On the Chow Ring of the Classifying Stack of Algebraic Tori

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Abstract. We investigate the structure of the Chow ring of the classifying stacks BT of algebraic tori, as it has been defined by B. Totaro. Some previous work of N. Karpenko, A. Merkurjev, S. Blinstein and F. Scavia has shed some light on the structure of such rings. In particular Karpenko showed the absence of torsion classes in the case of permutation tori, while Merkurjev and Blinstein described in a very effective way the second equivariant Chow group A^2_T in the general case. Building on this work, Scavia exhibited an example where $(A^2_T)_{\text{tors}} \neq 0$.
Here, by making use of a very elementary approach, we extend the result of Karpenko to special tori and we completely determine the Chow ring A^*_T when T is an algebraic torus admitting a resolution with special tori $0 \to T \to Q \to P$. In particular we show that there can be torsion in the Chow ring of such tori.

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1 Introduction

In this paper we give some new examples of Chow rings of classifying spaces of algebraic tori with non-trivial additive torsion; our example is more elementary than the first one due to F. Scavia ([10], Th. 1.2), and it lies in A^3_T; the main tool is a description of the Chow group of the total space for a torsor under a permutation torus, which is of independent interest.
Let G be an affine algebraic group over a field k. B. Totaro [11] gave a definition of the Chow ring A^*_G of the classifying space BG; when $k = \mathbb{C}$ there is a

natural ring homomorphism from A^*_G to the cohomology of BG, which is an isomorphism after tensoring with $\mathbb{Q}$, but not integrally ([2], [13]).
The ring A^*_G has been computed for many classes of reductive split groups, such as GL_n, SL_n, SO_n, PGL_p when p is a prime (see [13],[12]). On the other hand, not much is known about Chow rings of classifying spaces of non-inner forms of these groups.
When $T = \mathbb{G}_m^n$ is a split algebraic torus, the Chow ring A^*_T is a polynomial ring $\mathbb{Z}[t_1, \dots, t_n]$, where t_i is the first Chern class of the i-th projection $\mathbb{G}_m^n \to \mathbb{G}_m$, considered as a representation of T. In this paper we consider non-split algebraic tori; let T be such a torus.
Choose a finite Galois extension k' of k, with Galois group Γ, such that $T_{k'} \simeq \mathbb{G}^n_{m,k'}$. The group Γ acts on $A^*_{T_{k'}} \simeq \mathbb{Z}[t_1, \dots, t_n]$; the natural pullback map $A^*_T \to A^*_{T_{k'}}$ factors through the invariant subring $(A^*_{T_{k'}})^\Gamma$. We call the resulting homomorphism of graded rings $A^*_T \to (A^*_{T_{k'}})^\Gamma$ *the base-change homomorphism.* It is not hard to check that it is independent of the choice of k', and that it is an isomorphism when tensored with $\mathbb{Q}$ (it is, actually, true that $A^*(X/G)_\mathbb{Q} \simeq A^*(X)^G_\mathbb{Q}$ for any quotient by a finite group G, see [3] ex. 1.7.6). It is a natural question whether it is always an isomorphism.
N. Karpenko ([5], Prop. 4.6) showed that this is always the case when T is a permutation torus.
A. Merkurjev and S. Blinstein (see [7] and [6]) gave a cohomological interpretation of the kernel $(A^2_T)_{\text{tors}}$ of the base-change homomorphism in degree 2. By using this, Scavia gave an example of a torus T for which $(A^2_T)_{\text{tors}} = 0$, so that the base-change homomorphism is not always injective in degree 2.
Recall that an algebraic group G is said to be *special* if every $G-$torsor is Zariski-locally trivial; it is easy to see that permutation tori are special. In this paper we show that the base-change is an isomorphism for all special tori; then we investigate the simplest class of non-special algebraic tori, those tori T that can be embedded into a special torus P, in such a way that the quotient P/T is also special. For these tori Merkurjev ([6], Cor. 5.4) showed that $(A^2_T)_{\text{tors}}$ is always 0.
Here we give an example of tori of this special form for which the base-change homomorphism is not injective nor surjective.
In the first section we study the simplest representation of a permutation torus T, showing the following

THEOREM. *There exists an equivariant decomposition of the tautological representation of T in $N = \dim(T)$ strata. These strata are disjoint unions of permutation tori, defined over some suitable extension $k \subseteq L \subseteq k'$.*

In the second section we show a general result (Theorem 2.1) which allows to calculate the Chow ring $A^*(E)$ for a $T-$torsor $p : E \to M$ from the ring $A^*(M)$, valid for every permutation torus T.

THEOREM. *The pull-back map $p^* : A^*(M) \to A^*(E)$ is surjective. For any $L : k$ let $p_L : E_L \to M_L$ be the base change of the torsor along the map*

$\pi^L : M_L \to M$; let $c \in A^*(M_L)$ be a characteristic class relative to p_L. Then the kernel of p^* is generated by the elements of the form $\pi^L_*(c)$ as L, c vary.

This is our main new contribution.
In the third section we briefly recall the results which are known from the literature and extend Karpenko's result to special tori. We finally concentrate on the T's with a resolution by special tori $0 \to T \to P \to Q \to 0$. We give a formula for the Chow ring A^*_T as a function of A^*_P, A^*_Q, based on Theorem 2.1. Using this formula, in the fourth and last section we prove

Theorem. *There are examples of algebraic tori T with a resolution by special tori $0 \to T \to P \to Q \to 0$ such that the map $\phi : A^*_T \to (A^*_{T_{k'}})^\Gamma$ in not injective or not surjective.*

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2 Stratifying a representation of permutation tori

Here we begin to set up the tools which we will need to calculate the Chow ring of the classifying stack BT for a class of algebraic tori T.
In particular we start by considering quasi-split tori over a fixed field k, i.e. algebraic tori which arise as Weil restriction from a finite type étale $k-$algebra. If E is such an algebra, we thus denote $T := R_{E/k}\mathbb{G}_{m,E}$. If k' is a Galois closure of E, then $T_{k'}$ is a split torus, and the group $\Gamma := \mathrm{Gal}(k' : k)$ acts by permutation on the character lattice of $T_{k'}$ (or, in other words, this lattice has a $\Gamma-$invariant basis on which the action of Γ permutes the elements). Therefore, these groups are also classified under the name of *permutation tori*.
Indeed let $E = \prod_{i=1}^m k_i$, where each k_i is a finite separable extension of k of degree d_i (in particular k' will contain the normal closure over k of each k_i). If $T_i := R_{k_i/k}\mathbb{G}_{m,k_i}$ then $T = \prod_{i=1}^m T_i$.
We now prove that each factor $R_{k_i/k}\mathbb{G}_{m,k_i}$ is isomorphic - as a scheme - to $\mathbb{A}^{d_i}_k - \{N^{k_i}_k = 0\}$. Here $\mathbb{A}^{d_i}_k$ is Spec $k[x_1, \ldots, x_{d_i}]$ and $N^{k_i}_k$ is the polynomial in the x_j's given by $\prod_{l=1}^{d_i} \sigma_l(\sum \alpha_j x_j)$ - where $\alpha_1, \ldots, \alpha_{d_i}$ is a basis of k_i over k and the $\sigma_l : k_i \hookrightarrow k'$ are the immersions of k_i in k' leaving k fixed. This is known from basic field theory to have coefficients in k.
In order to prove the above claim let us fix an extension $k' : k$, let F be the Galois closure of k' over k and let J be the set of immersions $k' \hookrightarrow F$ fixing k. Then $\Gamma := \mathrm{Gal}(F : k)$ acts on J by permuting its elements, and we have a well-known

