ON THE BRAUER GROUP OF BIELLIPTIC SURFACES
(WITH AN APPENDIX BY JONAS BERGSTRÖM AND SOFIA TIRABASSI)

EUGENIA FERRARI, SOFIA TIRABASSI, MAGNUS VODRUP,
AND JONAS BERGSTRÖM

ABSTRACT. We provide explicit generators of the torsion of the second cohomology of bielliptic surfaces, and we use this to study the pullback map between the Brauer group of a bielliptic surface and that of its canonical cover.

2020 Mathematics Subject Classification: 14F22
Keywords and Phrases: Bielliptic surfaces, Brauer group, canonical covers

CONTENTS

1   Introduction

Given a smooth complex projective variety Z, its (cohomological) Brauer group is defined as $\mathrm{Br}(Z) := H^2_{\text{ét}}(Z, \mathcal{O}^*_Z)_{\text{tor}}$. A morphism of projective varieties $f : Z \to Y$ induces, via pullback, a homomorphism $f_{\mathrm{Br}} : \mathrm{Br}(Y) \to \mathrm{Br}(Z)$, which we call the *Brauer map induced by f*. The map f_{Br} is studied in [Bea09] in the setting where $f : Z \longrightarrow Y$ is the canonical K3 cover of a complex Enriques surface Y. More precisely, Beauville identifies the locus in the moduli space of Enriques surfaces where f_{Br} is not injective (and so trivial, since $\mathrm{Br}(Y)$ is simple in this case). In this paper we carry out a similar investigation for the canonical covers of bielliptic surfaces.

A bielliptic surface is constructed by taking the quotient of a product of elipic curves $A \times B$ by the action of a finite group G. They were classified in 7 different types by Bagnera–De Franchis ([BDF10], see also [Suw69] for a presentation in a more modern language), as illustrated in Table 1. Since the canonical

Type	G	Order of ω_S in $\mathrm{Pic}(S)$	$H^2(S, \mathbb{Z})_{\text{tor}}$
1	$\mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
2	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z}$
3	$\mathbb{Z}/4\mathbb{Z}$	4	$\mathbb{Z}/2\mathbb{Z}$
4	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	4	0
5	$\mathbb{Z}/3\mathbb{Z}$	3	$\mathbb{Z}/3\mathbb{Z}$
6	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	3	0
7	$\mathbb{Z}/6\mathbb{Z}$	6	0

Table 1: Types of bielliptic surfaces and torsion of their second cohomology.

bundle of a bielliptic surface S is a torsion element in $\mathrm{Pic}(S)$, it can be used to define an étale cyclic cover $\pi : X \to S$, where X is an abelian variety isogenous to $A \times B$. We then obtain a homomorphism between the respective Brauer groups: $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$. A very natural question is the following.

Question. When is π_{Br} injective? When is it trivial?

As for Enriques surfaces, using the long exact exponential sequence and Poincaré duality, we have a non canonical isomorphism

$$\mathrm{Br}(S) \simeq H^2(S, \mathbb{Z})_{\text{tor}},$$

so from the fourth column of Table 1, we easily see that this map is trivial when S is of type 4, 6 or 7. Thus we will limit ourselves to surfaces of type 1, 2, 3, and 5. We will find that the behavior of the Brauer map depends heavily on the geometry of the bielliptic surface S.

Our first step in this investigation is to focus on bielliptic surfaces of type 2 and 3. By construction, which is written explicitly only in [Nue], they admit a

degree 2 étale cover $\tilde{\pi} : \tilde{S} \to S$, with $\tilde{S}$ a bielliptic surface of type 1 (see Examples 2.4(a) and 2.4(b) below for more details). We investigate the properties of the induced Brauer map $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$, finding how this behaves differently in the two cases:

THEOREM A.   *(a) If S is of type 2, then $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ is trivial.*

(b) If S is of type 3, then $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ is injective.

The main tool behind our argument is a result of Beauville (see Section 2 for more details) which states that the kernel of the Brauer map of a cyclic étale cover $X \to X/\sigma$ is naturally isomorphic to the kernel of the norm map $\mathrm{Nm} : \mathrm{Pic}(X) \to \mathrm{Pic}(X/\sigma)$ quotiented by $\mathrm{Im}(1-\sigma^*)$. We prove that a line bundle on $\tilde{S}$ is in the kernel of the norm map only if it is numerically trivial. Then we reach our conclusion by carefully computing the norm map of numerically trivial line bundles. The different behavior of the two type of surfaces is motivated by the different "values" taken by the norm map on torsion elements of $H^2(\tilde{S}, \mathbb{Z})$: in the type 2 case they are sent to topologically trivial line bundles, while this is not true in the type 3 case.

Theorem A is interesting in itself, and some parts of its proof will be useful in order to study the Brauer map to the canonical cover for bielliptic surfaces of type 2.

We now turn our attention to the main focus of this paper: to give a complete description for the Brauer map to the canonical cover of any bielliptic surface. We first show, similarly to what happens for Enriques surfaces, that the Brauer map is injective for a general bielliptic surface. In particular, we show the following general statement:

THEOREM B. *Given a bielliptic surface S, let $\pi : X \to S$ be its canonical cover. If the two elliptic curves A and B are not isogenous, then the pullback map*

$$\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$$

is injective.

When A and B are isogenous one encounters the first examples of bielliptic surfaces with non injective Brauer map. It turns out that the behavior of the Brauer map depends on the explicit action of the group G on the product $A \times B$. This is showed in Theorems 5.3, 5.8, 5.10, 5.14, 5.19, and 5.21, where we give necessary and sufficient conditions for the Brauer map to be injective, trivial, and, in the case of type 1 surfaces (whose Brauer group is not simple), neither trivial nor injective. Our very explicit results allow us to construct examples of bielliptic surfaces exhibiting each of the possible behaviors of the Brauer map. Unfortunately, the statements are involved, and it is not possible reproduce them here without a lengthy explanation of the notation used. The rough geometric picture is the following:

Type 1: These surfaces are constructed by choosing two elliptic curves A and B and a 2-torsion point on A. Thus the moduli space has dimension 2. In order to have a non injective Brauer map one can choose freely the elliptic curve B, but has only finitely many possibilities for A and the 2-torsion point. Thus we obtain a 1- dimensional family. In Example 5.9(c), we show that uncountably many such surfaces exist. On the other hand, only countably many type 1 bielliptic surfaces can have a trivial Brauer map to their canonical cover. In fact, to obtain a trivial Brauer map one has to choose the ellipic curve B among those having complex multiplication.

Type 2: These surfaces are constructed by choosing two elliptic curves A and B and two 2-torsion points, one on A and one on B. Hence the moduli space has dimension 2. Similarly to what happens in the previous case, in order to have a non injective (and hence trivial) Brauer map, only the choice of the curve B can be made freely, while A must be taken among finitely many possibilities. We show in this case (cf. Example 5.25(a)) that if the curves A and B are isomorphic, regardless of the choice of the torsion points, the Brauer map is trivial.

Type 3: These surfaces are constructed by choosing one elliptic curve A and a 4-torsion point on it. Therefore the moduli space has dimension 1. In order to have a non injective (and hence trivial) Brauer map, A must be isogenous to the curve with j-invariant 1728. Thus there are only finitely many such surfaces.

Type 5: These surfaces are constructed by choosing one elliptic curve A and a 3-torsion point on it. We deduce that the moduli space has dimension 1. In order to have a non injective (and hence trivial) Brauer map, A must be isogenous to the curve with j-invariant 0. Thus, as in the previous case, there are only finitely many such surfaces.

The proof of Theorem B uses the same ideas as in the proof of Theorem A. In fact, we can leverage on the fact that X and S have the same Picard number (as for the case of a bielliptic cover) to show that line bundles in the kernel of the norm map are topologically trivial. The result is then obtained by showing that line bundles in $\mathrm{Pic}^0(X)$ which are also in the kernel of the norm map are always in $\mathrm{Im}(1 - \sigma^*)$. As a corollary of both Theorem A and B we find

Corollary C (Corollary 5.4). *Given an isogeny of abelian varieties* $\varphi : X \to Y$, *the corresponding group homomorphism* $\varphi_{\mathrm{Br}} : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ *is not necessarily injective.*

More precisely, we provide an explicit example of an isogeny between two abelian surfaces $\varphi : X \to Y$ such that the corresponding group homomorphism φ_{Br} is not injective, (see Section 5.2).

This paper is organized as follows. Section 2 contains all the background and

preliminary results. More precisely, we outline some classical facts on the geometry of bielliptic surfaces, and present the construction, due to Nuer, of the bielliptic covers of surfaces of type 2 and 3. We also expose the work of Beauville [Bea09] which allows us to study the kernel of the Brauer map in terms of the norm homomorphism of the cover. We conclude the section by describing the Néron–Severi group of a product of elliptic curves. In Section 3, we provide explicit generators for $H^2(S, \mathbb{Z})_{\mathrm{tor}}$, when S is a bielliptic surface of type 1, 2, 3 or 5. We prove Theorem A in Section 4, while we completely describe the norm map to the canonical cover in Section 5. Here we also construct examples of bielliptic surfaces of every type in which the Brauer map behaves differently. In the Appendix, which is joint work of the second author of the main paper with J. Bergström, a structure theorem for the homomorphism ring of two elliptic curves is given in the case of j-invariant 0 or 1728. This will give, in turn, a really useful description of the Picard group of the product of such curves, which is fundamental to study the Brauer map of bielliptic surfaces of type 3 and 5.

Notation. We are working over the field of complex numbers $\mathbb{C}$. If X is a complex abelian variety over $\mathbb{C}$, and $n \in \mathbb{Z}$, then $X[n]$ will denote the subscheme of n-torsion points of X, while $n_X : X \to X$ will stand for the "multiplication by n isogeny". Given $x \in X$ a point, then the translation by x will be denoted as t_x. In addition, if $\dim X = 1$ – that is, X is an elliptic curve – then P_x will be the line bundle $\mathcal{O}_X(x - p_0) \simeq t_{-x}^* \mathcal{O}_X(p_0) \otimes \mathcal{O}_X(-p_0)$ in $\mathrm{Pic}^0(X)$, where $p_0 \in X$ is the identity element.

For any smooth complex projective variety Y we will denote the identity homomorphism as 1_Y (or simply 1 if there is no chance of confusion), while K_Y and ω_Y will stand for the canonical divisor class and the dualizing sheaf on Y, respectively. If D and E are two linearly equivalent divisors on Y, we will write $D \sim E$; in addition, $\mathcal{O}_Y(D)$ will denote the line bundle associated to the divisor D.

2   Background and preliminary results

2.1   Bielliptic surfaces

A complex *bielliptic* (or *hyperelliptic*) surface S is a minimal smooth projective surface over the field of complex numbers with Kodaira dimension $\kappa(S) = 0$, irregularity $q(S) = 1$, and geometric genus $p_g(S) = 0$. By the work of Bagenera–De Franchis (see for example [Bad01, 10.24-10.27]), the canonical bundle ω_S has order either 2, 3, 4 or 6 in $\mathrm{Pic}(S)$, and S occurs as a finite étale quotient of a product $A \times B$ of elliptic curves by a finite group G acting on A by translations, and on B such that $B/G \simeq \mathbb{P}^1$. More precisely we have the following classification result.

THEOREM 2.1 (Bagnera–De Franchis [BDF10], [Suw69, Theorem at p. 473], [BM77, p. 37]). *A bielliptic surface is of the form $S = A \times B/G$, where A and B are elliptic curves and G a finite group of translations of A acting on B by automorphisms. They are divided into seven types according to G as shown in Table 1.*

There are natural maps $a_S \colon S \to A/G$ and $g \colon S \to B/G \simeq \mathbb{P}^1$ which are both elliptic fibrations. The morphism a_S is smooth, and coincides with the Albanese morphism of S. On the other hand, g admits multiple fibers, corresponding to the branch points of the quotient $B \to B/G$, with multiplicity equal to that of the associated branch point. The smooth fibers of a_S and g are isomorphic to B and A, respectively. We will denote by a and b the classes of these fibers in $\mathrm{Num}(S)$, $H^2(S, \mathbb{Z})$ and $H^2(S, \mathbb{Q})$.

It is well known (see for example [Ser90a, p. 529]) that a and b span $H^2(S, \mathbb{Q})$ and satisfy $a^2 = b^2 = 0$, $ab = |G|$. Furthermore, we have the following description of the second cohomology of S:

PROPOSITION 2.2. *The decomposition of $H^2(S, \mathbb{Z})$ is described according to the type of S and the multiplicities $(m_1, \ldots, m_s)$ of the singular fibers of $g \colon S \to \mathbb{P}^1$ as follows:*

Type	$(m_1, \ldots, m_s)$	$H^2(S, \mathbb{Z})$	$H^2(S, \mathbb{Z})_{\mathrm{tor}}$
1	$(2, 2, 2, 2)$	$\mathbb{Z}[\frac{1}{2}a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
2	$(2, 2, 2, 2)$	$\mathbb{Z}[\frac{1}{2}a] \oplus \mathbb{Z}[\frac{1}{2}b] \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
3	$(2, 4, 4)$	$\mathbb{Z}[\frac{1}{4}a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
4	$(2, 4, 4)$	$\mathbb{Z}[\frac{1}{4}a] \oplus \mathbb{Z}[\frac{1}{2}b]$	0
5	$(3, 3, 3)$	$\mathbb{Z}[\frac{1}{3}a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
6	$(3, 3, 3)$	$\mathbb{Z}[\frac{1}{3}a] \oplus \mathbb{Z}[\frac{1}{3}b]$	0
7	$(2, 3, 6)$	$\mathbb{Z}[\frac{1}{6}a] \oplus \mathbb{Z}[b]$	0

Proof. See [Ser90a, Tables 2 and 3]. The computation of the torsion of $H^2(S, \mathbb{Z})$ can be also found in [Iit70, Ser91, Suw69, Ume75]. $\qquad\square$

Since $H^2(S, \mathcal{O}_S) = 0$, the first Chern class map $c_1 \colon \text{Pic}(S) \to H^2(S, \mathbb{Z})$ is surjective, so the Néron-Severi group has $NS(S) \simeq H^2(S, \mathbb{Z})$. Modulo torsion we then get
$$\text{Num}(S) = \mathbb{Z}[a_0] \oplus \mathbb{Z}[b_0]$$
where $a_0 = \frac{1}{\text{ord}\,(\omega_S)} a$ and $b_0 = \frac{\text{ord}\,(\omega_S)}{|G|} b$.

2.2 Canonical covers

Given a smooth projective variety Y and a torsion line bundle $\mathcal{L}$ of order n, then it is possible to construct a degree n cyclic étale cover $\pi : X \to Y$ such that $\pi^* \mathcal{L} \simeq \mathcal{O}_X$. Roughly speaking (for more details see [BHPvdV15, I.17]), one considers a trivializing section $s \in H^0(Y, \mathcal{L}^{\otimes n})$, and the total space of $\mathcal{L}$, $p : |\mathcal{L}| \to Y$. Denoting by $t \in H^0(|\mathcal{L}|, p^* \mathcal{L})$ the tautological section, then the zero-locus of $p^* s - t^n$ defines a subvariety X of $|\mathcal{L}|$ such that the restriction $p_{|X} : X \to Y$ is a cyclic étale cover with the desired property. When the canonical line bundle of Y is torsion, as in the case of a bielliptic surface, then it is possible to execute the aforementioned construction with $\mathcal{L} \simeq \omega_X$. We, thus, obtain a cyclic étale cover $\pi_Y : X \to Y$, or simply $\pi : X \to S$, if there is no chance of confusion, called the *canonical cover of Y*.

Let now S be a bielliptic surface with canonical bundle of order n. If we let $\lambda_S := |G|/n$, we have that $G \simeq \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/\lambda_S\mathbb{Z}$. Denote by $H := \mathbb{Z}/\lambda_S\mathbb{Z}$, then the canonical cover X is the abelian surface sitting as an intermediate quotient

Thus, X comes with homomorphisms of abelian varieties $p_A : X \to A/H$ and $p_B : X \to B/H$ with kernels isomorphic to B and A, respectively. Denoting by a_X and b_X the classes of the fibers A and B in $\text{Num}(X)$, we have $a_X \cdot b_X = \lambda_S$ and the embedding $\pi^* \colon \text{Num}(S) \hookrightarrow \text{Num}(X)$ satisfies

$$\pi^* a_0 = a_X, \pi^* b_0 = \frac{n}{\lambda_S} b_X. \qquad (2.1)$$

There is a fixed-point-free action of the group $\mathbb{Z}/n\mathbb{Z}$ on the abelian variety X such that the quotient is exactly S. We will denote by $\sigma \in \text{Aut}(X)$ a generator of $\mathbb{Z}/n\mathbb{Z}$. In what follows, it will be useful to have an explicit description of σ when S is of type 1, 2, 3, or 5.

Suppose first that S is of type 1, 3, or 5, so G is cyclic, H is trivial, and $X \simeq A \times B$. If S is of type 3, then the j-invariant of B is 1728, and B

admits an automorphism $\omega : B \to B$ of order 4. If S is of type 5, B has j-invariant 0 and admits an automorphism ρ of order 3 (see for example [BM77, p. 37], [Bad01, List 10.27] or [BHPvdV15, p. 199]). With this notation, there are points in A τ, ϵ, of order 2, 4, and 3 respectively, such that the automorphism σ of $A \times B$ inducing the covering π is given by

$$\sigma(x,y) = \begin{cases} (x + \tau, -y), & \text{if } S \text{ is of type 1,} \\ (x + \epsilon, \omega(y)), & \text{if } S \text{ is of type 3,} \\ (x + \eta, \rho(y)), & \text{if } S \text{ is of type 5.} \end{cases} \qquad (2.2)$$

We remark that different choices for the automorphism ρ and ω - there are two possible choices in each case- will lead to isomorphic bielliptic surfaces.

