A Universal Rigid Abelian Tensor Category

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Abstract. We prove that any rigid additive symmetric monoidal category can be mapped to a rigid abelian symmetric monoidal category in a universal way. This sheds a new light on abelian $\otimes$-envelopes and on motivic conjectures such as Grothendieck's standard conjecture D and Voevodsky's smash nilpotence conjecture.

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1 Introduction

Recently there has been a lot of activity on embedding a given rigid additive symmetric monoidal category into an abelian one in a universal way: we can quote on the one hand Coulembier's monoidal abelian envelopes from [8] and his subsequent works alone and with coauthors [9, 10], on the other O'Sullivan's super Tannakian hulls [26]. Different but closely related is the work of Schäppi [31]. We may also quote that of Delpeuch in the non-additive context [13, App. B.3].

Both Coulembier and O'Sullivan impose the condition that their envelopes represent *faithful* monoidal functors (for very good reasons, see introduction to Section 7). In this paper, we take a different approach which relaxes this restriction, based on Freyd's free abelian category on a given additive category [16]. Freyd's construction associates to an additive category $\mathcal{C}$ an abelian category $\mathrm{Ab}(\mathcal{C})$ and a (fully faithful) additive functor $\iota_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ab}(\mathcal{C})$ such that any additive functor from $\mathcal{C}$ to an abelian category $\mathcal{A}$ extends uniquely to an exact functor from $\mathrm{Ab}(\mathcal{C})$ to $\mathcal{A}$. Suppose that $\mathcal{C}$ is symmetric monoidal. In [5], the authors provided $\mathrm{Ab}(\mathcal{C})$ with a right exact, symmetric monoidal structure such that $\iota_{\mathcal{C}}$ is strong monoidal and universal for strong monoidal functors to abelian categories provided with a right exact symmetric monoidal structure (with a technical condition, see *loc. cit.* Prop. 1.10).

Suppose now that $\mathcal{C}$ is rigid. We construct a localisation $T(\mathcal{C})$ of $\mathrm{Ab}(\mathcal{C})$ which is rigid and such that the induced functor $\mathcal{C} \to T(\mathcal{C})$ is, this time, universal for strong additive symmetric monoidal (not necessarily faithful) functors from $\mathcal{C}$ to rigid symmetric monoidal abelian categories: see Theorem 6.1. When $\mathcal{C}$ admits a $\otimes$-envelope, the latter turns out to be a localisation of $T(\mathcal{C})$ (Proposition 7.2). The novel thing here is that the ring of endomorphisms $Z(T(\mathcal{C}))$ of the unit object of $T(\mathcal{C})$ need not be a field even if this is the case for $\mathcal{C}$: if $\mathcal{C}$ is the category of representations of an affine group scheme G over a field of characteristic 0, then $Z(T(\mathcal{C}))$ is a field if and only if G is proreductive, see Example 8.10. This example is the only one where $T(\mathcal{C})$ can be computed so far (and only for certain G's), besides Example 6.5.

Our motivating example was the one where $\mathcal{C}$ is the $\mathbf{Q}$-linear category $\mathcal{M}_{\mathrm{rat}}(k)$ of Chow motives over a field k (motives with $\mathbf{Q}$ coefficients modulo rational equivalence). Write $T(k)$ for $T(\mathcal{M}_{\mathrm{rat}}(k))$. By Jannsen's work [18], the category $\mathcal{M}_{\mathrm{num}}(k)$ of motives modulo numerical equivalence is abelian semi-simple, whence a canonical $\mathbf{Q}$-linear exact $\otimes$-functor $T(k) \to \mathcal{M}_{\mathrm{num}}(k)$. We prove in Theorem 9.1 that this functor is a Serre localisation: it is an equivalence of categories if and only if $Z(T(k))$ is a field. In particular, the latter condition implies Grothendieck's standard conjecture D [21, 3.6 (i)]: homological and numerical equivalences agree for any Weil cohomology over k.

In Proposition 9.4, we give a related consequence of the existence of $T(k)$ on Schäppi's category mentioned earlier.

Conversely, Voevodsky's conjecture [32, Conj. 4.2], predicting that smash nilpotent and numerical equivalences agree, implies that $Z(T(k))$ is a field: see Proposition 9.5. To summarise the situation, we have the following chain of implications, noting that the canonical functor $\mathcal{M}_{\mathrm{rat}}(k) \to T(k)$ factors through $\mathcal{M}_{\mathrm{tnil}}(k)$ by Theorem 6.1, where tnil is smash-nilpotent equivalence:

Theorem. *Voevodsky's conjecture* $\iff$ $Z(T(k))$ *is a field and* $\mathcal{M}_{\mathrm{tnil}}(k) \to T(k)$ *is faithful* $\Rightarrow$ $Z(T(k))$ *is a field* $\iff$ $T(k) \xrightarrow{\sim} \mathcal{M}_{\mathrm{num}}(k) \Rightarrow$ *the standard conjecture D.*

If one wants to stay away from any conjecture, one can argue that $T(k)$ gives in some sense an answer to Grothendieck's quest for a universal abelian category representing all cohomology theories on smooth projective varieties. However, Grothendieck really thought of *Weil cohomologies*, and $T(k)$ does not carry *a priori* a grading: we plan to solve this issue in a further work, by adjoining such grading universally (and unconditionally).

2 Notation and terminology

An additive, symmetric, monoidal, unital category (with bilinear tensor product) will be briefly called a $\otimes$-*category*. A $\otimes$-*functor* between $\otimes$-categories is a strong symmetric, monoidal, unital additive functor. See [30] or [12] for the background.

Notation 2.1. For any $\otimes$-category $\mathcal{C}$, we write $Z(\mathcal{C})$ for $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$.

Recall that the ring $Z(\mathcal{C})$ is commutative [30, I.1.3.3.1] and that $\mathcal{C}$ is a $Z(\mathcal{C})$-linear category; it coincides with the ring of [17, III.5.d] (*cf.* [30, I.2.5.2]). If $F:\mathcal{C}\to\mathcal{D}$ is a $\otimes$-functor, we write $Z(F):Z(\mathcal{C})\to Z(\mathcal{D})$ for the induced ring homomorphism.

Notation 2.2. a) $\mathbf{Add}^{\otimes}$: the 2-category of $\otimes$-categories, $\otimes$-functors and $\otimes$-natural isomorphisms.
b) $\mathbf{Ex}^{\otimes}$: the 2-category of abelian $\otimes$-categories, exact $\otimes$-functors and $\otimes$-natural isomorphisms.
c) $\mathbf{Add}^{\mathrm{rig}}$ and $\mathbf{Ex}^{\mathrm{rig}}$: their 1-full and 2-full sub-2-categories of rigid categories.

We have the following basic lemma:

Lemma 2.3. *For $\mathcal{C}\in\mathbf{Add}^{\otimes}$, let $\mathcal{C}_{\mathrm{rig}}$ denote its strictly full subcategory of dualisable objects. Then $\mathcal{C}_{\mathrm{rig}}$ is closed under direct sums and direct summands, and $F(\mathcal{C}_{\mathrm{rig}})\subset\mathcal{D}_{\mathrm{rig}}$ for any $\otimes$-functor $F:\mathcal{C}\to\mathcal{D}$.*

Proof. Both points follow from [15, Th. 1.3]. □

3 Reminders and complements on rigid abelian $\otimes$-categories

Let $\mathcal{A}$ be a rigid abelian $\otimes$-category.