Proposition 2.1. *Let X' be a variety over k' and let $\overline{X} := \prod_{j\in J} X' \times_{k',j} F$. For every $\sigma \in \Gamma$ let ϕ_σ be the automorphism of $\overline{X}$ induced by the action of σ on J. Then the ϕ_σ define Galois descent data, and the variety over k defined by these data is $R_{k'/k}X'$.*

With this result we can easily construct $R_{k'/k}X'$ for a given $k'-$variety X'. In particular let us concentrate on $X' = \mathbb{G}_{m,k'}$.
Let $x_1, \dots, x_n$ be a set of indeterminates indexed by J, and let $e_1, \dots, e_n$ be a basis of k' over k. Then $\prod_{\sigma:k'\hookrightarrow F}(\sum_{j\in J}\sigma(e_j)x_j)$ is the homogeneous polynomial $N_k^{k'}$ in the x'_js which defines the norm over k of the element $\sum x_j e_j$. Let $X = \mathbb{A}_k^n - \{N_k^{k'} = 0\}$.
In order to prove our claim, we can just consider $X \times_k F$ (with F a Galois closure as above): this is equal to $\mathbb{A}_F^n - \{\prod_{\sigma:k'\hookrightarrow F}\sum e_j x_j = 0\} \simeq \mathbb{A}_F^n - \{y_1 \dots y_n = 0\}$ via the isomorphism sending $\sum\sigma(e_i)x_i$ to y_σ. But it is exactly $\prod_{j\in J}\mathbb{G}_{m,k'} \times_{k',j} F$, with the action of $\Gamma = \mathrm{Gal}(F : k)$ which permutes the factors as it does on J (corresponding to the action of Γ on the factors $\sum\sigma(e_i)x_i$). This completes the proof.
We note that each $R_{k_i/k}\mathbb{G}_{m,k_i}$ splits over $\overline{k}$ as a product of $[k_i : k]$ factors isomorphic to $\mathbb{G}_{m,\overline{k}}$, and Γ permutes them according to its action on the set of $k-$immersions $k_i \hookrightarrow \overline{k}$.
Now, $R_{k_i/k}\mathbb{G}_{m,k_i}$ has a natural representation V_i over k of degree d_i and the whole product T has a natural representation $V := \oplus V_i$ of degree $N := \sum_i d_i$. Then $V_{k'} := V \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ k'$ is a representation of the split torus $T_{k'}$ whose action is induced by the tautological inclusion $T_{k'} \hookrightarrow GL(V_{k'})$ of diagonal matrices.
From now on we will call I_i the set of integers in the interval $(\sum_{j<i} d_j, \sum_{j\le i} d_j]$, which can label the immersions $k_i \hookrightarrow \overline{k}$.
We have $T_i \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ k' \simeq \mathbb{A}_{k'}^{d_i} - \{\prod_{l=1}^{d_i}\sigma_l(\sum\alpha_j x_j) = 0\}$, which gives an isomorphism $T_i \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ k' \simeq \mathbb{G}_{m,k'}^{d_i} = \mathrm{Spec}\ k'[y_1^{\pm1}, \dots y_{d_i}^{\pm1}]$ sending y_l to $\sigma_l(\sum\alpha_j x_j)$ for each l. In particular $V_i \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ k' \simeq \mathrm{Spec}\ k'[y_1, \dots, y_{d_i}]$ inherits an action of Γ which is the obvious one on Spec k' and sends y_l to $y_{\tau(l)}$, where τ is the permutation of $\{1, \dots, d_i\}$ induced by the action of Γ on the σ_l's.
To summarize, the group Γ acts on the space $V \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ k'$ as follows: tautologically on Spec k', while for each i it permutes $\{x_j : j \in I_i\}$ according to the identification of the latter with the set of $k-$immersions $k_i \hookrightarrow \overline{k}$.
Clearly $V = V_{k'}/\Gamma$, where the action of Γ on V is the one described at the end of the preceding paragraph.
For each $j \in \{1, \dots N\}$ let H_j be the hyperplane in $V_{k'} = \mathrm{Spec}\ k'[y_1, \dots, y_N]$ defined by $\{y_j = 0\}$ and for each $J \subseteq \{1, \dots N\}$ let H_J be the linear subspace of $V_{k'}$ obtained by $\bigcap_{j\in J} H_j$.

Then if $\overline{Z_p} := \coprod_{|J|=p} H_J$, these closed subschemes are $T_{k'}$ and Γ-stable; moreover these two actions are compatible with the action of Γ on $T_{k'}$, in the sense that for $g \in \Gamma, t \in T_{k'}$ it holds $g(t \cdot x) = g(t) \cdot g(x)$ for every x. This implies that the closed subschemes $Z_p := \overline{Z_p}/\Gamma \hookrightarrow V$ are $T-$stable. Thus we have found a $T-$equivariant stratification of V, and the latter is the disjoint union of the $W_p := Z_p - Z_{p+1}$, which we call *the strata*.

In particular $\overline{Z_0} = T_{k'}$ and $Z_0 = T$.

We now describe quite explicitly the strata W_p, in terms of the action of Γ on $\{1, \dots, N\}$.

Let us take $J \subseteq \{1, \dots, N\}$ such that $|J| = p$ and consider the split $N-p$-torus $T_J := H_J - \bigcup_{J \subsetneq I} H_I$. Clearly Γ acts on the set $\{I \subseteq \{1, \dots, N\} : |I| = p\}$ and the orbit of J for this action $\mathrm{Orb}_\Gamma(J)$ determines the orbit of T_J for the action of Γ on $\overline{Z_p}$: $\mathrm{Orb}_\Gamma(T_J) = \coprod_{I \in \mathrm{Orb}_\Gamma(J)} T_I$.

We shall see, by Galois descent, how to characterize $\mathrm{Orb}_\Gamma(T_J)/\Gamma$; we will show that this is again a permutation torus defined over a finite extension $k_J : k$.