If S is otherwise of type 2, then there are points $\theta_1 \in A$ and $\theta_2 \in B$, both of order two, such that X is the quotient of $A \times B$ by the involution $(x,y) \mapsto (x + \theta_1, y + \theta_2)$. If we denote by $[x,y]$ the image of (x,y) through the quotient map, we have that there is a point τ of order 2 in A, $\tau \neq \theta_1$ such that

$$\sigma[x,y] = [x + \tau, -y], \qquad (2.3)$$

where $\tau \in A$ is a point of order 2, $\tau \neq \theta_1$.

2.3 Covers of bielliptic surfaces by other bielliptic surfaces

When G is not a cyclic group, or when G is cyclic, but the order of G is not a prime number, then the bielliptic surface S admits a cyclic cover $\tilde{\pi} : \tilde{S} \to S$, where $\tilde{S}$ is another bielliptic surface. This construction, together with the statement of Lemma 2.3, appears explicitly in the work of Nuer [Nue], and is implicit in the work of Suwa [Suw69, p. 475]. The main point that we will need in Section 4 is the description of the pull-back map $\mathrm{Num}(S) \to \mathrm{Num}(\tilde{S})$.

Lemma 2.3. (i) Let S be a bielliptic surface such that $\mathrm{ord}(\omega_S)$ is not a prime number and take d a proper divisor of n. Then there is a bielliptic surface $\tilde{S}$ sitting as an intermediate étale cover between S and X,

$$X \xrightarrow{\pi_{\tilde{S}}} \tilde{S} \xrightarrow{\tilde{\pi}} S$$
$$\underbrace{\phantom{X \xrightarrow{\pi_{\tilde{S}}} \tilde{S} \xrightarrow{\tilde{\pi}}}}_{\pi_S}$$

such that $\mathrm{ord}\,(\omega_{\tilde{S}}) = \frac{\mathrm{ord}\,(\omega_S)}{d}$ and

$$\tilde{\pi}^* a_0 = \tilde{a_0}, \tilde{\pi}^* b_0 = d\tilde{b_0},$$

where $\tilde{a_0}, \tilde{b_0}$ are the natural generators of $\mathrm{Num}\,(\tilde{S})$.

(ii) Let S be a bielliptic surface with $\lambda_S > 1$, i.e., with G not cyclic. Then there is a bielliptic surface $\tilde{S}$ sitting as an intermediate étale cover between S and $A \times B$

$$A \times B \xrightarrow{\pi_{\tilde{S}}} \tilde{S} \xrightarrow{\tilde{\pi}} S$$
$$\underbrace{\phantom{A \times B \xrightarrow{\pi_{\tilde{S}}} \tilde{S} \xrightarrow{\tilde{\pi}}}}_{\pi_S}$$

such that $\lambda_{\tilde{S}} = 1$, $\mathrm{ord}(\omega_{\tilde{S}}) = \mathrm{ord}(\omega_S)$ and

$$\tilde{\pi}^* a_0 = \lambda_S \tilde{a_0}, \tilde{\pi}^* b_0 = \tilde{b_0},$$

where $\tilde{a_0}, \tilde{b_0}$ are the natural generators of $\mathrm{Num}(\tilde{S})$.

In what follows, we will need a more explicit construction of $\tilde{S}$, when S is either of type 2 or 3.

Example 2.4. (a) Suppose that S is a bielliptic surface of type 3. Then the canonical bundle has order 4. In addition, the canonical cover X of S is a product of elliptic curves, that is $X \simeq A \times B$. Using the notation of (2.2), we obtain $\tilde{S}$ from $A \times B$ by taking the quotient with respect to the involution $(x, y) \mapsto (x + 2\epsilon, -y)$. Thus, we have that $\tilde{S}$ is a bielliptic surface of type 1. The map $\tilde{\pi} : \tilde{S} \to S$ is an étale double cover with associated involution $\tilde{\sigma}$. Hence, given $s \in \tilde{S}$, we can see it as an equivalence class $[x, y]$ of a point $(x, y) \in A \times B$. Then we have an explicit expression for $\tilde{\sigma}$:

$$\tilde{\sigma}(s) = [x + \epsilon, \omega(y)]. \tag{2.4}$$

(b) Suppose that S is a bielliptic surface of type 2, so the group G is isomorphic to the product $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then we obtain $\tilde{S}$ from $A \times B$ by taking the quotient with respect to $(x, y) \mapsto (x + \tau, -y)$, where we are using the notation of (2.3). Thus, as before, $\tilde{S}$ is a bielliptic surface of type 1 and each $s \in \tilde{S}$ can be written as an equivalence class $[x, y]$ of a point $(x, y) \in A \times B$. If we denote again by $\tilde{\sigma}$ the involution induced by the cover $\tilde{\pi} : \tilde{S} \to S$, we have the following:

$$\tilde{\sigma}(s) = [x + \theta_1, y + \theta_2]. \tag{2.5}$$

2.4 Norm homomorphisms

Let $\pi : X \to Y$ be a finite locally free morphism of projective varieties of degree n. To it we can associate a group homomorphism $\mathrm{Nm}_\pi : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ called the *norm homomorphism associated to π*. This is constructed in the following manner. First, one lets $\mathscr{B} := \pi_* \mathcal{O}_X$, and defines a morphism of sheaves of multiplicative monoids $N : \mathscr{B} \to \mathcal{O}_Y$: given s a section of $\mathscr{B}$ on an open set U, let m_s be the endomorphism of $\mathscr{B}(U)$ induced by the multiplication by s; we set $N(s) := \det(m_s) \in \mathcal{O}_Y(U)$ (see [Gro61, § 6.4, and §6.5] or [Sta19, Lemma 0BD2]). The restriction of N to invertible sections induces a morphism of sheaves of groups $N : \mathscr{B}^* \to \mathcal{O}_Y^*$. Now, given L an invertible sheaf on X, $\pi_* L$ is an invertible $\mathscr{B}$-module and, as such is represented by a cocycle $\{u_{ij}, U_i\}$ for an open cover $\{U_i\}$ of Y. Observe that $u_{ij} \in \mathscr{B}^*(U_{ij})$. The fact that N is multiplicative ensures that also the $v_{ij} := N(u_{ij})$ satisfies the cocycle condition and so uniquely identifies a line bundle $\mathrm{Nm}_\pi(L)$ on Y. The map $L \mapsto \mathrm{Nm}_\pi(L)$ is a group homomorphism by [Gro61, (6.5.2.1)]. In addition [Gro61, (6.5.2.4)] ensures that

$$\mathrm{Nm}_\pi(\pi^* M) \simeq M^{\otimes n}, \tag{2.6}$$

and we also have the following important property:

Proposition 2.5. *Given two finite locally free morphism $\pi_1 : X \to Y$ and $\pi_2 : Y \to Z$, then*

$$\mathrm{Nm}_{\pi_2 \circ \pi_1} = \mathrm{Nm}_{\pi_2} \circ \mathrm{Nm}_{\pi_1} .$$

Proof. See [Gro67, Lemma 21.5.7.2]. □

Suppose now that $\pi : X \to Y$ is an étale cyclic cover of degree n. Then there is a fixed-point-free automorphism $\sigma : X \to X$ of order n such that $Y \simeq X/\sigma$. In addition we can write $\mathscr{B} \simeq \bigoplus_{h=0}^{n-1} M^{\otimes h}$ with M a line bundle of order n in $\mathrm{Pic}(Y)$. In this particular setting, the norm homomorphism satisfies some additional useful properties. First, as Nm_π behaves well with base change ([Gro61, Proposition 6.5.8]), it is not difficult to see that

$$\mathrm{Nm}_\pi \circ (1_X - \sigma^*) = 0. \tag{2.7}$$

In addition, as discussed by Beauville in [Bea09], we have that

$$\pi^* \mathrm{Nm}_\pi(L) \simeq \bigotimes_{h=0}^{n} (\sigma^h)^* L. \tag{2.8}$$

In fact, by the definiton of pushforward of divisors ([Gro67, Definition 21.5.5]), if $L \simeq \mathcal{O}_X(\sum a_i \cdot D_i)$ where the D_i's are prime divisors on X, then $\mathrm{Nm}_\pi(L) \simeq \mathcal{O}_Y(\sum a_i \cdot \pi_* D_i)$. Therefore, (2.8) follows from the fact that for a prime divisor D we have that $\pi^* \pi_* D \sim \sum_{h=0}^{n-1} (\sigma^h)^* D$.

Remark 2.6 (Pic^0 trick). In what follows, it will be important to provide elements in the kernel of the norm homomorphism. We will often use the following trick. Let $\pi : X \to Y$ be an étale morphism of degree n and suppose that there is a line bundle L on X such that $\mathrm{Nm}_\pi(L) \in \mathrm{Pic}^0(Y)$. Then there is an element $\alpha \in \mathrm{Pic}^0(X)$ such that $\mathrm{Nm}_\pi(L \otimes \alpha)$ is trivial. In fact, as abelian varieties are divisible groups, it is possible to find $\beta \in \mathrm{Pic}^0(Y)$ such that $\beta^{\otimes n} \simeq \mathrm{Nm}_\pi(L)^{-1}$. Then, by (2.6) we get

$$\mathrm{Nm}_\pi(L \otimes \pi^* \beta) \simeq \mathrm{Nm}_\pi(L) \otimes \beta^{\otimes n} \simeq \mathcal{O}_Y.$$

We conclude this paragraph by saying that, from now on, if there is no possibility of confusion, we will omit the subscript when denoting the norm. That is we will write Nm instead of Nm_π

2.5   Brauer groups and Brauer maps

For a scheme X, the *cohomological Brauer group* $\mathrm{Br}'(X)$ is defined as the torsion part of the étale cohomology group $H^2_{\mathrm{et}}(X, \mathcal{O}_X^*)$. For complex varieties, this is isomorphic to the torsion of $H^2(X, \mathcal{O}_X^*)$ in the analytic topology. In addition, when X is quasi-compact and separated, by a theorem of Gabber (see, for example, [dJ] for more details) the cohomological Brauer group of X is canonically isomorphic to the *Brauer group* $\mathrm{Br}(X)$ of Morita-equivalence

classes of Azumaya algebras on X. In this paper, we will only be concerned with smooth complex projective varieties, therefore all these three groups will be isomorphic and will be denoted simply by $\mathrm{Br}(X)$. Furthermore, we will only speak of the *Brauer group of X*, without any additional connotation.

If S is a bielliptic surface, the exponential sequence yields that $H^3(S, \mathbb{Z}) \simeq H^2(S, \mathcal{O}_S^*)$, so that the Brauer group of S is isomorphic to the torsion of $H^3(S, \mathbb{Z})$. By Poincaré duality and the universal coefficients theorem, the torsion of $H^3(S, \mathbb{Z})$ is (non canonically) isomorphic to the torsion of $H^2(S, \mathbb{Z})$, so the isomorphism type of the Brauer group of S can be deduced in terms of Proposition 2.2.

Crucial to our purposes will be the following result of Beauville which describes the kernel of the Brauer map π_{Br} when π is a cyclic étale cover.

PROPOSITION 2.7 ([Bea09, Prop. 4.1]). *Let $\pi\colon X \to S$ be an étale cyclic covering of smooth projective varieties. Let σ be a generator of the Galois group of π, $\mathrm{Nm}\colon \mathrm{Pic}(X) \to \mathrm{Pic}(S)$ be the norm map and $\pi_{\mathrm{Br}}\colon \mathrm{Br}(S) \to \mathrm{Br}(X)$ be the pullback. Then we have a canonical isomorphism*

$$\mathrm{Ker}(\pi_{\mathrm{Br}}) \simeq \mathrm{Ker}\,\mathrm{Nm}/(1 - \sigma^*)\,\mathrm{Pic}(X).$$

2.6   The Neron–Severi of a product of elliptic curves

In this paragraph we want to describe $\mathrm{Num}(A \times B)$ when A and B are two elliptic curves. We will do so by using the identification of $\mathrm{Num}(X) \simeq NS(X)$ which holds for abelian surfaces. We believe that many of these topics might be well known by experts, but we were not able to find a rigorous literature, thus we wrote this for the reader convenience. In the first part of this paragraph, we will follow closely the narrative of [HLT19].

Let A be an elliptic curve over $\mathbb{C}$ with identity element p_0, then there is a lattice Λ such that $A \simeq \mathbb{C}/\Lambda$. Identify A with its dual and consider $\mathscr{P}_A$ the *normalized Poincaré bundle* on $A \times A$:

$$\mathscr{P}_A \simeq \mathcal{O}_{A \times A}(\Delta_A) \otimes \mathrm{pr}_1^* \mathcal{O}_A(-p_0) \otimes \mathrm{pr}_2^* \mathcal{O}_A(-p_0),$$

where $\Delta_A \subset A \times A$ is the diagonal divisor and pr_1, pr_2 are the projections of $A \times A$ onto the first and second factor respectively. Observe that if x is a point in A, then the topologically trivial line bundle P_x is simply $\mathscr{P}_{A|A \times \{x\}} \simeq \mathscr{P}_{A|\{x\} \times A}$.

Take another elliptic curve B, and consider the product $A \times B$, with projection pr_A and pr_B onto A and B respectively. Given two line bundles L_A and L_B on A and B respectively, and a morphism $\varphi : B \to A$, we define a line bundle on the product $A \times B$

$$L(L_A, L_B, \varphi) := (1_A \times \varphi)^* \mathscr{P}_A \otimes \mathrm{pr}_A^* L_A \otimes \mathrm{pr}_B^* L_B. \tag{2.9}$$

As a direct consequence of the see-saw principle (see, for example [Mum70, Section 10]), it is possible to see that, if M_A and M_B are two other line bundles

on A and B, and $\psi : B \to A$ is another homomorphism, then

$$L(L_A \otimes M_A, L_B \otimes M_B, \varphi + \psi) \simeq L(L_A, L_B, \varphi) \otimes L(M_A, M_B, \psi).$$

In addition, the universal property of the dual abelian variety ensures that every line bundle $L \in \mathrm{Pic}(A \times B)$ is of the form $L(L_A, L_B, \varphi)$ for some invertible sheaves L_A and L_B and a morphism φ. Therefore, we have an isomorphism

$$\mathrm{Pic}(A \times B) \simeq \mathrm{Pic}(A) \times \mathrm{Pic}(B) \times \mathrm{Hom}(B, A).$$

We now quotient by numerically trivial line bundles and let $[A]$ and $[B]$ denote the numerical classes of the fibers of the two projections. We find that

$$H^2(A \times B, \mathbb{Z}) \simeq \mathrm{Num}(A \times B) \simeq \mathbb{Z} \cdot [B] \times \mathbb{Z} \cdot [A] \times \mathrm{Hom}(B, A). \qquad (2.10)$$

Let us denote by $l(\deg(L_A), \deg(L_B), \varphi)$ the first Chern class of $L(L_A, L_B, \varphi)$. Then, every class in $\mathrm{Num}(A \times B)$ can be written as $l(m, n, \varphi)$ for some integers n and m and an isogeny φ. In what follows, we will often refer to line bundles (or numerical classes) in $\mathrm{Hom}(B, A)$ as elements of the *Hom-part* of $\mathrm{Pic}(A \times B)$ (or of $\mathrm{Num}(A \times B)$). For our purposes, it will be really important to pick explicit generators for $\mathrm{Num}(A \times B)$ to see how the automorphism σ acts on $H^2(A \times B, \mathbb{Z})$. In order to do that, we need to investigate the $\mathbb{Z}$-module structure on $\mathrm{Hom}(B, A)$.

So, suppose that there is a nontrivial isogeny $\varphi : B \to A$. Then, we know that $\mathrm{Hom}(B, A)$ has rank 1, if A does not have complex multiplication, and 2 otherwise (more details about elliptic curves with complex multiplication can be found in the Appendix).

Suppose the first, so that there exists an isogeny $\psi : B \to A$ such that $l(0, 0, \psi)$ generates the Hom-part of $H^2(A \times B, \mathbb{Z})$. We will call such isogeny a *generating isogeny for* $\mathrm{Num}(A \times B)$. Observe that, since $l(0, 0, \psi)$ is necessarily a primitive class, ψ cannot factor through any "multiplication by n" map. That is, we cannot write $\psi = n \cdot \psi'$ for any n. In particular, for any integer n we have that $\mathrm{Ker}\,\psi$ does not contain $B[n]$ as a subscheme.

Suppose now that A has complex multiplication, and again fix a nontrivial isogeny $\varphi : B \to A$. Then also B has complex multiplication, and $\mathrm{Hom}(B, A)$ is a rank 2 free $\mathbb{Z}$-module. We pick generators ψ_1 and ψ_2, and we have that, for any line bundle L on $A \times B$, there are two unique integers h and k, and two line bundles M_A and M_B on A and B respectively, such that

$$L \simeq L(M_A, M_B, h \cdot \psi_1 + k \cdot \psi_2). \qquad (2.11)$$

In addition, we can write

$$H^2(A \times B, \mathbb{Z}) = \langle l(1, 0, 0), l(0, 1, 0), l(0, 0, \psi_1), l(0, 0, \psi_2) \rangle. \qquad (2.12)$$

In the particular cases in which the j-invariant of B is either 0 or 1728, then Theorem A.1 in the Appendix yields a more accurate description. In fact, if

we denote by $\lambda_B : B \to B$ the automorphism ρ or ω (see again the Appendix or Paragraph 2.2), we have that there exist an isogeny $\psi : B \to A$ such that, in (2.11) and (2.12) we can take $\psi_1 = \psi$ and $\psi_2 = \psi \circ \lambda_B$. So we have that

$$H^2(A \times B, \mathbb{Z}) = \langle l(1,0,0), l(0,1,0), l(0,0,\psi), l(0,0,\psi \circ \lambda_B) \rangle. \qquad (2.13)$$

In this case, we say that ψ is again a *generating isogeny* for $H^2(A \times B, \mathbb{Z})$. Observe again the isogenies ψ_i, as well as ψ, cannot factor through the multiplication by an integer or they could not generate the whole $\mathrm{Hom}(B, A)$.

