Rank Inequalities for the Heegaard Floer Homology of Branched Covers

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Abstract. We show that if L is a nullhomologous link in a 3-manifold Y and $\Sigma(Y,L)$ is a double cover of Y branched along L then for each spin^c-structure $\mathfrak{s}$ on Y there is an inequality

$$\dim \widehat{HF}(\Sigma(Y,L),\pi^*\mathfrak{s};\mathbb{F}_2) \geq \dim \widehat{HF}(Y,\mathfrak{s};\mathbb{F}_2).$$

We discuss the relationship with the L-space conjecture and give some other topological applications, as well as an analogous result for sutured Floer homology.

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1 Introduction

Heegaard Floer homology is a collection of invariants of low-dimensional objects: 3-manifolds, 4-manifolds, knots, and so on. Its most basic component is $\widehat{HF}$, which associates an $\mathbb{F}_2$-vector space $\widehat{HF}(Y,\mathfrak{s})$ to a closed, connected, oriented 3-manifold Y together with a spin^c-structure $\mathfrak{s}\in \mathrm{spin}^c(Y)$ [OSz04b]. Our main theorem concerns the behavior of $\widehat{HF}(Y)$ under taking branched covers:

Theorem 1.1. *Let Y be a closed 3-manifold, $L\subset Y$ an oriented nullhomologous link of $\ell>0$ components with Seifert surface F, and $\mathfrak{s}$ a spin^c-structure on Y. Let $\pi\colon \Sigma(Y,L)\to Y$ be the double cover branched along L induced by the Seifert surface F. Let $\pi^*\mathfrak{s}$ denote the pullback of $\mathfrak{s}$ to $\Sigma(Y,L)$ (Definition 4.4). Then, there is a spectral sequence with E^1-page given by*

$$\widehat{HF}(\Sigma(Y,L),\pi^*\mathfrak{s})\otimes H_*(T^{\ell-1})\otimes \mathbb{F}_2[[\theta,\theta^{-1}]$$

converging to

$$\bigoplus_{\{\mathfrak{s}'\mid \pi^*\mathfrak{s}'=\pi^*\mathfrak{s}\}}\widehat{HF}(Y,\mathfrak{s}')\otimes H_*(T^{\ell-1})\otimes \mathbb{F}_2[[\theta,\theta^{-1}].$$

In particular,

$$\dim \widehat{HF}(\Sigma(Y,L),\pi^*\mathfrak{s})\geq \sum_{\{\mathfrak{s}'\mid \pi^*\mathfrak{s}'=\pi^*\mathfrak{s}\}}\dim \widehat{HF}(Y,\mathfrak{s}').$$

Here, $T^{\ell-1}$ denotes the $(\ell-1)$-dimensional torus, so $H_*(T^{\ell-1})$ is isomorphic to the exterior algebra on $\ell-1$ generators. An oriented link $L\subset Y$ is nullhomologous if $[L]=0\in H_1(Y)$; we do not require each component to be nullhomologous. The pullback spin^c-structure $\pi^*\mathfrak{s}$ is explained in Definition 4.4. Throughout this paper, Floer homology groups have coefficients in $\mathbb{F}_2$ or an $\mathbb{F}_2$-module, and tensor products are over $\mathbb{F}_2$ unless otherwise noted.

Theorem 1.1 is part of a growing literature on the behavior of Heegaard Floer homology under various kinds of covers. Previously, Hendricks [Hen12] used Seidel-Smith's localization theorem for Lagrangian intersection Floer theory [SS10] to prove a similar result for the knot Floer homology of the double point set, as well as a spectral sequence for the Floer homology of 2-periodic links in S^3 [Hen15] (see also [HLS16,Boy18]). Lidman-Manolescu [LM18b] used Manolescu's homotopical refinement of monopole Floer homology [Man03] (see also [LM18a]) to prove an analogue of Theorem 1.1 for unbranched p-fold regular covers between rational homology spheres. Lipshitz-Treumann [LT16] used bordered Floer homology, Hochschild homology, and a Yoneda-type argument to prove analogous results for certain 2-fold covers of 3-manifolds with $b_1>0$

as well as for the knot Floer homology of knots with genus ≤ 2. (See also Remark 4.14.) Hendricks-Lipshitz-Sarkar [HLS16] deduced the special case $Y = S^3$ of Theorem 1.1 from Seidel-Smith's localization theorem, and used it to construct concordance invariants of knots.

Most recently, Large proved a generalization of Seidel-Smith's localization theorem and used it to prove there are spectral sequences for the knot Floer homology of branched double covers and $\widehat{HF}$ of ordinary double covers under less restrictive hypotheses [Lar19]. We deduce Theorem 1.1 from Large's localization theorem. The main work is to check that the bundle-theoretic hypotheses his result requires hold in the setting of $\widehat{HF}$ of branched double covers (see Section 3).

Theorem 1.1 has a number of corollaries. Recall that a rational homology sphere Y is a (modulo-2) *L-space* if $\dim \widehat{HF}(Y) = |H_1(Y)|$, the minimum possible dimension of $\widehat{HF}(Y)$; this is equivalent to $HF_{\mathrm{red}}(Y) = 0$.

Corollary 1.2. *Let L be a nullhomologous link in Y. If $b_1(\Sigma(Y,L)) \leq 1$ and $HF_{\mathrm{red}}(\Sigma(Y,L)) = 0$, then $HF_{\mathrm{red}}(Y) = 0$. In particular, if $\Sigma(Y,L)$ is an L-space then Y is an L-space.*

Ni points out that when restricting to non-torsion spin^c structures, Corollary 1.2 follows easily from the Thurston norm detection of Floer homology without Theorem 1.1 and requires no constraints on b_1.

Boyer-Gordon-Watson [BGW13] conjectured that an irreducible rational homology sphere Y is an L-space if and only if $\pi_1(Y)$ does not admit a left-invariant total order. This is known as the *L-space conjecture*. By work of Boyer-Rolfsen-Wiest [BRW05, Theorem 1.1], if $\pi_1(Y)$ does not admit a left-invariant total order then neither does the fundamental group of any 3-manifold Y' which admits a non-zero degree map from Y. So, Corollary 1.2 provides some further evidence for Boyer-Gordon-Watson's conjecture. In particular, we have:

Corollary 1.3. *Let L be a nullhomologous link in an irreducible rational homology sphere Y. If $\Sigma(Y,L)$ is an irreducible L-space and satisfies the L-space conjecture, then so does Y.*

Remark 1.4. It has also been conjectured that an irreducible rational homology sphere Y is an L-space if and only if Y admits a co-orientable taut foliation. Note that if Y admits a co-orientable taut foliation and K is transverse to the foliation, then $\Sigma(Y,K)$ admits a co-orientable taut foliation as well. However, there are nullhomologous knots which cannot be transverse to the foliation (e.g. if the knot is nullhomotopic) and Theorem 1.1 still predicts that the branched double cover should admit a co-orientable taut foliation if it is irreducible. It would be interesting to see evidence of this through foliations.

Remark 1.5. We do not know if the restriction that L be nullhomologous in Theorem 1.1 is necessary: in light of the L-space conjecture, perhaps the condition that $[L] = 0 \in H_1(Y;\mathbb{Z}/2\mathbb{Z})$ suffices. (This condition is needed to define a branched double cover at all.) The main step where we use that L is nullhomologous is the proof of Lemma 3.8, which is used to prove Proposition 3.1.

Theorem 1.1 also has some corollaries pertaining to the structure of Floer homology.

Corollary 1.6. *If* $\dim \widehat{HF}(\Sigma(Y,L),\pi^*\mathfrak{s}) = \dim \widehat{HF}(Y,\mathfrak{s})$ *then the involution* τ_* *on the Floer homology* $\widehat{HF}(\Sigma(Y,L),\pi^*\mathfrak{s})$ *of the branched double cover is the identity.*

Remark 1.7. The above corollary does not require the use of the main theorem if $\Sigma(Y,L)$ is an L-space or L is the Borromean knot in $\#_{2g}S^2 \times S^1$. We do not know any examples satisfying the hypothesis of the corollary when Y has non-trivial reduced Floer homology.

Corollary 1.8. *Let* Y *be a homology sphere with a non-trivial surgery to* S^3*. Let* K *be a knot in* Y *such that* $\Sigma(Y,K)$ *is an L-space. Then* $Y = S^3$ *or the Poincaré homology sphere.*

Proof. Let Y be a homology sphere obtained by surgery on a knot in S^3. By work of Ghiggini [Ghi08] and Ozsváth-Szabó [OSz04a], either Y is not an L-space, Y is the Poincaré homology sphere, or $Y = S^3$ and K is the unknot. If Y is not an L-space, Corollary 1.2 implies that $\Sigma(Y,K)$ cannot be an L-space. □

Remark 1.9. If we additionally ask that K be a knot realizing the S^3 surgery, we obtain stronger constraints. If Y is S^3, then K is unknotted by Gordon-Luecke [GL89]. If Y is the Poincaré homology sphere, then by Ghiggini's theorem, K is the core of surgery on the right-handed trefoil, that is, the singular fiber of order 5 in the unique Seifert fibered structure on the Poincaré homology sphere. We can compute the double cover of $\Sigma(2,3,5)$ branched over the singular fiber of order 5: it is the Seifert fibered space $S^2(-1;1/3,1/3,2/5)$. This manifold is not an L-space (see for example [LS07]). Hence, if K is a knot in a homology sphere Y with a non-trivial surgery to S^3 and branched double cover an L-space then Y is S^3 and K is the unknot.

Here is another application of the main theorem:

Proposition 1.10. *Let* K *be a knot in a prime homology sphere* Y*. Assume that* K *has determinant 1 and is obtained from the unknot by a rational tangle replacement. If* $\Sigma(Y,K)$ *is an L-space then either* K *is isotopic to an unknot or* $\pm T_{3,5}$ *in an embedded* B^3*.*

We also prove an analogue of Theorem 1.1 for sutured Floer homology:

Proposition 1.11. *Let* (M,γ) *be a balanced sutured manifold and* $L \subset M$ *a nullhomologous link with* $\ell > 0$ *components, and let* $(\Sigma(M,L),\widetilde{\gamma})$ *denote a double cover of* M *branched over* L *with the induced sutures. Then, there is a spectral sequence with* E^1 *page* $SFH(\Sigma(M,L),\widetilde{\gamma}) \otimes H_*(T^\ell) \otimes \mathbb{F}_2[[\theta,\theta^{-1}]$ *converging to* $SFH(M,\gamma) \otimes H_*(T^\ell) \otimes \mathbb{F}_2[[\theta,\theta^{-1}]$*. In particular,*

$$\dim SFH(\Sigma(M,L),\widetilde{\gamma}) \geq \dim SFH(M,\gamma).$$

Note that here we have $H_*(T^\ell)$ instead of $H_*(T^{\ell-1})$ as in Theorem 1.1. Branched covers of sutured manifolds are discussed further in Section 5.1.

There is a relationship between Theorem 1.1 and the Smith conjecture [Smi39, MB84]. Specifically, the Smith conjecture implies that $\mathbb{Z}/p$-actions on S^3 with nonempty fixed sets are standard, so S^3 is not the branched cover of any other 3-manifold. Theorem 1.1 implies the weaker statement that if S^3 is the branched cover of Y then Y is an L-space integer homology sphere. Ozsváth-Szabó conjecture that the only irreducible integer homology sphere L-spaces are S^3 and the Poincaré homology sphere [OSz06, Section 1.5] (see also [HL16, Conjecture 1]); this is sometimes referred to, somewhat drolly, as the Heegaard Floer Poincaré Conjecture. Together with the Heegaard Floer Poincaré Conjecture, Theorem 1.1 implies that if S^3 or the Poincaré homology sphere is a branched cover of Y then Y is itself a connect sum of copies of the Poincaré sphere.

It would be interesting to obtain a similar result in Seiberg-Witten theory, extending Lidman-Manolescu's work [LM18b]. In particular, such a result would perhaps entail studying Seiberg-Witten solutions on the orbifold quotient of the branched double cover, and relating them with the underlying manifold. There have been a number of other results on the Heegaard or Seiberg-Witten Floer homology of branched covers with which it would also be interesting to compare [Kan18b, Kan18a, AKS20, KL15, LRS18, LRS20]. In particular, perhaps Lin-Ruberman-Saveliev's techniques [LRS20] could lead to a Seiberg-Witten-theoretic proof of Theorem 1.1.

This paper is organized as follows. Section 2 recalls Large's localization theorem and some background about K-theory and maps of stable vector bundles. Section 3 verifies the main hypothesis for Large's localization theorem, an isomorphism between the stable relative tangent and normal bundles to the fixed sets. Section 4 verifies the remaining hypotheses and deduces Theorem 1.1. Finally, Section 5 discusses applications of Theorem 1.1, as well as Proposition 1.11 for sutured Floer homology.

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2 Background

2.1 Polarization data

The following definitions are drawn from Large's paper [Lar19, Section 3.2].

Definition 2.1. *Let* (M, L_0, L_1) *be a symplectic manifold and two Lagrangian submanifolds. A* set of polarization data *for* (M, L_0, L_1) *is a triple* $\mathfrak{p} =$

(E, F_0, F_1) *where*

- E *is a symplectic vector bundle over* M
- F_i *is a Lagrangian subbundle of* $E|_{L_i}$ *for* $i = 0, 1$.

Given (M, L_0, L_1) and $\mathfrak{p} = (E, F_0, F_1)$ a set of polarization data for (M, L_0, L_1), we may stabilize to obtain $\mathfrak{p} \oplus \underline{\mathbb{C}} = (E \oplus \underline{\mathbb{C}}, F_0 \oplus \underline{\mathbb{R}}, F_1 \oplus i\underline{\mathbb{R}})$.

