

FIBERWISE KÄHLER-RICCI FLOWS ON FAMILIES OF
BOUNDED STRONGLY PSEUDOCONVEX DOMAINS

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ABSTRACT. Let $\pi : \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}$ be the projection map onto the second factor and let D be a domain in \mathbb{C}^{n+1} such that for $y \in \pi(D)$, every fiber $D_y := D \cap \pi^{-1}(y)$ is a smoothly bounded strongly pseudoconvex domain in \mathbb{C}^n and is diffeomorphic to each other. By Chau's theorem, the Kähler-Ricci flow has a long time solution $\omega_y(t)$ on each fiber D_y . This family of flows induces a smooth real $(1,1)$ -form $\omega(t)$ on the total space D whose restriction to the fiber D_y satisfies $\omega(t)|_{D_y} = \omega_y(t)$. In this paper, we prove that $\omega(t)$ is positive for all $t > 0$ in D if $\omega(0)$ is positive. As a corollary, we also prove that the fiberwise Kähler-Einstein metric is positive semi-definite on D if D is pseudoconvex in \mathbb{C}^{n+1} .

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1 INTRODUCTION

Let D be a domain in \mathbb{C}^{n+1} and $S := \pi(D) \subset \mathbb{C}$, where $\pi : \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}$ is the standard projection map onto the second factor. We say that D is a *holomorphic family of bounded strongly pseudoconvex domains* if it satisfies the following:

- (i) $\pi^{-1}(S) \cap \partial D$ is smooth and $\pi|_{\partial D} : \pi^{-1}(S) \cap \partial D \rightarrow S$ is a submersion.
- (ii) For $y \in S$, all fibers $D_y := \pi^{-1}(y) \cap D$ are smoothly bounded strongly pseudoconvex domains in \mathbb{C}^n .

In this case, there exists a defining function r of D such that $\omega := i\partial\bar{\partial}(-\log(-r))$ is a d -closed smooth real $(1, 1)$ -form on D whose restriction to the fibers $\omega|_{D_y}$ is a complete Kähler metric with bounded geometry (see Section 3.1).

Now we consider the following (normalized) Kähler-Ricci flow on each fiber D_y :

$$\begin{aligned}\frac{\partial}{\partial t}\omega_y(t) &= -\text{Ric}(\omega_y(t)) - (n+1)\omega_y(t), \\ \omega_y(0) &= \omega|_{D_y}.\end{aligned}\tag{1.1}$$

This flow has a long time solution $\omega_y(t)$ which converges to the unique complete Kähler-Einstein metric ω_y^{KE} with Ricci curvature $-(n+1)$ as $t \rightarrow \infty$ by Chau's theorem in [6]. In fact, $\omega_y(t)$ is given by the solution of a parabolic Monge-Ampère equation. As a consequence of the implicit function theorem for the Monge-Ampère operator, we obtain smooth real $(1, 1)$ -forms $\omega(t)$ on the total space D whose restriction to the fibers D_y satisfies $\omega(t)|_{D_y} = \omega_y(t)$ (see Proposition 3.3). Moreover, $\omega(t)$ evolves by the following equation, called the *fiberwise Kähler-Ricci flow*:

$$\begin{aligned}\frac{\partial}{\partial t}\omega(t) &= \Theta_{\omega(t)} - (n+1)\omega(t), \\ \omega(0) &= \omega,\end{aligned}\tag{1.2}$$

where $\Theta_{\omega(t)}$ is the relative curvature form of $\omega(t)$ (Theorem 3.5). This flow was first introduced by Berman in [2] for the case of compact fibrations with the name “*relative Kähler-Ricci flow*”. The main theorem of this paper is a non-compact version of Berman's theorem (cf. Corollary 4.9 in [2]).

THEOREM 1.1. *If ω is semi-positive in D and strictly positive at least one point on each fiber D_y , then $\omega(t)$ is positive in D for all $t > 0$.*

On the other hand, the family of Kähler-Einstein metrics ω_y^{KE} on D_y also induces a d -closed smooth real $(1, 1)$ -form ρ on the total space D . The form ρ is called the *fiberwise Kähler-Einstein metric* since it satisfies $\rho|_{D_y} = \omega_y^{KE}$ (cf. [9, 18]). Using the fact that $\omega_y(t)$ converges to ω_y^{KE} on each fiber D_y , one can show that the solution of the fiberwise Kähler-Ricci flow $\omega(t)$ smoothly converges to the fiberwise Kähler-Einstein metric ρ on the total space D (Theorem 3.8). Since the existence of initial form ω satisfying the hypothesis in Theorem 1.1 is guaranteed provided that D is pseudoconvex in \mathbb{C}^{n+1} (Proposition 3.1), we have the following

COROLLARY 1.2. *The fiberwise Kähler-Einstein metric ρ is semi-positive if D is pseudoconvex.*

Corollary 1.2 has already proved by the first named author in [9, 10]. In fact, he further proved that ρ is strictly positive if D is strongly pseudoconvex. In his papers [9, 10], he analyzed the boundary behavior of the variation of Kähler-Einstein metrics via the boundary behavior of Kähler-Einstein metric due to

Cheng and Yau. It is remarkable to note that the analysis for Corollary 1.2 in this paper is lighter than the one in [9, 10].

A study on the positive variation of Kähler-Einstein metrics is first developed by Schumacher [18]. More precisely, he has proved that the variation of Kähler-Einstein metrics on a family of canonically polarized compact Kähler manifolds is positive-definite on the total space. In [18], he showed that the geodesic curvature of the fiberwise Kähler-Einstein metric, which measures the positivity along the horizontal direction, satisfies a certain elliptic partial differential equation. A direct application of maximum principle says that the geodesic curvature is positive, which is equivalent to the positivity of the fiberwise Kähler-Einstein metric.

Later, Berman [2] proved the parabolic version of Schumacher's result in the same setting. On a canonically polarized compact Kähler manifold, the Kähler-Ricci flow has a long time solution which converges to the unique Kähler-Einstein metric by Cao's theorem in [5]. Using this result, Berman constructed the relative Kähler-Ricci flow on a family of canonically polarized compact Kähler manifolds. In [2], he proved the geodesic curvature of the relative Kähler-Ricci flow satisfies a parabolic version of Schumacher's elliptic PDE. A parabolic maximum principle implies that the positivity of the relative Kähler-Ricci flow is preserved. In particular, Berman's result implies the Schumacher's one since the relative Kähler-Ricci flow converges to the fiberwise Kähler-Einstein metric.

In this paper, we shall generalize Berman's results to a family of bounded strongly pseudoconvex domains, which is one of the most important examples for non-compact complete Kähler manifolds. In this case, the Kähler-Ricci flow has a long time solution which converges to the unique Kähler-Einstein metric due to Chau [6]. Moreover, the geodesic curvature of the fiberwise Kähler-Ricci flow still satisfies Berman's parabolic PDE. The difference comes from applying the parabolic maximum principle. In the previous case, since every fiber is compact, we can apply the standard weak and strong parabolic maximum principle. However, if the manifold is non-compact, the weak maximum principle does not hold in general.

To resolve this problem, we will use Ni's theorem in [15], which says that if the function does not blow up too fast at the point at infinity, then the weak maximum principle holds. To apply this, we have to investigate the boundary behavior of the geodesic curvature of the fiberwise Kähler-Ricci flow. In fact, we will show that it has a polynomial growth near the boundary with respect to the defining function.

Throughout this paper, $z = (z^1, \dots, z^n)$ will be a holomorphic local coordinate system for the fibers $D_y \subset \mathbb{C}^n$. For the base space $S \subset \mathbb{C}$, we will always use the standard Euclidean coordinate, denoted by s . We will use small Greek letters, $\alpha, \beta, \dots = 1, \dots, n$ for indices on z unless otherwise specified. For a properly differentiable function f on the total space $D \subset \mathbb{C}^n \times \mathbb{C}$, we denote by

$$f_\alpha = \frac{\partial f}{\partial z^\alpha}, \quad f_{\bar{\beta}} = \frac{\partial f}{\partial z^{\bar{\beta}}}, \quad \text{and} \quad f_s = \frac{\partial f}{\partial s}, \quad f_{\bar{s}} = \frac{\partial f}{\partial \bar{s}}, \quad (1.3)$$

where $z^{\bar{\beta}}$ mean $\overline{z^{\beta}}$. We will always use the Einstein convention and the same letter “ C ” to denote a generic constant, which may change from one line to another, but it is independent of the pertinent parameters involved.

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2 PRELIMINARIES

A compact Kähler manifold with negative first Chern class admits an unique Kähler-Einstein metric with negative constant Ricci curvature by Aubin [1] and Yau [21] using the continuity method. Later, Cao [5] gave another proof of the existence of Kähler-Einstein metric using the Kähler-Ricci flow.

On the other hand, these results can be generalized to non-compact complete Kähler manifolds which admit properties of bounded geometry due to Cheng-Yau and Chau. In this section, we recaptulate their results (for the details, see [6, 8]).

2.1 ELLIPTIC MONGE-AMPÈRE EQUATION AND KÄHLER-EINSTEIN METRIC

The existence of Kähler-Einstein metric comes from the solvability of the complex Monge-Ampère equation. For that purpose, Cheng and Yau introduced the notion of the bounded geometry of non-compact complete Kähler manifold.