3 Generators for the torsion of the second cohomology for bielliptic surfaces

In this section, we give explicit generators for the torsion of $H^2(S, \mathbb{Z})$ in terms of the reduced multiple fibers of the elliptic fibration $g : S \to \mathbb{P}^1$. More precisely we will prove the following statement:

PROPOSITION 3.1. *Let $S = A \times B/G$ be a bielliptic surface. Denote by D_i the reduced multiple fibers of $g : S \to \mathbb{P}^1$ with the same multiplicity. Then the torsion of $H^2(S, \mathbb{Z})$ is generated by the classes of differences $D_i - D_j$ for $i \neq j$.*

The reader who is familiar with the work of Serrano might find similarities between the above statement and Serrano's description of the torsion of $H^2(X, \mathbb{Z})$ when there is an elliptic fibration $\varphi : X \to C$ with multiple fibers (cfr. [Ser90b], Corollary 1.5 and Proposition 1.6]); however Serrano uses the additional assumption that $h^1(X, \mathcal{O}_X) = h^1(C, \mathcal{O}_C)$, which clearly does not hold in our context.

Before proving Proposition 3.1 we need two preliminary Lemmas.

LEMMA 3.2. *Let $g : S \to \mathbb{P}^1$ be a pencil with connected fibers. Let D_1 and D_2 be two distinct reduced multiple fibers. Let m_1 and m_2 be the corresponding multiplicities. Then, for all non negative integers n,*

$$D_1 \nsim nD_2. \qquad (3.1)$$

Proof. The statement is obvious for $n = 0$, so one has to check for $n > 0$. By contradiction, assume $D_1 \sim nD_2$, and let F be the generic fiber of g. Then

$$\begin{aligned}
h^0(S, \mathcal{O}_S(F)) &= h^0(\mathbb{P}^1, g_* \mathcal{O}_S(F)) \\
&= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \otimes g_* \mathcal{O}_S) \\
&= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 2.
\end{aligned}$$

Since $h^0(S, \mathcal{O}_S(D_1)) \leq h^0(S, \mathcal{O}_S(m_1 D_1)) = h^0(S, \mathcal{O}_S(F))$, it follows that $h^0(S, \mathcal{O}_S(D_1)) \leq 2$.
The absurd hypothesis is used here: if $D_1 \sim nD_2$, then, since the supports of D_1 and D_2 are disjoint, $H^0(S, \mathcal{O}_S(D_1))$ has at least two independent sections, and

therefore the dimension of $H^0(S, \mathcal{O}_S(D_1))$ is 2. Thus, since $D_1^2 = 0$ implies that the linear system $|D_1|$ has no basepoints (see for example [Bea96, II.5]), the map determined by $|D_1|$, $\varphi_{|D_1|} : S \longrightarrow \mathbb{P}^1$, is actually a morphism. Note that both D_1 and nD_2 are fibers of this morphism.
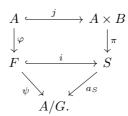
Let now C be the generic fiber of φ (which is irreducible by semicontinuity). Since $C \cdot D_1 = 0$, one gets $C \cdot F = 0$ for any fiber F of g. This implies that g and $\varphi_{|D_1|}$ have the same generic fiber. So one can write $C = F$ for a fiber F of g. But then

$$D_1 \sim F \sim m_1 D_1,$$

which in turn implies that $\mathcal{O}_S(D_1)^{\otimes(m_1-1)} \simeq \mathcal{O}_S$, which is a contradiction. $\square$

Lemma 3.3. *Let $S = A \times B/G$ be a bielliptic surface with its fibrations $a_S \colon S \to A/G$ and $g \colon S \to \mathbb{P}^1$. Let D_1 and D_2 be two reduced multiple fibers of g. Then the restriction of $\mathcal{O}_S(D_1 - D_2)$ to the generic fiber of a_S is trivial.*

Proof. Let $F = g^{-1}(p)$ be a smooth fiber of g. Here p is the orbit $G \cdot y$ of a point $y \in B$ not fixed under any element of G. Let i be natural inclusion of a general fiber F into S, and denote by $\pi : A \times B \to S$ the quotient map. We will choose an embedding of A into S via an isomorphism $\varphi \colon A \to F$ such that we get a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ j\ } & A \times B \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\pi} \\
F & \xrightarrow{\ i\ } & S \\
\end{array}
$$
$$
\begin{array}{c}
\quad\;\; {\scriptstyle\psi}\searrow \quad \swarrow {\scriptstyle a_S} \\
A/G.
\end{array}
$$

To this end we let $\varphi \colon A \to F$ be the isomorphism $x \mapsto G \cdot (x, y)$ and j be the embedding $x \mapsto (x, y)$. Let $y_i \in B$, for $i = 1, 2$, be points fixed under a subgroup of G of order m_i. Then the multiple fibers D_i are contained in $\pi(A \times \{y_i\})$, and have multiplicity m_i. Denote by p_B is the projection $A \times B \to B$ and $y_1, y_2 \in B$ are the points corresponding to D_1, D_2, respectively, we have that $\pi^* \mathcal{O}_S(D_1 - D_2) = p_B^* \mathcal{O}_B(y_1 - y_2)$. Then, we have

$$
\begin{aligned}
\varphi^* i^* \mathcal{O}_S(D_1 - D_2) &\simeq j^* \pi^* \mathcal{O}_S(D_1 - D_2) \\
&\simeq j^* p_B^* \mathcal{O}_B(y_1 - y_2).
\end{aligned}
$$

As $p_B \circ j$ is the constant map we have that this is clearly trivial. Hence, $\varphi^* i^* \mathcal{O}_S(D_1 - D_2)$ is trivial, and since φ is an isomorphism, we deduce the statement. $\square$

For the remainder, we identify F and A via the isomorphism φ defined in the

proof above. So we get the following commutative triangle.

$$\begin{array}{ccc} A & \xrightarrow{\quad i \quad} & S \\ & \psi \searrow \quad \swarrow a_S & \\ & A/G. & \end{array} \qquad (3.2)$$

Note that ψ is an isogeny of degree $|G|$. In particular we have also that the dual isogeny
$\psi^* : \mathrm{Pic}^0(S) \to \mathrm{Pic}^0(A)$ has degree $|G|$ (see, for example [BL13, Proposition 2.4.3]).
With these observations, we are now ready to start proving Proposition 3.1. Denote with D_k the multiple fibers of g, and let m_k be their multiplicity. We first remark that, by the canonical bundle formula for elliptic fibrations (see e.g. [Bad01, Thm. 7.15]) applied to $g : S \to \mathbb{P}^1$, we can write

$$\omega_S \simeq g^*\mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_S\Big(\sum_k (m_k - 1)D_k\Big).$$

Choosing points p, q on $\mathbb{P}^1$ giving rise to the fibers $m_i D_i$ and $m_j D_j$ we get that

$$K_S \sim -D_i - D_j + \sum_{k \neq i,j} (m_k - 1)D_k. \qquad (3.3)$$

Since ω_S is a nontrivial element in $\mathrm{Pic}^0(S)$ (see [BHPvdV15, p. 199]), we conclude that the classes of $D_i + D_j$ and $\sum_{k \neq i,j}(m_k-1)D_k$ coincide in $H^2(S, \mathbb{Z})$. Moreover, we observe that K_S restricts trivially to A, so ω_S yields a nontrivial element in $\mathrm{Ker}\,\psi^*$. Note that if D_i and D_j have the same multiplicity m, the difference $D_i - D_j$ induces a (possibly trivial) torsion element in $H^2(S, \mathbb{Z})$ of order m. We prove Proposition 3.1 by showing that a sufficient number of these is nontrivial so to generate the torsion of $H^2(S, \mathbb{Z})$. We proceed by a case by case analysis, studying separately bielliptic surfaces of type 1, 2, 3, and 5. The key point in the argument is the observation that, if $[D_i - D_j]$ is trivial, then the line bundle $\mathcal{O}_S(D_i - D_j)$ belongs to $\mathrm{Pic}^0(S)$. In addition, using Lemma 3.3 and the diagram (3.2), we would have that $\psi^*\mathcal{O}_S(D_i - D_j) \simeq \mathcal{O}_S$, in particular $\mathcal{O}_S(D_i - D_j) \in \mathrm{Ker}\,\psi^*$, while Lemma 3.2 ensures that $\mathcal{O}_S(D_i - D_j)$ cannot be $\mathcal{O}_S$. A closer study of the structure of $\mathrm{Ker}\,\psi^* \simeq \hat{G}$ will bring us to the desired conclusion.

3.1   Type 1 bielliptic surfaces

In this case, we have that $\mathrm{Ker}\,\psi^*$ is the reduced group scheme $\mathbb{Z}/2\mathbb{Z}$, and the fibration $g : S \to \mathbb{P}^1$ has four multiple fibers all of multiplicity 2. Hence, up to reordering the indices (3.3) yields

$$K_S \sim D_i - D_j + D_k - D_l. \qquad (3.4)$$

In particular, as the canonical divisor is algebraically equivalent to 0, for distinct indices i, j, k, and l we have that $D_j - D_i$ is algebraically equivalent to $D_k - D_l$. Thus, we get three classes in $H^2(S, \mathbb{Z})$

$$
\begin{aligned}
[D_1 - D_2] &= \{D_1 - D_2, \ D_3 - D_4\}, \\
[D_1 - D_3] &= \{D_1 - D_3, \ D_2 - D_4\}, \\
[D_1 - D_4] &= \{D_1 - D_4, \ D_2 - D_3\},
\end{aligned}
\tag{3.5}
$$

which a priori are neither distinct nor nontrivial. Since $H^2(S, \mathbb{Z})_{\mathrm{tors}}$ is isomorphic to the Klein 4-group, we need to show that they are indeed different classes and are not zero. Note that, if two classes are equal, since they both are 2-torsion and the third class is clearly equal to the sum of the first two, then the remaining class would be trivial. Thus, it will be enough to show that for any two distinct indices the divisor $D_i - D_j$ is not algebraically equivalent to 0. Suppose otherwise that for some indices we have that $\mathcal{O}_S(D_i - D_j) \in \mathrm{Pic}^0(S)$, then (3.4) would imply that also $\mathcal{O}_S(D_k - D_l)$ would be in $\mathrm{Pic}^0(S)$. The above discussion yields that both $\mathcal{O}_S(D_i - D_j)$ and $\mathcal{O}_S(D_i - D_j)$ are nontrivial elements of $\mathrm{Ker}\, \psi^*$, which has only one nontrivial element, ω_S. Then we can write

$$
\omega_S \simeq \mathcal{O}_S(D_i - D_j) \otimes \mathcal{O}_S(D_k - D_l) \simeq \omega_S^{\otimes 2} \simeq \mathcal{O}_S,
$$

which brings a contradiction, and thus we may conclude.

3.2   Type 2 bielliptic surfaces

Here $H^2(S, \mathbb{Z})_{\mathrm{tors}} \simeq \mathbb{Z}/2\mathbb{Z}$, $\mathrm{Ker}(\psi^*) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and like in the previous case there are four multiple fibers, each of multiplicity 2. As above we get the three classes induced by $D_1 - D_2$, $D_1 - D_3$ and $D_1 - D_4$, and we want to show that they cannot be all trivial. Suppose that two of these classes, say $[D_1 - D_2]$ and $[D_1 - D_3]$, are trivial in $H^2(S, \mathbb{Z})$. For $i = 2, 3$ set $L_i := \mathcal{O}_S(D_1 - D_i)$ and $M_i := \mathcal{O}_S(D_i - D_4)$, then the L_i's and the M_i's determine nontrivial elements of $\mathrm{Ker}\, \psi^*$, which has only three nonzero elements. We deduce that some of these must be the same line bundle. The only option which would not contradict Lemma 3.2 would be that $L_i \simeq M_j$ for some $i \neq j$. But then we would have

$$
\omega_S \simeq L_i \otimes M_j \simeq L_i^{\otimes 2} \simeq \mathcal{O}_S,
$$

which would be a contradiction. Hence at most one of the three classes can be trivial, and indeed one is actually trivial because the two nontrivial classes must coincide, implying the third is trivial.

3.3   Type 3 bielliptic surfaces

Here $H^2(S, \mathbb{Z})_{\mathrm{tors}} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Ker}(\psi^*) \simeq \mathbb{Z}/4\mathbb{Z}$, but now we have two fibers of multiplicity 4 and one of multiplicity 2. Denote by E the reduced multiple fiber

of multiplicity 2 and by D_1, D_2 the reduced multiple fibers of multiplicity 4. By the canonical bundle formula, we get

$$K_S \sim E - D_1 - D_2.$$

Then in $H^2(S, \mathbb{Z})$ we have the following equalities

$$[E - 2D_1] = [D_2 - D_1], \quad \text{and} \quad [E - 2D_2] = [D_1 - D_2].$$

We need to show that they are not both trivial. Suppose by contradiction they are both zero in $H^2(S, \mathbb{Z})$, then, as before we have that $\mathcal{O}_S(E - 2D_1)$ and $\mathcal{O}_S(E - 2D_2)$ are nontrivial elements of $\operatorname{Ker} \psi^*$. Since both these line bundles have order two in $\operatorname{Pic}(S)$, and $\operatorname{Ker} \psi^*$ has only one element of order 2, we deduce that

$$\mathcal{O}_S(E - 2D_1) \simeq \mathcal{O}_S(E - 2D_2).$$

But then

$$\begin{aligned}
\omega_S^{\otimes 2} &\simeq \mathcal{O}_S(E - D_1 - D_2)^{\otimes 2} \\
&\simeq \mathcal{O}_S(E - 2D_1) \otimes \mathcal{O}_S(E - 2D_2) \\
&\simeq \mathcal{O}_S(E - 2D_1)^{\otimes 2} \simeq \mathcal{O}_S,
\end{aligned}$$

which is impossible because ω_S is of order 4. Therefore $E - 2D_1$ and $E - 2D_2$ induce the same nontrivial torsion element of $H^2(S, \mathbb{Z})$.

3.4   Type 5 bielliptic surfaces

Here $H^2(S, \mathbb{Z})_{\text{tors}} \simeq \mathbb{Z}/3\mathbb{Z}$, $\operatorname{Ker}(\psi^*) \simeq \mathbb{Z}/3\mathbb{Z}$, and there are three multiple fibers, each of multiplicity 3. By the canonical bundle formula, we get

$$K_S \sim -D_i - D_j + 2D_k = (D_k - D_i) + (D_k - D_j).$$

Again, K_S is algebraically equivalent to zero, so we get that $[D_k - D_i] = [D_j - D_k]$ in $H^2(S, \mathbb{Z})$. Running through the indices we get the two classes

$$\begin{aligned}
[D_1 - D_2] &= \{D_1 - D_2,\ D_3 - D_1,\ D_2 - D_3\}, \\
[D_1 - D_3] &= \{D_1 - D_3,\ D_3 - D_2,\ D_2 - D_1\}.
\end{aligned}$$

We need to show that they are distinct and both nontrivial. Observe that if they were the same class then both classes would be trivial, so it is enough to show that they are not the zero class. Again, suppose by contradiction that $[D_k - D_i] = 0$ in $H^2(S, \mathbb{Z})$, then we can write

$$\omega_S \simeq \mathcal{O}_S(D_1 - D_2) \otimes \mathcal{O}_S(D_1 - D_3),$$

with $\mathcal{O}_S(D_1 - D_2)$ and $\mathcal{O}_S(D_1 - D_3)$ for nontrivial elements in $\operatorname{Ker}(\psi^*)$. Neither $\mathcal{O}_S(D_1 - D_2)$ nor $\mathcal{O}_S(D_1 - D_3)$ can be isomorphic to the canonical bundle ω_S,

or we would have $\mathcal{O}_S(D_k - D_i) \simeq \mathcal{O}_S$, contradicting Lemma 3.2. As $\operatorname{Ker} \psi^*$ has only two nontrivial elements, we necessarily have

$$\mathcal{O}_S(D_1 - D_2) \simeq \mathcal{O}_S(D_1 - D_3)$$

and so $\mathcal{O}_S(D_2 - D_3) \simeq \mathcal{O}_S$, which contradicts again Lemma 3.2, thus we can conclude.

4   The Brauer map to another bielliptic surface

Let S be a bielliptic surface of type 2 or 3. Then by Examples 2.4(a) and 2.4(b) there is a 2:1 cyclic cover $\tilde{\pi} : \tilde{S} \to S$, where $\tilde{S}$ is a bielliptic surface of type 1. As in paragraph 2.3 , we will denote by $\tilde{\sigma}$ the involution induced by $\tilde{\pi}$. In this section, we are concerned with studying the Brauer map $\tilde{\pi}_{\mathrm{Br}} : \operatorname{Br}(S) \to \operatorname{Br}(\tilde{S})$. Surpisingly we reach two antipodal conclusions, depending on the type of the bielliptic surface.