Lemma 3.1. *The $\otimes$-structure of $\mathcal{A}$ is exact.*

This is [12, Prop. 1.16]. □

Proposition 3.2. *a) If U is a subobject of $\mathbf{1}$, then $\mathbf{1}=U\oplus U^{\perp}$ where $U^{\perp}=\mathrm{Ker}(\mathbf{1}\to U^{\vee})$, and $U\otimes U=U$.*
b) There is a bijective correspondence between subobjects of $\mathbf{1}$, idempotents $e\in Z(\mathcal{A})$ and ($\otimes$-)decompositions $\mathcal{A}=\mathcal{A}_1\times\mathcal{A}_2$ in which an object is in $\mathcal{A}_1$ (resp. in $\mathcal{A}_2$) if e (resp. $1-e$) acts as the identity morphism on it.

Proof. a) The first fact is [12, Prop. 1.17], and the second one is contained in its proof. b) is [12, Rem. 1.18] complemented by [20, Lemma 4.4 b)]. □

Lemma 3.3. *Let $\mathcal{A}=\mathcal{A}_1\times\mathcal{A}_2$. Any $\otimes$-functor $F:\mathcal{A}\to\mathcal{B}$ such that $Z(\mathcal{B})$ is a field vanishes either on $\mathcal{A}_1$ or $\mathcal{A}_2$.*

Proof. Indeed, the idempotent corresponding to this decomposition in Proposition 3.2 b) must go to 0 or 1 in $Z(\mathcal{B})$. □

Proposition 3.4. *Suppose that $Z(\mathcal{A})$ is a field. Then a $\otimes$-functor from $\mathcal{A}$ to a (nonzero) abelian $\otimes$-category $\mathcal{B}$ is faithful if it is exact. The converse is true if $Z(\mathcal{B})$ is a field.*

Proof. "If" is [12, Prop. 1.19], and "only if" is [9, Th. 2.4.1] (see also [26, Lemma 10.7] in the case of super Tannakian categories). ☐

Proposition 3.5. *a) if $Z(\mathcal{A})$ is a field, $\mathcal{A}$ is integral: for two morphisms f, g, $f \otimes g = 0$ implies $f = 0$ or $g = 0$.*
b) In general, $\mathcal{A}$ is reduced: for a morphism f, $f^{\otimes 2} = 0$ implies $f = 0$. In particular, the ring $Z(\mathcal{A})$ is reduced.
c) $\mathcal{A}$ is fractionally closed in the sense of [26, Sect. 3, bot. p. 6]. In particular, $Z(\mathcal{A})$ is its own total ring of quotients.
d) If $Z(\mathcal{A})$ is a domain then it is a field.
e) $\mathbf{1}$ is Noetherian (equivalently Artinian) if and only if $Z(\mathcal{A})$ is, and then $\mathcal{A}$ is equivalent to a product $\prod_{i\in I} \mathcal{A}_i$, where I is finite and $Z(\mathcal{A}_i)$ is a field for each i.
f) $Z(\mathcal{A})$ is absolutely flat (= von Neumann regular).

Proof. a) is [19, Rem. 2.10]. b) was suggested to the second author by Peter O'Sullivan. By rigidity, we may assume that the domain of f is $\mathbf{1}$. Then f factors through a monomorphism $\operatorname{Coker}(f) \hookrightarrow A$, where A is the codomain of f. If $f \neq 0$, $\operatorname{Coker}(f) \neq 0$; but this is a subobject of $\mathbf{1}$ by Proposition 3.2 a), hence $\operatorname{Coker}(f)^{\otimes 2} = \operatorname{Coker}(f) \neq 0$ by the same Proposition, and $f \otimes f \neq 0$ by Lemma 3.1. c) is [26, Lemma 3.1]. d) is special case of c), but we give a direct proof: if $Z(\mathcal{A})$ is not a field, it contains an idempotent by Proposition 3.2 b) so it cannot be a domain. e) follows easily from the same proposition. The consequences of b) and c) on $Z(\mathcal{A})$ follow from the fact that, in this ring, composition and tensor product coincide, see [30, I.1.3.3.1]. For f), see [20, Prop. 4.2]. ☐

4 Complements on abelian ⊗-categories

In this section, $\mathcal{A}$ is an abelian (not necessarily rigid) ⊗-category. The following proposition completes the proof of [5, Prop. 1.13 (3)], removing its hypothesis of right exactness of the Homs.

Proposition 4.1. *If the tensor structure of $\mathcal{A}$ is exact, then the full subcategory $\mathcal{A}_{\mathrm{rig}}$ of dualisable objects is closed under kernels and cokernels. In particular, $\mathcal{A}_{\mathrm{rig}}$ is abelian and the inclusion functor $\mathcal{A}_{\mathrm{rig}} \hookrightarrow \mathcal{A}$ is exact.*

Proof. Let $A' \to A \to A'' \to 0$ be exact, with $A', A \in \mathcal{A}_{\mathrm{rig}}$, and write $K = \operatorname{Ker}(A^\vee \to A'^\vee)$. For any $B, C \in \mathcal{A}$, we have a commutative diagram of exact sequences

$$\begin{array}{ccccc}
0 \to \mathcal{A}(A'' \otimes B, C) & \longrightarrow & \mathcal{A}(A \otimes B, C) & \longrightarrow & \mathcal{A}(A' \otimes B, C) \\
 & & \wr\downarrow & & \wr\downarrow \\
0 \to \mathcal{A}(B, K \otimes C) & \longrightarrow & \mathcal{A}(B, A^\vee \otimes C) & \longrightarrow & \mathcal{A}(B, A'^\vee \otimes C)
\end{array}$$

due to the exactness of $\otimes$. It induces an isomorphism $\mathcal{A}(A'' \otimes B, C) \xrightarrow{\sim} \mathcal{A}(B, K \otimes C)$ natural in B and C, showing that K is right dual to A''. But then, it is also left dual since $\otimes$ is symmetric (see [7]).
For an exact sequence $0 \to A' \to A \to A''$ with $A, A'' \in \mathcal{A}_{\mathrm{rig}}$, we argue similarly by changing the side of the tensor products in the Hom groups. □

Proposition 4.2. *Suppose $\mathcal{A}$ essentially small with a right exact tensor structure. Let $\mathcal{I} \subseteq \mathcal{A}$ be a Serre subcategory containing the objects $K(f, A) := \operatorname{Ker}(1_A \otimes f)$ for all monomorphisms f and stable under (external) tensor products. Then $\mathcal{A}/\mathcal{I}$ inherits a tensor structure, which is exact. If $\mathcal{I}$ is minimal for those properties, $\mathcal{A} \to \mathcal{A}/\mathcal{I}$ is initial for exact $\otimes$-functors from $\mathcal{A}$ to an abelian $\otimes$-category with exact tensor structure.*

