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Differentiable Functions Equivalent to Analytic Functions

By

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1. Let f, g be real-valued functions of class C^∞ in $\mathbf{R}^1$. Functions f, g are called *equivalent* if there exists a diffeomorphism (of class C^∞) τ of $\mathbf{R}^1$ such that $f\circ\tau=g$. The main object of this paper is to show under what conditions a function is equivalent to an analytic function (Theorem 1).

In the case of polynomials, the corresponding result is proved in Thom [1]. The method of our proof is analogous to that in [1], and our Lemma 3,4 correspond to Theorem R in [1].

Theorem 2 refines Mittag-Lefler's theorem in the real case.

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2. A function is called *flat* at a point a if for each $n\geqq 0$ the n-th derived function $f^{(n)}$ vanishes at a.

Theorem 1. *A C^∞-function f (not constant) is equivalent to an analytic function if and only if the derived function f' is nowhere flat.*

If f' is nowhere flat, then we can see by Rolle's theorem that the set of critical points of f (i.e. the set of points where f' vanishes) has no accumulating points. Let $\{a_n\}$ denote the set. Adding regular points to the set (if necessary), we can assume $\{a_n\}$ satisfies the following conditions

(1) $a_n<a_{n+1}$,

(2) $a_n\to\infty(n\to\infty)\qquad a_n\to-\infty(n\to-\infty)$,

(3) $a_{-1}<0<a_0$.

For each integer n we define $k(f, n)$ the least non-negative integer $k-1$

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such that the k-th derived function $f^{(k)}$ does not vanish at a_n, and put

$$b(f,n)=\begin{cases}(-1)^{\sum_{l=0}^{n} k(f,l)}(f(a_{n+1})-f(a_n)) & n\geqq 0\\ f(a_0)-f(a_{-1}) & n=-1\\ (-1)^{\sum_{l=n+1}^{-1} k(f,l)}(f(a_{n+1})-f(a_n)) & n\leqq -2.\end{cases}$$

Then $b(f, n)$ have all the same sign.

For the proof we need the following lemmas.

Lemma 1. *Let c, d be real numbers $(0<c<d)$ and h a real-valued continuous function in $(-\infty, c]\cup[d, \infty)$. Then there is an entire holomorphic function ϕ in the complex plane C^1 which satisfies the following conditions*:

(1) *the restriction of ϕ on the real axis is real-valued,*

(2) $\phi(x)\leqq h(x)$ *on* $(-\infty, c]\cup[d, \infty)$,

$\phi(x)\geqq 0$ *on* $\left[\frac{2}{3}c+\frac{1}{3}d,\ \frac{1}{3}c+\frac{2}{3}d\right]$,

(3) $\operatorname{Re}\phi(z)\leqq h(c)$ *on* $|z|\leqq c$.

Proof. It is enough to prove the lemma for a function h' such that $h'\leqq h$. So, from the first we can assume

$$h=\begin{cases}K_n+1 & \text{on } [nd, (n+1)d) \quad \text{for } n\neq 0, -1\ (K_n<0),\\ K+1 & \text{on } [-d, c] \quad (K<0).\end{cases}$$

We put

$$\phi_0(z)=\frac{-K+1}{d-c}(3z-2c-d)+1,$$

$$\phi_n(z)=K_n e^{l_n\{-z+(n+1)d\}} \qquad n\leqq -2,$$

(l_n are taken large enough so that $|\phi_n(z)|\leqq 2^{-n}$ on $|z|\leqq \max(\frac{-n-1}{2}d, c)$)

$$\phi_n(z)=(K_n+\frac{K-1}{d-c}(3nd+2d-2c))e^{l_n(z-nd)} \qquad n\geqq 1,$$

(l_n are taken large enough so that $|\phi_n(z)| \leqq 2^{-n}$ on $|z| \leqq \max(\frac{nd}{2}, c)$).
Then for any compact set $K(\subset \mathbf{C}^1)$, $\sum_{n\neq -1} \phi_n$ converges uniformly on K, so $\phi = \sum_{n\neq -1} \phi_n$ is holomorphic. It is easily seen that ϕ satisfies the conditions in the lemma.