Recall that, as $\tilde{S}$ is a bielliptic surface of type 1, the elliptic fibration $q_B : \tilde{S} \to \mathbb{P}^1$ has four multiple fibers $D_1, \dots, D_4$ of multiplicity 2, corresponding to the four 2-torsion points of B. We will denote by τ_{ij} the line bundle $\mathcal{O}_{\tilde{S}}(D_i - D_j)$.

4.1   Bielliptic surfaces of type 2

Suppose that S is of type 2, and note that the involution $\tilde{\sigma}$ acts on the set of the D_i's by exchanging them pairwise. Up to relabeling we can assume that $\tilde{\sigma}^* D_1 \sim D_2$ and $\tilde{\sigma}^* D_3 \sim D_4$. By (2.8), we therefore have that

$$\tilde{\pi}^*(\operatorname{Nm}(\tau_{13})) \simeq \tau_{13} \otimes \sigma^* \tau_{13} \simeq \tau_{13} \otimes \tau_{24} \simeq \omega_{\tilde{S}}, \qquad (4.1)$$

where the last equality is a consequence of (3.4).Thus, if we denote by γ the generator of $\operatorname{Ker} \tilde{\pi}^*$, we get that

$$\operatorname{Nm}(\tau_{13}) \in \{\omega_S, \ \omega_S \otimes \gamma\} \subset \operatorname{Pic}^0(S).$$

Then we can use the Pic^0 trick (Remark 2.6), and find a $\beta \in \operatorname{Pic}^0(S)$ such that $\operatorname{Nm}(\tilde{\pi}^* \beta \otimes \tau_{13})$ is trivial.

LEMMA 4.1. *In the above notation, the line bundle $\tilde{\pi}^* \beta \otimes \tau_{13}$ does not belong to the image of $1 - \tilde{\sigma}^*$*

Before going forward with the proof, let us notice how, as an easy corollary, we get

COROLLARY 4.2. *If S is of type 2, then the induced map $\pi_{\mathrm{Br}} : \operatorname{Br}(S) \to \operatorname{Br}(\tilde{S})$ is trivial.*

Proof of Lemma 4.1. We will show that the class of τ_{13} in $H^2(\tilde{S}, \mathbb{Z})$ is not in the image of $1 - \tilde{\sigma}^*$. Denote by $[\tau_{ij}]$ the algebraic equivalence class of the line

bundle τ_{ij}. Then, by Proposition 2.2 and (3.5), for every L in $\mathrm{Pic}(\tilde{S})$ there are integers n, m, and h, and k such that

$$c_1(L) = \frac{n}{2} \cdot a + m \cdot b + h \cdot [\tau_{13}] + k \cdot [\tau_{14}].$$

Then it is easy to see that

$$(1 - \tilde{\sigma}^*)c_1(L) = 2h \cdot [\tau_{13}] + 2k \cdot [\tau_{14}] = 0.$$

But on the other side we have that $c_1(\tilde{\pi}^*\beta \otimes \tau_{13}) = [\tau_{13}]$ is not trivial, thus $\tilde{\pi}^*\beta \otimes \tau_{13}$ cannot possibly lie in the image of $(1 - \tilde{\sigma}^*)$, and the lemma is proved. $\square$

4.2   Bielliptic surface of type 3

In this paragraph we aim to show the following statement:

THEOREM 4.3. *If S is a bielliptic surface of type 3, then the Brauer map $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ induced by the cover $\tilde{\pi} : \tilde{S} \to S$, where $\tilde{S}$ is bielliptic of type 1, is injective.*

We will use 2.7 and show that $\mathrm{Ker}(\mathrm{Nm})/\mathrm{Im}(1 - \sigma^*)$ is trivial. There are two key steps:

1. We first study the norm map when applied to numerically trivial line bundles;

2. then we prove that all the line bundles L in $\mathrm{Ker}(\mathrm{Nm})$ are numerically trivial.

4.2.1   Norm of numerically trivial line bundles

We will use the notation of Example 2.4. Observe that we have the following diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\ \tilde{\pi}\ } & S \\ {\scriptstyle a_{\tilde{S}}}\big\downarrow & & \big\downarrow{\scriptstyle a_S} \\ A/G & \xrightarrow[\ \varphi\ ]{} & A/H, \end{array} \qquad (4.2)$$

where $G \simeq \mathbb{Z}/2\mathbb{Z}$, and H is $\mathbb{Z}/4\mathbb{Z}$.

Remark 4.4. Note that the bottom arrow, φ, is an isogeny of degree 2. As the vertical arrows are the Albanese maps of $\tilde{S}$ and S respectively, we have that $\tilde{\pi}^* : \mathrm{Pic}^0(S) \to \mathrm{Pic}^0(\tilde{S})$ coincides with the isogeny dual to φ. In particular it is surjective.

Our first step in the study of the norm homomorphism for numerically trivial line bundles is to see how it behaves when applied to the generator of the torsion of $H^2(\tilde{S}, \mathbb{Z})$. In order to do that, we remark that the automorphism ω

acts on $B[2]$ with at least one fixed point, the one corresponding to the identity element of B. Since ω has order 4, it cannot act transitively on the remaining three points on $B[2]$. Thus the action has at least two fixed points. We deduce that $\tilde{\sigma}$ acts on the set of the reduced multiple fibers by leaving fixed at least two of them, let us say D_1 and D_2. If the action were trivial, then we would have that all the line bundles τ_{ij} are invariant under the action of $\tilde{\sigma}$ and as a consequence they would be pullbacks of line bundles coming from S. We would deduce that all the torsion classes of $H^2(\tilde{S}, \mathbb{Z})$ are pullbacks of classes from $H^2(S, \mathbb{Z})$, which is impossible. Thus we know that $\tilde{\sigma}$ exchanges D_3 and D_4. Then we can prove the following Lemma.

Lemma 4.5. *Let n and m be two integers. Then the norm of the line bundle $\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}$ is zero if, and only if, n and m have the same parity. In addition, we have that $\mathrm{Nm}(\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m})$ is not in $\mathrm{Pic}^0(S)$ if n and m are not congruent modulo 2.*

Proof. Observe first of all that, thanks to the above discussion, the line bundle $\tau_{34} \simeq \tau_{13} \otimes \tau_{14}$ is invariant with respect to the action of $\tilde{\sigma}$. In particular we can write $\tau_{34} \simeq \tilde{\pi}^* \tau$ where τ is a line bundle on S whose algebraic equivalence class is the only nontrivial class in $H^2(S, \mathbb{Z})_{\mathrm{tors}}$.
Now, if n and m are both even, then $\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}$ is the trivial line bundle, and there is nothing to prove. Otherwise, if n and m are odd, then

$$\mathrm{Nm}(\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}) \simeq \mathrm{Nm}(\tau_{34}) \simeq \tau^{\otimes 2} \simeq \mathcal{O}_S.$$

Conversely, suppose that n and m are not congruent modulo 2. Up to exchanging n and m, we can assume that m is even, while n is odd. Then $\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m} \simeq \tau_{13}$. Again, by (2.8), we get

$$\tilde{\pi}^* \mathrm{Nm}(\tau_{13}) \simeq \tau_{13} \otimes \tilde{\sigma}^* \tau_{13} \simeq \tau_{34} \simeq \tilde{\pi}^* \tau.$$

We deduce that $\mathrm{Nm}(\tau_{13})$ is either equal to τ or to $\tau \otimes \omega_S^{\otimes 2}$. In any case it is not algebraically equivalent to zero and so the statement is proven. $\square$

Remark 4.6. (a) Observe that τ_{34} is in the image of $1 - \tilde{\sigma}^*$, as we have that $\tau_{34} \simeq \mathcal{O}_{\tilde{S}}(D_3) \otimes \tilde{\sigma}^* \mathcal{O}_{\tilde{S}}(-D_3)$.
(b) We will see in what follows that the different behavior of the norm map applied to torsion classes is what determines the contrast between the type 2 and type 3 bielliptic surfaces. In particular, the fact that the norm map of a torsion class is not necessarily algebraically trivial is what does not allow us to use Remark 2.6 in order to provide a nontrivial class in $\mathrm{Ker}(\mathrm{Nm})/\mathrm{Im}(1 - \tilde{\sigma}^*)$

Now we turn our attention to the elements of $\mathrm{Pic}^0(\tilde{S})$ whose norm is trivial. We will show that they never determine nonzero classes in $\mathrm{Ker}(\mathrm{Nm})/\mathrm{Im}(1 - \tilde{\sigma}^*)$.

Lemma 4.7. *Denote by $\mathrm{Nm} : \mathrm{Pic}(\tilde{S}) \to \mathrm{Pic}(S)$ the norm homomorphism. Let $L \in \mathrm{Pic}^0(\tilde{S})$, such that $\mathrm{Nm}(L) = \mathcal{O}_S$. Then the class of L in $\mathrm{Ker}(\mathrm{Nm})/\mathrm{Im}(1 - \sigma^*)$ is trivial.*

Proof. We have to show that such L is in the image of the morphism $1-\tilde{\sigma}^*$. By Remark 4.4, we can write $L \simeq \tilde{\pi}^* M$ with $M \in \mathrm{Pic}^0(S)$. Then our assumption warrants that

$$\mathcal{O}_S \simeq \mathrm{Nm}(L) \simeq M^{\otimes 2}.$$

We deduce that M is a 2-torsion point in $\mathrm{Pic}^0(S)$. Now we know that $\mathrm{Pic}^0(S)[2]$ is a group scheme isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let γ be the element $\omega_S^{\otimes 2} \in \mathrm{Pic}^0(S)[2]$ then we can find $\beta \in \mathrm{Pic}^0(S)[2]$, β nontrivial, such that

$$\mathrm{Pic}^0(S)[2] = \{\mathcal{O}_S, \gamma, \beta, \gamma \otimes \beta\}.$$

In particular, as $\tilde{\pi}^* \gamma \simeq \mathcal{O}_{\tilde{S}}$, we have

$$\mathrm{Ker}(\mathrm{Nm}) \cap \mathrm{Pic}^0(\tilde{S}) = \{\mathcal{O}_{\tilde{S}}, \tilde{\pi}^* \beta\}. \tag{4.3}$$

Now we aim at producing a line bundle $\alpha \in \mathrm{Pic}^0(\tilde{S}) \cap \mathrm{Im}(1-\tilde{\sigma}^*)$, $\alpha \not\simeq \mathcal{O}_{\tilde{S}}$. Thus we will have that $\mathrm{Pic}^0(\tilde{S}) \cap \mathrm{Im}(1-\tilde{\sigma}^*)$ is a nontrivial subgroup of $\mathrm{Ker}(\mathrm{Nm}) \cap \mathrm{Pic}^0(\tilde{S})$. From (4.3) we deduce that

$$\mathrm{Ker}(\mathrm{Nm}) \cap \mathrm{Pic}^0(\tilde{S}) = \mathrm{Pic}^0(\tilde{S}) \cap \mathrm{Im}(1-\tilde{\sigma}^*),$$

and so the statement.

To this aim, let $\overline{\epsilon} \in A' := A/G$ the image of the point $\epsilon \in A$ defining the involution $\tilde{\sigma}$ (see (2.4)). Denote also by p_0 the identity element of A'; observe that by the construction of bielliptic surfaces $\overline{\epsilon} \neq p_0$. Consider the following line bundle on $\tilde{S}$:

$$\alpha := a_{\tilde{S}}^*(\mathcal{O}_{A'}(p_0) \otimes t_{\overline{\epsilon}}^* \mathcal{O}_{A'}(-p_0)).$$

Clearly α is a nontrivial element in $\mathrm{Pic}^0(\tilde{S})$. In addition by (2.4) we see that

$$\alpha \simeq a_{\tilde{S}}^*(\mathcal{O}_{A'}(p_0)) \otimes \tilde{\sigma}^* a_{\tilde{S}}^*(\mathcal{O}_{A'}(-p_0)),$$

and therefore it is in the image of $1 - \tilde{\sigma}^*$. Thus we can conclude. $\qquad\square$

4.2.2 Injectivity of the Brauer map

We are now ready to prove Theorem 4.3. We will do so by showing the following statement.

PROPOSITION 4.8. *If $L \in \mathrm{Ker}(\mathrm{Nm})$ then L is numerically trivial.*

Before proceeding with the proof, let us show how this implies Theorem 4.3. Let L be a line bundle in the kernel of the norm map. Then Proposition 4.8 yields that

$$L \simeq \alpha \otimes \tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}$$

for some positive integers n and m, and for some $\alpha \in \mathrm{Pic}^0(\tilde{S})$. Write again $\alpha \simeq \tilde{\pi}^* \beta$, and observe that if n and m are not congruent modulo 2, then by Lemma 4.5,we get

$$\mathrm{Nm}(L) \simeq \beta^{\otimes 2} \otimes \tau,$$

which is not algebraically trivial. We deduce that n and m must have the same parity. Now we apply the first part of Lemma 4.5 and see that $\alpha \in \mathrm{Ker}(\mathrm{Nm})$. In particular, Lemma 4.7 implies that $\alpha \in \mathrm{Im}(1 - \tilde{\sigma}^*)$, and so the class of L in $\mathrm{Ker}(\mathrm{Nm})/\mathrm{Im}(1 - \tilde{\sigma}^*)$ is the same as the class of τ_{34}. But Remark 4.6(a) tells us that the latter is trivial and so Theorem 4.3 is proved.

Proof of Proposition 4.8. Let L be in the kernel of the norm map. Lemmas 2.3 and 2.3 imply that $\tilde{\pi}^* \mathrm{Num}(S)$ is a sublattice of index 2 of $\mathrm{Num}(\tilde{S})$. In particular, $L^{\otimes 2}$ is numerically equivalent to the pullback of a line bundle from S. Thus we can write

$$L^{\otimes 2} \simeq \tilde{\pi}^* M \otimes \alpha \otimes \tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}$$

for some positive integers n and m, and for some $\alpha \in \mathrm{Pic}^0(\tilde{S})$. Again, by Remark 4.4 we can write $\alpha \simeq \tilde{\pi}^* \beta$ for some $\beta \in \mathrm{Pic}^0(S)$, and so, up to substituting M with $M \otimes \beta$ we have that

$$L^{\otimes 2} \simeq \tilde{\pi}^* M \otimes \tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}.$$

If we show that M is numerically trivial, we can conclude. Observe that

$$\begin{aligned}
\mathcal{O}_S &\simeq \mathrm{Nm}(L) \otimes \mathrm{Nm}(L) \\
&\simeq \mathrm{Nm}(L^{\otimes 2}) \\
&\simeq M^{\otimes 2} \otimes \mathrm{Nm}(\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}) \\
&\simeq M^{\otimes 2} \otimes \tau^{\otimes(n+m)},
\end{aligned}$$

where the last equality is a consequence of Lemma 4.5. As τ is numerically trivial we conclude that the same is true for M. □

5   The Brauer map to the canonical cover

In this section we study the Brauer map $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$ when S is a bielliptic surface and X is its canonical cover. Then there is an n to 1 étale cyclic cover $\pi : X \to S$, where n denotes the order of the canonical bundle ω_S. Thus, as in the previous section, we can use Beauville's work [Bea09] to study the kernel of the π_{Br} via the norm homomorphism $\mathrm{Nm} : \mathrm{Pic}(X) \to \mathrm{Pic}(S)$. As in the other cases the Brauer group is trivial, we can assume that S is of type 1, 2, 3, or 5. Recall that, independently from the case at hand, there are two elliptic curves A and B such that X is isogenous to $A \times B$. In addition, for an abelian surface X, we have that $\mathrm{Num}(X)$ and $NS(X)$ coincide.

In what follows we will see that the geometry of the Brauer maps depends much on the geometry of $A \times B$, and in particular on which kind of isogenies there are between A and B. Throughout this section we will use the notation established in paragraph 2.2.

5.1   The norm of numerically trivial line bundles

Our first step will be proving the following proposition, which will allow us to study the norm map from a strictly numerical point of view.

PROPOSITION 5.1. *Let $L \in \mathrm{Pic}^0(X) \cap \mathrm{Ker}(\mathrm{Nm})$. Then L is in $\mathrm{Im}(1 - \sigma^*)$.*

Before going any further we need to describe more precisely our setting and introduce some notation.

Observe first that, if we let as in 2.2 $p_A : X \to A/H$ and $p_B : X \to B/H$ be the two elliptic fibrations of the abelian variety X, then $\mathrm{Pic}^0(X)$ is generated by $p_A^* \mathrm{Pic}^0(A/H)$ and $p_B^* \mathrm{Pic}^0(B/H)$; thus we can write any $L \in \mathrm{Pic}^0(X)$ as $p_A^*\alpha \otimes p_B^*\beta$, where $\alpha \in \mathrm{Pic}^0(A/H)$, $\beta \in \mathrm{Pic}^0(B/H)$. In this notation we have the following.

LEMMA 5.2. *For every $\beta \in \mathrm{Pic}^0(B/H)$ we have that $p_B^*\beta$ is in the image of $1 - \sigma^*$. In particular these line bundles are in the kernel of the norm homomorphism.*

Proof. We suppose first that G is cyclic and so the group H is trivial, and $X \simeq A \times B$. We proceed with a case by case analysis.

Type 1 case. Since abelian varieties are divisible groups, there exist $\gamma \in \mathrm{Pic}^0(B)$ such that $\gamma^{\otimes 2} \simeq \beta$. Then by (2.2) we have that

$$(1 - \sigma^*)p_B^*\gamma \simeq p_B^*\gamma \otimes (\sigma^* p_B^*\gamma)^{-1} \simeq p_B^* 2_B^* \gamma \simeq p_B^*\beta,$$

and the statement is proven in this case.

