Flat Cotorsion Modules over Noether Algebras

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Abstract. For a module-finite algebra over a commutative noetherian ring, we give a complete description of flat cotorsion modules in terms of prime ideals of the algebra, as a generalization of Enochs' result for a commutative noetherian ring. As a consequence, we show that pointwise Matlis duality gives a bijective correspondence between the isoclasses of indecomposable injective left modules and the isoclasses of indecomposable flat cotorsion right modules. This correspondence is an explicit realization of Herzog's homeomorphism induced from elementary duality of Ziegler spectra.

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1   Introduction

A right module M over a ring A is called *cotorsion* if $\operatorname{Ext}_A^1(F, M) = 0$ for
every flat right A-module F. This class of modules was originally studied in
the context of abelian groups (see [Fuc70, §54]), and Enochs [Eno84] extended
it to the current definition, in relation to the precedent work [Eno81] containing
the question whether flat covers exist for an arbitrary ring. This question, later
called the *flat cover conjecture*, was affirmatively solved by Bican, El Bashir,
and Enochs [BEBE01], showing that the class of flat modules and the class of
cotorsion modules form a complete cotorsion pair, i.e., given any module M,
there exists a surjection from a flat module to M with cotorsion kernel and an
injection from M into a cotorsion module with flat cokernel. This cotorsion
pair is called the *flat cotorsion pair*.
Like torsion pairs, cotorsion pairs are a general notion in abelian categories,
which initially appeared in [Sal79]. A *cotorsion pair* consists of two classes of
objects in an abelian category such that they are the orthogonal subcategory of
each other with respect to the first extension functor $\operatorname{Ext}^1(-, -)$. This notion
is closely related to abelian model structures ([Hov02], [Hov07]), and plays an
important role in homological algebra and representation theory (e.g., [AB89],
[AR91], [KS03], [AHMH09], [HJ19], [BBOS20], [ŠŠ20]), extending its scope to
exact and triangulated categories (e.g., [IY08], [Gil11], [Nak11], [NP19], [LN19],
[PZ20]).
Given a cotorsion pair, it is often important to consider the intersection of the
two classes, called the *core* of the cotorsion pair. For the flat cotorsion pair,
its core consists of all flat cotorsion modules, and they have nice homological
properties close to projective modules and injective modules. To explain such
aspects, let us pay our attention to complexes of modules.
Gillespie [Gil04] showed that the flat cotorsion pair induces two complete co-
torsion pairs in the category of complexes, and this fact, along with the work of
Bazzoni, Cortés-Izurdiaga, and Estrada [BCIE20], enables us to show that the
(unbounded) derived category of modules is equivalent to the homotopy cate-
gory of K-flat complexes of flat cotorsion modules; see [NT20, Appendix A]. In
fact, this remarkable equivalence can be regarded as a restriction of a bigger
equivalence, by identifying the derived category with the homotopy category of
K-projective complexes of projective modules. Indeed, Neeman [Nee08] proved
that the homotopy category of projective modules is equivalent to the pure
derived category of flat modules (in the sense of Murfet and Salarian [MS11]),
which turns out to be also equivalent to the homotopy category of flat co-

torsion modules as shown by Šťovíček [Što14, Corollary 5.8]; see also [NT20, Remark A.9].

If the ring is left coherent and all flat right modules have finite projective dimension, then the equivalence between the homotopy category of projective modules and that of flat cotorsion modules also induces an equivalence between their full subcategories consisting of totally acyclic complexes. Furthermore, the homotopy category of totally acyclic complexes of projective modules is equivalent to the stable category of Gorenstein-projective modules ([Buc86]; see also [Kra05, Proposition 7.2]), and the homotopy category of totally acyclic complexes of flat cotorsion modules is equivalent to the stable category of Gorenstein-flat cotorsion modules (studied in [Gil17]); see [CET20] for more details.

These facts motivate us to determine the structure of flat cotorsion modules. The aim of this paper is to give a noncommutative generalization of Enochs' structure theorem [Eno84] for flat cotorsion modules over a commutative noetherian ring R. Enochs showed that an R-module M is flat cotorsion if and only if M is isomorphic to

$$\prod_{\mathfrak{p}\in\operatorname{Spec}R} T_{\mathfrak{p}},$$

where each $T_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic completion of some free $R_{\mathfrak{p}}$-module, that is,

$$T_{\mathfrak{p}} = (R_{\mathfrak{p}}^{(B_{\mathfrak{p}})})_{\mathfrak{p}}^{\wedge} := \varprojlim_{n\geq 1}(R_{\mathfrak{p}}^{(B_{\mathfrak{p}})}\otimes_R R/\mathfrak{p}^n)$$

for a basis set $B_{\mathfrak{p}}$. The cardinality of $B_{\mathfrak{p}}$ for each $\mathfrak{p}\in\operatorname{Spec}R$ is determined by M. Enochs reached this formulation by using Matlis' result [Mat58] on the structure of injective R-modules and an isomorphism

$$T_{\mathfrak{p}} \cong \operatorname{Hom}_R(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{p})^{(B_{\mathfrak{p}})}), \tag{1.1}$$

where $E_R(R/\mathfrak{p})$ denotes the injective envelope of $R/\mathfrak{p}$. We generalize Enochs' structure theorem to Noether algebras, which are a simultaneous generalization of commutative noetherian rings and finite-dimensional algebras over a field. Noether algebras have been studied from various aspects (e.g., [AS81a, AS81b], [GN02], [IR08], [DK19], [IK20], [Kim20], [IK21]).

Let R be a commutative noetherian ring. A *Noether R-algebra* is a ring A together with a ring homomorphism $\varphi\colon R\to A$ such that the image of φ is contained in the center of A and A is finitely generated as an R-module. Denote by $\operatorname{Spec}A$ the set of prime (two-sided) ideals of A. The structure homomorphism $R\to A$ induces a canonical map $\operatorname{Spec}A\to\operatorname{Spec}R$ given by $P\mapsto\varphi^{-1}(P)$. For brevity, we write $P\cap R := \varphi^{-1}(P)$.

It is known that Matlis' result on injective R-modules is generalized to a Noether algebra A (see Theorem 2.22); there is a one-to-one correspondence

$$\operatorname{Spec}A \xrightarrow{\sim} \{\,\text{isoclasses of indecomposable injective right }A\text{-modules}\,\} \tag{1.2}$$

in which each $P \in \operatorname{Spec} A$ corresponds to $I_A(P)$, the unique indecomposable direct summand of the injective envelope of A/P. Using the injective module $I_{A^{\mathrm{op}}}(P)$ over the opposite ring A^{op}, we define

$$T_A(P) := \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})),$$

which is an indecomposable flat cotorsion right A-module (Remark 4.12) and also an indecomposable projective right module over $\widehat{A_{\mathfrak{p}}} := (A_{\mathfrak{p}})^{\wedge}_{\mathfrak{p}}$ (Proposition 5.2). The following is one of the main results of this paper:

THEOREM 1.1 (Theorem 6.1). *Let A be a Noether R-algebra. A right A-module M is flat cotorsion if and only if M is isomorphic to*

$$\prod_{P \in \operatorname{Spec} A} (T_A(P)^{(B_P)})^{\wedge}_{\mathfrak{p}}$$

for some family of sets $\{B_P\}_{P \in \operatorname{Spec} A}$, where $T_A(P)^{(B_P)}$ is the direct sum of B_P-indexed copies of $T_A(P)$ and $\mathfrak{p} := P \cap R$. The cardinality of each B_P is uniquely determined by M.

This theorem recovers Enochs' result because $T_R(\mathfrak{p}) \cong \widehat{R_{\mathfrak{p}}}$ and $(T_R(\mathfrak{p})^{(B_{\mathfrak{p}})})^{\wedge}_{\mathfrak{p}} \cong (R_{\mathfrak{p}}^{(B_{\mathfrak{p}})})^{\wedge}_{\mathfrak{p}}$ for each $\mathfrak{p} \in \operatorname{Spec} R$ and any set $B_{\mathfrak{p}}$. Moreover, each component of the direct product in Theorem 1.1 has a description

$$(T_A(P)^{(B_P)})^{\wedge}_{\mathfrak{p}} \cong \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B_P)}),$$

which recovers the isomorphism (1.1) (see Proposition 5.4).

As a consequence of Theorem 1.1, we obtain the following result:

COROLLARY 1.2 (Corollary 6.2). *Let A be a Noether R-algebra. Then there is a one-to-one correspondence*

$$\operatorname{Spec} A \xrightarrow{\sim} \{ \text{isoclasses of indecomposable flat cotorsion right } A\text{-modules} \}$$

given by $P \mapsto T_A(P)$.

We denote by inj_A (resp. flcot_A) the set of the isoclasses of indecomposable injective (resp. flat cotorsion) right A-modules. By (1.2) and Corollary 1.2, there is a bijection $\operatorname{inj}_{A^{\mathrm{op}}} \xrightarrow{\sim} \operatorname{flcot}_A$ given by $I_{A^{\mathrm{op}}}(P) \mapsto T_A(P)$. We interpret this bijection as a phenomenon on Ziegler spectra.

An exact sequence of right modules over a ring A is said to be *pure exact* if its exactness is preserved by the functor $- \otimes_A U$ for every left A-module U. A right A-module N is called *pure-injective* if the functor $\operatorname{Hom}_A(-, N)$ sends pure exact sequences to exact sequences. The isoclasses of indecomposable pure-injective right modules form a topological space Zg_A called the *Ziegler spectrum* of A. There is a bijection, called *elementary duality*, between the open subsets of Zg_A and those of $\operatorname{Zg}_{A^{\mathrm{op}}}$. Note that this does not mean that these topological spaces are homeomorphic in general. Our assumption that A is a Noether R-algebra ensures that inj_A and flcot_A are closed subsets of Zg_A. We endow inj_A and flcot_A with the topologies induced from Zg_A.

Theorem 1.3 (Theorems 8.8 and 8.14). *Let A be a Noether R-algebra. The bijection $\mathrm{inj}_{A^{\mathrm{op}}} \xrightarrow{\sim} \mathrm{flcot}_A$ given by $I_{A^{\mathrm{op}}}(P) \mapsto T_A(P)$ is a homeomorphism. The open sets of these topological spaces bijectively correspond to the specialization-closed subsets of $\mathrm{Spec}\, A$.*

It should be mentioned that Herzog [Her93] observed the existence of a homeomorphism $\mathrm{inj}_{A^{\mathrm{op}}} \xrightarrow{\sim} \mathrm{flcot}_A$ for a certain class of rings, which includes all left noetherian rings. The homeomorphism was obtained as a restriction of a bijection between certain points of Ziegler spectra, called *reflexive points*; for each reflexive point $N \in \mathrm{Zg}_{A^{\mathrm{op}}}$, the corresponding reflexive point $DN \in \mathrm{Zg}_A$ is determined by the property that the closure of N corresponds to the closure of DN by elementary duality (regarded as a bijection for closed subsets). The following result, together with Theorem 1.3, shows that our homeomorphism in Theorem 1.3 is an explicit realization of Herzog's homeomorphism for Noether algebras:

Corollary 1.4. *Let A be a Noether R-algebra. For each $P \in \mathrm{Spec}\, A$, the points $I_{A^{\mathrm{op}}}(P) \in \mathrm{Zg}_{A^{\mathrm{op}}}$ and $T_A(P) \in \mathrm{Zg}_A$ are the unique generic points in their closures, and these closed subsets correspond to each other by elementary duality.*

This paper is organized as follows. In section 2, we recall basic facts on Noether algebras, including those on the flat cotorsion pair and pure-injective modules. In section 3, we show that every flat cotorsion module over a Noether R-algebra A can be decomposed as a direct product of $\mathfrak{p}$-local $\mathfrak{p}$-complete modules for various $\mathfrak{p} \in \mathrm{Spec}\, R$. In section 4, we prove that each $\mathfrak{p}$-local $\mathfrak{p}$-complete flat (resp. $\mathfrak{p}$-local $\mathfrak{p}$-torsion injective) A-module is a flat cover (resp. injective envelope) of a semisimple $A_{\mathfrak{p}}$-module. Furthermore, we observe that the flat cover (resp. injective envelope) of a semisimple right $A_{\mathfrak{p}}$-module can be obtained by applying a variant of Matlis duality to the injective envelope (resp. flat cover) of a simple left $A_{\mathfrak{p}}$-module. In section 5, we show that every $\mathfrak{p}$-local $\mathfrak{p}$-complete flat A-module is cotorsion and such a module is characterized as the $\mathfrak{p}$-adic completion of a direct sum of indecomposable projective modules over $\widehat{A_{\mathfrak{p}}}$. In section 6, we complete the proofs of Theorem 1.1 and Corollary 1.2. In section 7, we give a result that realizes flat cotorsion A-modules as nontrivial flat covers and pure-injective (or cotorsion) envelopes. In section 8, we first recall some known results on Ziegler spectra and elementary duality, and then show that Herzog's homeomorphism applied to a Noether algebra coincides with the homeomorphism in Theorem 1.3. section A provides some basic facts on ideal-adic completion over Noether algebras, which are used throughout the paper.

2   Preliminaries

Throughout the paper, A will be a Noether R-algebra unless otherwise specified. That is, R is a commutative noetherian ring, A is a ring together with a ring homomorphism $R \to A$, called the *structure homomorphism*, whose image is contained in the center of A, and A is finitely generated as an R-module. It follows that A is a left and right noetherian ring. We denote by $\operatorname{Mod} A$ the category of right A-modules, and interpret $\operatorname{Mod} A^{\mathrm{op}}$ as the category of left A-modules, where A^{op} is the opposite ring of A.
In this section, we collect some known results, which we will use in later sections.

2.1   Cotorsion modules and pure-injective modules

A right A-module M is called *cotorsion* if $\operatorname{Ext}^1_A(F, M) = 0$ for all flat right A-modules F. A *flat cotorsion module* is a module that is flat and cotorsion. A short exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{Mod} A$ is said to be *pure exact* if it remains exact after applying $- \otimes_A U$ for every $U \in \operatorname{Mod} A^{\mathrm{op}}$. A right A-module N is called *pure-injective* if $\operatorname{Hom}_A(-, N)$ sends each pure exact sequence in $\operatorname{Mod} A$ to an exact sequence. Every injective module is pure-injective by definition.

Proposition 2.1. *Every pure-injective right A-module is cotorsion.*

Proof. See [EJ00, Lemma 5.3.23]. □

Proposition 2.2. *Fix an injective R-module E and consider the exact functor*

$$(-)^* := \operatorname{Hom}_R(-, E) \colon \operatorname{Mod} A \to \operatorname{Mod} A^{\mathrm{op}}.$$

For a right A-module M, the following hold:

(1) *M^* is pure-injective, and hence cotorsion.*

(2) *If E is an injective cogenerator, then the canonical morphism $M \to M^{**}$ is a pure monomorphism. In particular, this map splits if M is pure-injective.*

(3) *If M is flat, then M^* is injective. The converse holds if E is an injective cogenerator.*

(4) *If M is injective, then M^* is flat (cotorsion). The converse holds if E is an injective cogenerator.*

Proof. (1): See [EJ00, Proposition 5.3.7] or [Pre09, Proposition 4.3.29].
(2): See [EJ00, Proposition 5.3.9] or [GT12, Corollary 2.21(b)].
(3): See [EJ00, Theorem 3.2.9].
(4): See [EJ00, Theorem 3.2.16].   □

PROPOSITION 2.3. *Every flat cotorsion right A-module is pure-injective.*

Proof. This is [Xu96, Lemma 3.2.3], but we give a proof here as it will be used in the proof of Lemma 3.1.
Let E be an injective cogenerator in $\operatorname{Mod} R$, and put $(-)^* := \operatorname{Hom}_R(-, E)$. If M is a flat right A-module, then M^* is injective and M^{**} is flat by Proposition 2.2(3) and (4). Thus the cokernel of the pure monomorphism $M \to M^{**}$ in Proposition 2.2(2) is flat. If in addition M is cotorsion, then the pure monomorphism splits, so M is pure-injective by Proposition 2.2(1).   □

By Propositions 2.1 and 2.3, a flat cotorsion right A-module is nothing but a flat pure-injective right A-module.

REMARK 2.4. Although we are focusing on a Noether R-algebra, Proposition 2.1, and Proposition 2.5 and Lemma 2.7 below hold for an arbitrary ring (which is assumed to be associative and to have identity throughout this paper). Proposition 2.2(1)–(3) hold for a ring A together with a ring homomorphism from a commutative ring R to the center of A. The first claim of (4) holds if in addition A is right coherent, and the second claim holds if A is right noetherian; see [GT12, Corollary 2.18(b)]. Proposition 2.3 and Proposition 2.8 below hold for a left coherent ring.

2.2   COVERS AND ENVELOPES

Let $\mathcal{A}$ be an additive category and let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ closed under isomorphisms. A morphism $f \colon N \to M$ in $\mathcal{A}$ is called *right minimal* if every $g \in \operatorname{End}_{\mathcal{A}}(N)$ satisfying $fg = f$ is an isomorphism. A *left minimal* morphism is defined dually, that is, it is a morphism that is right minimal in the opposite category.
A morphism $f \colon X \to M$ in $\mathcal{A}$ is called an $\mathcal{X}$-*precover*, or a *right $\mathcal{X}$-approximation*, if $X \in \mathcal{X}$ and, for every $X' \in \mathcal{X}$, the induced map $\operatorname{Hom}_{\mathcal{A}}(X', X) \to \operatorname{Hom}_{\mathcal{A}}(X', M)$ is surjective. The latter condition means that every morphism from an object in $\mathcal{X}$ to M factors through f. An $\mathcal{X}$-*cover*, or a *right minimal $\mathcal{X}$-approximation*, is an $\mathcal{X}$-precover $X \to M$ that is right minimal. It is immediate that an $\mathcal{X}$-cover is unique up to isomorphism in the sense that, if $f \colon X \to M$ and $f' \colon X' \to M$ are $\mathcal{X}$-covers, then there exists an isomorphism $h \colon X' \to X$ such that $fh = f'$. An $\mathcal{X}$-*preenvelope* (or a *left $\mathcal{X}$-approximation*) and an $\mathcal{X}$-*envelope* (or a *left minimal $\mathcal{X}$-approximation*) are defined dually. If an $\mathcal{X}$-cover $X \to M$ (resp. an $\mathcal{X}$-envelope $M \to X$) exists,

then the object X is often called the $\mathcal{X}$-cover (resp. the $\mathcal{X}$-envelope) of M since the isoclass (i.e., isomorphism class) of X is uniquely determined by M.

Now let $\mathcal{A}$ be an abelian category. A *cotorsion pair* in $\mathcal{A}$ is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of $\mathcal{A}$ such that

$$\mathcal{X} = \{\, M \in \mathcal{A} \mid \operatorname{Ext}^1_{\mathcal{A}}(M, Y) = 0 \text{ for all } Y \in \mathcal{Y} \,\} \quad \text{and}$$
$$\mathcal{Y} = \{\, M \in \mathcal{A} \mid \operatorname{Ext}^1_{\mathcal{A}}(X, M) = 0 \text{ for all } X \in \mathcal{X} \,\}.$$

A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called *hereditary* if $\operatorname{Ext}^i_{\mathcal{A}}(X, Y) = 0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, and $i \geq 1$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called *complete* if, for every $M \in \mathcal{A}$, there exist exact sequences

$$0 \to Y \to X \to M \to 0 \quad \text{and} \quad 0 \to M \to Y' \to X' \to 0$$

with $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$. Morphisms $X \to M$ and $M \to Y'$ fitting into such exact sequences are often called a *special $\mathcal{X}$-precover* and a *special $\mathcal{Y}$-preenvelope*, respectively. It is easy to see that they are indeed an $\mathcal{X}$-precover and a $\mathcal{Y}$-preenvelope.

Denote by $\operatorname{Flat} A$ (resp. $\operatorname{Cot} A$) the full subcategory of $\operatorname{Mod} A$ consisting of all flat (resp. cotorsion) modules. If $\mathcal{X} = \operatorname{Flat} A$, then an $\mathcal{X}$-(pre)cover is called a *flat (pre)cover*, which is necessarily an epimorphism. If $\mathcal{Y} = \operatorname{Cot} A$, then a $\mathcal{Y}$-(pre)envelope is called a *cotorsion (pre)envelope*, which is necessarily a monomorphism. It is known that $(\operatorname{Flat} A, \operatorname{Cot} A)$ is a complete hereditary cotorsion pair in $\operatorname{Mod} A$ and every right A-module has a flat cover and a cotorsion envelope (see [Xu96, the proof of Proposition 3.1.2, Lemma 3.4.1, and Theorem 3.4.6] and [BEBE01]), where these facts are proved for an arbitrary ring. Given a right A-module M, we denote the flat cover of M by $F_A(M) \to M$ and the cotorsion envelope of M by $M \to C_A(M)$.

Projective (pre)covers, injective (pre)envelopes, and pure-injective (pre)envelopes can be defined in the same way. A projective precover is merely an epimorphism from a projective module, and an injective preenvelope is merely a monomorphism to an injective module. Recall that a monomorphism $f \colon L \to M$ is called an *essential monomorphism* if, for every morphism $h \colon M \to M'$ such that hf is a monomorphism, h is a monomorphism. An injective envelope is nothing but an essential monomorphism to an injective module. Pure-injective (pre)envelopes also have an alternative characterization, as in Proposition 2.5. A monomorphism $L \to M$ is called a *pure monomorphism* if it fits into a pure exact sequence. Moreover, a pure monomorphism $f \colon L \to M$ is called a *pure-essential monomorphism* if, for every morphism $h \colon M \to M'$ such that hf is a pure monomorphism, h is a pure monomorphism.