DEFINITION 1 (Bounded geometry). *Let (M, ω) be a complete Kähler manifold of dimension n . We say that (M, ω) has bounded geometry of order k if for each $p \in M$ there exists a holomorphic chart (U_p, ξ_p) centered at p satisfying following conditions:*

- (i) *There exist constant $r > 0$, independent of p satisfying*

$$\mathbb{B}_r(0) \subset V_p := \xi_p(U_p) \subset \mathbb{C}^n,$$

where $\mathbb{B}_r(0)$ denotes the ball of radius r centered at 0 in \mathbb{C}^n .

- (ii) *There exists a constant $C > 0$ independent of p satisfying*

$$\frac{1}{C} (\delta_{\alpha\bar{\beta}}) \leq (g_{\alpha\bar{\beta}}) \leq C (\delta_{\alpha\bar{\beta}}),$$

where $\omega = ig_{\alpha\bar{\beta}}d\xi^\alpha \wedge d\bar{\xi}^\beta$ for the coordinates $\xi_p = (\xi^1, \dots, \xi^n)$.

(iii) For any $l \leq k$, there exist constants $C_l > 0$ independent of p satisfying

$$\|g_{\alpha\bar{\beta}}\|_{C^l(V_p)} \leq C_l.$$

Suppose that a complete Kähler manifold (M, ω) has bounded geometry of order k . Let $\{(U_p, \xi_p)\}$ be a family of holomorphic charts covering M and satisfying the conditions in Definition 1. For any functions $u \in C^k(M)$, we define the Hölder norm by

$$\|u\|_{k+\epsilon} := \sup_{p \in M} \{ \|u \circ \xi_p^{-1}\|_{C^{k+\epsilon}(V_p)} \},$$

where $\|\cdot\|_{C^{k+\epsilon}(V_p)}$ is the standard elliptic Hölder norm on $V_p := \xi_p(U_p) \subset \mathbb{C}^n$. Define the *little Hölder space* $\tilde{C}^{k+\epsilon}(M)$ by the closure of the subspace $C^\infty(M)$ with respect to the norm $\|\cdot\|_{k+\epsilon}$. This is a Banach space. Now we can state the following theorem due to Cheng and Yau.

THEOREM 2.1 (Theorem 4.4 in [8]). *Suppose (M, ω) is a complete Kähler manifold with bounded geometry of order $k \geq 5$. Then, for any $K > 0$ and $F \in \tilde{C}^{k-2+\epsilon}(M)$, there exists a unique $\psi \in \tilde{C}^{k+\epsilon}(M)$ satisfying the following conditions:*

$$(\omega + i\partial\bar{\partial}\psi)^n = e^{K\psi+F}\omega^n, \tag{2.1}$$

$$\frac{1}{C}\omega \leq \omega + i\partial\bar{\partial}\psi \leq C\omega. \tag{2.2}$$

Moreover, if all the data are analytic, the solution is also analytic.

REMARK 2.2. *The equation (2.1) is called the elliptic complex Monge-Ampère equation. The inequality (2.2) implies that $(M, \omega + i\partial\bar{\partial}\psi)$ also has bounded geometry of order k (see Proposition 1.4 in [8]).*

We further assume that the Kähler form ω satisfies the following condition:

$$\text{Ric}(\omega) + K\omega = i\partial\bar{\partial}F, \tag{2.3}$$

for some constant $K \geq 0$ and function $F \in \tilde{C}^{k-2+\epsilon}(M)$. Consider the Kähler metric $\omega_{KE} := \omega + i\partial\bar{\partial}\psi$, where ψ is the solution of the Monge-Ampère equation (2.1) in Theorem 2.1. Then we have the following

THEOREM 2.3 (Cheng-Yau [8]). *The Kähler metric $\omega_{KE} := \omega + i\partial\bar{\partial}\psi$ is the unique complete Kähler-Einstein metric of M satisfying $\text{Ric}(\omega_{KE}) = -K\omega_{KE}$.*

2.2 PARABOLIC MONGE-AMPÈRE EQUATION AND KÄHLER-RICCI FLOW

There is an alternative proof of Theorem 2.3 using Hamilton’s Ricci flow due to Chau [6]. This flow is called the *Kähler-Ricci flow* since it preserves the Kähler-ness along the flow.

One of the advantages of the Kähler-Ricci flow approach is that one can prove the existence of Kähler-Einstein metric under weaker assumptions. More precisely, Chau proved the following parabolic version of Theorem 2.3.

THEOREM 2.4 (Chau [6]). *Let (M, ω) be a complete Kähler manifold with bounded curvature. Suppose that there exist a smooth bounded function F and a constant $K > 0$ satisfying*

$$\operatorname{Ric}(\omega) + K\omega = i\partial\bar{\partial}F.$$

Then there exist a time family of Kähler metrics $\omega(t)$ for all $t > 0$ satisfying

$$\begin{aligned} \frac{\partial}{\partial t}\omega(t) &= -\operatorname{Ric}(\omega(t)) - K\omega(t), \\ \omega(0) &= \omega. \end{aligned} \tag{2.4}$$

Moreover, $\omega(t)$ converges to the unique complete Kähler-Einstein metric ω_{KE} .

The equation (2.4) is called the (normalized) *Kähler-Ricci flow*. Note that here, we assumed neither the conditions of bounded geometry for ω nor $F \in \tilde{C}^{k-2+\varepsilon}(M)$. But one can always find such metrics using the short time existence of Kähler-Ricci flow due to Shi [19] so that the Kähler-Einstein metric exists by Theorem 2.3.

To prove the long time existence of Kähler-Ricci flow, Chau considered the functions $\varphi \in C^\infty(M \times [0, \infty))$ such that $\omega(t) := \omega + i\partial\bar{\partial}\varphi(t)$ satisfies the equation (2.4). Then the problem is reduced to the solvability of the following *parabolic complex Monge-Ampère equation*.

THEOREM 2.5 (Theorem 1.1 in [6]). *There exists a solution $\varphi \in \tilde{C}^{k+\varepsilon, \frac{k+\varepsilon}{2}}(M \times [0, \infty))$ of the following equation:*

$$\begin{cases} \frac{\partial}{\partial t}\varphi = \log \frac{(\omega + i\partial\bar{\partial}\varphi)^n}{\omega^n} - K\varphi - F, \\ \varphi|_{t=0} = 0. \end{cases} \tag{2.5}$$

Moreover, $\varphi(t)$ converges to the function ψ in $\tilde{C}^{k+\varepsilon}(M)$ as $t \rightarrow \infty$, where ψ is the unique solution of the equation (2.1) in Theorem 2.1.

Here, $\tilde{C}^{k+\varepsilon, \frac{k+\varepsilon}{2}}(M \times [0, T))$ is the *little (parabolic) Hölder space*, which is the Banach completion of the subspace $C^\infty(M \times [0, T))$ with respect to the norm

$$\|u\|_{k+\varepsilon, \frac{k+\varepsilon}{2}} := \sup_{p \in M} \left\{ \|u \circ \xi_p^{-1}\|_{C^{k+\varepsilon, \frac{k+\varepsilon}{2}}(V_p \times [0, T))} \right\},$$

where $\|\cdot\|_{C^{k+\varepsilon, \frac{k+\varepsilon}{2}}(V_p \times [0, T))}$ is the standard parabolic Hölder norm on $V_p \times [0, T)$.

3 FIBERWISE KÄHLER-RICCI FLOW

In this section, we discuss the variation of the Kähler-Ricci flows on a holomorphic family of bounded strongly pseudoconvex domains, which gives the fiberwise Kähler-Ricci flow. Moreover, we will prove that the fiberwise Kähler-Ricci flow converges to the fiberwise Kähler-Einstein metric.

3.1 CONSTRUCTION OF THE REFERENCE FORM

First recall the setting in Introduction : Let D be a domain in \mathbb{C}^{n+1} and $S := \pi(D) \subset \mathbb{C}$. Suppose that D is a holomorphic family of bounded strongly pseudoconvex domains, i.e., it satisfies the following:

- (i) $\pi^{-1}(S) \cap \partial D$ is smooth and $\pi|_{\partial D} : \pi^{-1}(S) \cap \partial D \rightarrow S$ is a submersion.
- (ii) For $y \in S$, all fibers $D_y := \pi^{-1}(y) \cap D$ are smoothly bounded strongly pseudoconvex domains in \mathbb{C}^n .

Note that the Condition (i) implies that all fibers are diffeomorphic by Ehresmann’s fibration theorem (cf. [17]). Together with the Condition (ii), there exists a defining function r of D such that $r|_{\overline{D}_y}$ is a strictly plurisubharmonic function on \overline{D}_y . Define a d -closed smooth $(1, 1)$ -form on the total space D by

$$\omega := i\partial\bar{\partial}(-\log(-r)),$$

where ∂ and $\bar{\partial}$ are the operators of the total space \mathbb{C}^{n+1} . Then one can check that (D_y, ω_y) is a complete Kähler manifold with bounded geometry of infinite order (for the details, see [8]). However, there is no information about the positivity of the reference form ω along the base direction. The following theorem says that positivity of ω on D is guaranteed by the pseudoconvexity of D in \mathbb{C}^{n+1} .