Definition 2.2. *Let* $\mathfrak{p} = (E, F_0, F_1)$ *and* $\mathfrak{p}' = (E', F_0', F_1')$ *be two sets of polarization data for* (M, L_0, L_1). *An* isomorphism of polarization data *is an isomorphism of symplectic vector bundles*

$$\alpha \colon E \to E'$$

such that there are homotopies of Lagrangian subbundles of $E'|_{L_i}$ *between* $\alpha(F_i)$ *and* F_i' *for* $i = 0, 1$ *(so that the subbundles stay Lagrangian throughout the homotopy). A* stable isomorphism of polarization data *between* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *is an isomorphism of polarization data between* $\mathfrak{p} \oplus \underline{\mathbb{C}}^n$ *and* $\mathfrak{p}' \oplus \underline{\mathbb{C}}^{n'}$ *for some* n, n'.

One special case of this definition will be of particular importance. Suppose (M, L_0, L_1) is equipped with a symplectic involution preserving L_0 and L_1 setwise. Let $(M^{\mathit{fix}}, L_0^{\mathit{fix}}, L_1^{\mathit{fix}})$ denote the fixed sets under the involution. Then there are two sets of polarization data for $(M^{\mathit{fix}}, L_0^{\mathit{fix}}, L_1^{\mathit{fix}})$: the *tangent polarization* $(TM^{\mathit{fix}}, TL_0^{\mathit{fix}}, TL_1^{\mathit{fix}})$ consisting of the tangent bundles to M^{fix} and L_i^{fix}, and the *normal polarization* $(NM^{\mathit{fix}}, NL_0^{\mathit{fix}}, NL_1^{\mathit{fix}})$ consisting of the normal bundles to $M^{\mathit{fix}} \subset M$ and $L_i^{\mathit{fix}} \subset L_i$.

Definition 2.3. *With notation as above, a* stable tangent-normal isomorphism *is a stable isomorphism of polarization data between the tangent polarization* $(TM^{\mathit{fix}}, TL_0^{\mathit{fix}}, TL_1^{\mathit{fix}})$ *and the normal polarization* $(NM^{\mathit{fix}}, NL_0^{\mathit{fix}}, NL_1^{\mathit{fix}})$.

2.2 Large's localization theorem

The following is an immediate consequence of Large's construction of equivariant Floer homology and its formal properties (including his localization isomorphism):

Theorem 2.4. *[Lar19] Suppose that*

(L1) M *is an exact symplectic manifold and convex at infinity, and* L_0, L_1 *are exact Lagrangians such that either* L_0 *and* L_1 *are compact or* M *is a symplectization near infinity and* L_0 *and* L_1 *are conical and disjoint near infinity;*

(L2) τ *is a symplectic involution of* M *preserving the* L_i *setwise, and* $(M^{\mathit{fix}}, L_0^{\mathit{fix}}, L_1^{\mathit{fix}})$ *are the fixed sets under* τ*; and*

(L3) there is a stable tangent-normal isomorphism between the data $(NM^{fix}, NL_0^{fix}, NL_1^{fix})$ *and* $(TM^{fix}, TL_0^{fix}, TL_1^{fix})$.

Then there is an ungraded spectral sequence with E_1*-page isomorphic to* $HF(L_0, L_1) \otimes_{\mathbb{F}_2} \mathbb{F}_2[[\theta, \theta^{-1}]$ *converging to* $HF(L_0^{fix}, L_1^{fix}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[[\theta, \theta^{-1}]$. *In particular, there is a rank inequality* $\dim_{\mathbb{F}_2} HF(L_0, L_1) \geq \dim_{\mathbb{F}_2} HF(L_0^{fix}, L_1^{fix})$.

Proof. This argument is essentially given by Large [Lar19, Proof of Theorem 1.4]; we summarize it here. First, under the hypotheses (L1) and (L2), Seidel-Smith [SS10, Section 3.2] couple the $\bar{\partial}$-equation on (M, L_0, L_1) to Morse theory on $\mathbb{R}P^\infty$ to construct $\mathbb{Z}/2\mathbb{Z}$-equivariant Floer homology groups $HF^{SS}_{\mathbb{Z}/2\mathbb{Z}}(L_0, L_1)$ and a spectral sequence

$$HF(L_0, L_1) \otimes_{\mathbb{F}_2} \mathbb{F}_2[[\theta]] \Rightarrow HF^{SS}_{\mathbb{Z}/2\mathbb{Z}}(L_0, L_1). \tag{2.5}$$

(See also [HLS16] for an equivalent construction.) Under the same hypotheses, Large uses a blow-up construction analogous to Kronheimer-Mrowka's construction of monopole Floer homology to define another equivariant cohomology group $HF^{KM}_{\mathbb{Z}/2\mathbb{Z}}(L_0, L_1)$. He then shows [Lar19, Theorem 1.2 or Theorem 8.1] that

$$HF^{KM}_{\mathbb{Z}/2\mathbb{Z}}(L_0, L_1) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[[\theta]] \cong HF^{SS}_{\mathbb{Z}/2\mathbb{Z}}(L_0, L_1). \tag{2.6}$$

Given a set of polarization data $\mathfrak{p}$, under hypothesis (L1) Large also constructs a Floer homology twisted by $\mathfrak{p}$, $HF_{tw}(L_0, L_1; \mathfrak{p})$. In the special case that $\mathfrak{p}_N$ is the normal polarization for $(M^{fix}, L_0^{fix}, L_1^{fix})$, he shows [Lar19, Theorem 1.1] that there is an isomorphism

$$HF^{KM}_{\mathbb{Z}/2\mathbb{Z}}(L_0, L_1) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}] \cong HF_{tw}(L_0^{fix}, L_1^{fix}; \mathfrak{p}_N). \tag{2.7}$$

On the other hand, using what he calls the total Steenrod square (coming from the $\mathbb{Z}/2\mathbb{Z}$-action on $M \times M$ exchanging the factors), he shows [Lar19, Proposition 9.5] that for the tangent polarization $\mathfrak{p}_T$,

$$HF_{tw}(L_0^{fix}, L_1^{fix}; \mathfrak{p}_T) \cong HF(L_0^{fix}, L_1^{fix}) \otimes_{\mathbb{F}_2} \mathbb{F}[\theta, \theta^{-1}]. \tag{2.8}$$

(This uses the action filtration. In particular, exactness of the Lagrangians is used here.) The existence of a tangent-normal isomorphism yields an isomorphism

$$HF_{tw}(L_0^{fix}, L_1^{fix}; \mathfrak{p}_N) \cong HF_{tw}(L_0^{fix}, L_1^{fix}; \mathfrak{p}_T). \tag{2.9}$$

Combining these formulas gives the spectral sequence. Finally, the rank inequality over $\mathbb{F}_2$ follows from the universal coefficient theorem, bearing in mind that $\mathbb{F}_2$ and $\mathbb{F}_2[[\theta, \theta^{-1}]$ are fields. $\square$

We note a minor refinement of Large's result. Let $P(L_0, L_1)$ denote the space of paths from L_0 to L_1. For $x \in L_0 \cap L_1$ there is a corresponding constant path $[x] \in P(L_0, L_1)$. Two points $x, y \in L_0 \cap L_1$ can be connected by a Whitney disk if and only if $[x]$ and $[y]$ lie in the same component of $P(L_0,L_1)$. So, the Floer complex $CF(L_0,L_1)$ decomposes as a direct sum

$$CF(L_0,L_1)=\bigoplus_{\mathfrak{s}\in\pi_0P(L_0,L_1)}CF(L_0,L_1;\mathfrak{s}). \tag{2.10}$$

The relevance for us is that, in Heegaard Floer homology, the path components of $P(\mathbb{T}_\alpha,\mathbb{T}_\beta)$ correspond to the spin^c-structures on Y.

In the setting of Theorem 2.4, there is an inclusion map $\iota\colon P(L_0^{fix},L_1^{fix})\hookrightarrow P(L_0,L_1)$, inducing a set map $\iota_*\colon\pi_0P(L_0^{fix},L_1^{fix})\to\pi_0P(L_0,L_1)$. The map ι_* is typically neither injective nor surjective. Large's invariant $HF_{tw}(L_0^{fix},L_1^{fix};\mathfrak{p})$ decomposes along $\pi_0P(L_0^{fix},L_1^{fix})$ as

$$HF_{tw}(L_0^{fix},L_1^{fix};\mathfrak{p})=\bigoplus_{\mathfrak{s}\in\pi_0P(L_0^{fix},L_1^{fix})}HF_{tw}(L_0^{fix},L_1^{fix};\mathfrak{p};\mathfrak{s}) \tag{2.11}$$

and hence also as

$$HF_{tw}(L_0^{fix},L_1^{fix};\mathfrak{p})=\bigoplus_{\widetilde{\mathfrak{s}}\in\pi_0P(L_0,L_1)}\bigoplus_{\mathfrak{s}\in\iota_*^{-1}(\widetilde{\mathfrak{s}})}HF_{tw}(L_0^{fix},L_1^{fix};\mathfrak{p};\mathfrak{s}). \tag{2.12}$$

Both the invariants $HF^{KM}_{\mathbb{Z}/2\mathbb{Z}}(L_0,L_1)$ and $HF^{SS}_{\mathbb{Z}/2\mathbb{Z}}(L_0,L_1)$ and the Seidel-Smith spectral sequence (2.5) decompose along τ-orbits in $\pi_0P(L_0,L_1)$ as

$$HF^{SS/KM}_{\mathbb{Z}/2\mathbb{Z}}(L_0,L_1)=\bigoplus_{[\widetilde{\mathfrak{s}}]\in\pi_0P(L_0,L_1)/\tau}HF^{SS/KM}_{\mathbb{Z}/2\mathbb{Z}}(L_0,L_1;[\widetilde{\mathfrak{s}}]) \tag{2.13}$$

$$\bigoplus_{\widetilde{\mathfrak{s}}\in[\widetilde{\mathfrak{s}}]}HF(L_0,L_1;\widetilde{\mathfrak{s}})\otimes_{\mathbb{F}_2}\mathbb{F}_2[[\theta]]\Rightarrow HF^{SS}_{\mathbb{Z}/2\mathbb{Z}}(L_0,L_1;[\widetilde{\mathfrak{s}}]). \tag{2.14}$$

Further, the equivariant Steenrod square, which comes from Floer theory on $M^{fix}\times M^{fix}$, respects the decompositions (2.10) and (2.11), and the localization isomorphism (2.7) respects the decompositions (2.12) and (2.13). (If $\widetilde{\mathfrak{s}}$ is not fixed by τ then $HF^{SS/KM}_{\mathbb{Z}/2\mathbb{Z}}(L_0,L_1;[\widetilde{\mathfrak{s}}])\cong HF(L_0,L_1;\widetilde{\mathfrak{s}})$ for either representative $\widetilde{\mathfrak{s}}$ of $[\widetilde{\mathfrak{s}}]$ and, in particular, is θ-torsion.)

So, we have:

Proposition 2.15. *Under the same hypotheses as Theorem 2.4, for each $\widetilde{\mathfrak{s}}\in\pi_0P(L_0,L_1)$ there is a spectral sequence*

$$HF(L_0,L_1;\widetilde{\mathfrak{s}})\otimes_{\mathbb{F}_2}\mathbb{F}_2[[\theta,\theta^{-1}]\Rightarrow\bigoplus_{\mathfrak{s}\in\iota_*^{-1}(\widetilde{\mathfrak{s}})}HF(L_0^{fix},L_1^{fix};\mathfrak{s})\otimes_{\mathbb{F}_2}\mathbb{F}_2[[\theta,\theta^{-1}]$$

and a rank inequality

$$\dim_{\mathbb{F}_2}HF(L_0,L_1;\widetilde{\mathfrak{s}})\geq\sum_{\mathfrak{s}\in\iota_*^{-1}(\widetilde{\mathfrak{s}})}\dim_{\mathbb{F}_2}HF(L_0^{fix},L_1^{fix};\mathfrak{s}).$$

2.3 K-theory and maps of stable vector bundles

In this section we recall some notions related to the K-theory of complex vector bundles. We consider bundles over a CW complex X which is homotopy equivalent to a finite CW complex.
We focus particularly on maps between stable bundles. The main goal is to recall that the set of homotopy classes of isomorphisms between stable bundles is an affine copy of $K^1(X)$ and hence, under favorable conditions, there is a Chern character isomorphism from this set to the odd cohomology of X.

Definition 2.16. *Let E, E' be complex vector bundles over a base X. A* stable isomorphism *from E to E' is a bundle isomorphism*

$$f\colon E \oplus \underline{\mathbb{C}}^N \to E' \oplus \underline{\mathbb{C}}^N$$

for some integer N. Stable isomorphisms compose in the obvious way.
Two stable isomorphisms $f_i\colon E \oplus \underline{\mathbb{C}}^{N_i} \to E' \oplus \underline{\mathbb{C}}^{N_i}$, $i = 1, 2$, are homotopic *if there is an integer $M \geq \max\{N_1, N_2\}$ and a homotopy between*

$$f_1 \oplus \mathbb{I}_{\underline{\mathbb{C}}^{M-N_1}}, f_2 \oplus \mathbb{I}_{\underline{\mathbb{C}}^{M-N_2}} : E \oplus \underline{\mathbb{C}}^M \to E' \oplus \underline{\mathbb{C}}^M.$$

Let $\mathrm{Iso}(E, E')$ denote the set of homotopy classes of stable isomorphisms from E to E'.