PROPOSITION 2.5. *Let $f \colon M \to N$ be a morphism in $\operatorname{Mod} A$ with N pure-injective.*

(1) *f is a pure-injective preenvelope if and only if f is a pure monomorphism.*

(2) f *is a pure-injective envelope if and only if f is a pure-essential monomorphism.*

Proof. (1): The "if" part is straightforward. To show the "only if" part, suppose that f is a pure-injective preenvelope. Let E be an injective cogenerator in $\operatorname{Mod} R$ and set $(-)^* := \operatorname{Hom}_R(-, E)$. Then we have the canonical pure monomorphism $g\colon M \to M^{**}$, where M^{**} is pure-injective; see Proposition 2.2(1) and (2). Then there is a morphism $h\colon N \to M^{**}$ such that $hf = g$. Since g is a pure monomorphism, it follows that f is a pure monomorphism.

(2): If f is a pure-essential monomorphism, then it is a pure-injective preenvelope by (1) and is also left minimal by the definition of pure-essentiality because every pure monomorphism from a pure-injective module splits. To show the "only if" part, suppose that f is a pure-injective envelope. Let $h\colon N \to N'$ be a morphism such that hf is a pure monomorphism. To observe that h is a pure monomorphism, it suffices to show that the composition of h with the canonical pure monomorphism $N' \to N'^{**}$ is a pure monomorphism. Therefore, replacing N'^{**} by N', we may assume that N' is pure-injective. Then the morphism $hf\colon M \to N'$ is a pure-injective preenvelope, but then h is a split monomorphism since f is a pure-injective envelope. $\square$

Remark 2.6. Our pure-essentiality is the same as that of [Pre09, p. 145], and this definition is, in general, strictly stronger than the classical definition, in which a pure monomorphism $f\colon L \to M$ is called a pure-essential monomorphism if, for every morphism $h\colon M \to M'$ such that hf is a pure monomorphism, h is a *monomorphism*. It has been known to experts that some of the proofs for the existence of pure-injective envelopes (pure-injective hulls) do not work due to this difference; see [GPGA00, p. 197, Remarks]. However, the notion of pure-injective envelopes is consistent in any case, and they do exist over any ring. For valid proofs on the existence of pure-injective envelopes, we refer the reader to [Pre09, Theorem 4.3.18] or [Dau94, §18-5]. The former uses a functor category, and the latter (based on the classical pure-essentiality) uses a cardinality argument. The definitions of pure-injective envelopes therein are given in different ways, but they both agree with ours defined as $\mathcal{X}$-envelopes for the class $\mathcal{X}$ of pure-injective modules; see [Pre09, Proposition 4.3.16] and [Dau94, Theorem 18-5.9].

As mentioned above, every right A-module M has a pure-injective envelope, which is unique up to isomorphism. It is denoted by $M \to H_A(M)$, following the notation in [Pre09, §4.3.3].

Flat precovers and cotorsion preenvelopes are not necessarily special, but flat covers and cotorsion envelopes are:

Lemma 2.7. *The kernel of a flat cover is cotorsion. The cokernel of a cotorsion envelope is flat.*

Proof. This is a consequence of Wakamatsu's lemma; see [Xu96, Lemmas 2.1.1 and 2.1.2] or [EJ00, Lemma 5.3.25 and Proposition 7.2.4]. $\square$

By this lemma and Proposition 2.3, a cotorsion envelope of a flat right A-module is a pure monomorphism into a pure-injective module, that is, a pure-injective preenvelope, by Proposition 2.5(1). It is a pure-injective envelope by the left minimality of the cotorsion envelope. Hence we have:

PROPOSITION 2.8. *For a flat right A-module, its cotorsion envelope and pure-injective envelope coincide.*

REMARK 2.9. Over a right artinian ring, all flat right (and also left) modules are projective ([AF92, Theorem 28.4 and Corollary 28.8]), and hence all right (and left) modules are cotorsion. So flat covers and projective covers are the same notion, and cotorsion envelopes are identity morphisms.

We observe when the scalar restriction functor along a ring homomorphism preserves various covers and envelopes. To this end, we make use of the following proposition:

PROPOSITION 2.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor that has a right adjoint $G\colon \mathcal{B} \to \mathcal{A}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be full subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively, that are closed under isomorphisms and satisfy $F(\mathcal{X}) \subseteq \mathcal{Y}$ and $G(\mathcal{Y}) \subseteq \mathcal{X}$. Then F sends $\mathcal{X}$-preenvelopes to $\mathcal{Y}$-preenvelopes and G sends $\mathcal{Y}$-precovers to $\mathcal{X}$-precovers.*

Proof. Straightforward. □

REMARK 2.11. Let $f\colon A \to B$ be a homomorphism of arbitrary rings. Since the scalar restriction functor $f_*\colon \operatorname{Mod} B \to \operatorname{Mod} A$ can be written as $\operatorname{Hom}_B(B, -)$, where B is regarded as an (A, B)-bimodule, the scalar extension functor $f^* = - \otimes_A B\colon \operatorname{Mod} A \to \operatorname{Mod} B$ is a left adjoint to f_*. Since f_* can also be written as $- \otimes_B B$, where B is regarded as a (B, A)-bimodule, the functor $f^! = \operatorname{Hom}_A(B, -)\colon \operatorname{Mod} A \to \operatorname{Mod} B$ is a right adjoint to f_*.
It is obvious that $f^* = - \otimes_A B$ and $f_* \cong - \otimes_B B$ preserve pure exact sequences, so it follows that their right adjoints f_* and $f^!$ preserve pure-injective modules. Hence, by Proposition 2.10 applied to the adjoint pair $(f_*, f^!)$, f_* preserves pure-injective preenvelopes. If B is flat as a left A-module, then the functor $f^* = - \otimes_A B$ is exact as well as f_*, so a similar argument, replacing pure exactness and pure-injectivity by exactness and injectivity, respectively, shows that f_* preserves injective preenvelopes.
If B is flat as a right A-module, then $f_* \cong - \otimes_B B$ preserves flat modules as well as its left adjoint f^*, so f_* preserves flat precovers by Proposition 2.10 applied to the adjoint pair (f^*, f_*). We observe that f_* preserves cotorsion preenvelopes under the same assumption. Let C be a cotorsion right B-module. For every flat right A-module F, take an exact sequence

$$0 \to F' \to P \to F \to 0 \tag{2.1}$$

in $\operatorname{Mod} A$, where P is projective. Since this sequence is pure exact, f^* applied to (2.1) is pure exact, so the flatness of $f^*(F)$ implies that $\operatorname{Hom}_B(f^*(-), C) \cong$

$\mathrm{Hom}_A(-, f_*(C))$ applied to (2.1) is exact since C is cotorsion. Therefore $\mathrm{Ext}^1_A(F, f_*(C)) = 0$, which means that f_* preserves cotorsion modules. If B is flat as a right A-module, f_* preserves flat modules, so a similar argument, using the adjoint pair $(f_*, f^!)$ instead of (f^*, f_*), shows that $f^!$ also preserves cotorsion modules. Therefore f_* preserves cotorsion preenvelopes by Proposition 2.10 applied to the adjoint pair $(f_*, f^!)$.

Recall that the functor $f_* \colon \mathrm{Mod}\, B \to \mathrm{Mod}\, A$ is fully faithful if and only if f is a *ring epimorphism* (i.e., an epimorphism in the category of rings); see [Ste75, Proposition XI.1.2]. When f is a ring epimorphism, a morphism g in $\mathrm{Mod}\, B$ is left minimal (resp. right minimal) if and only if so is $f_*(g)$ in $\mathrm{Mod}\, A$.

Consequently, if f is a ring epimorphism such that B is flat as a left and a right A-module, then f_* sends injective (pre)envelopes, pure-injective (pre)envelopes, cotorsion (pre)envelopes, and flat (pre)covers in $\mathrm{Mod}\, B$ to those in $\mathrm{Mod}\, A$.

2.3 Prime ideals and localization

An *ideal* means a two-sided ideal unless otherwise specified. A *prime ideal* of A is an ideal $P \subsetneq A$ such that, for any $a, b \in A$, the condition $aAb \subseteq P$ implies that $a \in P$ or $b \in P$. A *maximal ideal* of A is an ideal $Q \subsetneq A$ that is maximal among all ideals except A itself. Every maximal ideal is a prime ideal. Denote by $\mathrm{Spec}\, A$ (resp. $\mathrm{Max}\, A$) the set of all prime (resp. maximal) ideals of A.

Denote by $\varphi \colon R \to A$ the structure homomorphism of the Noether R-algebra A. This homomorphism induces a canonical map $\mathrm{Spec}\, A \to \mathrm{Spec}\, R$ which sends each $P \in \mathrm{Spec}\, A$ to its preimage $\varphi^{-1}(P)$; see Remark 2.17 below. Although R is not necessarily a subring of A, we write $P \cap R$ for $\varphi^{-1}(P)$.

LEMMA 2.12. *For every $P \in \mathrm{Spec}\, A$, we have $P \in \mathrm{Max}\, A$ if and only if $P \cap R \in \mathrm{Max}\, R$. In particular, the map $\mathrm{Spec}\, A \to \mathrm{Spec}\, R$ restricts to $\mathrm{Max}\, A \to \mathrm{Max}\, R$.*

Proof. This follows from [MR01, 10.2.12 and 10.2.13]. We give a more direct proof in Remark 2.16 below for the reader's convenience. □

For each $\mathfrak{p} \in \mathrm{Spec}\, R$, the $R_{\mathfrak{p}}$-module $A_{\mathfrak{p}}$, the localization of A at $\mathfrak{p}$ as an R-module, is naturally a Noether $R_{\mathfrak{p}}$-algebra. Moreover, for every right A-module M, the localization $M_{\mathfrak{p}}$ has a structure of a right $A_{\mathfrak{p}}$-module, and it holds that $M_{\mathfrak{p}} \cong M \otimes_R R_{\mathfrak{p}} \cong M \otimes_A A_{\mathfrak{p}}$. We say that M is $\mathfrak{p}$-*local* if the canonical A-homomorphism $M \to M_{\mathfrak{p}}$ is an isomorphism. In this case, M itself can be regarded as a right $A_{\mathfrak{p}}$-module.

REMARK 2.13. The localization functor $(-)_{\mathfrak{p}} \colon \mathrm{Mod}\, A \to \mathrm{Mod}\, A_{\mathfrak{p}}$ has a fully faithful right adjoint $\mathrm{Mod}\, A_{\mathfrak{p}} \to \mathrm{Mod}\, A$, which sends each $A_{\mathfrak{p}}$-module to itself but regarded as an A-module along the canonical ring homomorphism $A \to A_{\mathfrak{p}}$. In other words, the homomorphism $A \to A_{\mathfrak{p}}$ is a ring epimorphism (see Remark 2.11). The essential image of the right adjoint $\mathrm{Mod}\, A_{\mathfrak{p}} \to \mathrm{Mod}\, A$ consists of all $\mathfrak{p}$-local right A-modules.

PROPOSITION 2.14. *Let $\mathfrak{p} \in \mathrm{Spec}\, R$.*

(1) *There is an order-preserving bijection*

$$\{\, P \in \operatorname{Spec} A \mid P \cap R \subseteq \mathfrak{p} \,\} \xrightarrow{\sim} \operatorname{Spec} A_{\mathfrak{p}}$$

given by $P \mapsto P_{\mathfrak{p}} = PA_{\mathfrak{p}}$. *The inverse map is given by* $Q \mapsto f^{-1}(Q)$, *where* $f \colon A \to A_{\mathfrak{p}}$ *is the canonical ring homomorphism.*

(2) *The bijection in (1) restricts to a bijection*

$$\{\, P \in \operatorname{Spec} A \mid P \cap R = \mathfrak{p} \,\} \xrightarrow{\sim} \operatorname{Max} A_{\mathfrak{p}}.$$

Proof. (1): This follows from [MR01, 2.1.16, Proposition(vii)]. See also Remark 2.16 below.
(2): Let $P \in \operatorname{Spec} A$ such that $P \cap R \subseteq \mathfrak{p}$. Lemma 2.12 applied to the Noether $R_{\mathfrak{p}}$-algebra $A_{\mathfrak{p}}$ implies that $PA_{\mathfrak{p}} \in \operatorname{Max} A_{\mathfrak{p}}$ if and only if $PA_{\mathfrak{p}} \cap R_{\mathfrak{p}} \in \operatorname{Max} R_{\mathfrak{p}}$. Since $PA_{\mathfrak{p}} \cap R_{\mathfrak{p}} = (P \cap R)R_{\mathfrak{p}}$, the latter condition is equivalent to $P \cap R = \mathfrak{p}$. $\square$

For $\mathfrak{p} \in \operatorname{Spec} R$, the residue field at $\mathfrak{p}$ is denoted by $\kappa(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Note that $A \otimes_R \kappa(\mathfrak{p})$ is a *finite-dimensional* $\kappa(\mathfrak{p})$-algebra in the sense that the Noether $\kappa(\mathfrak{p})$-algebra $A \otimes_R \kappa(\mathfrak{p})$ is finite-dimensional as a $\kappa(\mathfrak{p})$-vector space. In particular, it is a left and right artinian ring.

PROPOSITION 2.15. *Let* $\mathfrak{p} \in \operatorname{Spec} R$. *There is a bijection*

$$\{\, P \in \operatorname{Spec} A \mid P \cap R = \mathfrak{p} \,\} \xrightarrow{\sim} \operatorname{Spec}(A \otimes_R \kappa(\mathfrak{p})) = \operatorname{Max}(A \otimes_R \kappa(\mathfrak{p}))$$

given by $P \mapsto P_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. *The inverse map is given by* $Q \mapsto f^{-1}(Q)$, *where* $f \colon A \to A \otimes_R \kappa(\mathfrak{p})$ *is the canonical ring homomorphism.*
Consequently, the fiber $\{\, P \in \operatorname{Spec} A \mid P \cap R = \mathfrak{p} \,\}$ *over each* $\mathfrak{p} \in \operatorname{Spec} R$ *is a (possibly empty) finite set.*

Proof. The bijection in Proposition 2.14(1) induces an injection

$$\{\, P \in \operatorname{Spec} A \mid P \cap R = \mathfrak{p} \,\} \to \{\, Q \in \operatorname{Spec} A_{\mathfrak{p}} \mid \mathfrak{p}A_{\mathfrak{p}} \subseteq Q \,\},$$

and an elementary argument shows that every $P \in \operatorname{Spec} A$ with $P \cap R \subseteq \mathfrak{p}$ and $\mathfrak{p}A_{\mathfrak{p}} \subseteq P_{\mathfrak{p}}$ satisfies $P \cap R = \mathfrak{p}$, so the above injection is bijective, and the right-hand side can naturally be identified with $\operatorname{Spec}(A \otimes_R \kappa(\mathfrak{p}))$ since $A \otimes_R \kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Thus we have the desired bijection. By [GW04, Theorem 3.4 and Proposition 4.19], $A \otimes_R \kappa(\mathfrak{p})$ has only finitely many prime ideals, which are all maximal. $\square$

If the structure homomorphism $R \to A$ is injective, then the induced map $\operatorname{Spec} A \to \operatorname{Spec} R$ is surjective, i.e., each fiber is nonempty ([MR01, 10.2.9, Theorem]).

Remark 2.16. Proposition 2.14(1) holds for a ring A together with a ring homomorphism from a commutative ring R to the center of A. In fact, it can be proved in a similar way to the case $A = R$ ([Mat89, p. 22, Theorem 4.1 and Example 2]); note that, if $P \in \operatorname{Spec} A$, $a \in A$, $s \in R \setminus (P \cap R)$, then $as \in P$ implies that $aAs \subseteq P$ and hence $a \in P$. Lemma 2.12, Proposition 2.14(2), and Proposition 2.15 also hold if in addition A is finitely generated as an R-module. We give here a proof of Lemma 2.12, which works in this setting.

The "only if" part of the lemma follows from [MR01, 10.2.10, Corollary(iii)], and it can also be proved as follows: Let $P \in \operatorname{Max} A$ and set $\mathfrak{p} := P \cap R$. Then we have an injection $R/\mathfrak{p} \hookrightarrow A/P$, so we may suppose $P = 0$ and $\mathfrak{p} = 0$, and hence R is a domain and A is a simple ring. Suppose that there is a prime ideal $0 \neq \mathfrak{q}$ of R. Localization of the injection $R \hookrightarrow A$ at $\mathfrak{q}$ yields an injection $R_{\mathfrak{q}} \hookrightarrow A_{\mathfrak{q}}$, where $A_{\mathfrak{q}}$ is a simple ring by Proposition 2.14(1). Since $A_{\mathfrak{q}}$ is a nonzero finitely generated $R_{\mathfrak{q}}$-module, $A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}$ is nonzero by Nakayama's lemma, so the ring $A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}$ contains a prime ideal. This means that $A_{\mathfrak{q}}$ contains a prime ideal Q with $\mathfrak{q}A_{\mathfrak{q}} \subseteq Q$, but $0 \neq \mathfrak{q}R_{\mathfrak{q}} \subseteq \mathfrak{q}A_{\mathfrak{q}}$, so this contradicts the fact that $A_{\mathfrak{q}}$ is simple.

To prove the "if" part, assume that $\mathfrak{p} := P \cap R \in \operatorname{Max} R$. Then $R/\mathfrak{p}$ is a field, and $A \otimes_R (R/\mathfrak{p}) = A/\mathfrak{p}A$ is a finite-dimensional $(R/\mathfrak{p})$-algebra. Since $\mathfrak{p}A \subseteq P$, we have a canonical surjective ring homomorphism $A/\mathfrak{p}A \to A/P$, and hence A/P is also a finite-dimensional $(R/\mathfrak{p})$-algebra. So all prime ideals of A/P are maximal ideals by [GW04, Proposition 4.19]. In particular, the zero ideal $0 = P/P$ of A/P is a maximal ideal. Therefore $P \in \operatorname{Max} A$.

Remark 2.17. In general, for a ring homomorphism $f\colon A \to B$ of noncommutative rings and a prime ideal Q of B, the ideal $f^{-1}(Q)$ of A is not necessarily a prime ideal; see [MR01, 10.2.3].

However, if we assume that B is a *centralizing* extension of $f(A)$ (cf. [MR01, 10.1.3]), that is, as a right (or equivalently, left) $f(A)$-module, B is generated by a (possibly infinite) subset $S \subseteq B$ such that every element of S commutes with every element of $f(A)$, then $f^{-1}(Q)$ is a prime ideal of A for every prime ideal Q of B (cf. [MR01, 10.2.4, Theorem]). The proof is straightforward. This assumption is satisfied if the homomorphism f is surjective or $f(A)$ is contained in the center of B.

2.4   Simple modules and injective modules

We will recall that (semi)simple modules and injective modules over a Noether R-algebra A are controlled by maximal ideals and prime ideals, respectively. First we assign a simple module to each prime ideal.

Let $P \in \operatorname{Spec} A$ and put $\mathfrak{p} := P \cap R$. The ring $A_{\mathfrak{p}}/P_{\mathfrak{p}}$ is a simple right artinian ring, and hence decomposes as a finite direct sum of copies of a simple right $A_{\mathfrak{p}}/P_{\mathfrak{p}}$-module, where the simple module is unique up to isomorphism ([Lam91, Theorems (3.3) and (3.10)]). We denote the simple right $A_{\mathfrak{p}}/P_{\mathfrak{p}}$-module by

$S_A(P)$ and its multiplicity in $A_{\mathfrak{p}}/P_{\mathfrak{p}}$ by n_P, that is,

$$A_{\mathfrak{p}}/P_{\mathfrak{p}} \cong S_A(P)^{n_P}. \tag{2.2}$$

By construction, $S_A(P) \cong S_{A_{\mathfrak{p}}}(P_{\mathfrak{p}})$ and $S_A(P)$ is also a simple right $A_{\mathfrak{p}}$-module. It is often regarded as a right A-module (which is not necessarily simple).

Denote by $\operatorname{rad} A$ the *Jacobson radical of A*, which is the intersection of all annihilators of simple right (or left) A-modules (or equivalently, the intersection of all maximal right (or left) ideals of A; see [GW04, Proposition 3.16]). In general, the annihilator of a simple module over an arbitrary ring is a prime ideal, and any maximal (two-sided) ideal is the annihilator of some simple module ([GW04, Proposition 3.15]). In particular, the Jacobson radical of a finite-dimensional algebra over a field (or more generally, a right artinian ring) equals to the intersection of all maximal ideals ([GW04, Corollary 4.16 and Proposition 4.19]). The following fact implies that the same characterization holds for a Noether R-algebra A:

THEOREM 2.18. *There is a bijection*

$$\operatorname{Max} A \xrightarrow{\sim} \{\, \text{isoclasses of simple right } A\text{-modules} \,\}$$

given by $P \mapsto S_A(P)$. The inverse map is given by $S \mapsto \operatorname{Ann}_A(S)$.

Proof. For a simple right A-module S, let $P := \operatorname{Ann}_A S$ and $\mathfrak{p} := P \cap R$. Then, by [GW04, Proposition 9.1(a) and Corollary 9.5], P is a maximal ideal of A and the right A-module A/P is a finite direct sum of copies of S. Since each $a \in R \setminus \mathfrak{p}$ does not annihilate S, it acts as an isomorphism on the simple A-module S. This means that S is $\mathfrak{p}$-local, and hence $A/P = A_{\mathfrak{p}}/P_{\mathfrak{p}}$ is a finite direct sum of copies of S. Therefore $S_A(P)$ is isomorphic to S by the definition of $S_A(P)$.

