

FINITE GROUPS SCHEME ACTIONS AND INCOMPRESSIBILITY
OF GALOIS COVERS: BEYOND THE ORDINARY CASE

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ABSTRACT. Inspired by recent work of Farb, Kisin and Wolfson, we develop a method for using actions of finite group schemes over a mixed characteristic dvr R to get lower bounds for the essential dimension of a cover of a variety over $K = \text{Frac}(R)$. We then apply this to prove p -incompressibility for congruence covers of a class of unitary Shimura varieties for primes p at which the reduction of the Shimura variety (at any prime of the reflex field over p) does not have any ordinary points. We also make some progress towards a conjecture of Brosnan on the p -incompressibility of the multiplication by p map of an abelian variety.

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1 INTRODUCTION

Let $f : X \rightarrow Y$ be a generically étale morphism of varieties over a field K with Y integral. We say that f is *incompressible* if there is no commutative Cartesian diagram

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ \downarrow f & & \downarrow f' \\ Y & \dashrightarrow & Y' \end{array}$$

with Y' integral, $\dim(Y') < \dim(Y)$, and the horizontal arrows are dominant rational maps. If p is a prime number, we say that f is *p -incompressible* if for

any integral Z with a dominant generically finite morphism $Z \rightarrow Y$ of degree prime to p , the morphism $X \times_Y Z \rightarrow Z$ is incompressible. For more on this definition (and the more general notion of *essential dimension* of f) the reader may consult [3], [8] or [16].

In the article [8], Farb, Kisin and Wolfson introduced a new method for proving p -incompressibility over fields of characteristic zero by using an elegant mixed characteristic method and applied it to prove the p -incompressibility of a large class of congruence covers of Shimura varieties of Hodge type under an *ordinarity* assumption on the reduction of the Shimura variety at a prime of the reflex field lying over p . Their result was reproved in many cases, and extended to positive characteristics and certain Shimura varieties not of Hodge type, using the “fixed-point method” by Brosnan and the first-named author in [2]. Moreover, the latter authors formulated a general conjecture [2, Conjecture 1] on the essential dimension of covers associated to arbitrary variations of Hodge structure which, in the context of Shimura varieties, predicts that the ordinarity assumption of [8] is unnecessary. The main goal of this paper is to develop methods which allow us to prove that this is indeed true for a certain class of unitary Shimura varieties (Theorem 5.1) and using related methods we also make progress towards a conjecture of P. Brosnan (Conjecture 6.1) on the p -incompressibility of the multiplication by p map $[p] : A \rightarrow A$, where A is an abelian variety over a field of characteristic zero (Theorem 6.4).

A special case of our theorem on unitary Shimura varieties is the following result, which we state somewhat informally here. For the precise (and more general) statement, see Theorem 5.1.

THEOREM 1.1. *Let F be an imaginary quadratic extension of \mathbb{Q} and let S be a PEL moduli space of abelian varieties of odd dimension $d = 2\delta + 1 \geq 3$ with endomorphism ring containing O_F . If the type of the O_F -action is $(\delta, \delta + 1)$, then for all odd primes p of good reduction of S the principal level p congruence cover $S(p) \rightarrow S$ is p -incompressible.*

If one specializes [8, Theorem 4.3.6] to this case, one obtains p -incompressibility only for primes p which split in O_F . Theorem 5.1 applies to PEL Shimura varieties corresponding to arbitrary CM fields F (with a similar restriction on the “type”) and though in this setting our results are strictly stronger than those of [8], they require that p split completely in the maximal totally real subfield F_0 of F .

A consequence of our main theorem (Theorem 6.4) on abelian varieties is the following (Corollary 6.7):

THEOREM 1.2. *For any abelian variety A of dimension $d \leq 3$ over a field of characteristic zero there exists a set $\mathfrak{P}(A)$ of rational primes of positive density such that $[p] : A \rightarrow A$ is p -incompressible for $p \in \mathfrak{P}(A)$.*

Our methods, which are inspired by and extend those of Farb, Kisin and Wolfson, depend on degenerating the covers to fields of characteristic p . The two key ingredients used in [8] are the Serre–Tate theorem relating deformations of

abelian varieties and their p -divisible groups, and Kummer theory, which gives a very concrete description of μ_p -torsors (over any local base). At an ordinary point, the Serre–Tate theorem gives a very explicit (formal) description of the degeneration of the congruence cover when one specializes to characteristic p and Kummer theory allows the authors of [8] to prove an extension result ([8, Lemma 3.1.5]) for $(\mu_p)^n$ -torsors which plays an essential role in their proofs of p -incompressibility.

For torsors under more general finite (possibly noncommutative) group schemes there is no good analogue of Kummer theory, so we change our point of view and consider actions of finite group schemes rather than torsors. In this context, our replacement for [8, Lemma 3.1.5] is Lemma 2.2, an extension result for actions of general finite flat group schemes. Not working with torsors has drawbacks though, since general actions of finite group schemes, even if they are faithful (which is not guaranteed), can have complicated stabilizers, and this is the main reason why we cannot prove incompressibility in the full expected generality (in the settings we consider). In §2 we consider two classes of group schemes for which Lemma 2.2 can be usefully applied to give lower bounds for essential dimension of covers; the first class is applied to prove our results on Shimura varieties and the second on abelian varieties. The main idea in both cases is the simple fact that we can bound from below the dimension of a smooth variety on which a (finite) group scheme acts using knowledge of the dimension of the Lie algebra of the group scheme and that of the stabilizer of a general point.

The group schemes used for the application to Shimura varieties are noncommutative (in the non-ordinary case, which is our main interest) and §3 is devoted to an analysis of the structure of these group schemes in the setting of PEL Shimura varieties of type A. Here we depend heavily on some results of Moonen from [19]. In §4 we construct certain integral models of the p -congruence cover of such Shimura varieties (for p an unramified prime) by a naive method, i.e., by normalisation of the Kottwitz integral model [13] in the function field of the p -congruence cover. We show that these integral models are generically smooth along the special fibre and admit an action of a suitable finite flat group scheme using a result of Cariani–Scholze from [4] and the results of §3. In §5 we apply Proposition 2.6 (our incompressibility criterion for certain noncommutative group scheme action) to this integral model to prove our main result on Shimura varieties, Theorem 5.1. Finally, in §6 we prove our results on abelian varieties. Here we use commutative group schemes which are “almost ordinary” via Proposition 2.12, a criterion giving a lower bound on the essential dimension for actions of such group schemes.

1.1 NOTATION

We usually denote by K a complete discretely valued field of characteristic 0, by R its ring of integers and by k its residue field which will always be perfect (and usually, but not always, of characteristic $p > 0$). We denote $\mathrm{Spec}(R)$ by \mathcal{T} and usually (but not always) denote schemes over \mathcal{T} by using calligraphic

fonts, e.g., $\mathcal{X}, \mathcal{G}, \dots$, and the generic fibre of such a scheme by the corresponding letter in ordinary font, i.e., $X = \mathcal{X}_K, G = \mathcal{G}_K, \dots$, and we will say that \mathcal{X} (resp. \mathcal{G}) is a model of X (resp. G), \dots . We will implicitly assume (except when explicitly mentioned otherwise) that the scheme \mathcal{X} over \mathcal{T} is flat and the closed fibre, which will usually be denoted by using a subscript 0, e.g., $\mathcal{X}_0, \mathcal{G}_0, \dots$, is non-empty. However, we will often use notation such as \mathcal{Z}_0 to denote a subscheme of \mathcal{X}_0 as above without assuming that it is the closed fibre of a flat subscheme \mathcal{Z} of \mathcal{X} .

For any finite connected (not necessarily flat) non-empty scheme \mathcal{X} over \mathcal{T} we will denote by $T(\mathcal{X})$ its reduced tangent space, i.e., the tangent space of \mathcal{X}_0 (at its unique point). We set $t(\mathcal{X}) = \dim(T(\mathcal{X}))$.

For a finite commutative group scheme (resp. p -divisible group) over any base, we use D as a superscript (e.g., \mathcal{G}^D) to denote its Cartier (resp. Serre) dual. We also recall that a finite flat group scheme (over any base) is *multiplicative* if its Cartier dual is (finite) étale.

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2 FINITE GROUP SCHEME ACTIONS AND ESSENTIAL DIMENSION

2.1 DESCENDING GROUP SCHEME ACTIONS

The lemma below is the key property of multiplicative group schemes that (from our point of view) makes them very useful for proving incompressibility of covers. Although the goal of this paper is to prove incompressibility using more general group schemes, multiplicative group schemes (as subgroup schemes of more general group schemes) still play a key role in all our results.

LEMMA 2.1. *Let \mathcal{G} be a finite flat connected multiplicative group scheme over \mathcal{T} . If \mathcal{G} acts on a scheme \mathcal{X} which is smooth and of finite type over \mathcal{T} (with $\mathcal{X}_0 \neq \emptyset$) so that G acts generically freely on X , then \mathcal{G} acts freely on an open subscheme $\mathcal{X}' \subset \mathcal{X}$ (with $\mathcal{X}'_0 \neq \emptyset$).*

Proof. It suffices to show that \mathcal{G}_0 acts generically freely on \mathcal{X}_0 , so after a base change we may assume that k is algebraically closed which implies that \mathcal{G}^D is constant. For the action of a finite group scheme freeness of the action is equivalent to the stabilizers of all geometric points being trivial, so if generic freeness does not hold then since \mathcal{X}_0 is smooth and \mathcal{G}_0 only has finitely many subgroup schemes it follows that there exists a non-empty open set of \mathcal{X}_0 such that all geometric points of this subset have the same non-trivial stabilizer. We choose any \mathcal{G} -invariant affine open $\mathcal{X}' \subset \mathcal{X}$ (with $\mathcal{X}'_0 \neq \emptyset$) such that all

geometric points of \mathcal{X}'_0 have the same non-trivial stabilizer in \mathcal{G}_0 . Since \mathcal{G}^D is constant, one then easily reduces to the case that $\mathcal{G} = \mu_p$ and the action of \mathcal{G}_0 on \mathcal{X}'_0 is trivial; here we use smoothness of \mathcal{X}'_0 to conclude that if μ_p stabilizes each geometric point then the action must be trivial.

Writing $\mathcal{X}' = \text{Spec}(B)$, the μ_p action translates into the structure of a $\mathbb{Z}/p\mathbb{Z}$ grading on the R -algebra B , i.e., $B = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} B_i$ with $B_i \cdot B_j \subset B_{i+j}$. Since this grading is compatible with base change, the assumption that the action on \mathcal{X}'_0 is trivial implies that $B_i \otimes_R k = \{0\}$ for $i \neq 0$. This implies that the G -action on the image of $\mathcal{X}'(R)$ in $X(K)$ is trivial, but the assumptions on \mathcal{X} and \mathcal{T} imply that this set is Zariski dense in X , thereby contradicting the generic freeness of the G -action on X . \square

LEMMA 2.2. *Let \mathcal{G} be a finite flat group scheme over \mathcal{T} acting on a separated irreducible normal scheme \mathcal{X} , faithfully flat and of finite type over \mathcal{T} , with $\mathcal{G}_0(k)$ acting trivially on the set of irreducible components of \mathcal{X}_0 . Assume further that $G = \mathcal{G}_K$ is a constant group scheme and there is a G -equivariant dominant morphism $f : X \rightarrow Y$, where Y is a smooth quasi-projective variety over K . Then there exist*

1. *a non-empty open G -invariant subvariety $Y' \subset Y$ and a normal model \mathcal{Y}' of Y' over \mathcal{T} (with $\mathcal{Y}'_0 \neq \emptyset$) such that the G -action on Y' extends to a \mathcal{G} -action on \mathcal{Y}' ,*
2. *a non-empty open \mathcal{G} -invariant subscheme $\mathcal{X}' \subset \mathcal{X}$,*

such that $f|_{\mathcal{X}'}$ extends to a \mathcal{G} -equivariant surjective flat morphism $\phi : \mathcal{X}' \rightarrow \mathcal{Y}'$. If \mathcal{X} is smooth over \mathcal{T} then we may also take \mathcal{Y}' to be smooth over \mathcal{T} .

Proof. Let $Z' = Y/G$ and let \mathcal{Z} be a normal projective (flat) scheme over \mathcal{T} containing Z' as an open subscheme. Since \mathcal{X} is normal and \mathcal{Z} is proper, the rational map $g : \mathcal{X} \dashrightarrow \mathcal{Z}$, induced by f and the quotient map $Y \rightarrow Z'$, is defined at the generic points of \mathcal{X}_0 . By replacing \mathcal{Z} by a suitable normalized blow up of \mathcal{Z} along a closed subscheme of \mathcal{Z}_0 (using [11, Theorem VI.1.3]), we may assume that there is an open \mathcal{G} -invariant subscheme \mathcal{X}' of \mathcal{X} on which g is defined and the morphism $\mathcal{X}'_0 \rightarrow \mathcal{Z}_0$ dominates at least one irreducible component of \mathcal{Z}_0 . Let $\tilde{\mathcal{Y}}$ be the normalisation of \mathcal{Z} in the function field of Y . The group $G_{\mathcal{T}}$ (the group G viewed as a constant group scheme over \mathcal{T}) acts on $\tilde{\mathcal{Y}}$ and it contains Y as a G -invariant open subset. The morphism f extends to a $G_{\mathcal{T}}$ -equivariant morphism $\phi : \mathcal{X}' \rightarrow \tilde{\mathcal{Y}}$ (since \mathcal{X}' is normal, $\tilde{\mathcal{Y}}$ is finite over \mathcal{Z} and its function field is contained in that of \mathcal{X}') and \mathcal{X}'_0 dominates an irreducible component of $\tilde{\mathcal{Y}}_0$. By the construction of \mathcal{Z} , the morphism ϕ is flat at at least one of the generic points of \mathcal{X}'_0 . By removing, if necessary, the components of \mathcal{X}'_0 at whose generic points ϕ is not flat, it follows from the openness of flatness and the condition on the $\mathcal{G}_0(k)$ -action on the irreducible components of \mathcal{X}'_0 , that we may assume that there exists an open $G_{\mathcal{T}}$ -invariant subscheme \mathcal{Y}' of $\tilde{\mathcal{Y}}$ (with $\mathcal{Y}'_0 \neq \emptyset$) such that ϕ restricted to \mathcal{X}' is flat over \mathcal{Y}' . By shrinking \mathcal{Y}' further and replacing \mathcal{X}'

by $\phi^{-1}(\mathcal{Y}')$ we may assume that $Y' \subset Y$ and the restriction of ϕ from $\mathcal{X}' \rightarrow \mathcal{Y}'$ is flat, surjective and $G_{\mathcal{T}}$ -equivariant.