Definition 2.17. *Let $\underline{\mathbb{C}}^0$ denote the trivial 0-dimensional vector bundle over X. Let E, E' be vector bundles over X so that $\mathrm{Iso}(E, E') \neq \varnothing$. Given $[f] \in \mathrm{Iso}(E, E')$ and $[g] \in \mathrm{Iso}(\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$ define $[f * g] \in \mathrm{Iso}(E, E')$ as follows. The map f is a bundle isomorphism $E \oplus \underline{\mathbb{C}}^N \to E' \oplus \underline{\mathbb{C}}^N$ and the map g is a bundle isomorphism $\underline{\mathbb{C}}^M \to \underline{\mathbb{C}}^M$, for some integers M, N. Then $[f * g]$ is the homotopy class of the bundle isomorphism $f \oplus g\colon E \oplus \underline{\mathbb{C}}^N \oplus \underline{\mathbb{C}}^M \to E' \oplus \underline{\mathbb{C}}^N \oplus \underline{\mathbb{C}}^M$.*

Proposition 2.18. *Let X be a CW complex homotopy equivalent to a finite CW complex. Then, given complex vector bundles E, E' over X with $\mathrm{Iso}(E, E') \neq \varnothing$, Definition 2.17 defines an action of $\mathrm{Iso}(\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$ on $\mathrm{Iso}(E, E')$. Further, this action makes $\mathrm{Iso}(E, E')$ into a torsor over $\mathrm{Iso}(\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$.*

Proof. The key point is that given a bundle E'' and bundle isomorphisms $k, \ell\colon E'' \to E''$ the bundle isomorphisms

$$k \oplus \ell,\ (k \circ \ell) \oplus \mathbb{I}\colon E'' \oplus E'' \to E'' \oplus E''$$

are homotopic. To see this, note that given an invertible 2×2 matrix A over $\mathbb{C}$ there is an induced automorphism $A\colon E'' \oplus E'' \to E'' \oplus E''$. The homotopy between $k \oplus \ell$ and $(k \circ \ell) \oplus \mathbb{I}$ is given by

$$\begin{pmatrix} k & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \ell \end{pmatrix} \begin{pmatrix} \cos(\pi t) & \sin(\pi t) \\ -\sin(\pi t) & \cos(\pi t) \end{pmatrix}$$

(cf. [Ati89]).

Using this observation, if $f' = f \oplus \mathbb{I}_{\mathbb{C}^K}$ then $f' \oplus g \sim f \oplus g \oplus \mathbb{I}_{\mathbb{C}^K}$. (Here, the bundle E'' in the key observation is a trivial bundle.) It follows easily that $[f] * [g]$ is independent of the choices of representatives f and g. Next, for appropriate choices of representatives, $[f] * ([g] * [h])$ and $([f] * [g]) * [h]$ agree on the nose. It remains to see that for any pair of elements $f, h \in \mathrm{Iso}(E, E')$ there is a $g \in \mathrm{Iso}(\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$ so that $f * g = h$.

Given $[f] \in \mathrm{Iso}(E, E')$, composition with f gives a bijection between the $\mathrm{Iso}(\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$-sets $\mathrm{Iso}(E, E)$ and $\mathrm{Iso}(E, E')$. So, it suffices to prove freeness and transitivity of the action in the case that $[f], [h] \in \mathrm{Iso}(E, E)$.

We start with transitivity of the action. To keep notation simple, replace E by its sum with a high-dimensional trivial bundle, so $f, h\colon E \to E$. Choose a bundle F so that $E \oplus F$ is isomorphic to a trivial bundle $\underline{\mathbb{C}}^N$. Let $\phi\colon E \oplus F \xrightarrow{\cong} \underline{\mathbb{C}}^N$ be an isomorphism. Then we have isomorphisms

$$\phi \circ (f \oplus \mathbb{I}_F) \circ \phi^{-1},\ \phi \circ (h \oplus \mathbb{I}_F) \circ \phi^{-1} \colon \underline{\mathbb{C}}^N \to \underline{\mathbb{C}}^N.$$

Let

$$g = \phi \circ (f \oplus \mathbb{I}_F)^{-1} \circ (h \oplus \mathbb{I}_F) \circ \phi^{-1} \colon \underline{\mathbb{C}}^N \to \underline{\mathbb{C}}^N.$$

We claim that $f * g \sim h$. Indeed, applying the key point above, we have

$$\begin{aligned} f * g &= (\mathbb{I}_E \oplus \phi) \circ (f \oplus f^{-1} \oplus \mathbb{I}_F) \circ (\mathbb{I}_E \oplus h \oplus \mathbb{I}_F) \circ (\mathbb{I}_E \oplus \phi^{-1}) \\ &\sim (\mathbb{I}_E \oplus \phi) \circ ((f \circ f^{-1} \circ h) \oplus \mathbb{I}_E \oplus \mathbb{I}_F) \circ (\mathbb{I}_E \oplus \phi^{-1}) \\ &= (\mathbb{I}_E \oplus \phi) \circ (h \oplus \mathbb{I}_E \oplus \mathbb{I}_F) \circ (\mathbb{I}_E \oplus \phi^{-1}) \\ &= h \oplus \mathbb{I}_{\mathbb{C}^N}, \end{aligned}$$

as desired.

Similarly, for freeness, suppose that $[f]*[g] = [f]*[g']$. By stabilizing as needed, we may assume that $f * g$ and $f * g'$ are homotopic maps $E \oplus \underline{\mathbb{C}}^M \to E \oplus \underline{\mathbb{C}}^M$. Let F be as above. Then

$$f \oplus \mathbb{I}_F \oplus g \sim f \oplus \mathbb{I}_F \oplus g' \colon E \oplus F \oplus \underline{\mathbb{C}}^M \to E \oplus F \oplus \underline{\mathbb{C}}^M,$$

so

$$[\phi \circ (f \oplus \mathbb{I}_F)] \oplus g \sim [\phi \circ (f \oplus \mathbb{I}_F)] \oplus g' \colon \underline{\mathbb{C}}^N \oplus \underline{\mathbb{C}}^M \to \underline{\mathbb{C}}^N \oplus \underline{\mathbb{C}}^M.$$

Thus, composing both sides with $(\phi \circ (f \oplus \mathbb{I}_F))^{-1} \oplus \mathbb{I}_{\mathbb{C}^M}$, the maps $\mathbb{I}_{\mathbb{C}}^N \oplus g$ and $\mathbb{I}_{\mathbb{C}}^N \oplus g'$ are homotopic, so $[g] = [g']$. This completes the proof. □

Remark 2.19. Here is an alternative understanding of Proposition 2.18. The stable automorphisms of the trivial bundle over X are the same as $\pi_1(\mathrm{Map}(X, BU))$, based at the constant map. The group of stable automorphisms of a nontrivial bundle is the fundamental group of a different path component of $\mathrm{Map}(X, BU)$. Since BU is an h-space, all path components of $\mathrm{Map}(X, BU)$ have isomorphic fundamental groups.

We can extend the Chern character to stable isomorphisms. Recall that given an automorphism f of the trivial bundle $\underline{\mathbb{C}}^N$ over X, the mapping cylinder $\mathrm{Cyl}(f)$ of f is a bundle over $X \times [0,1]$ equipped with a trivialization of $\mathrm{Cyl}(f)|_{X\times\{0,1\}}$. Specifically, $\mathrm{Cyl}(f) = \big((\underline{\mathbb{C}}^N \times [0,1]) \amalg \underline{\mathbb{C}}^N\big)/\sim$ where $(x,v,1) \in \underline{\mathbb{C}}^N \times \{1\}$ is identified to $(x, f(x)(v)) \in \underline{\mathbb{C}}^N$, and the trivializations over $X \times \{0\}$ and X are the standard ones. Equivalently, $\mathrm{Cyl}(f)$ is the trivial bundle over $X \times [0,1]$ where the trivializations over $X \times \{0\}$ and $X \times \{1\}$ are the standard trivialization and f, respectively.

A (stable) trivialization of the relative bundle $(\mathrm{Cyl}(f), \mathrm{Cyl}(f)|_{X\times\{0,1\}})$ is equivalent to a (stable) homotopy between f and the identity map. Consequently, the Chern character of $\mathrm{Cyl}(f)$ is an element

$$\mathrm{ch}(f) \in H^{\mathrm{even}}(X \times [0,1], X \times \{0,1\}; \mathbb{Q}) = H^{\mathrm{even}}(SX; \mathbb{Q}) = H^{\mathrm{odd}}(X; \mathbb{Q})$$

and the map

$$\mathrm{ch}\colon \mathrm{Iso}(\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0) \otimes \mathbb{Q} \to H^{\mathrm{odd}}(X; \mathbb{Q})$$

is an isomorphism.

By Proposition 2.18, given an element $[f] \in \mathrm{Iso}(E, E')$, any other element $[h] \in \mathrm{Iso}(E, E')$ can be written as $[h] = [f] * [g]$ for a unique $[g] \in \mathrm{Iso}(\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$. Define

$$\mathrm{ch}_f([h]) = \mathrm{ch}([g]) \in H^{\mathrm{odd}}(X; \mathbb{Q}).$$

In particular, in the case $E = E'$ we can take $f = \mathbb{I}$, and we have a canonical choice of Chern character $\mathrm{ch}\colon \mathrm{Iso}(E,E) \to H^{\mathrm{odd}}(X;\mathbb{Q})$. Here is an alternative description of the Chern character in this case. Given $h \in \mathrm{Iso}(E,E)$ the mapping torus T_h of h is a vector bundle over $X \times S^1$. The maps $X \hookrightarrow X \times S^1 \twoheadrightarrow X$ and the canonical generator $[S^1] \in H^1(S^1)$ identify $H^{\mathrm{even}}(X \times S^1) \cong H^{\mathrm{even}}(X) \oplus H^{\mathrm{odd}}(X)$; the map $H^{\mathrm{odd}}(X) \to H^{\mathrm{even}}(X \times S^1)$ is $a \mapsto a \times [S^1]$. We have:

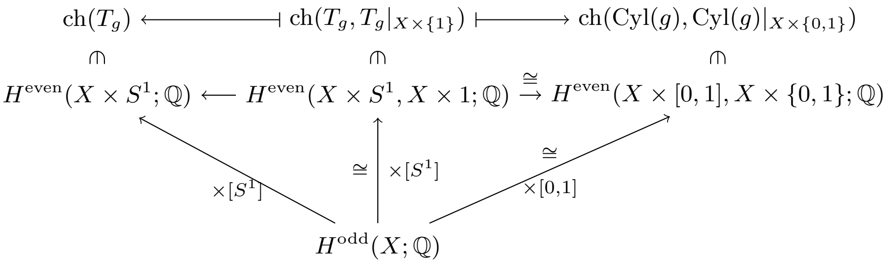
Lemma 2.20. *For $h \in \mathrm{Iso}(E,E)$, the Chern character $\mathrm{ch}(h)$ is the image of the Chern character of T_h in $H^{\mathrm{odd}}(X)$.*

Proof. We first reduce to the case that E is the trivial bundle. Write $h = \mathbb{I} * g$, where $g \in \mathrm{Iso}(\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$. On the one hand, $\mathrm{ch}(h) = \mathrm{ch}(g)$. On the other hand, T_h is stably isomorphic to $T_{\mathbb{I}} \oplus T_g$, so $\mathrm{ch}(T_{\mathbb{I}} \oplus T_g) = \mathrm{ch}(T_{\mathbb{I}}) + \mathrm{ch}(T_g)$. Since $T_{\mathbb{I}} = E \times S^1$, $\mathrm{ch}(T_{\mathbb{I}}) \in H^{\mathrm{even}}(X) \subset H^{\mathrm{even}}(X \times S^1)$. Hence, the image of $\mathrm{ch}(T_{\mathbb{I}})$ in $H^{\mathrm{odd}}(X)$ vanishes, so the image of $\mathrm{ch}(T_{\mathbb{I}} \oplus T_g)$ is the same as the image of $\mathrm{ch}(T_g)$.

So, it remains to show that for $g \in \mathrm{Iso}(\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$, the class $\mathrm{ch}(g)$ agrees with the image of $\mathrm{ch}(T_g)$. Fix a distinguished point $1 \in S^1$. There is a commutative diagram of bundles and trivializations

$$\begin{array}{ccccc}
(T_g, \varnothing) & \longrightarrow & (T_g, T_g|_{X\times\{1\}}) & \longleftarrow & (\mathrm{Cyl}(g), \mathrm{Cyl}(g)|_{X\times\{0,1\}}) \\
\downarrow & & \downarrow & & \downarrow \\
(X \times S^1, \varnothing) & \longrightarrow & (X \times S^1, X \times \{1\}) & \longleftarrow & (X \times [0,1], X \times \{0,1\}).
\end{array}$$

(In the top row, the entries $T_g|_{X\times\{1\}}$ and $\mathrm{Cyl}(g)|_{X\times\{0,1\}}$ are shorthand for the fixed trivializations of these bundles.) Further, naturality of the cohomology cross product, the definition of the fundamental class in cohomology, and naturality of the Chern character give a commutative diagram



The Chern character of g is the preimage of $\mathrm{ch}(\mathrm{Cyl}(g), \mathrm{Cyl}(g)|_{0,1})$ under the right diagonal isomorphism. Thus, the left diagonal arrow sends the Chern character of g to the Chern character of T_g, as claimed. □

This Chern character map is natural in the following sense:

Lemma 2.21. *Let $G\colon X\to Y$ be a continuous map, E, E' be complex vector bundles over Y, and $[f],[h]\in\mathrm{Iso}(E,E')$. There are induced isomorphisms $[G^*f],[G^*h]\in\mathrm{Iso}(G^*E,G^*E')$. Then,*

$$\mathrm{ch}_{G^*f}([G^*h]) = G^*\,\mathrm{ch}_f([h]).$$

In particular, if $E=E'$ then

$$\mathrm{ch}([G^*h]) = G^*\,\mathrm{ch}([h]).$$

Proof. This is immediate from the definitions. □

Finally, the Chern character respects composition:

Lemma 2.22. *If $[h_1],[h_2]\in\mathrm{Iso}(E,E)$ then*

$$\mathrm{ch}([h_2\circ h_1]) = \mathrm{ch}(h_1)+\mathrm{ch}(h_2).$$

More generally, given bundles E_1,E_2,E_3 and maps $[f_1],[h_1]\in\mathrm{Iso}(E_1,E_2)$ and $[f_2],[h_2]\in\mathrm{Iso}(E_2,E_3)$ we have

$$\mathrm{ch}_{f_2\circ f_1}([h_2\circ h_1]) = \mathrm{ch}_{f_1}([h_1])+\mathrm{ch}_{f_2}([h_2]).$$

Proof. We prove the more general statement; the special case follows by taking $f_1=f_2=\mathbb{I}$. Write $[h_1]=[f_1*g_1]$ and $[h_2]=[f_2*g_2]$. As in the beginning of the proof of Proposition 2.18, $[h_2\circ h_1]=[(f_2\circ f_1)*g_1*g_2]$. Hence

$$\mathrm{ch}_{f_2\circ f_1}([h_2\circ h_1]) = \mathrm{ch}(g_1*g_2).$$

It is immediate from the construction of the Chern character for maps of trivial bundles and additivity of the usual Chern character for complex vector bundles that for $g_1, g_2 \in \mathrm{Iso}(\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$, $\mathrm{ch}(g_1 * g_2) = \mathrm{ch}(g_1) + \mathrm{ch}(g_2)$. The result follows. □

Remark 2.23. Since the inclusion of the unitary group into the symplectic group is a homotopy equivalence, the K-theory of complex vector bundles is the same as the K-theory of symplectic vector bundles. In particular, one can take the Chern character of symplectic vector bundles and isomorphisms between them, and the results of this section hold in the symplectic case as well.