Let $Q \in \operatorname{Max} A$ and $\mathfrak{q} := Q \cap R$. Again by the definition of $S_A(Q)$, we have $\operatorname{Ann}_A S_A(Q) = \operatorname{Ann}_A(A_{\mathfrak{q}}/Q_{\mathfrak{q}}) = Q$. This completes the proof. □

It follows from Theorem 2.18 that

$$\operatorname{rad} A = \bigcap_{P \in \operatorname{Max} A} P. \tag{2.3}$$

Given $\mathfrak{p} \in \operatorname{Spec} R$, we have $\operatorname{Max} A_{\mathfrak{p}} = \{\, P_{\mathfrak{p}} \mid P \in \operatorname{Spec} A,\ P \cap R = \mathfrak{p} \,\}$ by Proposition 2.14. Thus (2.3) implies that

$$\operatorname{rad} A_{\mathfrak{p}} = \bigcap_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} P_{\mathfrak{p}}. \tag{2.4}$$

PROPOSITION 2.19. *For every $\mathfrak{p} \in \operatorname{Spec} R$, we have*

$$A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}} \cong \bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} S_A(P)^{n_P}$$

as right A-modules.

Proof. We have

$$A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}} \cong \prod_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} A_{\mathfrak{p}}/P_{\mathfrak{p}} \cong \bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} S_A(P)^{n_P},$$

where the first isomorphism of rings follows from (2.4) and the Chinese remainder theorem ([AF92, §7, Exercise 13]), and the second follows from Proposition 2.15 and (2.2). □

REMARK 2.20. By Proposition 2.14(2) and Proposition 2.15, the maximal ideals of $A_{\mathfrak{p}}$ naturally correspond to those of $\overline{A_{\mathfrak{p}}} := A \otimes_R \kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, and hence the canonical surjection $A_{\mathfrak{p}} \to \overline{A_{\mathfrak{p}}}$ induces the isomorphism

$$A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}} \xrightarrow{\sim} \overline{A_{\mathfrak{p}}}/\operatorname{rad} \overline{A_{\mathfrak{p}}}$$

of finite-dimensional $\kappa(\mathfrak{p})$-algebras.

Now we turn our attention to injective modules. The following remark will be used later:

REMARK 2.21. Let $\mathfrak{p} \in \operatorname{Spec} R$. Injective envelopes, pure-injective envelopes, cotorsion envelopes, and flat covers in $\operatorname{Mod} A_{\mathfrak{p}}$ are those in $\operatorname{Mod} A$; see Remarks 2.11 and 2.13. In particular, the full subcategory of $\operatorname{Mod} A$ formed by $\mathfrak{p}$-local modules is closed under taking such envelopes and covers.

Now we assign an indecomposable injective module to each prime ideal of A. Let $P \in \operatorname{Spec} A$ and put $\mathfrak{p} := P \cap R$. Take injective envelopes $A/P \to E_A(A/P)$ and $A_{\mathfrak{p}}/P_{\mathfrak{p}} \to E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/P_{\mathfrak{p}})$ in $\operatorname{Mod} A$ and $\operatorname{Mod} A_{\mathfrak{p}}$, respectively. As observed in Remark 2.21, $E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/P_{\mathfrak{p}}) \cong E_A(A_{\mathfrak{p}}/P_{\mathfrak{p}})$. The canonical A-homomorphism $A/P \to A_{\mathfrak{p}}/P_{\mathfrak{p}}$ is injective, and it extends to a monomorphism $E_A(A/P) \to E_A(A_{\mathfrak{p}}/P_{\mathfrak{p}})$, which splits. So $E_A(A/P)$ is $\mathfrak{p}$-local. Localizing the injective envelope $A/P \to E_A(A/P)$, we obtain an essential extension $A_{\mathfrak{p}}/P_{\mathfrak{p}} \to E_A(A/P)$ in $\operatorname{Mod} A$. Therefore

$$E_A(A/P) \cong E_A(A_{\mathfrak{p}}/P_{\mathfrak{p}}) \cong E_A(S_A(P))^{n_P}, \tag{2.5}$$

where the second isomorphism follows from (2.2). This fact is essentially observed in the proof of [GN02, Proposition 2.5.2].

We set

$$I_A(P) := E_A(S_A(P)),$$

which is an indecomposable injective right A-module. By construction, it holds that

$$I_A(P) \cong I_{A_{\mathfrak{p}}}(P_{\mathfrak{p}}), \tag{2.6}$$

so $I_A(P)$ is $\mathfrak{p}$-local. If $A = R$, then $P = \mathfrak{p}$, so $I_A(P) = E_R(R/\mathfrak{p})$ and $I_{A_{\mathfrak{p}}}(P_{\mathfrak{p}}) = E_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}))$.

Theorem 2.22 ([Gab62, Chapter V, Proposition 8]). *There is a bijection*

$$\operatorname{Spec} A \xrightarrow{\sim} \{ \text{ isoclasses of indecomposable injective right } A\text{-modules} \}$$

given by $P \mapsto I_A(P)$.

Proof. See [GW04, Lemma 5.14, Proposition 9.1(a), and Theorem 9.15]. □

2.5   Matlis duality

Completion and Matlis duality play a central role in the proof of our main results. Here, and also in section A, we collect some basic facts on these operations.

For an ideal $\mathfrak{a} \subseteq R$, define the $\mathfrak{a}$-*adic completion functor* $\Lambda^{\mathfrak{a}} = (-)^{\wedge}_{\mathfrak{a}} \colon \operatorname{Mod} A \to \operatorname{Mod} A$ by

$$\Lambda^{\mathfrak{a}} M = M^{\wedge}_{\mathfrak{a}} := \varprojlim_{n \geq 1} M/\mathfrak{a}^n M.$$

We say that a right A-module M is $\mathfrak{a}$-*adically complete* (or $\mathfrak{a}$-*complete* for short) if the canonical A-homomorphism $M \to M^{\wedge}_{\mathfrak{a}}$ is an isomorphism. In particular, $M^{\wedge}_{\mathfrak{a}}$ is $\mathfrak{a}$-complete (Proposition A.5). The $\mathfrak{a}$-adic completion $A^{\wedge}_{\mathfrak{a}}$ of A naturally has a ring structure, and the canonical map $A \to A^{\wedge}_{\mathfrak{a}}$ is a ring homomorphism. Moreover, for each right A-module M, $M^{\wedge}_{\mathfrak{a}}$ has a (unique) right $A^{\wedge}_{\mathfrak{a}}$-module structure that is compatible with the right A-module structure on $M^{\wedge}_{\mathfrak{a}}$ (see Remark A.11 and Proposition A.15). We often write $M^{\wedge}_{\mathfrak{a}}$ as $\widehat{M}$ when M is $\mathfrak{p}$-local and $\mathfrak{a} = \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec} R$. For example, $\widehat{A_{\mathfrak{p}}}$ means $(A_{\mathfrak{p}})^{\wedge}_{\mathfrak{p}}$.

If M is a finitely generated right A-module, then the canonical A-homomorphism $M \otimes_R R^{\wedge}_{\mathfrak{a}} \to M^{\wedge}_{\mathfrak{a}}$ is an isomorphism ([Mat89, Theorem 8.7]), so $M^{\wedge}_{\mathfrak{a}}$ is finitely generated as an $R^{\wedge}_{\mathfrak{a}}$-module. The structure homomorphism $R \to A$ induces a ring homomorphism $R^{\wedge}_{\mathfrak{a}} \to A \otimes_R R^{\wedge}_{\mathfrak{a}} \cong A^{\wedge}_{\mathfrak{a}}$ whose image is contained in the center, and $R^{\wedge}_{\mathfrak{a}}$ is a commutative noetherian ring ([Mat89, Theorem 8.12]). Thus $A^{\wedge}_{\mathfrak{a}}$ is a Noether $R^{\wedge}_{\mathfrak{a}}$-algebra. In particular, $\widehat{A_{\mathfrak{p}}}$ is a Noether $\widehat{R_{\mathfrak{p}}}$-algebra for each $\mathfrak{p} \in \operatorname{Spec} R$. Since $R^{\wedge}_{\mathfrak{a}}$ is flat over R ([Mat89, Theorem 8.8]) and $A \otimes_R R^{\wedge}_{\mathfrak{a}} \cong R^{\wedge}_{\mathfrak{a}} \otimes_R A$, $A^{\wedge}_{\mathfrak{a}}$ is flat as a left and ring A-module. If R is $\mathfrak{a}$-complete, then all finitely generated R-modules are $\mathfrak{a}$-complete ([Mat89, Theorem 8.7]), and hence all finitely generated right A-modules are $\mathfrak{a}$-complete.

When R is a local ring with maximal ideal $\mathfrak{m}$ and residue field k, the $\mathfrak{m}$-adic completion $\widehat{R}$ is a (commutative noetherian) local ring with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$, and $k = R/\mathfrak{m} \cong (R/\mathfrak{m}) \otimes_R \widehat{R} \cong \widehat{R}/\widehat{\mathfrak{m}}$; see [AM69, Proposition 10.16] or [Mat89, p. 63]. The completion map $R \to \widehat{R}$ is a faithfully flat ring homomorphism ([Mat89, Theorem 8.14]), thus a pure monomorphism in $\operatorname{Mod} R$ by [Mat89, Theorem 7.5(i)]. Applying $- \otimes_R A$ to the completion map, we conclude that $\widehat{A}$ is faithfully flat right A-module and the completion map $A \to \widehat{A}$ is a pure monomorphism in $\operatorname{Mod} A$.

Dually, the $\mathfrak{a}$-*torsion functor* $\Gamma_{\mathfrak{a}}\colon \operatorname{Mod} A \to \operatorname{Mod} A$ is defined by

$$\Gamma_{\mathfrak{a}} := \varinjlim_{n \geq 1} \operatorname{Hom}_R(R/\mathfrak{a}^n, -).$$

For a right A-module M, we have $\Gamma_{\mathfrak{a}} M = \bigcup_{n \geq 1}\{\, x \in M \mid x\mathfrak{a}^n = 0 \,\}$. We say that M is $\mathfrak{a}$-*torsion* if the canonical inclusion $\Gamma_{\mathfrak{a}} M \to M$ is an isomorphism. It is well known that $E_R(R/\mathfrak{p})$ is $\mathfrak{p}$-torsion for each $\mathfrak{p} \in \operatorname{Spec} R$ ([Mat89, Theorem 18.4(v)]), and more generally, $I_A(P)$ is $\mathfrak{p}$-torsion for each $P \in \operatorname{Spec} A$ and $\mathfrak{p} := P \cap R$; see Remark 2.26 below.

REMARK 2.23. The $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ is left exact and commutes with arbitrary direct sums. Moreover, we can regard $\Gamma_{\mathfrak{a}}$ as a right adjoint to the inclusion functor from the full subcategory consisting of $\mathfrak{a}$-torsion A-modules to $\operatorname{Mod} A$. See also Propositions A.7 to A.9 for analogous facts on $\Lambda^{\mathfrak{a}}$.
The $\mathfrak{a}$-torsion functor is a typical example of a left exact radical, so the class of $\mathfrak{a}$-torsion A-modules is a hereditary torsion class (localizing subcategory) of $\operatorname{Mod} A$; see [Ste75, Chapter VI, Propositions 1.7 and 3.1].

The following fact relates $\mathfrak{a}$-torsion modules with $\mathfrak{a}$-complete modules:

PROPOSITION 2.24. *Let E be an injective R-module. Then the functor $\operatorname{Hom}_R(-, E)\colon \operatorname{Mod} A \to \operatorname{Mod} A^{\mathrm{op}}$ sends $\mathfrak{a}$-torsion right A-modules to $\mathfrak{a}$-complete left A-modules.*

Proof. If M is an $\mathfrak{a}$-torsion right A-module, then it is isomorphic to $\varinjlim_{n \geq 1} \operatorname{Hom}_R(R/\mathfrak{a}^n, M)$, so we have

$$\begin{aligned}
\operatorname{Hom}_R(M, E) &\cong \varprojlim_{n \geq 1} \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{a}^n, M), E) \\
&\cong \varprojlim_{n \geq 1} \operatorname{Hom}_R(M, E) \otimes_R (R/\mathfrak{a}^n) \\
&= \operatorname{Hom}_R(M, E)^{\wedge}_{\mathfrak{a}}
\end{aligned}$$

as left A-modules; see [EJ00, Theorem 3.2.11] for the second isomorphism. Hence $\operatorname{Hom}_R(M, E)$ is $\mathfrak{a}$-complete by Proposition A.5.   □

In the rest of this section, we assume that R is local, and denote its maximal ideal and residue field by $\mathfrak{m}$ and k, respectively. The functor $\operatorname{Hom}_R(-, E_R(k))\colon \operatorname{Mod} R \to \operatorname{Mod} R$ gives rise to a duality, known as *Matlis duality*, between the category of finitely generated R-modules and the category of artinian R-modules, provided that R is $\mathfrak{m}$-adically complete. This duality naturally extends to the case of Noether algebras over complete local rings (Theorem 2.27) as shown in [GN02, Proposition 2.6.1]. Let us observe how the proof goes since techniques therein will be used in later sections. We refer the reader to [Mat89, Theorem 18.6], [BH98, Proposition 3.2.12 and Theorem 3.2.13], and [ILL$^+$07, Appendix A, §4] for classical results on Matlis duality for the commutative case; they will be used in the rest of the section.

First, for every $\mathfrak{m}$-torsion right A-module M, there is a canonical isomorphism

$$M \xrightarrow{\sim} M \otimes_R \widehat{R} \qquad (2.7)$$

of right A-modules (Lemma A.10). This makes M an $\widehat{\mathfrak{m}}$-torsion right $\widehat{A}$-module via the isomorphism $A \otimes_R \widehat{R} \cong \widehat{A}$. In fact, this is the unique right $\widehat{A}$-module structure on M that is compatible with the right A-module structure (Proposition A.15). Moreover, all A-submodules of M are also $\widehat{A}$-submodules by (2.7) (and vice versa), so M is artinian (resp. of finite length, simple) as a right A-module if and only if M is artinian (resp. of finite length, simple) as a right $\widehat{A}$-module; see also Proposition A.12.

A key to Matlis duality is that the injective envelope $E_R(k)$ is an artinian R-module (hence $\mathfrak{m}$-torsion). The above arguments applied to $A = R$ makes $E_R(k)$ an artinian $\widehat{R}$-module, and it coincides with the injective envelope of $k \cong \widehat{R}/\widehat{\mathfrak{m}}$ in $\operatorname{Mod} \widehat{R}$, that is, $E_R(k) \cong E_{\widehat{R}}(k)$.

If M is a finitely generated right A-module, then it is, as an R-module, a quotient of a finitely generated free R-module, so its Matlis dual $\operatorname{Hom}_R(M, E_R(k))$ is an R-submodule of a finite direct sum of copies of $E_R(k)$. This implies that $\operatorname{Hom}_R(M, E_R(k))$ is artinian as an R-module. Thus $\operatorname{Hom}_R(M, E_R(k))$ is an artinian left A-module. Consequently, we obtain a contravariant functor

$$\operatorname{Hom}_R(-, E_R(k))\colon \operatorname{mod} A \to \operatorname{artin} A^{\mathrm{op}}, \qquad (2.8)$$

where $\operatorname{mod} A$ is the category of finitely generated right A-modules and $\operatorname{artin} A^{\mathrm{op}}$ is the category of artinian left A-modules.

Another key to Matlis duality is that the completion map $R \to \widehat{R}$ is identified with the canonical ring homomorphism $R \to \operatorname{End}_R(E_R(k))$ via the isomorphism $\widehat{R} \xrightarrow{\sim} \operatorname{End}_R(E_R(k))$ given by the action of $\widehat{R}$ on $E_R(k)$. For every finitely generated right A-module M, the standard isomorphisms $M \otimes_R \widehat{R} \cong \widehat{M}$ and $M \otimes_R \operatorname{Hom}_R(E_R(k), E_R(k)) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, E_R(k)), E_R(k))$ of right $\widehat{A}$-modules give a natural isomorphism

$$\widehat{M} \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(M, E_R(k)), E_R(k)) \qquad (2.9)$$

of right $\widehat{A}$-modules.

On the other hand, if a given right A-module M is artinian as an R-module (we will later show that every artinian right A-module satisfies this), then M can be regarded as a right $\widehat{A}$-module via (2.7). Since such M can be embedded as an $\widehat{R}$-submodule into a finite direct sum of copies of $E_R(k) \cong E_{\widehat{R}}(k)$, the left $\widehat{A}$-module $\operatorname{Hom}_R(M, E_R(k))$ is finitely generated as an $\widehat{R}$-module, and hence as an $\widehat{A}$-module. Matlis duality for $\widehat{R}$ implies that there is a natural isomorphism

$$M \xrightarrow{\sim} \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_R(M, E_R(k)), E_{\widehat{R}}(k)) \qquad (2.10)$$

of $\widehat{R}$-modules. Since this isomorphism commutes with the action of A, this is an isomorphism of right $\widehat{A}$-modules.

It remains to see that every artinian A-module is artinian as an R-module. To this end, we prove the following fact, in which we make use of Theorem 2.18:

PROPOSITION 2.25. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring and let A be a Noether R-algebra. For every $P \in \operatorname{Max} A$, there is an isomorphism*

$$\operatorname{Hom}_R(S_{A^{\mathrm{op}}}(P), E_R(k)) \cong S_A(P)$$

of right A-modules. This realizes the bijection

$$\{ \text{ isoclasses of simple left } A\text{-modules} \}$$
$$\xrightarrow{\sim} \{ \text{ isoclasses of simple right } A\text{-modules} \}$$

defined by $S_{A^{\mathrm{op}}}(P) \mapsto S_A(P)$.

Proof. Let $P \in \operatorname{Max} A$. Then $P \cap R = \mathfrak{m}$ (Lemma 2.12). Recall that $S_{A^{\mathrm{op}}}(P)$ is of finite length as an R-module since it is a finite-dimensional k-vector space; see Lemma 2.12 and (2.2). By (2.10), we have an isomorphism $S_{A^{\mathrm{op}}}(P) \xrightarrow{\sim} \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_R(S_{A^{\mathrm{op}}}(P), E_R(k)), E_{\widehat{R}}(k))$ of left $\widehat{A}$-modules, where $\operatorname{Hom}_R(S_{A^{\mathrm{op}}}(P), E_R(k))$ is $\mathfrak{m}$-torsion. Then $\operatorname{Hom}_R(S_{A^{\mathrm{op}}}(P), E_R(k))$ has to be simple as a right $\widehat{A}$-module, or equivalently, as a right A-module. Thus, by Theorem 2.18, $\operatorname{Hom}_R(S_{A^{\mathrm{op}}}(P), E_R(k)) \cong S_A(Q)$ for some $Q \in \operatorname{Max} A$. Since the left-hand side is annihilated by P, we have $P \subseteq Q$. Hence $P = Q$ since P is also maximal. □

REMARK 2.26. As shown in [GN02, Proposition 2.5.5], the injective envelope $I_A(P)$ of $S_A(P)$ is a direct summand of $\operatorname{Hom}_R(A, E_R(k))$. Indeed, there is a surjection $A \to S_{A^{\mathrm{op}}}(P)$ in $\operatorname{Mod} A^{\mathrm{op}}$ by construction, and $\operatorname{Hom}_R(-, E_R(k))$ sends this map to an injection $S_A(P) \to \operatorname{Hom}_R(A, E_R(k))$ in $\operatorname{Mod} A$ by Proposition 2.25. Since $\operatorname{Hom}_R(A, E_R(k))$ is an injective right A-module by Proposition 2.2(3), it contains $I_A(P)$ as a direct summand.
Consequently, $I_A(P)$ is artinian as an R-module because $\operatorname{Hom}_R(A, E_R(k))$ is an artinian R-module as we observed before (2.8). In particular, $I_A(P)$ is $\mathfrak{m}$-torsion, and hence becomes a right $\widehat{A}$-module, which is artinian as a right A-module and as a right $\widehat{A}$-module.

Let us finally verify that every artinian right A-module M is artinian as an R-module. The socle $\operatorname{soc}_A M$ is a finite direct sum of simple A-modules and it is an essential A-submodule of M. Thus $E_A(M) \cong E_A(\operatorname{soc}_A M)$, and the right-hand side is a finite direct sum of copies of indecomposable injective modules $I_A(P)$ for various $P \in \operatorname{Max} A$; see Theorem 2.18. Hence $E_A(M)$ is artinian as a right R-module by Remark 2.26, and so is M. Therefore,

$$\operatorname{artin} A = \{ M \in \operatorname{Mod} A \mid M \text{ is artinian as an } R\text{-module} \}, \qquad (2.11)$$

where the inclusion "$\supseteq$" is trivial. We have observed that there is a contravariant functor

$$\operatorname{Hom}_R(-, E_R(k)) \colon \operatorname{artin} A^{\mathrm{op}} \to \operatorname{mod} \widehat{A}. \qquad (2.12)$$

Combining (2.8)–(2.12), we obtain Matlis duality for a Noether algebra over a complete local ring:

Theorem 2.27. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian complete local ring and let A be a Noether R-algebra. Then the contravariant functors $\operatorname{mod} A \to$ artin A^{op} and artin $A^{\mathrm{op}} \to \operatorname{mod} A$ induced by $\operatorname{Hom}_R(-, E_R(k))$ are mutually quasi-inverse equivalences.*

3 Decomposition of flat cotorsion modules into local complete modules

Let A be a Noether R-algebra. In this section, we show that every flat cotorsion right A-module is decomposed as a direct product of $\mathfrak{p}$-local $\mathfrak{p}$-complete flat cotorsion modules for various $\mathfrak{p} \in \operatorname{Spec} R$ (Proposition 3.7).

The argument in this section is based on Enochs' idea that was used to describe flat cotorsion modules over a commutative noetherian ring ([Eno84, p. 183]). However, we present our generalized proof in a more precise manner for the sake of clarity.