Lemma 2. *Let c, d be real numbers $(0<c<d)$ and h a positive continuous function in $(-\infty, c] \cup \left[\frac{2}{3}c+\frac{1}{3}d, \frac{1}{3}c+\frac{2}{3}d\right] \cup [d, \infty)$. Then there is an entire holomorphic function ϕ in $\mathbf{C}^1$ which satisfies the following conditions.*

(1) *the restriction of ϕ on the real axis is real- and positive-valued,*

(2) $\phi(x) \leqq h(x)$ *on* $(-\infty, c] \cup [d, \infty)$,

$\phi(x) \geqq h(x)$ *on* $\left[\frac{2}{3}c+\frac{1}{3}d, \frac{1}{3}c+\frac{2}{3}d\right]$,

(3) $|\phi(z)| \leqq h(c)$ *on* $|z| \leqq c$.

Proof. Applying Lemma 1 to

$$\log h(x) - \log \sup_{x \in \left[\frac{2}{3}c+\frac{1}{3}d,\ \frac{c}{3}+\frac{2}{3}d\right]} h(x) ,$$

we easily prove this lemma.

Lemma 3. *For any real numbers $a_n \neq 0$ (such that $a_n < a_{n+1}$, $a_n \to \infty (n\to\infty)$, and $a_n \to -\infty (n \to -\infty)$), non-negative integers k_n, and postive numbers b_n, there is an entire holomorphic function g in $\mathbf{C}^1$ which satisfies the following conditions.*

(0) *the restriction of g on the real axis is real-valued,*

(1) *the set of critical points of g in the real axis is contained in the sequence $\{a_n\}$,*

(2) $k(g, n) = k_n$ *where* $k(g, n)$ *means* $k(g|\mathbf{R}, n)$,

(3) $0 < b(g, n) \leqq b_n$ *for* $n \neq 0$

$b(g, 0) = b_0$ *for* $n = 0$,

where $b(g, n)$ means $b(g|\mathbf{R}, n)$.

Proof. We put

$$H_n(z) = \exp\left[k_n\left\{\left(\frac{z}{a_n}\right) + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{|n|}\left(\frac{z}{a_n}\right)^{|n|}\right\}\right], \text{ for each } n,$$

$$G(x)=\int_0^x \prod_{n=-\infty}^{\infty}\left(1-\frac{z}{a_n}\right)^{k_n} H_n(z)\,dz,$$

then by Mittag-Lefler's theorem $G(x)$ is an entire holomorphic function in $\mathbf{C}^1$ which satisfies the conditions (1), (2) in the lemma. Let h be a positive valued continuous function such that

small enough on $(-\infty,\ a_0]\cup[a_1,\ \infty)$,

large enough on $\left[\frac{2}{3}a_0+\frac{1}{3}a_1,\ \frac{1}{3}a_0+\frac{2}{3}a_1\right]$.

If we apply Lemma 2 to this h, then we get an entire holomorphic function $\phi(z)$ such that

$$g(x)=\delta\int_0^x \phi(z)\prod_{n=-\infty}^{\infty}\left(1-\frac{z}{a_n}\right)^{k_n} H_n(z)\,dz \qquad (\delta \text{ is a constant})$$

satisfies the conditions (1), (2), (3) in the lemma.

Lemma 4. *In Lemma* 3, *we can take* g *so as to satisfy moreover the following condition,*

$$(4) \qquad b(g,\ n)=b_n \quad \textit{for each } n.$$

Proof. For each m we have constructed an entire holomorphic function g_m in $\mathbf{C}^1$ which verifies (1), (2) in Lemma 3 and a condition

$$(3)' \qquad \begin{aligned} 0<b(g_m,\ n)&\leqq 2^{-|m|-3}b_n \qquad \text{for} \quad m\neq n,\\ b(g_m,\ n)&=b_n \qquad\qquad\quad\ \ \text{for} \quad m=n. \end{aligned}$$

