on ΣY, and let c be the composite

$$c\colon \Sigma\Omega X \wedge Y \xrightarrow{\Sigma 1 \wedge \eta} \Sigma\Omega X \wedge \Omega\Sigma Y \xrightarrow{a} \Omega X \ltimes \Sigma Y.$$

Let d be the composite

$$d\colon \Sigma Y \hookrightarrow \Omega X \times \Sigma Y \longrightarrow \Omega X \ltimes \Sigma Y$$

and let

$$e\colon (\Sigma\Omega X \wedge Y) \vee \Sigma Y \longrightarrow \Omega X \ltimes \Sigma Y$$

be the wedge sum of c and d.

We will show that e is a homotopy equivalence. Granting this for the moment, observe that when c is composed to $X \vee \Sigma Y$ the left square in (9) implies that we obtain $[ev_1, ev_2] \circ (\Sigma 1 \wedge \eta)$. By the naturality of the Whitehead product this is homotopic to $[ev_1, ev_2 \circ \Sigma\eta]$. As $\Sigma\eta$ is a left homotopy inverse for the evaluation map and, by definition, $ev_2 = j_2 \circ ev$, we obtain $[ev_1, ev_2 \circ \Sigma\eta] \simeq [ev_1, j_2]$. Next, by Lemma 2.1 and the homotopy equivalence ϵ in (7), the composite

$$\Sigma Y \xrightarrow{d} \Omega X \ltimes \Sigma Y \longrightarrow X \vee \Sigma Y \tag{10}$$

is the inclusion j_2 of the right wedge summand. Therefore, as e is the wedge sum of c and d, the composite $(\Sigma\Omega X \wedge Y) \vee \Sigma Y \xrightarrow{e} \Omega X \ltimes \Sigma Y \longrightarrow X \vee \Sigma Y$ is homotopic to $[ev_1, j_2] + j_2$, giving the homotopy commutative diagram asserted by the lemma. Naturality then follows by the naturality of the homotopy fibration diagram (9) and of the maps c and d.

It remains to show that e is a homotopy equivalence. To do so it suffices to show that Ωe is a homotopy equivalence, for then Ωe induces an isomorphism of homotopy groups and hence so does e, implying that e is a homotopy equivalence. (Technically, this is a weak homotopy equivalence, but as all spaces have the homotopy type of finite type CW-complexes, we obtain an honest homotopy equivalence.) To show that Ωe is a homotopy equivalence, by Dror's extension of Whitehead's Theorem to path-connected H-spaces [D], it suffices to show that Ωe induces an isomorphism in integral homology. Finally, to see this it suffices to show that Ωe induces an isomorphism in homology with field coefficients for any field. Assume from now on that homology is taken with field coefficients.

First, we show that Ωc has a left homotopy inverse. The homotopy commutativity of the upper left square in (9) implies it suffices to show that the composite

$$\Omega(\Sigma\Omega X \wedge Y) \xrightarrow{\Omega(\Sigma 1 \wedge \eta)} \Omega(\Sigma\Omega X \wedge \Omega\Sigma Y) \xrightarrow{\Omega[ev_1, ev_2]} \Omega(X \vee \Sigma Y)$$

has a left homotopy inverse. Since $\Omega[ev_1, ev_2]$ has a left homotopy inverse, it therefore suffices to show that $\Omega(\Sigma 1 \wedge \eta)$ has a left homotopy inverse, and to show this it suffices to show that $\Sigma 1 \wedge \eta$ has a left homotopy inverse. But this is the case since the evaluation map $\Sigma\Omega\Sigma Y \xrightarrow{ev} \Sigma Y$ is a left homotopy inverse for $\Sigma\eta$ and therefore $1 \wedge ev$ is a left homotopy inverse for $1 \wedge \Sigma\eta \simeq \Sigma 1 \wedge \eta$.

Second, we show that d is a left homotopy inverse for b. As noted above, the composite $\Sigma Y \xrightarrow{d} \Omega X \ltimes \Sigma Y \longrightarrow X \vee \Sigma Y$ is the inclusion of the right wedge summand, so composing further with $X \vee \Sigma Y \longrightarrow X \times \Sigma Y \xrightarrow{\pi_2} \Sigma Y$ gives the identity map on ΣY. The homotopy commutativity of the upper right square in (9) then implies that $\Sigma Y \xrightarrow{d} \Omega X \ltimes \Sigma Y \xrightarrow{b} \Sigma Y$ is homotopic to the identity map.

Now consider homology. In general, if B is a path-connected space and homology is taken with a coefficient ring that is a PID, then the Bott-Samelson Theorem states that there is an algebra isomorphism $H_*(\Omega\Sigma B) \cong T(\widetilde{H}_*(B))$, where $T(\ )$ is the free tensor algebra functor, and the suspension map $B \xrightarrow{\eta} \Omega\Sigma B$ induces the inclusion of the generating set in homology. In our case, Lemma 2.10 implies that $\Omega X \ltimes \Sigma Y$ is homotopy equivalent to the suspension $\Sigma((\Omega X \wedge Y) \vee Y)$ and the coefficient ring in homology is a field, so there is an algebra isomorphism $H_*(\Omega(\Omega X \ltimes \Sigma Y)) \cong T(V)$, where $V = \widetilde{H}_*((\Omega X \wedge Y) \vee Y)$. Since Ωc has a left homotopy inverse, $(\Omega c)_*$ is a monomorphism. Consequently, as the adjoint of c is homotopic to the composite

$$\tilde{c}\colon \Omega X \wedge Y \xrightarrow{\eta} \Omega\Sigma(\Omega X \wedge Y) \xrightarrow{\Omega c} \Omega(\Omega X \ltimes \Sigma Y)$$

and η_* is also a monomorphism, we obtain that $\tilde{c}_*$ is a monomorphism. Since Ωd has a left homotopy inverse, $(\Omega d)_*$ is a monomorphism. Consequently, as the adjoint of d is homotopic to the composite

$$\tilde{d}\colon Y \xrightarrow{\eta} \Omega\Sigma Y \xrightarrow{\Omega d} \Omega(\Omega X \ltimes \Sigma Y)$$

and η_* is also a monomorphism, we obtain that $\tilde{d}_*$ is a monomorphism. Further, since d is the inclusion of ΣY into $\Omega X \ltimes \Sigma Y$, the image of $\tilde{d}_*$ is the submodule $\widetilde{H}_*(Y) \subseteq V$. Since b is a left homotopy inverse for d, the composite $Y \xrightarrow{\tilde{d}} \Omega(\Omega X \ltimes \Sigma Y) \xrightarrow{\Omega b} \Omega\Sigma Y$ is a monomorphism with image $\widetilde{H}_*(Y)$. In (9) the composite $b \circ a$ is null homotopic since these are consecutive maps in a homotopy fibration, so as c factors through a, we have $\Omega b \circ \tilde{c}$ null homotopic, and hence $(\Omega b)_* \circ \tilde{c}_* = 0$. Therefore, in reduced homology, the images C and D of $\tilde{c}_*$ and $\tilde{d}$ are disjoint. Thus, as both $\tilde{c}_*$ and $\tilde{d}_*$ are monomorphisms in reduced homology, their wedge sum is as well and it has image $C \oplus D \subseteq T(V)$. As the tensor algebra is free, this implies that there is a monomorphism $T(C \oplus D) \longrightarrow T(V)$. But this monomorphism is realized by the multiplicative extension (via the James construction) of

$$(\Omega X \wedge Y) \vee Y \xrightarrow{\tilde{c} \vee \tilde{d}} \Omega(\Omega X \ltimes \Sigma Y)$$