Proof. Let Σ be the multiplicative system associated to $\mathcal{I}$, *i.e.* $s \in \Sigma \iff \operatorname{Ker} s, \operatorname{Coker} s \in \mathcal{I}$. The first point means that Σ is stable under tensor product with identities on the left and on the right, so, say, on the left by symmetry. We proceed as follows:
1) Let $s : C \hookrightarrow D$ be a monomorphism in Σ, and let $A \in \mathcal{A}$. The exact sequence

$$0 \to K(s, A) \to A \otimes C \xrightarrow{1_A \otimes s} A \otimes D \to A \otimes \operatorname{Coker} s \to 0$$

shows that $1_A \otimes s \in \Sigma$.
2) Let $s : C \twoheadrightarrow D$ be an epimorphism in Σ, and let $A \in \mathcal{A}$. The exact sequence

$$A \otimes \operatorname{Ker} s \to A \otimes C \xrightarrow{1_A \otimes s} A \otimes D \to 0$$

shows that $1_A \otimes s \in \Sigma$.
3) Any $s \in \Sigma$ can be written $t \circ u$, with u epi, t mono and $t, u \in \Sigma$. This, with 2) and 3), completes the proof of the first point.
Let us now prove exactness of the tensor product. Let $A \in \mathcal{A}/\mathcal{I}$ and let $(*) : 0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$ be a short exact sequence in $\mathcal{A}/\mathcal{I}$ (recall that $\mathcal{A}$ and $\mathcal{A}/\mathcal{I}$ have the same objects). By [17, p. 368, Cor. 1], $(*)$ is isomorphic to a short exact sequence of $\mathcal{A}$; to show that $A \otimes (*)$ is exact, we may therefore assume that f and g come from morphisms of $\mathcal{A}$. Since the projection functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$ is exact, we already have right exactness. But $1_A \otimes f$ is a monomorphism in $\mathcal{A}/\mathcal{I}$ by definition of $\mathcal{I}$, so the proof is complete.
Note that a minimal $\mathcal{I}$ exists: any intersection of Serre subcategories (*resp.* closed under external tensor product) is a Serre subcategory (*resp.* is closed under external tensor product). The initiality is now clear, since any exact $\otimes$-functor from $\mathcal{A}$ to an abelian $\otimes$-category with an exact tensor structure sends all objects $K(f, A)$ to 0. □

Remark 4.3. Let $\mathcal{I}_0$ be the minimal Serre subcategory as in Proposition 4.2, and let $\mathcal{I}' \subseteq \mathcal{I}_0$ denotes the smallest Serre subcategory of $\mathcal{A}$ containing the objects $K(f, A)$ (no $\otimes$-ideal condition). It is tempting to try and show that

$\mathcal{I}' = \mathcal{I}_0$. At least, $B \otimes K(f,A) \in \mathcal{I}'$ for any $A, B \in \mathcal{A}$ and any monomorphism $f : C \hookrightarrow D$, as follows from the exact sequence

$$0 \to K(g,B) \to B \otimes K(f,A) \to K(f, B \otimes A)$$

where g is the monomorphism $K(f,A) \hookrightarrow A \otimes C$. But this does not seem sufficient: since $\otimes$ is right exact, $\mathcal{C} \otimes \mathcal{I}'$ is stable under quotients and extensions, but maybe not under kernels.

Remark 4.4. In general, $Z(\mathcal{A}/\mathcal{I})$ need not be a field even if $Z(\mathcal{A})$ is: see Example 8.10 below. Here are two things one can say:
a) We have

$$Z(\mathcal{A}/\mathcal{I}) = \varinjlim_{A', A''} \mathcal{A}(A', \mathbf{1}/A'') \tag{1}$$

where A' (*resp.* A'' runs through subobjects of $\mathbf{1}$ such that $\mathbf{1}/A' \in \mathcal{I}$ (*resp.* $A'' \in \mathcal{I}$): this translates the definition of morphisms in $\mathcal{A}/\mathcal{I}$ as in [17, III.1].
b) $\mathbf{1}_{\mathcal{A}/\mathcal{I}}$ is irreducible provided any subobject or any quotient of $\mathbf{1}_{\mathcal{A}}$ belongs to $\mathcal{I}$: this follows from [17, p. 368, Cor. 1] which was already used in the proof of Proposition 4.2. In particular, $\mathbf{1}_{\mathcal{A}}$ irreducible $\Rightarrow$ $\mathbf{1}_{\mathcal{A}/\mathcal{I}}$ irreducible. See also the last point of the next proposition.

Proposition 4.5. *Let $\mathcal{A} \in \mathbf{Ex}^{\otimes}$ be essentially small and let $\mathcal{I}$ be a Serre subcategory stable under external tensor product. If the tensor structure of $\mathcal{A}$ is exact, there is a unique tensor structure on $\mathcal{A}/\mathcal{I}$ such that $p : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is a $\otimes$-functor, and this tensor structure is exact. If $\mathcal{A}$ is rigid, so is $\mathcal{A}/\mathcal{I}$, and $Z(\mathcal{A}) \to Z(\mathcal{A}/\mathcal{I})$ is surjective.*

Proof. The first two points are special cases of Proposition 4.2, since all objects $K(f,A)$ are 0 by assumption. The rigidity of $\mathcal{A}/\mathcal{I}$ follows from Lemma 2.3, since p is surjective. The last point follows from (1) and Proposition 3.2, which implies that $Z(\mathcal{A}) \to \mathcal{A}(A', \mathbf{1}/A'')$ is (split) surjective for all $A', A'' \subseteq \mathbf{1}$. □

5 Freyd's universal construction

Recall ([16], see also [22, 2.10], [28, Th. 4.1 and Cor. 4.2], [6], [5, §1]) that, for any additive category $\mathcal{C}$, there is an abelian category $\mathrm{Ab}(\mathcal{C})$ and an additive functor $\iota_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ab}(\mathcal{C})$ such that any additive functor $\mathcal{C} \to \mathcal{A}$, where $\mathcal{A}$ is an abelian category, extends through $\iota_{\mathcal{C}}$ to an exact functor $\mathrm{Ab}(\mathcal{C}) \to \mathcal{A}$, unique up to unique equivalence of categories. The functor $\iota_{\mathcal{C}}$ is fully faithful.

Lemma 5.1. *a) The category $\mathcal{C}$ generates $\mathrm{Ab}(\mathcal{C})$ as an abelian category in the following sense: if $\mathcal{A} \subseteq \mathrm{Ab}(\mathcal{C})$ is a strictly full abelian subcategory such that the inclusion functor is exact and $\mathcal{A}$ contains $\iota_{\mathcal{C}}(\mathcal{C})$, then $\mathcal{A} = \mathrm{Ab}(\mathcal{C})$. In particular, any object of $\mathrm{Ab}(\mathcal{C})$ is a subquotient of $\iota_{\mathcal{C}}(C)$ for some $C \in \mathcal{C}$.*
b) $Z(\mathcal{C}) \xrightarrow{\sim} Z(\mathrm{Ab}(\mathcal{C}))$.

Proof. a) is [28, Lemma 4.12], and b) is obvious by full faithfulness. In a) "In particular" holds because any subobject and any quotient of a subquotient is a subquotient. □

Definition 5.2. An abelian category $\mathcal{A}$ is *split* if $\mathcal{A} \xrightarrow{\iota_{\mathcal{A}}} \mathrm{Ab}(\mathcal{A})$ is an equivalence of categories.

Proposition 5.3. *For an abelian category $\mathcal{A}$, the following are equivalent:*

1. *$\mathcal{A}$ is split.*
2. *Every additive functor from $\mathcal{A}$ to an abelian category is exact.*
3. *Every short exact sequence splits.*
4. *Every object is projective.*
5. *Every object is injective.*

This holds in particular if $\mathcal{A}$ is semisimple[1].
If $\mathcal{A}$ is split, any full pseudo-abelian subcategory of $\mathcal{A}$ is a Serre subcategory (in particular, abelian) and is split, and any Serre localisation of $\mathcal{A}$ is split. The same holds when replacing "split" by "semisimple".