As the first step, we prove the following lemma. Note that for a module M and a set B, we denote by $M^{(B)}$ (resp. M^B) the direct sum (resp. direct product) of B-indexed copies of M.

Lemma 3.1. *A right A-module M is flat cotorsion if and only if M is a direct summand of*

$$\prod_{P \in \operatorname{Spec} A} \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B_P)}) \tag{3.1}$$

for some family of sets $\{B_P\}_{P \in \operatorname{Spec} A}$, where $\mathfrak{p} := P \cap R$ in each component.

Proof. Since R is noetherian, the direct sum $E_R(R/\mathfrak{p})^{(B_P)}$ of injective R-modules is again injective and, by Proposition 2.2(4), each $\operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B_P)})$ is a flat cotorsion right A-module. It is straightforward to see that the product (3.1) is cotorsion, and it is also flat because A is left coherent (see [Cha60, Theorem 2.1] or [EJ00, Theorem 3.2.24]). Therefore every direct summand of (3.1) is flat cotorsion.

Conversely, suppose that M is flat cotorsion. As in the proof of Proposition 2.3, M is a direct summand of $\operatorname{Hom}_R(I, E)$ for an injective cogenerator E in $\operatorname{Mod} R$, where $I := \operatorname{Hom}_R(M, E)$ is an injective left A-module. Since A is left noetherian, I decomposes as a direct sum of indecomposable injective left A-modules ([Mat58, Theorem 2.5]). Hence, using Theorem 2.22, we have

$$I \cong \bigoplus_{P \in \operatorname{Spec} A} I_{A^{\mathrm{op}}}(P)^{(C_P)} \tag{3.2}$$

for some family of sets $\{C_P\}_{P \in \operatorname{Spec} A}$. Then

$$\operatorname{Hom}_R(I, E) \cong \prod_{P \in \operatorname{Spec} A} \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E)^{C_P} \cong \prod_{P \in \operatorname{Spec} A} \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E^{C_P})$$

as right A-modules. Now fix $P \in \operatorname{Spec} A$ and let $\mathfrak{p} := P \cap R$. Since $I_{A^{\mathrm{op}}}(P)$ is $\mathfrak{p}$-local by (2.6), we have $I_{A^{\mathrm{op}}}(P) \cong I_{A^{\mathrm{op}}}(P) \otimes_R R_\mathfrak{p}$, and hence

$$\begin{aligned}
\operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E^{C_P}) &\cong \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P) \otimes_R R_\mathfrak{p}, E^{C_P}) \\
&\cong \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), \operatorname{Hom}_R(R_\mathfrak{p}, E^{C_P})).
\end{aligned}$$

Notice that $\operatorname{Hom}_R(R_\mathfrak{p}, E^{C_P})$ is an injective R-module by Proposition 2.2(3). Since it is $\mathfrak{p}$-local, it cannot contain any $\mathfrak{q}$-torsion submodule unless $\mathfrak{q} \subseteq \mathfrak{p}$. Therefore we have

$$\operatorname{Hom}_R(R_\mathfrak{p}, E^{C_P}) \cong \bigoplus_{\substack{\mathfrak{q} \in \operatorname{Spec} R \\ \mathfrak{q} \subseteq \mathfrak{p}}} E_R(R/\mathfrak{q})^{(C_P^\mathfrak{q})}$$

for some family of sets $\{C_P^\mathfrak{q}\}_{\mathfrak{q} \subseteq \mathfrak{p}}$. Then

$$\begin{aligned}
\operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E^{C_P}) &\cong \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), \bigoplus_{\substack{\mathfrak{q} \in \operatorname{Spec} R \\ \mathfrak{q} \subseteq \mathfrak{p}}} E_R(R/\mathfrak{q})^{(C_P^\mathfrak{q})}) \\
&\cong \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(C_P^\mathfrak{p})}),
\end{aligned}$$

where the last isomorphism follows from Remark 2.23 (and Remark 3.4(1) below) because $I_{A^{\mathrm{op}}}(P)$ is $\mathfrak{p}$-torsion (Remark 2.26) and each $E_R(R/\mathfrak{q})$ is $\mathfrak{q}$-local. Setting $B_P := C_P^\mathfrak{p}$, we conclude that $\operatorname{Hom}_R(I, E)$ is of the form (3.1). $\qquad\square$

REMARK 3.2. Each component $\operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B_P)})$ in (3.1) is $\mathfrak{p}$-local and $\mathfrak{p}$-complete, by Propositions 2.14 and 2.24, (2.6), and Remark 2.26. Moreover, we can rewrite (3.1) as $\prod_{\mathfrak{p} \in \operatorname{Spec} R} M(\mathfrak{p})$, where $M(\mathfrak{p})$ is

$$\bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B_P)}),$$

which is a finite direct sum due to Proposition 2.15.

By Lemma 3.1 and Remark 3.2, every flat cotorsion right A-module M is a direct product of $\mathfrak{p}$-local $\mathfrak{p}$-complete flat cotorsion modules for various $\mathfrak{p} \in \operatorname{Spec} R$. The next result shows that the isoclass of the component at $\mathfrak{p}$ is uniquely determined by M.

LEMMA 3.3. *Let $M(\mathfrak{p})$ be a $\mathfrak{p}$-local $\mathfrak{p}$-complete right A-module for each $\mathfrak{p} \in \operatorname{Spec} R$, and let $M := \prod_{\mathfrak{p} \in \operatorname{Spec} R} M(\mathfrak{p})$.*

(1) *For every ideal $\mathfrak{a} \subseteq R$, the canonical morphism $M \to \Lambda^\mathfrak{a} M$ is a split epimorphism, and*

$$\Lambda^\mathfrak{a} M = \prod_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \mathfrak{a} \subseteq \mathfrak{p}}} M(\mathfrak{p})$$

as quotient modules of M, where the right-hand side is regarded as a quotient module via the projection.

(2) *For every multiplicatively closed set $S \subseteq R$, the canonical morphism $\mathrm{Hom}_R(S^{-1}R, M) \to \mathrm{Hom}_R(R, M) \xrightarrow{\sim} M$ is a split monomorphism, and*

$$\mathrm{Hom}_R(S^{-1}R, M) = \prod_{\substack{\mathfrak{p} \in \mathrm{Spec}\, R \\ \mathfrak{p} \cap S = \emptyset}} M(\mathfrak{p})$$

as submodules of M, where the right-hand side is regarded as a submodule via the inclusion.

In particular, for every $\mathfrak{q} \in \mathrm{Spec}\, R$,

$$\mathrm{Hom}_R(R_\mathfrak{q}, M) = \prod_{\substack{\mathfrak{p} \in \mathrm{Spec}\, R \\ \mathfrak{p} \subseteq \mathfrak{q}}} M(\mathfrak{p}).$$

(3) *Let $\mathfrak{q} \in \mathrm{Spec}\, R$, and let $\iota_\mathfrak{q} \colon M(\mathfrak{q}) \to M$ and $\pi_\mathfrak{q} \colon M \to M(\mathfrak{q})$ be the inclusion and the projection, respectively. Then $\Lambda^\mathfrak{q} \mathrm{Hom}_R(R_\mathfrak{q}, \iota_\mathfrak{q})$ and $\Lambda^\mathfrak{q} \mathrm{Hom}_R(R_\mathfrak{q}, \pi_\mathfrak{q})$ are isomorphisms, and*

$$\Lambda^\mathfrak{q} \mathrm{Hom}_R(R_\mathfrak{q}, M) \cong M(\mathfrak{q}).$$

Proof. (1): Since $\Lambda^\mathfrak{a}$ commutes with arbitrary direct products (Proposition A.8), the canonical morphism $M \to \Lambda^\mathfrak{a} M$ can be naturally identified with the direct product of the canonical morphisms $M(\mathfrak{p}) \to \Lambda^\mathfrak{a} M(\mathfrak{p})$ for $\mathfrak{p} \in \mathrm{Spec}\, R$. If $\mathfrak{a} \subseteq \mathfrak{p}$, then the $\mathfrak{p}$-complete module $M(\mathfrak{p})$ is also $\mathfrak{a}$-complete (Remark A.6), so the canonical morphism $M(\mathfrak{p}) \to \Lambda^\mathfrak{a} M(\mathfrak{p})$ is an isomorphism. If $\mathfrak{a} \not\subseteq \mathfrak{p}$, then $\mathfrak{a}^n M(\mathfrak{p}) = M(\mathfrak{p})$ for all $n \geq 1$ because $M(\mathfrak{p})$ is $\mathfrak{p}$-local, so $\Lambda^\mathfrak{a} M(\mathfrak{p}) = 0$.

(2): Similarly, since the functor $\mathrm{Hom}_R(S^{-1}R, -)$ commutes with arbitrary direct products, the canonical morphism $\mathrm{Hom}_R(S^{-1}R, M) \to M$ can be naturally identified with the direct product of the canonical morphisms $\mathrm{Hom}_R(S^{-1}R, M(\mathfrak{p})) \to M(\mathfrak{p})$ for $\mathfrak{p} \in \mathrm{Spec}\, R$. If $\mathfrak{p} \cap S = \emptyset$, then $(S^{-1}R)_\mathfrak{p} \cong R_\mathfrak{p}$ so we can deduce from Remark 2.13 that $\mathrm{Hom}_R(S^{-1}R, M(\mathfrak{p})) \cong \mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}, M(\mathfrak{p})) \cong M(\mathfrak{p})$. If $\mathfrak{p} \cap S \neq \emptyset$, then $\Lambda^\mathfrak{p} S^{-1}R = 0$, so $\mathrm{Hom}_R(S^{-1}R, M(\mathfrak{p})) \cong \mathrm{Hom}_R(\Lambda^\mathfrak{p} S^{-1}R, M(\mathfrak{p})) = 0$ by Proposition A.9 since $M(\mathfrak{p})$ is $\mathfrak{p}$-complete.

(3): This follows from (1) and (2). □

There is a dual statement of Lemma 3.3 for a direct sum of local torsion modules. We state it as a remark because the proof is immediate in view of Remark 2.23.

REMARK 3.4. Let $M(\mathfrak{p})$ be a $\mathfrak{p}$-local $\mathfrak{p}$-torsion right A-module for each $\mathfrak{p} \in \mathrm{Spec}\, R$, and let $M := \bigoplus_{\mathfrak{p} \in \mathrm{Spec}\, R} M(\mathfrak{p})$.

(1) For every ideal $\mathfrak{a} \subseteq R$, the canonical morphism $\Gamma_\mathfrak{a} M \to M$ is a split monomorphism, and

$$\Gamma_\mathfrak{a} M = \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Spec}\, R \\ \mathfrak{a} \subseteq \mathfrak{p}}} M(\mathfrak{p})$$

as submodules of M, where the right-hand side is regarded as a submodule via the inclusion.

(2) For every multiplicatively closed set $S \subseteq R$, the canonical morphism $M \to M \otimes_R S^{-1}R$ is a split epimorphism, and

$$M \otimes_R S^{-1}R = \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \mathfrak{p} \cap S = \emptyset}} M(\mathfrak{p})$$

as quotient modules of M, where the right-hand side is regarded as a quotient module via the projection.

In particular, for every $\mathfrak{q} \in \operatorname{Spec} R$,

$$M_{\mathfrak{q}} = \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \mathfrak{p} \subseteq \mathfrak{q}}} M(\mathfrak{p}).$$

(3) Let $\mathfrak{q} \in \operatorname{Spec} R$, and let $\iota_{\mathfrak{q}} \colon M(\mathfrak{q}) \to M$ and $\pi_{\mathfrak{q}} \colon M \to M(\mathfrak{q})$ be the inclusion and the projection, respectively. Then $\Gamma_{\mathfrak{q}}(\iota_{\mathfrak{q}} \otimes_R R_{\mathfrak{q}})$ and $\Gamma_{\mathfrak{q}}(R_{\mathfrak{q}}, \pi_{\mathfrak{q}} \otimes_R R_{\mathfrak{q}})$ are isomorphisms, and

$$\Gamma_{\mathfrak{q}}(M \otimes_R R_{\mathfrak{q}}) \cong M(\mathfrak{q}).$$

Lemma 3.3 above and Lemma 3.5 below are shown in [Tho19, Lemma 2.2] and [Tho19, Lemma 3.1], respectively, but those were for direct products of $\mathfrak{p}$-local $\mathfrak{p}$-complete *flat* R-modules for various $\mathfrak{p} \in \operatorname{Spec} R$.

LEMMA 3.5. *Let $M(\mathfrak{p})$ and $N(\mathfrak{p})$ be $\mathfrak{p}$-local $\mathfrak{p}$-complete right A-modules for each $\mathfrak{p} \in \operatorname{Spec} R$. For an A-homomorphism $f \colon \prod_{\mathfrak{p} \in \operatorname{Spec} R} M(\mathfrak{p}) \to \prod_{\mathfrak{p} \in \operatorname{Spec} R} N(\mathfrak{p})$, the following are equivalent:*

(1) *f is an isomorphism.*

(2) *$\Lambda^{\mathfrak{q}} \operatorname{Hom}_R(R_{\mathfrak{q}}, f)$ is an isomorphism for all $\mathfrak{q} \in \operatorname{Spec} R$.*

(3) *The composition*

$$M(\mathfrak{q}) \hookrightarrow \prod_{\mathfrak{p} \in \operatorname{Spec} R} M(\mathfrak{p}) \xrightarrow{f} \prod_{\mathfrak{p} \in \operatorname{Spec} R} N(\mathfrak{p}) \twoheadrightarrow N(\mathfrak{q})$$

is an isomorphism for all $\mathfrak{q} \in \operatorname{Spec} R$, where the first morphism is the inclusion and the last one is the projection.

Proof. (1)$\Rightarrow$(2) is obvious. (2) and (3) are equivalent by Lemma 3.3. The proof of (3)$\Rightarrow$(1) is parallel to that of [Tho19, Lemma 3.1] with $T_{\mathfrak{p}}$ and $T'_{\mathfrak{p}}$ replaced by $M(\mathfrak{p})$ and $N(\mathfrak{p})$, respectively; [Tho19, (3.2)] is replaced by the fact that, for every $\mathfrak{p} \in \operatorname{Spec} R$,

$$\operatorname{Hom}_A(\prod_{\substack{\mathfrak{q} \in \operatorname{Spec} R \\ \mathfrak{p} \not\subseteq \mathfrak{q}}} M(\mathfrak{q}), N(\mathfrak{p})) = 0, \tag{3.3}$$

which follows from Proposition A.9 and Lemma 3.3(1). □

The next lemma recovers and generalizes [Xu96, Theorem 4.1.14], which deals with direct products of $\mathfrak{p}$-local $\mathfrak{p}$-complete flat R-modules for various $\mathfrak{p} \in \operatorname{Spec} R$.

LEMMA 3.6. *Let $M(\mathfrak{p})$ be a $\mathfrak{p}$-local $\mathfrak{p}$-complete right A-module for each $\mathfrak{p} \in \operatorname{Spec} R$. If we have a decomposition*

$$\prod_{\mathfrak{p} \in \operatorname{Spec} R} M(\mathfrak{p}) \cong M_1 \oplus M_2,$$

then there are right A-modules $M_i(\mathfrak{p})$, indexed by $\mathfrak{p} \in \operatorname{Spec} R$ and $i = 1, 2$, such that

$$M_i \cong \prod_{\mathfrak{p} \in \operatorname{Spec} R} M_i(\mathfrak{p})$$

and $M(\mathfrak{p}) \cong M_1(\mathfrak{p}) \oplus M_2(\mathfrak{p})$ for each $\mathfrak{p} \in \operatorname{Spec} R$.

Proof. Let $M := \prod_{\mathfrak{p} \in \operatorname{Spec} R} M(\mathfrak{p})$. For each $\mathfrak{q} \in \operatorname{Spec} R$, we have a canonical split monomorphism $\operatorname{Hom}_R(R_\mathfrak{q}, M) \to M$ and a canonical split epimorphism $\operatorname{Hom}_R(R_\mathfrak{q}, M) \to \Lambda^\mathfrak{q} \operatorname{Hom}_R(R_\mathfrak{q}, M)$, which both become isomorphisms upon application of $\Lambda^\mathfrak{q} \operatorname{Hom}_R(R_\mathfrak{q}, -)$; see Lemma 3.3. Since $M \cong M_1 \oplus M_2$, the same holds for M_1 and M_2. For each $i = 1, 2$, let $h_i^\mathfrak{q}$ be the composition

$$M_i \xrightarrow{g_i^\mathfrak{q}} \operatorname{Hom}_R(R_\mathfrak{q}, M_i) \longrightarrow \Lambda^\mathfrak{q} \operatorname{Hom}_R(R_\mathfrak{q}, M_i) =: M_i(\mathfrak{q}),$$

where $g_i^\mathfrak{q}$ is an arbitrary splitting of the canonical split monomorphism $f_i^\mathfrak{q} \colon \operatorname{Hom}_R(R_\mathfrak{q}, M_i) \to M_i$. Since $\Lambda^\mathfrak{q} \operatorname{Hom}_R(R_\mathfrak{q}, f_i^\mathfrak{q})$ is an isomorphism, so is $\Lambda^\mathfrak{q} \operatorname{Hom}_R(R_\mathfrak{q}, g_i^\mathfrak{q})$.

Let $h_i \colon M_i \to \prod_{\mathfrak{p} \in \operatorname{Spec} R} M_i(\mathfrak{p})$ be the morphism induced by the family $\{h_i^\mathfrak{p}\}_{\mathfrak{p} \in \operatorname{Spec} R}$. If we show that $h_1 \oplus h_2 \colon M_1 \oplus M_2 \to \left(\prod_\mathfrak{p} M_1(\mathfrak{p})\right) \oplus \left(\prod_\mathfrak{p} M_2(\mathfrak{p})\right) = \prod_\mathfrak{p} (M_1(\mathfrak{p}) \oplus M_2(\mathfrak{p}))$ is an isomorphism, then so are h_1 and h_2, and thus the desired conclusion follows since

$$M(\mathfrak{p}) \cong \Lambda^\mathfrak{p} \operatorname{Hom}_R(R_\mathfrak{p}, M) \cong \Lambda^\mathfrak{p} \operatorname{Hom}_R(R_\mathfrak{p}, M_1 \oplus M_2) \cong M_1(\mathfrak{p}) \oplus M_2(\mathfrak{p})$$

by Lemma 3.3(3).

By Lemma 3.5, it suffices to show that $\Lambda^\mathfrak{q} \operatorname{Hom}_R(R_\mathfrak{q}, h_1 \oplus h_2)$ is an isomorphism for each $\mathfrak{q} \in \operatorname{Spec} R$. Consider the commutative diagram

$$
\begin{array}{ccc}
M_1 \oplus M_2 & \xrightarrow{\ h_1 \oplus h_2\ } & \displaystyle\prod_{\mathfrak{p} \in \operatorname{Spec} R} (M_1(\mathfrak{p}) \oplus M_2(\mathfrak{p})) \\[2ex]
{\scriptstyle g_1^\mathfrak{q} \oplus g_2^\mathfrak{q}} \big\downarrow & & \big\downarrow {\scriptstyle \text{projection}} \\[2ex]
\operatorname{Hom}_R(R_\mathfrak{q}, M_1 \oplus M_2) & & \\[2ex]
\big\downarrow & & \big\downarrow \\[2ex]
\Lambda^\mathfrak{q} \operatorname{Hom}_R(R_\mathfrak{q}, M_1 \oplus M_2) & =\!=\!=\!= & M_1(\mathfrak{q}) \oplus M_2(\mathfrak{q}),
\end{array}
$$

where the vertical morphisms become isomorphisms upon application of $\Lambda^{\mathfrak{q}} \operatorname{Hom}_R(R_{\mathfrak{q}}, -)$. Therefore $\Lambda^{\mathfrak{q}} \operatorname{Hom}_R(R_{\mathfrak{q}}, h_1 \oplus h_2)$ is an isomorphism. $\square$

PROPOSITION 3.7. *A right A-module M is flat cotorsion if and only if $M \cong \prod_{\mathfrak{p} \in \operatorname{Spec} R} M(\mathfrak{p})$, where each $M(\mathfrak{p})$ is some $\mathfrak{p}$-local $\mathfrak{p}$-complete flat cotorsion module. The isoclass of $M(\mathfrak{p})$ is uniquely determined by M.*

Proof. The "if" part is clear; see the proof of Lemma 3.1. The "only if" part follows from Lemma 3.1, Remark 3.2, and Lemma 3.6. The uniqueness of $M(\mathfrak{p})$ follows from Lemma 3.3(3). $\square$

4   Local complete flat modules as flat covers

Let A be a Noether R-algebra. In section 3, we observed that every flat cotorsion module is uniquely decomposed as a direct product of $\mathfrak{p}$-local $\mathfrak{p}$-complete ones. The purpose of the present section is to realize each $\mathfrak{p}$-local $\mathfrak{p}$-complete flat module as a flat cover of a semisimple $A_{\mathfrak{p}}$-module (Theorem 4.9).

The key result in this section is Proposition 4.5. The next three results are necessary steps for this; Lemmas 4.1 and 4.2 are inspired by [NT20, Lemma 1.1], and Proposition 4.4(1) is a generalization of [Xu96, Proposition 4.1.6].

LEMMA 4.1. *Let $J \subseteq A$ be a nilpotent ideal.*

(1) *For every flat right A-module F, the canonical morphism $F \to F \otimes_A (A/J)$ is right minimal in $\operatorname{Mod} A$.*

(2) *For every injective right A-module I, the canonical morphism $\operatorname{Hom}_A(A/J, I) \to I$ is left minimal in $\operatorname{Mod} A$.*

Proof. This proof works for an arbitrary ring A.