Let us consider for a moment a field $\mathfrak{k}$ such that $\mathfrak{k} \subseteq k_i$ for all i, and let $\mathfrak{d} \subseteq \Gamma$ be the group $\mathrm{Gal}(k' : \mathfrak{k})$. Then Γ acts on $R_{E/\mathfrak{k}}\mathbb{G}_{m,E} \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$, which is a disjoint union of $n := [\mathfrak{k} : k]$ tori of dimension N/n, as follows: the orbits of $\mathfrak{d}$ partition $\{1, \dots, N\}$ into n pieces, and we can identify each of these n tori with one of them (where characters corresponds to elements $i \in \{1, \dots, N\}$); then Γ, acting on $\{1, \dots, N\}$ induces the sought action on these tori.

This variety embeds $\Gamma-$equivariantly into $V_{k'}$, as we can see immediately. Indeed if $\mathfrak{I} \subset \{1, \dots, N\}$ is the subset corresponding to the immersions $k_i \hookrightarrow k'$ (for any i) which leave $\mathfrak{k}$ fixed, and $\mathfrak{I}^c$ is its complement, then $\mathrm{Orb}_\Gamma(T_{\mathfrak{I}^c})$ is a disjoint union of n split tori of dimension N/n (since Γ partitions $\{1, \dots, N\}$ into N/n subsets isomorphic to $\mathfrak{I}$); in particular it is isomorphic to $R_{E/\mathfrak{k}}\mathbb{G}_{m,E} \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$. According to the observations we made in the preceding paragraphs, Γ acts on this orbit exactly as it does on $R_{E/\mathfrak{k}}\mathbb{G}_{m,E} \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$, so that the embedding of the latter into $V_{k'}$ is $\Gamma-$equivariant.

In particular by Galois descent we have that $(\mathrm{Orb}_\Gamma(T_{\mathfrak{I}^c}))/\Gamma \simeq R_{E/\mathfrak{k}}\mathbb{G}_{m,E}$. Moreover we note that the stabilizer of $\mathfrak{I}$ in Γ is exactly $\mathfrak{d}$, while the k_i's correspond to the subsets of $\mathfrak{d}$ which stabilize the orbits of its action on $\mathfrak{I}$ (or on $\mathfrak{I}^c$, which does not change them). Thus we can determine $\mathfrak{k}$ and all the k_i's just by looking at the action of Γ on $\mathrm{Orb}_\Gamma(T_{\mathfrak{I}^c})$.

The same reasoning allows us to determine $(\mathrm{Orb}_\Gamma T_J)/\Gamma$ for a general J.

First of all, let $S_J \subseteq \Gamma$ be the stabilizer of J, which corresponds to a field extension $K : k$. Moreover let $J_1, \dots, J_s$ be the orbits of the elements of J under the action of S_J, and $S_1, \dots, S_s$ the respective stabilizers. The S_i's induce extensions $K_i : K$, and let $\mathbb{E}$ be the étale $K-$algebra $\prod_{i=1}^s K_i$. Then we can compare the action of Γ on $R_{\mathbb{E}/K}\mathbb{G}_{m,\mathbb{E}} \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$ with the action of Γ on $\mathrm{Orb}_\Gamma(J)$, so that we are exactly in the situation of the last paragraph

(we just restrict to the vector subspace of $V_{k'}$ generated by $\{y_i : i \in \bigcup_{g\in\Gamma} g(J)\}$).
We conclude that $\mathrm{Orb}_\Gamma(T_J)/\Gamma \simeq R_{\mathbb{E}/K}\mathbb{G}_{m,\mathbb{E}}$. Let this torus be called $T_{\mathrm{Orb}(J)}$. So each of the W_p's is a disjoint union of algebraic tori defined over extensions $k \subseteq K \subseteq k'$: $Z_p = \coprod_{|J|=p} T_{\mathrm{Orb}(J)}$.
We conclude this section with the crucial

Lemma 2.2. *Let $J \subseteq \{1,\dots,N\}$, and $\mathfrak{k}$ be the fixed field inside k' of the stabilizer in Γ of J and J^c. Then $T_{Orb(J)} \times_{Spec\ \mathfrak{k}} T_{Orb(J^c)} = T \times_{Spec\ k}$ Spec $\mathfrak{k}$.*

Proof. Let $\mathfrak{d}$ be the group $\mathrm{Gal}(k' : \mathfrak{k})$. As before, we can take the product (over Spec $\mathfrak{k}$) of both sides with Spec k'. Then $T \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ \mathfrak{k} \simeq T_{k'}/\mathfrak{d}$, where the action of $\mathfrak{d}$ on $T_{k'}$ is just the one induced by the action of Γ. So by Galois descent we just need to verify that the action of $\mathfrak{d}$ on $(\mathrm{Orb}_\Gamma(T_J)/\Gamma \times_{\mathrm{Spec}\ \mathfrak{k}} \mathrm{Orb}_\Gamma(T_{J^c})/\Gamma) \times_{\mathrm{Spec}\ \mathfrak{d}} \mathrm{Spec}\ k'$ (which is a split $N-$torus) can be identified with the action of $\mathfrak{d}$ on $T_{k'}$. But this is clear since the action of $\mathfrak{d}$ on $\mathrm{Orb}_\Gamma(T_J)/\Gamma \times_{\mathrm{Spec}\ \mathfrak{d}} \mathrm{Spec}\ k'$ is induced by the restriction of the action of Γ on $T_{k'}$ to the subspace generated by $\{y_i : i \in J^c\}$; so the product of the J, J^c-tori gives a split $N-$torus with the entire action that $\mathfrak{d}$ had on $T_{k'}$. □

3 The Chow ring of $T-$torsors

We now exploit the results of the previous section to give a general formula for the Chow ring of a $T-$torsor, where T is a permutation torus.
We will keep the same notation as before. In particular, V will denote the standard representation of T, and the Z_p's will be the closed sets of the stratification we found. Moreover we have a set of permutation tori $W_p = Z_p - Z_{p+1}$ defined over field extensions of k.
From now on, we will also call $U_p := V - Z_p$ the open sets of the stratification.
Let $E \xrightarrow{p} M$ be a $T-$torsor of smooth schemes. Then we have a ring map $p^* : A^*(M) \to A^*(E)$ given by pull-back.
Clearly if $x \in A^*(M)$ is a characteristic class relative to E (or, more fancifully, an element in the image of the pull-back $A^*_T \to A^*(M)$ for the map which classifies E) we have $p^*(x) = 0$; indeed $p^*(x)$ is nothing but a characteristic class relative to the pull-back torsor $p^*E \to E$, which is the trivial torsor $E \times T \to E$.
Analogously, let us fix a field extension $L \supseteq k$, and let $E_L \xrightarrow{p_L} M_L$ be the corresponding T_L-torsor. Then, as before, we have that if $x \in A^*(M_L)$ is a T_L-characteristic class relative to E_L then $p_L^*(x) = 0$. Consider the commuting square

$$\begin{array}{ccc} A^*(M_L) & \longrightarrow & A^*(M) \\ \downarrow{\scriptstyle p_L^*} & & \downarrow{\scriptstyle p^*} \\ A^*(E_L) & \longrightarrow & A^*(E) \end{array}$$

where the horizontal arrows are the push-forwards for the obvious projections. Then, if $x' \in A^*(M_L)$ lies in the ideal generated by the T_L-characteristic classes induced by E_L and if x is its image in $A^*(M)$, we see immediately that $p^*(x) = 0$ in $A^*(E)$.
We claim that the classes we found so far are the only generators of $\ker(p^*)$:

THEOREM 3.1. *Let $E \xrightarrow{p} M$ be a $T-$torsor. Let $\mathcal{I} \subseteq A^*(M)$ be the ideal generated by the push-forwards $\pi^L_*(x \cdot m)$, where for some field $L \supseteq k$ we denote $\pi^L : M_L \to M$ the projection and $x, m \in A^*(M_L)$, x being a T_L-characteristic class induced by E_L.*
Then $A^*(E) = A^*(M)/\mathcal{I}$.

Proof. We will prove by descending induction on p that $A^*(M) \to A^*(E \times_T U_p)$ is surjective, and that its kernel lies in $\mathcal{I}$. This will be sufficient, since $U_0 = T$ and $E \times_T U_0 = E$.
The base case is clear, since $U_N = V - \{0\}$ and $A^*(E \times_T U_N) = A^*(E \times_T V)/c_N(V)$, where c_N is the $N-$th Chern class relative to V (which is a T-characteristic class coming from E).
Fix now some $J \subseteq \{1, \dots, N\}$ with $|J| = p$ and let $\mathfrak{k}$ be the field of definition of $T_{\mathrm{Orb}(J)}$ (which, we recall, is the subfield of $\overline{k}$ corresponding to the stabilizer of J with respect to the action on $\{1, \dots, N\}$ of the Galois group of k).
We have $W_p = \coprod_{|J|=p} T_{\mathrm{Orb}(J)}$, and by the localization exact sequence

$$A^*(E \times_T W_p) \xrightarrow{i_*} A^*(E \times_T U_{p+1}) \to A^*(E \times_T U_p) \to 0$$

we see that we just need to prove that for each J the image of $A^*(E \times_T T_{\mathrm{Orb}(J)})$ in $A^*(E \times_T U_{p+1})$ lies in the image of $\mathcal{I}$.
Now observe that if V_J is the standard representation over $\mathfrak{k}$ of $T_{\mathrm{Orb}(J)}$ we have an open immersion $j : E \times_T T_{\mathrm{Orb}(J)} \hookrightarrow E_{\mathfrak{k}} \times_{T_{\mathfrak{k}}} V_J$.
Moreover we can see $T_{\mathrm{Orb}(J)}$ as an irreducible component of $T_{\mathrm{Orb}(J)} \times_k \mathrm{Spec}(\mathfrak{k})$, since $T_{\mathrm{Orb}(J)}$ is defined over $\mathfrak{k}$; so we have a closed immersion i' given by

$$E \times_T T_{\mathrm{Orb}(J)} \hookrightarrow E_{\mathfrak{k}} \times_{T_{\mathfrak{k}}} (T_{\mathrm{Orb}(J)} \times_k \mathrm{Spec}(\mathfrak{k})) \hookrightarrow (E \times_T U_{p+1}) \times_k \mathrm{Spec}(\mathfrak{k}).$$

In particular we can factorize the closed immersion $i : E \times_T T_{\mathrm{Orb}(J)} \hookrightarrow E \times_T U_{p+1}$ as the composition

$$E \times_T T_{\mathrm{Orb}(J)} \xhookrightarrow{i'} (E \times_T U_{p+1}) \times_k \mathrm{Spec}(\mathfrak{k}) \xrightarrow{\pi^{\mathfrak{k}}} E \times_T U_{p+1} \tag{1}$$

Now, there is a fiber diagram

$$\begin{array}{ccc} E \times_T T_{\mathrm{Orb}(J)} & \xhookrightarrow{i'} & (E \times_T U_{p+1}) \times_k \mathrm{Spec}(\mathfrak{k}) \\ \downarrow j & & \downarrow j' \\ E_{\mathfrak{k}} \times_{T_{\mathfrak{k}}} V_J & \xhookrightarrow{k} & E_{\mathfrak{k}} \times_{T_{\mathfrak{k}}} V_{\mathfrak{k}} \end{array}$$

where k is the closed immersion induced by $V_J \hookrightarrow V_{\mathfrak{k}}$ and the vertical rows are open immersions.
Clearly we have that $j^* : A^*(M_{\mathfrak{k}}) \simeq A^*(E_{\mathfrak{k}} \times_{T_{\mathfrak{k}}} V_J) \to A^*(E \times_T T_{\mathrm{Orb}(J)})$ is surjective, and that j'^* is also surjective (it is true for every open immersion). But $i'_* j^* = j'^* k_*$, so that if $x' \in A^*(M_{\mathfrak{k}})$ lifts $x \in A^*(E \times_T T_{\mathrm{Orb}(J)})$, we can apply the self-intersection formula to see that $x' \cdot x_J$ lifts the push-forward of x in $A^*((E \times_T U_{p+1}) \times_k \mathrm{Spec}(\mathfrak{k}))$. Here x_J denotes the characteristic class in $A^*(M_{\mathfrak{k}})$ corresponding to $x_J \in A^*_{(}T_{\mathfrak{k}})$, in the notation of the previous section. Indeed, the conormal bundle of $E_{\mathfrak{k}} \times_{T_{\mathfrak{k}}} V_J$ in $E_{\mathfrak{k}} \times_{T_{\mathfrak{k}}} V_{\mathfrak{k}}$ is induced exactly by the standard representation V_{J^c} of $T_{\mathrm{Orb}(J^c)}$. This is easily seen recalling that $T_{\mathrm{Orb}(J)} \times T_{\mathrm{Orb}(J^c)} = T_{\mathfrak{k}}$ (Lemma 1.2). Its higher Chern class is represented precisely by x_J.
From the factorization (1), it follows that $\pi^{\mathfrak{k}}_*(x' \cdot x_J)$ lifts $i_*(x)$ in $A^*(M)$. This clearly lies in $\mathcal{I}$, so we have our claim. □