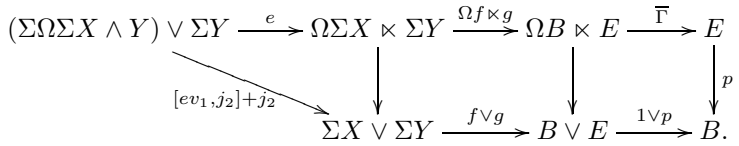
to

$$\upsilon\colon \Omega\Sigma((\Omega X \wedge Y) \vee Y) \longrightarrow \Omega(\Omega X \ltimes \Sigma Y).$$

Since the domain and range of υ have the same homotopy type, the fact that υ_* is a monomorphism implies that it is in fact an isomorphism. Finally, observe that the definitions of $\tilde{c}$ and $\tilde{d}$ as adjoints implies that $\upsilon \simeq \Omega(c \vee d)$, that is, $\upsilon \simeq \Omega e$. Hence Ωe induces an isomorphism in homology. □

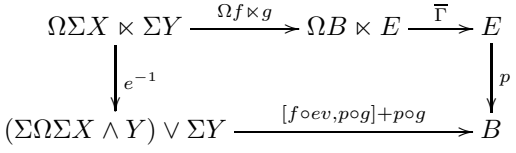
Returning to Lemma 2.8 and incorporating Lemma 2.11 we obtain a homotopy

commutative diagram



The naturality of the Whitehead product implies that the composite in the lower direction of this diagram is homotopic to $[f \circ ev, p \circ g] + p \circ g$. We record this as follows.

Corollary 2.12. *There is a homotopy commutative diagram*



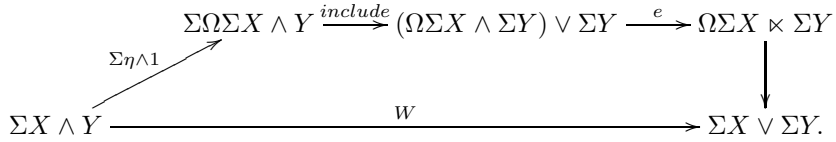
□

Corollary 2.12 says that the map $\overline{\Gamma} \circ (\Omega f \ltimes g)$ appearing in the definition of a relative Whitehead product is itself already a lift of Whitehead products, up to a homotopy equivalence.

Remark 2.13. *The naturality at each stage of the construction implies that the homotopy commutative diagram in Corollary 2.12 satisfies the same naturality properties listed in Remark 2.7.*

Finally, we better identify the map $\overline{\lambda}$ in the other part of the diagram in Lemma 2.8. Let $\eta\colon X \longrightarrow \Omega\Sigma X$ be the suspension map, which is the adjoint of the identity map on ΣX.

Proposition 2.14. *There is a homotopy commutative diagram*



Consequently, the map $\Sigma X \wedge Y \xrightarrow{\overline{\lambda}} \Omega\Sigma X \ltimes \Sigma Y$ *in Lemma 2.8 can be chosen to be the composite* $L\colon \Sigma X \wedge Y \xrightarrow{\Sigma\eta\wedge 1} \Sigma\Omega\Sigma X \wedge \Sigma Y \hookrightarrow (\Omega\Sigma X \wedge \Sigma Y) \vee \Sigma Y \xrightarrow{e} \Omega\Sigma X \ltimes Y$.

Proof. Recall that W is the Whitehead product $[j_1, j_2]$. Let L be the composite comprising the upper direction around the diagram in the statement of the proposition. By Lemma 2.11, $L \simeq [ev_1, j_2] \circ (\Sigma\eta \wedge 1)$. By the naturality of the Whitehead product, this is homotopic to $[ev_1 \circ \Sigma\eta, j_2]$. As the composite

$\Sigma X \xrightarrow{\Sigma\eta} \Sigma\Omega\Sigma X \xrightarrow{ev} \Sigma X$ is homotopic to the identity map, we obtain $ev_1 \circ \Sigma\eta = j_1 \circ ev \circ \Sigma\eta \simeq j_1$. Thus $L \simeq [j_1, j_2] = W$.
Consequently, L lifts W to the homotopy fibre of the pinch map $\Sigma X \vee \Sigma Y \longrightarrow \Sigma Y$. As the loop of the pinch map has a right homotopy inverse, the homotopy class of the lift of W to the homotopy fibre is uniquely determined by the homotopy class of W. Thus we may unambiguously choose the lift $\overline{\lambda}$ of W in Lemma (8) to be L. □

3 Relative Whitehead products and the homotopy type of certain pushouts

This section fuses a construction in [GT] with relative Whitehead products. Suppose that there is a cofibration $A \xrightarrow{f} Y \longrightarrow Y \cup CA$. Let $\Omega Z \longrightarrow E' \longrightarrow Y \cup CA$ be a principal fibration induced by a map $\varphi\colon Y \cup CA \longrightarrow Z$. Define spaces Q and E and maps p and e by the iterated pullback

$$\begin{array}{ccccc} Q & \longrightarrow & E & \xrightarrow{e} & E' \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\ A & \xrightarrow{f} & Y & \longrightarrow & Y \cup CA. \end{array} \tag{11}$$

Observe that there are principal fibrations $\Omega Z \longrightarrow Q \longrightarrow A$ and $\Omega Z \longrightarrow E \longrightarrow Y$ induced by the composites $A \xrightarrow{f} Y \longrightarrow Y \cup CA \xrightarrow{\varphi} Z$ and $Y \longrightarrow Y \cup CA \xrightarrow{\varphi} Z$ respectively. The map $A \longrightarrow Z$ is trivial since it factors through two consecutive maps in a cofibration. Thus $Q \simeq \Omega Z \times A$. However, there may be many inequivalent choices of a decomposition and we wish to choose one such that E' is the pushout of the projection $\Omega Z \times A \longrightarrow A$ and an "action" map $\Omega Z \times A \longrightarrow E$.

LEMMA 3.1. *There is a map of pairs $(CA, A) \longrightarrow (E', E)$ such that the composite $(CA, A) \longrightarrow (E', E) \longrightarrow (Y \cup CA, Y)$ is the standard inclusion.*

Proof. Start with the standard inclusion $(CA, A) \longrightarrow (Y \cup CA, Y)$. Consider the composite $h\colon A \times I \longrightarrow CA \longrightarrow Y \cup CA$, where the left map is the quotient map to $CA = A \wedge I$. The map h is a pointed homotopy which at $t = 0$ sends A to the base of the cone in $Y \cup CA$ and at $t = 1$ sends A to the basepoint. Thus h_1 lifts to E', so the homotopy extension property implies that h lifts to a map $\overline{h}\colon A \times I \longrightarrow E'$. As this occurs in the pointed category, $\overline{h}$ factors as a composite $A \times I \longrightarrow CA \xrightarrow{\overline{h}'} E'$. The pullback property of E then implies that there is a map $g\colon A \longrightarrow E$ such that $p \circ g$ is the identity on A and $e \circ g$ is $\overline{h'}$. Thus g and $e \circ g$ give a map of pairs $(CA, A) \longrightarrow (E', E)$ with the property that the composite $(CA, A) \longrightarrow (E', E) \longrightarrow (Y \cup CA, Y)$ is the standard inclusion. □

Since the homotopy fibration $\Omega Z \longrightarrow E \longrightarrow Y$ is principal, there is a homotopy action $\Omega Z \times E \xrightarrow{a} E$. Let ϑ be the composite

$$\vartheta\colon \Omega Z \times A \xrightarrow{1\times g} \Omega Z \times E \xrightarrow{a} E.$$

The following was shown in the first half of the proof of [GT, Theorem 2.2].