(1): Denote the canonical morphism by f and let $g \in \operatorname{End}_A(F)$ such that $fg = f$. Since $f \otimes_A (A/J)$ is an isomorphism, so is $g \otimes_A (A/J)$. Applying $- \otimes_A (A/J)$ to the exact sequence

$$F \xrightarrow{g} F \longrightarrow \operatorname{Cok} g \longrightarrow 0,$$

we obtain $(\operatorname{Cok} g) \otimes_A (A/J) = 0$. Hence $\operatorname{Cok} g = (\operatorname{Cok} g)J = (\operatorname{Cok} g)J^2 = \cdots$, but J is nilpotent, so $\operatorname{Cok} g = 0$. Applying $- \otimes_A (A/J)$ to

$$0 \longrightarrow \operatorname{Ker} g \longrightarrow F \xrightarrow{g} F \longrightarrow 0,$$

we obtain $(\operatorname{Ker} g) \otimes_A (A/J) = 0$ since F is flat. Hence $\operatorname{Ker} g = 0$ by the same argument.

(2): Given a right A-module M, $\operatorname{Hom}_A(A/J, M) = 0$ if and only if $\operatorname{Hom}_A(A/J^n, M) = 0$ for every $n \geq 1$, since $\operatorname{Hom}_A(A/J^n, M) = \{x \in M \mid xJ^n = 0\}$. Thus $\operatorname{Hom}_A(A/J, M) = 0$ implies $M = 0$, as J is nilpotent. The rest of the proof is parallel to (1). $\square$

Lemma 4.2. *Let $\mathfrak{a} \subseteq R$ be an ideal such that $R/\mathfrak{a}$ is an artinian ring. For every $\mathfrak{a}$-complete flat right A-module F, the canonical morphism $F \to F \otimes_R (R/\mathfrak{a})$ is a flat cover in* $\operatorname{Mod} A$.

Proof. Denote the canonical morphism by f. We first show that it is a flat precover. Let $h\colon G \to F \otimes_R (R/\mathfrak{a})$ be an A-homomorphism from a flat right A-module G. Then h naturally factors through an A-homomorphism $\overline{h}\colon G \otimes_R (R/\mathfrak{a}) \to F \otimes_R (R/\mathfrak{a})$. Set $g_1 := \overline{h}$. For each $n \geq 1$, $G \otimes_R (R/\mathfrak{a}^n)$ is a flat $A/\mathfrak{a}^n A$-module and it is actually projective since $A/\mathfrak{a}^n A$ is right artinian (Remark 2.9). Thus, there exist A-homomorphisms $g_n\colon G \otimes_R (R/\mathfrak{a}^n) \to F \otimes_R (R/\mathfrak{a}^n)$, for all $n \geq 2$, such that the diagram

$$
\begin{array}{ccccc}
\cdots \longrightarrow G \otimes_R (R/\mathfrak{a}^3) & \longrightarrow & G \otimes_R (R/\mathfrak{a}^2) & \longrightarrow & G \otimes_R (R/\mathfrak{a}) \\
\downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} \\
\cdots \longrightarrow F \otimes_R (R/\mathfrak{a}^3) & \longrightarrow & F \otimes_R (R/\mathfrak{a}^2) & \longrightarrow & F \otimes_R (R/\mathfrak{a})
\end{array}
\tag{4.1}
$$

commutes, where the horizontal maps are the canonical ones. Defining g to be the composition of $\varprojlim_{n \geq 1} g_n\colon G_{\mathfrak{a}}^\wedge \to F_{\mathfrak{a}}^\wedge$ and the canonical isomorphism $F_{\mathfrak{a}}^\wedge \cong F$, we have a commutative diagram

$$
\begin{array}{ccc}
G_{\mathfrak{a}}^\wedge & \longrightarrow & G \otimes_R (R/\mathfrak{a}) \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_1} \\
F & \xrightarrow{\ f\ } & F \otimes_R (R/\mathfrak{a}),
\end{array}
$$

where the horizontal map in the first row is the canonical one. The composition of the completion map $G \to G_{\mathfrak{a}}^\wedge$ and $g\colon G_{\mathfrak{a}}^\wedge \to F$ is a lifting of $h\colon G \to F \otimes_R (R/\mathfrak{a})$ because $g_1 = \overline{h}$. This shows that f is a flat precover.

Next we show that f is right minimal. Take an arbitrary $g \in \operatorname{End}_A(F)$ such that the diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\ f\ } & \\
{\scriptstyle g}\downarrow & & F \otimes_R (R/\mathfrak{a}) \\
F & \xrightarrow[\ f\ ]{} &
\end{array}
$$

commutes. Letting $g_n := g \otimes_R (R/\mathfrak{a}^n) = g \otimes_A (A/\mathfrak{a}^n A)$ for each $n \geq 1$, we obtain a diagram of the form (4.1) with $G = F$ and $g_1 = \operatorname{id}_{F/\mathfrak{a}F}$. Letting $A_n := A/\mathfrak{a}^n A$ and $J_n := \mathfrak{a}A/\mathfrak{a}^n A \subseteq A_n$, we have $g_n \otimes_{A_n} (A_n/J_n) = g_n \otimes_A (A/\mathfrak{a}A) = \operatorname{id}_{F/\mathfrak{a}F}$. Thus Lemma 4.1(1), applied to $J_n \subseteq A_n$, implies that g_n is an isomorphism for every $n \geq 2$. Then g is an isomorphism as $g = \varprojlim_{n \geq 1} g_n$. This concludes that f is a flat cover. $\square$

Remark 4.3. The assumption on $\mathfrak{a}$ in Lemma 4.2 is satisfied if and only if the Zariski closed subset $V(\mathfrak{a}) := \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p} \,\}$ consists of only (necessarily

finitely many) maximal ideals of R. If this is the case, then the $\mathfrak{a}$-adic completion functor $\Lambda^{\mathfrak{a}}$ decomposes as the finite direct product $\prod_{\mathfrak{m} \in V(\mathfrak{a})} \Lambda^{\mathfrak{m}}$. Indeed, setting $\mathfrak{b} := \bigcap_{\mathfrak{m} \in V(\mathfrak{a})} \mathfrak{m}$, we have $\mathfrak{b}^n \subseteq \mathfrak{a} \subseteq \mathfrak{b}$ for some positive integer n, so the proof goes in a similar way to that of [Mat89, Theorem 8.15].

PROPOSITION 4.4. *Let* $\mathfrak{p} \in \operatorname{Spec} R$.

(1) *For every $\mathfrak{p}$-local $\mathfrak{p}$-complete flat right A-module F, the canonical morphism $F \to F \otimes_R \kappa(\mathfrak{p})$ is a flat cover in* $\operatorname{Mod} A$.

(2) *For every $\mathfrak{p}$-local $\mathfrak{p}$-torsion injective right A-module I, the canonical morphism $\operatorname{Hom}_R(\kappa(\mathfrak{p}), I) \to I$ is an injective envelope in* $\operatorname{Mod} A$.

Proof. By Remarks 2.13 and 2.21 along with the standard isomorphism $F \otimes_R \kappa(\mathfrak{p}) \cong F \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p})$, we may assume that $(R, \mathfrak{m}, k)$ is a local ring and $\mathfrak{p} = \mathfrak{m}$. Then (1) follows from Lemma 4.2. To prove (2), notice that the canonical map $R \to \kappa(\mathfrak{p}) = k$ is surjective and it induces an injection $\operatorname{Hom}_R(k, I) \to I$, which is clearly an injective preenvelope in $\operatorname{Mod} A$. As I is $\mathfrak{m}$-torsion by assumption, every nonzero A-submodule M of I satisfies $\operatorname{Hom}_R(k, M) \neq 0$. This implies that the morphism $\operatorname{Hom}_R(k, I) \to I$ is an essential monomorphism, so it is an injective envelope in $\operatorname{Mod} A$. $\square$

PROPOSITION 4.5. *Let* $\mathfrak{p} \in \operatorname{Spec} R$.

(1) *Let F be a $\mathfrak{p}$-local $\mathfrak{p}$-complete flat right A-module. Then the canonical morphism*

$$F \to F \otimes_A (A_{\mathfrak{p}} / \operatorname{rad} A_{\mathfrak{p}})$$

is a flat cover in $\operatorname{Mod} A$.

(2) *Let I be a $\mathfrak{p}$-local $\mathfrak{p}$-torsion injective right A-module. The canonical morphism*

$$\operatorname{Hom}_A(A_{\mathfrak{p}} / \operatorname{rad} A_{\mathfrak{p}}, I) \to I$$

an injective envelope in $\operatorname{Mod} A$.

Proof. As we observed in the proof of Proposition 4.4, we may assume that $(R, \mathfrak{m}, k)$ is a local ring and $\mathfrak{p} = \mathfrak{m}$. Put $J := \operatorname{rad} A$. Since J is the intersection of all maximal ideals of A and $P \cap R = \mathfrak{m}$ for all $P \in \operatorname{Max} A$ (see Lemma 2.12 and (2.3)), we have $\mathfrak{m}A \subseteq J$. Consequently, for every right A-module M, the canonical morphism $M \otimes_R k \cong M \otimes_A (A/\mathfrak{m}A) \to M \otimes_A (A/J)$ is surjective and the canonical morphism $\operatorname{Hom}_A(A/J, M) \to \operatorname{Hom}_A(A/\mathfrak{m}A, M) \cong \operatorname{Hom}_R(k, M)$ is injective.

(1): Denote the given morphism by f. First we show that it is a flat precover. Let $h \colon G \to F \otimes_A (A/J)$ be a morphism from a flat right A-module G. It naturally factors through a morphism $\overline{h} \colon G \otimes_A (A/J) \to F \otimes_A (A/J)$. By the

projectivity of $G \otimes_R k$ in $\mathrm{Mod}(A \otimes_R k)$, there exists a morphism $\overline{g}$ such that the diagram

$$
\begin{array}{ccc}
G \otimes_R k & \longrightarrow & G \otimes_A (A/J) \\
{\scriptstyle \overline{g}} \downarrow & & \downarrow {\scriptstyle \overline{h}} \\
F \otimes_R k & \longrightarrow & F \otimes_A (A/J)
\end{array}
$$

commutes. Furthermore, by Proposition 4.4(1), there exists a morphism g such that the diagram

$$
\begin{array}{ccccc}
G & \longrightarrow & G \otimes_R k & \longrightarrow & G \otimes_A (A/J) \\
{\scriptstyle g} \downarrow & & {\scriptstyle \overline{g}} \downarrow & & \downarrow {\scriptstyle \overline{h}} \\
F & \longrightarrow & F \otimes_R k & \longrightarrow & F \otimes_A (A/J)
\end{array}
$$

commutes, where unadorned morphisms are the canonical ones. The composition of the morphisms in the first row together with $\overline{h}$ is h, so the diagram shows that h factors through the second row, which is f. Hence f is a flat precover.

Next, for every $s \in \mathrm{End}_A(F)$ such that $fs = f$, we have a commutative diagram

$$
\begin{array}{ccc}
F & \longrightarrow & F \otimes_R k \\
{\scriptstyle s} \downarrow & {\scriptstyle s \otimes_R k} \downarrow & \searrow \\
& & \qquad F \otimes_A (A/J), \\
F & \longrightarrow & F \otimes_R k \nearrow
\end{array}
$$

where unadorned morphisms are the canonical ones. Here $s \otimes_R k$ is an isomorphism by Lemma 4.1(1) as $J/\mathfrak{m}A = \mathrm{rad}(A \otimes_R k)$ (see Remark 2.20) is nilpotent. Therefore Proposition 4.4(1) implies that s is an isomorphism. This concludes that f is a flat cover.

(2): The injection $\mathrm{Hom}_A(A/J, I) \to I$ is clearly an injective preenvelope. The rest of the proof is parallel to (1). Use Lemma 4.1(2) and Proposition 4.4(2) instead. □

Lemma 4.6. *Let $\mathfrak{a}$ be an ideal of R and let M be a right A-module.*

(1) *If M is an $\mathfrak{a}$-complete right A-module, then its flat cover $F_A(M)$ is $\mathfrak{a}$-complete.*

(2) *If M is an $\mathfrak{a}$-torsion right A-module, then its injective envelope $E_A(M)$ is $\mathfrak{a}$-torsion.*

Proof. (1): Let $f \colon F_A(M) \to M$ be the flat cover. Since M is $\mathfrak{a}$-complete, we may identify M with $\Lambda^{\mathfrak{a}} M$. Then f is factorized as the composition of the completion map $F_A(M) \to \Lambda^{\mathfrak{a}} F_A(M)$ and $\Lambda^{\mathfrak{a}} f \colon \Lambda^{\mathfrak{a}} F_A(M) \to M$. By Proposition A.3, $\Lambda^{\mathfrak{a}} F_A(M)$ is flat. Since f is a flat cover, there exists a morphism g

such that the diagram

$$F_A(M) \longrightarrow \Lambda^{\mathfrak{a}} F_A(M) \xrightarrow{\ g\ } F_A(M)$$

with f, $\Lambda^{\mathfrak{a}} f$, f mapping to M

commutes. The right minimality of f implies that g is a split epimorphism, so $F_A(M)$ is a direct summand of $\Lambda^{\mathfrak{a}} F_A(M)$. Since $\Lambda^{\mathfrak{a}} F_A(M)$ is $\mathfrak{a}$-complete (Proposition A.5), the direct summand $F_A(M)$ is also $\mathfrak{a}$-complete.

(2): Let $g\colon M \to E_A(M)$ be the injective envelope. Since A is right noetherian, $E_A(M)$ decomposes as a direct sum of copies of $I_A(P)$ for various $P \in \operatorname{Spec} A$; see (3.2). The $\mathfrak{a}$-torsion A-submodule $\Gamma_{\mathfrak{a}} E_A(M)$ of $E_A(M)$ is injective by Remark 3.4(1), and the induced map $\Gamma_{\mathfrak{a}} g\colon M \to \Gamma_{\mathfrak{a}} E_A(M)$ is a monomorphism as $\Gamma_{\mathfrak{a}}$ is left exact (Remark 2.23). Thus we have $\Gamma_{\mathfrak{a}} E_A(M) = E_A(M)$. $\square$

REMARK 4.7. Let $\mathfrak{p} \in \operatorname{Spec} A$. Recall that $A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}$ is a semisimple ring (see Proposition 2.19). Hence every module over this ring is a direct sum of simple modules, and each simple module is isomorphic to $S_A(P)$ for some $P \in \operatorname{Spec} A$ with $P \cap R = \mathfrak{p}$. Moreover, the category $\operatorname{Mod}(A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}})$ is naturally equivalent to the subcategory of $\operatorname{Mod} A_{\mathfrak{p}}$ (or $\operatorname{Mod} A$) formed by semisimple $A_{\mathfrak{p}}$-modules (see Proposition 2.14(2) and Theorem 2.18).

With this remark, we obtain the following result:

PROPOSITION 4.8. Let $\mathfrak{p} \in \operatorname{Spec} R$ and let M be a semisimple right $A_{\mathfrak{p}}$-module.

(1) Let $f\colon F_A(M) \to M$ be a flat cover in $\operatorname{Mod} A$. Then $F_A(M)$ is $\mathfrak{p}$-local and $\mathfrak{p}$-complete. Moreover, the morphism f induces an isomorphism

$$F_A(M) \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}) \xrightarrow{\ \sim\ } M.$$

(2) Let $g\colon M \to E_A(M)$ be an injective envelope in $\operatorname{Mod} A$. Then $E_A(M)$ is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion. Moreover, the morphism g induces an isomorphism

$$M \xrightarrow{\ \sim\ } \operatorname{Hom}_A(A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}, E_A(M)).$$

Proof. We only prove (1) because the dual argument proves (2).
Note that the semisimple $A_{\mathfrak{p}}$-module M is $\mathfrak{p}$-complete as $\mathfrak{p}^n M = 0$ for every $n \geq 1$ (see Remark 4.7). Hence $F_A(M)$ is $\mathfrak{p}$-local $\mathfrak{p}$-complete by Remark 2.21 and Lemma 4.6(1). We have a commutative diagram

$$
\begin{array}{ccc}
F_A(M) & \xrightarrow{\ f\ } & M \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \wr} \\
F_A(M) \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}) & \xrightarrow{\ h\ } & M \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}),
\end{array}
$$

where the vertical morphisms are canonical and $h := f \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}})$. By Proposition 4.5(1), u is a flat cover. Since h is an epimorphism between semisimple $A_{\mathfrak{p}}$-modules, it is a split epimorphism. Thus the flat cover $F_A(\operatorname{Ker} h) \to \operatorname{Ker} h$ is a direct summand of the flat cover u. Since $F_A(\operatorname{Ker} h)$ is in the kernel of f, it should be zero by the right minimality of f. Therefore $\operatorname{Ker} h = 0$.   □

THEOREM 4.9. *For every $\mathfrak{p} \in \operatorname{Spec} R$, we have the following one-to-one correspondences:*

$$\{\text{ isoclasses of } \mathfrak{p}\text{-local } \mathfrak{p}\text{-complete flat right } A\text{-modules }\}$$

$$-\otimes_A(A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}) \Big\downarrow \wr \Big\uparrow F_A(-)$$

$$\{\text{ isoclasses of semisimple right } A_{\mathfrak{p}}\text{-modules }\}$$

$$E_A(-) \Big\downarrow \wr \Big\uparrow \operatorname{Hom}_A(A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}, -)$$

$$\{\text{ isoclasses of } \mathfrak{p}\text{-local } \mathfrak{p}\text{-torsion injective right } A\text{-modules }\},$$

where the maps between the first and the second classes (resp. between the second and the third classes) are mutually inverse.

Proof. This follows from Propositions 4.5 and 4.8 and Remark 4.7.   □

Let $\mathfrak{p} \in \operatorname{Spec} R$. As we observed in Remark 4.7, every semisimple right $A_{\mathfrak{p}}$-module M decomposes as

$$M \cong \bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} S_A(P)^{(B_P)} \tag{4.2}$$

for some family of sets $\{B_P\}_P$, where $\{\, P \in \operatorname{Spec} A \mid P \cap R = \mathfrak{p} \,\}$ is a finite set (Proposition 2.15). Proposition 4.10 (resp. Proposition 4.13) below shows that a flat cover (resp. injective envelope) of $S_A(P)^{(B_P)}$ in $\operatorname{Mod} A$ can be obtained by applying a variant of Matlis dual to an injective envelope (resp. flat cover) of $S_{A^{\mathrm{op}}}(P)$ in $\operatorname{Mod} A^{\mathrm{op}}$.

PROPOSITION 4.10. *Let $P \in \operatorname{Spec} A$ and $\mathfrak{p} := P \cap R$. For every set B, the injective envelope $S_{A^{\mathrm{op}}}(P) \to I_{A^{\mathrm{op}}}(P)$ induces a flat cover*

$$\operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B)}) \to \operatorname{Hom}_R(S_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B)}) \cong S_A(P)^{(B)} \tag{4.3}$$

in $\operatorname{Mod} A$.

Proof. We first recall that each Hom_R in (4.3) can be replaced by $\operatorname{Hom}_{R_{\mathfrak{p}}}$; see Remark 2.13 and (2.6). Moreover, the $\mathfrak{p}$-local right A-module $\operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B)})$ is $\mathfrak{p}$-complete and flat as well; see Lemma 3.1 and Remark 3.2.

Next, notice that the functor $\operatorname{Hom}_{R_{\mathfrak{p}}}(S_{A^{\mathrm{op}}}(P), -) \colon \operatorname{Mod} R_{\mathfrak{p}} \to \operatorname{Mod} A$ commutes with arbitrary direct sums, because $S_{A^{\mathrm{op}}}(P)$ is finitely generated over $R_{\mathfrak{p}}$. So the isomorphism in (4.3) follows from Proposition 2.25. Since $S_A(P) \cong S_A(P) \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}})$ by construction and we have Proposition 4.5(1), it only remains to show that (4.3) becomes an isomorphism upon application of $- \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}})$.

To this end, we remark that there is a canonical isomorphism

$$
\begin{aligned}
&\operatorname{Hom}_R(-, E_R(R/\mathfrak{p})^{(B)}) \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}) \\
&\cong \operatorname{Hom}_R(\operatorname{Hom}_{A^{\mathrm{op}}}(A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}, -), E_R(R/\mathfrak{p})^{(B)})
\end{aligned}
\tag{4.4}
$$

of functors $\operatorname{Mod} A_{\mathfrak{p}}^{\mathrm{op}} \to \operatorname{Mod} A$ (see [EJ00, Theorem 3.2.11]), because Hom_R, $\operatorname{Hom}_{A^{\mathrm{op}}}$, and $\otimes_A$ can be replaced by $\operatorname{Hom}_{R_{\mathfrak{p}}}$, $\operatorname{Hom}_{A_{\mathfrak{p}}^{\mathrm{op}}}$, and $\otimes_{A_{\mathfrak{p}}}$, respectively. Under (4.4) and the natural isomorphism $\operatorname{Hom}_{A^{\mathrm{op}}}(A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}, S_{A^{\mathrm{op}}}(P)) \xrightarrow{\sim} S_{A^{\mathrm{op}}}(P)$, application of $- \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}})$ to the first map in (4.3) yields a morphism

$$
\begin{aligned}
&\operatorname{Hom}_R(\operatorname{Hom}_{A^{\mathrm{op}}}(A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}, I_{A^{\mathrm{op}}}(P)), E_R(R/\mathfrak{p})^{(B)}) \\
&\to \operatorname{Hom}_R(S_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B)})
\end{aligned}
$$

of right A-modules. This is an isomorphism since it is induced by the isomorphism

$$
S_{A^{\mathrm{op}}}(P) \xrightarrow{\sim} \operatorname{Hom}_{A^{\mathrm{op}}}(A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}, I_{A^{\mathrm{op}}}(P))
$$

obtained by applying Proposition 4.8(2) to the injective envelope $S_{A^{\mathrm{op}}}(P) \to I_{A^{\mathrm{op}}}(P)$. $\qquad\square$

Proposition 4.10 leads us to the following definition, which is essential for the main results of this paper:

DEFINITION 4.11. Let $P \in \operatorname{Spec} A$ and $\mathfrak{p} := P \cap R$. We define

$$
T_A(P) := \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})),
$$

which is a flat cover of $S_A(P)$ in $\operatorname{Mod} A$ by Proposition 4.10, that is,

$$
T_A(P) = F_A(S_A(P))
$$

as isoclasses of right A-modules. As we recalled in the proof of Proposition 4.10, $T_A(P)$ is $\mathfrak{p}$-local and $\mathfrak{p}$-complete. Moreover, by Proposition 2.2(1), $T_A(P)$ is pure-injective, hence cotorsion.