4 Some remarks on other nice classes of tori

We can observe that the permutation tori $R_{E/k}\mathbb{G}_m$ are special groups, that is every $T-$torsor is Zariski-locally trivial: indeed by the main result (Theorem 1.1) of [9] it suffices to check that for every field extension $k \subseteq K$ with Galois group $\Gamma := \mathrm{Gal}(\overline{K} : K)$ we have $0 = H^1(K, K \otimes_k R_{E/k}\mathbb{G}_m) = H^1(K, R_{E\otimes_k K/K}\mathbb{G}_m) = H^1(\Gamma, (E \otimes_k \overline{K})^*)$; but $E \otimes_k \overline{K}$ as a $\Gamma-$module splits as a product of $\overline{K}$, so the latter group is 0 by Hilbert 90 theorem.
Let us briefly see what is the structure of A^*_T for such tori.
The ring $A^*_{T_{k'}}$ is easily seen (since $T_{k'}$ is a split torus) to be isomorphic to the polynomial ring $\mathbb{Z}[x_1, \ldots, x_N]$, where each of the x_i's corresponds to a character of $T_{k'}$.
Indeed Totaro defines A^*_G as the limit of $A^*(U/G)$ where U ranges over open sets of $G-$representations on which G acts freely; for the rest of the paper we will refer to these as *approximating varieties for* BG.
The space $BT_{k'}$ is then the "limit" of the quotients $(V_{k'} - \{0\})^{\otimes n}/T_{k'} \simeq (\mathbb{P}^{N-1}_{k'})^{\otimes n}$, whose Chow rings are generated exactly by the (image of) elements $x_1, \ldots, x_N$.
In particular we can identify the Chow ring $A^*_{T_{k'}}$ with the symmetric ring over the character module $\widehat{T}_{k'}$, which we will denote $S(\widehat{T}_{k'})$. We will call it *the ring of characters* of the torus.
Now, $BT \simeq BT_{k'}/\Gamma$ is approximated by the spaces $(\mathbb{P}^{N-1}_{k'})^{\otimes n}/\Gamma$, where Γ acts tautologically on Spec k' and permutes the factors $\mathbb{P}^{N-1}_k$ according to its action on V.
In his article [5], Karpenko considers the approximating varieties for the classifying space of a permutation torus. In particular he takes the tori $T = R_{k'/k}\mathbb{G}_{m,k}$ obtained by Weil restriction from the multiplicative group. These approximating varieties are thus equal to the Weil restrictions $R_{k'/k}\mathbb{P}^n$ (where as before k' is a splitting field for the permutation torus T).

He then shows how to calculate the class of these varieties in the ring of Chow motives (for a definition and the main properties of Grothendieck's theory of motives see [1]). In the ring of motives, setting $\mathrm{pt} = (\mathrm{Spec}\ L, \mathrm{Id})$ we have $\mathbb{P}^i_L = (\mathrm{pt}, 0) \oplus (\mathrm{pt}, 1) \cdots \oplus (\mathrm{pt}, i)$ ([1], ex. 6); in particular we have a decomposition for $\mathbb{P}^i_L \times \ldots \mathbb{P}^i_L$ in factors of the form (pt, k).
Karpenko determines the quotient of this by Γ; indeed, each factor of the above product is determined by a map $p : \{1, \ldots, n\} \to \{0, 1, \ldots, i\}$, and Γ permutes the factors corresponding to the $\Gamma-$orbits in the set of such maps. If S is an orbit, for each $p \in S$ the quantity $|p| = \sum_{k=1}^{n} p(k)$ is the same, and we call this sum $|S|$. Karpenko's result states that

PROPOSITION 4.1. *For each orbit S let $\mathfrak{b} \subseteq \Gamma$ be its stabilizer, and $L_S \subseteq L$ the corresponding subfield of L. The the motive of $R_{L/k}\mathbb{P}^i_L$ is equal to $\bigoplus_S (\mathrm{Spec}\ L_S, |S|)$.*

By using this it is immediate to conclude that

COROLLARY 4.2. *If T is the Weil restriction $T = R_{E/k}\mathbb{G}_{m,k'}$ and $n = E : k$ then the map $A^*_T \to \mathbb{Z}[x_1, \ldots, x_n]^\Gamma$ is an isomorphism.*

Actually with the results of Section 1 we could re-demonstrate this result.
Now, let us come back to special tori. We can characterize all the tori which are special (e.g. [4], Th. 5.1):

THEOREM 4.3. *An algebraic torus is special if and only if there exists a torus T_1 such that $T \times T_1$ is a permutation torus.*

Note that the sufficiency is clear, since in this case for any field extension $K : k$ we have $H^1(K, T) \oplus H^1(K, T_1) = H^1(K, T \times T_1) = 0$.
We can extend the result of COROLLARY 3.2 to the case where T is special:

PROPOSITION 4.4. *Let T be a special torus over an infinite field with splitting field k' and $S(\widehat{T_{k'}})$ be its ring of characters.*
If $\Gamma = \mathrm{Gal}(k' : k)$, then the map $A^*_T \to S(\widehat{T_{k'}})^\Gamma$ is an isomorphism.

Proof. Let T_1 be a torus such that $P := T \times T_1$ is quasi-split; since extending k' does not change the action of the Galois group on $S(\widehat{T_{k'}})$ we can assume that k' splits P, T_1. We have $BT \times BT_1 = BP$ (we mean that, for each classifying bundles $U/T, U_1/T_1$, the product $(U \times U_1)/(T \times T_1)$ is an approximating variety of BP).
The variety U_1/T_1 has a $k-$rational point (as k is infinite). In particular we have a regular immersion map $i : BT \to BP$ and a projection map $p : BP \to BT$ such that $p \circ i = \mathrm{Id}$, and $i^* \circ p^* = \mathrm{Id} : A^*_T \to A^*_T$. We conclude that $p^* : A^*_T \to A^*_P$ is injective and that $i^* : A^*_P \to A^*_T$ is surjective. Thus A^*_T does not contain torsion classes since A^*_P does not.
Now we want to show that the base change $A^*_T \to (A^*_{T_{k'}})^\Gamma$ (where k' is a splitting field for T and $\Gamma = \mathrm{Gal}(k' : k)$) is an isomorphism or precisely that,

passing to the limit, the image of $p^* : A^*_T \to A^*_P$ contains all the symmetric tensors coming from $S(\widehat{T}) \subset S(\widehat{P})$.
Consider the short exact sequence

$$0 \to R \to A^*_{P_{k'}} \xrightarrow{i^*} A^*_{T_{k'}} \to 0.$$

The maps i^*, p^* are $\Gamma-$equivariant, so the above sequence has an equivariant splitting and remains exact after passing to $\Gamma-$invariants.
Consider now the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q & \longrightarrow & A^*_P & \xrightarrow{i^*} & A^*_T & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \\ 0 & \longrightarrow & R^\Gamma & \longrightarrow & (A^*_{P_{k'}})^\Gamma & \xrightarrow{i^*} & (A^*_{T_{k'}})^\Gamma & \longrightarrow & 0 \end{array}$$