Theorem 3.2. *Using the map of pairs in Lemma 3.1, there is a homotopy pushout*

$$\begin{array}{ccc} \Omega Z \times A & \xrightarrow{\vartheta} & E \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ \Omega Z & \longrightarrow & E'. \end{array}$$

□

The value of Theorem 3.2 is that it allows for the homotopy type of E' to be identified in terms of known maps. In the previous section we related the action ϑ to relative Whitehead products. We now make this explicit in the context of the pushout in Theorem 3.2.

Let h be the composite $Y \longrightarrow Y \cup CA \xrightarrow{\varphi} Z$. Consider the homotopy fibration sequence

$$\Omega Y \xrightarrow{\Omega h} \Omega Z \longrightarrow E \xrightarrow{p} Y \xrightarrow{h} Z.$$

Suppose that the map $\Omega Z \longrightarrow E$ is null homotopic. Then Ωh has a right homotopy inverse $s\colon \Omega Z \longrightarrow \Omega Y$. Let $g\colon A \longrightarrow E$ be the restriction of $\overline{g}$ to A. Consider the diagram

$$\begin{array}{ccccccccc} \Omega Z \times A & \xrightarrow{1\times g} & \Omega Z \times E & \xrightarrow{s\times 1} & \Omega Y \times E & \xrightarrow{\Omega h\times 1} & \Omega Z \times E & \xrightarrow{a} & E \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & & \nearrow_{\overline{\Gamma}} & \\ \Omega Z \ltimes A & \xrightarrow{1\ltimes g} & \Omega Z \ltimes E & \xrightarrow{s\ltimes 1} & \Omega Y \ltimes E. & & & & \end{array}$$

The left and middle squares commute by naturality and the right triangle homotopy commutes by Proposition 2.9. Since $\Omega h \circ s$ is homotopic to the identity map on ΩZ, the composite along the top row is homotopic to $a\circ(1\times g)$, which by definition, is ϑ. Thus we obtain a homotopy commutative diagram

$$\begin{array}{ccccc} \Omega Z \times A & & \xrightarrow{\vartheta} & & E \\ \downarrow{\scriptstyle q} & & & & \| \\ \Omega Z \ltimes A & \xrightarrow{s\ltimes g} & \Omega Y \ltimes E & \xrightarrow{\overline{\Gamma}} & E. \end{array} \tag{12}$$

The factorization of ϑ through the half-smash $\Omega Z \ltimes A$ implies that, in the homotopy pushout in Theorem 3.2, the space ΩZ may be pinched out, proving the following.

THEOREM 3.3. *Given the same hypotheses as in Theorem 3.2, suppose in addition that the map* $\emptyset 1u11megaY \xrightarrow{\Omega h} \Omega Z$ *has a right homotopy inverse* $s\colon \Omega Z \longrightarrow \Omega Y$*. Then there is a homotopy cofibration*

$$\Omega Z \ltimes A \xrightarrow{\overline{\Gamma}\circ(s\ltimes g)} E \longrightarrow E'. \qquad \square$$

Ideally, if the homotopy types of ΩZ, A and E are known, and the homotopy class of $\overline{\Gamma}\circ(s\ltimes g)$ can be identified, then the homotopy type of E' could also be identified. Since Ωh has a right homotopy inverse, the homotopy class of $\overline{\Gamma}\circ(s\ltimes g)$ is determined by the homotopy class of its composition with $E \xrightarrow{p} Y$. In Theorem 3.4 the homotopy class of $p\circ\overline{\Gamma}\circ(s\ltimes g)$ is described more precisely, provided that A is a suspension. Let γ be the composite

$$\gamma\colon \Sigma\Omega Z \xrightarrow{\Sigma s} \Sigma\Omega Y \xrightarrow{ev} Y.$$

THEOREM 3.4. *Given the same hypotheses as Theorem 3.3. Suppose in addition that* $A \simeq \Sigma\overline{A}$*. Then there is a homotopy commutative diagram*

$$\begin{array}{ccccc}
\Omega Z \ltimes A & \xrightarrow{s\ltimes g} & \Omega Y \ltimes E & \xrightarrow{\overline{\Gamma}} & E \\
\downarrow{\scriptstyle e^{-1}} & & & & \downarrow{\scriptstyle p} \\
(\Sigma\Omega Z \wedge \overline{A}) \vee A & & \xrightarrow{[\gamma,f]+f} & & Y
\end{array}$$

where e *is the homotopy equivalence from Lemma 2.11.*

Proof. Generically, let $e_1\colon \Sigma\Omega Q \wedge R \longrightarrow \Omega Q \ltimes \Sigma R$ be the restriction of the homotopy equivalence $(\Sigma\Omega Q \wedge R) \vee \Sigma R \xrightarrow{e} \Omega Q \ltimes \Sigma R$ in Lemma 2.11 to the first wedge summand. Consider the diagram

$$\begin{array}{ccccccccccc}
\Omega Z \wedge A & = & \Omega Z \wedge A & \xrightarrow{e_1} & \Omega Z \ltimes A & \xrightarrow{s\ltimes 1} & \Omega Y \ltimes A & & & & \\
\downarrow & & \downarrow{\scriptstyle \eta\wedge 1} & & \downarrow{\scriptstyle \eta\ltimes 1} & & \downarrow{\scriptstyle \eta\ltimes 1} & \searrow{\scriptstyle 1\ltimes g} & & & \\
\simeq & & \Omega\Sigma\Omega Z \wedge A & \xrightarrow{e_1} & \Omega\Sigma\Omega Z \ltimes A & \xrightarrow{\Omega\Sigma s\ltimes 1} & \Omega\Sigma\Omega Y \ltimes A & \xrightarrow{\Omega ev\ltimes g} & \Omega Y \ltimes E & \xrightarrow{\overline{\Gamma}} & E \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\
\Sigma\Omega Z \wedge \overline{A} & & & \xrightarrow{W} & \Sigma\Omega Z \vee A & \xrightarrow{\Sigma s\vee 1} & \Sigma\Omega Y \vee A & \xrightarrow{ev\vee g} & Y \vee E & \xrightarrow{1\vee p} & Y.
\end{array}$$