REMARK 4.12. In (4.3) and Definition 4.11, each Hom_R can also be replaced by $\operatorname{Hom}_{\widehat{R_{\mathfrak{p}}}}$; see (2.6), Proposition A.12 (applied to $A = R$), and Remark 2.26. On the other hand, the second $\operatorname{Hom}_{\widehat{R_{\mathfrak{p}}}}$ in (4.5) below cannot be replaced either by Hom_R or by $\operatorname{Hom}_{R_{\mathfrak{p}}}$. To see this, consider the case where $(R, \mathfrak{m}, k)$ is local, $A = R$, $P = \mathfrak{m}$, and B is a set consisting of one element. Then $T_R(\mathfrak{m}) \cong \widehat{R}$,

and $E_R(k)$ naturally becomes an $\widehat{R}$-module (see section 2.5), so we have a canonical injection $f\colon E_R(k) = \operatorname{Hom}_{\widehat{R}}(\widehat{R}, E_R(k)) \to \operatorname{Hom}_R(\widehat{R}, E_R(k))$. This injection is not surjective as far as R is not $\mathfrak{m}$-complete, because the surjection $g\colon \operatorname{Hom}_R(\widehat{R}, E_R(k)) \to \operatorname{Hom}_R(R, E_R(k)) = E_R(k)$ induced by the completion map $R \to \widehat{R}$ is not injective, and gf is the identity map.

It should also be noticed that the last isomorphism of (4.5) shows that $T_A(P)$ is indecomposable.

PROPOSITION 4.13. *Let $P \in \operatorname{Spec} A$ and $\mathfrak{p} := P \cap R$. For every set B, the flat cover $T_A(P) \to S_A(P)$ induces an injective envelope*

$$
\begin{aligned}
S_{A^{\mathrm{op}}}(P)^{(B)} &\cong \operatorname{Hom}_{\widehat{R_{\mathfrak{p}}}}(S_A(P), E_R(R/\mathfrak{p})^{(B)}) \\
&\to \operatorname{Hom}_{\widehat{R_{\mathfrak{p}}}}(T_A(P), E_R(R/\mathfrak{p})^{(B)}) \cong I_{A^{\mathrm{op}}}(P)^{(B)}
\end{aligned}
\tag{4.5}
$$

in $\operatorname{Mod} A^{\mathrm{op}}$.

Proof. The first isomorphism in (4.5) follows from Proposition 4.10 and Remark 4.12.

Let $(-)^* := \operatorname{Hom}_{\widehat{R_{\mathfrak{p}}}}(-, E_R(R/\mathfrak{p}))$. Since $I_{A^{\mathrm{op}}}(P)$ is an artinian left $\widehat{A_{\mathfrak{p}}}$-module (see (2.6) and Remark 2.26), $T_A(P) = (I_{A^{\mathrm{op}}}(P))^*$ is a finitely generated right $\widehat{A_{\mathfrak{p}}}$-module by Theorem 2.27, and thus we have a canonical isomorphism

$$
(T_A(P)^*)^{(B)} \overset{\sim}{\longrightarrow} \operatorname{Hom}_{\widehat{R_{\mathfrak{p}}}}(T_A(P), E_R(R/\mathfrak{p})^{(B)}).
$$

Theorem 2.27 also yields a canonical isomorphism

$$
I_{A^{\mathrm{op}}}(P)^{(B)} \overset{\sim}{\longrightarrow} (I_{A^{\mathrm{op}}}(P)^{**})^{(B)} = (T_A(P)^*)^{(B)}
$$

of left A-modules. Therefore the last isomorphism in (4.5) holds.

Recall that $I_{A^{\mathrm{op}}}(P)^{(B)}$ is a $\mathfrak{p}$-local $\mathfrak{p}$-torsion injective left A-module (see Remark 2.26). Since we have Proposition 4.5(2) and $\operatorname{Hom}_{A^{\mathrm{op}}}(A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}, S_{A^{\mathrm{op}}}(P)^{(B)}) \cong S_{A^{\mathrm{op}}}(P)^{(B)}$, it suffices to show that the morphism in (4.5) becomes an isomorphism upon application of $\operatorname{Hom}_{A^{\mathrm{op}}}(A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}, -)$. By the tensor-hom adjunction, we have a canonical isomorphism

$$
\begin{aligned}
&\operatorname{Hom}_{A^{\mathrm{op}}}(A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}, \operatorname{Hom}_{\widehat{R_{\mathfrak{p}}}}(-, E_R(R/\mathfrak{p})^{(B)})) \\
&\cong \operatorname{Hom}_{\widehat{R_{\mathfrak{p}}}}(- \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}), E_R(R/\mathfrak{p})^{(B)})
\end{aligned}
$$

of functors $\operatorname{Mod} \widehat{A_{\mathfrak{p}}} \to \operatorname{Mod} A^{\mathrm{op}}$. In addition, the flat cover $T_A(P) \to S_A(P)$ induces an isomorphism $T_A(P) \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}) \overset{\sim}{\longrightarrow} S_A(P) \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}) \cong S_A(P)$ by Proposition 4.8(1). Therefore application of $\operatorname{Hom}_{A^{\mathrm{op}}}(A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}, -)$ makes the morphism in (4.5) an isomorphism. $\qquad\square$

In general, the character dual of a flat precover over an arbitrary ring is an injective preenvelope; see Proposition 2.2(3) and Remark 2.4. Conversely, if the ring is right coherent, then the character dual of an injective preenvelope of a right module is a flat precover; see [EJ00, Proposition 5.3.5].

5   Descriptions of local complete flat modules

In this section, we give various descriptions of local complete flat right modules over a Noether algebra. We first look back to some classical facts for a commutative noetherian ring R.

Let $\mathfrak{p} \in \operatorname{Spec} R$. Gruson and Raynaud [RG71, Part II, Proposition 2.4.3.1] showed that every $\mathfrak{p}$-local $\mathfrak{p}$-complete flat R-module is isomorphic to the $\mathfrak{p}$-adic completion of some free $R_{\mathfrak{p}}$-module. More precisely, it is shown that, given a flat R-module F, there is an isomorphism $\widehat{F_{\mathfrak{p}}} \cong (R_{\mathfrak{p}}^{(B)})_{\mathfrak{p}}^{\wedge}$, where $B := \dim_{\kappa(\mathfrak{p})} F \otimes_R \kappa(\mathfrak{p})$ (see also [EJ00, Lemma 6.7.4]). It is also shown that $(R_{\mathfrak{p}}^{(B)})_{\mathfrak{p}}^{\wedge}$ is a flat R-module ([RG71, Part II, (2.4.2)]). Furthermore, Enochs pointed out in [Eno84, p. 181, Example] that $(R_{\mathfrak{p}}^{(B)})_{\mathfrak{p}}^{\wedge}$ is isomorphic to $\operatorname{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})^{(B)})$ (see also [EJ00, Theorem 3.4.1(7)]). In particular, $(R_{\mathfrak{p}}^{(B)})_{\mathfrak{p}}^{\wedge}$ is a flat cotorsion R-module (Proposition 2.2(4)). It then follows that the following conditions are equivalent for an arbitrary R-module M:

(1) M is a $\mathfrak{p}$-local $\mathfrak{p}$-complete flat R-module.

(2) M is a $\mathfrak{p}$-local $\mathfrak{p}$-complete flat cotorsion R-module.

(3) M is isomorphic to the $\mathfrak{p}$-adic completion of a free $R_{\mathfrak{p}}$-module.

The term "free $R_{\mathfrak{p}}$-module" in (3) can be replaced by "projective $R_{\mathfrak{p}}$-module", "free $\widehat{R_{\mathfrak{p}}}$-module", or "projective $\widehat{R_{\mathfrak{p}}}$-module" because $R_{\mathfrak{p}}$ and $\widehat{R_{\mathfrak{p}}}$ are local rings and $(R_{\mathfrak{p}}^{(B)})_{\mathfrak{p}}^{\wedge} \cong (\widehat{R_{\mathfrak{p}}}^{(B)})_{\mathfrak{p}}^{\wedge}$ by Lemma A.4.

This section is devoted to generalizing these classical facts to an arbitrary Noether R-algebra A. We start with the next lemma, which slightly refines [GN02, Proposition 2.5.5] and is known when A is commutative (see [Rah09, Theorem 1.1]). Recall that n_P for each $P \in \operatorname{Spec} A$ is the number defined by (2.2).

Lemma 5.1. *For every $\mathfrak{p} \in \operatorname{Spec} R$, there is an isomorphism*

$$\operatorname{Hom}_R(A, E_R(R/\mathfrak{p})) \cong \bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} I_{A^{\mathrm{op}}}(P)^{n_P}$$

of left A-modules.

Proof. By Proposition 2.2(3), $\operatorname{Hom}_R(A, E_R(R/\mathfrak{p}))$ is an injective left A-module. As $E_R(R/\mathfrak{p}) \cong E_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}))$ by (2.6), the functor $\operatorname{Hom}_R(-, E_R(R/\mathfrak{p}))$ sends finitely generated right A-modules to $\mathfrak{p}$-local $\mathfrak{p}$-torsion left A-modules; see Remark 2.13 and (2.8). Thus, Proposition 4.5(2) applied to $I := \operatorname{Hom}_R(A, E_R(R/\mathfrak{p}))$ implies that the canonical morphism

$\operatorname{Hom}_{A^{\mathrm{op}}}(A_{\mathfrak{p}}/\operatorname{rad}A_{\mathfrak{p}}, I) \to I$ is an injective envelope in $\operatorname{Mod}A^{\mathrm{op}}$. Now we have

$$
\begin{aligned}
\operatorname{Hom}_{A^{\mathrm{op}}}(A_{\mathfrak{p}}/\operatorname{rad}A_{\mathfrak{p}}, I) &\cong \operatorname{Hom}_R(A \otimes_A (A_{\mathfrak{p}}/\operatorname{rad}A_{\mathfrak{p}}), E_R(R/\mathfrak{p})) \\
&\cong \operatorname{Hom}_R(\bigoplus_{\substack{P \in \operatorname{Spec}A \\ P \cap R = \mathfrak{p}}} S_A(P)^{n_P}, E_R(R/\mathfrak{p})) \\
&\cong \bigoplus_{\substack{P \in \operatorname{Spec}A \\ P \cap R = \mathfrak{p}}} S_{A^{\mathrm{op}}}(P)^{n_P},
\end{aligned}
$$

where the second isomorphism follows from Proposition 2.19, and the third follows from Propositions 2.15 and 2.25. Since each $I_{A^{\mathrm{op}}}(P)$ is the injective envelope of $S_{A^{\mathrm{op}}}(P)$, we obtain the desired isomorphism. □

PROPOSITION 5.2. *For every $\mathfrak{p} \in \operatorname{Spec}R$, there is an isomorphism*

$$
\widehat{A_{\mathfrak{p}}} \cong \bigoplus_{\substack{P \in \operatorname{Spec}A \\ P \cap R = \mathfrak{p}}} T_A(P)^{n_P}
$$

of right A-modules.

Proof. By Remark 2.13, (2.6), and (2.9), there is a canonical isomorphism

$$
\widehat{A_{\mathfrak{p}}} \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(A, E_R(R/\mathfrak{p})), E_R(R/\mathfrak{p})) \tag{5.1}
$$

of right $\widehat{A_{\mathfrak{p}}}$-modules. Thus the result follows from Proposition 2.15 and Lemma 5.1. □

REMARK 5.3. By Remark 2.13 and Proposition A.14, all A-homomorphism between $\mathfrak{p}$-local $\mathfrak{p}$-complete right A-modules are $\widehat{A_{\mathfrak{p}}}$-homomorphisms. The isomorphism in Proposition 5.2 is therefore an isomorphism of right $\widehat{A_{\mathfrak{p}}}$-modules. This implies that each $T_A(P)$ is a projective right $\widehat{A_{\mathfrak{p}}}$-module.
Similarly, by Remark 2.13 and Proposition A.12, all A-homomorphism between $\mathfrak{p}$-local $\mathfrak{p}$-torsion right A-modules are also $\widehat{A_{\mathfrak{p}}}$-homomorphisms. So the isomorphism in Lemma 5.1 is an isomorphism of left $\widehat{A_{\mathfrak{p}}}$-modules.

We will observe in Remark 7.4 that a direct sum of infinite copies of $T_A(P)$ is not necessarily cotorsion, but its $\mathfrak{p}$-adic completion is cotorsion by the next result.

PROPOSITION 5.4. *Let $P \in \operatorname{Spec}A$ and $\mathfrak{p} := P \cap R$. For every set B, there exists a canonical isomorphism*

$$
(T_A(P)^{(B)})_{\mathfrak{p}}^{\wedge} \xrightarrow{\sim} \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B)}).
$$

of A-modules. In particular, $(T_A(P)^{(B)})_{\mathfrak{p}}^{\wedge}$ is flat and pure-injective, that is, flat cotorsion.

Proof. Let $S_{A^{\mathrm{op}}}(P) \rightarrow I_{A^{\mathrm{op}}}(P)$ be the injective envelope. Applying $\operatorname{Hom}_R(-, E_R(R/\mathfrak{p}))$ to this, we obtain the flat cover $\operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})) \rightarrow \operatorname{Hom}_R(S_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p}))$ by Proposition 4.10. Taking the direct sum of B-indexed copies of the flat cover, we obtain the first row of the following diagram:

$$
\begin{array}{ccccc}
T_A(P)^{(B)} & =\!=\!= & \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p}))^{(B)} & \longrightarrow & \operatorname{Hom}_R(S_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p}))^{(B)} \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle\wr} \\
(T_A(P)^{(B)})_{\mathfrak{p}}^{\wedge} & \longrightarrow & \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B)}) & \longrightarrow & \operatorname{Hom}_R(S_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B)})
\end{array}
$$

The vertical morphisms are canonical ones, and the third is an isomorphism by the proof of Proposition 4.10. The first morphism in the second row is the unique morphism making the left square commutative; this exists since $\operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p})^{(B)})$ is $\mathfrak{p}$-complete; see Remark 3.2 and Proposition A.9. The second morphism in the second row is the one induced by the injective envelope $S_{A^{\mathrm{op}}}(P) \rightarrow I_{A^{\mathrm{op}}}(P)$, so it is a flat cover by Proposition 4.10. Moreover, the right square is commutative as well.

If we apply $- \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}})$ to the above commutative diagram, then the second morphism in each row becomes an isomorphism by the proof of Proposition 4.10, and the first vertical morphism becomes an isomorphism by Lemma A.4 and Remark 2.20, so the other morphisms in the diagram are also isomorphisms, where the third vertical morphism remains the same morphism. Therefore, it follows from Proposition 4.5(1) that the second row of the above diagram is a flat cover because $(T_A(P)^{(B)})_{\mathfrak{p}}^{\wedge}$ is a $\mathfrak{p}$-local $\mathfrak{p}$-complete flat right A-module; see Definition 4.11 and Propositions A.3 and A.5. □

REMARK 5.5. Let $\mathfrak{p} \in \operatorname{Spec} R$ and let B be a set. We can recover the known isomorphism

$$(R_{\mathfrak{p}}^{(B)})_{\mathfrak{p}}^{\wedge} \cong \operatorname{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})^{(B)}) \tag{5.2}$$

from Propositions 5.2 and 5.4. Indeed, Proposition 5.2 applied to $A = R$ simply identifies $\widehat{R_{\mathfrak{p}}}$ with $T_R(\mathfrak{p}) = \operatorname{Hom}_R(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{p}))$, and then Proposition 5.4 gives an isomorphism

$$(\widehat{R_{\mathfrak{p}}}^{(B)})_{\mathfrak{p}}^{\wedge} \xrightarrow{\sim} \operatorname{Hom}_R(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{p})^{(B)}),$$

where the left-hand side coincides with $(R_{\mathfrak{p}}^{(B)})_{\mathfrak{p}}^{\wedge}$ by Lemma A.4.

This isomorphism can be generalized to A. Applying the functor $- \otimes_R A$ to (5.2) and using Proposition A.2, we obtain an isomorphism

$$(A_{\mathfrak{p}}^{(B)})_{\mathfrak{p}}^{\wedge} \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_R(A, E_R(R/\mathfrak{p})), E_R(R/\mathfrak{p})^{(B)}),$$

as we deduced (2.9). In particular, it follows that $(A_{\mathfrak{p}}^{(B)})_{\mathfrak{p}}^{\wedge}$ is a $\mathfrak{p}$-local $\mathfrak{p}$-complete flat cotorsion right A-module; see Propositions A.5 and 2.2.

Theorem 5.6. *Let $\mathfrak{p} \in \operatorname{Spec} R$. For a right A-module M, the following are equivalent:*

(1) *M is a $\mathfrak{p}$-local $\mathfrak{p}$-complete flat right A-module.*

(2) *M is a $\mathfrak{p}$-local $\mathfrak{p}$-complete flat cotorsion right A-module.*

(3) *M is a direct summand of $(A_{\mathfrak{p}}^{(B)})_{\mathfrak{p}}^{\wedge}$ for some set B.*

(4) *M is isomorphic to*

$$\bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} (T_A(P)^{(B_P)})_{\mathfrak{p}}^{\wedge}$$

for some family of sets $\{B_P\}_P$.

The cardinality of each B_P in (4) is uniquely determined by M.

Proof. Assume (1). By Proposition 4.5(1), M is a flat cover of $M \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}})$ in $\operatorname{Mod} A$, where

$$M \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}) \cong \bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} S(P)^{(B_P)}$$

for a family of sets $\{B_P\}_P$; see Remark 4.7 and (4.2). Then (4) follows from the above decomposition and Propositions 4.10 and 5.4, along with Proposition 2.15 and the elementary fact that a finite direct sum of flat covers is a flat cover ([Xu96, Theorem 1.2.10]).

Assume (4). We first show the uniqueness of each B_P. By Propositions 4.10 and 5.4, $(T_A(P)^{(B_P)})_{\mathfrak{p}}^{\wedge}$ is the flat cover of $S_A(P)^{(B_P)}$. By Theorem 4.9, we have an isomorphism $(T_A(P)^{(B_P)})_{\mathfrak{p}}^{\wedge} \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}) \cong S_A(P)^{(B_P)}$. It then follows that

$$M \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}) \cong \left(\bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} (T_A(P)^{(B_P)})_{\mathfrak{p}}^{\wedge} \right) \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}})$$

$$\cong \bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} S_A(P)^{(B_P)}.$$

Therefore, the cardinality of each B_P is uniquely determined by M, due to the Krull-Remak-Schmidt-Azumaya theorem; see [Pre09, Theorem E.1.24] for example.

Let us next show that (4) implies (3). We know from Proposition 5.2 that $T_A(P)^{(B_P)}$ is a direct summand of $\widehat{A_{\mathfrak{p}}}^{(B_P)}$, so $(T_A(P)^{(B_P)})_{\mathfrak{p}}^{\wedge}$ is a direct summand of $(\widehat{A_{\mathfrak{p}}}^{(B_P)})_{\mathfrak{p}}^{\wedge}$, which is isomorphic to $(A_{\mathfrak{p}}^{(B_P)})_{\mathfrak{p}}^{\wedge}$ by Lemma A.4. Let B be the disjoint union of all B_P. Then

$$(A_{\mathfrak{p}}^{(B)})_{\mathfrak{p}}^{\wedge} \cong \bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} (A_{\mathfrak{p}}^{(B_P)})_{\mathfrak{p}}^{\wedge},$$

so (3) holds.

The implication $(3) \Rightarrow (2)$ is clear in view of the last sentence of Remark 5.5. The remaining implication $(2) \Rightarrow (1)$ is trivial. $\qquad\qquad\qquad\square$

REMARK 5.7. By Proposition 5.2, Remarks 5.3 and 5.5, and Theorem 5.6, the following conditions are equivalent for a right A-module M:

(1) M is a $\mathfrak{p}$-local $\mathfrak{p}$-complete flat right A-module.

(2) M is a $\mathfrak{p}$-local $\mathfrak{p}$-complete flat cotorsion right A-modules.

(3) M is isomorphic to the $\mathfrak{p}$-adic completion of a projective $\widehat{A_{\mathfrak{p}}}$-module.

If this is the case, then the projective $\widehat{A_{\mathfrak{p}}}$-module in (3) can be taken as a direct sum of indecomposable projective $\widehat{A_{\mathfrak{p}}}$-modules.