Let $O(\mathcal{G})$ denote the affine algebra of \mathcal{G} . The \mathcal{G} action on \mathcal{X}' then corresponds to a map of sheaves $O_{\mathcal{X}'} \rightarrow O(\mathcal{G}) \otimes_R O_{\mathcal{X}'}$ satisfying the usual identities. The $G_{\mathcal{T}}$ -action on \mathcal{Y}' induces a rational map $O_{\mathcal{Y}'} \dashrightarrow O(\mathcal{G}) \otimes_R O_{\mathcal{Y}'}$ of sheaves, i.e., a map defined after tensoring with K . The action of \mathcal{G} extends to \mathcal{Y}' if this map is in fact defined without tensoring with K (since the identities needed for a group action will automatically hold by flatness over \mathcal{T} .)

Since $O(\mathcal{G})$ is a finite free module, by using an R -basis and the $G_{\mathcal{T}}$ equivariance of the map $\mathcal{X}' \rightarrow \mathcal{Y}'$ we are reduced to showing that if $\sigma \in \Gamma(Y', O_{Y'})$ is such that $f^*(\sigma)$ extends to an element of $\Gamma(\mathcal{X}', O_{\mathcal{X}'})$ then σ extends to an element of $\Gamma(\mathcal{Y}', O_{\mathcal{Y}'})$. This follows immediately from the surjectivity of $\mathcal{X}' \rightarrow \mathcal{Y}'$ and the normality of \mathcal{Y}' (see, e.g., [7, Lemma 2.1]).

For the smoothness statement, since k is perfect it suffices to observe that if \mathcal{X}'_0 is reduced then so is \mathcal{Y}'_0 since the map ϕ is flat. \square

Remark 2.3. The lemma above is the basis for all our results on incompressibility. Note that although \mathcal{G} is arbitrary, in this generality even if \mathcal{G} acts freely on \mathcal{X} and Y we cannot conclude that it acts freely on \mathcal{Y}' (see Example 2.4 below). This is the fundamental difficulty one encounters when trying to apply this lemma, and the rest of this section is devoted to developing methods which will allow us to overcome this difficulty in certain special cases.

Example 2.4. Let E be an elliptic curve over K with good supersingular reduction and let $\mathcal{E}_{/\mathcal{T}}$ be its Neron model. Let $\mathcal{E}[p]$ be the p -torsion subscheme of \mathcal{E} and assume that its generic fibre is a constant group scheme (so isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$). Let $\mathcal{G}', \mathcal{G}'' \subset \mathcal{E}[p]$ be distinct finite flat subgroup schemes of order p and let $\mathcal{G} = \mathcal{G}' \times_{\mathcal{T}} \mathcal{G}''$. Clearly $t(\mathcal{G}) = 2$ (since \mathcal{E}_0 is supersingular) and \mathcal{G} acts freely on $\mathcal{E} \times_{\mathcal{T}} \mathcal{E}$. However, the sum map $\mathcal{E}^2 \rightarrow \mathcal{E}$ is \mathcal{G} -equivariant and G acts freely on E but there is a non-trivial subgroup scheme $\mathcal{H}_0 \subset \mathcal{G}_0$ which acts trivially on \mathcal{E}_0 .

2.2 GROUP SCHEME ACTIONS, STABILIZERS AND ESSENTIAL DIMENSION OF COVERS

LEMMA 2.5. *Let \mathcal{G} be a finite flat connected group scheme acting on a smooth scheme \mathcal{X} over \mathcal{T} . Suppose that for some closed point $x \in \mathcal{X}_0(k)$, $t(\mathcal{G}_0/\text{Stab}(x)) \geq n$. Then $\dim(X) \geq n$.*

Proof. This is clear since the \mathcal{G} action induces a closed embedding of $\mathcal{G}_0/\text{Stab}(x_0)$ into \mathcal{X}_0 . The hypothesis on the dimension and the smoothness of \mathcal{X}_0 implies that $\dim(\mathcal{X}_0) \geq n$ and so (by flatness) $\dim(X) \geq n$ as well. \square

The proposition below allows us to use non-free \mathcal{G} -actions to find lower bounds on the essential dimension of covers, but only for very special \mathcal{G} . Note that for any non-trivial application \mathcal{G} has to be noncommutative.

PROPOSITION 2.6. *Let G be a finite group acting faithfully on a smooth quasiprojective variety X over K . Let \mathcal{G} be a flat connected model of G over \mathcal{T} acting on a smooth model \mathcal{X} of X over \mathcal{T} (extending the G -action on X). Let \mathcal{H} be a finite flat subgroup scheme of \mathcal{G} which is of multiplicative type. Suppose that for all subgroup schemes \mathcal{K}_0 of \mathcal{G}_0 which intersects \mathcal{H}_0 trivially, we have $t(\mathcal{G}_0/\mathcal{K}_0) \geq e$. Then*

1. $\text{ed}(X) \geq e$.
2. If \mathcal{G} acts freely on \mathcal{X} then $\text{ed}(X;p) \geq e$.

Proof. We first reduce (2) to (1), so let us assume that the \mathcal{G} action is free. Then let \mathcal{Z} be the quotient of \mathcal{X} by \mathcal{G} ; since \mathcal{G} is finite and \mathcal{X} is quasiprojective this exists (by the theory of Hilbert schemes) and the freeness of the \mathcal{G} -action implies that it is also smooth over \mathcal{T} ; this can be checked fibre-wise and by [20, §12, Theorem 1] the quotient map is flat which implies that if \mathcal{X} is smooth then so is the quotient. Let $L/K(\mathcal{Z})$ be a finite extension of degree prime to p and let \mathcal{Z}' be the normalisation of \mathcal{Z} in L . Using Abhyankar's lemma as in the proof of [8, Theorem 3.2.6], it follows that after replacing K by a finite extension we may assume that the map $\mathcal{Z}' \rightarrow \mathcal{Z}$ is étale at at least one generic point of \mathcal{Z}'_0 . By shrinking \mathcal{Z}' we may assume that the map $\mathcal{Z}' \rightarrow \mathcal{Z}$ is étale and, in particular, \mathcal{Z}' is smooth. Let $\tilde{\mathcal{X}} = \mathcal{X} \times_{\mathcal{Z}} \mathcal{Z}'$. The given \mathcal{G} action on \mathcal{X} and the trivial action on \mathcal{Z}' induces a free action of \mathcal{G} on $\tilde{\mathcal{X}}$. By replacing \mathcal{X} by $\tilde{\mathcal{X}}$ we see that it suffices to prove that $\text{ed}(X) \geq e$.

If $X \rightarrow Y$ is a compression of the G -action on X then using Lemma 2.2 we get a smooth scheme \mathcal{Y}' over \mathcal{T} with a \mathcal{G} -action such that $Y' \subset \mathcal{Y}'$ compatibly with the G -action on Y . By Lemma 2.1, \mathcal{H}_0 acts freely on a non-empty open subset of \mathcal{Y}_0 , so the stabilizer in \mathcal{G}_0 of a general point of \mathcal{Y}_0 must intersect \mathcal{H}_0 trivially. The proposition then follows immediately from Lemma 2.5. \square

QUESTION 2.7. Can one characterize all finite p -groups G for which there exists a finite flat *connected* group scheme \mathcal{G} over some \mathcal{T} such that $G = \mathcal{G}_K$?

The technical lemma below will be used (via Lemma 3.10) to check that the dimension hypothesis of Proposition 2.6 holds in the proof of Theorem 5.1.

LEMMA 2.8. *Let k be any field and let G be an affine group scheme over k . Let H be a closed subgroup scheme of G and let $\pi : G \rightarrow G/H$ be the quotient map. Let Z be an affine scheme over k such that π factors through a morphism $\pi' : G \rightarrow Z$. If the scheme-theoretic image $\pi'(H)$ of H in Z is a reduced k -rational point z , then the map on tangent spaces $T_z Z \rightarrow T_{[H]}(G/H)$ is surjective.*

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be the maps of k -algebras corresponding to the maps $G \xrightarrow{\pi'} Z \rightarrow G/H$ and let $h = gf$. Let m_A be the maximal ideal of A corresponding to $[H]$, m_B the maximal ideal of B corresponding to z and let $I_H \subset C$ be the ideal of H . The quotient map π is faithfully flat and

$h(m_A)C = I_H$. We need to prove that the map $m_A/m_A^2 \rightarrow m_B/m_B^2$ induced by f is injective.

The assumption on the map π' implies that $g(m_B)C \subset I_H$. Thus, if an element $x \in m_A$ is such that $f(x) \in m_B^2$, then $h(x) \in I_H^2$. Since h is faithfully flat, the map $A/m_A^2 \rightarrow C \otimes_A (A/m_A^2) = C/I_H^2$ induced by h is injective. This implies that $x \in m_A^2$ as desired. \square

Remark 2.9. We have assumed that G is affine only for convenience since this is the case we will need later: it is easy to see that the lemma can be extended to any G (and Z) of finite type over k .

2.3 ALMOST MULTIPLICATIVE GROUP SCHEMES

As already noted above, in any non-trivial application of Proposition 2.6 the group scheme \mathcal{G} has to be noncommutative. In this subsection we discuss a special class of commutative group schemes for which we can also overcome the non-freeness problem mentioned in Remark 2.3.

LEMMA 2.10. *Let \mathcal{G} be a finite flat group scheme over \mathcal{T} such that $G \cong (\mathbb{Z}/p\mathbb{Z})^r$. Then $\mathcal{G} \cong \mathcal{G}' \times \mathcal{G}''$, where \mathcal{G}' is connected and $\mathcal{G}'' \cong (\mathbb{Z}/p\mathbb{Z})^s$ for some $s \leq r$.*

Proof. We let \mathcal{G}' be the connected component of \mathcal{G} containing the image of the identity section $\mathcal{T} \rightarrow \mathcal{G}$. It is clearly a closed connected subgroup scheme of \mathcal{G} which is finite flat over \mathcal{T} . The group scheme \mathcal{G}/\mathcal{G}' is then étale and the structure of G implies that it is isomorphic to the constant group scheme $(\mathbb{Z}/p\mathbb{Z})_s^{\mathcal{T}}$ for some $s \leq r$. We let \mathcal{G}'' be any closed subgroup scheme of \mathcal{G} such that the induced map $\mathcal{G}'' \rightarrow (\mathcal{G}/\mathcal{G}')_K$ is an isomorphism. Such a group scheme exists by the structure of G and since \mathcal{G}/\mathcal{G}' is étale it follows that the map $\mathcal{G}'' \rightarrow \mathcal{G}/\mathcal{G}'$ is an isomorphism, so $\mathcal{G}'' \cong (\mathbb{Z}/p\mathbb{Z})^s$. The map $\mathcal{G}' \times \mathcal{G}'' \rightarrow \mathcal{G}$ induced by the two inclusions gives the desired isomorphism. \square

LEMMA 2.11. *Let \mathcal{G} be a finite flat connected group scheme over \mathcal{T} such that $\mathcal{G} \cong \mathcal{G}' \times \mu_p^{r-1}$, where $G' \cong \mathbb{Z}/p\mathbb{Z}$.*

1. *If there exists a morphism of group schemes $\mathcal{G} \rightarrow \mathcal{H}$, with \mathcal{H} also finite flat, which is an isomorphism on generic fibres, then $\mathcal{H} \cong \mathcal{H}' \times \mu_p^{r-1}$ where $H' \cong \mathbb{Z}/p\mathbb{Z}$. In particular, $t(\mathcal{H}) = r$.*
2. *If \mathcal{G} acts faithfully on a smooth scheme \mathcal{X} with \mathcal{X}_0 irreducible, then \mathcal{G}_0 acts generically freely on \mathcal{X}_0 .*

Proof. We have $\mathcal{G}^D \cong (\mathcal{G}')^D \times (\mathbb{Z}/p\mathbb{Z})^{r-1}$ and since \mathcal{H}^D maps generically isomorphically to \mathcal{G}^D , it follows by applying Lemma 2.10 to \mathcal{H}^D and dualising that $\mathcal{H} \cong \mathcal{H}' \times \mu_p^{r-1}$ where \mathcal{H}' is connected (since we have a map $\mathcal{G} \rightarrow \mathcal{H}$) and $H' \cong \mathbb{Z}/p\mathbb{Z}$. Since \mathcal{H}' is connected and $H' \cong \mathbb{Z}/p\mathbb{Z}$, it follows from the classification of connected group schemes of order p over k that $t(\mathcal{H}) = 1$, so $t(\mathcal{H}) \geq 1 + (r - 1) = r$. This proves (1).

For (2), we use that \mathcal{G}'_0 is isomorphic to μ_p or α_p so \mathcal{G}_0 is isomorphic to μ_p^r or $\alpha_p \times \mu_p^{r-1}$. In either case, \mathcal{G}_0 has only finitely many subgroup schemes (bounded independently of k , as may be seen easily using Cartier duality) and (2) follows from this since the stabilizer of a general point of \mathcal{X}_0 must be constant. \square

PROPOSITION 2.12. *Let $G = (\mathbb{Z}/p\mathbb{Z})^r$ and suppose G acts faithfully on a smooth quasiprojective variety X over K . Let \mathcal{G} be a finite flat model of G over \mathcal{T} and suppose it acts on \mathcal{X} , a smooth quasiprojective model of X , with \mathcal{X}_0 geometrically irreducible.*

1. *If $\mathcal{G} \cong \mathcal{G}' \times \mu_p^{r-1}$ with \mathcal{G}' connected, then $\text{ed}(X) = r$.*
2. *If the \mathcal{G} action is also free then $\text{ed}(X; p) = r$.*

Proof. We may reduce (2) to (1) by Abhyankar's lemma as in the proof of Proposition 2.6, so it suffices to prove (1).

Using the definition of essential dimension and Lemma 2.2, if $\text{ed}(X) = s < r$, we may find a smooth affine scheme \mathcal{Y} over \mathcal{T} of relative dimension s with \mathcal{Y}_0 geometrically irreducible on which \mathcal{G} acts with the G action on Y being (generically) free.

We let \mathcal{H} be the effective model for the action of \mathcal{G} on \mathcal{Y} given by [21, Theorem A (ii)]; to see that this applies we can use [21, Theorem B] since one easily checks that the assumptions on \mathcal{X} imply that it is pure over \mathcal{T} in the sense of [21, Definition 2.1.1]. The group scheme \mathcal{H} acts faithfully on \mathcal{Y} and there is a morphism $\mathcal{G} \rightarrow \mathcal{H}$ which is an isomorphism on generic fibres. It then follows from Lemma 2.11 that \mathcal{H} acts generically freely on \mathcal{Y}_0 and so $s = \dim(Y) = \dim(\mathcal{Y}_0) \geq t(\mathcal{H}) = r$, a contradiction. \square

Remark 2.13. Proposition 2.12 does not extend in a simple way to more general integral models \mathcal{G} of $(\mathbb{Z}/p\mathbb{Z})^r$, as shown by Example 2.4. It would be very interesting to find (or even classify all) other \mathcal{G} for which (1) and (2) hold.