PROPOSITION 3.1. *If D is pseudoconvex on \mathbb{C}^{n+1} , then there exists a defining function r of D such that $\omega := i\partial\bar{\partial}(-\log(-r))$ satisfies the following conditions*

- $\omega_y := \omega|_{D_y}$ is complete Kähler form on each fiber D_y .
- $\omega \geq 0$ on D , and ω is strictly positive at least one point on each fiber D_y .

Proof. Note that D is a holomorphic family of bounded strongly pseudoconvex domains, which is pseudoconvex in \mathbb{C}^{n+1} . Then there exists a smooth plurisubharmonic defining function \tilde{r} of D such that for $y \in S$, $\tilde{r}|_{\overline{D}_y}$ is a strictly plurisubharmonic function on $\overline{D}_y \cap U$, where U is a neighborhood of $\pi^{-1}(S) \cap \partial D$. Let ϵ_1, ϵ_2 be negative constants satisfying $\{x \in U : \epsilon_1 < \tilde{r}(x) < \epsilon_2 < 0\} \subset\subset U \cap D$. Choose $\chi \in C^\infty(\mathbb{R})$ such that χ is negative constant for $t \leq \epsilon_1$, $\chi(t) = t$ for $t \geq \epsilon_2$, and $\chi', \chi'' > 0$ for $\epsilon_1 < t < \epsilon_2$. Then for a suitable cutoff function λ , $r := \chi \circ \tilde{r} + \lambda|z|^2$ is a smooth defining function of D satisfying the conditions in the statement of the proposition. □

Since $\omega_y > 0$ for each fiber, the relative curvature form of ω can be defined by

$$\Theta_\omega := i\partial\bar{\partial}\log(\omega^n \wedge dV_s),$$

where $dV_s := ids \wedge d\bar{s}$ is the volume form on the base space \mathbb{C} (cf. [11]). In fact, this is the curvature form of the relative canonical line bundle. The following proposition will be used later to prove Theorem 3.5 and Theorem 4.2.

PROPOSITION 3.2. *There exists a bounded smooth function F on D satisfying*

$$-\Theta_\omega + (n + 1)\omega = i\partial\bar{\partial}F. \tag{3.1}$$

Moreover, F is smoothly extended up to ∂D .

Proof. Let (z^1, \dots, z^n, s) be the Euclidean coordinate for \mathbb{C}^{n+1} . Then we have

$$\Theta_\omega = i\partial\bar{\partial} \log \det(g_{\alpha\bar{\beta}}),$$

where $g := -\log(-r)$ is a function on D . The computations in [8] show that

$$\det(g_{\alpha\bar{\beta}}) = \left(\frac{1}{-r}\right)^{n+1} \det(r_{\alpha\bar{\beta}}) \left(-r + |\partial r|^2\right),$$

where $|\partial r|^2 := r^{\alpha\bar{\beta}}r_\alpha r_{\bar{\beta}}$ with $r^{\alpha\bar{\beta}} = (r_{\alpha\bar{\beta}})^{-1}$ as matrices. It follows that

$$\Theta_\omega = i\partial\bar{\partial} \log \det(g_{\alpha\bar{\beta}}) = (n + 1)\omega + i\partial\bar{\partial} \log \left(\det(r_{\alpha\bar{\beta}}) \left(-r + |\partial r|^2\right)\right).$$

If we define the function $F : D \rightarrow \mathbb{R}$ by

$$F := -\log \left(\det(r_{\alpha\bar{\beta}}) \left(-r + |\partial r|^2\right)\right),$$

then F is a bounded smooth function satisfying the equation (3.1). Since r is smooth on \bar{D} and $|\partial r| \neq 0$ on ∂D , the second assertion follows. \square

3.2 FIBERWISE KÄHLER-RICCI FLOW

Note that $\Theta_\omega|_{D_y} = -\text{Ric}(\omega_y)$. Restricting the equation (3.1) to the fiber D_y , we have

$$\text{Ric}(\omega_y) + (n + 1)\omega_y = i\partial\bar{\partial}F_y,$$

where $F_y := F|_{D_y}$. Therefore, Theorem 2.5 implies that for $y \in S$, there exists a solution φ_y on $D_y \times [0, \infty)$ of the parabolic Monge-Ampère equation:

$$\begin{aligned} \frac{\partial}{\partial t} \varphi_y &= \log \frac{(\omega_y + i\partial\bar{\partial}\varphi_y)^n}{\omega_y^n} - (n + 1)\varphi_y - F_y, \\ \varphi_y|_{t=0} &= 0 \end{aligned} \tag{3.2}$$

Hence $\omega_y(t) := \omega_y + i\partial\bar{\partial}\varphi_y(t)$ is the solution of the (normalized) *Kähler-Ricci flow*:

$$\begin{aligned} \frac{\partial}{\partial t} \omega_y(t) &= -\text{Ric}(\omega_y(t)) - (n + 1)\omega_y(t), \\ \omega_y(0) &= \omega_y. \end{aligned} \tag{3.3}$$

The following proposition yields that the solution $\varphi_y(t)$ of the equations (3.2) vary smoothly along the base direction s .

PROPOSITION 3.3. For $t \in [0, \infty)$, the function $\varphi(t)$ given by

$$\varphi(x; t) := \varphi_y(x; t)$$

where $y = \pi(x)$ and $x \in D$, is smooth on the total space D .

Proof. For a fixed point $y_0 \in S$, denote by $\Omega := D_{y_0}$. Ehresmann’s fibration theorem implies that there exists a fiber-preserving diffeomorphism $\Phi : D \rightarrow \Omega \times S$ which is smoothly extended up to the boundary $\pi^{-1}(S) \cap \partial D$. Hence for $y \in S$, all Banach spaces $\tilde{C}^{k+\epsilon, \frac{k+\epsilon}{2}}(D_y \times [0, T])$ can be identified with the space $\tilde{C}^{k+\epsilon, \frac{k+\epsilon}{2}}(\Omega \times [0, T])$.

Now we define the following *parabolic Monge-Ampère operator*

$$\mathcal{M} : U \times \tilde{C}^{k+\epsilon, \frac{k+\epsilon}{2}}(\Omega \times [0, T]) \rightarrow \tilde{C}^{k-2+\epsilon, \frac{k-2+\epsilon}{2}}(\Omega \times [0, T])$$

by

$$\mathcal{M}(y, \phi) = \frac{\partial}{\partial t} \phi - \log \frac{(\omega_y + i\partial\bar{\partial}\phi_y)^n}{\omega_y^n} + (n+1)\phi + F_y,$$

where $i\partial\bar{\partial}$ is the operator of D_y pulled back under the chosen diffeomorphism Φ .

By Theorem 2.5, there exists $\varphi_{y_0} \in \tilde{C}^{k+\epsilon, \frac{k+\epsilon}{2}}(\Omega \times [0, T])$ such that

$$\mathcal{M}(y_0, \varphi_{y_0}) = 0.$$

Then, the partial Fréchet derivative of \mathcal{M} at the point (y_0, φ_{y_0}) is an operator

$$D_2\mathcal{M}(y_0, \varphi_{y_0}) : \tilde{C}^{k+\epsilon, \frac{k+\epsilon}{2}}(\Omega \times [0, T]) \rightarrow \tilde{C}^{k-2+\epsilon, \frac{k-2+\epsilon}{2}}(\Omega \times [0, T])$$

which is defined by for any $\phi \in \tilde{C}^{k+\epsilon, \frac{k+\epsilon}{2}}(\Omega \times [0, T])$,

$$D_2\mathcal{M}(y_0, \varphi_{y_0})(\phi) = \left(\frac{\partial}{\partial t} - \Delta_t + (n+1) \cdot id\right)\phi,$$

where Δ_t is the Laplacian with respect to $\omega_{y_0}(t) = \omega_{y_0} + i\partial\bar{\partial}\varphi_{y_0}(t)$.