The L-shaped part of the diagram on the left homotopy commutes by Proposition 2.14, and the square in its upper right corner homotopy commutes by the naturality of e_1. The upper middle square homotopy commutes by the naturality of η and the upper triangle homotopy commutes since η is a right homotopy inverse of Ωev. The three remaining squares on the bottom homotopy commute by the construction of relative Whitehead products. Thus the entire diagram homotopy commutes. Observe that the upper direction around the diagram is the restriction of $\overline{\Gamma}\circ(s\ltimes g)\circ e$ to $\Omega Z \wedge A$, while the naturality

of the Whitehead product implies that the lower direction around the diagram is homotopic to $[ev \circ \Sigma s, p \circ g] = [\gamma, f]$. Thus the diagram in the statement of the theorem homotopy commutes when restricted to $\Sigma\Omega Z \wedge \overline{A} \simeq \Omega Z \wedge A$. On the other hand, the restriction of the diagram in the statement of the theorem to A clearly homotopy commutes since the restriction of $\overline{\Gamma}(s \ltimes g)$ to A is g. This completes the proof. □

A special case of Theorems 3.3 and 3.4 that will be used in the next section is given by taking $Z = Y \cup CA$ and $Y \xrightarrow{\varphi} Z$ as the identity map. This gives the following diagram of spaces and maps that collects the data going into Theorem 3.3:

$$\begin{array}{ccccc} & & E & \longrightarrow & E' \\ & & \downarrow{\scriptstyle p} & & \downarrow \\ A & \xrightarrow{f} & Y & \longrightarrow & Y \cup CA \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle =} \\ & & Y \cup CA & = & Y \cup CA. \end{array} \tag{13}$$

Here, the middle row is a cofibration, the two columns are homotopy fibrations, and the upper square is a homotopy pullback. The map h in general is the composite $Y \longrightarrow Y \cup CA \xrightarrow{\varphi} Z$, so in this case it is simply the inclusion $Y \longrightarrow Y \cup CA$. Thus the space E is the homotopy fibre of this inclusion and the space E' is contractible. From Theorems 3.3 and 3.4 we obtain the following.

Proposition 3.5. *Let $A \xrightarrow{f} Y \xrightarrow{h} Y \cup CA$ be a cofibration and consider the homotopy fibration $E \xrightarrow{p} Y \xrightarrow{h} Y \cup CA$. Suppose that Ωh has a right homotopy inverse $s\colon \Omega(Y \cup CA) \longrightarrow \Omega Y$. Then there is a homotopy equivalence*

$$\Omega(Y \cup CA) \ltimes A \xrightarrow{\overline{\Gamma} \circ (s \ltimes g)} E.$$

Further, if $A \simeq \Sigma\overline{A}$ then there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega(Y \cup CA) \ltimes A & \xrightarrow[\simeq]{\overline{\Gamma} \circ (s \ltimes g)} & E \\ \simeq \downarrow{\scriptstyle e^{-1}} & & \downarrow{\scriptstyle p} \\ (\Sigma\Omega(Y \cup CA) \wedge \overline{A}) \vee A & \xrightarrow{[\gamma, f] + f} & Y. \end{array}$$

Consequently, there is a homotopy fibration

$$\Sigma\Omega(Y \cup CA) \wedge \overline{A}) \vee A \xrightarrow{[\gamma, f] + f} Y \xrightarrow{h} Y \cup CA. \qquad \square$$

That is, Proposition 3.5 identifies the homotopy type of E and, if A is a suspension, identifies the map $E \xrightarrow{p} Y$ in terms of f and a Whitehead product.

4 The based loops on highly connected Poincaré Duality complexes I

In this section Proposition 3.5 is used to analyze the homotopy theory of certain cell attachments, as described in Theorem 4.1. This is then applied to identify the homotopy types of the based loops on certain Poincaré Duality complexes. By the Hilton-Milnor Theorem the inclusion $S^m \vee S^n \xrightarrow{i} S^m \times S^n$ has a right homotopy inverse after looping; that is, there is a map $t\colon \Omega S^m \times \Omega S^n \longrightarrow \Omega(S^m \vee S^n)$ such that $\Omega i \circ t$ is homotopic to the identity map.

Theorem 4.1. *Suppose that there is a cofibration*

$$S^{m+n-1} \longrightarrow Y \longrightarrow Y' = Y \cup e^{m+n}$$

where $m, n \geq 2$. Suppose also that:

(i) there is a homotopy equivalence $Y \simeq S^m \vee S^n \vee \Sigma X$;

(ii) the composite of inclusions $f\colon \Sigma X \hookrightarrow Y \hookrightarrow Y'$ has homotopy cofibre D with the property that $H^*(D) \cong H^*(S^m \times S^n)$.

Let $q\colon Y' \longrightarrow D$ be the map to the homotopy cofibre and let s be the composite $\Omega S^m \times \Omega S^n \xrightarrow{t} \Omega(S^m \vee S^n) \longrightarrow \Omega Y'$. Then:

(a) the composite $\Omega S^m \times \Omega S^n \xrightarrow{s} \Omega Y' \xrightarrow{\Omega q} \Omega D$ is a homotopy equivalence;

(b) there is a homotopy fibration

$$(\Sigma(\Omega S^m \times \Omega S^n) \wedge X) \vee \Sigma X \xrightarrow{[\gamma, f]+f} Y' \xrightarrow{q} D$$

where $\gamma = ev \circ \Sigma s$;

(c) there is a homotopy equivalence

$$\Omega Y' \simeq \Omega S^m \times \Omega S^n \times \Omega((\Omega S^m \times \Omega S^n) \ltimes \Sigma X).$$

Proof. For part (a), let r be the composite $r\colon S^m \vee S^n \hookrightarrow Y' \xrightarrow{q} D$. The definitions of r and s imply that $\Omega r \circ t = \Omega q \circ s$. So to prove part (a) it is equivalent to show that $\Omega r \circ t$ is a homotopy equivalence. This is proved by directly copying the argument in [BT, Lemmas 2.2 and 2.3].

For part (b), consider the homotopy cofibration $\Sigma X \xrightarrow{f} Y' \xrightarrow{q} D$. Define the space E and the map p by the homotopy fibration $E \xrightarrow{p} Y' \xrightarrow{q} D$. Part (a) implies the map Ωq has a right homotopy inverse $s'\colon \Omega D \longrightarrow \Omega Y'$. Therefore Proposition 3.5 implies that there is a homotopy fibration

$$(\Sigma \Omega D \wedge X) \vee \Sigma X \xrightarrow{[\gamma', f]+f} Y' \xrightarrow{q} D$$

where $\gamma' = ev \circ \Sigma s'$. Substituting in the homotopy equivalence for ΩD in part (a), which also lets us substutite $\gamma = ev \circ s$ for $\gamma' = ev \circ s'$, we obtain a homotopy fibration

$$(\Sigma(\Omega S^m \times \Omega S^n) \wedge X) \vee \Sigma X \xrightarrow{[\gamma,f]+f} Y' \xrightarrow{q} D,$$

proving part (b).
Finally, for part (c), since Ωq has a right homotopy inverse by part (a), the homotopy fibration in part (b) splits after looping, giving a homotopy equivalence

$$\Omega Y' \simeq \Omega D \times \Omega((\Omega S^m \times \Omega S^n) \ltimes \Sigma X).$$

Substituting in the homotopy equivalence $\Omega D \simeq \Omega S^m \times \Omega S^n$ from part (a) then completes the proof. □