3 AUTOMORPHISMS OF (TRUNCATED) μ -ORDINARY p -DIVISIBLE GROUPS

The goal of this section is to prove some basic facts about automorphisms of truncated μ -ordinary p -divisible groups (with extra structure) which will be the key to applying the results of §2 to (certain) unitary Shimura varieties. We assume throughout that $p > 2$.

3.1 ENDOMORPHISMS OF μ -ORDINARY (TRUNCATED) p -DIVISIBLE GROUPS

Let κ be a finite extension of \mathbb{F}_p and $W(\kappa)$ the ring of Witt vectors over κ . In this subsection, we recall some basic facts and results about the building blocks of μ -ordinary group schemes and p -divisible groups from [19] and then prove a lemma (Lemma 3.1) about morphisms between such group schemes. This lemma is the basis for our computation of automorphism groups of more

general μ -ordinary group schemes in the next two subsections. To describe these objects we use contravariant Dieudonné theory as in [19].

Let I be the set of homomorphisms from κ to k (which we assume is algebraically closed in this section). This set has a natural cyclic structure induced by the Frobenius automorphism on κ and we denote the successor of $\tau \in I$ by $\tau + 1$. To any function $f : I \rightarrow \{0, 1\}$ we associate a Dieudonné module $M(f)$ over $W(k)$ with basis $\{e_\tau\}_{\tau \in I}$ with a $W(\kappa)$ -action defined by $x \cdot e_\tau = \tau(x)e_\tau$ for $x \in W(\kappa)$. We define the Frobenius and Verschiebung on $M(f)$ by defining them on basis elements as follows:

$$F(e_\tau) = \begin{cases} e_{\tau+1} & \text{if } f(\tau) = 0, \\ p \cdot e_{\tau+1} & \text{if } f(\tau) = 1, \end{cases} \quad V(e_{\tau+1}) = \begin{cases} p \cdot e_\tau & \text{if } f(\tau) = 0, \\ e_\tau & \text{if } f(\tau) = 1. \end{cases} \quad (1)$$

The Dieudonné module $M(f)$ corresponds to a p -divisible group over k with an action of $W(\kappa)$ which we call $\mathbb{X}(f)$ and we denote the p^n -torsion of this p -divisible group by $\mathbb{X}(f)_n$. By [19, Corollary 2.1.5], $\mathbb{X}(f)$ has a canonical lift to $W(k)$ which we denote by $\mathbb{X}^{\text{can}}(f)$ and we denote its p^n -torsion by $\mathbb{X}^{\text{can}}(f)_n$. Let $f^j : I \rightarrow \{0, 1\}$, $j = 1, 2$, be two functions, so we have $\mathbb{X}^{\text{can}}(f^j)_n$, $j = 1, 2$. Our first goal is to compute the group scheme over $W(k)$ which represents¹ the functor $\mathbf{Hom}(\mathbb{X}^{\text{can}}(f^1)_n, \mathbb{X}^{\text{can}}(f^2)_n)$ on $W(k)$ -algebras given by $A \mapsto \text{Hom}(\mathbb{X}^{\text{can}}(f^1)_{n/A}, \mathbb{X}^{\text{can}}(f^2)_{n/A})$; here, and later, we always assume that all maps are compatible with the $W(\kappa)$ -action. Given f^j , $j = 1, 2$, we define (following [19, 2.3.2]) $f' : I \rightarrow \{0, 1\}$ by

$$f'(\tau) = \begin{cases} 0 & \text{if } f^1(\tau) = f^2(\tau), \\ 1 & \text{if } f^1(\tau) \neq f^2(\tau). \end{cases} \quad (2)$$

We will say that $f^1 \leq f^2$ if $f^1(\tau) \leq f^2(\tau)$ for all $\tau \in I$ and say that $f^1 < f^2$ if $f^1 \leq f^2$ and $f^1 \neq f^2$.

LEMMA 3.1. *Assume $f^1 \leq f^2$ or $f^2 \leq f^1$. Then $\mathbf{Hom}(\mathbb{X}(f^1)_n, \mathbb{X}(f^2)_n) = 0$ unless $f^1 \leq f^2$. If this holds then*

$$\mathbf{Hom}(\mathbb{X}(f^1)_n, \mathbb{X}(f^2)_n) = \mathbb{X}(f')_n. \quad (3)$$

Furthermore, if $f^1 < f^2$ then

$$\mathbf{Hom}(\mathbb{X}^{\text{can}}(f^1)_n, \mathbb{X}^{\text{can}}(f^2)_n) = \mathbb{X}^{\text{can}}(f')_n. \quad (4)$$

The lemma is a mild strengthening of [19, Remarks 2.3.4(ii)] and the proof uses the same methods.

Proof. We first note that if $f^1 \leq f^2$, then $X(f^1) \times X(f^2)$ is an ordinary p -divisible group with $W(\kappa)$ -structure in the sense of [19, §1], so we may use all the results proved therein.

¹Since the group schemes involved are finite, these functors, as well as the various **Aut** functors used in the sequel, are representable by (an elementary case of) the theory of Hilbert schemes.

Now since $\mathbf{Hom}(\mathbb{X}(f^1)_n, \mathbb{X}(f^2)_n)$ is representable by a group scheme of finite type over k , it is determined by its points in Artin local k -algebras (with residue field k). We use the exact sequence (of sheaves in the fppf topology over $\mathrm{Spec}(k)$)

$$0 \rightarrow \mathbb{X}(f^1)_n \rightarrow \mathbb{X}(f^1) \xrightarrow{p^n} \mathbb{X}(f^1) \rightarrow 0$$

which for any k -algebra A gives rise to a long exact sequence

$$\begin{aligned} 0 \rightarrow \mathbf{Hom}(\mathbb{X}(f^1), \mathbb{X}(f^2))(A) &\xrightarrow{p^n} \mathbf{Hom}(\mathbb{X}(f^1), \mathbb{X}(f^2))(A) \\ &\rightarrow \mathbf{Hom}(\mathbb{X}(f^1)_n, \mathbb{X}(f^2))(A) \rightarrow \mathbf{Ext}^1(\mathbb{X}(f^1), \mathbb{X}(f^2))(A) \\ &\xrightarrow{p^n} \mathbf{Ext}^1(\mathbb{X}(f^1), \mathbb{X}(f^2))(A) \rightarrow \dots \end{aligned} \quad (5)$$

On the category of Artin local k -algebras we have from (the proof of) [19, Theorem 2.3.3] that $\mathbf{Ext}^1(\mathbb{X}(f^1), \mathbb{X}(f^2))$ is the trivial sheaf unless $f^1 < f^2$ in which case it is represented by $\mathbb{X}(f')$. Clearly $\mathbf{Hom}(\mathbb{X}(f^1)_n, \mathbb{X}(f^2))(A) = \mathbf{Hom}(\mathbb{X}(f^1)_n, \mathbb{X}(f^2)_n)(A)$ and by [9, Corollaire 4.3 a)] we see that $\mathbf{Hom}(\mathbb{X}(f^1), \mathbb{X}(f^2))(A) = 0$ whenever $f^1 \neq f^2$.

If $f := f^1 = f^2$ then $\mathbf{Hom}(\mathbb{X}(f), \mathbb{X}(f))(k) = W(k)$ by Lemma 3.2. We claim that in fact $\mathbf{Hom}(\mathbb{X}(f), \mathbb{X}(f))(A) = W(k)$ for any Artin local k -algebra A . To see this we note that there is a tautological map $W(k) \rightarrow \mathbf{Hom}(\mathbb{X}(f), \mathbb{X}(f))(A)$ for any such A , so it suffices to show that any element of $\mathbf{Hom}(\mathbb{X}(f), \mathbb{X}(f))(A)$ which reduces to 0 modulo the maximal ideal of A must be 0, but this again follows from [9, Corollaire 4.3 a)]. It then follows from (5) that $\mathbf{Hom}(\mathbb{X}(f)_n, \mathbb{X}(f)_n)$ is represented by the constant group scheme $W_n(\kappa)$ over k . Since this is $\mathbb{X}(\mathfrak{o})_n$, where \mathfrak{o} is the zero function on I , and f' is also the zero function if $f^1 = f^2$, it follows that all the statements up to (3) hold.

We now prove that (4) also holds. We can run through the above argument with \mathbb{X} replaced by $\mathbb{X}^{\mathrm{can}}$ and A running over Artin local $W(k)$ -algebras (all the results we have used from [19] and [9] still hold in this context) to conclude that the group scheme $\mathbf{Hom}(\mathbb{X}^{\mathrm{can}}(f^1)_n, \mathbb{X}^{\mathrm{can}}(f^2)_n)$ is a closed subscheme of $\mathbb{X}^{\mathrm{can}}(f')$ over $W(k)$. By (3) we know that this inclusion induces an isomorphism on special fibres. On the other hand, it is clear that the generic fibre of $\mathbf{Hom}(\mathbb{X}^{\mathrm{can}}(f^1)_n, \mathbb{X}^{\mathrm{can}}(f^2)_n)$ is étale of order equal to the order $|\kappa|$ (it is a form of κ viewed as a constant group scheme over \bar{K}) so in fact the inclusion must be an equality. We conclude that (4) holds. \square

LEMMA 3.2. *For any f and $n > 0$, the tautological map $W_n(\kappa) \rightarrow \mathbf{Hom}(\mathbb{X}(f)_n, \mathbb{X}(f)_n)$ is an isomorphism.*

Proof. This is an easy computation using Dieudonné modules: Let $M(f)_n$ be the Dieudonné module of $\mathbb{X}(f)_n$. Since the action of $W_n(\kappa)$ on each e_i is induced by different embeddings of κ in k , one easily sees by induction on n that the $W_n(k)$ -linear endomorphisms of $M(f)_n$ consists of diagonal matrices (using the basis $\{e_i\}_{i \in I}$ with entries in $W_n(k)$). Now since F on $M(f)_n$ is σ -linear and V is σ^{-1} linear, the fact that for each i , either $F(e_i) = e_{i+1}$ or

$V(e_{i+1}) = e_i$ easily implies that any such endomorphism must come from an element of $W_n(\kappa)$. \square

Remark 3.3. We recall here for later use the standard fact (see, e.g., [17, I, Lemma 1.5]) that for any (m -truncated) p -divisible group \mathbb{Y} over any base the multiplication by p map from \mathbb{Y}_n to itself induces a faithfully flat map $\mathbb{Y}_n \rightarrow \mathbb{Y}_{n-1}$ for all $n \leq m$.

3.2 AUTOMORPHISMS OF μ -ORDINARY (TRUNCATED) p -DIVISIBLE GROUPS

In this section we recall the classification of μ -ordinary truncated p -divisible groups with $W(\kappa)$ -structure from [19, §1] and describe their automorphism groups.

We keep the notation from §3.1. The classification depends on an integer $d > 0$ and a function $f : I \rightarrow \{0, 1, \dots, d\}$. Let $M(f)$ be the free $W_n(\kappa)$ -module with basis $\{e_{\tau,j}\}$ with $\tau \in I$ and $j \in \{1, 2, \dots, d\}$. Define F and V on $M(f)$ by

$$F(e_{\tau,j}) = \begin{cases} e_{\tau+1,j} & \text{if } j \leq d - f(\tau), \\ p \cdot e_{\tau+1} & \text{if } j > d - f(\tau), \end{cases} \quad V(e_{\tau+1,j}) = \begin{cases} p \cdot e_{\tau,j} & \text{if } j \leq d - f(\tau), \\ e_{\tau,j} & \text{if } j > d - f(\tau). \end{cases} \tag{6}$$

We define a $W_n(\kappa)$ -action on $M(f)$ by $b \cdot e_{\tau,j} = \tau(b)e_{\tau,j}$ where we denote the map $W_n(\kappa) \rightarrow W_n(k)$ induced by τ also by τ . The Dieudonné module $M(f)$ corresponds to a p -divisible group over k with an action of $W(\kappa)$ which we call $\mathbb{X}(f)$ and we denote the p^n -torsion of this p -divisible group by $\mathbb{X}(f)_n$. By [19, Corollary 2.1.5], $\mathbb{X}(f)$ has a canonical lift to $W(k)$ which we denote by $\mathbb{X}^{\text{can}}(f)$ and we denote its p^n -torsion by $\mathbb{X}^{\text{can}}(f)_n$.

If we denote by $M^j(f)$ the $W(k)$ -submodule of $M(f)$ spanned by the $\{e_{\tau,j}\}$ for $\tau \in I$ then it is clear from the definitions that it is a direct summand of $M(f)$ as a Dieudonné module and is a module of the form $M(f^j)$ as defined in §3.1. Here the function $f^j : I \rightarrow \{0, 1\}$ is given by $f^j(\tau) = 0$ if $j \leq d - f(\tau)$ and $f^j(\tau) = 1$ otherwise. Moreover, it is clear that $f^j \leq f^{j'}$ if $j \leq j'$. Thus $\mathbb{X}(f)_n$ is a direct product $\prod_{j \in \{1, \dots, d\}} \mathbb{X}(f^j)_n$ (with $\mathbb{X}(f^j)$ as defined in §3.1).

The product decomposition of $\mathbb{X}(f)_n$ and $\mathbb{X}^{\text{can}}(f)_n$ as described above is not intrinsic, but we can make it so by grouping together the factors which are isomorphic. Let $\sigma : \{1, \dots, d\} \rightarrow \{1, \dots, r\}$ be the unique surjective non-decreasing function such that the j, j' are in a fibre of σ iff $f^j = f^{j'}$. We then set $\mathbb{X}_i(f)$ for $i \in \{1, \dots, r\}$ to be $\prod_{j \in \sigma^{-1}(i)} \mathbb{X}(f^j)_n$ and similarly for $\mathbb{X}^{\text{can}}(f)_n$. So we have

$$\mathbb{X}(f)_n = \prod_{i \in \{1, \dots, r\}} \mathbb{X}_i(f)_n \quad , \quad \mathbb{X}^{\text{can}}(f)_n = \prod_{i \in \{1, \dots, r\}} \mathbb{X}_i^{\text{can}}(f)_n. \tag{7}$$

For finite n , we would like to have a precise description of the group scheme $\mathbf{Aut}(\mathbb{X}^{\text{can}}(f)_n)$ of the automorphisms of $\mathbb{X}^{\text{can}}(f)_n$ preserving the $W(\kappa)$ -action. However, this is not finite over $W(k)$ in general and we will restrict ourselves to describing two closed subgroup schemes: the connected component of the

identity section $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathbf{f})_n)^0$ which is finite over $W(k)$ and the special fibre $\mathbf{Aut}(\mathbb{X}(\mathbf{f})_n)$ which is finite over k .