3 The stable tangent-normal isomorphism

Let $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ be a doubly-pointed Heegaard diagram for a nullhomologous knot K in a 3-manifold Y, $\pi\colon \Sigma(Y,K) \to Y$ a double cover of Y branched along K, and $\widetilde{K} = \pi^{-1}(Y)$. There is an induced doubly-pointed Heegaard diagram $\widetilde{\mathcal{H}} = (\widetilde{\Sigma}_{2g}, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{z}, \widetilde{w})$ for $(\Sigma(Y,K), \widetilde{K})$ as follows. Viewing Σ as a subset of Y, $\widetilde{\Sigma} = \pi^{-1}(\Sigma)$. The preimage of $\boldsymbol{\alpha}$ (respectively $\boldsymbol{\beta}$) is a collection of $2g$ circles $\widetilde{\boldsymbol{\alpha}}$ (respectively $\widetilde{\boldsymbol{\beta}}$) in $\widetilde{\Sigma}$, and the preimage of z (respectively w) is a point $\widetilde{z}$ (respectively $\widetilde{w}$) in $\widetilde{\Sigma}$.
The covering involution $\tau\colon \Sigma(Y,K) \to \Sigma(Y,K)$ induces an involution τ of $\widetilde{\mathcal{H}}$. A complex structure on Σ induces a τ-equivariant complex structure on $\widetilde{\Sigma}$, which makes $\mathrm{Sym}^{2g}(\widetilde{\Sigma})$ into a smooth complex manifold. The involution τ induces a smooth involution of $\mathrm{Sym}^{2g}(\widetilde{\Sigma})$, by

$$\tau(\{x_1, \dots, x_{2g}\}) = \{\tau(x_1), \dots, \tau(x_{2g})\}.$$

The goal of this section is to prove:

PROPOSITION 3.1. *Let $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ be a doubly-pointed Heegaard diagram for a nullhomologous knot K in a closed 3-manifold Y and let $\widetilde{\mathcal{H}} = (\widetilde{\Sigma}, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{z}, \widetilde{w})$ be the branched double cover of $\mathcal{H}$, which is a doubly-pointed Heegaard diagram for $(\Sigma(Y,K), \widetilde{K})$. Then there is a stable tangent-normal isomorphism*

$$\left(T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}\})^{\mathit{fix}}, T\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}}, T\mathbb{T}_{\widetilde{\beta}}^{\mathit{fix}}\right) \cong \left(N\,\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}\})^{\mathit{fix}}, N\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}}, N\mathbb{T}_{\widetilde{\beta}}^{\mathit{fix}}\right).$$

We start by noting that the fixed set of the involution is familiar:

LEMMA 3.2. *There is a τ-equivariant Kähler form on $\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}\})$ and a Kähler form on $\mathrm{Sym}^{g}(\Sigma \setminus \{z\})$, so that the fixed set $\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}\})^{\mathit{fix}}$ is symplectomorphic to $\mathrm{Sym}^{g}(\Sigma \setminus \{z\})$, and the symplectomorphism takes the fixed sets $(\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}}, \mathbb{T}_{\widetilde{\beta}}^{\mathit{fix}})$ of the Lagrangian tori to the Lagrangian tori $\mathbb{T}_{\alpha}$ and $\mathbb{T}_{\beta}$.*

Proof. The proof is the same as the analogous result for branched double covers of genus 0 multi-pointed Heegaard diagrams for links in S^3 [Hen12, Section 4 and Appendix A]. □

Lemma 3.3. *Let $\bigvee_{i=1}^k S_i^1$ be a bouquet of circles. Choose coordinates on each S_i^1 such that the wedge point is $1 \in S^1 \subset \mathbb{C}$. Then $\mathrm{Sym}^r(\bigvee_{i=1}^k S_i^1)$ deformation retracts onto its subspace*

$$\Big\{(z_1,\dots,z_k) \in \prod_{i=1}^k S_i^1 \mid \text{at most } r \text{ coordinates satisfy } z_i \neq 1\Big\}.$$

In particular, if $r \geq k$, $\mathrm{Sym}^r(\bigvee_{i=1}^k S_i^1)$ is homotopy equivalent to the k-torus $\prod_{i=1}^k S_i^1$, while if $r < k$, then $\mathrm{Sym}^r(\bigvee_{i=1}^k S_i^1)$ is homotopy equivalent to the r-skeleton of the k-torus $\prod_{i=1}^k S_i^1$ with respect to the standard product CW decomposition of the torus.

Proof. The map $\mathrm{Sym}^r(S^1) \to S^1$ given by multiplication $\{z_1,\dots,z_r\} \mapsto z_1 \cdots z_r$ is a homotopy equivalence (see, e.g., the proof of [Hen12, Lemma 5.1] or, for the essence of the argument, [Hat02, Example 4K.4]). Work of Ong [Ong03] (see also [Hen12, Lemma 5.1]) shows that this map can be used to construct the desired deformation retract from $\mathrm{Sym}^r(\bigvee_{i=1}^k S_i^1)$ to the r-skeleton of the torus. □

Corollary 3.4. *Given a complex vector bundle $E \to \mathrm{Sym}^g(\Sigma \setminus \{z\})$, the Chern character map $\mathrm{ch}\colon \mathrm{Iso}(E,E) \to H^{\mathrm{odd}}(\mathrm{Sym}^g(\Sigma\setminus\{z\});\mathbb{Q})$ (Section 2.3) is injective with image $H^{\mathrm{odd}}(\mathrm{Sym}^g(\Sigma\setminus\{z\});\mathbb{Z})$ and hence induces an isomorphism $\mathrm{ch}\colon \mathrm{Iso}(E,E) \to H^{\mathrm{odd}}(\mathrm{Sym}^g(\Sigma\setminus\{z\});\mathbb{Z})$.*

Proof. If X is a wedge sum of spheres then the Chern character map is an isomorphism $K^0(X) \to H^{\mathrm{even}}(X)$ [May99, pp. 212]. So, since the Chern character map under consideration is induced from the usual Chern character map on the suspension of X, the result follows from Lemma 3.3 and the fact that the suspension of a skeleton of a torus is a wedge sum of spheres. □

Given a doubly-pointed Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z,w)$ for a nullhomologous knot K, with branched double cover diagram $(\widetilde{\Sigma},\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}},\widetilde{z},\widetilde{w})$, Large [Lar19, Proposition 10.2] constructed a stable tangent-normal isomorphism

$$\begin{aligned}\Phi_1\colon (T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma}\setminus\{\widetilde{z},\widetilde{w}\})^{\mathit{fix}}, T\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}}, T\mathbb{T}_{\widetilde{\beta}}^{\mathit{fix}}) \\ \xrightarrow{\cong} (N\,\mathrm{Sym}^{2g}(\widetilde{\Sigma}\setminus\{\widetilde{z},\widetilde{w}\})^{\mathit{fix}}, N\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}}, N\mathbb{T}_{\widetilde{\beta}}^{\mathit{fix}}).\end{aligned} \tag{3.5}$$

Eventually, we will modify Φ_1 so that it extends over $\{w\}\times\mathrm{Sym}^{g-1}(\Sigma)$, without changing Φ_1 on $\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}}$ and $\mathbb{T}_{\widetilde{\beta}}^{\mathit{fix}}$ (up to homotopy). As a first step we have:

Lemma 3.6. *There is a stable isomorphism of complex vector bundles*

$$\Phi_2\colon T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma}\setminus\{\widetilde{z}\})^{\mathit{fix}} \xrightarrow{\cong} N\,\mathrm{Sym}^{2g}(\widetilde{\Sigma}\setminus\{\widetilde{z}\})^{\mathit{fix}}.$$

Proof. Let E be a disk in Σ containing z and w, so that $\Sigma \setminus E$ is a deformation retract of $\Sigma \setminus \{z\}$. Let Y be the image of $\mathrm{Sym}^g(\Sigma \setminus E)$ in $\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}, \widetilde{w}\})^{\mathit{fix}}$. Large's isomorphism Φ_1 restricts to an isomorphism $TY \simeq NY$. Since Y is a deformation retract of $\widetilde{\Sigma} \setminus \{\widetilde{z}\}$, this implies the existence of the isomorphism Φ_2. □

Note that, in the proof of Lemma 3.6, since E may intersect the α- and β-curves, we have no control over Φ_2 on $T\mathbb{T}^{\mathit{fix}}_{\widetilde{\alpha}}$ and $T\mathbb{T}^{\mathit{fix}}_{\widetilde{\beta}}$.

Remark 3.7. One can alternately prove Lemma 3.6 by using Macdonald's computation of the Chern classes of symmetric products of surfaces [Mac62], along with the fact that over spaces with torsion-free cohomology the Chern classes of a vector bundle determine its stable isomorphism class.

Lemma 3.8. *Let V be a closed tubular neighborhood of $\{w\} \times \mathrm{Sym}^{g-1}(\Sigma \setminus \{z\}) \subset \mathrm{Sym}^g(\Sigma \setminus \{z\})$. Consider the commutative diagram*

$$\begin{array}{ccccc} G & & & & \\ & \searrow & & & \\ H^*(\mathrm{Sym}^g(\Sigma \setminus \{z\})) & \longrightarrow & H^*(\mathrm{Sym}^g(\Sigma \setminus \{z,w\})) & & \\ \downarrow & & \downarrow & \searrow & \\ H^*(V) & \longrightarrow & H^*(\partial V) & & H^*(\mathbb{T}_\alpha) \oplus H^*(\mathbb{T}_\beta) \end{array}$$

where G is the kernel of the map $H^*(\mathrm{Sym}^g(\Sigma \setminus \{z,w\})) \to H^*(\mathbb{T}_\alpha) \oplus H^*(\mathbb{T}_\beta)$ (so the diagonal line is exact). Given any class $a \in H^*(\mathrm{Sym}^g(\Sigma \setminus \{z,w\}))$ there is a class $b \in G$ so that the image of $a+b$ in $H^*(\partial V)$ is in the image of $H^*(V)$.

Proof. Let $\gamma \subset \Sigma \setminus \{z,w\}$ be a small circle around w. Since K is nullhomologous there is a class $c \in H^1(\Sigma \setminus \{z,w\})$ so that $c([\alpha_i]) = c([\beta_i]) = 0$ for all i and $c([\gamma]) = 1$. Specifically, since K is nullhomologous, K bounds a Seifert surface F in Y. The Poincaré-Lefschetz dual $\mathrm{PD}([F]) \in H^1(Y \setminus K)$ evaluates to 1 on a meridian of K. Since each α_i and β_i is nullhomologous in $Y \setminus K$ (they bound disks), $\mathrm{PD}([F])$ evaluates to 0 on $[\alpha_i]$ and $[\beta_i]$. Hence, the image of $\mathrm{PD}[F]$ in $H^1(\Sigma \setminus \{z,w\})$ is the desired class c.
Projection to $\{w\} \times \mathrm{Sym}^{g-1}(\Sigma \setminus \{z\})$ gives a homotopy equivalence $V \simeq \mathrm{Sym}^{g-1}(\Sigma \setminus \{z\})$. Further, since the restriction of $c_1(T\,\mathrm{Sym}^g(\Sigma \setminus \{z\}))$ to $\{w\} \times \mathrm{Sym}^{g-1}(\Sigma \setminus \{z\})$ is exactly $c_1(T\,\mathrm{Sym}^{g-1}(\Sigma \setminus \{z\}))$ [Mac62, Formula 14.5], the normal bundle to $\{w\} \times \mathrm{Sym}^{g-1}(\Sigma \setminus \{z\})$ is trivial so $\partial V \cong \mathrm{Sym}^{g-1}(\Sigma \setminus \{z\}) \times S^1$. (The restriction of the cohomology class $\eta \in H^2$ appearing in MacDonald's formula to the symmetric product of $\Sigma \setminus \{z\}$ vanishes.) From Lemma 3.3, the cohomology $H^i(\mathrm{Sym}^{g-1}(\Sigma \setminus \{z,w\}))$ vanishes for $i > g-1$ and the inclusion map $\mathrm{Sym}^{g-1}(\Sigma \setminus \{z,w\})) \hookrightarrow \mathrm{Sym}^g(\Sigma \setminus \{z,w\}))$ induces an isomorphism on H^i for $i \leq g-1$. By a small abuse of notation, let c denote the image of the class

$c \in H^1(\Sigma \setminus \{z, w\})$ in $H^1(\mathrm{Sym}^n(\Sigma \setminus \{z, w\}))$ under this string of isomorphisms for any n. We thus have a diagram

$$\begin{array}{ccc}
H^*(\mathrm{Sym}^{g-1}(\Sigma \setminus \{z,w\}))^{\oplus 2} & \xleftarrow{\quad\approx\quad} & H^*(\mathrm{Sym}^{g}(\Sigma \setminus \{z,w\}))^{\oplus 2} \\
\uparrow{\scriptstyle i^*} & & \downarrow{\scriptstyle (x,y)\mapsto x+c\cup y} \\
H^*(\mathrm{Sym}^{g-1}(\Sigma \setminus \{z\}))^{\oplus 2} & & H^*(\mathrm{Sym}^{g}(\Sigma \setminus \{z,w\})) \\
\downarrow{\scriptstyle \cong} & & \downarrow \\
H^*(V)^{\oplus 2} & \xrightarrow[(x,y)\mapsto i^*x+c\cup i^*y]{\cong} & H^*(\partial V)
\end{array}$$

where the map labeled $\approx$ is an isomorphism for $* < g$ (and the target vanishes for $* \geq g$), and the maps i^* are induced by the inclusion $\partial V \hookrightarrow V$ and $\mathrm{Sym}^{g-1}(\Sigma \setminus \{z, w\}) \hookrightarrow \mathrm{Sym}^{g-1}(\Sigma \setminus \{z\})$.