Example 4.2. *For $n \geq 2$, let M be an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex. By Poincaré Duality,*

$$H^m(M) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0 \text{ or } m = 2n \\ \mathbb{Z}^d & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

for some integer $d \geq 0$. Assume $d \geq 2$ and $n \notin \{2, 4, 8\}$. By [BT, Lemma 3.3] generators $x_1, \ldots, x_d$ of $H^n(M)$ can be chosen such that $x_1 \cup x_2$ generates $H^{2n}(M)$ for some $x_1 \neq x_2$. (Note that if $n \in \{2, 4, 8\}$ then the existence of an element of Hopf invariant one allows for the possibility that only $x_1 \cup x_1$ generates $H^{2n}(M)$.) Now give M a CW-structure by corresponding one n-cell to each x_k and attaching the top cell. Then there is a homotopy cofibration

$$S^{2n-1} \xrightarrow{g} \bigvee_{i=1}^{d} S^n \xrightarrow{j} M$$

where j^* sends x_k to the generator of the k^{th} sphere in the wedge, and g attaches the top cell of M. Let $\Sigma X = \bigvee_{i=3}^{d} S^n$ and let f be the composite $f\colon \Sigma X \hookrightarrow \bigvee_{i=1}^{d} S^n \xrightarrow{j} M$. Define the space D and the map q by the homotopy cofibration

$$\Sigma X \xrightarrow{f} M \xrightarrow{q} D.$$

Since $x_1 \cup x_2$ generates $H^{2n}(M)$ and x_1, x_2 correspond to $\bigvee_{i=1}^{2} S^n$, the space D satisfies $H^*(D) \cong H^*(S^n \times S^n)$. Therefore, Theorem 4.1 applies, and we obtain a homotopy fibration

$$(\Sigma(\Omega S^n \times \Omega S^n) \wedge X) \vee \Sigma X \xrightarrow{[\gamma,f]+f} M \xrightarrow{q} D$$

where $\gamma = ev \circ \Sigma s$ and a homotopy equivalence

$$\Omega M \simeq \Omega S^n \times \Omega S^n \times \Omega((\Omega S^n \times \Omega S^n) \ltimes \Sigma X). \tag{14}$$

Example 4.2 improves on some of the relevant results in [BB, BT]. Using different methods, in [BT] the same decomposition for ΩM was obtained if $n \notin \{2, 4, 8\}$ and using yet another set of methods, in [BB] the same decomposition for ΩM was obtained for all n. But in neither case was the map from the fibre of q into M identified.

REMARK 4.3. *In general, there are homotopy equivalences* $B \ltimes \Sigma A \simeq (\Sigma B \wedge A) \vee \Sigma A$ *and* $\Sigma(B \times A) \simeq \Sigma B \vee \Sigma A \vee (\Sigma A \wedge B)$*, and a property of the James construction is that* $\Sigma\Omega S^n$ *is homotopy equivalent to a wedge of spheres. Combining these facts shows that* $(\Omega S^n \times \Omega S^n) \ltimes \Sigma X$ *is homotopy equivalent to a wedge of spheres, and if desired, a precise enumeration of these spheres can be made. The Hilton-Milnor Theorem then implies that* $\Omega((\Omega S^n \times \Omega S^n) \ltimes \Sigma X)$ *is homotopy equivalent to an infinite product of spheres. Hence the decomposition (14) implies that the homotopy groups of* ΩM *can be determined to exactly the same extent as can the homotopy groups of spheres.*

EXAMPLE 4.4. *For* $n \geq 2$*, let* M *be an* $(n-1)$*-connected* $(2n+1)$*-dimensional Poincaré Duality complex. By Poincaré Duality,*

$$H^m(M) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0 \text{ or } m = 2n+1 \\ \mathbb{Z}^d & \text{if } m = n \\ \mathbb{Z}^d \oplus G & \text{if } m = n+1 \\ 0 & \text{otherwise} \end{cases}$$

for some integer $d \geq 0$ *and some finite abelian group* G*. Assume that* $d \geq 1$*. Rationally,* M *still satisfies Poincaré Duality, so we can choose generators* $x_1, \ldots, x_d$ *of the subgroup* $\mathbb{Z}^d$ *in* $H^n(M)$ *and* $y_1, \ldots, y_d$ *of the subgroup* $\mathbb{Z}^d$ *in* $H^{n+1}(M)$ *such that* $x_1 \cup y_1$ *generates* $H^{2n+1}(M)$*. Give* $\overline{M}$ *a CW-structure by associating an* S^n *to each* x_k*, an* S^{n+1} *to each* y_k*, and an* $(n+1)$*-dimensional Moore space* $P^{n+1}(t_j)$ *to each cyclic direct summand* $\mathbb{Z}/t_j\mathbb{Z}$ *of* G*. Write*

$$\overline{M} \simeq S^n \vee S^{n+1} \vee \Sigma X \tag{15}$$

where the S^n *corresponds to* x_1*, the* S^{n+1} *corresponds to* y_1*, and* $\Sigma X = (\bigvee_{i=2}^{d} S^n \vee S^{n+1}) \vee (\bigvee_{j=1}^{s} P^{n+1}(t_j))$*. Give* M *a CW-structure by attaching the top cell to* $\overline{M}$*. Then there is a homotopy cofibration*

$$S^{2n} \xrightarrow{g} S^n \vee S^{n+1} \vee \Sigma X \xrightarrow{j} M.$$

Let f *be the composite* $f\colon \Sigma X \hookrightarrow S^n \vee S^{n+1} \vee \Sigma X \xrightarrow{j} M$*. Define the space* D *and the map* q *by the homotopy cofibration*

$$\Sigma X \xrightarrow{f} M \xrightarrow{q} D.$$

Since $x_1 \cup y_1$ *generates* $H^{2n+1}(M)$ *and* x_1, y_1 *correspond to* $S^n \vee S^{n+1}$ *in (15), the space* D *satisfies* $H^*(D) \cong H^*(S^n \times S^{n+1})$*. Therefore, Theorem 4.1 applies, and we obtain a homotopy fibration*

$$(\Sigma(\Omega S^n \times \Omega S^{n+1}) \wedge X) \vee \Sigma X \xrightarrow{[\gamma, f]+f} M \xrightarrow{q} D$$

where $\gamma = ev \circ \Sigma s$ *and a homotopy equivalence*

$$\Omega M \simeq \Omega S^n \times \Omega S^{n+1} \times \Omega((\Omega S^n \times \Omega S^{n+1}) \ltimes \Sigma X). \tag{16}$$

Example 4.4 improves on the result in [B]. That paper used different methods to obtain the same homotopy decomposition, but did not describe the map from the fibre of q into M.