Let F be a flat right A-module. Then its localization $F_{\mathfrak{p}}$ is also a flat right A-module, so its $\mathfrak{p}$-adic completion $\widehat{F_{\mathfrak{p}}}$ is a $\mathfrak{p}$-local $\mathfrak{p}$-complete flat right A-module (see Propositions A.3 and A.5). Thus Theorem 5.6 yields an isomorphism $\widehat{F_{\mathfrak{p}}} \cong \bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} (T_A(P)^{(B_P)})_{\mathfrak{p}}^{\wedge}$, and the proof of the theorem shows that the index sets B_P are determined by a decomposition

$$F \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}) \cong \widehat{F_{\mathfrak{p}}} \otimes_A (A_{\mathfrak{p}}/\operatorname{rad} A_{\mathfrak{p}}) \cong \bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} S(P)^{(B_P)},$$

where the first isomorphism follows from Remark 2.20 and $\widehat{F_{\mathfrak{p}}} \otimes_R \kappa(\mathfrak{p}) \cong F \otimes_R \kappa(\mathfrak{p})$ (see Lemma A.4). If $A = R$, then the left-most side is $F \otimes_R \kappa(\mathfrak{p})$. Therefore, all the classical facts mentioned at the beginning of this section have been generalized to Noether algebras.

REMARK 5.8. Contrary to the classical case, the term "projective $\widehat{A_{\mathfrak{p}}}$-module" in Remark 5.7(3) cannot be replaced either by "free $A_{\mathfrak{p}}$-module", "projective $A_{\mathfrak{p}}$-module", or "free $\widehat{A_{\mathfrak{p}}}$-module", even if A is commutative. We give a counter-example to all of these at the same time.

Let k be a field and $R := k[x, y]/(y^2 - x^2(x+1))$. The ring R can be embedded into the polynomial ring $A := k[t]$ by $x \mapsto t^2 - 1$ and $y \mapsto t(t^2 - 1)$. Then R and $k[t]$ have the same quotient field $k(t)$, and $k[t]$ is the integral closure of R in the quotient field $k(t)$. Note that $A = k[t]$ is a Noether R-algebra since $A = R + Rt$.

Consider the maximal ideal $\mathfrak{m} := (x, y) \subseteq R$. We have $\mathfrak{m}^n A = (\mathfrak{n}_1 \mathfrak{n}_{-1})^n$ for each $n \geq 1$, where $\mathfrak{n}_i = (t - i) \in \operatorname{Max} A$. The $\mathfrak{m}$-adic completion of $A_{\mathfrak{m}}$ has a decomposition $\widehat{A_{\mathfrak{m}}} = A_{\mathfrak{m}}^{\wedge} \cong A_{\mathfrak{n}_1}^{\wedge} \times A_{\mathfrak{n}_{-1}}^{\wedge}$ as a ring (Remark 4.3). Letting $M := A_{\mathfrak{n}_1}^{\wedge}$, we have $M = \widehat{A_{\mathfrak{n}_1}} \cong T_A(\mathfrak{n}_1)$, so this is an indecomposable flat cotorsion A-module and also is an indecomposable projective $\widehat{A_{\mathfrak{m}}}$-module, which is $\mathfrak{m}$-complete. Thus M satisfies the equivalent conditions in Remark 5.7, setting $\mathfrak{p} := \mathfrak{m}$.

However, M is not isomorphic to the $\mathfrak{m}$-adic completion of any free $\widehat{A_{\mathfrak{m}}}$-module since such a completion is a direct sum of copies of $A_{\mathfrak{n}_1}^\wedge \times A_{\mathfrak{n}_{-1}}^\wedge$ (which is decomposable or zero). We also show that M is not isomorphic to the $\mathfrak{m}$-adic completion of any projective $A_{\mathfrak{m}}$-module, either. Given a nonzero projective $A_{\mathfrak{m}}$-module P, we have $P_{\mathfrak{m}}^\wedge \cong P_{\mathfrak{n}_1}^\wedge \oplus P_{\mathfrak{n}_{-1}}^\wedge$ (Remark 4.3). Since $\bigcap_{n \geq 1} \mathfrak{n}_i^n F = 0$ for every free $A_{\mathfrak{m}}$-module F (see [Mat89, Theorem 8.10]), the canonical map $F \to F_{\mathfrak{n}_i}^\wedge$ is injective, so the same holds for P. Therefore each $P_{\mathfrak{n}_i}^\wedge$ is nonzero, and hence $P_{\mathfrak{m}}^\wedge$ is not indecomposable.

6 Structure of flat cotorsion modules

Let us now complete the proof of the structure theorem for flat cotorsion modules (Theorem 1.1):

THEOREM 6.1. *Let A be a Noether R-algebra. A right A-module M is flat cotorsion if and only if M is isomorphic to*

$$\prod_{P \in \mathrm{Spec}\, A} (T_A(P)^{(B_P)})_{P \cap R}^\wedge \tag{6.1}$$

for some family of sets $\{B_P\}_{P \in \mathrm{Spec}\, A}$. The cardinality of each B_P is uniquely determined by M.

Proof. This follows from Proposition 3.7 and Theorem 5.6. $\square$

Consequently, we obtain a complete description of indecomposable flat cotorsion modules (Corollary 1.2):

COROLLARY 6.2. *Let A be a Noether R-algebra. Then there is a bijection*

$$\mathrm{Spec}\, A \xrightarrow{\sim} \{ \text{ isoclasses of indecomposable flat cotorsion right } A\text{-modules} \}$$

given by $P \mapsto T_A(P) = \mathrm{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/P \cap R))$.

Proof. By Remark 4.12, $T_A(P)$ is indecomposable. The uniqueness of the cardinalities of B_P in Theorem 6.1 implies that the map in the statement is injective. To observe the surjectivity, take an indecomposable flat cotorsion right A-module M. By Theorem 6.1, M is isomorphic to $(T_A(P)^{(B)})_{\mathfrak{p}}^\wedge$ for some $P \in \mathrm{Spec}\, A$ and a nonempty set B, where $\mathfrak{p} := P \cap R$. Since $T_A(P) \cong T_A(P)_{\mathfrak{p}}^\wedge$ is a direct summand of $(T_A(P)^{(B)})_{\mathfrak{p}}^\wedge$, the indecomposability of M implies that the cardinality of B is one, and hence $M \cong T_A(P)$. $\square$

EXAMPLE 6.3. Let R be a commutative noetherian ring and let A be the 2×2 lower triangular matrix algebra over R, that is,

$$A = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}.$$

Then A is a Noether R-algebra. We describe all isoclasses of simple, indecomposable injective, and indecomposable flat cotorsion right A-modules. The algebra A has a decomposition

$$A = \begin{pmatrix} R & 0 \end{pmatrix} \oplus \begin{pmatrix} R & R \end{pmatrix}$$

as a right A-module, where the action of A is matrix multiplication. For each $\mathfrak{p} \in \operatorname{Spec} R$,

$$P_1(\mathfrak{p}) := \begin{pmatrix} \mathfrak{p} & 0 \\ R & R \end{pmatrix} \quad \text{and} \quad P_2(\mathfrak{p}) := \begin{pmatrix} R & 0 \\ R & \mathfrak{p} \end{pmatrix}$$

are prime ideals of A, and varying $\mathfrak{p}$, these are all the prime ideals of A (see Proposition 2.15). The simple right $A_\mathfrak{p}$-modules are

$$S_A(P_1(\mathfrak{p})) = \begin{pmatrix} \kappa(\mathfrak{p}) & 0 \end{pmatrix} \quad \text{and} \quad S_A(P_2(\mathfrak{p})) = \frac{\begin{pmatrix} \kappa(\mathfrak{p}) & \kappa(\mathfrak{p}) \end{pmatrix}}{\begin{pmatrix} \kappa(\mathfrak{p}) & 0 \end{pmatrix}}.$$

By (2.2), $n_{P_i(\mathfrak{p})} = 1$ for $i = 1, 2$. On the other hand, the algebra A has a decomposition

$$A = \begin{pmatrix} R \\ R \end{pmatrix} \oplus \begin{pmatrix} 0 \\ R \end{pmatrix}$$

as a left A-module, and we have

$$\operatorname{Hom}_R(A, E_R(R/\mathfrak{p})) \cong \begin{pmatrix} E_R(R/\mathfrak{p}) & E_R(R/\mathfrak{p}) \end{pmatrix} \oplus \frac{\begin{pmatrix} E_R(R/\mathfrak{p}) & E_R(R/\mathfrak{p}) \end{pmatrix}}{\begin{pmatrix} E_R(R/\mathfrak{p}) & 0 \end{pmatrix}}$$

as right A-modules. Hence, by Lemma 5.1,

$$I_A(P_1(\mathfrak{p})) = \begin{pmatrix} E_R(R/\mathfrak{p}) & E_R(R/\mathfrak{p}) \end{pmatrix} \text{ and } I_A(P_2(\mathfrak{p})) = \frac{\begin{pmatrix} E_R(R/\mathfrak{p}) & E_R(R/\mathfrak{p}) \end{pmatrix}}{\begin{pmatrix} E_R(R/\mathfrak{p}) & 0 \end{pmatrix}}$$

because each $I_A(P_i(\mathfrak{p}))$ should have $S_A(P_i(\mathfrak{p}))$ as a right A-submodule. Similarly, by Proposition 5.2,

$$T_A(P_1(\mathfrak{p})) = \begin{pmatrix} \widehat{R_\mathfrak{p}} & 0 \end{pmatrix} \quad \text{and} \quad T_A(P_2(\mathfrak{p})) = \begin{pmatrix} \widehat{R_\mathfrak{p}} & \widehat{R_\mathfrak{p}} \end{pmatrix}$$

since each $T_A(P_i(\mathfrak{p}))$ should have $S_A(P_i(\mathfrak{p}))$ as a quotient A-module.

REMARK 6.4. Let us consider the case where $R = k$ is a field, that is, A is a finite-dimensional k-algebra. As mentioned in Remark 2.9, all flat right A-modules are projective and all right A-modules are cotorsion. Thus the flat cotorsion right A-modules are precisely the projective right A-modules. For every $P \in \operatorname{Spec} A$,

$$T_A(P) = \operatorname{Hom}_k(I_{A^{\mathrm{op}}}(P), k)$$

is the projective cover of $S_A(P)$ (see Proposition 4.10), and the product in Theorem 6.1 can be written as

$$\bigoplus_{P \in \operatorname{Spec} A} T_A(P)^{(B_P)}$$

since $\operatorname{Spec} A$ is a finite set by Proposition 2.15 and $P \cap k = 0$ for each $P \in \operatorname{Spec} A$.

7 Flat cotorsion modules as flat covers and pure-injective envelopes

Let A be a Noether R-algebra. In this section, we prove Theorem 7.6, which gives other descriptions of each flat cotorsion right A-module in terms of a flat cover and a pure-injective envelope.

LEMMA 7.1. *For each $\mathfrak{p} \in \operatorname{Spec} R$, let $f(\mathfrak{p})\colon F(\mathfrak{p}) \to M(\mathfrak{p})$ be a flat cover in* $\operatorname{Mod} A$ *such that $F(\mathfrak{p})$ is $\mathfrak{p}$-local and $\mathfrak{p}$-complete. Then the product*

$$\prod_{\mathfrak{p}\in\operatorname{Spec} R} f(\mathfrak{p})\colon \prod_{\mathfrak{p}\in\operatorname{Spec} R} F(\mathfrak{p}) \to \prod_{\mathfrak{p}\in\operatorname{Spec} R} M(\mathfrak{p})$$

is a flat cover.

Proof. Denote the product of morphisms by $f\colon F \to M$, where F is a flat right A-module since A is left noetherian. For every flat right A-module F', the morphism $\operatorname{Hom}_A(F', f(\mathfrak{p}))$ is an epimorphism since $f(\mathfrak{p})$ is a flat (pre)cover. Hence the product $\operatorname{Hom}_A(F', f) = \prod_{\mathfrak{p}\in\operatorname{Spec} R} \operatorname{Hom}_A(F', f(\mathfrak{p}))$ is also an epimorphism. This shows that f is a flat precover.

It remains to show that f is right minimal. Let $g \in \operatorname{End}_A(F)$ such that $fg = f$. For each $\mathfrak{q} \in \operatorname{Spec} R$, we have a commutative diagram
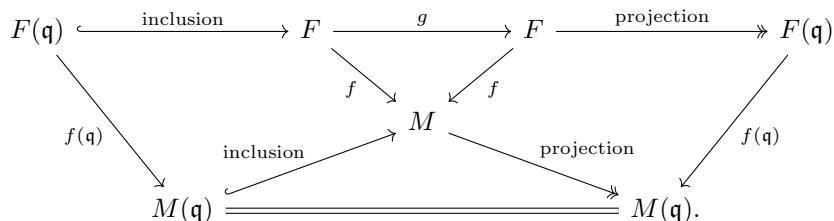
Since $f(\mathfrak{q})$ is a flat cover, the composition in the first row is an isomorphism. Therefore, g is an isomorphism by Lemma 3.5. $\qquad\square$

A special case of Lemma 7.1 is discussed in the third paragraph of the proof of [Eno84, p. 183, Theorem].

The assumption in Lemma 7.1 that each $F(\mathfrak{p})$ is $\mathfrak{p}$-local and $\mathfrak{p}$-complete is satisfied if each $M(\mathfrak{p})$ is $\mathfrak{p}$-local and $\mathfrak{p}$-complete, by Remark 2.21 and Lemma 4.6(1).

PROPOSITION 7.2. *Let M be a right A-module that is finitely generated or projective. Then the morphism $M \to \prod_{\mathfrak{m}\in\operatorname{Max} R} M_{\mathfrak{m}}^{\wedge}$ induced by the completion maps $M \to M_{\mathfrak{m}}^{\wedge}$ is a pure-injective envelope.*

When $A = R$, this is shown in [EJ00, Proposition 6.7.3 and Remark 6.7.12]. Let us first recall an elementary fact before giving a proof.

REMARK 7.3. For every right A-module M, the morphism $g\colon M \to \prod_{\mathfrak{m}\in\operatorname{Max} R} M_{\mathfrak{m}}$ induced by the localization maps $M \to M_{\mathfrak{m}}$ is a pure monomorphism, or equivalently, $g \otimes_A N$ is a monomorphism in $\operatorname{Mod} R$ for every finitely

generated (presented) left A-module N (see [GT12, Lemma 2.19]). Indeed, the functor $- \otimes_A N$ commutes with arbitrary direct products (see [EJ00, Theorem 3.2.22]) and the localization functors $(-)_{\mathfrak{m}}$, so the morphism $g \otimes_A N$ can be written as $M \otimes_R N \to \prod_{\mathfrak{m} \in \operatorname{Max} R}(M \otimes_R N)_{\mathfrak{m}}$. This is a monomorphism, because if a given element x of $M \otimes_R N$ becomes zero in $(M \otimes_R N)_{\mathfrak{m}}$ for all $\mathfrak{m} \in \operatorname{Max} R$, then x is zero in $M \otimes_R N$; see [Mat89, Theorem 4.6].

It is also seen from the above argument that, given a family $\{M_b\}_{b \in B}$ of right A-modules, the canonical inclusion $\bigoplus_{b \in B} M_b \hookrightarrow \prod_{b \in B} M_b$ is a pure monomorphism ([Pre09, Lemma 2.1.10]).

Proof of Proposition 7.2. Denote by f the morphism $M \to \prod_{\mathfrak{m} \in \operatorname{Max} R} M_{\mathfrak{m}}^{\wedge}$. To see that f is left minimal, it suffices, by (3.3), to show that each morphism $M \to M_{\mathfrak{m}}^{\wedge}$ is left minimal for all $\mathfrak{m} \in \operatorname{Max} R$ (since $M_{\mathfrak{m}}^{\wedge}$ is $\mathfrak{m}$-local and $\mathfrak{m}$-complete), and this follows from the adjoint property of the $\mathfrak{m}$-adic completion functor (Proposition A.9).

It remains to check that f is a pure-injective preenvelope. If M is finitely generated, then each $M_{\mathfrak{m}}^{\wedge}$ is pure-injective by Proposition 2.2(1), Remark 2.21, and (2.9). If M is projective, then $\prod_{\mathfrak{m} \in \operatorname{Max} R} M_{\mathfrak{m}}^{\wedge}$ is flat cotorsion (Remark 5.7) and hence pure-injective by Proposition 2.3. Therefore, it suffices to check that f is a pure monomorphism; see Proposition 2.5(1). By Remark 7.3, we only need to check that the completion map $M_{\mathfrak{m}} \to M_{\mathfrak{m}}^{\wedge}$ is a pure monomorphism for each $\mathfrak{m} \in \operatorname{Max} R$, so we may assume that A is a Noether algebra over a local ring R with maximal ideal $\mathfrak{m}$.

If M is finitely generated, then the completion map $M \to \widehat{M}$ is a pure monomorphism since it coincides with the map induced by the pure monomorphism $R \to \widehat{R}$; see section 2.5.

If M is projective, then we may replace it by a free module $A^{(B)}$ with basis B. The inclusion $g \colon A^{(B)} \hookrightarrow A^B$ and the canonical morphism $\eta \colon \operatorname{id}_{\operatorname{Mod} A} \to \Lambda^{\mathfrak{m}}$ of functors $\operatorname{Mod} A \to \operatorname{Mod} A$ yield a commutative diagram:

$$
\begin{array}{ccc}
A^{(B)} & \xrightarrow{\ \eta(A^{(B)})\ } & \Lambda^{\mathfrak{m}}(A^{(B)}) \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \Lambda^{\mathfrak{m}} g} \\
A^B & \xrightarrow[\ \eta(A^B)\ ]{} & \Lambda^{\mathfrak{m}}(A^B).
\end{array}
$$

The functor $\Lambda^{\mathfrak{m}}$ commutes with arbitrary direct products (Proposition A.8), so $\Lambda^{\mathfrak{m}}(A^B) \cong (\Lambda^{\mathfrak{m}} A)^B$ and $\eta(A^B)$ is identified with $\eta(A)^B$.

Now, g is a pure monomorphism (Remark 7.3). The completion map $\eta(A) \colon A \to \Lambda^{\mathfrak{m}} A = \widehat{A}$ is also a pure monomorphism as we recalled above, and hence so is $\eta(A)^B = \eta(A^B)$ (because tensoring with a finitely generated module commutes with arbitrary direct products; see Remark 7.3). Therefore the commutative diagram above implies that $\eta(A^{(B)})$ is a pure monomorphism, as desired. $\qquad\square$

As a consequence of Proposition 7.2, we obtain the following remark:

Remark 7.4. Let $P \in \operatorname{Spec} A$ and $\mathfrak{p} := P \cap R$. Recall that $T_A(P)$ is a projective right $\widehat{A_{\mathfrak{p}}}$-module (Remark 5.3). It then follows from Proposition 7.2 that the completion map

$$f \colon T_A(P)^{(B)} \to (T_A(P)^{(B)})_{\mathfrak{p}}^{\wedge}$$

is a pure-injective envelope in $\operatorname{Mod} \widehat{A_{\mathfrak{p}}}$, for every set B. Hence the cokernel of this map is a flat right $\widehat{A_{\mathfrak{p}}}$-module (see Proposition 2.8 and Lemma 2.7). The cokernel is also a flat right A-module, as the canonical maps $A \to A_{\mathfrak{p}} \to \widehat{A_{\mathfrak{p}}}$ are flat ring homomorphisms. This shows that f is a pure monomorphism in $\operatorname{Mod} A$ as well. Moreover, f is left minimal in $\operatorname{Mod} A$ by Proposition A.9, and $(T_A(P)^{(B)})_{\mathfrak{p}}^{\wedge}$ is pure-injective in $\operatorname{Mod} A$ by Proposition 5.4. Therefore, f is a pure-injective envelope in $\operatorname{Mod} A$.

Let us consider the case where $A = R$ is a local ring and $P = \mathfrak{m}$ is its maximal ideal. Then $T_R(\mathfrak{m}) \cong \widehat{R}$, so the pure-injective envelope of $\widehat{R}^{(B)}$ (which is also the cotorsion envelope) is the completion map $f \colon \widehat{R}^{(B)} \to \widehat{\widehat{R}^{(B)}}$. If B is an infinite set and the Krull dimension of R is greater than 0, then f is not an isomorphism. This shows that a direct sum of copies of $T_A(P)$ is neither pure-injective nor cotorsion in general.

Lemma 7.5. For each $\mathfrak{p} \in \operatorname{Spec} R$, let $g(\mathfrak{p}) \colon M(\mathfrak{p}) \to H(\mathfrak{p})$ be a pure-injective envelope in $\operatorname{Mod} A$ such that $H(\mathfrak{p})$ is $\mathfrak{p}$-local and $\mathfrak{p}$-complete. Then the morphism

$$g \colon \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} M(\mathfrak{p}) \to \prod_{\mathfrak{p} \in \operatorname{Spec} R} H(\mathfrak{p})$$

induced by $\{g(\mathfrak{p})\}_{\mathfrak{p} \in \operatorname{Spec} R}$ is a pure-injective envelope.

Proof. We first remark that g is factorized as the composition of the direct sum

$$\bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} g(\mathfrak{p}) \colon \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} M(\mathfrak{p}) \to \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} H(\mathfrak{p})$$

and the canonical map $\bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} H(\mathfrak{p}) \to \prod_{\mathfrak{p} \in \operatorname{Spec} R} H(\mathfrak{p})$, where the former map is evidently a pure monomorphism, and so is the latter by Remark 7.3. It follows from the definition of pure-injective modules that the direct product $\prod_{\mathfrak{p} \in \operatorname{Spec} R} H(\mathfrak{p})$ is pure-injective. Thus, g is a pure monomorphism into a pure-injective module, that is, g is a pure-injective preenvelope; see Proposition 2.5(1).

It remains to show the left minimality of $g \colon M \to H$, where $M := \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} M(\mathfrak{p})$ and $H := \prod_{\mathfrak{p} \in \operatorname{Spec} R} H(\mathfrak{p})$. Let $h \in \operatorname{End}_A(H)$ with $hg = g$.