We claim that if we invert the arrow labeled $\approx$ in the degrees where it is an isomorphism, the diagram commutes. This is a consequence of Lemma 3.3, as follows. Let $S^1_1, \dots, S^1_{2g}$ be a collection of $2g$ circles in $\Sigma \setminus \{z\}$ such that the punctured surface $\Sigma \setminus \{z\}$ deformation retracts onto $\bigvee_{i=1}^{2g} S^1_i$, the surface $\Sigma \setminus \{z, w\}$ deformation retracts onto $(\bigvee_{i=1}^{2g} S^1_i) \vee \gamma$, and the inclusion map $\Sigma \setminus \{z, w\} \hookrightarrow \Sigma \setminus \{z\}$ goes by filling in a disk D_γ containing w whose boundary is γ.

Lemma 3.3 shows that $\mathrm{Sym}^g(\Sigma \setminus \{z\})$ deformation retracts onto the g-skeleton

$$\Big\{(z_1, \dots, z_{2g}) \in S^1_1 \times \cdots \times S^1_{2g} \mid z_i \neq 1 \text{ for at most } r \text{ coordinates}\Big\}$$

of $\prod_{i=1}^{2g} S^1_i$. However, for this argument we wish to apply a milder deformation retraction, starting with the fact that $\Sigma \setminus \{z\}$ deformation retracts onto $\left(\bigvee_{i=1}^{2g} S^1_i\right) \vee D_\gamma$. The same argument as Lemma 3.3 shows that $\mathrm{Sym}^g\left(\left(\bigvee_{i=1}^{2g} S^1_i\right) \vee D_\gamma\right)$ deformation retracts onto

$$\Big\{(z_1, \dots, z_{2g+1}) \in S^1_1 \times \cdots \times S^1_{2g} \times D_\gamma \mid z_i \neq 1 \text{ for at most } r \text{ coordinates}\Big\}.$$

This deformation retraction takes the subspace $\mathrm{Sym}^g(\Sigma \setminus \{z, w\})$ onto

$$\Big\{(z_1, \dots, z_{2g+1}) \in S^1_1 \times \cdots \times S^1_{2g} \times (D_\gamma \setminus \{w\}) \mid z_i \neq 1 \text{ for at most } r \text{ coords.}\Big\}$$

which itself deformation retracts onto the g-skeleton of $\left(\prod_{i=1}^{2g} S^1_i\right) \times \gamma$. Furthermore, it carries V to the product of the $g-1$ skeleton of $\prod_{i=1}^{2g} S^1_i$ with D_γ, and ∂V onto the product of the $g-1$ skeleton of $\prod_{i=1}^{2g} S^1_i$ with γ. It is now simple to see from this description of the spaces in terms of tori that the diagram above commutes.

Write the image of a in $H^*(\partial V)$ as $i^*a_1 + c \cup i^*a_2$ for some $a_1, a_2 \in H^*(V)$. Let $a_i' \in H^*(\mathrm{Sym}^{g-1}(\Sigma \setminus \{z\}))$ be a preimage of a_i under the isomorphism. Since the top horizontal map is an isomorphism whenever $H^*(\mathrm{Sym}^{g-1}(\Sigma \setminus \{z, w\}))$ (or equivalently $H^*(V)$) is non-zero, there are elements $\widetilde{a}_i \in H^*(\mathrm{Sym}^g(\Sigma \setminus \{z, w\}))$ mapping to i^*a_i'. Take $b = -c \cup \widetilde{a}_2 \in H^*(\mathrm{Sym}^g(\Sigma \setminus \{z, w\}))$. Since $c|_{\alpha_i}$ and $c|_{\beta_i}$ vanish, b lies in the kernel G. The image of $a + b \in H^*(\mathrm{Sym}^g(\Sigma \setminus \{z, w\}))$ in $H^*(\partial V)$ is the same as the image of $a_1 \in H^*(V)$. This proves the result. ☐

Proof of Proposition 3.1. The composition

$$\Phi_2^{-1} \circ \Phi_1 \colon T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}, \widetilde{w}\})^{fix} \to T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}, \widetilde{w}\})^{fix}$$

is an element of

$$\mathrm{Iso}\big(T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}, \widetilde{w}\})^{fix}, T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}, \widetilde{w}\})^{fix}\big).$$

Identify $\mathrm{Sym}^g(\Sigma \setminus \{z, w\})$ with $\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}, \widetilde{w}\})^{fix}$ as in Lemma 3.2 and let $a = \mathrm{ch}[\Phi_2^{-1} \circ \Phi_1] \in H^{\mathrm{odd}}(\mathrm{Sym}^g(\Sigma \setminus \{z, w\}))$ (see Section 2.3 and Remark 2.23). By Lemma 3.8, there exists $b \in H^{\mathrm{odd}}(\mathrm{Sym}^g(\Sigma \setminus \{z, w\}))$ such that b is in the kernel of the map $H^*(\mathrm{Sym}^g(\Sigma \setminus \{z, w\})) \to H^*(\mathbb{T}_\alpha) \oplus H^*(\mathbb{T}_\beta)$ and the image of $a + b$ in $H^*(\partial V)$ is in the image of the map $H^*(V) \to H^*(\partial V)$. By Corollary 3.4, $b = \mathrm{ch}[\Phi_3]$ for some

$$\Phi_3 \in \mathrm{Iso}(T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}, \widetilde{w}\})^{fix}, T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}, \widetilde{w}\})^{fix}).$$

Functoriality of the Chern character implies that $\mathrm{ch}[\Phi_3|_{T\mathbb{T}^{fix}_{\widetilde{\alpha}} \otimes \mathbb{C}}] = 0$. Hence, the restriction $\Phi_3|_{T\mathbb{T}^{fix}_{\widetilde{\alpha}} \otimes \mathbb{C}}$ is stably homotopic to the identity isomorphism. Likewise, $\Phi_3|_{T\mathbb{T}^{fix}_{\widetilde{\beta}} \otimes \mathbb{C}}$ is stably homotopic to the identity isomorphism.

Consider

$$\Phi_2^{-1} \circ \Phi_1 \circ \Phi_3 : T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}, \widetilde{w}\})^{fix} \to T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}, \widetilde{w}\})^{fix}.$$

Since $\mathrm{ch}[\Phi_2^{-1} \circ \Phi_1 \circ \Phi_3] = \mathrm{ch}[\Phi_2^{-1} \circ \Phi_1] + \mathrm{ch}[\Phi_3] = a + b$ and the Chern character is functorial, we see that $\mathrm{ch}[(\Phi_2^{-1} \circ \Phi_1 \circ \Phi_3)|_{\partial V}]$ is the image of $a + b$ in $H^*(\partial V)$ and therefore lies in the image of the bottom horizontal map in the following commutative diagram:

$$\begin{array}{ccccc}
 & & & & (\Phi_2^{-1} \circ \Phi_1 \circ \Phi_3)|_{\partial V} \\
 & & & \rotatebox{-45}{$\in$} & \Big\downarrow \\
 & \mathrm{Iso}(TV, TV) & \longrightarrow & \mathrm{Iso}(TV|_{\partial V}, TV|_{\partial V}) & \\
 & \mathrm{ch} \Big\downarrow \cong & & \mathrm{ch} \Big\downarrow \cong & \\
 & H^{\mathrm{odd}}(V) & \longrightarrow & H^{\mathrm{odd}}(\partial V) & \\
 \in & & & \ni & \\
(a+b)|_V & & \longmapsto & & \mathrm{ch}\big((\Phi_2^{-1} \circ \Phi_1 \circ \Phi_3)|_{\partial V}\big).
\end{array}$$

Corollary 3.4 implies that the vertical maps in this diagram are isomorphisms, so the isomorphism $(\Phi_2^{-1}\circ\Phi_1\circ\Phi_3)|_{\partial V}$ extends over V. There is therefore an extension

$$\Phi_4\colon T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma}\setminus\{\widetilde{z}\})^{\mathit{fix}}\to T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma}\setminus\{\widetilde{z}\})^{\mathit{fix}}$$

of $\Phi_2^{-1}\circ\Phi_1\circ\Phi_3$. Our final isomorphism Φ_5 is the composition

$$\Phi_5 \coloneqq \Phi_2\circ\Phi_4 : T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma}\setminus\{\widetilde{z}\})^{\mathit{fix}}\to N\,\mathrm{Sym}^{2g}(\widetilde{\Sigma}\setminus\{\widetilde{z}\})^{\mathit{fix}}.$$

This map Φ_5 agrees with $\Phi_1\circ\Phi_3$ away from the divisor $\{w\}\times\mathrm{Sym}^{g-1}(\Sigma)$. Since the restriction of Φ_3 to $T\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}}\otimes\mathbb{C}$ is homotopic to the identity and there is a homotopy of Lagrangian subbundles from $\Phi_1(T\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}})$ to $N\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}}$, there is a homotopy of Lagrangian subbundles from $\Phi_5(T\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}})$ to $N\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}}$, and similarly for $T\mathbb{T}_{\widetilde{\beta}}^{\mathit{fix}}$. Therefore the map Φ_5 is the desired stable tangent-normal isomorphism

$$\big(T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma}\setminus\{\widetilde{z}\})^{\mathit{fix}},T\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}},T\mathbb{T}_{\widetilde{\beta}}^{\mathit{fix}}\big)\cong\big(N\,\mathrm{Sym}^{2g}(\widetilde{\Sigma}\setminus\{\widetilde{z}\})^{\mathit{fix}},N\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}},N\mathbb{T}_{\widetilde{\beta}}^{\mathit{fix}}\big).\ \square$$

4 Proof of the main theorem

In this section, we prove Theorem 1.1. We begin with the simplest version of the spectral sequence, and then prove a spin^c-refined statement in Section 4.1 and the generalization from knots to links in Section 4.2.

Theorem 4.1. *Let Y be a closed 3-manifold and $K\subset Y$ an oriented nullhomologous knot with Seifert surface F. Let $\pi\colon\Sigma(Y,K)\to Y$ be the double cover branched along K induced by the Seifert surface F. Then, there is a spectral sequence with E^1-page given by*

$$\widehat{HF}(\Sigma(Y,K))\otimes\mathbb{F}_2[[\theta,\theta^{-1}]$$

converging to

$$\widehat{HF}(Y)\otimes\mathbb{F}_2[[\theta,\theta^{-1}].$$

In particular,

$$\dim\widehat{HF}(\Sigma(Y,K))\geq\dim\widehat{HF}(Y).$$

Proof. Fix $\mathcal{H}=(\Sigma_g,\boldsymbol{\alpha},\boldsymbol{\beta},z,w)$ a weakly admissible doubly-pointed Heegaard diagram for a nullhomologous knot K in Y and let $\widetilde{\mathcal{H}}=(\widetilde{\Sigma}_{2g},\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}},\widetilde{z},\widetilde{w})$ denote a doubly-pointed Heegaard diagram for $(\Sigma(Y,K),\widetilde{K})$ obtained by taking the branched double cover of $\mathcal{H}$. By Proposition 4.2 below, $\widetilde{\mathcal{H}}$ is also weakly admissible. By Proposition 3.1, there is a stable tangent-normal isomorphism

$$\big(T\,\mathrm{Sym}^{2g}(\widetilde{\Sigma}\setminus\{\widetilde{z}\})^{\mathit{fix}},T\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}},T\mathbb{T}_{\widetilde{\beta}}^{\mathit{fix}}\big)\cong\big(N\,\mathrm{Sym}^{2g}(\widetilde{\Sigma}\setminus\{\widetilde{z}\})^{\mathit{fix}},N\mathbb{T}_{\widetilde{\alpha}}^{\mathit{fix}},N\mathbb{T}_{\widetilde{\beta}}^{\mathit{fix}}\big).$$

By Proposition 4.2 again, the remaining hypotheses of Theorem 2.4 are satisfied. So, Theorem 2.4 implies the result. $\square$

Proposition 4.2. *Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ be a Heegaard diagram for a null-homologous knot K in a closed 3-manifold Y and let $\widetilde{\mathcal{H}} = (\widetilde{\Sigma}, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{z}, \widetilde{w})$ be a branched double cover of $\mathcal{H}$. Assume that $\mathcal{H}$ is weakly admissible for all* spin^c*-structures. Then $(\widetilde{\Sigma}, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{z}, \widetilde{w})$ is weakly admissible for all* spin^c*-structures. Further, there is a choice of symplectic form on* $\mathrm{Sym}^g(\widetilde{\Sigma} \setminus \{\widetilde{z}\})$ *satisfying hypotheses (L1) and (L2) from Theorem 2.4 (and inducing the polarization data studied in Section 3).*

Proof. Weak admissibility is equivalent to the existence of an area form ω on Σ so that the signed area of every periodic domain with multiplicity 0 at z is zero [OSz04b, Lemma 4.12]. Since K is nullhomologous, every periodic domain for $(\widetilde{\Sigma}, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}})$ with multiplicity 0 at $\widetilde{z}$ also has multiplicity 0 at $\widetilde{w}$, and hence projects to a periodic domain in Σ with multiplicity 0 at z (and w). Hence, the pullback $\widetilde{\omega}$ of ω (smoothed out at $\widetilde{z}$ and $\widetilde{w}$) has the property that every periodic domain with multiplicity 0 at $\widetilde{z}$ has signed area 0. In particular, $(\widetilde{\Sigma}, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{z})$ is also weakly admissible for all spin^c-structures.