Type 3 case. In this case the j-invariant of B is 1728 and there is an automorphism ω of B of order 4. Consider the map $1 - \omega : B \to B$. Since this is not trivial it is an isogeny, and in particular $(1 - \omega)^* : \mathrm{Pic}^0(B) \to \mathrm{Pic}^0(B)$ is surjective. Let $\gamma \in \mathrm{Pic}^0(B)$ such that $(1 - \omega)^*\gamma \simeq \beta$, then by (2.2) we have

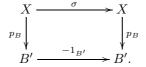$$(1 - \sigma^*)p_B^*\gamma \simeq p_B^*(1 - \omega)^*\gamma \simeq p_B^*\beta,$$

and the statement is proven in this case.

Type 5 case. This case is similar to the previous one in which instead of ω we use the automorphism ρ. We note that $(1 - \rho) : B \to B$ is nontrivial, and so an isogeny. In particular the dual map $(1 - \rho)^* : \mathrm{Pic}^0(B) \to \mathrm{Pic}^0(B)$ is surjective and we can find γ such that $(1 - \rho)^*\gamma \simeq \beta$. Again (2.2) yields:

$$(1 - \sigma^*)p_B^*\gamma \simeq p_B^*(1 - \rho)^*\gamma \simeq p_B^*\beta,$$

and the statement is proven.

Type 2 case. In this case, the group H is not trivial but it is cyclic of order 2. Let $B' := B/H$, and observe that we have the following diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;\sigma\;} & X \\
{\scriptstyle p_B}\big\downarrow & & \big\downarrow{\scriptstyle p_B} \\
B' & \xrightarrow{-1_{B'}} & B'.
\end{array}
$$

So let, as in the type 1 case, $\gamma \in \mathrm{Pic}^0(B')$ be such that $2^*_{B'}\gamma \simeq \beta$, then we will have again that $(1 - \sigma^*)p^*_B\gamma \simeq p^*_B\beta$ and the proof is concluded. $\square$

Proof of Proposition 5.1. Now let $L = p^*_A\alpha \otimes p^*_B\beta \in \mathrm{Pic}^0(X)$ such that $\mathrm{Nm}(L) \simeq \mathcal{O}_S$. Lemma 5.2 implies that also $p^*_A\alpha$ is in the kernel of the norm homomorphism. In addition, we have that the class of L in $\mathrm{Ker}(\mathrm{Nm})/\mathrm{Im}(1 - \sigma^*)$ is just the class of $p^*_A\alpha$. We have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \pi\ } & S \\
{\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle a_S} \\
A/H & \xrightarrow{\ \varphi\ } & A/G,
\end{array}
$$

where the bottom arrow is an isogeny of degree n. In particular, we can write $p^*_A\alpha \simeq \pi^*M$ with $M \in \mathrm{Pic}^0(S)$. Moreover, we have that

$$\mathcal{O}_S \simeq \mathrm{Nm}(p^*_A\alpha) \simeq M^{\otimes n},$$

thus we have that

$$p^*_A\left(\mathrm{Pic}^0(A/H)\right) \cap \mathrm{Ker}(\mathrm{Nm}) = \pi^*\left(\mathrm{Pic}^0(S)[n]\right).$$

It easy to see that the right-hand side above is a cyclic group of order n. Since $\mathrm{Im}(1 - \sigma^*)$ is a subgroup of the kernel of the norm, if we provide an element of order n in $p^*_A\left(\mathrm{Pic}^0(A/H)\right) \cap \mathrm{Im}(1 - \sigma^*)$, we would conclude that

$$p^*_A\left(\mathrm{Pic}^0(A/H)\right) \cap \mathrm{Im}(1 - \sigma^*) = p^*_A\left(\mathrm{Pic}^0(A/H)\right) \cap \mathrm{Ker}(\mathrm{Nm})$$

and consequently the statement of Proposition 5.1. Let p_0 be the identity element of A/H, using the notation of 2.2 and (2.3) we set

$$
\gamma := \begin{cases}
\mathcal{O}_A(p_0) \otimes t^*_\tau(\mathcal{O}_A(-p_0)), & \text{if } S \text{ is of type 1,} \\
\mathcal{O}_{A/H}(p_0) \otimes t^*_{\tau'}(\mathcal{O}_{A/H}(-p_0)), & \text{if } S \text{ is of type 2,} \\
\mathcal{O}_A(p_0) \otimes t^*_\epsilon(\mathcal{O}_A(-p_0)), & \text{if } S \text{ is of type 3,} \\
\mathcal{O}_A(p_0) \otimes t^*_\eta(\mathcal{O}_A(-p_0)), & \text{if } S \text{ is of type 5;}
\end{cases}
$$

where τ' is the image of τ under the isogeny $A \to A/H$. Then γ is a nontrivial element of $\mathrm{Pic}^0(A/H)$ with the desired property. In addition, by (2.2) (2.3), we have that $p^*_A\gamma \simeq (1 - \sigma^*)p^*_A\mathcal{O}_A(p_0)$, and so we can conclude. $\square$

Now we are ready to start our investigation of the Brauer map $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$. We first put ourselves in generic situation in which there are no nontrivial morphisms between A and B.

If there are no isogenies between A and B, then the lattice $\mathrm{Num}(X)$ has rank 2 and it is generated by the classes of the two fibers, a_X and b_X. In addition, $\pi^* \mathrm{Num}(S)$ is a sublattice of $\mathrm{Num}(X)$ of index n. So, let L be in the kernel of the norm map. We have that $L^{\otimes n}$ is numerically equivalent to the pullback of a line bundle from S. More precisely we can write

$$L^{\otimes n} \simeq \pi^* L' \otimes p_A^* \alpha \otimes p_B^* \beta \simeq \pi^* M \otimes p_B^* \beta,$$

with $\beta \in \mathrm{Pic}^0(B/H)$. Lemma 5.2 ensures that $\pi^* M$ is in the kernel of the norm map. In particular, M is an n-torsion element in $\mathrm{Pic}(S)$. We deduce that it is numerically trivial, and so L was numerically trivial to start with. Now we apply Proposition 5.1 and deduce the following statement.

THEOREM 5.3. *If $S := A \times B/G$ is a bielliptic surface such that the elliptic curves A and B are not isogenous, then the Brauer map to the canonical cover $\pi_{Br} : \mathrm{Br}(S) \to \mathrm{Br}(X)$ is injective.*

Before going to the next case, observe that if S is a bielliptic surface of type 2, then we have the following diagram

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\ \pi_{\tilde{S}}\ } & \tilde{S} \\
\varphi \downarrow & & \downarrow \tilde{\pi} \\
X & \xrightarrow[\ \pi_S\ ]{} & S.
\end{array}
$$

If A and B are not isogenous, Theorem 5.3) above implies that the Brauer map induced by π_S is injective. On the other side, Corollary 4.2 implies that the Brauer map induced by $\pi_S \circ \varphi$ is trivial. Then the Brauer map induced by φ cannot be injective and we have

COROLLARY 5.4. *If $\varphi : X \to Y$ is an isogeny of abelian varieties, the map $\varphi_{\mathrm{Br}} : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is not necessarily injective.*

5.3   The Brauer map when the two elliptic curves are isogenous

Suppose now that A and B are isogenous. Our first step will be to use the description of the Picard group and of the Néron–Severi of $A \times B$, that we outlined in 2.6, in order to find the image of $1 - \sigma^*$, and the numerical type of line bundles in the kernel of the norm homomorphism when S is a cyclic bielliptic surface. We begin with the following Lemma.

LEMMA 5.5. *Suppose that G is a cyclic group, so that $X \simeq A \times B$. If $L \in \mathrm{Pic}(A \times B)$ is in the kernel of the norm map, then $c_1(L) = l(0, 0, \varphi)$ for some isogeny $\varphi : B \to A$.*

Proof. By the result of 2.6, we have that $c_1(L) = l(m, n, \varphi)$ for two integers n and m and an isogeny φ. Suppose that S is of type 1 and L is in the kernel of the norm map. By (2.8), we have that $L \otimes \sigma^* L$ is trivial. In particular, $c_1(L \otimes \sigma^* L)$ is zero. But then we get the following

$$
\begin{aligned}
0 = c_1(L \otimes \sigma^* L) \\
= c_1(L) + \sigma^* c_1(L) \\
= l(m, n, \varphi) + l(m, n, -\varphi) = l(2m, 2n, 0).
\end{aligned}
$$

We conclude that $n = m = 0$. At the same time, if S is of type 5, we have that

$$
0 = c_1(L) + \sigma^* c_1(L) + (\sigma^2)^* c_1(L) = l(3m, 3n, 0).
$$

Finally, if S is of type 3 we get

$$
0 = c_1(L) + \sigma^* c_1(L) + (\sigma^2)^* c_1(L) + (\sigma^3)^* c_1(L) = l(4m, 4n, 0),
$$

so the statement is proven. $\square$

We turn now our attention to the Brauer map in general, and we study it by performing a case by case analysis on the different type of bielliptic surfaces.

5.3.1   Bielliptic surfaces of type 1

In this paragraph, we study the Brauer map to the canonical cover of bielliptic surfaces of type 1. If B does not have complex multiplication, we fix, once and for all, $\psi : B \to A$ be a generating isogeny. Otherwise we fix $\psi_i : B \to A$, for $i = 1, 2$ two generators of $\mathrm{Hom}(B, A)$. Our first step is to describe $(1 - \sigma^*) \mathrm{Pic}(A \times B)$.

LEMMA 5.6. *Let S be a bielliptic surface of type 1, and consider $L \in (1 - \sigma^*) \mathrm{Pic}(A \times B)$, then there exist three integers m, h and k, and a line bundle $\beta \in \mathrm{Pic}^0(B)$ such that*

$$
L \simeq \begin{cases} L(P_\tau^{\otimes m}, \beta, 2h \cdot \psi_1 + 2k \cdot \psi_2) & \text{if } B \text{ has complex multiplication (CM).} \\ L(P_\tau^{\otimes m}, \beta, 2h \cdot \psi) & \text{if } B \text{ does not have CM;} \end{cases}
$$

Proof. We do the complex multiplication case, the other is similar. Let $M \in \mathrm{Pic}(A \times B)$, then by the results of 2.6 we have that $M \simeq L(M_A, M_B, h \cdot \psi_1 + k \cdot \psi_2)$. We can write $M_A \simeq \mathcal{O}_A(n \cdot p_0) \otimes \alpha$ and $M_B \simeq \mathcal{O}_B(m \cdot q_0) \otimes \gamma$ for q_0 the identity element of B, some integers n and m and some topologically trivial line bundles α and γ. We recall that, by the Theorem of the Square, topologically trivial line bundles on abelian varieties are translation invariant (see for example [Mum70, p. 74]. Thus we can write

$$
\begin{aligned}
\sigma^* M \simeq L\,(t_\tau^* \mathcal{O}_A(n \cdot p_0) \otimes \alpha, \\
\mathcal{O}_B(m \cdot q_0) \otimes \gamma^{-1} \otimes (-h \cdot \psi_1 - k \cdot \psi_2)^* P_\tau, -h \cdot \psi_1 - k \cdot \psi_2)\,.
\end{aligned}
$$

Observe that, as γ ranges in all $\mathrm{Pic}^0(B)$, also $\beta := \gamma^{\otimes 2} \otimes (h \cdot \psi_1 + k \cdot \psi_2)^* P_\tau$ ranges in the whole $\mathrm{Pic}^0(B)$. In addition, we have that

$$(1 - \sigma^*)M \simeq L\left(P_\tau^{\otimes n}, \beta, 2h \cdot \psi_1 + 2k \cdot \psi_2\right).$$

$\square$

Remark 5.7. It is not difficult to check that, for any two integers h and k,

$$L(0, 0, 2h \cdot \psi_1 + 2k \cdot \psi_2) = L(0, 0, h \cdot \psi_1 + k \cdot \psi_2) \otimes \sigma^* L(0, 0, h \cdot \psi_1 + k \cdot \psi_2)^{-1},$$

and so it is in $\mathrm{Im}(1 - \sigma^*)$.

We are now ready to prove one of the main statements of this section:

THEOREM 5.8. *Suppose that S is a bielliptic surface of type 1 whose canonical cover is $A \times B$ with A and B isogenous elliptic curves. Then the Brauer map to the canonical cover of S is not injective if, and only if, one of the following mutually exclusive conditions is satisfied:*

1. *the elliptic curve B (and so A) does not have complex multiplication and $\psi^* P_\tau$ is trivial;*

2. *the elliptic curve B (and so A) has complex multiplication and we have that at least one of the following line bundles is trivial*

$$L_1 := \psi_1^* P_\tau, \quad L_2 := \psi_2^* P_\tau, \quad L_3 := (\psi_1 + \psi_2)^* P_\tau. \tag{5.1}$$

Proof. We deal with the complex multiplication case that is slightly more involved. The argument for the other case is very similar.

Before explaining the details of our reasoning we would like to give, for the reader convenience, a quick outline of the proof. The key observation is that the assumption on the line bundles (5.1) are equivalent to the norm of one of the following invertible sheaves to lies in $\mathrm{Pic}^0(S)$.

$$M_1 := (1 \times \psi_1)^* \mathscr{P}_A, \ M_2 := (1 \times \psi_2)^* \mathscr{P}_A, \ M_3 := (1 \times (\psi_1 + \psi_2))^* \mathscr{P}_A. \tag{5.2}$$

Therefore, if the assumptions are verified, we can use the Pic^0 trick (Remark 2.6) to produce an element in the kernel of the norm map. Such an element will give, by construction, a nontrivial class in $\mathrm{Ker\,Nm}\,/\,\mathrm{Im}(1 - \sigma^*)$. Conversely, if none of the line bundles is trivial, then an element in the kernel of the norm map will be forced to be numerically equivalent to $(1 \times 2 \cdot \varphi)^* \mathscr{P}_A$ for some isogeny $\varphi \in \mathrm{Hom}(B, A)$. Then we will apply Lemma 5.6 and see that such a line bundle lies in $\mathrm{Im}(1 - \sigma^*)$, so no element of $\mathrm{Pic}(A \times B)$ yields a nontrivial class in $\mathrm{Ker\,Nm}\,/\,\mathrm{Im}(1 - \sigma^*)$.

Now, for the complete argument, observe first that by (2.8) and the see-saw principle, it is easy to check that, for every α in $\mathrm{Pic}^0(A)$ and every isogeny

$\varphi : B \to A$,

$$\begin{aligned}
\pi^* \operatorname{Nm}((1 \times \varphi)^* \mathscr{P}_A \otimes p_A^* \alpha) &\simeq (1 \times \varphi)^* \mathscr{P}_A \otimes (1 \times \varphi)^* \mathscr{P}_A^{-1} \otimes \\
&\quad \otimes p_A^* \alpha^{\otimes 2} \otimes p_B^* \varphi^* P_\tau \\
&\simeq p_A^* \alpha^{\otimes 2} \otimes p_B^* \varphi^* P_\tau.
\end{aligned} \tag{5.3}$$

Suppose first that one of the three line bundles in (5.1) is trivial. To fix the ideas we can assume that $\psi_1^* P_\tau$ is trivial, the argument is identical in the other cases. Then, by (5.3), we have that $\operatorname{Nm}((1 \times \psi_1)^* \mathscr{P}_A)$ is in the kernel of π^*, so in particular it is in $\operatorname{Pic}^0(S)$. We can therefore apply Pic^0 trick (Remark 2.6), and find $\gamma \in \operatorname{Pic}^0(S)$ such that the norm of $(1 \times \psi_1)^* \mathscr{P}_A \otimes \pi^* \gamma$ is trivial. But, by Lemma 5.6, we have that $(1 \times \psi_1)^* \mathscr{P}_A \otimes \pi^* \gamma$ is not in the image of $1 - \sigma^*$ and so it defines a nontrivial class in $\operatorname{Ker} \operatorname{Nm} / \operatorname{Im}(1 - \sigma^*)$, and one direction of the statement is proven.

Conversely, suppose that there is a line bundle L on X which identifies a nontrivial class in $\operatorname{Ker} \operatorname{Nm} / \operatorname{Im}(1 - \sigma^*)$. By Lemmas 5.5 and 5.6, we can write

$$L \simeq (1 \times h \cdot \psi_1 + k \cdot \psi_2)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes p_B^* \beta,$$

for two integers h and k, and two topologically trivial line bundles α and β. Note that h and k cannot be both even, for otherwise Lemma 5.6 and Remark 5.7 yield that $[L] = [p_A^* \alpha \otimes p_B^* \beta] \in \operatorname{Ker} \operatorname{Nm} / \operatorname{Im}(1 - \sigma^*)$ which, by Proposition 5.1, implies that $[L] = 0$. Thus we can assume that one between h and k is odd. Then, by Lemma 5.2 and Lemma 5.6, we have that

$$L \simeq (1 \times \psi_i)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes M, \quad \text{or} \quad L \simeq (1 \times \psi_1 + \psi_2)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes M,$$

with M in $\operatorname{Im}(1 - \sigma^*)$. In particular, a twist of a line bundle in 5.2 has trivial norm. We deduce, by (5.3), that one of the line bundles in (5.1) is trivial and the statement is proved.   □

Example 5.9.   (a) Suppose that $A \simeq B$. If A does not have complex multiplication, then we can take $\psi = \pm 1_A$. In particular we have that $\psi^* P_\tau$ is never trivial and the Brauer map is injective.

(b) Suppose again that $A \simeq B$, and that the j-invariant of A is 1728. Then $\operatorname{End}(A) \simeq \mathbb{Z}[i]$ and the multiplication by i induces an automorphism ω of A of order 4, and we can take 1_A and ω as generators of $\operatorname{End}(A)$. Suppose that P_τ is a fixed point[†] of the dual automorphism ω^*. Then $(1_A + \omega)^* P_\tau$ is trivial, and the Brauer map is not injective.