Since the middle vertical map is an isomorphism, we immediately conclude that the vertical map on the right is a surjection, as we wanted to see. □

This result also allows us to calculate the Chow ring of BT when T is an algebraic torus of the simplest kind apart from special tori, that is when there exists a resolution

$$0 \to T \to Q \to P \to 0$$

with P, Q special tori. Of course, without loss of generality we can assume that P is a permutation torus: if this is not the case then we can choose a special torus P' such that $P \oplus P'$ is permutation, and then

$$0 \to T \to Q \oplus P' \to P \oplus P' \to 0$$

is a resolution of the required kind.
Now, the point is that we have a map $BT \to BQ$ which is a $P-$torsor (to be precise: for every approximating torsor $U \to U/Q$ we have that T acts freely on U and $U/T \to U/Q$ is a $Q/T = P-$torsor).
In particular we can apply Theorem 2.1 and Corollary 3.2 to calculate A^*_T in terms of A^*_P, A^*_Q. From what we know so far we can infer that $A^*_P = S(\widehat{P})^\Gamma$, where Γ is the Galois group for some common splitting field over k of T, Q, P, and that $A^*_Q \to A^*_T$ is surjective.
The ideal $\mathcal{I} = \ker(A^*_Q \to A^*_T)$ can be easily calculated. Indeed by the previous theorem we have that $\mathcal{I}$ contains, for every subgroup $\Gamma' < \Gamma$ corresponding to a field extension L, the image of $(A^*_{P_L})^+ A^*_{Q_L} \to A^*_Q$ (where M^+ denotes the ideal of elements with positive degree in a graded ring). The push-forward from $A^*_{Q_L}$ to A^*_Q is just $\mathrm{Ind}^\Gamma_{\Gamma'}(-)$, where "$\mathrm{Ind}^\Gamma_{\Gamma'}$" means the sum over representatives of the right cosets of Γ/Γ'; this is well-defined since $A^*_{Q_L}$ is invariant under the action of Γ'. These elements are a set of generators, and we can conclude the following

COROLLARY 4.5. *If T is an algebraic torus with a resolution $0 \to T \to Q \to P \to 0$, Q special and P quasi-split, the ideal $\mathcal{I} = \ker(A_Q^* \twoheadrightarrow A_T^*)$ can be written as $\mathcal{I} = \sum_{\Gamma' < \Gamma} \mathrm{Ind}_{\Gamma'}^{\Gamma}((S(\widehat{P})^+)^{\Gamma'} S(\widehat{Q})^{\Gamma'})$.*

This comes almost trivially from THEOREM 2.1, since it implies that we have

$$\mathcal{I} = \sum_{\Gamma' < \Gamma} S(\widehat{Q})^{\Gamma} \cdot \mathrm{Ind}_{\Gamma'}^{\Gamma}((S(\widehat{P})^+)^{\Gamma'} S(\widehat{Q})^{\Gamma'}),$$

but for any $x \in S(\widehat{Q})^{\Gamma}$ and any $y \in (S(\widehat{P})^+)^{\Gamma'} S(\widehat{Q})^{\Gamma'}$ we have $x \cdot \mathrm{Ind}_{\Gamma'}^{\Gamma}(y) = \mathrm{Ind}_{\Gamma'}^{\Gamma}(x \cdot y)$.

Let $\mathcal{J} := S(\widehat{P})^+ S(\widehat{Q})$ be the kernel of the projection map $S(\widehat{Q}) \to S(\widehat{T})$; then we have a commutative square

$$\begin{array}{ccc} A_Q^* & \xrightarrow{\simeq} & S(\widehat{Q})^{\Gamma} \\ \downarrow & & \downarrow \\ A_T^* & \longrightarrow & S(\widehat{T})^{\Gamma} \end{array}$$

which gives us that $\mathrm{coker}(A_T^* \to S(\widehat{T}))^{\Gamma} = \mathrm{coker}(S(\widehat{Q})^{\Gamma} \to S(\widehat{T})^{\Gamma}) = H^1(\Gamma, \mathcal{J})$.
The last step follows from the exact sequence

$$S(\widehat{Q})^{\Gamma} \to S(\widehat{T})^{\Gamma} \to H^1(\Gamma, \mathcal{J}) \to H^1(\Gamma, S(\widehat{Q})) = 0$$

and the final identity is just Shapiro's lemma.
Finally $\ker(A_T^* \to S(\widehat{T})^{\Gamma}) = \ker(A_Q^*/\mathcal{I} \to S(\widehat{T})^{\Gamma}) = \mathcal{J}^{\Gamma}/\mathcal{I}$.
Let us summarize the above results:

PROPOSITION 4.6. *Let us be given a short exact sequence of tori $0 \to T \to Q \to P \to 0$, with Q special and P quasi-split. Then if ϕ is the map $A_T^* \to (A_{T_{k'}}^*)^{\Gamma}$ and $\mathcal{I}, \mathcal{J}$ are defined as above we have that*

$$\ker(\phi) = \mathcal{J}^{\Gamma}/\mathcal{I}$$

and

$$\mathrm{coker}(\phi) = H^1(\Gamma, \mathcal{I}).$$

By using these formulas we will be able to show that in some cases we have nonzero kernels and cokernels for the map $A_T^* \to S(\widehat{T})^{\Gamma}$.

5 TORSION AND COTORSION CLASSES

By using our Proposition 3.6 we will see how the kernel and cokernel of the map $A_T^* \to A_{T_{\overline{k}}}^{*\ \Gamma}$ behave.

Special tori are not the largest class for which the base change map is an isomorphism. A very easy example is that of norm one tori, that is the closed subschemes $T = \ker N_k^{k'} \colon R_{k'/k}\mathbb{G}_{m,k'} \to \mathbb{G}_{m,k}$, whose $k-$rational points correspond to the elements in k'^* with unitary norm over k. These are tori of the type considered at the end of the previous section, but let us nonetheless calculate directly their A_T^*:

Proposition 5.1. *If T is a norm one torus the base-change map $A_T^* \to S(\widehat{T})^\Gamma$ is an isomorphism.*

Proof. Let the character lattice of T be $(x_1\mathbb{Z} \oplus \cdots \oplus x_n\mathbb{Z})/(x_1 + \cdots + x_n)$. If $P = R_{k'/k}\mathbb{G}_{m,k'}$, then the norm map gives an isomorphism $P/T \simeq \mathbb{G}_{m,k}$; then BT is a $P/T \simeq \mathbb{G}_{m,k}-$bundle over BP, so his Chow ring is equal to A_P^*/c_1, where c_1 is the first Chern class of this bundle.
Let $R := \mathbb{Z}[x_1, \ldots, x_n]$ and $x := c_1 = x_1 + \cdots + x_n$. Then $A_P^* = R^\Gamma$, so $A_T^* = R^\Gamma/(x)$.
To conclude, we just note that the invariants of the character ring of T are exactly $(R/(x))^\Gamma = R^\Gamma/(x)$, as we can see from the exact sequence

$$R^\Gamma \xrightarrow{\cdot x} R^\Gamma \to \Big(\frac{R}{(x)}\Big)^\Gamma \to H^1(\Gamma, R) = 0$$

where the last group is zero by Shapiro's lemma. □

Now, by using our calculation of A_T^* for a torus T with a resolution by special tori $0 \to T \to Q \to P \to 0$ we give some example of tori with nonzero kernel and cokernel for $A_T^* \to S(\widehat{T})^\Gamma$.