Perutz's techniques [Per08, Section 7], as applied by Hendricks to the case of punctured Heegaard surfaces [Hen12, Section 4], show that if ϕ is an exhausting function on $\Sigma \setminus \{z\}$ such that $\omega = -dd^{\mathbb{C}}\phi$ and $\widetilde{\phi}$ is the lift of ϕ to $\widetilde{\Sigma} \setminus \{\widetilde{z}\}$, then there is an equivariant smooth exhausting function ψ on $\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}\})$ which agrees with $\widetilde{\phi}^{\times 2g}$ away from a neighborhood of the diagonal. In particular, if $\widetilde{\omega} = -dd^{\mathbb{C}}\widetilde{\phi}$ is the symplectic form on $\widetilde{\Sigma} \setminus \{\widetilde{z}\}$, then $-dd^{\mathbb{C}}\psi$ is an exact equivariant symplectic form on $M = \mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}\})$ which agrees with $\widetilde{\omega}^{\times 2g}$ away from a neighborhood of the diagonal. This shows that M is an exact symplectic manifold and convex at infinity. Further, if $\lambda = -d^{\mathbb{C}}\widetilde{\phi}$ then $-dd^{\mathbb{C}}\psi$ has a primitive $-d^{\mathbb{C}}\psi$ that agrees with $\lambda^{\times 2g}$ away from the diagonal.

To establish that $L_0 = \mathbb{T}_{\widetilde{\alpha}}$ and $L_1 = \mathbb{T}_{\widetilde{\beta}}$ are exact Lagrangians in M, we first check that the curves $\widetilde{\alpha}_i$ and $\widetilde{\beta}_j$ are exact with respect to a suitable primitive of $\widetilde{\omega}$ in $\widetilde{\Sigma} \setminus \{\widetilde{z}\}$. Consider the primitive $\lambda = -d^{\mathbb{C}}\widetilde{\phi}$ of $\widetilde{\omega}$. We will adjust λ on $\widetilde{\Sigma} \setminus \{\widetilde{z}\}$ so that for all i, $\int_{\widetilde{\alpha}_i} \lambda = \int_{\widetilde{\beta}_i} \lambda = 0$, and then adjust $-d^{\mathbb{C}}\psi$ correspondingly on $\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}\})$. Reordering the $\widetilde{\beta}_i$, arrange that $[\widetilde{\alpha}_1], \dots, [\widetilde{\alpha}_{2g}], [\widetilde{\beta}_1], \dots, [\widetilde{\beta}_k] \in H_1(\widetilde{\Sigma}; \mathbb{Q})$ are linearly independent and

$$[\widetilde{\beta}_{k+1}], \dots, [\widetilde{\beta}_{2g}] \in \mathrm{Span}([\widetilde{\alpha}_1], \dots, [\widetilde{\alpha}_{2g}], [\widetilde{\beta}_1], \dots, [\widetilde{\beta}_k]) \subset H_1(\widetilde{\Sigma}; \mathbb{Q}). \tag{4.3}$$

There is a cohomology class $[a] \in H^1(\widetilde{\Sigma}; \mathbb{R})$ so that for all $i = 1, \dots, 2g$, $\langle [a], [\widetilde{\alpha}_i] \rangle = \int_{\widetilde{\alpha}_i} \lambda$, and for $i = 1, \dots, k$, $\langle [a], [\widetilde{\beta}_i] \rangle = \int_{\widetilde{\beta}_i} \lambda$. Choose a closed 1-form a representing $[a]$ and let $\lambda' = \lambda - a$. Then λ' is still a primitive of $\widetilde{\omega}$ and $\int_{\widetilde{\alpha}_i} \lambda' = \int_{\widetilde{\beta}_j} \lambda' = 0$ for $1 \le i \le 2g$ and $1 \le j \le k$. We claim that in fact $\int_{\widetilde{\beta}_j} \lambda' = 0$ for $j = k+1, \dots, 2g$ as well. By Equation (4.3) there is a periodic domain P with boundary

$$\partial P = m_1[\widetilde{\alpha}_1] + \cdots + m_{2g}[\widetilde{\alpha}_{2g}] + n_1[\widetilde{\beta}_1] + \cdots + n_k[\widetilde{\beta}_k] + p[\widetilde{\beta}_j]$$

for some $m_1, \dots, m_{2g}, n_1, \dots, n_k, p \in \mathbb{Z}$, $p \neq 0$. By Stokes' theorem,

$$p \int_{\widetilde{\beta}_j} \lambda' = \int_P \widetilde{\omega} - m_1 \int_{\widetilde{\alpha}_1} \lambda' - \cdots - n_k \int_{\widetilde{\beta}_k} \lambda',$$

but by construction every term on the right-hand side vanishes.

Now, let $[b] \in H^1(\mathrm{Sym}^g(\widetilde{\Sigma} \setminus \{\widetilde{z}\}); \mathbb{R})$ be the image of the class $[a]$ under the isomorphism $H^1(\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}\}); \mathbb{R}) \cong H^1(\widetilde{\Sigma} \setminus \{\widetilde{z}\}; \mathbb{R})$ induced by the inclusion $\widetilde{\Sigma} \hookrightarrow \mathrm{Sym}^{2g}(\widetilde{\Sigma})$, and let b be a closed 1-form representing $[b]$. Then $-d^{\mathbb{C}}\psi - b$ is a primitive for the symplectic form on $\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}\})$ and, from the computation in the previous paragraph, the restriction of $-d^{\mathbb{C}}\psi - b$ to $\mathbb{T}_{\widetilde{\alpha}}$ and $\mathbb{T}_{\widetilde{\beta}}$ is exact. This concludes the proof. □

4.1 The spin^c refinement

In this section, we will refine Theorem 4.1 to respect spin^c structures. First, we must discuss spin^c structures on branched covers.

Definition 4.4. *Let $\mathfrak{s}$ be a* spin^c*-structure on Y and $\pi\colon (\Sigma(Y,K), \widetilde{K}) \to (Y,K)$ be a double cover branched along a nullhomologous knot K. The* pullback spin^c*-structure $\pi^*\mathfrak{s}$ is characterized as follows. If $\widetilde{K} = \pi^{-1}(K)$ denotes the double point set then on $\Sigma(Y,K) \setminus \mathrm{nbd}(\widetilde{K})$, the map π is a local diffeomorphism, so $T\big(\Sigma(Y,K) \setminus \mathrm{nbd}(\widetilde{K})\big) \cong \pi^* T\big(Y \setminus \mathrm{nbd}(K)\big)$. Thus, $\mathfrak{s} \in \mathrm{spin}^c(Y)$ induces a* spin^c*-structure $\pi^*\mathfrak{s}$ on $\Sigma(Y,K) \setminus \mathrm{nbd}(\widetilde{K})$. The obstruction to extending $\pi^*\mathfrak{s}|_{\partial\,\mathrm{nbd}(\widetilde{K})}$ over $\mathrm{nbd}(\widetilde{K})$ is $c_1(\pi^*\mathfrak{s}|_{\partial\,\mathrm{nbd}(\widetilde{K})}) = \pi^* c_1(\mathfrak{s}|_{\partial\,\mathrm{nbd}(\widetilde{K})})$, which is the pullback of the obstruction to extending $\mathfrak{s}$ over $\mathrm{nbd}(K)$ and hence vanishes. Any two extensions differ by a multiple of $\mathrm{PD}[\widetilde{K}] = 0$, so the extension of $\pi^*\mathfrak{s}$ to all of $\Sigma(Y,K)$ is unique.*

For the branched double cover of an (oriented) nullhomologous link L where some components are homologically essential, the uniqueness step above fails. For links, define $\pi^*\mathfrak{s}$ as follows. Identify a neighborhood of L with $D^2 \times L$ so that the Seifert surface is given by $[0,1) \times \{0\} \times L$. Choose a vector field v on Y representing $\mathfrak{s}$, and so that in this neighborhood v is given by $\partial/\partial\theta$, where θ is a coordinate on L. In particular, v is positively tangent to L. From the construction of the branched double cover, there is an induced vector field $\widetilde{v}$ on $\Sigma(Y,L)$ so that on $\Sigma(Y,L) \setminus \widetilde{L}$, $d\pi(\widetilde{v}) = v$, and $\widetilde{v}$ is positively tangent to $\widetilde{L}$. Then $\pi^*\mathfrak{s}$ is the spin^c*-structure represented by $\widetilde{v}$.*

It is immediate from the construction that, for knots, these two definitions of $\pi^*\mathfrak{s}$ agree. It follows from Proposition 4.12 below that for links the second construction is independent of the choice of v representing $\mathfrak{s}$. It also follows that reversing the orientation of all components of L gives the same map π^* on spin^c-structures.

We note next that the definition of pullback spin^c structures behaves well with respect to the association of spin^c structures to intersection points in Heegaard diagrams. Fix $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ a doubly-pointed Heegaard diagram for a nullhomologous knot K in Y and let $\widetilde{\mathcal{H}} = (\widetilde{\Sigma}, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{z}, \widetilde{w})$ be a branched double cover of $\mathcal{H}$, which is a doubly-pointed Heegaard diagram for $(\Sigma(Y,K), \widetilde{K})$. Recall that Ozsváth-Szabó [OSz04b] gave an association $\mathfrak{s}_z\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \text{spin}^c(Y)$. For $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we will sometimes write $\widetilde{x}$ for the intersection point $\pi^{-1}(x)$ in $\mathbb{T}_{\widetilde{\alpha}} \cap \mathbb{T}_{\widetilde{\beta}}$.

Lemma 4.5. *Let K be a nullhomologous knot in Y. Then for $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we have $\pi^*(\mathfrak{s}_z(x)) = \mathfrak{s}_{\widetilde{z}}(\pi^{-1}(x))$.*

Proof. Choose a Morse function f on (Y,K) compatible with the doubly-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$. Represent $\mathfrak{s}_z(x)|_{Y\setminus \text{nbd}(K)}$ by a non-vanishing vector field by modifying ∇f on $Y \setminus \text{nbd}(K)$ in a neighborhood of the trajectories of ∇f through x. Consider $\pi^* f = f \circ \pi$ and the induced homology class of vector field on $\Sigma(Y,K) \setminus \text{nbd}(\widetilde{K})$. This class is precisely the spin^c structure on $\Sigma(Y,K) \setminus \text{nbd}(\widetilde{K})$ corresponding to $\tilde{x}$. Now, define a spin^c structure on $\Sigma(Y,K)$ by extending over $\Sigma(Y,K) \setminus \text{nbd}(\widetilde{K})$. As discussed in Definition 4.4, the extension is unique because K is nullhomologous. Hence, this spin^c structure is exactly $\mathfrak{s}_{\widetilde{z}}(\pi^{-1}(x))$. However, this spin^c-structure is also $\pi^*(\mathfrak{s}_z(x))$ as constructed in Definition 4.4. ☐

Remark 4.6. By Lemma 4.5, if we change the intersection point x for Y without changing the corresponding spin^c structure on Y, then the lifted elements represent the same spin^c structure on $\Sigma(Y,K)$. Another way to see this is as follows. Given a Whitney disk $u \in \pi_2(x,y)$ in $Sym^g(\Sigma \setminus \{z\})$, this naturally induces a Whitney disk $\widetilde{u} \in \pi_2(\widetilde{x}, \widetilde{y})$ in $Sym^{\widetilde{g}}(\widetilde{\Sigma} \setminus \{\widetilde{z}\})$ by $\widetilde{u}(q) = \pi^{-1}(u(q))$.

The alternative description of pullback spin^c structures described in the proof of Lemma 4.5 is also a useful viewpoint for studying the connection between spin^c structures and cohomology classes.

Lemma 4.7. *For $K \subset Y$ a nullhomologous knot, the pullback* spin^c *structure satisfies*

$$\pi^*\overline{\mathfrak{s}} = \overline{\pi^*\mathfrak{s}} \tag{4.8}$$

$$\pi^*(\mathfrak{s} + a) = \pi^*(\mathfrak{s}) + \pi^*(a) \tag{4.9}$$

$$c_1(\pi^*\mathfrak{s}) = \pi^* c_1(\mathfrak{s}). \tag{4.10}$$

for any $\mathfrak{s} \in \text{spin}^c(Y)$ and $a \in H^2(Y)$.

Proof. (4.8) Recall that if v is a non-vanishing vector field corresponding to a spin^c structure $\mathfrak{s}$, then $-v$ corresponds to $\overline{\mathfrak{s}}$. So, the claim follows easily from Definition 4.4, since if v corresponds to $\mathfrak{s}$ on Y, then $v|_{Y\setminus \text{nbd}(K)}$ corresponds to $\mathfrak{s}|_{Y\setminus \text{nbd}(K)}$ on $Y \setminus \text{nbd}(K)$, and $\pi^* v|_{\Sigma(Y,K)\setminus \text{nbd}(\widetilde{K})}$ corresponds to $\pi^*\mathfrak{s}|_{\Sigma(Y,K)\setminus \text{nbd}(\widetilde{K})}$.

(4.9) This is equivalent to showing that $\pi^*(\mathfrak{s}'-\mathfrak{s}) = \pi^*\mathfrak{s}' - \pi^*\mathfrak{s}$. Let $\mathfrak{s}$ and $\mathfrak{s}'$ be represented by $x, x' \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ respectively, so

$$\mathfrak{s}_z(x') - \mathfrak{s}_z(x) = PD[\epsilon(x,x')],$$

and

$$\mathfrak{s}_{\widetilde{z}}(\widetilde{x}') - \mathfrak{s}_{\widetilde{z}}(\widetilde{x}) = PD[\epsilon(\widetilde{x},\widetilde{x}')].$$

The transfer map $\pi^!$ sends $\epsilon(x,x')$ to $\epsilon(\widetilde{x},\widetilde{x}')$, i.e., $\pi^!\epsilon(x,x') = \epsilon(\widetilde{x},\widetilde{x}')$. (If we represent $\epsilon(x,x')$ by a 1-manifold in $\Sigma \setminus \{z,w\}$ then $\pi^!\epsilon(x,x')$ is the total preimage of that 1-manifold.) It follows that

$$\begin{aligned}\pi^*(\mathfrak{s}_z(x') - \mathfrak{s}_z(x)) &= \pi^* PD[\epsilon(x,x')] \\ &= PD[\pi^!\epsilon(x,x')] \\ &= PD[\epsilon(\widetilde{x},\widetilde{x}')] \\ &= \mathfrak{s}_{\widetilde{z}}(\widetilde{x}') - \mathfrak{s}_{\widetilde{z}}(\widetilde{x}) \\ &= \pi^*(\mathfrak{s}_z(x')) - \pi^*(\mathfrak{s}_z(x)),\end{aligned}$$

by Lemma 4.5.