Remark 4.5. *As in Remark 4.3, the fact that* ΣX *is homotopy equivalent to a wedge of spheres and Moore spaces implies that* $(\Omega S^n \times \Omega S^{n+1}) \ltimes \Sigma X$ *is homotopy equivalent to a wedge of spheres and Moore spaces. The Hilton-Milnor Theorem can then be applied to* $\Omega((\Omega S^n \times \Omega S^{n+1}) \ltimes \Sigma X)$ *to decompose further. In particular, the smash product of two mod-*p^r *Moore spaces is homotopy equivalent to a wedge of two mod-*p^r *Moore spaces for* p *a prime and* $r \neq 2$*, so if the* 2*-torsion in* $H^n(M)$ *is controlled in this way then the output of the Hilton-Milnor Theorem is a product of looped spheres and looped Moore spaces. Therefore the decomposition (16) implies that the homotopy groups of* M *can be calculated to the same extent as can the homotopy groups of spheres and Moore spaces.*

Remark 4.6. *The* $n = 2$ *case of simply-connected* 4*-dimensional Poincaré Duality complexes can be recovered. In this case, we rely on the arugment in [BT, Section 4]; this is phrased in terms of simply-connected* 4*-manifolds but works equally well for simply-connected Poincaré Duality complexes. If* M *is such a space then* $\Omega M \simeq S^1 \times \Omega Z$ *where* Z *is a simply-connected torsion-free* 5*-dimensional Poincar'e Duality complex. If* $H^3(Z) = 0$ *then* $Z \simeq S^5$ *and otherwise* Z *is one of the cases considered in Example 4.4. Therefore, in all cases, we obtain a decomposition of* ΩM*.*

5 The based loops on highly connected Poincaré Duality complexes II

Theorem 4.1 can be pushed further. Consider again the cofibration

$$S^{m+n-1} \xrightarrow{g} Y \xrightarrow{j} Y' = Y \cup e^{m+n}$$

where g attaches the $(m+n)$-cell and j is the inclusion. In Theorem 5.1 we identify the homotopy fibre of j and show that Ωj has a right homotopy inverse.

Theorem 5.1. *Assume that there is a cofibration* $S^{m+n-1} \xrightarrow{g} Y \xrightarrow{j} Y'$ *as in Theorem 4.1. Then the following hold:*

(a) the map Ωj *has a right homotopy inverse* $t\colon \Omega M \longrightarrow \Omega Y$*;*

(b) there is a homotopy fibration

$$(\Sigma\Omega Y' \wedge S^{m+n-2}) \vee S^{m+n-1} \xrightarrow{[\gamma,g]+q} Y \xrightarrow{j} Y'$$

where $\gamma = ev \circ \Sigma t$*.*

Proof. Throughout the proof we write Y as $S^m \vee S^n \vee \Sigma X$.
For part (a), Theorem 4.1 gives a homotopy decomposition of $\Omega Y'$ via the map $Y' \xrightarrow{q} D$ but does not immediately relate this to j. To do so, let $\overline{q}$ be the composite

$$\overline{q}\colon S^m \vee S^n \vee \Sigma X \xrightarrow{j} Y' \xrightarrow{q} D.$$

Recall that there is a homotopy cofibration $\Sigma X \xrightarrow{f} Y' \xrightarrow{q} D$. We claim that there is a homotopy cofibration $S^{m+n-1} \vee \Sigma X \xrightarrow{g+i} S^m \vee S^n \vee \Sigma X \xrightarrow{\overline{q}} D$ where i is the inclusion of ΣX. To see this, consider the homotopy pushout diagram

$$\begin{array}{ccccc} S^{m+n-1} & = & S^{m+n-1} & & \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle g} & & \\ S^{m+n-1} \vee \Sigma X & \xrightarrow{g+i} & S^m \vee S^n \vee \Sigma X & \longrightarrow & \overline{D} \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle j} & & \| \\ \Sigma X & \longrightarrow & Y' & \longrightarrow & \overline{D}. \end{array} \tag{17}$$

Here, i_1 is the inclusion of the first wedge summand, q_2 is the pinch onto the second wedge summand, and $\overline{D}$ is the homotopy cofibre of $g + i$. The map $\Sigma X \longrightarrow Y'$ along the bottom row can be identified by restricting $j \circ (g + i)$ to ΣX: this is the definition of f. Hence $\overline{D} \simeq D$ and the homotopy cofibration along the bottom row of the diagram is $\Sigma X \xrightarrow{f} Y' \xrightarrow{q} D$. By definition, $\overline{q} = q \circ j$, so from the middle row of the diagram we obtain a homotopy cofibration $S^{m+n-1} \vee \Sigma X \xrightarrow{g+i} S^m \vee S^n \vee \Sigma X \xrightarrow{\overline{q}} D$.
Next, recall from Theorem 4.1 that the right homotopy inverse s of Ωq is defined as the composite $\Omega S^m \times \Omega S^n \xrightarrow{t} \Omega(S^m \vee \Omega S^n) \longrightarrow \Omega Y'$. The latter map is the loops on the composite $S^m \vee S^n \hookrightarrow S^m \vee S^n \vee \Sigma X \xrightarrow{j} Y'$. Thus if $\overline{s}$ is the composite

$$\overline{s}\colon \Omega S^m \times \Omega S^n \xrightarrow{t} \Omega(S^m \vee S^n) \longrightarrow \Omega(S^m \vee S^n \vee \Sigma X)$$

then $\overline{s}$ is a right homotopy inverse for $\Omega q \circ \Omega j = \Omega \overline{q}$. Therefore, if $t = m + n - 1$ to compress notation, applying Theorem 3.5 to the homotopy cofibration $S^t \vee \Sigma X \xrightarrow{g+i} S^m \vee S^n \vee \Sigma X \xrightarrow{\overline{q}} D$ gives a homotopy fibration

$$(\Sigma(\Omega S^m \times \Omega S^n) \wedge (S^t \vee \Sigma X)) \vee (S^t \vee \Sigma X) \xrightarrow{[\overline{\gamma}, g+i]+(g+i)} S^m \vee S^n \vee \Sigma X \xrightarrow{\overline{q}} D$$

where $\overline{\gamma} = ev \circ \Sigma \overline{s}$. By Theorem 4.1, there is a homotopy fibration

$$(\Sigma(\Omega S^m \times \Omega S^n) \wedge X) \vee \Sigma X \xrightarrow{[\gamma, f]+f} Y' \xrightarrow{q} D$$

where $\gamma = ev \circ \Sigma s$. The two fibrations are compatible: (i) by definition, $\overline{q} = q \circ j$, (ii) by (17), $j \circ (g + i) \simeq f \circ q_2$, (iii) note that $s = \Omega j \circ \overline{s}$ so $\gamma = j \circ \overline{\gamma}$, and

(iv) the Whitehead product is natural. Thus there is a homotopy commutative diagram

$$\begin{array}{ccccc}
(\Sigma(\Omega S^m \times \Omega S^n) \wedge (S^t \vee \Sigma X)) \vee (S^t \vee \Sigma X) & \xrightarrow{[\overline{\gamma}, g+i]+(g+i)} & S^m \vee S^n \vee \Sigma X & \xrightarrow{\overline{q}} & D \\
\downarrow{\scriptstyle (1\wedge q_2)\vee q_2} & & \downarrow{\scriptstyle j} & & \| \\
(\Sigma(\Omega S^m \times \Omega S^n) \wedge X) \vee \Sigma X & \xrightarrow{[\gamma, f]+f} & Y' & \xrightarrow{q} & D
\end{array}$$

where the two rows are homotopy fibrations. If $i_2 \colon \Sigma X \longrightarrow S^m \vee S^n \vee \Sigma X$ is the inclusion, then i_2 is a right homotopy inverse for q_2. Therefore $(1 \wedge i_2) \vee i_2$ is a right homotopy inverse for $(1 \wedge q_2) \vee q_2$. Consequently, letting

$$A = (\Sigma(\Omega S^m \times \Omega S^n) \wedge X) \vee \Sigma X \qquad B = S^m \vee S^n \vee \Sigma X \qquad \psi = (1 \wedge i_2) \vee i_2$$