Using the decompositions given in (7) we may view $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathbf{f})_n)$ as an open subscheme of the space of $r \times r$ matrices with the (i, i') entry being in $\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f}), \mathbb{X}_{i'}^{\text{can}}(\mathbf{f}))$, the group operation then being “matrix multiplication”. Similar statements hold for $\mathbf{Aut}(\mathbb{X}(\mathbf{f})_n)$.

LEMMA 3.4.

1. For all finite n , $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathbf{f})_n)^0$ is a finite flat group scheme over $W(k)$. Its matricial description is given as follows:
 - (a) if $i > i'$ then the (i, i') -entry is 0.
 - (b) the (i, i) -entry is the identity of $\mathbf{Aut}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n)$.
 - (c) if $i < i'$ then the (i, i') -entry is $\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n)$.
2. For all finite n , $\mathbf{Aut}(\mathbb{X}(\mathbf{f})_n)$ is a finite group scheme over k . Its matricial description is given as follows:
 - (a) if $i > i'$ then the (i, i') -entry is 0.
 - (b) the (i, i) -entry is $\mathbf{Aut}(\mathbb{X}_i(\mathbf{f})_n)$.
 - (c) if $i < i'$ then the (i, i') -entry is $\mathbf{Hom}(\mathbb{X}_i(\mathbf{f})_n, \mathbb{X}_{i'}(\mathbf{f})_n)$.

Proof. This follows immediately from the definition of the decomposition $\mathbb{X}^{\text{can}}(\mathbf{f})_n$ in (7) and Lemma 3.1: the schemes $\mathbf{Hom}(\mathbb{X}^{\text{can}}(f^j), \mathbb{X}^{\text{can}}(f^{j'}))$ are étale if $f^{j'} \geq f^j$ so do not contribute to the identity component of $\mathbf{Aut}'(\mathbb{X}^{\text{can}}(\mathbf{f})_n)$. Thus, the decreasing filtration of $\mathbb{X}^{\text{can}}(\mathbf{f})$ given by $\prod_{i \geq s} \mathbb{X}_i^{\text{can}}(\mathbf{f})$ is preserved by $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathbf{f})_n)^0$ and it must act as the identity on the associated graded. Furthermore, it follows from Lemma 3.1 that all the matrix entries are finite and flat over $W(k)$, so (1) is proved.

Part (2) is proved in essentially the same way, using that $\mathbf{Hom}(\mathbb{X}(f^j), \mathbb{X}(f^{j'})) = 0$ if $f^j > f^{j'}$. \square

Remark 3.5. The group schemes $\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f}), \mathbb{X}_{i'}^{\text{can}}(\mathbf{f}))$ and $\mathbf{Hom}(\mathbb{X}_i(\mathbf{f}), \mathbb{X}_{i'}(\mathbf{f}))$ occurring in Lemma 3.4 can be determined precisely by using Lemma 3.1. In particular, they are always n -truncated p -divisible groups. The structure of $\mathbf{Aut}(\mathbb{X}_i(\mathbf{f})_n)$ is also easy to determine: it is the constant group scheme over k of invertible $|\sigma^{-1}(i)| \times |\sigma^{-1}(i)|$ matrices with coefficients in $W_n(\kappa)$.

COROLLARY 3.6. For all $n > 0$ the maps

$$\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathbf{f})_{n+1})^0 \rightarrow \mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathbf{f})_n)^0, \quad \mathbf{Aut}(\mathbb{X}(\mathbf{f})_{n+1}) \rightarrow \mathbf{Aut}(\mathbb{X}(\mathbf{f})_n)$$

induced by restriction are surjective maps of group schemes (i.e., they are surjective maps of sheaves in the fppf topology) and are faithfully flat.

Proof. Surjectivity in the case of $\mathbb{X}^{\text{can}}(\mathfrak{f})$ follows from the explicit description of these group schemes given in Lemma 3.4 together with the surjectivity of the maps

$$\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathfrak{f})_{n+1}, \mathbb{X}_i^{\text{can}}(\mathfrak{f})_{n+1}) \rightarrow \mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathfrak{f})_n, \mathbb{X}_i^{\text{can}}(\mathfrak{f})_n)$$

which follows from Lemma 3.1. In the case of $\mathbb{X}(\mathfrak{f})$ we also need that the maps

$$\mathbf{Aut}(\mathbb{X}_i(\mathfrak{f})_{n+1}) \rightarrow \mathbf{Aut}(\mathbb{X}_i(\mathfrak{f})_n)$$

are surjective which is also an easy consequence of Lemma 3.1.

The faithful flatness in the case of $\mathbb{X}^{\text{can}}(\mathfrak{f})$ can be checked fibrewise since both the source and target are flat over $W(k)$; the statement for the generic fibre being obvious (since it is étale), this reduces us to the statement in the case of $\mathbb{X}(\mathfrak{f})$. Any surjective map of affine group schemes over a field is flat by [22, 14.2] so flatness follows. \square

3.3 AUTOMORPHISMS OF POLARISED μ -ORDINARY (TRUNCATED) p -DIVISIBLE GROUPS

In this section we restrict ourselves to the polarised μ -ordinary group schemes corresponding to unitary Shimura varieties, since these are the ones for which we will prove lower bounds for the essential dimension of congruence covers in §5. We begin by establishing some notation and recalling the definition and classification of such group schemes from [19, §3].

As before κ is a finite extension of \mathbb{F}_p and $\tilde{\kappa}$ is a quadratic extension of κ . The field $\tilde{\kappa}$ has a unique involution $x \mapsto x^*$ whose fixed field is κ . This involution lifts to $W(\tilde{\kappa})$ with the ring of $*$ -invariants being $W(\kappa)$. Let \mathbb{X}_n be an n -truncated p -divisible group over any base. We allow $n = \infty$ in which case \mathbb{X} is simply a p -divisible group. We denote by $c : \mathbb{X}_n \rightarrow \mathbb{X}_n^{DD}$ the canonical double duality isomorphism. By a *polarisation* or *duality* of \mathbb{X}_n we shall mean an isomorphism (of group schemes or p -divisible groups) $\lambda : \mathbb{X}_n \rightarrow \mathbb{X}_n^D$ such that $\lambda = \lambda^D \circ c$. Such a duality induces an involution $f \mapsto f^\dagger$ on $\text{End}(\mathbb{X}_n)$. Now suppose we have a $W(\tilde{\kappa})$ structure on \mathbb{X}_n , i.e., a ring homomorphism $\iota : W(\tilde{\kappa}) \rightarrow \text{End}(\mathbb{X}_n)$. We impose the compatibility condition on λ and ι that $\iota(b^*) = \iota(b)^\dagger$ for all $b \in W(\tilde{\kappa})$ and call such a triple $(\mathbb{X}_n, \iota, \lambda)$ a *polarized (n -truncated) p -divisible group (or BT_n) with $W(\tilde{\kappa})$ -structure.*² Note that \mathbb{X}_n^D acquires a $W(\tilde{\kappa})$ -structure by duality and the compatibility condition means that λ is antilinear as a map of group schemes with $W(\tilde{\kappa})$ -structure.

For our purposes we shall need an explicit classification of those objects over k which Moonen sometimes calls ordinary in [19], but we shall always call them μ -ordinary in order to avoid any confusion. We recall this in terms of their Dieudonné modules from ([19, §3.2.3], Case AU).

²These are the ones of type AU in the terminology of [19] which are the only ones we shall consider here.

Let \tilde{I} be the set of homomorphisms from $\tilde{\kappa} \rightarrow k$. The classification depends on two parameters, a positive integer d and a function $\mathfrak{f} : \tilde{I} \rightarrow \mathbb{Z}_{\geq 0}$ such that $\mathfrak{f}(\tau) + \mathfrak{f}(\bar{\tau}) = d$ for all $\tau \in \tilde{I}$, where for $\tau \in \tilde{I}$, $\bar{\tau} := \tau \circ *$. As in §3.1 for I , the set \tilde{I} has a cyclic ordering and we denote the successor of τ for this ordering by $\tau + 1$. Let $M(\mathfrak{f})$ be the Dieudonné module defined exactly as in §3.2 but now with $\tau \in \tilde{I}$ instead of I . We continue to use the same notation $\mathbb{X}(\mathfrak{f})$, $X(\mathfrak{f})_n$, $\mathbb{X}^{\text{can}}(\mathfrak{f})$, \dots , as in §3.2 for the associated p -divisible group, etc., which are objects with $W(\tilde{\kappa})$ -action.

The polarisation λ on the n -truncated p -divisible group $\mathbb{X}(\mathfrak{f})_n$ corresponding to $M(\mathfrak{f})_n$ can be chosen to be the map deduced from the pairing $\varphi : M(\mathfrak{f}) \times M(\mathfrak{f}) \rightarrow W(k)$ given on basis elements by $\varphi(e_{\tau,j}, e_{\tau',j'}) = 0$ unless $\tau' = \bar{\tau}$ and $j' = d - j + 1$ and $\varphi(e_{\tau,j}, e_{\bar{\tau},d-j+1}) = 1$.³ The tuple $(\mathbb{X}(\mathfrak{f})_n, \iota, \lambda)$ is then an n -truncated polarised p -divisible group with $W(\tilde{\kappa})$ -structure as defined above.

The polarisation λ lifts canonically to a polarisation on $\mathbb{X}^{\text{can}}(\mathfrak{f})_n$ which we still denote by λ , and then (together with its natural $W(\tilde{\kappa})$ -action) $\mathbb{X}^{\text{can}}(\mathfrak{f})_n$ also acquires the structure of a polarised truncated p -divisible group with $W(\tilde{\kappa})$ -structure.

We have a product decomposition $\mathbb{X}(\mathfrak{f})_n \cong \prod_{j \in \{1, \dots, d\}} \mathbb{X}(f^j)_n$ as in §3.2 and the definition of λ shows that it is a direct sum of isomorphisms $\mathbb{X}(f^j) \rightarrow \mathbb{X}(f^{d-j+1})^D$ for all j . A similar statement holds for $\mathbb{X}^{\text{can}}(\mathfrak{f})$.

We also have product decompositions as in (7) and then one sees from the definitions that λ induces isomorphisms $\mathbb{X}_i(\mathfrak{f})_n \rightarrow \mathbb{X}_{r-i+1}(\mathfrak{f})_n$ and the same with $\mathbb{X}^{\text{can}}(\mathfrak{f})$.

We now define $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathfrak{f})_n)$ to be the group scheme of all structure preserving automorphisms of $\mathbb{X}^{\text{can}}(\mathfrak{f})_n$, i.e., automorphisms preserving the $W(\tilde{\kappa})$ -action as well as the polarisation. As in §3.2 we will restrict ourselves to describing two closed subgroup schemes: the connected component of the identity section $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathfrak{f})_n)^0$ which is finite over $W(k)$ and the special fibre $\mathbf{Aut}(\mathbb{X}(\mathfrak{f})_n)$ which is finite over k .

We let $\mathbf{Aut}'(\mathbb{X}^{\text{can}}(\mathfrak{f})_n)^0$ (resp. $\mathbf{Aut}'(\mathbb{X}(\mathfrak{f})_n)$) be the group scheme of automorphisms preserving only the $W(\tilde{\kappa})$ -structure, i.e., we ignore the polarisation, so it is the group scheme denoted by $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathfrak{f})_n)^0$ (resp. $\mathbf{Aut}(\mathbb{X}(\mathfrak{f})_n)$) in §3.2. We will describe $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathfrak{f})_n)^0$ as a closed subscheme of $\mathbf{Aut}'(\mathbb{X}^{\text{can}}(\mathfrak{f})_n)^0$ and $\mathbf{Aut}(\mathbb{X}(\mathfrak{f})_n)$ as a closed subscheme of $\mathbf{Aut}'(\mathbb{X}(\mathfrak{f})_n)$ using the matricial descriptions from Lemma 3.4. The subgroup scheme we wish to determine consists of elements α of $\mathbf{Aut}'(\mathbb{X}^{\text{can}}(\mathfrak{f})_n)^0$ such that $\lambda \circ \alpha = (\alpha^D)^{-1} \circ \lambda$ and similarly for $\mathbf{Aut}'(\mathbb{X}(\mathfrak{f})_n)$. In terms of the involution \dagger on $\mathbf{End}(\mathbb{X}^{\text{can}}(\mathfrak{f}))$, we wish to determine all automorphisms α such that $\alpha^\dagger = \alpha^{-1}$.

As noted earlier, λ restricts to an isomorphism $\lambda_j : \mathbb{X}^{\text{can}}(f^j)_n \rightarrow \mathbb{X}^{\text{can}}(f^{d-j+1})_n^D$ and so also an isomorphism $\lambda_i : \mathbb{X}_i^{\text{can}}(\mathfrak{f})_n \rightarrow \mathbb{X}_{r-i+1}^{\text{can}}(\mathfrak{f})_n^D$. (Similarly, we have an isomorphism $\lambda_i : \mathbb{X}_i(\mathfrak{f})_n \rightarrow \mathbb{X}_{r-i+1}(\mathfrak{f})_n^D$.) This implies that

³The description of the pairing in [19, p. 251] has some typos which we have corrected.

the involution \dagger induces isomorphisms

$$\dagger_{i,i'} : \mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n) \rightarrow \mathbf{Hom}(\mathbb{X}_{r-i'+1}^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{r-i+1}^{\text{can}}(\mathbf{f})_n)$$

such that $\dagger_{r-i'+1, r-i+1} \circ \dagger_{i,i'}$ is the identity of $\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n)$ for all $i \leq i'$. (Similar statements hold for $\mathbb{X}(\mathbf{f})$ instead of $\mathbb{X}^{\text{can}}(\mathbf{f})$.)

If $i + i' = r + 1$ then $\dagger_{i,i'}$ is an involution of $\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n)$ and in this case we let $\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n)^+$ be the invariants of $\dagger_{i,i'}$ and $\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n)^-$ be the anti-invariants, i.e., subgroup scheme on which $\dagger_{i,i'}$ acts as -1 . (We use similar notation with $\mathbb{X}(\mathbf{f})$ instead of $\mathbb{X}^{\text{can}}(\mathbf{f})$.)

LEMMA 3.7. *If $i \leq i'$ and $i + i' = r + 1$, consider the endomorphism of $\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n)$ given by $\text{id} + \dagger_{i,i'}$. Then the kernel (resp. image) of $\text{id} + \dagger_{i,i'}$ is finite flat over $W(k)$ and equal to $\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n)^-$ (resp. $\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n)^+$) giving rise to a split exact sequence of n -truncated p -divisible groups over $W(k)$ (with $W(\kappa)$ -action):*

$$0 \rightarrow \mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n)^- \rightarrow \mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n) \rightarrow \mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n)^+ \rightarrow 0.$$

The analogous statement with $\mathbb{X}(\mathbf{f})$ instead of $\mathbb{X}^{\text{can}}(\mathbf{f})$ also holds.