Proof. The only possibly nonobvious point is (2) $\Rightarrow$ (4): use the Hom functor. □

If $\mathcal{C}$ has a $\otimes$-structure, then $\mathrm{Ab}(\mathcal{C})$ inherits a right exact $\otimes$-structure for which $\iota_{\mathcal{C}}$ is a $\otimes$-functor [5, Prop. 1.8].

Proposition 5.4. *Let $\mathcal{I} \subseteq \mathrm{Ab}(\mathcal{C})$ be minimal in Proposition 4.2, and write $T(\mathcal{C}) = \mathrm{Ab}(\mathcal{C})/\mathcal{I}$. Then the composition $\mathcal{C} \xrightarrow{\iota_{\mathcal{C}}} \mathrm{Ab}(\mathcal{C}) \to T(\mathcal{C})$ is 2-universal for $\otimes$-functors from $\mathcal{C}$ to abelian $\otimes$-categories with exact tensor structure (with respect to exact $\otimes$-functors). In particular, $\mathcal{C}$ generates $T(\mathcal{C})$ in the same sense as in Lemma 5.1 a).*

Proof. Let $F : \mathcal{C} \to \mathcal{A}$ be a $\otimes$-functor, with $\mathcal{A} \in \mathbf{Ex}^{\otimes}$. If the tensor structure of $\mathcal{A}$ is exact, F factors uniquely through an exact $\otimes$-functor $\tilde{F} : \mathrm{Ab}(\mathcal{C}) \to \mathcal{A}$. Indeed, we apply [5, Prop. 1.10]: by the exactness of the tensor product, we may take $\mathcal{A}^\flat = \mathcal{A}$ in *loc. cit.* so its hypothesis is trivially verified. Then $\tilde{F}$ factors uniquely through $T(\mathcal{C})$ by Proposition 4.2. □

Proposition 5.5. *Let $\mathcal{A} \in \mathbf{Ex}^{\mathrm{rig}}$. Then $\mathcal{A}$ is split if and only if $\mathbf{1}$ is projective.*

Proof. If $\mathbf{1}$ is projective, the functor $A \mapsto \mathcal{A}(B, A) \simeq \mathcal{A}(\mathbf{1}, B^\vee \otimes A)$ is right exact, since $\otimes$ is exact by Lemma 3.1. □

[1] Here we adopt the terminology of [2, 2.1.1]: a preadditive category $\mathcal{A}$ is semisimple if every left $\mathcal{A}$-module is a direct sum of simple objects. If $\mathcal{A}$ is abelian, this means [2, A.2.10 (10)] that every object of $\mathcal{A}$ is a finite direct sum of simple objects.

6 Main theorem

Theorem 6.1. *Let $\mathcal{C}$ be a rigid additive $\otimes$-category. Then the 2-functor*

$$\mathcal{A} \mapsto \mathbf{Add}^{\otimes}(\mathcal{C}, \mathcal{A})$$

from $\mathbf{Ex}^{\mathrm{rig}}$ to $\mathbf{Cat}$ is 2-representable by the category $T(\mathcal{C})$ of Proposition 5.4. Moreover, the obvious functors

$$T(\mathcal{C}) \to T(\mathcal{C}/\sqrt[\otimes]{0}) \to T((\mathcal{C}/\sqrt[\otimes]{0})_{\mathrm{fr}}) \to T(((\mathcal{C}/\sqrt[\otimes]{0})_{\mathrm{fr}})^{\natural})$$

are equivalences of categories, where $\sqrt[\otimes]{0}$ is the $\otimes$-ideal of $\otimes$-nilpotent morphisms [2, Def. 7.4.1], $\mathcal{D}_{\mathrm{fr}}$ (resp. $\mathcal{D}^{\natural}$) is the fractional closure of a $\otimes$-category $\mathcal{D}$ [26, p. 8] (resp. its pseudo-abelian envelope).

Proof. Let $T(\mathcal{C})_{\mathrm{rig}}$ be the strictly full subcategory of dualisable objects: by Proposition 4.1, it is abelian and the full embedding $T(\mathcal{C})_{\mathrm{rig}} \hookrightarrow T(\mathcal{C})$ is exact. Since $\mathcal{C}$ is rigid, its image in $T(\mathcal{C})$ lands into $T(\mathcal{C})_{\mathrm{rig}}$ by Lemma 2.3; therefore, $T(\mathcal{C})_{\mathrm{rig}} = T(\mathcal{C})$ by Proposition 5.4. Let now $\mathcal{A} \in \mathbf{Ex}^{\mathrm{rig}}$. Its tensor structure is exact by Lemma 3.1, hence, again by Proposition 5.4, any $\otimes$-functor $\mathcal{C} \to \mathcal{A}$ factors through $T(\mathcal{C})$, uniquely up to unique $\otimes$-equivalence.
In the last claim, the first equivalence follows from Proposition 3.5 b) and the second (*resp.* third) one from the fact that rigid abelian $\otimes$-categories are fractionally closed [26, Lemma 3.1] (*resp.* that abelian categories are pseudo-abelian). □

Corollary 6.2. *If $\mathcal{C}$ is abelian in Theorem 6.1, the canonical functor $\mathcal{C} \to T(\mathcal{C})$ has an exact $\otimes$-retraction $\sigma_{\mathcal{C}}$. If moreover $\mathcal{C}$ is split, this functor is an equivalence of $\otimes$-categories.*

Proof. The first claim follows from the universal property of $T(\mathcal{C})$ applied to $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. The second one follows from the definition of split (Definition 5.2). □

Remarks 6.3. a) One can perhaps extend Theorem 6.1 to not necessarily rigid additive $\otimes$-categories, see [13, App. B.3].
b) For $\mathcal{C}$ as in Theorem 6.1, $\mathrm{Ab}(\mathcal{C})$ is rigid if and only if its tensor structure is exact. Necessity follows from Lemma 3.1; if conversely the tensor structure of $\mathrm{Ab}(\mathcal{C})$ is exact, then $\mathrm{Ab}(\mathcal{C}) \xrightarrow{\sim} T(\mathcal{C})$ which is rigid by (the proof of) Theorem 6.1.
c) The category $(\mathcal{C}/\sqrt[\otimes]{0})_{\mathrm{fr}})^{\natural}$ of Theorem 6.1 is reduced, fractionally closed and pseudo-abelian.[2]

[2] We thank Peter O'Sullivan for confirming that the fractional closure of a reduced $\otimes$-category is reduced.

Example 6.4. Suppose that, in $\mathcal{C}$, there exists a nilpotent endomorphism with nonzero trace. Then $T(\mathcal{C}) = 0$ because, in rigid abelian $\otimes$-categories, any nilpotent endomorphism has trace 0 [2, Prop. 7.3.3]. An example where this happens is given in [11, §5.8].