Using a version of maximum principle, we can show that $D_2\mathcal{M}(y_0, \varphi_{y_0})$ is a Banach space isomorphism between $\tilde{C}^{k+\epsilon, \frac{k+\epsilon}{2}}(\Omega \times [0, T])$ and $\tilde{C}^{k-2+\epsilon, \frac{k-2+\epsilon}{2}}(\Omega \times [0, T])$ (for details, see the proof of Claim 1 of Lemma 2.2 in [7]). Hence the Implicit Function Theorem implies that there exists a Fréchet differentiable function $\mu : U \rightarrow \tilde{C}^{k+\epsilon, \frac{k+\epsilon}{2}}(\Omega \times [0, T])$ such that

$$\mathcal{M}(y, \mu(y)) = 0.$$

The uniqueness of the solution implies $\mu(y) = \varphi_y$ so that $\varphi_y \in \tilde{C}^{k+\epsilon, \frac{k+\epsilon}{2}}(\Omega \times [0, T])$. Since the Monge-Ampère operator \mathcal{M} is smooth, the implicit function theorem implies that φ_y vary smoothly along the base direction s . \square

REMARK 3.4. The proof of Proposition 3.3 implies that for any $l_1, l_2 = 0, 1, 2, \dots$,

$$\left(\frac{\partial}{\partial s}\right)^{l_1} \left(\frac{\partial}{\partial \bar{s}}\right)^{l_2} \varphi \Big|_{D_y} \in \tilde{C}^{k+\epsilon, \frac{k+\epsilon}{2}}(D_y \times [0, \infty)).$$

By Proposition 3.3, we can define d -closed smooth real $(1, 1)$ -forms $\omega(t)$ on the total space D by

$$\omega(t) := \omega + i\partial\bar{\partial}\varphi(t). \quad (3.4)$$

Since $\omega_y(t) := \omega(t)|_{D_y}$ is Kähler on each fiber D_y , one can consider relative curvature form $\Theta_{\omega(t)}$ of $\omega(t)$ on D , given by

$$\Theta_{\omega(t)} = i\partial\bar{\partial}\log(\omega(t)^n \wedge dV_s).$$

THEOREM 3.5. *The form $\omega(t)$ on D satisfies the following equation:*

$$\begin{aligned} \frac{\partial}{\partial t}\omega(t) &= \Theta_{\omega(t)} - (n+1)\omega(t), \\ \omega(0) &= \omega. \end{aligned} \quad (3.5)$$

Proof. It follows from (3.2) that

$$\frac{\partial}{\partial t}\varphi = \log \frac{(\omega + i\partial\bar{\partial}\varphi)^n \wedge dV_s}{\omega^n \wedge dV_s} - (n+1)\varphi - F.$$

Taking $i\partial\bar{\partial}$, we have

$$i\partial\bar{\partial}\left(\frac{\partial}{\partial t}\varphi\right) = \Theta_{\omega(t)} - \Theta_{\omega} - (n+1)i\partial\bar{\partial}\varphi - i\partial\bar{\partial}F.$$

Since ω does not depend on t , (3.1) implies that

$$\frac{\partial}{\partial t}(\omega + i\partial\bar{\partial}\varphi) = \Theta_{\omega(t)} - (n+1)\omega.$$

This completes the proof. \square

REMARK 3.6. *We will call the equation (3.5) the fiberwise Kähler-Ricci flow on D , since the restriction of it to the fiber D_y is equal to the equation (3.3). This flow was first introduced by Berman in [2] with the name “relative Kähler-Ricci flow”.*

3.3 FIBERWISE KÄHLER-EINSTEIN METRIC

On the other hand, Theorem 2.5 implies that for all fibers D_y , the solution $\varphi_y(t)$ of the parabolic Monge-Ampère equation (3.2) converges to the solution ψ_y of the elliptic Monge-Ampère equation:

$$\begin{aligned} (\omega_y + i\partial\bar{\partial}\psi_y)^n &= e^{(n+1)\psi_y + F_y}\omega_y^n, \\ \frac{1}{C}\omega_y &\leq \omega_y + i\partial\bar{\partial}\psi_y \leq C\omega_y. \end{aligned} \quad (3.6)$$

By the uniqueness of the Kähler-Einstein metric, we have

$$\omega_y + i\partial\bar{\partial}\psi_y = \omega_y^{KE},$$

where ω_y^{KE} is the unique Kähler-Einstein metric with Ricci curvature $-(n+1)$. As in Proposition 3.3, the implicit function theorem for the elliptic Monge-Ampère operator implies the following

PROPOSITION 3.7 (cf. Section 3 in [9]). *The function $\psi : D \rightarrow \mathbb{R}$, defined by*

$$\psi(x) := \psi_y(x)$$

where $y = \pi(x)$, is smooth on the total space D .

Define a d -closed smooth $(1, 1)$ -form ρ on the total space D by

$$\rho := \omega + i\partial\bar{\partial}\psi.$$

The form ρ is called the *fiberwise Kähler-Einstein metric*, since $\rho|_{D_y} = \omega_y^{KE}$.

THEOREM 3.8. *The solution of fiberwise Kähler-Ricci flow $\omega(t)$ locally uniformly converges to the fiberwise Kähler metric ρ on D as $t \rightarrow \infty$. More precisely, we have that $\varphi(t) \rightarrow \psi$ in $C_{loc}^\infty(D)$.*

Proof. It is enough to show that $\varphi(t)$ smoothly converges to ψ on any compact subset of D . More precisely, we will show that for each point $x \in D$, there exists a neighborhood U of x in D such that

$$\|\varphi(t) - \psi\|_{C^k(U)} \rightarrow 0$$

as $t \rightarrow \infty$, for all $k \geq 0$. Before going to the proof, note that for $y \in S$, we already know that as $t \rightarrow \infty$,

$$\|\varphi(t)|_{D_y} - \psi|_{D_y}\|_{\bar{C}^{k,\epsilon}(D_y)} \rightarrow 0. \tag{3.7}$$

First consider the C^0 -convergence. Differentiating (3.2) with respect to t , we get

$$\begin{aligned} \frac{\partial}{\partial t}\dot{\varphi}_y &= \Delta_t\dot{\varphi}_y - (n+1)\dot{\varphi}_y, \\ \dot{\varphi}_y|_{t=0} &= F_y. \end{aligned} \tag{3.8}$$

It follows that

$$\frac{\partial}{\partial t}(e^{(n+1)t}\dot{\varphi}_y) = \Delta_t(e^{(n+1)t}\dot{\varphi}_y).$$

A maximum principle implies that

$$\left|e^{(n+1)t}\dot{\varphi}_y\right| \leq \sup_{D_y}|F_y| \leq C$$

for some uniform constant $C > 0$, independent of y . For $0 < t' < t''$, we have

$$\begin{aligned} |\varphi_y(x, t') - \varphi_y(x, t'')| &\leq \left|\int_{t'}^{t''} \dot{\varphi}_y(x, u) du\right| \leq \int_{t'}^{t''} |\dot{\varphi}_y(x, u)| du \\ &\leq \int_{t'}^{t''} C e^{-(n+1)u} du \leq C \left(e^{-(n+1)t'} - e^{-(n+1)t''}\right). \end{aligned}$$

By (3.7), this implies that

$$\|\varphi(t) - \psi\|_{C^0(D)} \leq Ce^{-(n+1)t}.$$

Now we consider the C^k -convergence for any fixed $k \in \mathbb{N}$. For each $l_1, l_2 \in \mathbb{N}$ with $l_1 + l_2 \leq k$, the proof of Proposition 3.3 implies that

$$U \ni y \rightarrow D_s^{l_1, l_2} \varphi_y(t) := \left(\frac{\partial}{\partial s}\right)^{l_1} \left(\frac{\partial}{\partial \bar{s}}\right)^{l_2} \varphi(t) \Big|_{D_y} \in \tilde{C}^{k+\epsilon, \frac{k+\epsilon}{2}}(D_y \times [0, \infty)).$$

is smooth where U is a neighborhood of y . Hence there exists a uniform constant C which depends only on l_1, l_2, k such that

$$\sup_{y \in U} \sum_{l_1+l_2 \leq k} \|D_s^{l_1, l_2} \varphi_y(t)\|_{\tilde{C}^{k+\epsilon, \frac{k+\epsilon}{2}}(D_y \times [0, \infty))} < C.$$

This implies that there exists a neighborhood V of x in D and an uniform constant C which does not depend on t such that

$$\|\varphi(t)\|_{C^k(V)} < C$$

where $C^k(V)$ -norm means the usual C^k -norm on $V \subset \mathbb{C}^{n+1}$. Therefore, the proof is completed by the Arzela-Ascoli theorem and the uniqueness of limit (for the details, see [5, 4]). \square

Theorem 3.5 and Theorem 3.8 imply the following

COROLLARY 3.9 (cf. Remark 3.4 in [12]). *The fiberwise Kähler-Einstein metric ρ satisfies the equation*

$$\Theta_\rho = (n+1)\rho, \tag{3.9}$$

where Θ_ρ is the relative curvature form of ρ .

4 GEODESIC CURVATURE OF THE FIBERWISE KÄHLER-RICCI FLOW

In this section, we introduce the horizontal lift, which is developed by Siu and Schumacher (cf. [20, 18]), and the geodesic curvature which measures the positivity of a fiberwise Kähler form. We also discuss Berman's parabolic PDE which the geodesic curvature of the fiberwise Kähler-Ricci flow satisfies.

4.1 HORIZONTAL LIFT AND GEODESIC CURVATURE

Let D be a domain in \mathbb{C}^{n+1} such that every fiber D_y is a domain in \mathbb{C}^n for $y \in S := \pi(D)$. Denote by $v := \frac{\partial}{\partial s} \in T'_y S$ the coordinate vector field in the base.

DEFINITION 2. *Let τ be a d -closed smooth real $(1, 1)$ -form on D whose restriction to the fibers $\tau|_{D_y}$ is positive definite.*

(1) A vector field v_τ of type $(1, 0)$ is called the horizontal lift along D_y of v with respect to τ if v_τ satisfies the following:

- (i) $\langle v_\tau, w \rangle_\tau = 0$ for all $w \in T'D_y$,
- (ii) $d\pi(v_\tau) = v$.

(2) The geodesic curvature $c(\tau)(v)$ of τ along v is defined by the norm of v_τ with respect to the sesquilinear form $\langle \cdot, \cdot \rangle_\tau$ induced by τ , namely,

$$c(\tau) := c(\tau)(v) = \langle v_\tau, v_\tau \rangle_\tau.$$

REMARK 4.1. We have the following remarks.