Proof. Given the definition of $\mathbb{X}_i^{\text{can}}(\mathbf{f})_n$, it follows from Lemma 3.1 that $\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathbf{f})_n)$ is a finite product of group schemes $\mathbb{X}^{\text{can}}(f')_n$ where $f' : \tilde{I} \rightarrow \{0, 1\}$ is a function such that $f'(\tau) = f'(\bar{\tau})$ for all τ . Furthermore, the involution $\dagger_{i,i'}$ either permutes pairs of factors or preserves one factor on which it acts by an involution. For a pair of factors which is permuted it is clear that the invariants and anti-invariants are isomorphic to $\mathbb{X}^{\text{can}}(f')_n$ and we have a direct sum decomposition.

For a factor which is preserved, by the definitions in (1) we see that at the level of $\mathbb{X}(f')_n$ this involution is given on the Dieudonné module $M(f')_n$ by $e_\tau \mapsto e_{\bar{\tau}}$ for all τ . The invariants correspond to the Dieudonné module spanned by all $e_\tau + e_{\bar{\tau}}$ and the anti-invariants to the one spanned by all $e_\tau - e_{\bar{\tau}}$, both of which correspond to a truncated p -divisible group over k with a $W(\kappa)$ -action (given by the function $f'' : I \rightarrow \{0, 1\}$ defined by extending an element of I to \tilde{I} and then applying f'). The geometric generic fibre of $\mathbb{X}^{\text{can}}(f')_n$ is étale, isomorphic to the constant group $W_n(\bar{\kappa})$ (with the involution being the map $*$). This implies that the ranks of the generic and special fibres are the same (in both cases), and so the invariants and anti-invariants of the action on such a factor are (finite) flat over $W(k)$ and the structure of the special fibre shows that it is an n -truncated p -divisible group.

The first part of the lemma follows immediately from these observations. The statement about $\mathbb{X}(\mathbf{f})$ follows from this by restriction to the special fibre. \square

PROPOSITION 3.8. *The group scheme $\mathcal{G} = \mathcal{G}(\mathbf{f})_n := \mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathbf{f})_n)^0$ has a decreasing filtration by finite flat closed normal subgroup schemes $\mathcal{G} = \mathcal{G}(1) \supset$*

$\mathcal{G}(2) \supset \cdots \supset \mathcal{G}(s) \supset \cdots \supset \mathcal{G}(r) = \{1\}$ such that for $1 \leq s < r$ we have

$$\mathcal{G}(s)/\mathcal{G}(s+1) \cong \left(\prod_{t=1}^{\lfloor \frac{r-s}{2} \rfloor} \mathbf{Hom}(\mathbb{X}_t^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{t+s}^{\text{can}}(\mathbf{f})_n) \right) \times \mathcal{P}$$

where

$$\mathcal{P} = \begin{cases} \{1\} & \text{if } r-s \text{ is even,} \\ \mathbf{Hom}(\mathbb{X}_{\lfloor \frac{r-s}{2} \rfloor}^{\text{can}}(\mathbf{f})_n, \mathbb{X}_{\lfloor \frac{r-s}{2} \rfloor + s}^{\text{can}}(\mathbf{f})_n)^- & \text{if } r-s \text{ is odd.} \end{cases}$$

The group scheme $G = G(\mathbf{f})_n := \mathbf{Aut}(\mathbb{X}(\mathbf{f})_n)$ has a decreasing filtration by finite closed normal subgroup schemes $G = G(0) \supset G(1) \supset G(2) \supset \cdots \supset G(s) \supset \cdots \supset G(r) = \{1\}$ such that for $1 \leq s < r$ we have

$$G(s)/G(s+1) \cong \left(\prod_{t=0}^{-1 + \lfloor \frac{r-s}{2} \rfloor} \mathbf{Hom}(\mathbb{X}_{t+1}(\mathbf{f})_n, \mathbb{X}_{t+s+1}(\mathbf{f})_n) \right) \times P$$

where

$$P = \begin{cases} \{1\} & \text{if } r-s \text{ is even,} \\ \mathbf{Hom}(\mathbb{X}_{\lfloor \frac{r-s}{2} \rfloor}(\mathbf{f})_n, \mathbb{X}_{\lfloor \frac{r-s}{2} \rfloor + s}(\mathbf{f})_n)^- & \text{if } r-s \text{ is odd.} \end{cases}$$

Furthermore,

$$G(0)/G(1) \cong \left(\prod_{t=1}^{\lfloor \frac{r}{2} \rfloor} \mathbf{Aut}(\mathbb{X}_t(\mathbf{f})_n) \right) \times Q$$

where

$$Q = \begin{cases} \{1\} & \text{if } r \text{ is even,} \\ \mathbf{UAut}(\mathbb{X}_{\lfloor \frac{r}{2} \rfloor}(\mathbf{f})_n) & \text{if } r \text{ is odd.} \end{cases}$$

Here \mathbf{UAut} denotes the group of unitary automorphisms, i.e., the subgroup of \mathbf{Aut} consisting of elements x such that $x^{-1} = x^\dagger$.

Proof. We use the matricial representation from Lemma 3.4 and write elements of $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathbf{f})_n)^0$ as matrices $\Phi = (\phi_{i,i'})$ and the condition we need to be satisfied is $\Phi \cdot \Phi^\dagger = \text{id}$. Multiplying out the matrices gives relations:

$$\sum_{k=1}^r \phi_{i,k} \phi_{r-i'+1, r-k+1}^\dagger = 0 \quad (8)$$

if $i < i'$ (where for ease of notation we have dropped the subscript from the \dagger). Using that $\phi_{i,i'} = 0$ if $i > i'$ and $\phi_{i,i} = \text{id}$, we can solve these equations

inductively in $i' - i$. Doing this we see that the $\phi_{i,i'}$ can be chosen to be arbitrary elements of $\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathfrak{f})_n, \mathbb{X}_{i'}^{\text{can}}(\mathfrak{f})_n)$ if $i < r/2$ and $i' < r + 1 - i$ and then $\phi_{r+i'-1, r+i-1}$ is uniquely determined (since the set of equations is preserved by \dagger). When $i + i' = r + 1$ the equation (8) becomes

$$\phi_{i,i'} + \phi_{i,i'}^\dagger + \sum_{k=i+1}^{r-i} \phi_{i,k} \phi_{i,r-k+1}^\dagger = 0.$$

The sum is invariant under \dagger and since the map $\text{id} + \dagger_{i,i'}$ is faithfully flat onto the $\dagger_{i,i'}$ -invariants by Lemma 3.7, we can solve for $\phi_{i,i'}$, the space of solutions being a torsor over the $\dagger_{i,i'}$ -anti-invariants.

We define the filtration $\mathcal{G}(s)$ to be the subgroup scheme with $\phi_{i,i'} = 0$ if $0 < i - i' < s$ and then the above description of all elements of $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathfrak{f})_n)^0$ makes the claims about the filtration in this case clear.

For the group G , we take $G(1)$ to be the identity component. This is precisely the special fibre of the group \mathcal{G} and so the claim about G follows immediately from the already proved claim for \mathcal{G} and the structure of the étale quotient, which follows easily from Lemma 3.1. □

COROLLARY 3.9. *For all $n > 0$ the maps*

$$\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathfrak{f})_{n+1})^0 \rightarrow \mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathfrak{f})_n)^0, \quad \mathbf{Aut}(\mathbb{X}(\mathfrak{f})_{n+1}) \rightarrow \mathbf{Aut}(\mathbb{X}(\mathfrak{f})_n)$$

induced by restriction are surjective maps of group schemes (i.e., they are surjective maps of sheaves in the fppf topology) and are faithfully flat.

Proof. Surjectivity in the case of $\mathbb{X}^{\text{can}}(\mathfrak{f})$ follows by induction using the filtration of these group schemes given in Proposition 3.8 together with the surjectivity of the maps

$$\mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathfrak{f})_{n+1}, \mathbb{X}_i^{\text{can}}(\mathfrak{f})_{n+1}) \rightarrow \mathbf{Hom}(\mathbb{X}_i^{\text{can}}(\mathfrak{f})_n, \mathbb{X}_i^{\text{can}}(\mathfrak{f})_n)$$

which follows from Lemma 3.1. In the case of $\mathbb{X}(\mathfrak{f})$ we also need that the maps

$$\mathbf{Aut}(\mathbb{X}_i(\mathfrak{f})_{n+1}) \rightarrow \mathbf{Aut}(\mathbb{X}_i(\mathfrak{f})_n)$$

are surjective which is also an easy consequence of Lemma 3.1.

The faithful flatness can be proved in the same way as in Corollary 3.6. □

3.4 THE MAIN EXAMPLE

We now analyze in detail the structure of the automorphism group for a particular class of examples that will be essential for our incompressibility result for unitary Shimura varieties (Theorem 5.1). We assume that $\kappa = \mathbb{F}_p$ (so $\tilde{\kappa} = \mathbb{F}_{p^2}$) and $d = 2\delta + 1$ is odd. \tilde{I} consists of two elements which we call τ and $\bar{\tau}$. Define the function $\mathfrak{f} : \tilde{I} \rightarrow \{0, 1, \dots, d\}$ by $\mathfrak{f}(\tau) = \delta$ and $\mathfrak{f}(\bar{\tau}) = \delta + 1$.

One easily checks that the functions $f^j : \tilde{I} \rightarrow \{0, 1\}$, $j \in \{1, 2, \dots, d\}$, associated to \mathfrak{f} in §3.2 are given as follows:

- f^j is identically 0 if $j \leq \delta$,
- $f^{\delta+1}(\tau) = 0$ and $f^{\delta+1}(\bar{\tau}) = 1$,
- f^j is identically 1 if $j > \delta + 1$.

Thus $\mathbb{X}^{\text{can}}(f^j)_1 \cong \tilde{\kappa}$ (viewed as a constant group scheme with the tautological $\tilde{\kappa}$ -action) for $j \leq \delta$ and $\mathbb{X}^{\text{can}}(f^j)_1 \cong (\tilde{\kappa})^D$ for $j > \delta + 1$. To simplify notation in what follows, we will denote the group scheme $\mathbb{X}^{\text{can}}(f^{\delta+1})_1$ by \mathcal{E} . It follows from the description of its Dieudonné module that \mathcal{E}_0 is an extension of the group scheme α_p by itself, and there is a unique subgroup scheme \mathcal{F}_0 of \mathcal{E}_0 of order p which is the kernel of the Frobenius map. In particular, $t(\mathcal{E}) = 1$. (It is well-known that \mathcal{E}_0 is isomorphic to the p -torsion subscheme of any supersingular elliptic curve over k .)

It follows from the description of the f^j that the integer r associated to \mathfrak{f} defined in §3.2 is 3. By Proposition 3.8, the group scheme $\mathcal{G}(\mathfrak{f})_1 = \mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathfrak{f})_1)^0$ has a two step filtration, with a normal subgroup scheme $\mathcal{H}(\mathfrak{f})_1$ equal to $\mathbf{Hom}((\tilde{\kappa})^\delta, ((\tilde{\kappa})^D)^\delta)^- \subset \mathbf{Hom}((\tilde{\kappa})^\delta, ((\tilde{\kappa})^D)^\delta) \cong M_{\delta \times \delta}(\tilde{\kappa}^D)$. The involution \dagger induces an involution on $M_{\delta \times \delta}(\tilde{\kappa}^D)$ which we also denote by \dagger and is given by applying the non-trivial automorphism of $\tilde{\kappa}$ and taking the anti-transpose, i.e., the (i, j) -entry and the $(\delta - j + 1, \delta - i + 1)$ -entry are permuted. One sees from this that $\mathcal{H}(\mathfrak{f})_1 \cong (\mu_p)^{\delta^2}$. The quotient $\mathcal{G}(\mathfrak{f})_1/\mathcal{H}(\mathfrak{f})_1$ is equal to $\mathbf{Hom}((\tilde{\kappa})^\delta, \mathcal{E}) \cong \mathcal{E}^\delta$.

LEMMA 3.10. *Let $\mathcal{G} = (\mathcal{G}(\mathfrak{f})_1)^N \times \mathcal{H}'$, where \mathcal{H}' is a multiplicative group scheme. Let $\mathcal{H} = (\mathcal{H}(\mathfrak{f})_1)^N \times \mathcal{H}'$, so \mathcal{H} is a normal multiplicative subgroup scheme of \mathcal{G} . Then any subgroup scheme \mathcal{K}_0 of \mathcal{G}_0 which intersects \mathcal{H}_0 trivially projects to a subgroup scheme of $\mathcal{G}_0/\mathcal{H}_0$ which lies in the Frobenius kernel. Furthermore, $t(\mathcal{G}(\mathfrak{f})_1) = \delta^2 + \delta$ and $t(\mathcal{G}_0/\mathcal{K}_0) = t(\mathcal{G}_0) = N(\delta^2 + \delta) + t(\mathcal{H}')$.*

Proof. The first statement easily reduces to the case that $\mathcal{G} = \mathcal{G}(\mathfrak{f})_1$ since Frobenius kernels are compatible with products. We now describe the group scheme $\mathcal{G}(\mathfrak{f})_1$ more explicitly. By the results of §3.2 and §3.3 it can be represented as a closed subgroup scheme of the group scheme given in matricial form as

$$\begin{bmatrix} 1 & \mathcal{E}^\delta & M_{\delta \times \delta}(\tilde{\kappa}^D) \\ 0 & 1 & (\mathcal{E}^D)^\delta \\ 0 & 0 & 1 \end{bmatrix}$$

with multiplication being matrix multiplication, using the tautological pairing $\mathcal{E} \times \mathcal{E}^D \rightarrow \tilde{\kappa}^D$. We drop indices and write the isomorphism $\mathcal{E}^\delta \rightarrow (\mathcal{E}^D)^\delta$ induced by the involution \dagger simply as λ . This is a sum of component-wise isomorphisms, whose reduction modulo p can be described explicitly in terms of Dieudonné modules.