Example 6.5. [3] Let R^+ be the additive completion of a commutative ring R considered as a preadditive category (the objects of R^+ are R^n), provided with its canonical $\otimes$-structure. By [23, Prop. 5], there exists a ring homomorphism $R \to R^{\mathrm{abs}}$ which is universal for homomorphisms from R to absolutely flat rings. We claim that $T(R^+) = R^{\mathrm{abs}}$-mod, where the latter is the (abelian) rigid $\otimes$-category of finitely presented (equivalently, finitely generated projective) R^{abs}-modules. If R is absolutely flat, this follows from [29, Prop. 10.2.38] and Remark 6.3 b), the first reference showing that $\mathrm{Ab}(R^+) \xrightarrow{\sim} R$-mod in this case. In general, we use the fact that $Z(\mathcal{A})$ is absolutely flat for any $\mathcal{A} \in \mathbf{Ex}^{\mathrm{rig}}$, Proposition 3.5 f). For such $\mathcal{A}$, a $\otimes$-functor $F : R^+ \to \mathcal{A}$ amounts to an R-module structure on $\mathbf{1}_{\mathcal{A}}$. Thus $Z(\mathcal{A})$ is an R^{abs}-algebra, so F factors uniquely through $(R^{\mathrm{abs}})^+$, hence through R^{abs}-mod as promised.

7 Comparisons

7.1 Abelian $\otimes$-envelopes

Let $\mathbf{Add}_f^{\mathrm{rig}}$ be the sub-2-category of $\mathbf{Add}^{\mathrm{rig}}$ restricted to the $\mathcal{C}$'s such that $Z(\mathcal{C})$ is a field and to *faithful* functors. Similarly, let $\mathbf{Ex}_f^{\mathrm{rig}}$ be the 1-full, 2-full sub-2-category of $\mathbf{Ex}^{\mathrm{rig}}$ determined by those $\mathcal{A} \in \mathbf{Ex}^{\mathrm{rig}}$ such that $Z(\mathcal{A})$ is a field; note that in $\mathbf{Ex}_f^{\mathrm{rig}}$ all exact $\otimes$-functors are automatically faithful by Proposition 3.4, so $\mathbf{Ex}_f^{\mathrm{rig}}$ is contained in $\mathbf{Add}_f^{\mathrm{rig}}$. Coulembier as well as O'Sullivan consider the universal property of Theorem 6.1 restricted to these 2-categories. This has an advantage and a drawback:

- By Proposition 3.4, a solution to this universal problem is automatically an envelope (an idempotent construction).
- Such a solution is much more difficult to construct (when it exists, see Example 6.4).

Nevertheless, both authors provide a solution in special cases, by very different methods. To formalise things, let us set up a definition:

Definition 7.1. Let $\mathcal{C} \in \mathbf{Add}_f^{\mathrm{rig}}$. An *abelian $\otimes$-envelope* of $\mathcal{C}$ is a category $E(\mathcal{C}) \in \mathbf{Ex}_f^{\mathrm{rig}}$ which 2-represents the 2-functor

$$\mathcal{A} \mapsto \mathbf{Add}_f^{\mathrm{rig}}(\mathcal{C}, \mathcal{A})$$

from $\mathbf{Ex}_f^{\mathrm{rig}}$ to $\mathbf{Cat}$.

[3] This example was found independently by P. O'Sullivan [27].

Note that $Z(E(\mathcal{C}))$ must be a field extension of $Z(C)$ by definition. If $\mathcal{C} \to E(\mathcal{C})$ is also full then we have

$$Z(E(\mathcal{C})) = Z(\mathcal{C}). \tag{2}$$

Proposition 7.2. *a) $E(\mathcal{C})$ exists if and only if*

(i) *there exists a faithful $\otimes$-functor $F : \mathcal{C} \to \mathcal{A}$ with $\mathcal{A} \in \mathbf{Ex}_f^{\mathrm{rig}}$;*

(ii) *the (Serre) kernel $\mathcal{S}$ of the induced functor $T(\mathcal{C}) \to \mathcal{A}$ does not depend on $(\mathcal{A}, F)$.*

In this case, $\mathcal{C}$ is integral and $\mathcal{C} \to T(\mathcal{C})$ is faithful.
b) Any abelian $\mathcal{C}$ is its own abelian $\otimes$-envelope.

In particular, if $E(\mathcal{C})$ exists we have a canonical localisation $\otimes$-functor

$$T(\mathcal{C}) \to E(\mathcal{C}). \tag{3}$$

Proof. a) Conditions (i) and (ii) are obviously necessary. Conversely, assume (i) and (ii). Then $E(\mathcal{C}) = T(\mathcal{C})/\mathcal{S}$ is rigid by Proposition 4.5. Moreover, $Z(E(\mathcal{C}))$ is a subring of the field $Z(\mathcal{A})$ and therefore it is a field by Proposition 3.5 d). Thus $E(\mathcal{C}) \in \mathbf{Ex}_f^{\mathrm{rig}}$. Clearly, any functor F as in (i) factors uniquely through $E(\mathcal{C})$, so $E(\mathcal{C})$ satisfies the universal property. Finally, the induced functor $\mathcal{C} \to E(\mathcal{C})$ is faithful because its composition with $E(\mathcal{C}) \to \mathcal{A}$ is faithful for $\mathcal{A}$ as in (i).
The integrality of $\mathcal{C}$ follows from Proposition 3.5 a) and the faithfulness is obvious.
b) As explained above, this follows from Proposition 3.4.
The last remark follows from the proof of a). □

Suppose that $E(\mathcal{C})$ exists. If (3) is an equivalence, then *any* additive $\otimes$-functor from $\mathcal{C}$ to $\mathcal{A} \in \mathbf{Ex}^{\mathrm{rig}}$ is faithful. For example, the category of representations of an affine group scheme G is abelian hence its own envelope by Proposition 7.2 b), but (3) is not an equivalence if G is not proreductive by Example 8.10 below.

Proposition 7.3. *a) If $E(\mathcal{C})$ exists, (3) is an equivalence of categories if and only if $Z(T(\mathcal{C}))$ is a field.*
b) Suppose that $Z(T(\mathcal{C}))$ is a field. Then $T(\mathcal{C})$ is the abelian $\otimes$-envelope of $\mathcal{C}/I$, where I is the (additive) kernel of $\mathcal{C} \to T(\mathcal{C})$. In particular, $T(\mathcal{C})$ is an abelian $\otimes$-envelope of $\mathcal{C}$ if and only if $\mathcal{C} \to T(\mathcal{C})$ is faithful.

Proof. All this follows once again from Proposition 3.4. □

7.2 Coulembier's work

Coulembier's condition for an envelope [8, Def. 1.3.4] is Definition 7.1 plus the requirement that (2) holds. He proves:

THEOREM 7.4 ([8, Th. A]). *Let $\mathcal{C} \in \mathbf{Add}_f^{\mathrm{rig}}$. Then $E(\mathcal{C})$ exists, with property (2), provided every morphism f in $\mathcal{C}$ is split by a strongly faithful object in $\mathcal{C}$.*

Here, $X \in \mathcal{C}$ is *strongly faithful* if $X \otimes - : \mathcal{C} \to \mathcal{C}$ reflects all kernels and cokernels in $\mathcal{C}$, and a morphism $f : X \to Y$ in $\mathcal{C}$ is *split* if there exists $g : Y \to X$ such that $f \circ g \circ f = f$. "Split by X" means that $1_X \otimes f$ is split.
See [8, Th. 4.1.1 (a)] for another sheaf-theoretic sufficient condition.