(c) We can also use a similar argument to construct uncountably many type 1 bielliptic surfaces with non injective Brauer map. Let B any elliptic curve without complex multiplication, and chose θ a point of order 2 on B. Let

---

[†]For example we can identify A with its dual and ω^* with ω and take $\tau = (\frac{1}{2}, \frac{1}{2}) + \Lambda$, where $\Lambda = <1, i> A \simeq \mathbb{C}/\Lambda$.

$A := B/<\theta>$ and $\psi : B \to A$ the quotient map. This is a degree 2 isogeny, so it is primitive, and hence generating. If τ denotes the only point of order 2 in $\operatorname{Ker}\psi^*$, then we have that the data A, τ, B uniquely identify a type 1 bielliptic surface which has a non injective Brauer map.

In order to complete our description of the Brauer map for type 1 bielliptic surfaces, we need to give necessary and sufficient conditions for it to be trivial. To this aim, we want to provide two distinct non-zero classes in $\operatorname{Ker}\operatorname{Nm}/\operatorname{Im}(1-\sigma^*)$. We can assume that the Brauer map is already non-injective, and so one of the condition of Theorem 5.8 is satisfied. Suppose first that B does not have complex multiplication, and consider L in the kernel of the norm map, yielding a nontrivial class in $\operatorname{Ker}\operatorname{Nm}/\operatorname{Im}(1-\sigma^*)$. Then, as before, we have that

$$L \simeq (1 \times h \cdot \psi)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes p_B^* \beta.$$

Again, by Lemma 5.6, we can assume that h is odd, and the same result also yields that in $\operatorname{Ker}\operatorname{Nm}/\operatorname{Im}(1-\sigma^*)$ the class of L and that of $(1 \times \psi)^* \mathscr{P}_A \otimes p_A^* \alpha$ are the same. Since $\psi^* P_\tau$ is trivial, (5.3) implies that $(1 \times \psi)^* \mathscr{P}_A \otimes p_A^* \gamma$ is in the kernel of the norm map for some $\gamma \in \operatorname{Pic}^0(A)$. So $\operatorname{Nm}(p_A^*(\alpha \otimes \gamma^{-1})) \simeq \mathcal{O}_S$ and as before $p_A^*(\alpha \otimes \gamma^{-1})$ lies in the image of $(1-\sigma^*)$. We deduce that, in $\operatorname{Ker}\operatorname{Nm}/\operatorname{Im}(1-\sigma^*)$,

$$[L] = [(1 \times \psi)^* \mathscr{P}_A \otimes p_A^* \gamma] = [(1 \times \psi)^* \mathscr{P}_A \otimes p_A^* \delta],$$

for every $\delta \in \operatorname{Pic}^0(A)$ such that $(1 \times \psi)^* \mathscr{P}_A \otimes p_A^* \delta$ is in the kernel of the norm homomorphism. In particular there is only one non-trivial element in $\operatorname{Ker}\pi_{\mathrm{Br}}$. Thus we can assume that B has complex multiplication and that, as before, we have fixed ψ_1 and ψ_2 a system of generators for $\operatorname{Hom}(A,B)$. Suppose that only one among the line bundles (5.1) is trivial, for example L_1. As usual, we can take L in the kernel of the norm map, and we can write $L \simeq M_i \otimes p_A^* \alpha \otimes M$, with M in the image of $(1-\sigma^*)$, and M_i one of the line bundles appearing in (5.2). We deduce that $i = 1$, and that the class of L in $\operatorname{Ker}\operatorname{Nm}/\operatorname{Im}(1-\sigma^*)$ is equal to the class of $M_1 \otimes p_A^* \gamma$ for every $\gamma \in \operatorname{Pic}^0(A)$ such that $\operatorname{Nm}(M_1 \otimes p_A^* \gamma)$ is trivial. Thus, there is just one non-zero class, and the Brauer map is again nontrivial. Finally suppose that two (and so all) line bundles in (5.1) are trivial. We have that both M_1 and M_2 are in the kernel of the norm map. In addition,

$$M_1 \otimes M_2^{-1} \simeq (1 \times (\psi_1 - \psi_2))^* \mathscr{P}_A,$$

which by Lemma 5.6 is not in the image of $(1-\sigma^*)$. Therefore, we deduce that they determine two different classes in $\operatorname{Ker}\operatorname{Nm}/\operatorname{Im}(1-\sigma^*)$, and hence the Brauer map is trivial. We have thus proven the following statement:

THEOREM 5.10. *The Brauer map to the canonical cover of a type 1 bielliptic surface is trivial if, and only if, the elliptic curves A and B are isogenous, B has complex multiplication, and all the line bundles in (5.1) are trivial.*

Example 5.11.   (a) If $A \simeq B$ then the Brauer map is never trivial. Suppose otherwise that there are ψ_1 and ψ_2 generators of $\mathrm{End}(A)$ such that both $\psi_1^* P_\tau$ and $\psi_2^* P_\tau$ are zero. Then we can write $1_A = h \cdot \psi_1 + k \cdot \psi_2$ and we would get that $P_\tau \simeq 1_A^* P_\tau$ is trivial, reaching an obvious contradiction.

(b) Let now $A \simeq \mathbb{C}/\mathbb{Z}[2i]$ and let τ the point $(0, i) + \mathbb{Z}[2i]$. The elliptic curve $B := A/<\tau>$ has j-invariant 1728 and $\mathrm{Hom}(B, A)$ is generated by the isogenies $\psi_1 := \varphi_2$ and $\psi_2 := \varphi_2 \circ \lambda_B$, where $\varphi_2 : B \to A$ denotes the isogeny induced by multiplication by 2 (see Example A.8 in the Appendix). Observe that

$$\varphi_2^*(P_\tau) \simeq \varphi_2^*(\mathcal{O}_A(\tau - p_0)) \simeq \mathcal{O}_A(\varphi_2(\tau) - \varphi(p_0)) \simeq \mathcal{O}_B.$$

Thus we have that $\psi_1^* P_\tau \simeq \psi_2^* P_\tau \simeq \mathcal{O}_B$ and the Brauer map is trivial.

5.3.2 Bielliptic surfaces of type 3

Let now S be a bielliptic surface of type 3. Then the canonical cover of S is isomorphic to $A \times B$ with $j(B) = 1728$ and multiplication by i induces and automorphism ω of B of order 4, ω. By the discussion in 2.6, it is possible to find a generating isogeny ψ such that

$$\mathrm{Num}(X) = \langle l(1,0,0),\ l(0,1,0)\ l(0,0,\psi),\ l(0,0,\psi \circ \omega) \rangle,$$

We fix, once and for all, such a ψ and prove the following Lemma, which yields a precise description of $(1 - \sigma^*)\,\mathrm{Pic}(X)$.

Lemma 5.12. *Let $\varphi : B \to A$ be an isogeny, and let h and k be the two unique integers such that $\varphi = h \cdot \psi + k \cdot \psi \circ \omega$. Then the line bundle $(1 \times \varphi)^* \mathscr{P}_A \in \mathrm{Im}(1 - \sigma^*)$ if and only if $h + k$ is even.*

Proof. Let $T : \mathrm{Hom}(B, A) \to \mathrm{Hom}(B, A)$ be the linear operator obtained by pre-composing with $(1_B - \omega)$. Then, using that $\omega^2 = -1_B$, it is not difficult to show that an isogeny φ as in the statement is in the image of T if, and only if, $h + k$ is even. Hence if $h + k$ is not even, $(1 \times \varphi)^* \mathscr{P}_A \notin \mathrm{Im}(1 - \sigma^*)$.
Suppose now that $\varphi = h \cdot \psi + k \cdot \psi \circ \omega$ with $h + k$ an even number. Then, by the above argument, we can find an isogeny $\gamma : B \to A$ such that $\varphi = \gamma \circ (1_B - \omega)$. Then we have

$$\begin{aligned}
(1 - \sigma^*)(1 \times \gamma)^* \mathscr{P}_A &\simeq (1 \times \gamma \circ (1_B - \omega))^* \mathscr{P}_A \otimes p_B^* \omega^* \gamma^* P_\epsilon^{-1} \\
&\simeq (1 \times \varphi)^* \mathscr{P}_A \otimes p_B^* \omega^* \gamma^* P_\epsilon^{-1}.
\end{aligned}$$

By Lemma 5.2, elements of the form $p_B^* \beta$ with $\beta \in \mathrm{Pic}^0(B)$ are in the image of $(1 - \sigma^*)$, so we conclude that $(1 \times \varphi)^* \mathscr{P}_A \in \mathrm{Im}(1 - \sigma^*)$. $\square$

Remark 5.13. Observe that this Lemma implies easily that the quotient $\mathrm{Hom}(B, A)/\mathrm{Im}(1 - \sigma^*)$, where we are identifying $\mathrm{Hom}(B, A)$ with the corresponding subgroup of $\mathrm{Num}(A \times B)$, is cyclic generated by the coset $(1_A \times \psi)^* \mathscr{P}_A + \mathrm{Im}(1 - \sigma^*)$.

Now we are ready to start studying the kernel for the Brauer map $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$. Our main result is the following

THEOREM 5.14. *Let S is a bielliptic surface of type 3 with canonical cover $A \times B$ such that A and B are isogenous. Then the Brauer map to the canonical cover is identically zero if, and only if, $(1_B + \omega)^* \psi^* P_{2\epsilon}$ is trivial*

Proof. For any isogeny $\varphi : B \to A$, $\alpha \in \mathrm{Pic}^0(A)$ and $\beta \in \mathrm{Pic}^0(B)$, using that the norm of $p_B^* \beta$ is trivial by Lemma 5.2, we have that

$$\pi^* \mathrm{Nm}((1 \times \varphi)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes p_B^* \beta) \simeq$$
$$(1 \times \varphi)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes$$
$$(1 \times \varphi \circ \omega)^* \mathscr{P}_A \otimes p_B^* \omega^* \varphi^* P_\epsilon \otimes p_A^* \alpha \otimes$$
$$(1 \times -\varphi)^* \mathscr{P}_A \otimes p_B^* (-1_B)^* \varphi^* P_{2\epsilon} \otimes p_A^* \alpha \otimes$$
$$(1 \times -\varphi \circ \omega)^* \mathscr{P}_A \otimes p_B^* (-\omega)^* \varphi^* P_{3\epsilon} \otimes p_A^* \alpha \otimes$$
$$\simeq p_A^* \alpha^{\otimes 4} \otimes p_B^* (1 + \omega)^* \varphi^* P_{2\epsilon}.$$
$$(5.4)$$

Suppose that $(1_B + \omega)^* \psi^* P_{2\epsilon} \simeq \mathcal{O}_B$. Since $P_{2\epsilon}$ is a two torsion point, this is equivalent to asking that $(1_B - \omega)^* \psi^* P_{2\epsilon}$ is also trivial. Then (5.4) implies that the norms of $(1 \times \psi)^* \mathscr{P}_A$ and of $(1 \times \psi \circ \omega)^* \mathscr{P}_A$ lie in $\mathrm{Pic}^0(S)$. Then, using the Pic^0-trick (Remark 2.6) and Lemma 5.12, we can find a non zero class in $\mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1 - \sigma^*)$, and the Brauer map is trivial.

Conversely, let L be a line bundle defining a nontrivial class in $\mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1 - \sigma^*)$. Then as we did in the case of type 1 surfaces, we can write

$$L \simeq (1 \times h \cdot \psi + k \cdot \psi \circ \omega)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes p_B^* \beta,$$

with α and β in $\mathrm{Pic}^0(A)$ and $\mathrm{Pic}^0(B)$. Lemma 5.12 implies that the integer $h + k$ is odd, or we would have that $p_A^* \alpha$ is in the kernel of the norm map, and consequently, by Proposition 5.1, $L \in \mathrm{Im}(1 - \sigma^*)$. Thus, we can write

$$L \simeq M \otimes M',$$

where M' is in the image of $1 - \sigma^*$, and M is numerically equivalent to $(1 \times \psi)^* \mathscr{P}_A$ (this is a consequence of Lemma 5.12 and Remark 5.13). We deduce that M is in the kernel of the norm map. But then (5.4) implies that $(1 + \omega)^* \psi^* P_{2\epsilon}$ is trivial, proving the statement. $\square$

Example 5.15. Suppose that $A \simeq B$, so we can take $\psi = 1_A$. If $P_{2\epsilon}$ is a fixed point of ω, then we have that $\mathscr{P}_A$ yields a nontrivial element in $\mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1 - \sigma^*)$. Conversely, if $P_{2\epsilon}$ is not a fixed point of ω we will have that the Brauer map is injective.

5.3.3   Bielliptic surfaces of type 5

Let S be a bielliptic surface of type 5. We will solve this case in a similar fashion as for bielliptic surfaces of type 3. In the type 5 case, the canonical

cover is isomorphic to an abelian surface $A \times B$ with $j(B) = 0$. As already seen, B admits an automorphism ρ of order 3 such that $\rho^2 + \rho + 1 = 0$. Again, thanks to Theorem 5.3, we need to study only the case in which A and B are isogenous. Also in this case, by the results of 2.6, there is generating isogeny $\psi : B \to A$ such that

$$\operatorname{Num}(X) = \langle l(1,0,0),\ l(0,1,0)\ l(0,0,\psi),\ l(0,0,\psi \circ \rho)\rangle\,.$$

With this notation, we prove a statement analogous to Lemma 5.12:

LEMMA 5.16. *Let $\varphi : B \to A$ be an isogeny and let h and k be the two uniques integers such that $\varphi = h \cdot \psi + k \cdot \psi \circ \rho$. If $h + k$ is not divisible by 3, then $(1 \times \varphi)^* \mathscr{P}_A \notin \operatorname{Im}(1-\sigma^*)$. Conversely if 3 divides $h+k$, then $(1 \times \varphi)^* \mathscr{P}_A \otimes p_B^* \beta \in \operatorname{Im}(1-\sigma^*)$, for every $\beta \in \operatorname{Pic}^0(B)$.*

Proof. The argument is completely analogous to the proof of Lemma 5.12, after observing that, if $T : \operatorname{Hom}(B,A) \to \operatorname{Hom}(B,A)$ is the operator defined by pre-composing with $1_B - \rho$. Then the image of T consits of exactly the homomorphisms $h \cdot \psi + k \cdot \psi \circ \rho$ such that 3 divides $k + h$. □

Remark 5.17. This Lemma implies easily that the quotient of the Hom-part of $\operatorname{Num}(A \times B)$ by the action of $1 - \sigma^*$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ with nontrivial elements $(1_A \times \psi)^* \mathscr{P}_A + \operatorname{Im}(1-\sigma^*)$ and $(1_A \times \psi + \psi \circ \rho)^* \mathscr{P}_A + \operatorname{Im}(1-\sigma^*) = (1_A \times 2 \cdot \psi)^* \mathscr{P}_A + \operatorname{Im}(1-\sigma^*)$.

We will also need the following statement:

LEMMA 5.18. *Let B an elliptic curve with j-invariant 0 and β an element $\operatorname{Pic}^0(B)$. Consider the following line bundles*

$$P_1 := (2 \cdot \rho + 1_B)^* \beta,\quad , P_\rho := (2 \cdot \rho + 1_B)^* \rho^* \beta,$$

$$and \quad P_{1+\rho} := (2 \cdot \rho + 1_B)^* (1_B + \rho)^* \beta.$$

If any of them is trivial then they are all trivial.

Proof. Observe first that $(2 \cdot \rho + 1_B)^* \rho^* \beta \simeq \rho^* (2 \cdot \rho + 1_B)^* \beta$. Since ρ is an automorphism, the triviality of P_ρ is equivalent to the triviality of P_1. In addition as $P_{1+\rho} \simeq P_1 \otimes P_\rho$ we have that if P_1 and P_ρ are both trivial, then also $P_{1+\rho}$ is trivial. It remains to show that if $P_{1+\rho} \simeq \mathcal{O}_B$, then also P_1 and P_ρ are trivial. We note that $P_{1+\rho} \simeq \mathcal{O}_B$ if, and only if, $\rho^* P_{1+\rho} \simeq \mathcal{O}_B$. On the other side we have

$$\rho^* P_{1+\rho} \simeq \rho^* (2 \cdot \rho + 1_B)^* (1_B + \rho)^* \beta$$
$$\simeq \rho^* (\rho - 1_B)^* \beta \simeq (-2 \cdot \rho - 1_B)^* \beta \simeq P_1^{-1}.$$

We conclude that the triviality of $P_{1+\rho}$ is equivalent to the triviality of P_1, as required by the statement. □

Now we are ready to prove the main result of this paragraph:

Theorem 5.19. *Let S be an bielliptic surface of type 5 such that the two elliptic curves A and B are isogenous. Let ψ be a generating isogeny, then we have that the Brauer map $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(A \times B)$ is trivial if, and only if, the line bundle $(2 \cdot \rho + 1_B)^* \psi^* P_\eta \simeq \mathcal{O}_B$.*

Proof. The argument is really similar to what happens for type 3 bielliptic surfaces. We first note that, for any isogeny $\varphi : B \to A$, and every α and β in $\mathrm{Pic}^0(A)$ and $\mathrm{Pic}^0(B)$ respectively, we have that

$$\pi^* \mathrm{Nm}((1 \times \varphi)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes p_B^* \beta) \simeq p_A^* \alpha^{\otimes 3} \otimes p_B^* (2 \cdot \rho + 1_B)^* \varphi^* P_\eta. \qquad (5.5)$$

Suppose first that $(2 \cdot \rho + 1_B)^* \psi^* P_\eta$ is trivial. Then (5.5) ensures that the norm of $M_1 := (1 \times \psi)^* \mathscr{P}_A$ is topologically trivial. By Lemma 5.16 we know that no line bundle numerically equivalent to M_1 is in the image of $1 - \sigma^*$. Thus we use the Remark 2.6 to provide an element in $\mathrm{Ker} \, \mathrm{Nm}$ inducing a nontrivial class in $\mathrm{Ker} \, \mathrm{Nm} / \mathrm{Im}(1 - \sigma^*)$.