Using (8) one sees that the subgroup $\mathcal{G}(\mathfrak{f})_1$ is then given (in terms of points valued in any $W(k)$ -algebra) by matrices of the form

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

such that $z = -\lambda(x)$ and $y + y^\dagger = x \cdot \lambda(x)$. We therefore get a scheme theoretic section σ of the quotient map $\mathcal{G}(\mathfrak{f})_1 \rightarrow \mathcal{G}(\mathfrak{f})_1/\mathcal{H}(\mathfrak{f})_1 \cong \mathcal{E}^\delta$ by

$$x \in \mathcal{E}^\delta \mapsto \begin{bmatrix} 1 & x & x \cdot \lambda(x)/2 \\ 0 & 1 & -\lambda(x) \\ 0 & 0 & 1 \end{bmatrix}$$

since $(x \cdot \lambda(x))^\dagger = x \cdot \lambda(x)$ and division by 2 makes sense because $p \neq 2$. This implies that the map $T(\mathcal{G}(\mathfrak{f})_1) \rightarrow T(\mathcal{G}(\mathfrak{f})_1/\mathcal{H}(\mathfrak{f})_1)$ is surjective, so $T(\mathcal{G}(\mathfrak{f})_1) = T(\mathcal{H}(\mathfrak{f})_1) + T(\mathcal{E}^\delta) = \delta^2 + \delta$. We also note that for any point x (resp. z) in the Frobenius kernel of \mathcal{E}_0 (resp. of \mathcal{E}_0^D), we have $x \cdot z = 1$, the unit element in $\tilde{\kappa}^D$, so σ restricted to \mathcal{F}_0^δ , the Frobenius kernel of \mathcal{E}_0^δ , is a morphism of group schemes.

Now suppose that \mathcal{K}_0 is a subgroup scheme of \mathcal{G}_0 which intersects \mathcal{H}_0 trivially and consider its image \mathcal{K}'_0 in $\mathcal{G}_0/\mathcal{H}_0$ (which we identify with \mathcal{E}_0^δ using the $(1, 2)$ -component in the matricial description). The inverse map $s : \mathcal{K}'_0 \rightarrow \mathcal{K}_0$ can be written as

$$x \in \mathcal{K}'_0 \mapsto \begin{bmatrix} 1 & x & x \cdot \lambda(x)/2 + \nu(x) \\ 0 & 1 & -\lambda(x) \\ 0 & 0 & 1 \end{bmatrix}$$

where $\nu : \mathcal{K}'_0 \rightarrow \mathcal{H}_0$ is a map of k -schemes. Letting x, x' be any (valued) points of \mathcal{K}'_0 , by matrix multiplication we see that $s(x) \cdot s(x')$ is equal to

$$\begin{bmatrix} 1 & x + x' & (x' \cdot \lambda(x')/2 + \nu(x')) - x \cdot \lambda(x') + (x \cdot \lambda(x)/2 + \nu(x)) \\ 0 & 1 & -\lambda(x + x') \\ 0 & 0 & 1 \end{bmatrix}$$

Since \mathcal{E}_0^δ , hence also \mathcal{K}'_0 , is commutative it follows that we must have $x \cdot \lambda(x') = x' \cdot \lambda(x)$ for all x, x' as above.

If \mathcal{K}'_0 is not contained in $\mathcal{F}_0^\delta \subset \mathcal{E}_0^\delta$, the structure of \mathcal{E}_0 shows that it must surject onto one of the factors. Since the pairing $\mathcal{E}^\delta \times (\mathcal{E}^D)^\delta \rightarrow M_{\delta \times \delta}(\tilde{\kappa}^D)$ is defined component-wise, the existence of \mathcal{K}_0 as above and the commutativity of \mathcal{E}_0^δ would imply that for all $x, x' \in \mathcal{E}_0(\text{Spec}(A))$, where A is any k -algebra, we have $x \cdot \lambda(x') = x' \cdot \lambda(x)$. Here $\lambda_{\mathcal{E}_0} : \mathcal{E}_0 \rightarrow \mathcal{E}_0^D$ is the isomorphism induced by the polarisation restricted to \mathcal{E}_0 . However, since λ is $\tilde{\kappa}$ -antilinear, the (perfect) pairing $(x, x') \mapsto x \cdot \lambda(x')$ is hermitian (with respect to the $\tilde{\kappa}$ -structure) and not symmetric, so we must have $\mathcal{K}'_0 \subset \mathcal{F}_0^\delta$ as claimed. We also note that any subgroup scheme of \mathcal{F}_0^δ has a unique lift to a subgroup of \mathcal{G}_0 (given by the restriction of σ) since \mathcal{F}_0^δ is a unipotent group scheme and \mathcal{H}_0 is multiplicative. We now prove the last statement about dimensions (so now \mathcal{G} is once again as in the statement of the lemma). By what we have seen above, \mathcal{K}_0 projects (isomorphically) into the Frobenius kernel of $\mathcal{Q}_0 := \mathcal{G}_0/\mathcal{H}_0$ and is contained in the image of the section $\sigma : \mathcal{Q}_0 \rightarrow \mathcal{G}_0$ (as defined above for $\mathcal{G}(\mathfrak{f})_1$ and extended component-wise). Now consider the quotient map $q : \mathcal{G}_0 \rightarrow \mathcal{G}_0/\mathcal{K}_0$. We have a

commutative diagram

$$\begin{array}{ccccc}
 \sigma(\mathcal{Q}_0) & \longrightarrow & \mathcal{X}_0 & & \\
 \downarrow & & \downarrow & \searrow & \\
 \mathcal{G}_0 & \longrightarrow & \mathcal{G}_0/\mathcal{K}_0 & \longrightarrow & \mathcal{Q}_0/q(\mathcal{K}_0)
 \end{array}$$

Here \mathcal{X}_0 is defined to be the scheme-theoretic image of $\sigma(\mathcal{Q}_0)$ in $\mathcal{G}_0/\mathcal{K}_0$ and all the other maps are induced by the inclusions or quotient maps. Now \mathcal{K}_0 is contained in $\sigma(\mathcal{Q}_0)$, and since the scheme theoretic image of \mathcal{K}_0 in $\mathcal{G}_0/\mathcal{K}_0$ is trivial, so is its image in \mathcal{X}_0 . We may thus apply Lemma 2.8 (with G there being \mathcal{Q}_0 , H there being \mathcal{K}_0 , and X there being \mathcal{X}_0) to conclude that the map on tangent spaces $T(\mathcal{X}_0) \rightarrow T(\mathcal{Q}_0/q(\mathcal{K}_0))$ is surjective. A fortiori, the map $T(\mathcal{G}_0/\mathcal{K}_0) \rightarrow T(\mathcal{Q}_0/q(\mathcal{K}_0))$ is also surjective.

Since $\mathcal{K}_0 \cap \mathcal{H}_0$ is trivial, it follows that we have an injection $T(\mathcal{H}_0) \rightarrow T(\mathcal{G}_0/\mathcal{K}_0)$. As $T(\mathcal{H}_0)$ maps to 0 in $T(\mathcal{Q}_0/q(\mathcal{K}_0))$, we conclude that $t(\mathcal{G}_0/\mathcal{K}_0) \geq t(\mathcal{H}_0) + t(\mathcal{Q}_0/q(\mathcal{K}_0))$. Since $\mathcal{G}_0/\mathcal{K}_0$ is a principal \mathcal{H}_0 bundle over $\mathcal{Q}_0/q(\mathcal{K}_0)$ and $\mathcal{H}_0 \cong (\mu_p)^b$ for some b , $\mathcal{G}_0/\mathcal{K}_0$ embeds in a principal \mathbb{G}_m^b -bundle over $\mathcal{Q}_0/q(\mathcal{K}_0)$ (which must be trivial), so it follows that the inequality is actually an equality. Now $t(\mathcal{G}_0) = t(\mathcal{H}_0) + t(\mathcal{Q}_0)$ (by the existence of the section σ) so it suffices to show that $t(\mathcal{Q}_0) = t(\mathcal{Q}_0/q(\mathcal{K}_0))$. This follows from Lemma 3.11 since $\mathcal{Q}_0 \cong \mathcal{E}_0^m$ for some m and $q(\mathcal{K}_0)$ is contained in its Frobenius kernel. \square

LEMMA 3.11. *Let m be any positive integer and \mathcal{E}'_0 a closed subscheme of \mathcal{E}_0^m , contained in \mathcal{F}_0^m , its Frobenius kernel. Then $t(\mathcal{E}_0^m/\mathcal{E}'_0) = m$.*

Proof. Note that the Frobenius map F of \mathcal{E}_0^m has image \mathcal{F}_0^m , so $\mathcal{E}''_0 := F^{-1}(\mathcal{E}'_0)$ is a closed subscheme of \mathcal{E}_0^m whose image in $\mathcal{E}_0^m/\mathcal{E}'_0$ is precisely its Frobenius kernel. Since the image has order p^m (as the kernel is $\mathcal{E}'_0 \subset \mathcal{E}''_0$), the lemma follows. \square

4 SHIMURA VARIETIES

4.1 PEL SHIMURA VARIETIES

Let (\mathbf{G}, X) be a Shimura datum which we assume to be of PEL type in the sense of Kottwitz [13, §5, §7]. We assume that $p \neq 2$, the group $\mathbf{G}_{\mathbb{Q}_p}$ is unramified and we choose a compact open subgroup $C = C_p \times C^p \subset \mathbf{G}(\mathbb{Q}_p) \times \mathbf{G}(\mathbb{A}_f^p) = \mathbf{G}(\mathbb{A}_f)$ such that C_p is hyperspecial. We assume that C^p is small enough and R is large enough so that by *op. cit.* the Shimura variety corresponding to this data has a smooth integral model \mathcal{S}_C over \mathcal{T} (corresponding to some embedding of the reflex field E of the Shimura datum into the field K). Henceforth, the compact subgroup C will be fixed and since none of our statements will depend on the choice of C (assuming C_p is hyperspecial) we drop it from the notation, denoting \mathcal{S}_C simply by \mathcal{S} .

The scheme \mathcal{S} carries a universal family of abelian varieties $\mathcal{A} \rightarrow \mathcal{S}$ with (certain extra structures)⁴. Its special fibre \mathcal{S}_0 has a dense open subscheme $\mathcal{S}_0^{\text{ord}}$ called the μ -ordinary locus [23]; its points can be characterised in terms of the Newton polygon of the associated p -divisible group or even just the structure of its p -torsion (with the extra structure) [19]. If k is algebraically closed, for all k -points in the μ -ordinary locus the associated p -divisible group with extra structure is isomorphic and we denote it by \mathbb{X} . For any $n > 0$, we denote the p^n -torsion of \mathbb{X} by \mathbb{X}_n .

Let Ig^{ord} be the Igusa “variety” over k as in [4, Definition 4.3.1]⁵ with the b there corresponding to the μ -ordinary locus (we drop the μ for notational convenience). It is the inverse limit of the schemes Ig_n^{ord} (so Ig^{ord} is not really a variety) over $\mathcal{S}_0^{\text{ord}}$ parametrizing isomorphisms of $\mathcal{A}[p^n]|_{\mathcal{S}_0^{\text{ord}}}$ with $\mathbb{X}_n \times_k \mathcal{S}_0$ compatible with all extra structures. The maps $\text{Ig}_n^{\text{ord}} \rightarrow \mathcal{S}_0$ are all affine morphisms hence so is the map $\text{Ig}^{\text{ord}} \rightarrow \mathcal{S}_0^{\text{ord}}$.

Let $\mathbf{Aut}(\mathbb{X}_n)$ denote the group scheme of automorphisms of \mathbb{X}_n (preserving the extra structure). This is always an affine group scheme and if $n = 1$ it follows from [18, Theorem 2.1.2 (ii)] that it is a finite group scheme.

We would like to know whether Ig_n^{ord} is smooth for all n . The following lemma gives a simple criterion for this which we shall check holds in certain cases of interest to us.

LEMMA 4.1. *If all the maps $\mathbf{Aut}(\mathbb{X}_{n+1}) \rightarrow \mathbf{Aut}(\mathbb{X}_n)$ induced by restricting an automorphism of \mathbb{X}_{n+1} to \mathbb{X}_n are faithfully flat, then each Ig_n^{ord} is smooth.*

Proof. By [4, Corollary 4.3.9], the map $\text{Ig}^{\text{ord}} \rightarrow \mathcal{S}_0^{\text{ord}}$ is faithfully flat and since it is an affine morphism, it is an fpqc covering. This implies that for all $n > 0$, Ig_n^{ord} is an $\mathbf{Aut}(\mathbb{X}_n)$ -torsor over $\mathcal{S}_0^{\text{ord}}$. If all the maps $\mathbf{Aut}(\mathbb{X}_{n+1}) \rightarrow \mathbf{Aut}(\mathbb{X}_n)$ induced by restriction are faithfully flat then so are all the maps $\text{Ig}_{n+1}^{\text{ord}} \rightarrow \text{Ig}_n^{\text{ord}}$. Since tensor products commute with direct limits, this implies that all the maps $\text{Ig}^{\text{ord}} \rightarrow \text{Ig}_n^{\text{ord}}$ are flat and since each map $\text{Ig}_{r+1}^{\text{ord}} \rightarrow \text{Ig}_r^{\text{ord}}$ is surjective for all r (being faithfully flat) the maps $\text{Ig}^{\text{ord}} \rightarrow \text{Ig}_n^{\text{ord}}$ are also surjective and therefore faithfully flat.

By [4, Corollary 4.3.5], Ig^{ord} is a perfect scheme, i.e., the Frobenius map is an automorphism. In particular, Ig^{ord} is reduced. Since any faithfully flat map of commutative rings must be injective, it follows that Ig_n^{ord} is also reduced. Since k is a perfect field, it follows that Ig_n^{ord} is generically smooth. By the Serre–Tate theorem, the completion of the local ring of Ig_n^{ord} at any k -valued point only depends on the p -divisible group \mathbb{X} (with extra structure), so we deduce that Ig_n^{ord} , being generically smooth, is smooth at all points. \square

Let $C'_p \subset C_p$ be the kernel of the reduction map from $\mathcal{G}(\mathbb{Z}_p) \rightarrow \mathcal{G}(\mathbb{F}_p)$ and set

⁴We use the “isomorphism” version of the moduli functor, see [14, §1.4.1], rather than the “isogeny” version used by Kottwitz.

⁵Note that our definition below appears to be slightly different from that of *loc. cit.*, but that is not actually the case: the apparent difference is due to the fact that we are using the isomorphism version of the moduli functor.

$C' = C'_p \times C^p$. Corresponding to C' we have a finite étale cover $S(p)$ of $S = S_K$ (defined over K).

Let \mathbb{X}^{can} be the canonical lift of \mathbb{X} to $W(k)$ and let $\mathbb{X}_n^{\text{can}}$ denote its p^n -torsion. For any $s \in S_0^{\text{ord}}(k)$, \mathbb{X}^{can} is isomorphic to the p -divisible group of the canonical lift of s which gives a \mathcal{T} -valued point of \mathcal{S} . We replace R by a finite extension so that the generic fibre of $\mathbb{X}_1^{\text{can}}$ is a constant group.

Let $\mathbf{Aut}(\mathbb{X}_1^{\text{can}})^0$ be the connected component of the identity section of the group scheme $\mathbf{Aut}(\mathbb{X}_1^{\text{can}})$. It follows from the finiteness of $\mathbf{Aut}(\mathbb{X}_1)$ cited earlier that $\mathbf{Aut}(\mathbb{X}_1^{\text{can}})^0$ is finite over $W(k)$. We do not know if it is always flat over $W(k)$, but this seems likely.