(4.10) Recall that the first Chern class of a spin^c structure $\mathfrak{t}$ on a closed 3-manifold can be computed by $\mathfrak{t} - \bar{\mathfrak{t}}$. So, the claim follows from Equations (4.8) and (4.9). ☐

We are now ready to state the spin^c-refinement of Theorem 1.1.

Proposition 4.11. *Let Y be a closed, connected, oriented 3-manifold, $K \subset Y$ a nullhomologous knot, and $\mathfrak{s}$ a* spin^c*-structure on Y. Then, the spectral sequence from Theorem 4.1 splits along τ-invariant* spin^c*-structures on $\Sigma(Y,K)$. In particular, there is an inequality*

$$\dim \widehat{HF}(\Sigma(Y,K), \pi^*\mathfrak{s}) \geq \sum_{\pi^*\mathfrak{s}' = \pi^*\mathfrak{s}} \dim \widehat{HF}(Y, \mathfrak{s}').$$

Proof. Choose a doubly-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ for $K \subset Y$ which is weakly admissible for all spin^c-structures. As before, the fixed point sets of the $\mathbb{Z}/2\mathbb{Z}$-action on $(\mathrm{Sym}^{2g}(\widetilde{\Sigma} \setminus \{\widetilde{z}\}), \mathbb{T}_{\widetilde{\alpha}}, \mathbb{T}_{\widetilde{\beta}})$ are identified with $(\mathrm{Sym}^g(\Sigma \setminus \{\widetilde{z}\}), \mathbb{T}_\alpha, \mathbb{T}_\beta)$. Under this identification, the map

$$\iota_*\colon \pi_0 P(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to \pi_0 P(\mathbb{T}_{\widetilde{\alpha}}, \mathbb{T}_{\widetilde{\beta}})$$

from Section 2.2 sends the constant path $[x]$ associated to a point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to the constant path $[\pi^{-1}(x)]$ associated to the point $\pi^{-1}(x) \in \mathbb{T}_{\widetilde{\alpha}} \cap \mathbb{T}_{\widetilde{\beta}}$.

Recall that two elements $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ have $[x], [y]$ in the same path component in $P(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ (inside $\mathrm{Sym}^g(\Sigma \setminus \{z\})$) if and only if $\mathfrak{s}_z(x) = \mathfrak{s}_z(y)$ [OSz04b, Section 2]. Similarly, two elements $\widetilde{x}, \widetilde{y} \in \mathbb{T}_{\widetilde{\alpha}} \cap \mathbb{T}_{\widetilde{\beta}}$ have $[\widetilde{x}], [\widetilde{y}]$ in the same path component of $P(\mathbb{T}_{\widetilde{\alpha}}, \mathbb{T}_{\widetilde{\beta}})$ if and only if $\mathfrak{s}_{\widetilde{z}}(\widetilde{x}) = \mathfrak{s}_{\widetilde{z}}(\widetilde{y})$. Finally, by Lemma 4.5,

$\pi^*\mathfrak{s}_z(x) = \mathfrak{s}_{\widetilde{z}}(\pi^{-1}(x))$. Putting this all together, if an element of $\pi_0 P(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ corresponds to $\mathfrak{s}$, then the image under ι_* corresponds to $\pi^*\mathfrak{s}$. (See Remark 4.6 for an alternate viewpoint.)

Thus, Proposition 2.15 (together with Propositions 3.1 and 4.2) implies the desired splitting of spectral sequences and inequality

$$\dim \widehat{HF}(\Sigma(Y,K), \pi^*\mathfrak{s}) \geq \sum_{\pi^*\mathfrak{s}'=\pi^*\mathfrak{s}} \dim \widehat{HF}(Y, \mathfrak{s}'). \qquad \square$$

4.2 From knots to links

In this section, we use Ozsváth-Szabó's knotification procedure to deduce Theorem 1.1 for links with an arbitrary number of components from Proposition 4.11.

Suppose $L \subset Y$ has two components L_1, L_2. Let B_i be a ball intersecting L_i in a trivial arc A_i. Note that $Y \# S^2 \times S^1$ can be produced by identifying the boundary components of $Y \setminus (B_1 \cup B_2)$ so that the endpoints of A_1 and A_2 are identified. The link $(L_1 \setminus A_1) \cup (L_2 \setminus A_2) \subset Y \# S^2 \times S^1$ is the *knotification* of L. More generally, the knotification of an ℓ-component link is obtained by doing this process $\ell - 1$ times until a single component remains in $Y \#_{\ell-1} S^2 \times S^1$. We denote the knotification of L by κ_L. It turns out that the knotification operation behaves well with respect to branched double covers. Letting $\mathfrak{t}$ denote the unique torsion spin^c structure on $\#_{\ell-1} S^2 \times S^1$, we have:

Proposition 4.12. *Let L be a nullhomologous link in Y with ℓ components and let κ_L be its knotification. Fix a Seifert surface F for L, let $\pi\colon \Sigma(Y,L) \to Y$ denote the corresponding double cover of Y branched along L, and let $\widetilde{L} = \pi^{-1}(L)$ be the double point set. Then, $\Sigma(Y,L)\#_{\ell-1} S^2 \times S^1$ is homeomorphic to $\Sigma(Y \#_{\ell-1} S^2 \times S^1, \kappa_L)$ and the knotification of $\widetilde{L}$ is the preimage of κ_L. Furthermore, given a* spin^c *structure $\mathfrak{s}$ on Y, the pullback of $\mathfrak{s}\#\mathfrak{t}$ under π': $\Sigma(Y \#_{\ell-1} S^2 \times S^1, \kappa_L) \to Y \#_{\ell-1} S^2 \times S^1$ is $(\pi^*\mathfrak{s})\#\mathfrak{t}$.*

Proof. Recall that the branched double cover of a 3-ball over a trivial arc is again a 3-ball and the double point set is a trivial arc. So, if B_1 and B_2 are small balls around points on two components of L then $\pi^{-1}(B_1)$ and $\pi^{-1}(B_2)$ are small balls around points on two components of $\widetilde{L}$, and knotifying L using B_1 and B_2 corresponds to knotifying $\widetilde{L}$ using $\pi^{-1}(B_1)$ and $\pi^{-1}(B_2)$.

It remains to identify the spin^c structures. For notational simplicity, we consider the case of a 2-component link. Let $B_3 \subset Y$ be the union of B_1, B_2, and an arc connecting them. A spin^c-structure $\mathfrak{s}'$ on $Y \#(S^2 \times S^1)$ is determined by its restriction to $Y \setminus B_3$ and the evaluation of $c_1(\mathfrak{s}')$ on $S^2 \times \{pt\} = \partial B_1$. The same remarks hold for $\Sigma(Y,L)\#(S^2 \times S^1)$. Now, $(\pi^*\mathfrak{s})\#\mathfrak{t}$ and $(\pi')^*(\mathfrak{s}\#\mathfrak{t})$ agree on $Y \setminus B_3$ and $\langle c_1((\pi^*\mathfrak{s})\#\mathfrak{t}), [S^2]\rangle = 0$. Since $(\pi')^* c_1(\mathfrak{s}\#\mathfrak{t}) = c_1((\pi')^*(\mathfrak{s}\#\mathfrak{t}))$, we have $\langle c_1((\pi')^*(\mathfrak{s}\#\mathfrak{t})), [S^2]\rangle = 0$ also. It follows that the spin^c-structures $(\pi^*\mathfrak{s})\#\mathfrak{t}$ and $(\pi')^*(\mathfrak{s}\#\mathfrak{t})$ agree. $\square$

Proof of Theorem 1.1. This is immediate from Propositions 4.11 and 4.12 and the Künneth theorem for $\widehat{HF}$ of connected sums. □

Remark 4.13. The spectral sequence from Theorem 1.1 is an invariant of (Y, K) in the following sense. Given other choices in its construction (Heegaard diagrams, almost complex structures, and so on) there is an isomorphism between each page of the resulting spectral sequence. This follows from the fact that the spectral sequence is isomorphic to Seidel-Smith's spectral sequence for equivariant Floer cohomology [SS10, Section 3.2] (and hence to the spectral sequence one obtains by applying the techniques in [HLS16] to an equivariant Heegaard diagram for the branched double cover) and the proof of the analogous result for $\widehat{HFK}$ [HLS16, Corollary 1.10]. On the other hand, it is not clear that the isomorphism between the E^∞-page of the spectral sequence and $\widehat{HF}(Y) \otimes \mathbb{F}_2[[\theta, \theta^{-1}]$ is independent of choices.

Remark 4.14. Theorem 1.1 allows one to recover a result about ordinary double covers, by taking L to be the unknot and choosing an interesting Seifert surface. Specifically, a double cover $\widetilde{Y} \to Y$ is induced by a $\mathbb{Z}$ cover if the corresponding element in $H^1(Y; \mathbb{Z}/2\mathbb{Z})$ is the image of an element of $H^1(Y; \mathbb{Z})$. In that case, the double cover is obtained by cutting Y along a closed, orientable surface F and gluing two copies of the result together. Let F' be the complement of a small disk in F, and $U = \partial F'$. It is not hard to see that the double cover branched along U with respect to the Seifert surface F' is $\widetilde{Y} \#(S^2 \times S^1)$. So, Theorem 1.1 gives a spectral sequence relating $\widehat{HF}(\widetilde{Y} \#(S^2 \times S^1)) \cong \widehat{HF}(\widetilde{Y}) \otimes H_*(S^1)$ and $\widehat{HF}(Y)$. Such a spectral sequence was obtained by different techniques by Lipshitz-Treumann [LT16, Theorem 3] (for torsion spin^c-structures); this construction gives another explanation of the appearance of the $H_*(S^1)$ factor. This spectral sequence was also proved by Large [Lar19, Theorem 1.4], using his localization theorem; the argument we have just given essentially reduces to his. (This remark was suggested to us by the referee.)

5 Applications

Proof of Corollary 1.2. First, recall that, by Poincaré duality, a non-zero degree map $f: N_1 \to N_2$ between closed, connected, oriented 3-manifolds induces an injection on cohomology with rational coefficients. So, it follows from Lemma 4.7 that if $\pi: \Sigma(Y, L) \to Y$ is a branched double cover, then $\mathfrak{s} \in \mathrm{spin}^c(Y)$ is torsion if and only if $\pi^*\mathfrak{s}$ is. Also, of course, $b_1(N_2) \leq b_1(N_1)$. Suppose that $b_1(\Sigma(Y, L)) = 0$, so $b_1(Y) = 0$ as well. If $HF_{\mathrm{red}}(\Sigma(Y, L)) = 0$, then Theorem 1.1 implies that

$$1 = \dim \widehat{HF}(\Sigma(Y, L), \pi^*\mathfrak{s}) \geq \dim \widehat{HF}(Y, \mathfrak{s}) \geq 1$$

for all $\mathfrak{s} \in \mathrm{spin}^c(Y)$, so $HF_{\mathrm{red}}(Y) = 0$. Hence, if $\Sigma(Y, L)$ is an L-space, so is Y.

Next, suppose that $b_1(\Sigma(Y, L)) = 1$. If N is a 3-manifold with $b_1(N) = 1$, then $HF_{\text{red}}(N) = 0$ if and only if $\widehat{HF}(N, \mathfrak{t}) = 0$ for non-torsion $\mathfrak{t}$ and $\dim \widehat{HF}(N, \mathfrak{t}) = 2$ for all torsion $\mathfrak{t}$. (Recall that 2 is the lower bound for $\dim \widehat{HF}(N, \mathfrak{t})$ for torsion $\mathfrak{t}$, regardless of whether HF_{red} is non-trivial.) We now consider two cases: $b_1(Y) = 0$ or $b_1(Y) = 1$. First, assume $b_1(Y) = 0$. By Theorem 1.1, we see that $\dim \widehat{HF}(Y, \mathfrak{s}) \leq 2$ for all $\mathfrak{s} \in \text{spin}^c(Y)$. Since $\chi(\widehat{HF}(Y, \mathfrak{s})) = 1$, we must in fact have $\dim \widehat{HF}(Y, \mathfrak{s}) = 1$ for all $\mathfrak{s}$. This is equivalent to $HF_{\text{red}}(Y) = 0$.
Finally, assume $b_1(Y) = b_1(\Sigma(Y, L)) = 1$. As in the previous case, Theorem 1.1 guarantees

$$\dim \widehat{HF}(Y, \mathfrak{s}) \leq \begin{cases} 2 \text{ if } \pi^*\mathfrak{s} \text{ is torsion} \\ 0 \text{ if } \pi^*\mathfrak{s} \text{ is non-torsion.} \end{cases}$$

Since $\mathfrak{s}$ is torsion if and only if $\pi^*\mathfrak{s}$ is torsion, we have the desired constraints on $\widehat{HF}(Y)$ to guarantee that $HF_{\text{red}}(Y) = 0$. □

Proof of Corollary 1.6. In the Seidel-Smith spectral sequence (see Section 2.2), the E^1 page is $\widehat{HF}(\Sigma(Y, K), \pi^*\mathfrak{s}) \otimes \mathbb{F}_2[[\theta, \theta^{-1}]$, and the d_1 differential is given by $(1 + \tau_*)\theta$. If τ_* was not the identity, the d_1 differential would not be identically 0, and we would deduce that $\dim \widehat{HF}(Y, \mathfrak{s})$ is strictly less than $\dim \widehat{HF}(\Sigma(Y, K), \pi^*\mathfrak{s})$, contradicting Theorem 1.1. □

Proof of Proposition 1.10. Since K has determinant 1, $\Sigma(Y, K)$ is a homology sphere. As K is obtained by a rational tangle replacement, $\Sigma(Y, K)$ is obtained by surgery on a knot J in $\Sigma(Y, U) = Y \# Y$. Note that the surgery coefficient must be $1/n$ for some $n \in \mathbb{Z}$ to produce a homology sphere. Since $\Sigma(Y, K)$ is an L-space, Y is an L-space by Corollary 1.2, and so $Y \# Y$ is an L-space. In what follows, recall that if Z is a homology sphere L-space and a surgery $Z_\alpha(P)$ is an L-space then $|\alpha| \geq 2g(P) - 1$ (cf. [OSz11, Proposition 9.6]).
First, assume that $|n| \geq 2$, so by the previous remark $g(J) = 0$, i.e., J is unknotted in $Y \# Y$. Therefore, $\Sigma(Y, K)$ is a homology sphere obtained by surgery along an unknot in $Y \# Y$, so $\Sigma(Y, K)$ is $Y \# Y$ as well. By a result of Kim-Tollefson [KT80, Corollary 1], because Y is prime, the covering involution on $Y \# Y$ is either a connected sum of involutions on Y or comes from taking the branched double cover of an unknot in an embedded B^3 in Y. We must rule out the former. In order for a connected sum of involutions on Y to have a quotient to Y, we must be able to write $Y = \Sigma(Y, K')$ and $Y = \Sigma(S^3, K'')$; again, we are using the irreducibility of Y. If $Y = S^3$, then $K' = K'' = U$ and so K is unknotted. If $Y \neq S^3$, then Y cannot admit a self-map of degree 2. Indeed, if Y is a prime L-space other than S^3, then Y is the Poincaré homology sphere or is hyperbolic [Eft18, HRW16]. The case of the Poincaré homology sphere is handled by Boileau-Otal [BO91, Proposition 3.1] and the hyperbolic case follows from supermultiplicativity of the Gromov norm, which is positive for hyperbolic manifolds, under non-zero degree maps. Thus, in this case, K is unknotted.