7.3 O'Sullivan's work

Let $\mathcal{C} \in \mathbf{Add}^{\mathrm{rig}}$ be essentially small, $\mathbf{Q}$-linear, integral (see Proposition 3.5 a)) and Schur-finite. (In particular, the integrality hypothesis implies that $Z(\mathcal{C})$ is a field.) According to [26, Def. 10.2], we call such a category *pseudo-Tannakian*, and *super Tannakian* if it is abelian.
Write $\mathbf{Add}_t^{\mathrm{rig}}$ for the 1-full and 2-full subcategory of $\mathbf{Add}_f^{\mathrm{rig}}$ formed of pseudo-Tannakian categories, and $\mathbf{Ex}_t^{\mathrm{rig}}$ or the 1-full and 2-full subcategory of $\mathbf{Ex}^{\mathrm{rig}}$ formed of super Tannakian categories. We have:

THEOREM 7.5. *For any $\mathcal{C} \in \mathbf{Add}_t^{\mathrm{rig}}$, the 2-functor*

$$\mathcal{A} \mapsto \mathbf{Add}_t^{\mathrm{rig}}(\mathcal{C}, \mathcal{A})$$

from $\mathbf{Ex}_t^{\mathrm{rig}}$ to $\mathbf{Cat}$ is 2-representable.

Proof. This is [26, Lemma 10.7 and Th. 10.10]. □

REMARK 7.6. Let $ST(\mathcal{C})$ be the solution of the above universal problem. It can be proven that $ST(\mathcal{C}) = E(\mathcal{C})$. See [20, Thm. 7.8].

REMARKS 7.7. a) As pointed out by O'Sullivan, any Kimura category verifying the conditions of Theorem 7.4 is semi-simple.
b) As far as we know, and in spite of Propositions 7.2 and 7.3, Theorem 6.1 does not imply either Theorem 7.4 or Theorem 7.5 in any obvious way!

8 The split quotient of $T(\mathcal{C})$

8.1 The ideal $\mathcal{N}$

Let $\mathcal{C} \in \mathbf{Add}^{\mathrm{rig}}$. Recall from [2, 7.1] the $\otimes$-ideal $\mathcal{N}_\mathcal{C} \subseteq \mathcal{C}$ of morphisms universally of trace 0: for $A, B \in \mathcal{C}$,

$$\mathcal{N}_\mathcal{C}(A, B) = \{f \in \mathcal{C}(A, B) \mid \mathrm{tr}(gf) = 0 \ \forall g \in \mathcal{C}(B, A)\}$$

where tr is the categorical trace.

LEMMA 8.1. *For any split $\mathcal{A} \in \mathbf{Ex}^{\mathrm{rig}}$ (Definition 5.2), we have $\mathcal{N}_\mathcal{A} = 0$. Conversely, if $\mathcal{N}_\mathcal{A} = 0$ and $Z(\mathcal{A})$ is Noetherian, then $\mathcal{A}$ is split.*

Proof. By [2, 6.1.5], it suffices to show that $\mathcal{N}_{\mathcal{A}}(\mathbf{1}, A) = 0$ for any $A \in \mathcal{A}$. Let $f : \mathbf{1} \to A$ be a nonzero morphism. If $U = \operatorname{Im} f$, U is injective (Proposition 5.3 (5)), hence the induced monomorphism $U \to A$ has a retraction. Since U is a direct summand of $\mathbf{1}$, this retraction yields a morphism $g : A \to \mathbf{1}$ such that $gf \neq 0$.
For the converse, it suffices by Proposition 5.5 to show that $\mathbf{1}$ is projective. Let $f : A \to \mathbf{1}$ be an epimorphism, with $A \in \mathcal{A}$. Since $\mathcal{N}(A, \mathbf{1}) = 0$, there is $g : \mathbf{1} \to A$ such that $fg \neq 0$. Let $U \neq 0$ be the image of fg. Replace f by $f_1 = (1 - e)f$, where e is the idempotent with image U in the decomposition $\mathbf{1} = U \oplus U^\perp$ of Proposition 3.2 a). Since f was epi, f_1 is epi on $U^\perp$, hence nonzero if $U^\perp \neq 0$. Using $\mathcal{N}(A, U^\perp) = 0$, we find $g_1 : U^\perp \to A$ such that $g_1 f_1 \neq 0$. Iterating, we get a strictly increasing sequence of subobjects of $\mathbf{1}$, which must stop at $\mathbf{1}$ at a finite step. Collecting everything, we get a section of f. ☐

Lemma 8.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a full $\otimes$-functor with $\mathcal{C}, \mathcal{D} \in \mathbf{Add}^{\mathrm{rig}}$. Let f be a morphism of $\mathcal{C}$. Then $f \in \mathcal{N}_{\mathcal{C}} \Rightarrow F(f) \in \mathcal{N}_{\mathcal{D}}$, and the converse is true if $Z(F) : Z(\mathcal{C}) \to Z(\mathcal{D})$ is injective. Under this condition, the induced full $\otimes$-functor*

$$\mathcal{C}/\mathcal{N}_{\mathcal{C}} \to \mathcal{D}/\mathcal{N}_{\mathcal{D}}$$

is faithful.

Proof. Since

$$\operatorname{tr}(F(f)) = F(\operatorname{tr}(f))$$

for any $f \in \mathcal{C}$, this lemma is trivial. ☐

Proposition 8.3. *Let $\mathcal{C} \in \mathbf{Add}^{\mathrm{rig}}$ be such that $Z(\mathcal{C})$ is a field. Then the following conditions are equivalent:*

(i) $(\mathcal{C}/\mathcal{N}_{\mathcal{C}})^\natural$ *is abelian.*

(ii) $(\mathcal{C}/\mathcal{N}_{\mathcal{C}})^\natural$ *is (abelian and) split.*

(iii) *There exists $F : \mathcal{C} \to \mathcal{D}$ as in Lemma 8.2, with $\mathcal{D}$ split.*

Proof. (i) $\Rightarrow$ (ii) follows from the second part of Lemma 8.1, since $Z((\mathcal{C}/\mathcal{N}_{\mathcal{C}})^\natural) = Z(\mathcal{C})$ is a field. (ii) $\Rightarrow$ (iii) is trivial. If (iii) holds, the hypotheses imply that $Z(F)$ is injective and $\mathcal{N}_{\mathcal{D}} = 0$ (by the first part of Lemma 8.1), thus Lemma 8.2 gives us a fully faithful $\otimes$-functor $\mathcal{C}/\mathcal{N}_{\mathcal{C}} \to \mathcal{D}$. Since $\mathcal{D}$ is pseudo-abelian, it extends to a full embedding $(\mathcal{C}/\mathcal{N}_{\mathcal{C}})^\natural \hookrightarrow \mathcal{D}$. Finally, $(\mathcal{C}/\mathcal{N}_{\mathcal{C}})^\natural$ is abelian and split thanks to Proposition 5.3. So (iii) $\Rightarrow$ (ii). ☐

Remark 8.4. Proposition 8.3 means that $\mathcal{C} \to (\mathcal{C}/\mathcal{N}_{\mathcal{C}})^\natural$ is universal with respect to full $\otimes$-functors to split abelian $\otimes$-categories – assumimg such functors exist, which fails *e.g.* in Example 6.4. The situation is parallel to that of Proposition 7.2. When $\mathcal{C}$ is the category of pure motives over a field (see Section 9 below), we recover the classical fact that the Hodge conjecture or the Tate conjecture (plus semi-simplicity) implies the standard conjecture D.