In doing this, if we let h take values small enough at a_m and a_{m+1}, from the condition 3 in Lemma 2 g_m is constructed so that for any compact set $K(\subset\mathbf{C}^1)$ $\sum c_m g_m$ (for any $0<c_m\leqq 1$) converges uniformly on K. Here we should remark that $\sum c_m g_m (0<c_m\leqq 1)$ satisfies the (1), (2) in Lemma 3. We put

$$g_1^\circ=\sum_{m=-\infty}^{\infty} g_m, \text{ then } 1<\frac{b(g_1^\circ,\ n)}{b_n}<\frac{3}{2};$$

$$g_2^\circ=g_1^\circ+\sum_{m=-\infty}^{\infty} c_{m,1}g_m, \left(c_{m,1}=\sup_n\frac{b(g_1^\circ,\ n)}{b_n}-\frac{b(g_1^\circ,\ m)}{b_m}\right)$$

then

$$\sup_m \frac{b(g_1^\circ, m)}{b_m} < \frac{b(g_2^\circ, n)}{b_n} < \frac{3}{2} \sup_m \frac{b(g_1^\circ, m)}{b_m} - \frac{1}{2};$$

generally we put

$$g_k^\circ = g_{k-1}^\circ + \sum_{m=-\infty}^{\infty} c_{m,k-1} g_m,$$

$$\left(c_{m,k-1} = \sup_n \frac{b(g_{k-1}^\circ, n)}{b_n} - \frac{b(g_{k-1}^\circ, m)}{b_m}\right)$$

then

$$\sup_m \frac{b(g_{k-1}^\circ, m)}{b_m} < \frac{b(g_k^\circ, n)}{b_n} < \frac{3}{2} \sup_m \frac{b(g_{k-1}^\circ, m)}{b_m} - \frac{1}{2} \sup_m \frac{b(g_{k-2}^\circ, m)}{b_m}.$$

From this we can see $0 \leqq c_{m,k} \leqq \left(\frac{1}{2}\right)^k$. If we put

$$c_m = 1 + \sum_{k=1}^{\infty} c_{m,k},$$

$$g^\circ = \sum_{m=-\infty}^{\infty} c_m g_m.$$

Then g° has the property

$$\frac{b(g^\circ, n)}{b_n} = \frac{b(g^\circ, m)}{b_m}, \text{ for any } n, m.$$

So $g = \frac{b_n}{b(g^\circ, n)} g^\circ$ satisfies the condition (4) in the lemma.

Proof of Theorem 1. The necessity of the condition is trivial. We shall prove its sufficiency. From Lemma 3, 4 there is an analytic function g in $\mathbf{R}^1$ such that

(1) the set of critical points of g is contained in the sequence $\{a_n\}$,

(2) $k(g, n) = k(f, n)$,

(3) $b(g, n) = b(f, n)$,

(4) (by adding a constant) $g(a_n) = f(a_n)$.

Let τ be a function in $\mathbf{R}^1$ such that

$$\tau(x) = f^{-1} g(x) \cap [a_n, a_{n+1}] \text{ on } [a_n, a_{n+1}].$$

Then τ is diffeomorphic on (a_n, a_{n+1}). Around a_n there are certain C^∞-

functions F, G such that

$$f(x)=\delta\{(x-a_n)F(x)\}^{k(f,n)+1}+f(a_n),\ (\delta=\text{a constant},\ F(a_n)>0)$$

$$g(x)=\delta\{(x-a_n)G(x)\}^{k(f,n)+1}+f(a_n).\ (G(a_n)>0)$$

From this, τ is locally diffeomorphic around a_n. So τ is a diffeomorphism of $\mathbf{R}^1$, and satisfies $f\circ\tau=g$.

3. Applications of the lemmas

Next lemmas result from the corresponding previous lemmas and proofs.

Lemma 2′. *Let c, d, e be real numbers $(0<c<d, 0<e)$ h a positive-valued continuous function ($h(x)>e$ on $[c, d]$). Then there is an entire holomorphic function ϕ in $\mathbf{C}^1$ which satisfies the following conditions*

(1) *the restriction of ϕ on the real axis is real- and positive-valued,*

(2) *$\phi(x)\leqq h(x)$ on the real axis,*

(3) *$\phi(x)\geqq e$ on $[c, d]$,*

(4) $|\phi(z)|\leqq h\left(\frac{c}{2}\right)$ *on* $|z|\leqq\frac{c}{2}$.