PROPOSITION 4.2. *Assume that the hypotheses of Lemma 4.1 are satisfied. Let $S^\nu(p)$ be the normalisation of \mathcal{S} in $S(p)$. Then $S^\nu(p)_0$ is reduced and smooth at all μ -ordinary points (i.e., points lying over S_0^{ord}). There is a rational action of $\mathbf{Aut}(\mathbb{X}_1^{\text{can}})$ on $S^\nu(p)$ such that the restriction of the action to the subgroup scheme $\mathbf{Aut}(\mathbb{X}_1^{\text{can}})^0$ is regular and free on an open subset $S^o(p)$ of $S^\nu(p)$ which dominates \mathcal{S} and is smooth (and surjective) over \mathcal{T} .*

Proof. Let $\pi : S(p) \rightarrow \mathcal{S}$ be the scheme parametrising isomorphisms $\mathcal{A}[p] \rightarrow \mathbb{X}_1^{\text{can}} \times_{\mathcal{T}} \mathcal{S}$ (preserving the extra structure). The group scheme $\mathbf{Aut}(\mathbb{X}_1^{\text{can}})$ is quasifinite over \mathcal{T} by [18, Theorem 2.1.2 (ii)] (since its generic fibre is clearly finite), so the morphism π is quasifinite. The induced morphism $S(p)_K \rightarrow S$ is finite and étale. By the choice of R , it follows that $S(p)_K$ is isomorphic to $S(p)$ as defined earlier, justifying our notation. Furthermore, $S(p)_0$ is equal to Ig_1^{ord} (by definition), so it is smooth over S_0 .

Let $S^o(p)$ be the Zariski closure of $S(p)$ in $S(p)$ (with the reduced induced scheme structure). It is flat over \mathcal{T} and the closed fibre $S^o(p)_0$ of $S^o(p)$ dominates S_0 : for any point $x \in S_0^{\text{ord}}(k)$ the canonical lift of the abelian variety associated to x , i.e., the lift corresponding to the lift \mathbb{X}^{can} of \mathbb{X} by the Serre–Tate theorem [10, Theorem 1.2.1], gives rise, using [19, Proposition 2.3.12], to a morphism $x^{\text{can}} : \mathcal{T} \rightarrow \mathcal{S}$ lifting the point x . If we let \mathcal{B} be the abelian scheme over \mathcal{T} corresponding to x^{can} , then there exists an isomorphism $\mathcal{B}[p] \rightarrow \mathbb{X}_1^{\text{can}}$ (preserving the extra structure) by definition of the canonical lift, so x^{can} can be lifted to a map $\widehat{x}^{\text{can}} : \mathcal{T} \rightarrow S(p)$ whose image lies in $S^o(p)$. Since $\dim(\text{Ig}_1^{\text{ord}}) = \dim(S(p))$, it follows that $S^o(p)_0$ is a union of irreducible components of Ig_1^{ord} . Then since Ig_1^{ord} is smooth by Lemma 4.1, it follows that $S^o(p)$ is a smooth open subscheme of $S(p)$.

Let $S^\nu(p)$ be the normalisation of \mathcal{S} in $S(p)$. Since $S^o(p)$ is normal and quasifinite over \mathcal{S} , by Zariski's main theorem there exists an open embedding $S^o(p) \rightarrow S^\nu(p)$ (extending the identity map on the generic fibres). It follows from the smoothness of $S^o(p)_0$ that $S^\nu(p)_0$ has at least one component which is generically smooth. Since $S(p) \rightarrow S$ is Galois (by our assumption on R) and $S^\nu(p)$ is normal and finite over \mathcal{S} , the Galois group acts transitively on the components of $S^\nu(p)_0$ by [15, Theorem 9.3], so all these components are smooth at all μ -ordinary points.

The tautological action of $\mathbf{Aut}(\mathbb{X}_1^{\text{can}})$ on $\mathcal{S}(p)$ induces a rational action of $\mathbf{Aut}(\mathbb{X}_1^{\text{can}})^0$ on $\mathcal{S}^\nu(p)$ which is regular on $\mathcal{S}^o(p)$ (which dominates \mathcal{S} and is smooth and surjective over \mathcal{T}) and is free by definition of the scheme $\mathcal{S}(p)$ and the finiteness of $\mathbf{Aut}(\mathbb{X}_1^{\text{can}})^0$. \square

4.2 UNITARY SHIMURA VARIETIES

In this section we prove that the hypotheses of Proposition 4.2 hold in the cases of unitary PEL Shimura varieties at unramified primes. By this we mean the ones of type A in the sense of [13, §5] at primes p at which the algebraic group G of §4.1 is unramified and the subgroup $C_p \subset G(\mathbb{Q}_p)$ is hyperspecial. For a summary of the data needed to define these Shimura varieties the reader may also refer to [19, §4.1.1-4.1.3]

We continue with the notation of §4.1 (except we now assume that the PEL data is of type A), so we denote the integral model simply by \mathcal{S} and the corresponding μ -ordinary p -divisible group by \mathbb{X} and we would like to verify the assumptions on $\mathbf{Aut}(\mathbb{X}_n)$ and $\mathbf{Aut}(\mathbb{X}_n^{\text{can}})^0$. As explained in [19, §4.3], because of our assumption that p is unramified, the category of p -divisible groups (with extra structure) associated to the Shimura data is, by Morita theory, isomorphic to a product of categories of the sort considered in §3.2 and §3.3. Thus,

$$\mathbf{Aut}(\mathbb{X}_n) \cong \prod_{s=1}^m \mathbf{Aut}(\mathbb{X}(f_s)_n) \times \prod_{s'=1}^{m'} \mathbf{Aut}(\mathbb{X}(f'_{s'})_n) \tag{9}$$

where

- for each $s \in 1, \dots, m$, there is
 - a finite extension κ_s of \mathbb{F}_p and a positive integer d_s ;
 - setting $I_s := \text{Hom}(\kappa_s, k)$, a function $f_s : I_s \rightarrow \{0, 1, 2, \dots, d_s\}$;
 and then $\mathbf{Aut}(\mathbb{X}(f_s)_n)$ is the group scheme considered in §3.2, and
- for each $s' \in 1, \dots, m'$ there is
 - a finite extension $\tilde{\kappa}_{s'}$ of \mathbb{F}_p of even degree and a positive integer $d'_{s'}$;
 - setting $\tilde{I}_{s'} := \text{Hom}(\tilde{\kappa}_{s'}, k)$, a function $f'_{s'} : \tilde{I}_{s'} \rightarrow \{0, 1, 2, \dots, d'_{s'}\}$ satisfying the assumption of §3.3,
 and then $\mathbf{Aut}(\mathbb{X}(f'_{s'})_n)$ is the group scheme considered in §3.3.

An analogous isomorphism also holds in the case of $\mathbf{Aut}(\mathbb{X}_n^{\text{can}})^0$. With the isomorphism (9) at our disposal we are ready to prove the following:

COROLLARY 4.3. *Let \mathcal{S} be the integral model of a PEL Shimura of type A over \mathcal{T} where p is a prime at which the corresponding group G is unramified and the group C_p is hyperspecial. Let $\mathcal{S}(p)$ be the principal level p cover of \mathcal{S} with Galois*

group $G(\mathbb{F}_p)$ and let $S^\nu(p)$ be the normalisation of \mathcal{S} in $S(p)$. Then $S^\nu(p)$ is smooth at all μ -ordinary points of $S^\nu(p)_0$ and the restriction of the rational action of $\mathbf{Aut}(\mathbb{X}_1^{\text{can}})$ on $S^\nu(p)$ to the finite flat subgroup scheme $\mathbf{Aut}(\mathbb{X}_1^{\text{can}})^0$ is regular and free on the open subset $S^o(p)$ of $S^\nu(p)$ which dominates \mathcal{S} and is smooth (and surjective) over \mathcal{T} .

Proof. It follows from Corollaries 3.6 and 3.9 and the isomorphism (9) that in the type A case $\mathbf{Aut}(\mathbb{X}_1^{\text{can}})^0$ is finite and flat and the assumptions of Proposition 4.2 are satisfied, so the corollary follows from that proposition. \square

5 INCOMPRESSIBILITY OF CONGRUENCE COVERS

Let F_0 be a totally real number field, let F be a CM extension of F_0 and let c be the non-trivial element of $\text{Gal}(F/F_0)$. We consider a PEL Shimura datum (G, X) where G is a group of unitary similitudes corresponding to a d -dimensional Hermitian form h over F or to a central division algebra D over F of degree d^2 with an involution $*$ of the second kind, i.e, it fixes F_0 and acts non-trivially on F . For each embedding $\tau : F \rightarrow \mathbb{C}$ the Hermitian form or the involution $*$ gives rise to a non-negative integer $n(\tau)$: for a Hermitian form this is the dimension of the $+$ -part of the Hermitian form over \mathbb{C} induced by τ and in the division algebra case we refer to [12, §1] for the definition. These integers satisfy $n(\tau) + n(\bar{\tau}) = d$ for all τ , where $\bar{\tau} := c \circ \tau$.

Let L be the reflex field of the Shimura datum. The group $\text{Aut}(\mathbb{C}/\mathbb{Q})$ acts on the set of integer valued functions on $\text{Hom}(F, \mathbb{C})$ and in the cases we consider the reflex field is the fixed field of the stabilizer in $\text{Aut}(\mathbb{C}/\mathbb{Q})$ of the function $\tau \mapsto n(\tau)$ (we call this function the *type* of the PEL datum). The field L is clearly contained in the Galois closure of $\tau(F)$ (for any τ), so it is either totally real or a CM field. We let L_0 be the maximal totally real subfield of L . In the cases we are most interested in we will always have that $L \neq L_0$.

We fix the subgroup $C \subset G(\mathbb{A}^f)$ as in §4.1 and we let $S = S_C$ be the corresponding Shimura variety. Let $C' \subset G(\mathbb{A}^f)$ also be as in §4.1, giving a Shimura variety $S(p) = S_{C'}$, which is naturally a Galois cover of S . Our main theorem is the following:

THEOREM 5.1. *Let $p \neq 2$ be a prime which splits completely in F_0 . If $d = 2\delta + 1$ is odd and for each $\tau \in \text{Hom}(F, \mathbb{C})$, $n(\tau) \in \{0, \delta, \delta + 1, d\}$, then the cover $S(p)/S$ is p -incompressible. More precisely, for each irreducible component S' of S and $S(p)'$ of $S(p)$ lying over S' , the map $S(p)' \rightarrow S'$ is incompressible.*

Proof. By the definition of a Shimura variety, the derived group of $G(\mathbb{R})$ is not compact, so there exists a τ such that $n(\tau) = \delta$. This implies that $L \neq L_0$ as then $n(\bar{\tau}) = \delta + 1$ so the function $\tau \mapsto n(\tau)$ is not invariant under c .

Let P be a prime of L lying over p . Since p splits completely in F_0 , it also splits completely in L_0 because L_0 is contained in the Galois closure of F_0 . If the residue field at P is also \mathbb{F}_p then incompressibility follows from [8] so we assume that it is \mathbb{F}_{p^2} . Let \mathcal{S} be the integral model over R as in 4.1, where R is

a dvr containing the localisation of O_L at P . Since the scheme \mathcal{S} is a moduli space of abelian schemes of PEL type, by a result of Wedhorn [23] (see also [19]) the μ -ordinary locus is open and dense in \mathcal{S}_0 , hence there is a corresponding μ -ordinary polarised p -divisible group with suitable endomorphism structure. Up to Morita equivalence (as in [19, §3.1.2]) this is a product of groups of the type we have considered in §3.3, with factors being parametrised by the c -orbits on the set of simple factors of the \mathbb{Q}_p -algebra $F \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Our assumption on p implies that the simple factors of this semisimple algebra are either \mathbb{Q}_p itself or the unramified quadratic extension E of \mathbb{Q}_p , and c acts on this by permuting pairs of factors isomorphic to \mathbb{Q}_p and acting by the non-trivial involution on the factors isomorphic to E . Thus our factors are of type AL and AU in the sense of *loc. cit.*, where the endomorphism ring is just $\mathbb{Z}_p \times \mathbb{Z}_p = W(\mathbb{F}_p) \times W(\mathbb{F}_p)$ in the AL case and $\mathcal{O}_E = W(\mathbb{F}_{p^2})$ in the AU case. In the first case, since the involution permutes the factors we are reduced to considering p -divisible groups without any polarisation, i.e., as in §3.2, and in the second case p -divisible groups with endomorphisms by $W(\mathbb{F}_{p^2})$ and a polarisation as in §3.3.

We now use the classifications of such p -divisible groups which are ordinary, so we need to determine the function \mathfrak{f} corresponding to each factor. In the case AL it follows from the assumption on the function $n(\tau)$, that the function \mathfrak{f} , which is just a number since I is a singleton, is any element of the set $\{0, \delta, \delta + 1, d\}$. In the case AU, the same assumption implies that the values of \mathfrak{f} on \tilde{I} (which is a two element set) are either $\{0, d\}$ or $\{\delta, \delta + 1\}$.

We shall prove the incompressibility by combining Corollary 4.3 with Proposition 2.6, so we must analyze the groups $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathfrak{f})_1)^0$ in each case, as the group \mathcal{G} acting on the $\mathcal{S}^o(p)$ produced by Corollary 4.3 is the product of all these groups over all \mathfrak{f} occurring above.

In case AL, since two factors are interchanged we may assume that \mathfrak{f} is 0 or δ . When it is 0, $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathfrak{f})_1)^0$ is trivial. When it is δ , then $r = 2$ and $\mathbb{X}_1(\mathfrak{f})$ is étale and $\mathbb{X}_2(\mathfrak{f})$ is multiplicative. The group $\mathbf{Aut}(\mathbb{X}^{\text{can}}(\mathfrak{f})_1)^0$ is then isomorphic to $\mu_p^{\delta(\delta+1)}$ as a special case of the Lemma 3.4.

In the case AU, the function \mathfrak{f} is the one considered in §3.4 so we may use the computations we have made there. In particular, we see that \mathcal{G} above is exactly of the form considered in Lemma 3.10 with \mathcal{H}' being the product of all the multiplicative groups corresponding to the factors of type AL and t the number of factors of type AU.