8.2 A splitting

In this subsection, we assume that $(\mathcal{C}/\mathcal{N}_{\mathcal{C}})^\natural$ is abelian split. We write $S(\mathcal{C})$ for this category.

Example 8.5. By Lemma 2.3 and [3, Th. 1 a)], the above hypothesis is satisfied provided $K = Z(\mathcal{C})$ is a field and there exists an extension L/K and a K-linear $\otimes$-functor $H : \mathcal{C} \to \mathcal{V}$ to a nonzero rigid L-linear abelian $\otimes$-category $\mathcal{V}$ in which Hom groups have finite L-dimension. In this case, $S(\mathcal{C})$ is even semisimple.

Write π for the $\otimes$-functor $\mathcal{C} \to S(\mathcal{C})$. By Theorem 6.1, π induces an exact $\otimes$-functor

$$\bar{\pi} : T(\mathcal{C}) \to S(\mathcal{C}). \tag{4}$$

Let $T_0(\mathcal{C})$ be the quotient $T(\mathcal{C})/\operatorname{Ker}\bar{\pi}$ and let

$$\bar{\pi}_0 : T_0(\mathcal{C}) \to S(\mathcal{C})$$

be the induced faithful exact $\otimes$-functor. Note that $T_0(\mathcal{C})$ is still rigid by Proposition 4.5.

Theorem 8.6. *$\bar{\pi}_0$ is an equivalence of categories.*

We first prove that $\bar{\pi}_0$ is full. By rigidity, this is equivalent to

Lemma 8.7. *The injection*

$$T_0(\mathcal{C})(\mathbf{1}, X) \xrightarrow{\bar{\pi}_0} S(\mathcal{C})(\mathbf{1}, \bar{\pi}_0(X))$$

is surjective for any $X \in T_0(\mathcal{C})$.

Proof. Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence in $T_0(\mathcal{C})$. We then get the following commutative diagram

$$\begin{array}{ccccc}
0 \to T_0(\mathcal{C})(\mathbf{1}, X') & \longrightarrow & T_0(\mathcal{C})(\mathbf{1}, X) & \longrightarrow & T_0(\mathcal{C})(\mathbf{1}, X'') \\
{\scriptstyle \bar{\pi}_0}\downarrow & & {\scriptstyle \bar{\pi}_0}\downarrow & & {\scriptstyle \bar{\pi}_0}\downarrow \\
0 \to S(\mathcal{C})(\mathbf{1}, \bar{\pi}_0(X')) & \longrightarrow & S(\mathcal{C})(\mathbf{1}, \bar{\pi}_0(X)) & \longrightarrow & S(\mathcal{C})(\mathbf{1}, \bar{\pi}_0(X'')) \to 0
\end{array}$$

with exact rows and vertical injections. The bottom exact row follows from the exactness of $\bar{\pi}_0$ and the splitness of $S(\mathcal{C})$, granting that $S(\mathcal{C})(\mathbf{1}, -)$ is right exact. If the middle vertical arrow is an epimorphism then the left-most one is by diagram chase; moreover, the right-most is an epimorphism. Therefore, the statement is stable under passing to subquotients: since it is true for objects coming from $\mathcal{C}$, it is true in general by Proposition 5.4. □

To conclude the proof of Theorem 8.6, we show that $\bar{\pi}_0$ is essentially surjective. Let $Y = (\pi(C), e) \in S(\mathcal{C})$, where $C \in \mathcal{C}$ and e is an idempotent of $\operatorname{End}_{\mathcal{C}/\mathcal{N}_{\mathcal{C}}}(\pi(C))$. By the full faithfulness of $\bar{\pi}_0$, e lifts to an idempotent of $p(C)$, where $p : \mathcal{C} \to T_0(\mathcal{C})$ is the canonical functor. □

Corollary 8.8. *Let $F : \mathcal{C} \to \mathcal{A}$ be a $\otimes$-functor with $\mathcal{A} \in \mathbf{Ex}^{\mathrm{rig}}$. Assume that $Z(\mathcal{C})$ is a field. If F is full and $\mathcal{A}$ is split, then the functor*

$$\bar{F} : T(\mathcal{C}) \to \mathcal{A}$$

induced from Theorem 6.1 factors through $S(\mathcal{C})$ and $\operatorname{Ker} \bar{F} = \operatorname{Ker} \bar{\pi}$.

Proof. This follows from Remark 8.4 and Theorem 8.6. □

Corollary 8.9. *If $Z(\mathcal{C})$ is a field, consider the following conditions:*

(i) *The functor $\bar{\pi}$ of (4) is an equivalence of $\otimes$-categories.*

(ii) *$Z(T(\mathcal{C}))$ is a field.*

(iii) *$\mathcal{C} \to T(\mathcal{C})$ is faithful.*

(iv) *$\mathcal{N} = 0$.*

Then (i) $\iff$ (ii), and (i) + (iii) $\iff$ (iv).

Proof. (i) $\iff$ (ii) is obvious in view of Proposition 3.4. The implication (i) + (iii) $\Rightarrow$ (iv) is also trivial. Assume (iv). Then $\mathcal{C}^\natural$ is abelian and split. By Corollary 6.2, the functor $\mathcal{C}^\natural \to T(\mathcal{C}^\natural)$ is an equivalence of $\otimes$-categories. By Theorem 6.1, $T(\mathcal{C}) \to T(\mathcal{C}^\natural)$ is also an equivalence of $\otimes$-categories. This implies (i), and obviously (iii) as well. □

Example 8.10. Suppose $\operatorname{char} K = 0$. Let $\mathcal{C} = \mathbf{Rep}_K(G)$ where G is an affine K-group. In Example 8.5, we may take $L = K$, $\mathcal{V} = \mathbf{Vec}_K$ and for H the forgetful functor. Here $\mathcal{C}$ and $\mathcal{C}/\mathcal{N}$ are abelian. The functor $\bar{\pi}$ from (4) and the $\otimes$-retraction $\sigma_{\mathcal{C}}$ of Corollary 6.2 yield an exact $\otimes$-functor

$$T(\mathcal{C}) \to S(\mathcal{C}) \times \mathcal{C}. \tag{5}$$

If G is proreductive, then $\mathcal{N} = 0$, $\mathcal{C} = T(\mathcal{C}) = S(\mathcal{C})$ and (5) factors through the diagonal functor. On the other hand, if G is not proreductive, then $\mathcal{N} \neq 0$ and $\bar{\pi}$ *does not* factor through the retraction $\sigma_{\mathcal{C}}$.
If $G = \mathbb{G}_a$ (or more generally if its prounipotent radical is $\mathbb{G}_a$, as in [2, App. C]), O'Sullivan has proven that (5) is an equivalence [27]. Can one compute $T(\mathcal{C})$ in more complicated cases?