and letting μ be the loop multiplication, the composite

$$\Omega A \times (\Omega S^m \times \Omega S^n) \xrightarrow{\Omega\psi \times \overline{s}} \Omega B \times \Omega B \xrightarrow{\mu} \Omega B \xrightarrow{\Omega j} \Omega Y'$$

is a homotopy equivalence. Therefore Ωj has a right homotopy inverse.
Part (b) is now straightforward. The right homotopy inverse for Ωj implies that Theorem 4.1 can be applied to the homotopy cofibration $S^{m+n-1} \xrightarrow{g} S^m \vee S^n \vee \Sigma X \xrightarrow{j} Y'$ to obtain a homotopy fibration

$$(\Sigma\Omega Y' \wedge S^{m+n-2}) \vee S^{m+n-1} \xrightarrow{[\gamma, g]+g} S^m \vee S^n \vee \Sigma X \xrightarrow{j} Y'$$

where $\gamma = ev \circ \Sigma t$. □

Example 5.2. *Let $n \geq 2$ but $n \notin \{2, 4, 8\}$. Let M be an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex with $H^n(M) \cong \mathbb{Z}^d$ for $d \geq 2$. As in Example 4.2, there is a homotopy cofibration*

$$S^{2n-1} \xrightarrow{g} \bigvee_{i=1}^{d} S^n \xrightarrow{j} M$$

where g is the attaching map for the top cell. By Theorem 5.1, Ωj has a right homotopy inverse $t \colon \Omega M \longrightarrow \Omega(\bigvee_{i=1}^{d} S^n)$ and there is a homotopy fibration

$$(\Sigma\Omega M \wedge S^{2n-2}) \vee S^{2n-1} \xrightarrow{[\gamma, g]+g} \bigvee_{i=1}^{d} S^n \xrightarrow{j} M$$

where $\gamma = ev \circ \Sigma t$.

All cases in Example 5.2 are new. Also new are all cases in the following.

EXAMPLE 5.3. *For $n \geq 2$, let M be an $(n-1)$-connected $(2n+1)$-dimensional Poincaré Duality complex with $H^{n+1}(M) \cong \mathbb{Z}^d$ for $d \geq 1$. As in Example 4.4, there is a homotopy cofibration*

$$S^{2n} \xrightarrow{g} S^n \vee S^{n+1} \vee \Sigma X \xrightarrow{j} M$$

where g is the attaching map for the top cell. By Theorem 5.1, Ωj has a right homotopy inverse $t\colon \Omega M \longrightarrow \Omega(S^n \vee S^{n+1} \vee \Sigma X)$ and there is a homotopy fibration

$$(\Sigma\Omega M \wedge S^{2n-1}) \vee S^{2n} \xrightarrow{[\gamma,g]+g} S^n \vee S^{n+1} \vee \Sigma X \xrightarrow{\quad j \quad} M$$

where $\gamma = ev \circ \Sigma t$.

One useful application of Theorem 5.1 is to show that certain maps are null homotopic after looping.

LEMMA 5.4. *Let $S^{m+n-1} \xrightarrow{g} Y \xrightarrow{j} Y'$ be a cofibration as in Theorem 4.1. Suppose that there is a map $a\colon Y' \longrightarrow Z$ such that $a \circ j$ is null homotopic. Then Ωa is null homotopic.*

Proof. By Theorem 5.1, Ωj has a right homotopy inverse $t\colon \Omega Y' \longrightarrow \Omega Y$. So $\Omega a \simeq \Omega a \circ \Omega j \circ t$ but $\Omega a \circ \Omega j$ is null homotopic, implying that Ωa is null homotopic. □

EXAMPLE 5.5. *Let M be an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex and let G be a topological group group. Let $P \longrightarrow M$ be a principal G-bundle classified by a map of the form $a\colon M \xrightarrow{\pi} S^{2n} \xrightarrow{\epsilon} BG$ where π is the pinch map to the top cell and ϵ represents a generator of $\pi_{2n-1}(G)$. Depending on G, there may be a finite or countably infinite number of such principal bundles which are inequivalent. Since a factors through π, the composite $a \circ j$ is null homotopic. Therefore Ωa is null homotopic by Lemma 5.4. Consequently, there is a homotopy equivalence $\Omega P \simeq \Omega M \times \Omega G$, and this holds independently of the bundle type.*
Carrying on, observe that a map $f\colon M \longrightarrow BG$ classifying a principal G-bundle has a factorization in the form of a if and only if the restriction of f to the n-skeleton $\bigvee_{i=1}^{d} S^n$ of M is null homotopic. This would occur, for example, if $\pi_n(BG) \cong \pi_{n-1}(G) = 0$. In particular, if M is a 2-connected 6-manifold and G is a simply-connected simple compact Lie group then $\pi_2(G) = 0$ so every principal G-bundle P over M has the property that $\Omega P \simeq \Omega M \times \Omega G$. Specializing further, if $G = SU(n)$ then the pinch map π induces an isomorphism $[M, BSU(n)] \cong [S^6, BSU(n)]$ and if $n \geq 3$ then $\pi_5(SU(n)) \cong \mathbb{Z}$, so there are countably many distinct principal $SU(n)$-bundles over M but all become homotopy equivalent after looping.

EXAMPLE 5.6. *This is a variation on the previous example. Let M be an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex and let $a\colon M \xrightarrow{\pi}$*

$S^{2n} \xrightarrow{\epsilon} M$ be a self-map where π is the pinch map to the top cell and ϵ is any element of $\pi_{2n}(M)$. Since a factors through π, the composite $a \circ j$ is null homotopic. Therefore Ωa is null homotopic by Lemma 5.4. In particular, a induces the zero map in homotopy groups. This is despite the fact that a itself need not be null homotopic.

6 The collar map for a connected sum

In this section we give an entirely new example of our methods to demonstrate their strength. Let M and N be simply-connected closed n-dimensional Poincaré Duality complexes. Let $\overline{M}$ and $\overline{N}$ be the $(n-1)$-skeletons of M and N respectively, and let

$$S^{n-1} \xrightarrow{f_1} \overline{M} \xrightarrow{j_1} M$$
$$S^{n-1} \xrightarrow{f_2} \overline{N} \xrightarrow{j_2} N$$

be the homotopy cofibrations that attach the top cells to M and N. The connected sum $M\#N$ is given by the homotopy cofibration

$$S^{n-1} \xrightarrow{f_1+f_2} \overline{M} \vee \overline{N} \longrightarrow M\#N.$$

Geometrically, $M\#N$ is obtained by cutting an n-disc out of the interior of the top cell in each of M and N and then gluing the two together along the boundary of the removed n-disc. We can then collapse that collar (the boundary of the n-disc) to a point to obtain a cofibration

$$S^{n-1} \xrightarrow{g} M\#N \xrightarrow{j} M \vee N.$$

In [HaL, Theorem 5.1], Halperin and Lemaire showed that if M is any simply-connected closed, compact Poincaré Duality complex then the attaching map for the top cell of M is rationally inert, implying that the inclusion $\overline{M} \xrightarrow{j_1} M$ has the property that, rationally, Ωj_1 has a right homotopy inverse. Halperin and Lemaire go on [HaL, Theorem 5.4] to show that if at least one of $H^*(M;\mathbb{Q})$ or $H^*(N;\mathbb{Q})$ is not monogenic (generated by a single element) then the map $M\#N \xrightarrow{j} M \vee N$ also has the property that, rationally, Ωj has a right homotopy inverse.
We will prove an integral variation of this statement. Integrally, it may not be true that Ωj_1 and Ωj_2 have right homotopy inverses. However, we show that if they do then Ωj does as well.