Theorem 5.2. *There exists a torus T such that the base-change map has nonzero cokernel in degree 2.*

Proof. Let k, L be fields such that $\mathrm{Gal}(L : k) = S_n$ (the symmetric group of rank $n > 3$); we pick a torus with character lattice $\mathbb{Z}^n = \langle a_1, a_2, \ldots, a_n\rangle$ equipped with an action of the Galois group $\Gamma = S_n$ so that $\sigma \in S_n$ sends each element of the basis $(a_1, \ldots, a_n)$ to $\mathrm{sgn}(\sigma)a_{\sigma(i)}$ (where $\sigma(-)$ denotes the tautological action on a n-element set).
This has a resolution $0 \to \mathbb{Z}^n \to \mathbb{Z}^{2n} \to \widehat{T} \to 0$; the middle term $\mathbb{Z}^{2n}$ is generated by elements a_i^+, a_i^- for $i = 1, \ldots, n$ with a S_n-action $\sigma(a_i^\pm) = a_{\sigma(i)}^{\pm\mathrm{sgn}(\sigma)}$, and the projection to $\widehat{T}$ is just $a_i^\pm \to \pm a_i$.
The kernel of this projection is the $n-$dimensional lattice generated by the $a_i^+ + a_i^-$, with the S_n-action inherited by its inclusion in $\mathbb{Z}^{2n}$.
This resolution corresponds to a reverse resolution $0 \to T \to Q \to P \to 0$, where P, Q are permutation tori of dimension $n, 2n$ respectively.
With the notations of the previous section we claim that $H^1(S_n, \mathcal{J})$ is nonzero, and in particular the map $A_T^* \to S(\widehat{T})^\Gamma$ has a nonzero cokernel.

This can be seen very easily at first glance: indeed the element $x = \sum_{i<j} a_i a_j \in S(\widehat{T})^{S_n}$ cannot be lifted to any element of $S(\widehat{Q})^{S_n}$; on the other hand $2x$ can be lifted to $\sum_{i<j}(a_i^+ a_j^+ + a_i^- a_j^-) \in S(\widehat{Q})^{S_n}$.

In order to see it, let us calculate $S(\widehat{Q})_2^{S_n}$ (where the subscript labels the degree); the symmetric group permutes the monomials $a_i^\pm a_j^\pm$ which are a basis of $S(\widehat{Q})_2$, so an invariant element containing a monomial $n_{ij} a_i^\pm a_j^\pm$ must also contain all the monomials $n_{ij} a_{\sigma(i)}^{\pm\mathrm{sgn}(i)} a_{\sigma(j)}^{\pm\mathrm{sgn}(\sigma)}$. However if $n > 3$ then S_n acts transitively on the ordered pairs (i,j), and moreover for every (i,j) there is an odd permutation leaving the ordered pair fixed (just take a trasposition (a,b) with $\{a,b\}$ disjoint from $\{i,j\}$). We conclude that all the invariants are of the form $\sum_i((a_i^\pm)^2 + (a_i^\mp)^2)$ or $\sum_i a_i^+ a_i^-$ or $\sum_{i<j}(a_i^{\epsilon_1} a_j^{\epsilon_2} + a_i^{-\epsilon_1} a_j^{-\epsilon_2})$ with $\epsilon_1, \epsilon_2 \in \{-1, 1\}$. Thus the image of $S(\widehat{Q})_2^{S_n}$ in $S(\widehat{T})_2^{S_n}$ consists of elements of the form $a\sum_i a_i^2 + 2b\sum_{i<j} a_i a_j$ $(a, b \in \mathbb{Z})$, and none of these can be x. □

Now let us investigate more closely the problem of the existence of torsion classes in A_T^*. This is quite harder than the analogous problem on cotorsion, but we can nonetheless build an example.

THEOREM 5.3. *There exists an algebraic torus T such that the base-change map has nonzero kernel in degree* 3.

Proof. Let us fix a field extension $L : k$ with Galois group $\Gamma = Q_8$, the group of quaternions. We recall that it is a group of order 8 generated by elements $-1, i, j, k$ subject to the relations $(-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1$.
The group Q_8 has a natural action on $\mathbb{Z}^4$, and we can consider it as the character lattice of a $4-$dimensional torus T. There is a surjection $\mathbb{Z}^8 \to \widehat{T}$, where Q_8 acts on $\mathbb{Z}^8$ as in its regular representation (or, more concretely, we identify a basis of $\mathbb{Z}^8$ with the eight elements of Q_8, which has a tautological action on them). We will call these elements $e, e', x, x', y, y', z, z'$ (corresponding respectively to $1, -1, i, -i, j, -j, k, -k$). This defines a permutation torus Q and the kernel of $\widehat{Q} \to \widehat{T}$ is the sublattice of $\widehat{Q}$ generated by $e+e', x+x', y+y', z+z'$, on which Q_8 has a permutation action induced by $Q_8/\{1,-1\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. It gives a permutation torus P.
Let us consider the element

$$w = \mathrm{Orb}_{Q_8}(xyz) = xyz + x'y'z' + e'zy' + ez'y + z'e'x + zex' + yx'e' + y'xe$$

in $S(\widehat{Q})^{Q_8}$, which lies in the kernel of $S(\widehat{Q}) \to S(\widehat{T})$. We claim that this element does not lie in the ideal $\mathcal{I}$.
First of all we prove that $\mathcal{I}_3$ (the set of elements in degree 3) is generated (additively) by $\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{Q_8}((S(\widehat{P})^+)^{\mathbb{Z}/2\mathbb{Z}} S(\widehat{Q})^{\mathbb{Z}/2\mathbb{Z}})_3$, where $\mathbb{Z}/2\mathbb{Z} = \{-1, 1\}$.
We begin by enumerating these elements. We note that $S(\widehat{P})$ is already invariant by the action of $\{-1, 1\}$. The elements of $S(\widehat{Q})_1^{\mathbb{Z}/2\mathbb{Z}}$ are just those coming

from $\widehat{P}$, while the elements of $S(\widehat{Q})_2^{\mathbb{Z}/2\mathbb{Z}}$ are generated by ee', xx', yy', zz' and the polynomials of the form $ab + a'b', ab' + a'b, a^2 + a'^2$, where $a \neq b$ are in $\{e, i, j, k\}$.