Lemma 2″. *Let c, d, e, h be the same ones as in Lemma 2′. Let $\{a_n\}$, $\{b'_n\}$ be sets of real numbers which satisfy*

(a) $a_n<a_{n+1}$, $a_{-1}<0<a_0$, $a_n\to\infty(n\to\infty)$, $a_n\to-\infty(n\to-\infty)$,

(b) $0<b'_n<h(a_n)$ *when* $a_0\geqq c$ *and* $-a_{-1}\geqq c$, *the set of numbers* $\dfrac{b'_n}{h(a_n)}$ *is bounded, otherwise,*

(c) $\{a_n\}\cap[c, d]=\phi$.

Then, there are an entire holomorphic function ϕ in $\mathbf{C}^1$ and a constant δ (>0) which satisfy the condition (1), (2), (3), (4) *in Lemma* 2′, *and a condition*

(5) $\phi(a_n)=b'_n$ *when* $a_0\geqq c$ and $-a_{-1}\geqq c$,

$\phi(a_n)=\delta b'_n$ *otherwise.*

Lemma 4′. *For positive numbers c_n, the g (in Lemma 4) can be* *chosen to satisfy moreover*

$$g^{(k_n+1)}(a_n)=\begin{cases}(-1)^{\sum_{i=0}^{n} k_i} c_n & n\geqq 0,\\ c_{-1} & n=-1,\\ (-1)^{\sum_{i=n+1}^{-1} k_i} c_n & n\leqq -2.\end{cases}$$

Lemma 4''. *On the same conditions as in Lemma* 4', *for any* $\delta>0$ *there are entire holomorphic functions* $g_{N,\varepsilon}(z)$ *in* $\mathbf{C}^1$ (N: *positive integer* $0<\varepsilon\leqq\varepsilon(N)$ *where* $\varepsilon(N)>0$ *is defined on positive integers*) *which satisfy the conditions* (1), (2) *in Lemma* 3 *and the following conditions*

(i) $$g_{N,\varepsilon}^{(k_n+1)}(a_n)=\begin{cases}(-1)^{\sum_{i=0}^{n} k_i} c_n & n\geqq N+1\\ (-1)^{\sum_{i=n+1}^{-1} k_i} c_n & n\leqq -N+1,\end{cases}$$

(ii) $$b(g_{N,\varepsilon},\ n)=\begin{cases}b_n & |n|\geqq N\\ \varepsilon b_n & |n|<N,\end{cases}$$

(iii) $$|g_{N,\varepsilon}(z)|<\delta \qquad |z|<\frac{\min(a_N,\ -a_{-N})}{2}.$$

Theorem 2. *Let* $\{a_n\}$, $\{l_n\}$, $\{c_n\}$ *be sets of real numbers which satisfy*

(a) $\{n\}=\{\textit{integer}\}$, *or* $\{n\geqq N\}$ *for some* N, *or* $\{n\leqq N\}$ *for some* N,

(b) $a_n<a_{n+1}$, $a_n\to\infty$ (as $n\to\infty$), $a_n\to-\infty$ (as $n\to-\infty$),

(c) l_n *are positive integers*,

(d) c_0, $(-1)^{\sum_{i=1}^{n} l_i} c_n (n>0)$, $(-1)^{\sum_{i=n+1}^{-1} l_i} c_n\ (n<0)$

have the same sign.

Then there is an analytic function f *in* $\mathbf{R}^1$ *such that*

(i) *the set of zero points of* f *is* $\{a_n\}$,

(ii) *for each* n, a_n *is a zero point of* l_n-th *order of* f,

(iii) *for each* $n, f^{(l_n)}(a_n)=c_n$.