(1) Under a local coordinate system (z^1, \dots, z^n, s) , τ can be written as

$$\tau = i \left(\tau_{s\bar{s}} ds \wedge d\bar{s} + \tau_{\alpha\bar{s}} dz^\alpha \wedge d\bar{s} + \tau_{s\bar{\beta}} ds \wedge dz^{\bar{\beta}} + \tau_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}} \right).$$

Then the horizontal lift v_τ and the geodesic curvature $c(\tau)$ are given by

$$v_\tau = \frac{\partial}{\partial s} - \tau_{s\bar{\beta}} \tau^{\bar{\beta}\alpha} \frac{\partial}{\partial z^\alpha} \quad \text{and} \quad c(\tau) = \tau_{s\bar{s}} - \tau_{s\bar{\beta}} \tau^{\bar{\beta}\alpha} \tau_{\alpha\bar{s}},$$

where $(\tau^{\bar{\beta}\alpha})$ is the inverse matrix of $(\tau_{\alpha\bar{\beta}})$.

(2) The following identity is well-known and important (cf. [18]):

$$\frac{\tau^{n+1}}{(n+1)!} = c(\tau) \cdot \frac{\tau^n}{n!} \wedge ids \wedge d\bar{s}.$$

Since $\tau|_{D_y} > 0$, this implies that $c(\tau) \geq 0$ if and only if τ is a semi-positive real $(1, 1)$ -form on D . Furthermore, $c(\tau) > 0$ if and only if τ is positive.

4.2 BERMAN'S PARABOLIC PDE

Let D be a holomorphic family of bounded strongly pseudoconvex domains. Then the geodesic curvature $c(\omega(t))$ satisfies a certain parabolic PDE, which was first computed by Berman for a family of canonically polarized compact Kähler manifolds. The following theorem is essentially the same with Berman's one, but we will give a precise proof for the reader's convenience.

THEOREM 4.2 (cf. Theorem 4.7 in [2]). For each fiber D_y , $c(\omega(t))|_{D_y}$ evolves by

$$\left(\frac{\partial}{\partial t} - \Delta_t \right) c(\omega(t)) + (n+1)c(\omega(t)) = \|\bar{\partial}v_{\omega(t)}\|^2, \tag{4.1}$$

where Δ_t is the Laplace-Beltrami operator of the Kähler metric $\omega_y(t) := \omega(t)|_{D_y}$.

Proof. Note that $\omega(t) = i\partial\bar{\partial}g(t)$ on D , where $g(t) := -\log(-r) + \varphi(t)$. During this proof, for simplicity, we will omit t for the function $g(t) =: g$. Then $\omega(t)$ can be written as follows:

$$\omega(t) = i \left(g_{s\bar{s}} ds \wedge d\bar{s} + g_{\alpha\bar{s}} dz^\alpha \wedge d\bar{s} + g_{s\bar{\beta}} ds \wedge dz^{\bar{\beta}} + g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}} \right).$$

As we saw in Remark 4.1, the geodesic curvature is given by

$$c(\omega(t)) = g_{s\bar{s}} - g_{s\bar{\beta}} g^{\bar{\beta}\alpha} g_{\alpha\bar{s}}.$$

Thus we have

$$\frac{\partial}{\partial t} c(\omega(t)) = \left(\frac{\partial}{\partial t} g \right)_{s\bar{s}} - \left(\frac{\partial}{\partial t} g \right)_{s\bar{\beta}} g^{\bar{\beta}\alpha} g_{\alpha\bar{s}} - g_{s\bar{\beta}} \frac{\partial}{\partial t} g^{\bar{\beta}\alpha} g_{\alpha\bar{s}} - g_{s\bar{\beta}} g^{\bar{\beta}\alpha} \left(\frac{\partial}{\partial t} g \right)_{\alpha\bar{s}}.$$

On the other hand,

$$\begin{aligned} \Delta_t c(\omega(t)) &= \Delta_t g_{s\bar{s}} - \Delta_t (g_{s\bar{\beta}} g^{\bar{\beta}\alpha}) g_{\alpha\bar{s}} - g^{\bar{\delta}\gamma} (g_{s\bar{\beta}} g^{\bar{\beta}\alpha})_\gamma (g_{\alpha\bar{s}})_{\bar{\delta}} \\ &\quad - g^{\bar{\delta}\gamma} (g_{s\bar{\beta}} g^{\bar{\beta}\alpha})_{\bar{\delta}} (g_{\alpha\bar{s}})_\gamma - (g_{s\bar{\beta}} g^{\bar{\beta}\alpha}) \Delta_t g_{\alpha\bar{s}} \\ &= I_0 - I_1 - I_2 - I_3 - I_4. \end{aligned}$$

Notice that $(\log \det(g_{\alpha\bar{\beta}}))_{\bar{s}} = g^{\bar{\delta}\gamma} (g_{\bar{s}})_{\gamma\bar{\delta}} = \Delta_t g_{\bar{s}}$. This implies that

$$\begin{aligned} I_0 &:= \Delta_t g_{s\bar{s}} = g^{\bar{\delta}\gamma} (g_{s\bar{s}})_{\gamma\bar{\delta}} = g^{\bar{\delta}\gamma} (g_{\gamma\bar{\delta}})_{s\bar{s}} = (g^{\bar{\delta}\gamma} (g_{\gamma\bar{\delta}})_{\bar{s}})_s - (g^{\bar{\delta}\gamma})_s (g_{\gamma\bar{\delta}})_{\bar{s}} \\ &= (g^{\bar{\delta}\gamma} (g_{\bar{s}})_{\gamma\bar{\delta}})_s + g^{\bar{\delta}\alpha} (g_{\alpha\bar{\beta}})_s g^{\bar{\beta}\gamma} (g_{\gamma\bar{\delta}})_{\bar{s}} \\ &= (\log \det(g_{\alpha\bar{\beta}}))_{s\bar{s}} + g^{\bar{\delta}\alpha} (g_{\alpha\bar{\beta}})_s g^{\bar{\beta}\gamma} (g_{\gamma\bar{\delta}})_{\bar{s}} \\ &= \left(\frac{\partial}{\partial t} g \right)_{s\bar{s}} + (n+1)g_{s\bar{s}} + g^{\bar{\delta}\alpha} (g_{\alpha\bar{\beta}})_s g^{\bar{\beta}\gamma} (g_{\gamma\bar{\delta}})_{\bar{s}}. \end{aligned}$$

In the last equality, we used the fact that $\omega(t)$ satisfies the equation (3.5) so that

$$\left(\frac{\partial}{\partial t} g \right)_{s\bar{s}} = (\log \det(g_{\alpha\bar{\beta}}))_{s\bar{s}} - (n+1)g_{s\bar{s}}.$$

From now on, we fix a point and choose a normal coordinate (z^1, \dots, z^n) such that

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial z^\gamma}(x) = 0 = \frac{\partial g_{\alpha\bar{\beta}}}{\partial z^{\bar{\delta}}}(x).$$

Then the term $I_1 := \Delta_t (g_{s\bar{\beta}} g^{\bar{\beta}\alpha}) g_{\alpha\bar{s}}$ can be simplified as follows:

$$\begin{aligned} I_1 &= g_{s\bar{\alpha}} \Delta_t g^{\bar{\beta}\alpha} g_{\gamma\bar{s}} + g^{\bar{\delta}\gamma} (g_{s\bar{\alpha}})_{\bar{\delta}} (g^{\bar{\beta}\alpha})_\gamma g_{\gamma\bar{s}} + g^{\bar{\delta}\gamma} (g_{s\bar{\beta}})_\gamma (g^{\bar{\beta}\alpha})_{\bar{\delta}} g_{\alpha\bar{s}} + \Delta_t g_{s\bar{\beta}} g^{\bar{\beta}\alpha} g_{\alpha\bar{s}} \\ &= g_{s\bar{\beta}} \Delta_t (g^{\bar{\beta}\alpha}) g_{\alpha\bar{s}} + g^{\bar{\beta}\alpha} \Delta_t (g_{s\bar{\beta}}) g_{\alpha\bar{s}}. \end{aligned}$$

To compute the first term, note that

$$\begin{aligned} (g^{\bar{\beta}\alpha})_{\gamma\bar{\delta}} &= (-g^{\bar{\beta}\sigma}(g_{\sigma\bar{\tau}})_{\bar{\delta}}g^{\bar{\tau}\alpha})_{\gamma} \\ &= -(g^{\bar{\beta}\sigma})_{\gamma}(g_{\sigma\bar{\tau}})_{\bar{\delta}}g^{\bar{\tau}\alpha} - g^{\bar{\beta}\sigma}(g_{\sigma\bar{\tau}})_{\gamma\bar{\delta}}g^{\bar{\tau}\alpha} - g^{\bar{\beta}\sigma}(g_{\sigma\bar{\tau}})_{\bar{\delta}}(g^{\bar{\tau}\alpha})_{\gamma} \\ &= -g^{\bar{\beta}\sigma}(g_{\sigma\bar{\tau}})_{\gamma\bar{\delta}}g^{\bar{\tau}\alpha}. \end{aligned}$$