Next, assume that $n = \pm 1$. In this case, there are two options. The first is that J is unknotted, and by the previous argument, so is K. The other is that $g(J) = 1$. While a knot in S^3 with a non-trivial L-space surgery is fibered, a knot P in a homology sphere L-space Z with a non-trivial L-space surgery has the property that P is fibered in some (not necessarily prime or proper) connected-summand of Z. (The statement for knots in S^3 is due to Ghiggini [Ghi08]. The statement for knots in arbitrary homology spheres with irreducible exteriors follows from Ni's work [Ni07, Theorem 1.1 and Proof of Corollary 1.3].) Therefore, in our case, J is a genus one fibered knot in a summand Q of $Y\#Y$, which is necessarily a homology sphere L-space. Of course, viewed as a knot in Q, $1/n$-surgery on J is again an L-space homology sphere, since it is a summand of $\Sigma(Y, K)$. By Baldwin's work [Bal08], the only homology sphere L-space, genus one fibered L-space knot pairs are $(S^3, \pm T_{2,3})$ and $\mp(\Sigma(2, 3, 5), F_5)$, where F_5 denotes the singular fiber of order 5, i.e. the core of $+1$-surgery on $T_{2,3}$. (Here, the signs are chosen based on the sign of n.) Note that in the former case, $1/n$-surgery produces $\pm\Sigma(2, 3, 5)$, while in the latter case, $1/n$-surgery produces S^3.

In the first case, J is a copy of $\pm T_{2,3}$ contained in an embedded 3-ball in $Y\#Y$, and so $\Sigma(Y, K) = Y\#Y\# \pm \Sigma(2, 3, 5)$. Since Y is prime, it follows from Kim-Tollefson [KT80, Corollary 1] that K must be a knot in an embedded 3-ball in Y with branched double cover $\pm\Sigma(2, 3, 5)$. (Here we are using that $\Sigma(2, 3, 5)$ is not a branched or unbranched double cover of itself, which follows from [BO91, Proposition 3.1].) By a result of Watson [Wat12, Theorem 6.2] K is a copy of $\mp T_{3,5}$ in an embedded B^3 in Y. In the second case, we see that the Poincaré homology sphere is a summand of $Y\#Y$ and hence of Y. Because we assumed Y is irreducible, Y is the Poincaré homology sphere, and $\Sigma(Y, K)$ is one copy of the Poincaré homology sphere. Since there is no knot in the Poincaré homology sphere whose branched double cover is again the Poincaré homology sphere, this last case does not arise. □

Remark 5.1. If Y is not prime, similar characterizations can likely be obtained, but it requires a more tedious analysis of the possible involutions on the relevant 3-manifolds.

Remark 5.2. Assuming the Heegaard Floer Poincaré conjecture, this proposition can be proved without requiring the results from this paper, since the involutions on S^3 and connected sums of the Poincaré homology sphere are well understood.

5.1 Analogue in sutured Floer homology

In this section we prove an analogue of Theorem 1.1 for sutured Floer homology. Let (M, γ) be a balanced sutured manifold and $L \subset M$ a nullhomologous link in the interior of M. Then, there is a natural sutured structure $\widetilde{\gamma}$ on $\partial\Sigma(M, L)$: the sutures are the preimage of the sutures of M under the covering map $\pi\colon \partial\Sigma(M, L) \to \partial M$, and the positive / negative regions $\widetilde{R}_\pm$ are the preimages of the positive / negative regions in ∂M. Since $\chi(\widetilde{R}_+) = 2\chi(R_+) = 2\chi(R_-) = \chi(\widetilde{R}_-)$, $(\Sigma(M,L),\widetilde{\gamma})$ is also balanced.

Proof of Proposition 1.11. For simplicity, we assume that K is a knot. The extension from knots to links is analogous to the closed case.

By a *doubly-pointed sutured Heegaard diagram* for (M,γ,K) we mean a sutured Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ for (M,γ) together with a pair of points $z,w \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ so that $(\Sigma \setminus \mathrm{nbd}(\{z,w\}),\boldsymbol{\alpha},\boldsymbol{\beta})$ is a sutured Heegaard diagram for $M \setminus \mathrm{nbd}(K)$, with two meridional sutures around K. Call $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z,w)$ *admissible* if the sutured Heegaard diagram $(\Sigma \setminus \mathrm{nbd}(z),\boldsymbol{\alpha},\boldsymbol{\beta})$ is admissible.

A simple Morse-theory argument shows that every knot in the interior of M is represented by some doubly-pointed Heegaard diagram (compare [Juh06, Proposition 2.3]). Further, any doubly-pointed Heegaard diagram can be made weakly admissible by an isotopy of the α-circles (cf. [Juh06, Proposition 3.15]).

So, choose an admissible doubly-pointed sutured Heegaard diagram $\mathcal{H} = (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z,w)$ for $K \subset (M,\gamma)$. A Seifert surface for K transverse to Σ induces a branched double cover $\widetilde{\Sigma}$ of Σ, branched over $\{z,w\}$. If we let $\widetilde{\boldsymbol{\alpha}}$, $\widetilde{\boldsymbol{\beta}}$, $\widetilde{z}$, and $\widetilde{w}$ be the preimages of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, z, and w under the branched covering map then $(\widetilde{\Sigma},\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}},\widetilde{z},\widetilde{w})$ is a doubly-pointed sutured Heegaard diagram representing $\widetilde{K} = \pi^{-1}(K)$ in $(\Sigma(M,K),\widetilde{\gamma})$. (This is clear, for example, by considering a Morse-theoretic interpretation of sutured Heegaard diagrams.)

Let d be the number of α-circles in the Heegaard diagram $\mathcal{H}$. We apply Large's theorem to prove Lemma 5.4 below, which yields a spectral sequence with E^1 page the Floer homology $HF(\mathbb{T}_{\widetilde{\alpha}},\mathbb{T}_{\widetilde{\beta}}) \otimes \mathbb{F}_2[[\theta,\theta^{-1}]$ computed in $\mathrm{Sym}^{2d}(\widetilde{\Sigma} \setminus \{\widetilde{z}\})$ and E^∞ page the Floer homology $HF(\mathbb{T}_\alpha,\mathbb{T}_\beta) \otimes \mathbb{F}_2[[\theta,\theta^{-1}]$ computed in $\mathrm{Sym}^d(\Sigma \setminus \{z\})$.

Note that these Lagrangian Floer homologies are not describing the sutured Floer homologies in the proposition. Given a balanced sutured manifold (Z,η), define (Z°,η°) to be the balanced sutured manifold obtained by removing an embedded 3-ball from M and adding an equatorial suture on the additional 2-sphere component in the boundary. So,

$$HF(\mathbb{T}_{\widetilde{\alpha}},\mathbb{T}_{\widetilde{\beta}}) \cong SFH(\Sigma(M,K)^\circ,\widetilde{\gamma}^\circ)$$
$$HF(\mathbb{T}_\alpha,\mathbb{T}_\beta) \cong SFH(M^\circ,\gamma^\circ).$$

The Künneth theorem for sutured Floer homology [Juh06, Proposition 9.15] implies that

$$SFH(Z^\circ,\eta^\circ) \cong SFH(Z,\eta) \otimes H_*(S^1), \tag{5.3}$$

which gives the desired result. □

Lemma 5.4. *Consider an admissible doubly-pointed sutured Heegaard diagram* $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z,w)$ *for* $K \subset (M,\gamma)$, *and let* $(\widetilde{\Sigma},\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}},\widetilde{z},\widetilde{w})$ *be the associated diagram for* $(\Sigma(M,K),\gamma)$. *Then, there is a spectral sequence with* E^1*-page the Floer homology* $HF(\mathbb{T}_{\widetilde{\alpha}},\mathbb{T}_{\widetilde{\beta}}) \otimes \mathbb{F}_2[[\theta,\theta^{-1}]$ *inside* $\mathrm{Sym}^{2d}(\widetilde{\Sigma} \setminus \{\widetilde{z}\})$ *and* E^∞*-page the Floer homology* $HF(\mathbb{T}_\alpha,\mathbb{T}_\beta) \otimes \mathbb{F}_2[[\theta,\theta^{-1}]$ *inside* $\mathrm{Sym}^d(\Sigma \setminus \{z\})$.

Proof. The proof that the symplectic hypotheses of Theorem 2.4 are satisfied is similar to the proof of Proposition 4.2, and is left to the reader. It remains to show that there is a tangent-normal isomorphism

$$(T\,\mathrm{Sym}^d(\Sigma\setminus\{z\}), T\mathbb{T}_\alpha, T\mathbb{T}_\beta) \cong (N\,\mathrm{Sym}^d(\Sigma\setminus\{z\}), N\mathbb{T}_\alpha, N\mathbb{T}_\beta).$$

The argument proceeds in two steps as in the closed case. First, Large's argument [Lar19, Proof of Propositions 10.1 and 10.2] establishes an isomorphism

$$\Phi_1\colon (T\,\mathrm{Sym}^d(\Sigma\setminus\{w,z\}), T\mathbb{T}_\alpha, T\mathbb{T}_\beta) \cong (N\,\mathrm{Sym}^d(\Sigma\setminus\{w,z\}), N\mathbb{T}_\alpha, N\mathbb{T}_\beta).$$

Since z and w lie in the same component of Σ, as a special case we again get an isomorphism

$$\Phi_2\colon T\,\mathrm{Sym}^d(\Sigma\setminus\{z\}) \cong N\,\mathrm{Sym}^d(\Sigma\setminus\{z\})$$

which may not respect the tangent and normal bundles to the tori (cf. Lemma 3.6).
We show that the first of these isomorphisms can be modified to extend this over $\{w\}\times\mathrm{Sym}^{d-1}(\Sigma\setminus\{z\})$. As in Section 3, the space $\Sigma\setminus\{z\}$ deformation retracts onto a wedge of circles, so by Lemma 3.3 any g-fold symmetric product $\mathrm{Sym}^d(\Sigma\setminus\{z\})$ has the homotopy type of a skeleton of a torus. It follows by the same argument as Corollary 3.4 that the Chern character is an integral isomorphism $\mathrm{ch}\colon \mathrm{Iso}(E,E)\to H^{\mathrm{odd}}(\mathrm{Sym}^d(\Sigma\setminus\{z\}))$ for any complex vector bundle E over $\mathrm{Sym}^d(\Sigma\setminus\{z\})$, and similarly for $\mathrm{Sym}^d(\Sigma\setminus\{z,w\})$. Lemma 3.8 still holds in this context, with the same proof. So, the proof of Proposition 3.1 applies to show that $\Phi_2\circ\Phi_4$ gives a tangent-normal isomorphism, as desired. ☐

Corollary 5.5. *Let (M,γ) be a balanced sutured manifold and $L\subset M$ a nullhomologous link. If (M,γ) is a taut sutured manifold and $\Sigma(M,L)$ is irreducible, then $(\Sigma(M,L),\widetilde{\gamma})$ is taut as well.*

Proof. An irreducible balanced sutured manifold has non-vanishing sutured Floer homology if and only if it is taut [Juh06, Proposition 9.18], [Juh08, Theorem 1.4]. The theorem therefore follows from Proposition 1.11. ☐

Corollary 5.6. *Let Y be a closed, connected, oriented 3-manifold, $L\subset Y$ a link, and $Q\subset Y\setminus L$ a link which is nullhomologous in $Y\setminus L$. Let $\widetilde{L}$ be the preimage of L inside $\Sigma(Y,Q)$. Then, there is a rank inequality*

$$\dim \widehat{HFL}(\Sigma(Y,Q),\widetilde{L}) \geq \dim \widehat{HFL}(Y,L).$$

Here, if L is not nullhomologous, by $\widehat{HFL}(Y,L)$ we mean the sutured Floer homology of $Y\setminus \mathrm{nbd}(L)$ with meridional sutures. For a more concrete case, if $Y=S^3$ and Q is an unknot, then this gives a rank inequality for the knot Floer homology of certain 2-periodic links, which was proved by the first author. In this case, the condition that Q be nullhomologous in the exterior of L is equivalent to the quotient link having linking number 0 with the axis of symmetry.

Proof. Let M denote the exterior of L and γ consist of a pair of meridional sutures for each toral boundary component, so $SFH(M,\gamma) \cong \widehat{HFL}(Y,L)$. Similarly, $SFH(\Sigma(M,Q),\widetilde{\gamma}) \cong \widehat{HFL}(\Sigma(Y,Q),\widetilde{L})$. Thus, the result follows from Proposition 1.11, since Q is nullhomologous in M by assumption. □

Remark 5.7. Perhaps one could use Proposition 1.11 to recover classical theorems in equivariant 3-manifold topology for involutions (with suitable constraints on the branch set), such as the equivariant Dehn's lemma [MY81, Edm86].

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