Proof. We assume $\{n\}=\{\text{integer}\}$, otherwise we can prove in a similar way. If c_{-1} is positive, then we prove about $\{-c_n\}$. So we assume c_{-1} is negative, and $a_{-1}<0<a_0$. We put

$$a'_{2n}=a_n,\ a'_{2n+1}=\frac{a_n+a_{n+1}}{2}$$

$$k_{2n}=l_n-1,\ k_{2n+1}=1,$$

$$b_n=1$$

$$c'_{2n}=\begin{cases} -(-1)^{\sum_{i=0}^{n} l_i}\,c_n & n\geqq 0\\ -c_1 & n=-1\\ -(-1)^{\sum_{i=n+1}^{-1} l_i}\,c_n & n\leqq -2,\end{cases}$$

$$c'_{2n+1}=1,$$

from the above assumption, c'_n are positive. If we apply Lemma 4′ to $\{a'_n\}$, $\{k_n\}$, $\{c'_n\}$, then there is an analytic function g in $\mathbf{R}^1$ such that

(1) the set of critical points of g is contained in the sequence $\{a_n\}$,

(2) $k(g, n)=k_n$,

(3) $b(g, n)=1$,

$$(4)\quad g^{(k_n+1)}(a'_n)=\begin{cases} (-1)^{\sum_{i=0}^{n} k_i}\,c'_n & n\geqq 0\\ c'_{-1} & n=-1\\ (-1)^{\sum_{i=n+1}^{-1} k_i}\,c'_n & n\leqq -2.\end{cases}$$

From this it is easily seen that $g^{(l_n)}(a_n)=c_n$, $g(a'_{2n+2})=g(a'_{2n})$ and g is a monotone function on $[a'_n, a'_{n+1}]$. If we put $f=g-g(a_0)$ then the set of zero points of f is $\{a'_{2n}\}=\{a_n\}$, for each n a_n is a zero point of l_n-th order of f, and for each n $f^{(l_n)}(a_n)=c_n$.

Theorem 3. *Let* $\{a_n\}$ *be a set of real numbers (such that* $a_n < a_{n+1}$, $a_n \to \infty (n \to \infty)$, $a_n \to -\infty (n \to -\infty)$) $\{l_n\}$ *be a set of positive integers, and* $\{p_n(x)\}$ *be a set of polynomials (such that for each* n *the degree of* $p_n(x)$ *is less than* l_n). *Then there is an analytic function* f *in* $\mathbf{R}^1$ *such that for each* n, a_n *is a zero point of* l_n-th *order of* $f(x) - p_n(x)$.

Proof. In the same way as the proof of Theorem 2, if we define adequate values of derivatives (of the function which we want) at each point $\frac{a_n + a_{n+1}}{2}$, then we get an entire holomorphic function f_0 in $\mathbf{C}^1$ such that for each n $f_0(a_n) = p_n(a_n)$. We put

$$p_{n,1}(x) = p_n(x) - \{f_0(a_n) + (x - a_n) f_0'(a_n) + \cdots + (x - a_n)^{l_n} (l_n!)^{-1} f_0^{(l_n)}(a_n)\}$$

and defining adequate values of derivatives (of the function which we want) at one or two points of each (a_n, a_{n+1}), we get an entire holomorphic function f_1 in $\mathbf{C}^1$ such that

$$f_1(a_n) = p_{n,1}(a_n) = 0,$$

$$f_1'(a_n) = p_{n,1}'(a_n).$$

Repeating this and applying Lemma 4″, we get entire holomorphic functions f_m and polynomials $p_{n,m}$ which satisfy the following conditions

(1) $f_n^{(p)}(a_n) = p_{n,m}^{(p)}(a_n)$ for $p \leqq m$,

(2) $p_{n,m}(x) = p_{n,m-1}(x) - \{f_{m-1}(a_n) + \cdots + (x - a_n)^{l_n} (l_n!)^{-1} f_{m-1}^{(l_n)}(a_n)\}$

(3) for any compact set K $(\subset \mathbf{C}^1)$ $\sum f_n$ converges uniformly on K.

The function $\sum f_n$ is what we want.

Reference

[1] Thom, R., L'équivalence d'une fonction différentiable et d'un polynome, *Topology* **3**. Suppl. **2** (1965), 297-307.

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