This implies that

$$\begin{aligned} \Delta_t (g^{\bar{\beta}\alpha})_{\gamma\bar{\delta}} &= g^{\bar{\delta}\gamma} (g^{\bar{\beta}\alpha})_{\gamma\bar{\delta}} = -g^{\bar{\beta}\sigma} g^{\bar{\tau}\alpha} g^{\bar{\delta}\gamma} (g_{\sigma\bar{\tau}})_{\gamma\bar{\delta}} = -g^{\bar{\beta}\sigma} g^{\bar{\tau}\alpha} (\log \det (g_{\gamma\bar{\delta}}))_{\sigma\bar{\tau}} \\ &= -g^{\bar{\beta}\sigma} g^{\bar{\tau}\alpha} \frac{\partial}{\partial t} g_{\sigma\bar{\tau}} - (n+1)g^{\bar{\beta}\sigma} g^{\bar{\tau}\alpha} g_{\sigma\bar{\tau}} \\ &= -g^{\bar{\beta}\sigma} g^{\bar{\tau}\alpha} \frac{\partial}{\partial t} g_{\sigma\bar{\tau}} - (n+1)g^{\bar{\beta}\alpha}. \end{aligned}$$

In the last second equality, we used the fact that the equation (3.5) implies that

$$\left(\frac{\partial}{\partial t} g\right)_{\sigma\bar{\tau}} = (\log \det (g_{\alpha\bar{\beta}}))_{\sigma\bar{\tau}} - (n+1)g_{\sigma\bar{\tau}}.$$

The equation (3.5) also implies that $\Delta_t (g_{s\bar{\beta}}) = \frac{\partial}{\partial t} g_{s\bar{\beta}} + (n+1)g_{s\bar{\beta}}$. Then we have

$$\begin{aligned} I_1 &= g_{s\bar{\beta}} \Delta_t (g^{\bar{\beta}\alpha})_{g_{\alpha\bar{s}}} + \Delta_t (g_{s\bar{\beta}}) g^{\bar{\beta}\alpha} g_{\alpha\bar{s}} \\ &= -g_{s\bar{\beta}} \left(g^{\bar{\beta}\sigma} g^{\bar{\tau}\alpha} \left(\frac{\partial}{\partial t} g_{\sigma\bar{\tau}} \right) + (n+1)g^{\bar{\beta}\alpha} \right) g_{\alpha\bar{s}} + \left(\frac{\partial}{\partial t} g_{s\bar{\beta}} + (n+1)g_{s\bar{\beta}} \right) g^{\bar{\beta}\alpha} g_{\alpha\bar{s}} \\ &= -\left(\frac{\partial}{\partial t} g_{\sigma\bar{\tau}} \right) g^{\bar{\beta}\sigma} g^{\bar{\tau}\alpha} g_{s\bar{\beta}} g_{\alpha\bar{s}} + \left(\frac{\partial}{\partial t} g_{s\bar{\beta}} \right) g^{\bar{\beta}\alpha} g_{\alpha\bar{s}}, \end{aligned}$$

and

$$I_4 := g_{s\bar{\beta}} g^{\bar{\beta}\alpha} \Delta_t g_{\alpha\bar{s}} = g_{s\bar{\beta}} g^{\bar{\beta}\alpha} \left(\frac{\partial}{\partial t} g_{\alpha\bar{s}} \right) + (n+1)g_{s\bar{\beta}} g^{\bar{\beta}\alpha} g_{\alpha\bar{s}}.$$

Since our coordinate is normal,

$$I_2 := g^{\bar{\delta}\gamma} (g_{s\bar{\beta}} g^{\bar{\beta}\alpha})_{\gamma} (g_{\alpha\bar{s}})_{\bar{\delta}} = g^{\bar{\delta}\gamma} (g_{s\bar{\alpha}})_{\gamma} g^{\bar{\beta}\alpha} (g_{\gamma\bar{s}})_{\bar{\delta}} = g^{\bar{\delta}\gamma} g^{\bar{\beta}\alpha} (g_{s\bar{\alpha}})_{\gamma} (g_{\gamma\bar{s}})_{\bar{\delta}}.$$

Therefore, we have

$$\begin{aligned} \Delta_t c(\omega(t)) &= I_0 - I_1 - I_2 - I_3 - I_4 \\ &= \left(\frac{\partial}{\partial t} g \right)_{s\bar{s}} + (n+1)g_{s\bar{s}} + g^{\bar{\delta}\alpha}(g_{\alpha\bar{\beta}})_s g^{\bar{\beta}\gamma}(g_{\gamma\bar{\delta}})_{\bar{s}} \\ &\quad + \frac{\partial}{\partial t} g_{\sigma\bar{\tau}} g^{\bar{\beta}\sigma} g^{\bar{\tau}\alpha} g_{s\bar{\beta}} g_{\alpha\bar{s}} - \frac{\partial}{\partial t} g_{s\bar{\beta}} g^{\bar{\beta}\alpha} g_{\alpha\bar{s}} - g^{\bar{\delta}\gamma} g^{\bar{\beta}\alpha} (g_{s\bar{\alpha}})_{\gamma} (g_{\gamma\bar{s}})_{\bar{\delta}} \\ &\quad - I_3 - g_{s\bar{\beta}} g^{\bar{\beta}\alpha} \frac{\partial}{\partial t} g_{\alpha\bar{s}} - (n+1)g_{s\bar{\beta}} g^{\bar{\beta}\alpha} g_{\alpha\bar{s}} \\ &= \left(\frac{\partial}{\partial t} g \right)_{s\bar{s}} - g_{s\bar{\beta}} \frac{\partial}{\partial t} g^{\bar{\beta}\alpha} g_{\alpha\bar{s}} - \frac{\partial}{\partial t} g_{s\bar{\beta}} g^{\bar{\beta}\alpha} g_{\alpha\bar{s}} - g_{s\bar{\beta}} g^{\bar{\beta}\alpha} \frac{\partial}{\partial t} g_{\alpha\bar{s}} \\ &\quad + (n+1) \left(g_{s\bar{s}} - g_{s\bar{\beta}} g^{\bar{\beta}\alpha} g_{\alpha\bar{s}} \right) - I_3 \\ &= \frac{\partial}{\partial t} c(\omega(t)) + (n+1)c(\omega(t)) - I_3. \end{aligned}$$

Hence it is enough to show that $I_3 = \|\bar{\partial}v_{\omega(t)}\|^2$. Remark 4.1 says that

$$\bar{\partial}v_{\omega(t)} = \left(-(g_{s\bar{\beta}})_{\bar{\delta}} g^{\bar{\beta}\alpha} - g_{s\bar{\beta}} (g^{\bar{\beta}\alpha})_{\bar{\delta}} \right) dz^{\bar{\delta}} \otimes \frac{\partial}{\partial z^{\alpha}}.$$

In the normal coordinates, we have

$$\|\bar{\partial}v_{\omega(t)}\|^2 = g^{\bar{\tau}\alpha} (g_{s\bar{\tau}})_{\bar{\delta}} \overline{g^{\bar{\sigma}\beta} (g_{s\bar{\sigma}})_{\bar{\gamma}} g_{\alpha\bar{\beta}} g^{\bar{\delta}\gamma}} = g^{\bar{\tau}\alpha} g^{\bar{\delta}\gamma} (g_{s\bar{\tau}})_{\bar{\delta}} (g_{\bar{s}\alpha})_{\gamma}.$$

On the other hand,

$$I_3 = g^{\bar{\delta}\gamma} (g_{s\bar{\beta}} g^{\bar{\beta}\alpha})_{\bar{\delta}} (g_{\alpha\bar{s}})_{\gamma} = g^{\bar{\delta}\gamma} (g_{s\bar{\beta}})_{\bar{\delta}} g^{\bar{\beta}\alpha} (g_{\alpha\bar{s}})_{\gamma} = g^{\bar{\delta}\gamma} g^{\bar{\beta}\alpha} (g_{s\bar{\beta}})_{\bar{\delta}} (g_{\alpha\bar{s}})_{\gamma}.$$

This completes the proof. □

REMARK 4.3. *For a holomorphic family of canonically polarized compact Kähler manifolds, the positivity of $\omega(t)$ can be immediately proved by applying the standard weak and strong parabolic maximum principle to the equation (4.1) (see Corollary 4.9 in [2]). The standard weak maximum principle, however, does not hold in general if the manifold is non-compact. In the next section, we will use a version of weak parabolic maximum principle for non-compact manifolds due to Ni.*

5 POSITIVITY OF FIBERWISE KÄHLER-RICCI FLOWS

Fix an arbitrary point $y \in S$. Denote its fiber by $\Omega := D_y$. Throughout this section, we will omit the index y for the defining function r_y and the Kähler metrics ω_y and $\omega_y(t)$. Let $\underline{g} := -\log(-r)$ be the strictly plurisubharmonic function on Ω . Then $\omega = i\partial\bar{\partial}g$ is a complete Kähler metric on Ω satisfying

$$|dg|_{\omega}^2 = g^{\alpha\bar{\beta}} g_{\alpha} g_{\bar{\beta}} = \frac{|\partial r|^2}{|\partial r|^2 - r} \leq 1.$$

By Theorem 2.5, there exist one parameter family of Kähler metrics $\omega(t) := i\partial\bar{\partial}g(t)$ on Ω satisfying the Kähler-Ricci flow:

$$\begin{aligned} \frac{\partial}{\partial t}\omega(t) &= -\text{Ric}(\omega(t)) - (n + 1)\omega(t), \\ \omega(0) &= \omega. \end{aligned} \tag{5.1}$$

We also know that $\omega(t)$ converges to the unique complete Kähler-Einstein metric as $t \rightarrow \infty$. Moreover, there exists a constant $C > 0$ (independent of t) such that

$$\frac{1}{C}\omega \leq \omega(t) \leq C\omega. \tag{5.2}$$

We denote the volume forms by $dV_t := \frac{\omega(t)^n}{n!}$ and $dV_0 := \frac{\omega^n}{n!}$.