9 Application to motivic conjectures

9.1 The standard conjecture D

Theorem 9.1. *Let $\mathcal{M}_{\mathrm{rat}}(k)$ be the $\mathbf{Q}$-linear category of Chow motives over a field k.*
a) Let $\pi : \mathcal{M}_{\mathrm{rat}}(k) \to \mathcal{M}_{\mathrm{num}}(k)$ be the canonical functor to the $\mathbf{Q}$-linear abelian

category of motives modulo numerical equivalence. The functor π *induces a* $\otimes$-*functor*

$$\bar{\pi} : T(\mathcal{M}_{\mathrm{rat}}(k)) \to \mathcal{M}_{\mathrm{num}}(k) \tag{6}$$

which is a Serre localisation.
b) Let $F : \mathcal{M}_{\mathrm{rat}}(k) \to \mathcal{A}$ *be a* $\otimes$-*functor where* $\mathcal{A}$ *is abelian and rigid. If* $\mathcal{A}$ *is split and* F *is full, then* F *factors through numerical equivalence.*
c) We have: $\bar{\pi}$ *is an equivalence* $\iff$ $Z(T(\mathcal{M}_{\mathrm{rat}}(k))$ *is a field.*

Proof. Note that $Z(\mathcal{M}_{\mathrm{rat}}(k)) = \mathbf{Q}$.
a) In Example 8.5, take $\mathcal{C} = \mathcal{M}_{\mathrm{rat}}(k)$ and for H a Weil cohomology. Here, $\mathcal{V} = \mathbf{Vec}_L^{\pm}$, the abelian $\otimes$-category of finite-dimensional $\mathbf{Z}/2$-graded L-vector spaces, where L is the field of coefficients of H; the commutativity constraint is given by the Kozsul rule. We apply Theorem 8.6. The category $S(\mathcal{C}) = (\mathcal{C}/\mathcal{N})^{\natural}$ is the category $\mathcal{M}_{\mathrm{num}}(k)$ ([18], which inspired Example 8.5).
b) follow from Corollary 8.8.
c) follows from Corollary 8.9. □

Corollary 9.2. *If* $Z(T(\mathcal{M}_{\mathrm{rat}}(k))$ *is a field, the standard conjecture* D *is true.* □

In Theorem 9.1, we can replace rational equivalence by any adequate equivalence relation which is coarser than the homological equivalence given by a Weil cohomology H as in the proof of a). For example, this homological equivalence itself yields the category of motives $\mathcal{M}_H(k)$ together with the faithful functor $H : \mathcal{M}_H(k) \to \mathbf{Vec}_L^{\pm}$. Applying Corollary 8.9, we find:

Theorem 9.3. *The standard conjecture* D *holds for the Weil cohomology* H $\iff$ $Z(T(\mathcal{M}_H(k))$ *is a field.* □

In [31], Schäppi constructs a graded-Tannakian category $\mathbf{M}_H(k)$ along with a $\otimes$-functor $S : \mathcal{M}_H(k) \to \mathbf{M}_H(k)$ lifting H, and proves that S is an equivalence of categories under the standard conjecture D. From Theorem 6.1, we obtain a $\otimes$-functor

$$\bar{S} : T(\mathcal{M}_H(k)) \to \mathbf{M}_H(k)$$

extending S. Using $\bar{S}$, we get:

Proposition 9.4. *The standard conjecture* D *holds for* H *if and only if* $\mathbf{M}_H(k)$ *is split (e.g. semi-simple) and* S *is full.*

Proof. If: by [31, Th. 3.2.1 (i)], the standard conjecture D implies that S is an equivalence of categories. Since it also implies that $\mathcal{M}_H(k)$ is abelian semi-simple, we conclude. *Only if*: by Remark 8.4, the hypothesis implies that $\bar{S}$ factors through $S(\mathcal{M}_H(k))$; this in turn implies that H factors through numerical equivalence. □

(See [31, Prop. 3.3.4] for a consequence of the semi-simplicity of $\mathbf{M}_H(k)$, assumed alone.)

9.2 Voevodsky's conjecture

Let k be a field. Recall that an algebraic cycle z on a smooth projective variety X is *smash-nilpotent* if $z^{\times n}$ is rationally equivalent to 0 on X^n for $n \gg 0$. This defines an adequate equivalence relation tnil. In [32, Conj. 4.2], Voevodsky conjectured that $\mathcal{M}_{\text{tnil}}(k) \xrightarrow{\sim} \mathcal{M}_{\text{num}}(k)$.
By Theorem 6.1, we have $T(\mathcal{M}_{\text{rat}}(k)) \xrightarrow{\sim} T(\mathcal{M}_{\text{tnil}}(k))$. Let us write $T(k)$ for this rigid abelian $\otimes$-category.

Proposition 9.5. *Voevodsky's conjecture is equivalent to the following two statements put together:*

(i) *$Z(T(k))$ is a field;*

(ii) *the functor $\mathcal{M}_{\text{tnil}}(k) \to T(k)$ is faithful.*

Proof. This follows from Corollary 8.9 applied to $\mathcal{C} = \mathcal{M}_{\text{tnil}}(k)$, noting that this category is pseudo-abelian by definition. □

9.3 Motives over a base

Let S be a nonempty, connected, separated regular excellent scheme of finite Krull dimension. We then have the Deninger-Murre–O'Sullivan rigid $\otimes$-category of relative Chow motives over S [14], [25, §5.1]. For coherence with the classical definition of motives, we shall restrict to the thick subcategory of O'Sullivan's category defined by motives of smooth projective S-schemes (O'Sullivan considers more generally smooth proper S-schemes): we denote this category by $\mathcal{M}_{\text{rat}}(S)$. We have $Z(\mathcal{M}_{\text{rat}}(S)) = \mathbf{Q}$.
Let K be the function field of S. If $j : \operatorname{Spec} K \to S$ is the corresponding inclusion, we have the restriction $\otimes$-functor

$$j^* : \mathcal{M}_{\text{rat}}(S) \to \mathcal{M}_{\text{rat}}(K).$$

We write $\mathcal{M}_{\text{rat}}(K, S)$ for its essential image (motives with good reduction relatively to S).

Theorem 9.6. *The functor j^* is full and its kernel is smash-nilpotent. It induces an equivalence of categories*

$$T(S) \xrightarrow{\sim} T(K, S)$$

where $T(S) := T(\mathcal{M}_{\text{rat}}(S))$ and $T(K, S) := T(\mathcal{M}_{\text{rat}}(K, S))$, and a full embedding

$$\mathcal{M}_{\text{num}}(S) \hookrightarrow \mathcal{M}_{\text{num}}(K)$$

where $\mathcal{M}_{\text{num}}(S) := (\mathcal{M}_{\text{rat}}(S)/\mathcal{N})^\natural$.

Proof. The first assertions are [25, Prop. 5.1.1]. The equivalence of categories then follows fromTheorem 6.1, while the full embedding follows from Lemma 8.2. □

Let now $i : Z \hookrightarrow S$ be a closed subscheme of S, also connected and regular, with function field L; we have a pull-back functor $i^* : \mathcal{M}_{\mathrm{rat}}(S) \to \mathcal{M}_{\mathrm{rat}}(Z)$. Theorem 9.6 then yields a "specialisation" functor

$$i^! : T(K, S) \to T(L, Z).$$

Since i^* is not full, it does not *a priori* induce a functor $\mathcal{M}_{\mathrm{num}}(S) \to \mathcal{M}_{\mathrm{num}}(Z)$. Using a Weil cohomology H verifying the smooth and proper base change theorem (e.g. l-adic cohomology for a prime l invertible on S) and using the monoidal section theorem, we can construct as in [1, Th. 11] a "specialisation" $\otimes$-functor $\mathcal{M}_{\mathrm{num}}(S) \to \mathcal{M}_{\mathrm{num}}(Z)$, depending *a priori* on H; we leave details to the interested reader.

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