Thus there are elements of the form $\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{Q_8}(S(\widehat{P})_3)$ (first type) and elements of the form $\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{Q_8}((a+a')y)$ where $a \in \{e, x, y, z\}$ and $y \in S(\widehat{Q})_2^{\mathbb{Z}/2\mathbb{Z}}$ (second type). Actually, any element of the first type is also a sum of elements of the second type, hence we will restrict our attention to the latter.

Let $\mathcal{L}$ be this group of degree 3 elements; we have to verify that $\mathcal{I}_3 = \mathcal{L}$.

Let us check that these elements generate each $\mathrm{Ind}_{\Gamma'}^{Q_8}((S(\widehat{P})^+)^{\Gamma'} S(\widehat{Q})^{\Gamma'})_3$ with $\Gamma' < Q_8$.

Let $\Gamma' = Q_8$. We note that all the elements of $(S(\widehat{P})^+)_{\leq 3}^{Q_8}$ in degree $1, 3$ are $\mathbb{Z}/2\mathbb{Z}$-induced elements from $S(\widehat{P})^+$. Indeed the three elements of order two in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ send $e + e'$ to $x + x', y + y', z + z'$ respectively, so that the full action on the basis $\widehat{P}$ is transitive; in particular an invariant monomial in $e+e', x+x', y+y', z+z'$ must contain all of them and thus must have degree at least four. Moreover, if a monomial is invariant under the action of one of the three generators, suppose without loss of generality that this generator swaps $e + e', x + x'$ (as well as $y + y', z + z'$): then the monomial must have factors of the form $(e + e')(x + x')$ and $(y + y')(z + z')$; the only possibility is that it has degree 2.

Now, every element of the form $x \cdot y$ with $x \in (S(\widehat{P})^+)_1^{Q_8}, y \in S(\widehat{Q})_2^{Q_8}$ or $x \in (S(\widehat{P})^+)_3^{Q_8}, y \in \mathbb{Z}$ must be of the form $\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{Q_8}(x'y) \in \mathcal{L}$ with $x' \in S(\widehat{P})^+$, since x is induced by some x'.

Finally, an element of $S(\widehat{Q})_1^{Q_8}$ must be a multiple of $e+e'+x+x'+y+y'+z+z'$, so every element of $(S(\widehat{P})^+)_2^{Q_8} \cdot S(\widehat{Q})_1^{Q_8}$ is a sum of elements of the previous form. This concludes the case $\Gamma' = Q_8$.

Suppose $\Gamma' = \mathbb{Z}/4\mathbb{Z} = \{1, -1, i, -i\}$ (or some other subgroup obtained swapping i with j or k, which can be checked exactly in the same way).

This case is similar to the preceding one. Indeed in degree 1 the $\Gamma'-$invariants of $S(\widehat{P})^+$ are $e+e'+x+x'$ and $y+y'+z+z'$, while in degree 3 the invariants are $(e + e')(x + x')(y + y' + z + z')$ and $(e + e' + x + x')(y + y')(z + z')$, which are all $\{-1, 1\}$-induced from $S(\widehat{P})^+$. So every element $x \cdot y$ with $x \in (S(\widehat{P})^+)_1^{\Gamma'}, y \in S(\widehat{Q})_2^{\Gamma'}$ or $x \in (S(\widehat{P})^+)_3^{\Gamma'}, y \in \mathbb{Z}$ is $\mathbb{Z}/2\mathbb{Z}$-induced from an element of $S(\widehat{P})^+ S(\widehat{Q})^{\Gamma'}$. This implies that $\mathrm{Ind}_{\Gamma'}^{Q_8}(xy)$ is $\mathbb{Z}/2\mathbb{Z}$-induced from $S(\widehat{P})^+ S(\widehat{Q})^{\Gamma'}$.

Finally every element of $(S(\widehat{P})^+)_2^{\Gamma'} \cdot S(\widehat{Q})_1^{\Gamma'}$ is of the form above, since $S(\widehat{Q})_1^{\Gamma'}$ is generated by $e + e' + x + x'$ and $y + y' + z + z'$.

If $\Gamma' = \{-1, 1\}$ the claim is tautological, so we are left with checking the case $\Gamma' = \{1\}$. This is almost trivial: indeed $\mathrm{Ind}_1^{\mathbb{Z}/2\mathbb{Z}}(S(\widehat{P})^+ S(\widehat{Q})) \subseteq S(\widehat{P})^+ S(\widehat{Q})^{\mathbb{Z}/2\mathbb{Z}}$, since $S(\widehat{P})^+$ is $\mathbb{Z}/2\mathbb{Z}$-invariant. In particular

$$\mathrm{Ind}_1^{Q_8}(S(\widehat{P})^+ S(\widehat{Q})) = \mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{Q_8}(\mathrm{Ind}_1^{\mathbb{Z}/2\mathbb{Z}}(S(\widehat{P})^+ S(\widehat{Q}))) \subseteq \mathcal{L}.$$

Now we can finally prove that $w \notin \mathcal{L}$. Note that we can write $\mathcal{L} = A \oplus B$, where A is generated by $\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{Q_8}((a+a')y)$ with $a \in \{e,x,y,z\}$ and $y \in S(\widehat{Q})_2^{\mathbb{Z}/2\mathbb{Z}}$ is an element such that either a appears in some of its monomials or y is of the form $bb', b^2 + b'^2$; then we have B, generated by $\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{Q_8}((a+a')y)$, where $a \in \{e,x,y,z\}$ and $y \in S(\widehat{Q})_2^{\mathbb{Z}/2\mathbb{Z}}$ is of the form $pq + p'q'$ or $pq' + p'q$, with $p, q \in \{e,x,y,z\}\backslash\{a\}$ distinct from each other. In other words A is generated by the elements such that every monomial has factors intersecting at most two of the couples $\{e,e'\},\{x,x'\},\{y,y'\},\{z,z'\}$; similarly the elements of B have monomials with factors in distinct couples.
Clearly if $w \in \mathcal{L}$ then $w \in B$. However we immediately see that the sum of the coefficients of each generator of B is 16, and so the sum of the coefficients of w should be a multiple of 16. This is obviously not true, since the sum is 8. This gives us our sought torsion class.
Let us finally note that $2w \in \mathcal{L}$, so that the class has order 2. Indeed

$$\begin{aligned} 2w &= 2 \cdot \mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{Q_8}(xyz + x'y'z') \\ &= 2 \cdot \mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{Q_8}(yz(x+x') - x'z(y+y') + x'y'(z+z')) \\ &= \mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{Q_8}((x+x')(yz+y'z') - (y+y')(x'z+xz') + (z+z')(xy+x'y')) \end{aligned}$$

□

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