5.1 PARABOLIC MAXIMUM PRINCIPLE

The following theorem is essentially the same with Ni’s parabolic maximum principle in [15], except it is expressed by a plurisubharmonic exhaustion function instead of the distance function.

THEOREM 5.1 (cf. Theorem 2.1 in [15]). *Let f be a smooth function on $\Omega \times [0, T)$ satisfying*

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) f \geq 0 \quad \text{whenever } f \leq 0.$$

Assume that there exists a constant $b > 0$ such that

$$\int_0^T \int_{\Omega} (-r)^b (f_-)^2 dV_t dt < \infty, \tag{5.3}$$

where $f_- := -\min\{f, 0\}$. If $f \geq 0$ on Ω at $t = 0$, then $f \geq 0$ on $\Omega \times [0, T)$.

Proof. Let $S(t)$ be the scalar curvature of $\omega(t)$, defined by

$$S(z, t) := g(t)^{\alpha\bar{\beta}} (\log \omega(t)^n)_{\alpha\bar{\beta}}.$$

Denote by $S_*(t) := \inf_{z \in \Omega} S(z, t)$ and let

$$\tilde{f}(z, t) := \exp\left(\int_0^t \frac{1}{2}(S_*(s) + n(n + 1)) ds\right) f(z, t).$$

A direct computation gives that

$$\left(\frac{\partial}{\partial t} - \Delta_t - \frac{1}{2}(S_*(t) + n(n + 1))\right) \tilde{f}(z, t) \geq 0$$

whenever $\tilde{f}(z, t) \leq 0$. For any T' with $0 < T' < T$, let

$$\tilde{g}(z, t) := -\frac{g(z)^2}{4C(2T' - t)}.$$

Without the loss of the generality we may assume that $T' \leq \frac{1}{2b^2C}$, since we can always split $[0, T']$ into smaller intervals (such that each has length less than $\frac{1}{2b^2C}$) and apply the induction. Therefore near the boundary of Ω and $t \in [0, T']$, we have

$$e^{\tilde{g}} \leq e^{-\frac{b^2}{4}g^2} = e^{-(\frac{b}{2} \log(\frac{1}{-r}))^2} \leq (-r)^b,$$

since $e^{-x^2} \leq e^{-2x}$ for any large enough x . Now the condition (5.3) implies that

$$\int_0^{T'} \int_{\Omega} e^{\tilde{g}} \tilde{f}_-^2 dV_t dt < \infty. \tag{5.4}$$

Using the inequality (5.2), we have $|\nabla g|_{\omega(t)}^2 \leq C$. Hence it follows that

$$|\nabla \tilde{g}|^2 + \frac{\partial}{\partial t} \tilde{g} \leq 0.$$

Let $\chi : [0, \infty) \rightarrow [0, 1]$ be a cut-off function so that $\chi(s) = 0$ for $s \geq 1$ and $\chi(s) = 1$ for $s \leq 1$. Set $\eta(z) := \chi\left(\frac{g(z)}{a}\right)$. Using the inequality (5.2), it is easy to see that there exists a constant $C_1 > 0$ independent of a such that

$$|\nabla \eta|^2 \leq \frac{C_1}{a^2}. \tag{5.5}$$

Now Stoke’s theorem implies the following:

$$\begin{aligned} \int_{\Omega} \eta^2 e^{\tilde{g}} \tilde{f}_- \Delta_t \tilde{f} dV_t &= - \int_{\Omega} \langle \nabla (\eta^2 e^{\tilde{g}} \tilde{f}_-), \nabla \tilde{f} \rangle dV_t \\ &= - \int_{\Omega} \left(2 \langle \nabla \eta, \nabla \tilde{f}_- \rangle \eta e^{\tilde{g}} \tilde{f}_- + \langle \nabla \tilde{f}, \nabla \tilde{g} \rangle \eta^2 e^{\tilde{g}} \tilde{f}_- + |\nabla \tilde{f}_-|^2 \eta^2 e^{\tilde{g}} \right) dV_t \\ &\leq \int_{\Omega} \left(2 |\nabla \eta|^2 e^{\tilde{g}} \tilde{f}_-^2 + \frac{1}{2} |\nabla \tilde{g}|^2 \eta^2 e^{\tilde{g}} \tilde{f}_-^2 \right) dV_t. \end{aligned}$$

On the other hand, integration by parts implies that

$$\begin{aligned} \int_0^{T'} \int_{\Omega} \eta^2 e^{\tilde{g}} \tilde{f}_- \frac{\partial \tilde{f}}{\partial t} dV_t dt &= -\frac{1}{2} \int_0^{T'} \int_{\Omega} \eta^2 e^{\tilde{g}} \frac{\partial}{\partial t} (\tilde{f}_-^2) dV_t dt \\ &= - \int_{\Omega} \frac{1}{2} \eta^2 e^{\tilde{g}} \tilde{f}_-^2 dV_t \Big|_0^{T'} + \frac{1}{2} \int_0^{T'} \int_{\Omega} \frac{\partial}{\partial t} \left(\eta^2 e^{\tilde{g}} \frac{dV_t}{dV_0} \right) \tilde{f}_-^2 dV_0 dt. \end{aligned} \tag{5.6}$$

Taking the trace of the equation (5.1), we have

$$\frac{\partial g(t)_{\alpha\bar{\beta}}}{\partial t} g(t)^{\alpha\bar{\beta}} = -S(t) - n(n+1).$$

Using Cramer’s rule, we obtain

$$\frac{\partial}{\partial t} \left(\frac{dV_t}{dV_0} \right) = \frac{\det(g(t)_{\alpha\bar{\beta}})}{\det(g_{\alpha\bar{\beta}})} \frac{\partial g(t)_{\alpha\bar{\beta}}}{\partial t} g(t)^{\alpha\bar{\beta}} = \frac{dV_t}{dV_0} (-S(t) - n(n+1)).$$

Altogether, it follows that

$$\begin{aligned} 0 &\leq \int_0^{T'} \int_{\Omega} \eta^2 e^{\tilde{g}} \tilde{f}_- \left(\frac{\partial}{\partial t} - \Delta_t - \frac{1}{2}(S_*(t) + n(n+1)) \right) \tilde{f}_- dV_t dt \\ &\leq \int_0^{T'} \int_{\Omega} \left(2|\nabla \eta|^2 e^{\tilde{g}} \tilde{f}_-^2 + \frac{1}{2} |\nabla \tilde{g}|^2 \eta^2 e^{\tilde{g}} \tilde{f}_-^2 + \frac{1}{2} \frac{\partial \tilde{g}}{\partial t} \eta^2 e^{\tilde{g}} \tilde{f}_-^2 \right) dV_t dt \\ &\quad - \int_{\Omega} \frac{1}{2} \eta^2 e^{\tilde{g}} \tilde{f}_-^2 dV_t \Big|_0^{T'} + \int_0^{T'} \int_{\Omega} \frac{1}{2} \left(\eta^2 e^{\tilde{g}} \tilde{f}_-^2 (-S(t) + S_*(t)) \right) dV_t dt \\ &\leq 2 \int_0^{T'} \int_{\Omega} |\nabla \eta|^2 e^{\tilde{g}} \tilde{f}_-^2 dV_t dt - \left(\frac{1}{2} \int_{\Omega} \eta^2 e^{\tilde{g}} \tilde{f}_-^2 dV_t \right) (T') \end{aligned}$$

Letting $a \rightarrow \infty$, the inequalities (5.4) and (5.5) imply that

$$\left(\int_{\Omega} e^{\tilde{g}} \tilde{f}_-^2 dV_t \right) (T') \leq 0$$

This implies that $\tilde{f}_- \equiv 0$, therefore we have $f \geq 0$ on $\Omega \times [0, T']$. □

5.2 PROOF OF THEOREM 1.1

By Remark 4.1, it is enough to show that the restriction of the geodesic curvature of the fiberwise Kähler-Ricci flow $c(\omega(t)) := c(\omega(t)|_{\Omega})$ is positive on Ω . We will apply Theorem 5.1 to the function $c(\omega(t))$ on $\Omega \times [0, \infty)$. Note that Berman’s parabolic equation (4.1) says that

$$\left(\frac{\partial}{\partial t} - \Delta_t \right) c(\omega(t)) \geq 0 \quad \text{whenever } c(\omega(t)) \leq 0.$$

On the other hand, the computation in the proof of Proposition 3.2 implies that

$$dV_0 = \det(r_{\gamma\bar{\delta}})(-r + |\partial r|^2) \left(\frac{1}{-r} \right)^{n+1} dV,$$

where dV is the Euclidean volume form of \mathbb{C}^n . Since $\det(r_{\gamma\bar{\delta}})(-r + |\partial r|^2) = e^{-F_y}$ is bounded function on $\bar{\Omega}$, this together with the quasi-isometry (5.2) implies that

$$\int_{\Omega} (-r)^b c(\omega(t))^2 dV_t \lesssim \int_{\Omega} \left(\frac{1}{-r} \right)^{n+1-b} c(\omega(t))^2 dV.$$

To satisfy the condition (5.3) in Theorem 5.1, we only need to show that the geodesic curvature $c(\omega(t))$ has a polynomial growth near the boundary with respect to the defining function. More precisely, we will show that $|c(\omega(t))| = O((-r)^{-2})$.