The dimension of any unitary Shimura variety as at the beginning of this section is given by $\sum_{\{\tau, \bar{\tau}\}} n(\tau) \cdot n(\bar{\tau})$ since each pair $\{n(\tau), n(\bar{\tau})\}$ contributes a factor $SU(n(\tau), n(\bar{\tau}))$ to the adjoint group of $\mathbf{G}(\mathbb{R})$. It follows from this and Lemma 3.10 that $t(\mathcal{G}) = \dim(S) = \dim(S(p))$. We are now ready to apply Proposition 2.6. Let \mathcal{G} be as above and let \mathcal{H} be the subgroup $\mathcal{H}(\mathfrak{f})_1^t \times \mathcal{H}'$ of \mathcal{G} . Lemma 3.10 shows that the hypotheses of Proposition 2.6 are satisfied for the \mathcal{G} -action on $\mathcal{S}^o(p)$ with $e = \dim(S)$ so the theorem is proved. \square

Example 5.2. If $F_0 = \mathbb{Q}$, so F is an imaginary quadratic field, then all primes p satisfy the condition in 5.1 but since $L = F$, the results of [8] only apply for

the primes p which split in F . As a consequence, one can add the case of odd special unitary groups, albeit only over \mathbb{F}_p , to the list of groups occurring in [8, Corollary 4.3.12]. These groups are also inaccessible by the methods of [2] since the Hermitian symmetric space in the corresponding Shimura datum (G, X) is not of tube type.

Remark 5.3.

1. The reason that we restrict the values of $n(\tau)$ and also only consider primes split in F_0 is that the analogue of Lemma 3.10 does not hold for any \mathfrak{f} except for the one considered there. However, one can derive explicit inequalities for $t(\mathcal{G}_0/\mathcal{K}_0)$ in many other cases and these can be used to give non-trivial lower bounds, not obtainable by other known methods, for the essential dimension at p of p -congruence covers of more general unitary Shimura varieties.
2. In the case the Shimura variety corresponds to a division algebra, it is proper. The results of [8] also apply to proper Shimura varieties—this is one of the main advantages of their method compared to the method of [2]—but in that (ordinary) setting the incompressibility essentially comes about from the action of an elementary abelian p -group (a subgroup of the full Galois group) on the congruence cover $S(p)$. In the non-ordinary setting the use of noncommutative group schemes seems to be unavoidable.
3. Our method also applies in the case of PEL Shimura varieties at primes of ordinary reduction, thereby giving a mild variant of the proof of some of the results in [8], but this case is considerably simpler: since we need only deal with multiplicative group schemes, we do not need Proposition 2.6 and the analogues of the results of §3 needed are elementary.

6 ABELIAN VARIETIES

In this section we apply Proposition 2.12 to prove some results towards the following conjecture due to P. Brosnan.⁶

CONJECTURE 6.1. *Let A be an abelian variety over a field L of characteristic zero. Then for all primes p , the multiplication by p map $[p] : A \rightarrow A$ is p -incompressible.*

It is clear that this conjecture implies that the multiplication by n map $[n] : A \rightarrow A$ is incompressible for all $n > 1$. If $\dim(A) = 1$ then the conjecture is trivially true and if $\dim(A) = 2$ incompressibility—but not p -incompressibility—can be proved in an elementary way using the known structure of the automorphism groups of \mathbb{P}^1 and elliptic curves. We also note that

⁶Personal communication

if $\dim(A)$ is arbitrary but A is sufficiently generic then the conjecture can be proved by using results of Gabber [5, Appendice] (or from the results below).

We first note a simple reduction:

LEMMA 6.2. *Suppose the field L has a discrete valuation at which A has good reduction and the residue characteristic is also zero. If the conjecture for a fixed prime p holds for the special fibre of the Neron model of A , then it also holds for A .*

Proof. We let K be the completion of L with respect to the given discrete valuation. Let \mathcal{G} be the finite étale group scheme over \mathcal{T} given by $\mathcal{A}[p]$ (which we may assume is constant by increasing K), where \mathcal{A} is the Neron model of A over \mathcal{T} . By Lemma 2.2, given a compression $f : A \dashrightarrow B$ of the G -torsor A we get a \mathcal{G} -equivariant surjective flat morphism $\mathcal{A}' \rightarrow \mathcal{B}'$ where \mathcal{A}' is open in \mathcal{A} , \mathcal{A}'_0 is non-empty, \mathcal{B}' is affine and smooth over \mathcal{T} and on the generic fibres the map on function fields is equal to that induced by f . It suffices to show that the G action on \mathcal{B}'_0 is generically free.

Let $O(\mathcal{B}')$ be the affine ring of \mathcal{B}' . The group G acts on it and $O(\mathcal{B}')^G$ is finitely generated over R since $O(\mathcal{B}')$ is so (by Hilbert's argument). It is then clearly normal and $O(\mathcal{B}')$ is finite over $O(\mathcal{B}')^G$ and we set $\mathcal{B}'/G := \text{Spec}(O(\mathcal{B}')^G)$. Since \mathcal{B}' is smooth over \mathcal{T} it follows that the local ring of \mathcal{B}'/G at the generic point of its special fibre is a dvr which is unramified over R . It follows that the map of function fields induced by the map $\mathcal{B}'_0 \rightarrow (\mathcal{B}'/G)_0$ must be Galois with Galois group G , so the G -action on \mathcal{B}'_0 is generically free. \square

Remark 6.3. The proof of the lemma gives a very general statement comparing essential dimension of the generic fibre and special fibre of a G -torsor over a dvr.

Lemma 6.2 easily allows one to reduce Conjecture 6.5 to the case when A is the base change to L of an abelian variety over a number field, but we do not use this in the following:

THEOREM 6.4. *Let A be an abelian variety over a field L of characteristic zero. Suppose L has a discrete valuation of prime residue characteristic p at which A has semi-stable reduction and the special fibre of the Neron model of A associated to this valuation has a closed subgroup scheme of the form $(\mu_p)^n$ for some $n < d := \dim(A)$. Then the essential dimension at p of $[p] : A \rightarrow A$ over any extension field of L is at least $n + 1$.*

Proof. We may replace L by an extension K which is complete with respect to a discrete valuation extending the one on L . Let R be the ring of integers of K and k the residue field which we may assume is algebraically closed. We may also assume that K is large enough so that $\mathcal{A}[p]$ (over K) is a constant group scheme. Let \mathcal{A} be the Neron model of A over \mathcal{T} and let $\mathcal{A}[p]$ be its p -torsion subscheme. Since A has semi-stable reduction, $\mathcal{A}[p]$ is flat over \mathcal{T} , so its identity component $\mathcal{A}[p]^0$ is finite flat over \mathcal{T} . The closed fibre $\mathcal{A}[p]^0_0$ is a finite group scheme of order at least p^{n+1} (since $n < d$).

By applying Lemma 2.10 to the Cartier dual of $\mathcal{A}[p]_0^0$ (and then dualising again), we see that $\mathcal{A}[p]_0^0$ contains a finite flat subgroup scheme \mathcal{G} of rank $n + 1$ such that \mathcal{G}_0 contains a subgroup scheme isomorphic to $(\mu_p)^n$. The Neron model \mathcal{A} is smooth over \mathcal{T} and $\mathcal{A}[p]_0^0$, hence also \mathcal{G} , acts freely on it by translation. We may therefore apply Proposition 2.12 to conclude the proof. \square

COROLLARY 6.5. *Let A be an abelian variety of dimension d over a field L of characteristic zero. Suppose L has a discrete valuation of prime residue characteristic p at which A has good reduction and the special fibre of the Neron model \mathcal{A} of A associated to this valuation has p -rank at least $d - 1$. Then the essential dimension at p of $[p] : A \rightarrow A$ over any extension field of L is d .*

Proof. This is an almost immediate consequence of Theorem 6.4. If the p -rank of the special fibre is at least $d - 1$, then since the special fibre is an abelian variety it must contain a subgroup scheme isomorphic to $(\mu_p)^{d-1}$. \square

LEMMA 6.6. *Let A be an abelian variety of dimension $d > 1$ over a number field L . The p -rank of the reduction of a g -dimensional abelian variety A at a prime P of O_L above p is least two for a set of rational primes p of positive density (depending on A).*

Proof. This is well-known if $d = 2$ [6, VI, Corollary 2.9] and a similar argument works in general. We first note that the set of primes P of O_L lying above rational primes p which split completely in O_L is of density one. We consider the characteristic polynomial

$$\Phi_P(x) = x^{2d} + a_1(P)x^{2d-1} + a_2(P)x^{2d-2} + \cdots + a_{2d}(P)$$

of geometric Frobenius for such primes P acting on the l -adic Tate module of A for some fixed rational prime l . The $a_i(P) \in \mathbb{Z}$ and by Weil's theorem $|a_i(P)| \leq c_i p^{i/2}$, where c_i is a constant depending only on d . If the p -rank of the reduction of A at P is at most one, then a_2 is divisible by p so a_2/p lies in a finite set of integers. On the other hand, since $a_{2d}(P) = p^d$ for all P as above, we see that $a_2(P)^d/a_{2d}(P)$ takes on only finitely many values for such P .

By enlarging L if necessary, we may assume that the Zariski closure G of the image of $\text{Gal}(\bar{L}/L)$ (acting on the Tate module $\otimes \mathbb{Q}_l$) is connected. Each a_i , viewed as a function on $\text{Gal}(\bar{L}/L)$ is the restriction of an algebraic function on G to the image of $\text{Gal}(\bar{L}/L)$, hence so is $(a_2)^d/a_d$. The image of $\text{Gal}(\bar{L}/L)$ is an open subgroup of $G(\mathbb{Q}_l)$ and G contains the homotheties by [1] so $(a_2)^d/a_d$ is not a constant function on G . Thus, the level sets of the function have measure 0 in $G(\mathbb{Q}_l)$, hence by the Chebotarev density theorem the set of all P as above with the reduction of A at P being of p -rank at most one has density 0. \square

COROLLARY 6.7. *Brosnan's conjecture in the case $\dim(A) \leq 3$ holds for a set of primes p (depending on A) of positive density.*

Proof. This follows by combining Lemma 6.6, Lemma 6.2 and Corollary 6.5. \square

Remark 6.8.

1. If $\dim(A) = 2$ then, as already noted earlier, incompressibility of $[p]$ holds for all p but we do not know whether $[p]$ is p -incompressible for all p (but see [5, Proposition 11] for a related result in a special case).
2. The difficulty in proving Brosnan’s conjecture in general by our method lies in the problem of the (possible) non-freeness of the descended \mathcal{G} -action mentioned in Remark 2.3. We do not actually have such an example when \mathcal{G} is the p -torsion of an abelian scheme over \mathcal{T} (or even a 1-truncated p -divisible group) and it would be very interesting to know whether in this case the conclusion of Lemma 2.2 can be made stronger, i.e., if the \mathcal{G} -action is free at a general point of \mathcal{X}_0 , is it also free at a general point of \mathcal{Y}'_0 ?
3. It is natural to extend Brosnan’s conjecture to abelian varieties over arbitrary fields—it suffices to assume algebraically closed—of characteristic $l \neq p$, but our methods do not apply: if $\dim(A) = 2$ then incompressibility of $[p]$ does hold for all $p \neq l$, but for a fixed A we do not know whether p -incompressibility holds for even a single p if $l > 0$. It would also be interesting to consider the case $p = l$, when $[p] : A \rightarrow A$ is a torsor under a nonreduced group scheme.

REFERENCES

- [1] F. A. BOGOMOLOV, *Sur l’algébricité des représentations l -adiques*, C. R. Acad. Sci. Paris Sér. A-B 290 (1980), A701–A703.
- [2] P. BROSINAN AND N. FAKHRUDDIN, *Fixed points, local monodromy, and incompressibility of congruence covers*. Preprint, 2020. <https://arxiv.org/abs/2007.01752>.
- [3] J. BUHLER AND Z. REICHSTEIN, *On the essential dimension of a finite group*, Compositio Math. 106 (1997), 159–179.
- [4] A. CARAIANI AND P. SCHOLZE, *On the generic part of the cohomology of compact unitary Shimura varieties*, Ann. of Math. (2) 186 (2017), 649–766.
- [5] J.-L. COLLIOT-THÉLÈNE, *Exposant et indice d’algèbres simples centrales non ramifiées*, Enseign. Math. (2) 48 (2002), 127–146. With an appendix by Ofer Gabber.
- [6] P. DELIGNE, J. S. MILNE, A. OGUS, AND K.-Y. SHIH, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, 900, Springer, Berlin-New York, 1982.
- [7] N. FAKHRUDDIN, *Finite group schemes of essential dimension one*, Doc. Math. 25 (2020), 55–64.

- [8] B. FARB, M. KISIN, AND J. WOLFSON, *The essential dimension of congruence covers*, *Compos. Math.* 157 (2021), no. 11, 2407–2432.
- [9] L. ILLUSIE, *Déformations de groupes de Barsotti–Tate (d’après A. Grothendieck)*. Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84). *Astérisque* 127 (1985), 151–198.
- [10] N. KATZ, *Serre–Tate local moduli*, in *Algebraic surfaces (Orsay, 1976–78)*, *Lecture Notes in Mathematics*, 868, Springer, Berlin-New York, 1981, 138–202.
- [11] J. KOLLÁR, *Rational curves on algebraic varieties*, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, 32, Springer, Berlin, 1996.
- [12] R. E. KOTTWITZ, *On the λ -adic representations associated to some simple Shimura varieties*, *Invent. Math.* 108 (1992), 653–665.
- [13] R. E. KOTTWITZ, *Points on some Shimura varieties over finite fields*, *J. Amer. Math. Soc.* 5 (1992), 373–444.
- [14] K.-W. LAN, *Arithmetic compactifications of PEL-type Shimura varieties*, *London Mathematical Society Monographs Series*, 36, Princeton University Press, Princeton, NJ, 2013.
- [15] H. MATSUMURA, *Commutative ring theory*, *Cambridge Studies in Advanced Mathematics*, 8, Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid.
- [16] A. S. MERKURJEV, *Essential dimension: a survey*, *Transform. Groups* 18 (2013), 415–481.
- [17] W. MESSING, *The crystals associated to Barsotti–Tate groups: with applications to abelian schemes*, *Lecture Notes in Mathematics*, 264, Springer, Berlin-New York, 1972.
- [18] B. MOONEN, *A dimension formula for Ekedahl–Oort strata*, *Ann. Inst. Fourier (Grenoble)* 54 (2004), 666–698.
- [19] B. MOONEN, *Serre–Tate theory for moduli spaces of PEL type*, *Ann. Sci. École Norm. Sup. (4)* 37 (2004), 223–269.
- [20] D. MUMFORD, *Abelian varieties*, *Tata Institute of Fundamental Research Studies in Mathematics*, 5, Oxford University Press, 1970.
- [21] M. ROMAGNY, *Effective models of group schemes*, *J. Algebraic Geom.* 21 (2012), 643–682.

- [22] W. C. WATERHOUSE, *Introduction to affine group schemes*, Graduate Texts in Mathematics, 66, Springer, New York-Berlin, 1979.
- [23] T. WEDHORN, *Ordinariness in good reductions of Shimura varieties of PEL-type*, Ann. Sci. École Norm. Sup. (4) 32 (1999), 575–618.

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