Conversely, assume that L is a line bundle in $\mathrm{Ker} \, \mathrm{Nm}$ whose class in $\mathrm{Ker} \, \mathrm{Nm} / \mathrm{Im}(1 - \sigma^*)$ is not trivial. As before we can write

$$L \simeq (1 \times h \cdot \psi + k \cdot \psi \circ \rho)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes p_B^* \beta.$$

We apply Lemma 5.16 and write $L \simeq M \otimes M'$ with $M' \in \mathrm{Im}(1 - \sigma^*)$ and M a line bundle numerically equivalent to one of the following

$$M_1 := (2 \cdot \rho + 1_B)^* \psi^* P_\eta, \quad \text{and} \quad M_{1+\rho} := (2 \cdot \rho + 1_B)^* (1 + \rho)^* \psi^* P_\eta. \qquad (5.6)$$

Clearly M is in the kernel of the norm map, which, by (5.5) implies that one among the following is trivial:

$$P_1 := (2 \cdot \rho + 1_B)^* \psi^* P_\eta, \quad \text{and} \quad P_{1+\rho} := (2 \cdot \rho + 1_B)^* (1_B + \rho)^* \psi^* P_\eta.$$

We conclude by applying Lemma 5.18 and deducing that $P_1 \simeq \mathcal{O}_B$. $\qquad \square$

Example 5.20. Suppose that $A \simeq B$. Note that the self-isogeny $\varphi := (2 \cdot \rho + 1_B) : B \to B$ has degree 3, and its kernel is contained in $B[3]$, which has order 9. If $\eta \in B[3]$ nontrivial is in the kernel of φ, then the bielliptic surface obtained by the action of $\sigma(x, y) = (x + \eta, \rho(y))$ has trivial Brauer map. Otherwise, the Brauer map is injective.

5.3.4 Bielliptic surfaces of type 2

We kept last the bielliptic surfaces of type two since for them we need an *ad hoc* argument. Let therefore S be a bielliptic surface of type 2 and denote by X its canonical cover. Then $X \simeq A \times B / < t_{(\theta_1, \theta_2)} >$ for two elliptic curves A and B and θ_1 and θ_2 points of order 2 in A and B respectively. Let us fix generators for $\mathrm{Hom}(B, A)$: if B does not have complex multiplication then $\mathrm{Hom}(B, A) = < \psi >$ with $\psi : B \to A$ an isogeny; otherwise there are two isogenies $\psi_1, \psi_2 : B \to A$ such that $\mathrm{Hom}(B, A) = < \psi_1, \psi_2 > .$ Our goal is to prove the following statement.

THEOREM 5.21. *In the above notation the Brauer map* $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$
is not injective if, and only if, one of the following conditions is satisfied:

1. *the elliptic curve B does not have complex multiplication and either $\psi(\theta_2)$ is not the identity element of A or $\psi^* P_{\theta_1}$ is not trivial.*

2. *the elliptic curve B has complex multiplication and at least one of the following line bundles is nontrivial:*

$$\left\{ P_{\psi_1(\theta_2)},\ P_{\psi_2(\theta_2)},\ \psi_1^* P_{\theta_1}, \psi_2^* P_{\theta_1},\ P_{(\psi_1+\psi_2)(\theta_2)},\ (\psi_1+\psi_2)^*(P_{\theta_1}) \right\}$$

Before proceeding with the proof we need to set up some notation. Recall that
we have the following diagram

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\ \pi_{\tilde{S}}\ } & \tilde{S} \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\
X & \xrightarrow{\ \pi_S\ } & S,
\end{array}
$$

where $\tilde{S}$ is a bielliptic surface of type 1. We have that $S \simeq X/\sigma$, $\tilde{S} \simeq A \times B/\tilde{\sigma}$
and $X \simeq A \times B/\Sigma$, where Σ denotes the translation $t_{(\theta_1,\theta_2)}$. We are going to
deal just with the case in which B hax complex multiplication. The proof in the
other case will be identical, provided that one drops one of the two generators.
We first observe the following fact:

LEMMA 5.22. *In the notation above suppose that B has complex multiplication
and let L_i be the line bundle $(1 \times \psi_i)^* \mathscr{P}_A$, for $i = 1, 2$. Then the conditions of
Theorem 5.21 are satisfied if, and only if, for every $\gamma \in \mathrm{Pic}^0(A \times B)$ one of the
following line bundles is not Σ-invariant:*

$$L_1 \otimes \gamma,\ L_2 \otimes \gamma, L_1 \otimes L_2 \otimes \gamma. \tag{5.7}$$

Proof. By see-saw, it is easy to see that

$$\Sigma^*[(1 \times \psi_i)^* \mathscr{P}_A \otimes \gamma] \simeq (1 \times \psi_i)^* \mathscr{P}_A \otimes \gamma \otimes p_A^* P_{\psi_i(\theta_2)} \otimes p_B^* \psi_i^* P_{\theta_1},$$

$$
\begin{aligned}
\Sigma^*[(1 \times (\psi_1 + \psi_2))^* \mathscr{P}_A \otimes \gamma] \simeq {} & (1 \times (\psi_1 + \psi_2))^* \mathscr{P}_A \otimes \gamma \\
& \otimes p_A^* P_{\psi_1+\psi_2(\theta_2)} \otimes p_B^* (\psi_1 + \psi_2)^* P_{\theta_1};
\end{aligned}
$$

the statement follows directly. $\qquad\square$

Proof of the sufficiency of the conditions of the Theorem 5.21. Suppose that
the conditions of the statement are satisfied. Then, by Lemma 5.22, one of
the line bundles (5.7) is not Σ-invariant. Suppose first that $L_1 \otimes \gamma$ is not
Σ-invariant for every topologically trivial γ. Thus we have that $l(0, 0, \psi_1)$ is
not in $\phi^* \mathrm{Num}(X)$. We deduce that

$$2 \cdot \psi_1 \notin (1 - \tilde{\sigma})^* \phi^* \mathrm{Num}(X). \tag{5.8}$$

Otherwise, we would have, for some $\varphi = h\psi_1 + k\psi_2 \in \mathrm{Hom}(A,B)$,

$$
\begin{aligned}
2 \cdot \psi_1 &= (1 - \tilde{\sigma})^* \phi^* \varphi \\
&= (1 - \tilde{\sigma})^* (h \cdot \psi_1 + k \cdot \psi_2) \\
&= 2h \cdot \psi_1 + 2k \cdot \psi_2.
\end{aligned}
$$

Therefore $h = 1$, $k = 0$ and $\phi^* \varphi = \psi_1$, contradicting our previous conclusion. Now consider the line bundle $L := \mathrm{Nm}_\phi((1 \times \psi_1)^* \mathscr{P}_A)$. We want to show that there is $\beta \in \mathrm{Pic}^0(X)$ such that $\mathrm{Nm}_{\pi_S}(L \otimes \beta)$ is trivial. We use the functoriality of the norm map (Proposition 2.5) and we obtain that

$$
\mathrm{Nm}_{\pi_S}(L) \simeq \mathrm{Nm}_{\tilde{\pi}} \circ \mathrm{Nm}_{\pi_{\tilde{S}}}((1 \times \psi_1)^* \mathscr{P}_A).
$$

Observe that, by (5.3), we have that $\pi_{\tilde{S}}^* \mathrm{Nm}_{\pi_{\tilde{S}}}((1 \times \psi_1)^* \mathscr{P}_A)$ is numerically trivial. Therefore we have that $\mathrm{Nm}_{\pi_{\tilde{S}}}((1 \times \psi_1)^* \mathscr{P}_A)$ is itself numerically trivial. This implies that

$$
\mathrm{Nm}_{\tilde{\pi}} \circ \mathrm{Nm}_{\pi_{\tilde{S}}}((1 \times \psi)^* \mathscr{P}_A) \in \mathrm{Pic}^0(S).
$$

In fact, if we have that $\mathrm{Nm}_{\pi_{\tilde{S}}}((1 \times \psi_1)^* \mathscr{P}_A) := \alpha \in \mathrm{Pic}^0(\tilde{S})$, then we write $\alpha \simeq \tilde{\pi}^* \gamma$ and we have that

$$
\mathrm{Nm}_{\pi_S}(L) \simeq \mathrm{Nm}_{\tilde{\pi}} \circ \mathrm{Nm}_{\pi_{\tilde{S}}}((1 \times \psi_1)^* \mathscr{P}_A) \simeq \gamma^{\otimes 2}.
$$

On the other side, if $\mathrm{Nm}_{\pi_{\tilde{S}}}((1 \times \psi_1)^* \mathscr{P}_A) =: T$ is numerically trivial, but not algebraically trivial, then as in (4.1) we have that $\mathrm{Nm}_{\tilde{\pi}}(T)$ is topologically trivial. Thus, as before, we obtain β such that $\mathrm{Nm}_{\pi_S}(L \otimes \beta) \simeq \mathcal{O}_S$ via the Pic^0 trick (Remark 2.6).

In order to determine the non injectivity of the Brauer map, we have to ensure that $L \otimes \beta$ is not in $\mathrm{Im}(1 - \sigma^*)$. Suppose that this were not the case, and consider the following commutative diagram

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\ \tilde{\sigma}\ } & A \times B \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\
X & \xrightarrow{\ \sigma\ } & X.
\end{array}
$$

Then $c_1(\phi^* L) \in (1 - \tilde{\sigma})^* \phi^* \mathrm{Num}(X)$. However, the properties of the norm (see (2.8)) ensures that $c_1(\phi^* L) = l(0, 0, 2 \cdot \psi_1)$, thus we would have that $l(0, 0, 2 \cdot \psi_1) \in \phi^* \mathrm{Num}(X)$, contradicting (5.8).

If $L_2 \otimes \gamma$ is not Σ-invariant for every $\gamma \in \mathrm{Pic}^0(A \times B)$, then we proceed as before by exchanging the role of ψ_1 and ψ_2. Thus, it remain only to see what happen if $L_1 \otimes L_2 \otimes \gamma$ is not Σ-invariant for every γ. In this case we will have that $l(0, 0, \psi_1 + \psi_2) \notin \phi^* \mathrm{Num}(A \times B)$, and so either $l(0, 0, \psi_1)$ or $l(0, 0, \psi_2)$ are not in the image of ϕ^*. Without loss of generality we can assume the first. Then we will still have (5.8) and we can repeat the above argument. $\square$

In order to complete the proof of Theorem 5.21 we need to show that if all $(1 \times \psi_1)^* \mathscr{P}_A$, $(1 \times \psi_2)^* \mathscr{P}_A$, and $(1 \times (\psi_1 + \psi_2))^* \mathscr{P}_A$ are Σ-invariant then the Brauer map to X is injective. Observe that, under these assumptions, we can write

$$(1 \times \psi_1)^* \mathscr{P}_A \simeq \phi^* L_1, \quad (1 \times \psi_2)^* \mathscr{P}_A \simeq \phi^* L_2,$$

$$\text{and} \quad (1 \times (\psi_1 + \psi_2))^* \mathscr{P}_A \simeq \phi^* L_3.$$

for some line bundles L_1, L_2, and L_3 in $\mathrm{Pic}(X)$. Observe that we can take $L_3 \simeq L_1 \otimes L_2$. Then, for $\alpha \in \mathrm{Pic}^0(X)$, with $\phi^* \alpha \simeq p_A^* \alpha_1 \otimes p_B^* \alpha_2$ we have

$$\phi^*(\pi_S^* \mathrm{Nm}_{\pi_S}(L_i \otimes \alpha)) \simeq \phi^*(L_i \otimes \alpha \otimes \sigma^*(L_i \otimes \alpha)) \simeq p_A^* \alpha_1^{\otimes 2} \otimes p_B^*(\psi_i^* P_\tau),$$

for $i = 1, 2$. Similarly, we get

$$\phi^*(\pi_S^* \mathrm{Nm}_{\pi_S}(L_3 \otimes \alpha)) \simeq \phi^*(L_1 \otimes L_2 \otimes \alpha \otimes \sigma^*(L_1 \otimes L_2 \otimes \alpha))$$
$$\simeq p_A^* \alpha_1^{\otimes 2} \otimes p_B^*(\psi_1^* P_\tau \otimes \psi_2^* P_\tau).$$

In both computations, the last isomorphism is again given by (5.3). Observe that neither the ψ_i's nor $\psi_1 + \psi_2$ can factor through the multiplication by 2 isogeny, or we would have that ψ_1 and ψ_2 cannot generate $\mathrm{Hom}(B, A)$. In particular, we must have that both $\psi_i^* P_\tau$ and $(\psi_1 + \psi_2)^* P_\tau$ are nontrivial. We deduce that

$$\phi^*(\pi_S^* \mathrm{Nm}_{\pi_S}(L_i \otimes \alpha)) \not\simeq \mathcal{O}_{A \times B},$$

for $i = 1, 2$, and 3. In particular we obtained the following lemma:

LEMMA 5.23. *In the above notation, if the conditions of Theorem 5.21 are not satisfied, then line bundles numerically equivalent to the L_i's are not in the kernel of the norm map Nm_{π_S}.*

Before going further we need an intermediate step:

LEMMA 5.24. *For any integer n, and $i = 1, 2$, and 3, $L_i^{\otimes 2n}$ are in $\mathrm{Im}(1 - \sigma^*)$.*

Proof. Obviously it is enough to show that $L_i^{\otimes 2}$ is in the image of $(1 - \sigma^*)$. To this aim, we pull $L_i \otimes \sigma^* L_i$ back to $A \times B$ and apply (5.3). We see that

$$\phi^*(L_i \otimes \sigma^*(L_i)) \in p_B^* \mathrm{Pic}^0(B) \subseteq A \times B,$$

and we deduce that $\gamma := L_i \otimes \sigma^*(L_i)$ is a line bundle in $p_B^* \mathrm{Pic}^0(B/H)$. By Lemma 5.2 we know that $\gamma \in \mathrm{Im}(1 - \sigma^*)$. Thus we can write

$$L_i^{\otimes 2} \simeq \gamma \otimes (\sigma^* L_i)^{-1} \otimes L_i.$$

$\square$

Conclusion of the Proof of Theorem 5.21. Let M is a line bundle such that $\mathrm{Nm}_{\pi_S}(M) \simeq \mathcal{O}_S$, we will show that M is in the image of $(1 - \sigma^*)$. Using (2.8), we know that $M \otimes \sigma^* M \simeq \mathcal{O}_X$. By pulling back via ϕ we get that $\phi^* M \otimes \tilde{\sigma}^* \phi^* M$ is again trivial and by the proof of Lemma 5.5 we see that $c_1(\phi^* M) = l(0, 0, h \cdot \psi_1 + k \cdot \psi_2)$ for two integers h and k. Then we can write

$$\phi^* M \simeq (1 \times h \cdot \psi_1)^* \mathscr{P}_A \otimes (1 \times k \cdot \psi_2)^* \mathscr{P}_A \otimes \gamma \simeq \phi^*(L_1^{\otimes h} \otimes L_2^{\otimes k}) \otimes \gamma,$$

for some γ in $\mathrm{Pic}^0(A \times B)$. Therefore $\phi^*(M \otimes L_1^{\otimes -h} \otimes L_2^{\otimes -k}) \simeq \gamma$, and we deduce that $M \simeq L_1^{\otimes h} \otimes L_2^{\otimes k} \otimes \alpha$ for some $\alpha \in \mathrm{Pic}^0(X)$. If h and k are both even, then by Lemma 5.24 we know that $\alpha \in \mathrm{Ker}\, \mathrm{Nm}_{\pi_S}$, and the class of M in $\mathrm{Ker}\, \mathrm{Nm}_{\pi_S} / \mathrm{Im}(1 - \sigma^*)$ is exactly $[\alpha]$. We apply Proposition 5.1 and deduce that $[M] = 0$.

We will now show that neither one between h and k can be odd. Suppose otherwise that h and k are not both even. For example, assume that h is odd and k is even, the proof in the other cases is very similar. Under this hypothesis, Lemma 5.24 ensures that $L_1 \otimes \alpha$ is in the kernel of the norm map. But this contradicts Lemma 5.23, and our proof is complete $\qquad\square$

Example 5.25.   (a) Suppose that $A \simeq B$, then the isogenies ψ_1 and ψ_2 are indeed isomorphisms and thus the Brauer map can never be injective.

(b) Let B be an elliptic curve without complex multiplication and consider θ_2 a point of order 2 in B. Let A be the elliptic curve $B/<\theta_2>$ and $\psi : B \to A$ the quotient map. The dual map ψ^* has degree 2. Let $\theta_1 \in A$ be the point such that $\psi^* P_{\theta_1}$ is trivial and let τ be another order 2 element of A. All this data identify a bielliptic surface of type 2 whose Brauer map to the canonical cover is injective.

A   The homomorphism lattice of two elliptic curves

Jonas Bergström and Sofia Tirabassi

The main goal of this appendix is to give a structure theorem for the $\mathbb{Z}$-module $\mathrm{Hom}(B, A)$ where A and B are two complex elliptic curves with $j(B) = 0, 1728$. This result has been used in 2.6 above in order to make a clever choice of generators for $\mathrm{Num}(A \times B)$ which in turn has allowed an accurate description of the action of the automorphism σ^* on the Neron–Severi group of the product $A \times B$ when S is a bielliptic surface of type 3 or 5.