First consider the initial data $c(\omega)$. Since $\omega = i\partial\bar{\partial}g$ with $g := -\log(-r)$, we have

$$c(\omega) := \langle v_{\omega}, v_{\omega} \rangle_{\omega} = \frac{1}{-r} i\partial\bar{\partial}r(v_{\omega}, \bar{v}_{\omega}) + \frac{1}{r^2} |\partial r(v_{\omega})|^2 = O((-r)^{-2}). \tag{5.7}$$

Hence it suffices to show the following proposition.

PROPOSITION 5.2. *There exists constant $C > 0$ independent of t such that*

$$|c(\omega(t)) - c(\omega)| \leq \frac{C}{-r}. \tag{5.8}$$

Proof. Recall that the fiberwise Kähler-Ricci flow $\omega(t)$ on D is given by $\omega(t) := i\partial\bar{\partial}g(t)$ where $g(t) := g + \varphi(t)$. For a fixed $t \in (0, \infty)$, denote by $\varphi := \varphi(t)$. Under the Euclidean coordinate system (z^1, \dots, z^n, s) , $c(\omega(t))$ can be expressed as

$$c(\omega(t)) = g_{s\bar{s}} + \varphi_{s\bar{s}} - (g_{s\bar{\beta}} + \varphi_{s\bar{\beta}})g(t)^{\bar{\beta}\alpha}(g_{\alpha\bar{s}} + \varphi_{\alpha\bar{s}})$$

Since $c(\omega) = g_{s\bar{s}} - g_{s\bar{\beta}}g^{\bar{\beta}\alpha}g_{\alpha\bar{s}}$ and $\frac{1}{C}\omega \leq \omega(t) \leq C\omega$, we have

$$|c(\omega(t)) - c(\omega)| \lesssim |\varphi_{s\bar{s}} - \varphi_{s\bar{\beta}}g^{\bar{\beta}\alpha}\varphi_{\alpha\bar{s}} - g_{s\bar{\beta}}g^{\bar{\beta}\alpha}\varphi_{\alpha\bar{s}} - \varphi_{s\bar{\beta}}g^{\bar{\beta}\alpha}g_{\alpha\bar{s}}|.$$

Moreover, an explicit calculation of derivatives of g implies that $g^{\bar{\beta}\alpha} = O(-r)$, $g_{s\bar{\beta}}g^{\bar{\beta}\alpha}$ and $g^{\bar{\beta}\alpha}g_{\alpha\bar{s}}$ are bounded functions on Ω (cf. Section 5 in [9]). Remark 3.4 implies that $\varphi_{s\bar{s}}$ is bounded. Thus it is enough to estimate functions $\varphi_{\alpha\bar{s}}$ and $\varphi_{s\bar{\beta}}$. Note that Remark 3.4 implies that

$$\|\xi_p^* \varphi_s\|_{C^{k+\epsilon, \frac{k+\epsilon}{2}}(V_p \times [0, \infty))} \leq C_k$$

for some constant $C_k > 0$. In particular, this implies that there exist a constant $C > 0$ independent of t such that $\left| \frac{\partial}{\partial \xi^j} \varphi_s \right| \leq C$, where $\xi_p = (\xi^1, \dots, \xi^n)$ is the coordinate system satisfying the conditions of bounded geometry. By the construction of the coordinate system for the strongly pseudoconvex domain (see Section 1 in [8]), we obtain the estimate

$$|\varphi_{s\bar{\beta}}| = \left| \frac{\partial}{\partial z^{\bar{\beta}}} \varphi_s \right| \leq \frac{C}{(-r)} \sum_{j=1}^n \left| \frac{\partial}{\partial \xi^j} \varphi_s \right| \leq \frac{C}{-r}$$

on the Euclidean coordinates (z^1, \dots, z^n) . The same argument for the function $\varphi_{\alpha\bar{s}}$ shows that $|\varphi_{\alpha\bar{s}}| \leq \frac{C}{-r}$. This completes the proof. \square

Equations (5.7) and (5.8) imply that $|c(\omega(t))| = O((-r)^{-2})$ as we required. Now the following strong maximum principle completes the proof of Theorem 1.1.

THEOREM 5.3 (cf. Theorem 6.54 in [13]). *Let f be a smooth function on $\Omega \times [0, T)$ satisfying*

$$\left(\frac{\partial}{\partial t} - \Delta_t \right) f \geq 0.$$

Suppose that $f \geq 0$ on $\Omega \times [0, T)$. If $f(x, 0) > 0$ for some point $x \in \Omega$ at the initial time $t = 0$, then $f > 0$ on $\Omega \times (0, T)$.

Finally, Theorem 1.1 and Theorem 3.8 imply Corollary 1.2.

REFERENCES

- [1] T. Aubin, *Some nonlinear problems in Riemannian Geometry*. Springer, Berlin, 1998.
- [2] R. J. Berman, *Relative Kähler-Ricci flows and their quantization*, Anal. PDE 6 (2013), 131–180.
- [3] B. Berndtsson, M. Păun, *Quantitative extensions of pluricanonical forms and closed positive currents*, Nagoya Math. J. 205 (2012), 25–65.
- [4] J. Song, B. Weinkove, *An Introduction to the Kähler–Ricci Flow*. In Bouncksom S., Eyssidieux P., Guedj V. (eds), *An Introduction to the Kähler-Ricci Flow*, Lecture Notes in Mathematics, 2086. 189–237. Springer, Cham, 2013.
- [5] H.-D. Cao, *Deformation of Kähler metrics to Kähler Einstein metrics on compact Kähler manifolds*, Invent. Math. 81 (1985), no. 2 359–372.
- [6] A. Chau, *Convergence of the Kähler-Ricci flow on noncompact Kähler manifolds*, J. Differential Geom. 66 (2004), no. 2, 211–232.
- [7] A. Chau, L.-F. Tam, *On a modified parabolic complex Monge-Ampère equation with applications*, Math. Z. 269 (2011), no. 3–4, 777–800.
- [8] S.-Y. Cheng, S.-T. Yau, *On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman’s equation*, Comm. Pure Appl. Math. 33 (1980), no. 4, 507–544.
- [9] Y.-J. Choi, *Variations of Kähler-Einstein metrics on strongly pseudoconvex domains*, Math. Ann. 362 (2015), 121–146.
- [10] Y.-J. Choi, *A study of variations of pseudoconvex domains via Kähler-Einstein metrics*, Math. Z. 281 (2015), 299–314.
- [11] Y.-J. Choi, G. Schumacher, *Extension of the curvature form of the relative canonical line bundle on family of Calabi-Yau manifolds and applications*, Ann. Inst. Fourier, Grenoble 71 (2021), no. 1, 393–406.
- [12] Y.-J. Choi, S. Yoo, *Holomorphic families of strongly pseudoconvex domains in a Kähler manifold*, J. Geom. Anal. 31 (2021), 2639–2655.
- [13] B. Chow, P. Lu, L. Ni, *Hamilton’s Ricci flow*, Graduate Studies in Mathematics, 77. American Mathematical Society, 2006.
- [14] G. Lieberman, *Second order parabolic differential equations*, World Scientific, 1996.
- [15] L. Ni, *Kähler-Ricci flow on complete manifolds*, Clay Mathematics Proceedings, 2005.

- [16] M. Păun, *Relative adjoint transcendental classes and Albanese map of compact Kähler manifolds with nef Ricci curvature*, Higher Dimensional Algebraic Geometry: In honour of Professor Yujiro Kawamata's sixtieth birthday. Mathematical Society of Japan, 2017.
- [17] O. Saeki, *Topology of singular fibers of differentiable maps*, Lecture Notes in Mathematics, 1854. Springer, Berlin, 2004.
- [18] G. Schumacher, *Positivity of relative canonical bundles and applications*, Invent. Math. 190 (2012), no. 1, 1–56.
- [19] W.-X. Shi, *Ricci flow and the uniformization on complete noncompact Kähler manifolds*, J. Differential Geom. 45 (1997), no. 1, 94–220.
- [20] Y.-T. Siu, *Curvature of the Weil-Petersson metric in the moduli space of compact Kähler-Einstein manifolds of negative first Chern class*, Contributions to several complex variables, Hon. W. Stoll, Proc. Conf. Complex Analysis, Notre Dame/Indiana 1984, Aspects Math. E9, 261–298, 1986.
- [21] S.-T. Yau, *On the Ricci curvature of a compact Kähler Manifold and the complex Monge-Ampere equation. I*, Comment. Pure Appl. Math. 31 (1978), 339–411.

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