Proposition 6.1. *Let M and N be simply-connected closed n-dimensional Poincaré Duality complexes. If the inclusions $\overline{M} \xrightarrow{j_1} M$ and $\overline{N} \xrightarrow{j_2} N$ have right homotopy inverses after looping, then so does the map $M\#N \xrightarrow{j} M \vee N$. In particular, there is a homotopy equivalence*

$$\Omega(M\#N) \simeq \Omega(M \vee N) \times \Omega(\Omega(M \vee N) \ltimes S^{n-1}).$$

Proof. Observe that the composite $\overline{M} \vee \overline{N} \longrightarrow M\#N \xrightarrow{j} M \vee N$ is $j_1 \vee j_2$. We will show that $\Omega(j_1 \vee j_2)$ has a right homotopy inverse, implying that Ωj also has a right homotopy inverse. Granting this, the homotopy cofibration $S^{n-1} \xrightarrow{g} M\#N \xrightarrow{j} M \vee N$ satisfies the hypotheses of Proposition 3.5, and therefore we obtain the asserted homotopy equivalence for $\Omega(M\#N)$.
It remains to show that $\Omega(j_1 \vee j_2)$ has a right homotopy inverse. In general, the inclusion of the wedge of simply-connected spaces into their product induces a natural homotopy fibration sequence $\Omega X * \Omega Y \longrightarrow X \vee Y \longrightarrow X \times Y$ which splits after looping to give a natural homotopy equivalence

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega(\Omega X * \Omega Y).$$

In our case we obtain a homotopy fibration diagram

$$\begin{array}{ccccc}
\Omega\overline{M} * \Omega\overline{N} & \longrightarrow & \overline{M} \vee \overline{N} & \longrightarrow & \overline{M} \times \overline{N} \\
\downarrow{\scriptstyle \Omega j_1 * \Omega j_2} & & \downarrow{\scriptstyle j_1 \vee j_2} & & \downarrow{\scriptstyle j_1 \times j_2} \\
\Omega M * \Omega N & \longrightarrow & M \vee N & \longrightarrow & M \times N
\end{array}$$

and a homotopy commutative diagram

$$\begin{array}{ccc}
\Omega\overline{M} \times \Omega\overline{N} \times \Omega(\Omega\overline{M} * \Omega\overline{N}) & \longrightarrow & \Omega(\overline{M} \vee \overline{N}) \\
\downarrow{\scriptstyle \Omega j_1 \times \Omega j_2 \times \Omega(\Omega j_1 * \Omega j_2)} & & \downarrow{\scriptstyle \Omega(j_1 \vee j_2)} \\
\Omega M \times \Omega N \times \Omega(\Omega M * \Omega N) & \longrightarrow & \Omega(M \vee N)
\end{array}$$

where the horizontal maps are homotopy equivalences. By hypothesis, Ωj_1 and Ωj_2 have right homotopy inverses, and therefore so does $\Omega j_1 \times \Omega j_2 \times \Omega(\Omega j_1 * \Omega j_2)$. Hence $\Omega(j_1 \vee j_2)$ also has a right homotopy inverse. □

For example, if $m \geq 2$ then $S^m \times S^{2n-m}$ is a Poincaré Duality complex of dimension $2n$, $\overline{S^m \times S^{2n-m}} \simeq S^m \vee S^{2n-m}$, and the inclusion $S^m \vee S^{2n-m} \longrightarrow S^m \times S^{2n-m}$ has a right homotopy inverse after looping. If $m \neq n$ then $S^m \times S^{2n-m}$ is not an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex, so it is different from the spaces in Example 4.2. Given an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex M, by Proposition 6.1 there is a homotopy equivalence

$$\begin{aligned}\Omega(M\#(S^m \times S^{2n-m})) \simeq \\ \Omega(M \vee (S^m \times S^{2n-m})) \times \Omega(\Omega(M \vee (S^m \times S^{2n-m})) \ltimes S^{2n-1}).\end{aligned}$$

Similarly, one can consider the connected sum of $S^m \times S^{2n+1-m}$ and one of the $(n-1)$-connected $(2n+1)$-dimensional manifolds in Example 4.4.

References

[A] D. J. Anick, Homotopy exponents for spaces of category two, *Algebraic topology (Arcata, CA, 1986)*, 24–52, Lecture Notes in Math., 1370, Springer, Berlin, 1989.

[B] S. Basu, The homotopy type of the loops on $(n-1)$-connected $(2n+1)$-manifolds, *Algebraic topology and related topics*, 1–25, Trends. Math., Birkhäuser/Springer, Singapore, 2019.

[BB] Sa. Basu and So. Basu, Homotopy groups of highly connected manifolds, *Adv. Math.* 337 (2018), 363–416.

[BT] P. Beben and S. Theriault, The loop space homotopy type of simply-connected four-manifolds and their generalizations, *Adv. Math.* 262 (2014), 213–238.

[BW] P. Beben and J. Wu, The homotopy type of a Poincaré duality complex after looping, *Proc. Edinb. Math. Soc. (2)* 58 (2015), 581–616.

[D] E. Dror, A generalization of the Whitehead theorem, *Symposium on Algebraic Topology (Battelle Seattle Res. Center, Seattle, Wash., 1971)*, 13–22, Lecture Notes in Math., 249, Springer, Berlin, 1971.

[FT] Y. Félix and J.-C. Thomas, Effet d'un attachement cellulaire dans l'homologie de l'espace des lacets, *Ann. Inst. Fourier (Grenoble)* 39 (1989), 207–224.

[Ga] T. Ganea, A generalization of the homology and homotopy suspension, *Comment. Math. Helv.* 39 (1965), 295–322.

[Gr] B. Gray, Abelian properties of Anick spaces, *Mem. Amer. Math. Soc.* 246 (2017), no. 1162.

[GT] B. Gray and S. Theriault, An elementary construction of Anick's space, *Geom. Topol.* 14 (2010), 243–275.

[HaL] S. Halperin and J.-M. Lemaire, Suites inertes dans les algèbres de Lie graduées ("Autopsie d'un meurtre. II"), *Math. Scand.* 61 (1987), 39–67.

[HeL] K. Hess and J.-M. Lemaire, Nice and lazy cell attachments, *J. Pure Appl. Algebra* 112 (1996), 29–39.

[M] M. Mather, Pull-backs in homotopy theory, *Canad. J. Math.* 28 (1976), 225–263.

[N] J. A. Neisendorfer, *Algebraic methods in unstable homotopy theory*, New Mathematical Monographs, 12, Cambridge University Press, Cambridge, 2010.

[S] J. Stasheff, *H-spaces from a homotopy point of view*, Lecture Notes in Math., 161, Springer, Berlin-New York, 1970.

Piotr Beben
International TechneGroup Ltd
Cambridge CB24 4UQ
United Kingdom
pdbcas@gmail.com

Stephen Theriault
Mathematical Sciences
University of Southampton
Southampton SO17 1BJ
United Kingdom
s.d.theriault@soton.ac.uk