If B is an elliptic curve with j-invariant 0 or 1728, then B admits an automorphism λ_B of order 3 or 4 respectively. The main result of this Appendix is that the group $\mathrm{Hom}(B, A)$ can be completely described in terms of λ_B and an isogeny $\psi : B \to A$. More precisely we have the following statement:

THEOREM A.1. *Let A and B two isogenous complex elliptic curves with $j(B)$ is either 0 or 1728. Then there exist an isogeny $\psi : B \to A$ such that*

$$\mathrm{Hom}(B, A) = < \psi, \psi \circ \lambda_B > .$$

This appendix is organized in three main parts. In the first we outline some classical results about imaginary quadratic fields and their orders. The second is concerned with complex elliptic curves with complex multiplication. Theorem A.1 is proven in A.3. The key idea of our argument is to describe $\mathrm{Hom}(B, A)$ as a fractional ideal of $\mathrm{End}(B)$ homothetic to $\mathrm{End}(B)$. This is done by observing that the class number of $\mathrm{End}(B)$ is 1.

A.1   Preliminaries on orders in imaginary quadratic fields

An *imaginary quadratic field* is a subfield $K \subseteq \mathbb{C}$ of the form $\mathbb{Q}(\sqrt{-d})$, with d a positive, square-free integer. The *discriminant of K* is the integer d_k defined as

$$d_K = \begin{cases} -d, & \text{if } d \equiv 3 \mod 4, \\ -4d, & \text{otherwise.} \end{cases}$$

The *ring of integers of K*, $\mathcal{O}_K$ is the largest subring of K which is a finitely generated abelian group. Then we have that $\mathcal{O}_K = \mathbb{Z}[\delta]$, where

$$\delta = \begin{cases} \frac{1+\sqrt{-d}}{2}, & \text{if } d \equiv 3 \mod 4, \\ \sqrt{-d}, & \text{otherwise.} \end{cases} \tag{A.1}$$

An *order* in an imaginary quadratic field K is a subring $\mathcal{O}$ of $\mathcal{O}_K$ which properly contains $\mathbb{Z}$. It turns out that $\mathcal{O} \simeq \mathbb{Z} + \mathbb{Z} \cdot (n\delta)$ for some positive integer n.

Given an order $\mathcal{O}$ in an imaginary quadratic field K, a *fractional ideal* of $\mathcal{O}$ is a non-zero finitely generated sub $\mathcal{O}$-module of K. For every fractional ideal M of $\mathcal{O}$ there is an $\alpha \in K^*$ and an ideal $\mathfrak{a}$ of $\mathcal{O}$ such that $M = \alpha \cdot \mathfrak{a}$. We will need the following notions.

DEFINITION A.2. (i) Two fractional $\mathcal{O}$-ideals M and M' are *homothetic* if there is $\alpha \in K^*$ such that $M = \alpha M'$.

(ii) A fractional $\mathcal{O}$-ideal is *invertible* if there is a fractional ideal M' such that $M \cdot M' = \mathcal{O}$. The set of invertible $\mathcal{O}$-ideals is denoted by $I(\mathcal{O})$.

(iii) A fractional $\mathcal{O}$-ideal M is principal if it is of the form $\alpha \cdot \mathcal{O}$ for some $\alpha \in K^*$. So principal ideals are precisely the fractional ideals homothetic to $\mathcal{O}$. The set of principal $\mathcal{O}$-ideals is denoted by $P(\mathcal{O})$.

Principal ideals are clearly invertible. In general not all fractional ideals are invertible, but they are so if $\mathcal{O} = \mathcal{O}_K$ (see also [Cox11, Proposition 5.7]). The quotient

$$\mathfrak{Cl}(\mathcal{O}) := I(\mathcal{O})/P(\mathcal{O})$$

describes the homothety classes of invertible $\mathcal{O}$-ideals. It is a group with the product and it is called the *ideal class group of $\mathcal{O}$*. Its order is finite and is called the *class number of $\mathcal{O}$*. When $\mathcal{O} = \mathcal{O}_K$, then the class number of $\mathcal{O}$ is exactly the class number of the field K, which is a function of the discriminant of K (see [Cox11, Theorem 5.30(ii)]). More generally the class number of $\mathcal{O}$ is a general function of d_K and $[\mathcal{O}_K : \mathcal{O}]$.

Example A.3. If K is either $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, then all the fractional ideals of $\mathcal{O}_K$ are homothetic to $\mathcal{O}_K$. In fact the class number of the field K in this cases is 1, as it was computed by Gauss in his book *Disquisitiones arithmeticae* .

A.2 Elliptic curves with complex multiplication

The importance of orders in the study of the geometry of elliptic curves is that they describe the endomorphism rings of a complex elliptic curve:

THEOREM A.4. *Let A be an elliptic curve over $\mathbb{C}$, then $\mathrm{End}(A)$ is either isomorphic to $\mathbb{Z}$ or to an order in an imaginary quadratic field.*

Proof. See [Sil09, Theorem VI.5.5]. $\qquad\square$

We say that a (complex) elliptic curve has *complex multiplication* if its endomorphism ring is larger than $\mathbb{Z}$. Observe that in this case $\mathrm{End}(A) \otimes \mathbb{Q}$ is a quadratic field K and $\mathrm{End}(A)$ is an order in K.

Given a complex elliptic curve A there is a canonical way to identify its endomorphism ring with a subring of $\mathbb{C}$. More generally let A and B two elliptic curves, then there are two lattices Λ_A and Λ_B in $\mathbb{C}$ such that $A \simeq \mathbb{C}/\Lambda_A$ and $B \simeq \mathbb{C}/\Lambda_B$. Given a complex number ζ such that $\zeta \cdot \Lambda_B \subseteq \Lambda_A$, the map $\Phi_\zeta : \mathbb{C} \to \mathbb{C}$ defined by $z \mapsto \zeta \cdot z$ descends to an (algebraic) homomorphism $\varphi_\zeta : B \to A$. It is possible to show (see [Sil09, VI.5.3(d)]) that any morphism of elliptic curves preserving the origin is obtained in this way, and in particular we get an isomorphism of abelian groups

$$\mathrm{Hom}(B, A) \simeq \{\zeta \in \mathbb{C} \mid \zeta \cdot \Lambda_B \subseteq \Lambda_A\} \subseteq \mathbb{C}. \tag{A.2}$$

By setting $B = A$ we get a ring isomorphism

$$\operatorname{End}(A) \simeq \mathcal{O} := \{\zeta \in \mathbb{C} \mid \zeta \cdot \Lambda_A \subseteq \Lambda_A\} \subseteq \mathbb{C}.$$

The isomorphism $\zeta \mapsto \varphi_\zeta$ is characterized as the unique isomorphism $f : \mathcal{O} \to \operatorname{End}(A)$ such that, for any $\zeta \in \mathcal{O}$ and for every invariant form ω on A we have that $f(\zeta)^* \omega = \zeta \cdot \omega$ ([Sil13, II.1.1]).

Notation A.5. For an elliptic curve with complex multiplication A such that $\operatorname{End}(A) \simeq \mathbb{Z} + \mathbb{Z} \cdot n\delta$, we will denote by λ_A the isogeny $\varphi_{n\delta} : A \to A$ and we will say that *A has complex multiplication by λ_A*.

It is clear that, with this identification, $\operatorname{End}(A) = <1_A, \lambda_A>$ as a $\mathbb{Z}$-module.

Example A.6.   (a) Suppose that B is an elliptic curve with j-invariant 0. Then we can write $B \simeq \mathbb{C}/\Lambda_B$, with $\Lambda_B = <1, e^{\frac{2\pi i}{3}}>$. Then $\operatorname{End}(B) \otimes \mathbb{Q} \simeq \mathbb{Q}(\sqrt{-3})$ and $\operatorname{End}(B) \simeq \mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. We have that λ_B is induced by the multiplication by $\frac{1+\sqrt{-3}}{2}$ and is an automorphism of B satisfying $\lambda_B^2 + \lambda_B + 1_B = 0$. This is exactly the automorphism which in 2.2 was denoted by ρ and which was used to construct bielliptic surfaces of type 5.

(b) Suppose now that the j-invariant of B is 1728. Then we can take $\Lambda_B = <1, i>$ and we have that $\operatorname{End}(B) \otimes \mathbb{Q} \simeq \mathbb{Q}(i)$. The endomorphism ring of B is isomorphic to $\mathbb{Z}[i]$ and the multiplication by i induces an automorphism λ_B such that $\lambda_B^2 = -1_B$. This is the automorphism ω of B used to construct bielliptic surfaces of type 3 in 2.2.

A.3   Proof of the main result

We are now ready to provide a proof for Theorem A.1. Our key point will be the following:

Claim: the $\mathbb{Z}$-module $\operatorname{Hom}(B, A)$ is isomorphic to a fractional ideal of $\mathcal{O}_K$.

Before proceeding with showing that this Claim is true, let us see how it implies the statement. We do this applying Example A.3 and deducing that all fractional $\mathcal{O}_K$-ideals are homothetic to $\mathcal{O}_K$. Therefore there exist $\alpha \in K^*$ such that

$$M \simeq \alpha \cdot \mathcal{O}_K = \alpha \cdot <1, \delta> = <\alpha, \alpha \cdot \delta>,$$

where δ is like in (A.1). But then we have that $\operatorname{Hom}(B, A) = <\varphi_\alpha, \varphi_\alpha \circ \lambda_B>$, and the statement is true.

Proof of the Claim. Let $\Lambda_A = <1, \tau>$ a lattice in $\mathbb{C}$ such that $A \simeq \mathbb{C}/\Lambda_A$, and denote by $K \subseteq \mathbb{C}$ the quadratic field $\operatorname{End}(B) \otimes \mathbb{Q}$. Then the ring $\operatorname{End}(B)$ is exactly the ring of integers $\mathcal{O}_K$. Observe that this is isomorphic to a lattice in $\mathbb{C}$, and that $B \simeq \mathbb{C}/\mathcal{O}_K$ (See Example A.6).

By (A.2) we can identify $M := \operatorname{Hom}(B, A)$ as a finitely generated subgroup of $\mathbb{C}$. Composition on the right with endomorphism of B gives to M a structure

of $\mathcal{O}_K$-module. Let $\alpha \neq 0$ denote an element of $\mathfrak{a} := \mathrm{Hom}(A, B)$, identifyied with a subgroup of $\mathbb{C}$. Then clearly $\alpha \cdot M \subseteq \mathcal{O}_K$. We deduce that $M \subseteq K$ is a fractional ideal of $\mathcal{O}_K$, and the Claim is proven. $\qquad\square$

Remark A.7.   (a) For any order $\mathcal{O}$ in a quadratic extension of $\mathbb{Q}$ a representative of each homothety class of fractional ideals can be given as $I \cdot \mathcal{O}'$, where $\mathcal{O}' \supseteq \mathcal{O}$ is an extension of orders, and I is an invertible fractional ideal (see [Mar18]). The over order S can be given a $\mathbb{Z}$-basis of the form $\{1, \delta \cdot f\}$ where f is a positive integer.

For any pair of isogenous complex elliptic curves with complex multiplication we have that $\mathrm{Hom}(B, A)$ is a fractional $\mathrm{End}(B)$ ideal. In addition, if we assume that $\mathrm{End}(B)$ has class number 1, we have that $B \simeq \mathbb{C}/\mathrm{End}(B)$. In fact, under this assumption [Cox11, Corollary 10.20] yields that, there is just one elliptic curve up to isomorphism with endomorphism ring End(B).

In conclusion, demanding that $\mathrm{End}(B)$ has class number 1 (instead of $j(B)$ being either 0 or 1728) is sufficient for Theorem A.1 to hold.

So Theorem A.1 will hold for the 13 isomorphism classes of complex elliptic curves B for which $\mathrm{End}(B)$ has discriminant -3, -4,-7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -164 (see [Sil13, Example 11.3.2]).

(b) It is clear from the proof that the role of A and B can be exchanged, so we have proven a structure theorem for $\mathrm{Hom}(A, B)$ when the endomorphism ring of one of the two curves has class number 1.

Theorem A.1 is not constructive, in the sense that it does not provide a way to determine the isogeny ψ such that ψ and $\psi \circ \lambda_B$ generate $\mathrm{Hom}(B, A)$. We conclude this appendix by constructing ψ in an easy example.

Example A.8. Let Λ be the lattice $< 1, 2i > \subseteq \mathbb{C}$, and consider $A := \mathbb{C}/\Lambda$. Consider the 2-torsion point $\tau := (0, i) + \Lambda$ of A and let B be the quotient $A/<\tau>$. It is clear that B has j-invariant 1728. We claim that $\mathrm{Hom}(B, A) =< \varphi_2, \varphi_2 \circ \lambda_B >$.
We use first (A.2) and identify $\mathrm{Hom}(B, A)$ with a lattice in $\mathbb{C}$. Given $\alpha = (a + bi) \in \mathrm{Hom}(B, A)$, we have that both α and $\alpha \cdot i$ must be elements of Λ. We deduce that both a and b must be even integers and so $\mathrm{Hom}(B, A) =< 2, 2 \cdot i >$. We conclude by observing that λ_B is the automorphism of B induced by multiplication by i.

References

[Bad01]   L. Badescu. *Algebraic surfaces.* Springer, New York, 2001.

[BDF10]   G. Bagnera and M. de Franchis. Le nombre ρ de Picard pour les surfaces hyperelliptiques. *Rend. Circ. Mat. Palermo*, 10, 1910.

[Bea96]   A. Beauville. *Complex algebraic surfaces.* Cambridge University Press, 1996.

[Bea09]   A. Beauville. On the Brauer group of Enriques surfaces. *Math. Res. Lett.* 16(6):927–934, 2009.

[BHPvdV15] W. Barth, K. Hulek, C. Peters, and A. van de Ven. *Compact complex surfaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer, Berlin, Heidelberg, 2004.

[BL13]   C. Birkenhake and H. Lange. *Complex abelian varieties*, Grundlehren der Mathematischen Wissenschaften, 302. Springer, Berlin, 2004.

[BM77]   E. Bombieri and D. Mumford. Enriques' classification of surfaces in char. p. II. In *Complex analysis and algebraic geometry, Collect. pap. dedic. K. Kodaira*, 23–42, 1977.

[Cox11]   D. A. Cox. *Primes of the form $x^2 + ny^2$: Fermat, class field theory, and complex multiplication*, In *Pure and Applied Mathematics*, John Wiley & Sons, 2013.

[dJ]   J. de Jong. A result of Gabber, 2005. https://www.math.columbia.edu/~dejong/papers/2-gabber.pdf.

[Gau66]   C. F. Gauss. *Disquisitiones arithmeticae.* Yale University Press, 1966.

[Gro61]   A. Grothendieck. Éléments de géométrie algébrique : II. Étude globale élémentaire de quelques classes de morphismes. *Publications Mathématiques de l'IHÉS* 8:5–222, 1961.

[Gro67]   A. Grothendieck. éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, quatrième partie. *Publications Mathématiques de l'IHÉS* 32:5–361, 1967.

[HLT19]   K. Honigs, L. Lombardi, and S. Tirabassi. Derived equivalences of canonical covers of hyperelliptic and Enriques surfaces in positive characteristic. *Math. Z.* 295(1-2):727–749, 2020.

[Iit70]   S. Iitaka. Deformations of compact complex surfaces. II. *J. Math. Soc. Japan* 22(2):247–261, 1970.

[Mar18]   S. Marseglia. Computing the ideal class monoid of an order. *J. Lond. Math. Soc., II. Ser.* 101(3):984–1007, 2020.

[Mum70]   D. Mumford. Abelian varieties, 1970. Published for the Tata Institute of Fundamental Research, Bombay, by Oxford University Press, London, 1970.

[Nue]   H. Nuer. Stable sheaves on bielliptic surfaces: from the classical to the modern. Preprint, 2021. arXiv preprint math/2107.13370.

[Ser90a]   F. Serrano. Divisors of bielliptic surfaces and embeddings in $\mathbf{P}^4$. *Math. Z.* 203(3):527–533, 1990.

[Ser90b]   F. Serrano. Multiple fibres of a morphism. *Comment. Math. Helv.* 65(2):287–298, 1990.

[Ser91]   F. Serrano. The Picard group of a quasi-bundle. *Manuscripta Math.* 73(1):63–82, 1991.

[Sil09]   J. H. Silverman. *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, 106. Springer, 2009.

[Sil13]   J. H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, 151. Springer, New York, 1994.

[Sta19]   The Stacks project authors. The Stacks project, 2019. https://stacks.math.columbia.edu.

[Suw69]   T. Suwa. On hyperelliptic surfaces. *J. Fac. Sci. Univ. Tokyo Sect. I* 16:469–476, 1970.

[Ume75]   H. Umemura. Stable vector bundles with numerically trivial Chern classes over a hyperelliptic surface. *Nagoya Math. J.* 59:107–134, 1975.

Eugenia Ferrari   Magnus Vodrup
Department of Mathematics   Department of Mathematics
University of Bergen   University of Bergen
Allégaten 41   Allégaten 41
Bergen   Bergen
Norway   Norway
eugenia.ferrari@uib.no   magnus.vodrup@uib.no

Sofia Tirabassi   Jonas Bergström
Department of Mathematics   Department of Mathematics
Stockholm University   Stockholm University
Kraftriket hus 6   Kraftriket hus 6
Stockholm   Stockholm
Sweden   Sweden
tirabassi@math.su.se   jonasb@math.su.se