For each $\mathfrak{q} \in \operatorname{Spec} R$, we have a commutative diagram
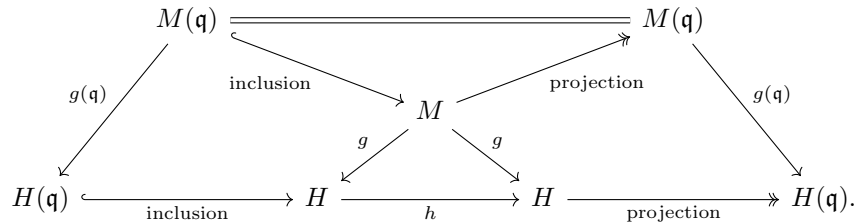
Since $g(\mathfrak{q})$ is a pure-injective envelope, the composition in the second row is an isomorphism. Therefore, h is an isomorphism by Lemma 3.5. $\qquad\square$

We can now prove the main result in this section. Recall that, for a right A-module M, its pure-injective envelope and cotorsion envelope are denoted by $H_A(M)$ and $C_A(M)$, respectively.

THEOREM 7.6. *For every family of sets $\{B_P\}_{P \in \operatorname{Spec} A}$, we have isomorphisms of right A-modules*

$$F_A\Big(\prod_{P \in \operatorname{Spec} A} S_A(P)^{(B_P)} \Big) \cong \prod_{P \in \operatorname{Spec} A} (T_A(P)^{(B_P)})^{\wedge}_{P \cap R}$$

$$\cong H_A\Big(\bigoplus_{P \in \operatorname{Spec} A} T_A(P)^{(B_P)} \Big),$$

where H_A can be replaced by C_A.

Proof. For every $P \in \operatorname{Spec} A$, we have a flat cover

$$f(P)\colon (T_A(P)^{(B_P)})^{\wedge}_{P \cap R} \to S_A(P)^{(B_P)}$$

and a pure-injective envelope

$$g(P)\colon T_A(P)^{(B_P)} \to (T_A(P)^{(B_P)})^{\wedge}_{P \cap R}$$

by Proposition 4.10, Proposition 5.4, and Remark 7.4. Fix $\mathfrak{p} \in \operatorname{Spec} R$, and define $f(\mathfrak{p})\colon F(\mathfrak{p}) \to M(\mathfrak{p})$ and $g(\mathfrak{p})\colon N(\mathfrak{p}) \to H(\mathfrak{p})$ as the direct sum of $f(P)$ and the direct sum of $g(P)$, respectively, for all $P \in \operatorname{Spec} A$ with $P \cap R = \mathfrak{p}$. Since there are only finitely many such P (Proposition 2.15), $f(\mathfrak{p})$ and $g(\mathfrak{p})$ are a flat cover and a pure-injective envelope, respectively (see [Xu96, Theorems 1.2.5 and 1.2.10]). Moreover, $F(\mathfrak{p}) = H(\mathfrak{p})$ is $\mathfrak{p}$-local and $\mathfrak{p}$-complete. Thus the first and the second isomorphisms in the theorem follow from Lemma 7.1 and Lemma 7.5, respectively. Proposition 2.8 shows that H_A can be replaced by C_A. $\qquad\square$

REMARK 7.7. It is known that each pure-injective right module M over an arbitrary ring A has a decomposition $M \cong H_A(\bigoplus_{c \in C} M_c^{(B_c)}) \oplus N$, where

$\{M_c\}_{c\in C}$ is a family of indecomposable pure-injective modules such that $M_c \not\cong M_{c'}$ whenever $c \neq c'$, $\{B_c\}_{c\in C}$ is a family of sets, and N is a *superdecomposable* module, that is, a module having no indecomposable direct summands. The cardinality of each B_c and the isoclass of N are uniquely determined by M. See [Pre09, Theorem 4.4.2].

For a flat cotorsion right module M over a Noether R-algebra A, this fact has been explicitly realized as

$$M \cong H_A\left(\bigoplus_{P\in\operatorname{Spec} A} T_A(P)^{(B_P)} \right)$$

by Theorem 6.1 and the second isomorphism in Theorem 7.6. Note that the superdecomposable summand N is interpreted as the zero module.

REMARK 7.8. Another general fact we should mention is that every flat right module over an arbitrary ring admits a pure monomorphism into a direct product of indecomposable flat cotorsion right modules; this was shown by Guil Asensio and Herzog [GAH07, Corollary 10]. In particular, if the ring is left coherent, then this result implies that every flat cotorsion right module is a direct summand of a direct product of indecomposable flat cotorsion right modules, as flat cotorsion right modules are pure-injective (see Remark 2.4).

In the case of a Noether R-algebra A, we can recover the result ([GAH07, Corollary 10]) as follows: Given a flat right A-module M, the pure-injective envelope $M \to H_A(M)$ is a pure monomorphism into a flat cotorsion module (see Propositions 2.5 and 2.8 and Lemma 2.7), so we may assume that M itself is flat cotorsion, and thus

$$M \cong \prod_{P\in\operatorname{Spec} A} (T_A(P)^{(B_P)})^\wedge_{P\cap R}$$

by Theorem 6.1. We show that the canonical morphism

$$\prod_{P\in\operatorname{Spec} A} (T_A(P)^{(B_P)})^\wedge_{P\cap R} \to \prod_{P\in\operatorname{Spec} A} (T_A(P)^{B_P})^\wedge_{P\cap R} \tag{7.1}$$

is a pure monomorphism. As we observed in Remark 7.3, it suffices to see that $-\otimes_A N$ applied to (7.1) is a monomorphism for every finitely generated left A-module N. We also observed that $-\otimes_A N$ commutes with direct products. Since $T_A(P)^{(B_P)}$ and $T_A(P)^{B_P}$ are flat, Proposition A.2 implies that $-\otimes_A N$ applied to (7.1) becomes

$$\prod_{P\in\operatorname{Spec} A} ((T_A(P)\otimes_A N)^{(B_P)})^\wedge_{P\cap R} \to \prod_{P\in\operatorname{Spec} A} ((T_A(P)\otimes_A N)^{B_P})^\wedge_{P\cap R},$$

which is clearly a monomorphism. Thus (7.1) is a pure monomorphism. Since completion commutes with direct products (Proposition A.8) and each $T_A(P)$ is $(P\cap R)$-complete, the right-hand side of (7.1) is $\prod_{P\in\operatorname{Spec} A} T_A(P)^{B_P}$, which is a direct product of indecomposable flat cotorsion right A-modules.

8 Ziegler spectra and elementary duality

Let A be a Noether R-algebra. Combining Theorem 2.22 and Corollary 6.2, it follows that there exists a one-to-one correspondence between the isoclasses of indecomposable injective left A-modules and the isoclasses of indecomposable flat cotorsion right A-modules, given by $I_{A^{\mathrm{op}}}(P) \mapsto T_A(P)$ for each $P \in \operatorname{Spec} A$. In this section, we observe that this one-to-one correspondence is compatible, and is actually induced from, elementary duality between the Ziegler spectrum of A^{op} and that of A (Theorem 8.14).

For a while, let A be an arbitrary ring. Denote by $\mathrm{fp}(\operatorname{mod} A, \mathrm{Ab})$ the category of finitely presented additive functors $\operatorname{mod} A \to \mathrm{Ab}$, where $\operatorname{mod} A$ is the category of finitely presented right A-modules and Ab is the category of abelian groups. Each functor $F \in \mathrm{fp}(\operatorname{mod} A, \mathrm{Ab})$ admits a unique extension $\overrightarrow{F} \colon \operatorname{Mod} A \to \mathrm{Ab}$ (up to isomorphism) that commutes with (filtered) direct limits. By definition, there exists an exact sequence $\operatorname{Hom}_A(M, -) \xrightarrow{\operatorname{Hom}_A(f, -)} \operatorname{Hom}_A(L, -) \to F \to 0$ in $\mathrm{fp}(\operatorname{mod} A, \mathrm{Ab})$, where $f \colon L \to M$ is a morphism in $\operatorname{mod} A$, and the extension $\overrightarrow{F}$ can be defined as the cokernel of the same morphism $\operatorname{Hom}_A(M, -) \xrightarrow{\operatorname{Hom}_A(f, -)} \operatorname{Hom}_A(L, -)$ but regarded as a morphism of functors $\operatorname{Mod} A \to \mathrm{Ab}$; see [Pre09, Corollary 10.2.42].

Denote by Zg_A the *Ziegler spectrum* of A, which is a topological space whose points are the isoclasses of indecomposable pure-injective right A-modules (they actually form a small set; see [Pre09, Corollary 4.3.38]). The topology on Zg_A is defined so that $\{\, (F) \mid F \in \mathrm{fp}(\operatorname{mod} A, \mathrm{Ab}) \,\}$ is an open basis, where

$$(F) := \{\, N \in \mathrm{Zg}_A \mid \overrightarrow{F}(N) \neq 0 \,\}$$

for each $F \in \mathrm{fp}(\operatorname{mod} A, \mathrm{Ab})$; see [Pre09, Corollary 10.2.45].

Although this definition of the topology is convenient, it should be mentioned that the topology was originally introduced in terms of model theory for modules, and such a viewpoint helps us to understand elementary duality, particularly via Lemma 8.3. For this reason, we interpret the topology on the Ziegler spectrum via model theoretic language.

Let l, m, and n be nonnegative integers. Let H and H' be matrices whose entries are elements of A, where H is an $n \times l$ matrix and H' is an $m \times l$ matrix. A *pp-formula* (*positive primitive formula*) ϕ for right A-modules is a formula of the form $\exists y(xH = yH')$, where $x = (x_1, \ldots, x_n)$ is a tuple of free variables and $y = (y_1, \ldots, y_m)$ is a tuple of variables bound by the existential quantifier. So n is referred to as the number of free variables in ϕ.

For each right A-module M, the pp-formula ϕ defines an abelian subgroup of M^n as

$$F_\phi(M) := \{\, x \in M^n \mid \text{there exists } y \in M^m \text{ such that } xH = yH' \,\},$$

where x and y are regarded as row vectors. A subgroup of M^n of this form (for some l and m) is called a subgroup of M^n *pp-definable* in M. For every

morphism $f\colon M \to N$ in $\operatorname{Mod} A$, the direct sum $f^n\colon M^n \to N^n$ restricts to a homomorphism $F_\phi(M) \to F_\phi(N)$ of abelian groups. So we obtain an additive functor $F_\phi\colon \operatorname{Mod} A \to \operatorname{Ab}$.

Let ϕ and ψ be pp-formulas for right A-modules, and suppose that they have the same number, say n, of free variables. Then both $F_\phi(M)$ and $F_\psi(M)$ are subgroups of M^n for each $M \in \operatorname{Mod} A$. We write $\phi \leq \psi$ if $F_\phi(M) \subseteq F_\psi(M)$ for all right A-modules M. Moreover, ϕ and ψ are said to be *equivalent* if $\phi \leq \psi$ and $\phi \geq \psi$, in which case we have equality of functors $F_\phi = F_\psi$.

A *pp-pair* ϕ/ψ is a pair of pp-formulas with $\phi \geq \psi$. Each pp-pair ϕ/ψ defines an additive functor $F_{\phi/\psi}\colon \operatorname{Mod} A \to \operatorname{Ab}$ by the assignment $M \mapsto F_\phi(M)/F_\psi(M)$. The following remarkable fact allows us to understand the topology on Zg_A via pp-pairs.

THEOREM 8.1. *Let A be a ring. For each pp-pair ϕ/ψ, the functor $F_{\phi/\psi}\colon \operatorname{Mod} A \to \operatorname{Ab}$ commutes with direct limits and its restriction to $\operatorname{mod} A$ belongs to $\operatorname{fp}(\operatorname{mod} A, \operatorname{Ab})$. Conversely, for each $F \in \operatorname{fp}(\operatorname{mod} A, \operatorname{Ab})$, there exists a pp-pair $\phi \geq \psi$ such that $\overrightarrow{F} \cong F_{\phi/\psi}$ as functors $\operatorname{Mod} A \to \operatorname{Ab}$.*

Proof. See [Pre09, Lemma 1.2.31 and Remark 10.2.29] for the first statement, and [Pre09, Proposition 10.2.43] for the second. $\square$

In fact, the category $\operatorname{fp}(\operatorname{mod} A, \operatorname{Ab})$ is equivalent to the *category of pp-pairs* for right A-modules; see [Pre09, Theorem 10.2.30].

It follows from Theorem 8.1 that

$$\{ (F_{\phi/\psi}) \mid \phi/\psi \text{ is a pp-pair for right } A\text{-modules} \}$$

is an open basis for Zg_A.

We now explain elementary duality, first in terms of pp-formulas. Let ϕ be a pp-formula $\exists y(xH = yH')$ for right A-modules, where H is an $n \times l$ matrix and H' is an $m \times l$ matrix. Regarding the transposes H^{t} and H'^{t} as matrices over A^{op}, we can define the pp-formula $D\phi$ for right A^{op}-modules to be $\exists z(xK = zK')$, where $x = (x_1, \ldots, x_n)$ is a tuple of free variables, $z = (z_1 \ldots, z_l)$ is a tuple of bound variables, $K := \begin{pmatrix} I & 0 \end{pmatrix}$, $K' := \begin{pmatrix} H^{\mathrm{t}} & H'^{\mathrm{t}} \end{pmatrix}$, I is the $n \times n$ identity matrix, and 0 is the $n \times m$ zero matrix. The pp-formula $D\phi$ is called the *elementary dual* of ϕ. For each right A^{op}-module M,

$$F_{D\phi}(M) = \{ x \in M^n \mid \text{there exists } z \in M^l \text{ such that } xK = zK' \}$$
$$= \{ x \in M^n \mid \text{there exists } z \in M^l \text{ such that } x = zH^{\mathrm{t}} \text{ and } zH'^{\mathrm{t}} = 0 \}.$$

If we regard M as a left A-module and x and z as column vectors, then the equations in the second line can be written as $x = Hz$ and $H'z = 0$.

We can apply the same construction to $D\phi$ and obtain the pp-formula $D^2\phi = DD\phi$ for right A-modules. Elementary duality claims that $D^2\phi$ is equivalent to ϕ, and moreover, two pp-formulas ϕ and ψ satisfy $\phi \geq \psi$ if and only if $D\psi \geq D\phi$. Denote by pp_A^n the poset of equivalent classes of pp-formulas in n free variables for right A-modules. In fact, this is a modular lattice; see [Pre09, §1.1.3].

Theorem 8.2 (Elementary duality of pp-formulas). *Let A be a ring. The operator D is an anti-isomorphism from pp_A^n to $\mathrm{pp}_{A^{\mathrm{op}}}^n$ for each $n \geq 0$.*

Proof. See [Pre09, Proposition 1.3.1]. □

The following fact is the key to describe elementary duality of Ziegler spectra:

Lemma 8.3. *Let A be a ring and let M be a right A-module. Fix a ring homomorphism $S \to \mathrm{End}_A(M)$ from a ring S. Let E be an injective cogenerator in $\mathrm{Mod}\, S^{\mathrm{op}}$ and set $M^* := \mathrm{Hom}_{S^{\mathrm{op}}}(M, E) \in \mathrm{Mod}\, A^{\mathrm{op}}$. For each pp-pair ϕ/ψ, we have $F_{\phi/\psi}(M) = 0$ if and only if $F_{D\psi/D\phi}(M^*) = 0$.*

Proof. See [Pre09, Theorem 1.3.15]. □

Let U be an open subset of Zg_A such that $U = (F_{\phi/\psi})$ for some pp-pair ϕ/ψ for right A-modules. Since $D\psi/D\phi$ is a pp-pair for right A^{op}-modules, $(F_{D\psi/D\phi})$ is an open subset of $\mathrm{Zg}_{A^{\mathrm{op}}}$, which does not depend on the choice of the pp-pair ϕ/ψ for U. Indeed, for each $N \in \mathrm{Zg}_{A^{\mathrm{op}}}$, Lemma 8.3 (applied to $S := \mathrm{End}_{A^{\mathrm{op}}}(N)$ and arbitrary E) implies that

$$N \in (F_{D\psi/D\phi}) \iff F_{D\psi/D\phi}(N) \neq 0 \iff F_{\phi/\psi}(N^*) \neq 0 \iff N^* \in U.$$

Therefore we can write $DU := (F_{D\psi/D\phi})$. Then $D^2 U = U$ by Theorem 8.2, so this gives an order-preserving bijection between open bases of Zg_A and $\mathrm{Zg}_{A^{\mathrm{op}}}$, so it extends uniquely to an order-preserving bijection between all open subsets of Zg_A and those of $\mathrm{Zg}_{A^{\mathrm{op}}}$. We summarize these facts in the next theorem, which was originally shown by Herzog [Her93, Proposition 4.4].

Theorem 8.4 (Elementary duality of Ziegler spectra). *For every ring A, there is an order-preserving bijection*

$$D \colon \{\, \text{open subsets of } \mathrm{Zg}_A \,\} \xrightarrow{\sim} \{\, \text{open subsets of } \mathrm{Zg}_{A^{\mathrm{op}}} \,\},$$

which sends $(F_{\phi/\psi})$ to $(F_{D\psi/D\phi})$ for each pp-pair ϕ/ψ.

Proof. See [Pre09, Theorem 5.4.1]. □

We may also interpret elementary duality as an order-preserving bijection between the closed subsets of Zg_A and those of $\mathrm{Zg}_{A^{\mathrm{op}}}$ in an obvious way; that is, given a closed subset $C \subseteq \mathrm{Zg}_A$, its complement $C^{\mathrm{c}} = \mathrm{Zg}_A \setminus C$ is open, so send C to $DC := (D(C^{\mathrm{c}}))^{\mathrm{c}} = \mathrm{Zg}_{A^{\mathrm{op}}} \setminus D(C^{\mathrm{c}})$. Note that, if C is also open, then elementary duality for open subsets and that for closed subsets send C to the same open closed subset of $\mathrm{Zg}_{A^{\mathrm{op}}}$. Thus we can safely denote both bijections by D.

Remark 8.5. Elementary duality does not mean that there is a homeomorphism between Zg_A and $\mathrm{Zg}_{A^{\mathrm{op}}}$. This is due to the fact that the Ziegler spectrum is not necessarily a T_0-space, that is, it may contain topologically indistinguishable points (see [Pre09, p. 267]). It is not known in general whether Zg_A is homeomorphic to $\mathrm{Zg}_{A^{\mathrm{op}}}$; see [Pre09, Question 5.4.8].

We next explain how elementary duality is interpreted in terms of finitely presented functors.

Theorem 8.6 (Auslander-Gruson-Jensen duality). *For every ring A, there is a duality of categories*

$$d\colon \mathrm{fp}(\operatorname{mod} A, \mathrm{Ab}) \xrightarrow{\;\sim\;} \mathrm{fp}(\operatorname{mod} A^{\mathrm{op}}, \mathrm{Ab})$$

given by $F \mapsto dF$, where $(dF)(L) := \operatorname{Hom}(F, - \otimes_A L)$ for $L \in \operatorname{mod} A^{\mathrm{op}}$. Its quasi-inverse is given by $G \mapsto dG$, where $(dG)(M) := \operatorname{Hom}(G, M \otimes_A -)$ for $M \in \operatorname{mod} A$.

Proof. [Pre09, Theorem 10.3.4]. $\square$

It is known that the equivalence d in Theorem 8.6 sends $F_{\phi/\psi}$ to $F_{D\psi/D\phi}$ for each pp-pair ϕ/ψ; see [Pre09, Corollary 10.3.8] and its proof. Thus the bijection in Theorem 8.4 can also be written as $(F) \mapsto (dF)$ for $F \in \mathrm{fp}(\operatorname{mod} A, \mathrm{Ab})$.

It would be worth noting that there is an order-preserving bijection between the open subsets of Zg_A and the Serre subcategories of $\mathrm{fp}(\operatorname{mod} A, \mathrm{Ab})$ ([Her97, Theorem 3.8] and [Kra97, Theorem 4.2]). Hence the bijection in Theorem 8.4 induces a bijection between the Serre subcategories of $\mathrm{fp}(\operatorname{mod} A, \mathrm{Ab})$ and those of $\mathrm{fp}(\operatorname{mod} A^{\mathrm{op}}, \mathrm{Ab})$; this also follows from Theorem 8.6.

On the other hand, there is an order-preserving bijection between the closed subsets of Zg_A and the definable subcategories of $\operatorname{Mod} A$, that is, full subcategories of $\operatorname{Mod} A$ closed under direct limits, direct products, and pure submodules (see [Pre09, Corollary 5.1.6]). Typical examples are the subcategory of injective right A-modules when A is right noetherian and the subcategory of flat right A-modules when A is left coherent (see [Pre09, Theorem 3.4.28(a) and Theorem 3.4.24]). Given a definable subcategory, its corresponding closed subset is obtained by collecting the isoclasses of indecomposable pure-injective modules in the subcategory.

Now let A be a Noether R-algebra. Denote by inj_A (resp. flcot_A) the set of isoclasses of indecomposable injective (resp. indecomposable flat cotorsion) right A-modules. As we observed in Propositions 2.1 and 2.3, the flat cotorsion right A-modules are precisely the flat pure-injective right A-modules. So inj_A and flcot_A are closed subsets of Zg_A by the above observation. We endow inj_A and flcot_A with the topologies induced from Zg_A.

Lemma 8.7. *Let A be a Noether R-algebra and let $P \in \operatorname{Spec} A$. For each open subset $U \subseteq \mathrm{Zg}_{A^{\mathrm{op}}}$, we have $I_{A^{\mathrm{op}}}(P) \in U$ if and only if $T_A(P) \in DU$.*

Proof. Since elementary duality D is order-preserving and $\mathrm{Zg}_{A^{\mathrm{op}}}$ has an open basis $\{(F_{\phi/\psi})\}$, we may assume that $U = (F_{\phi/\psi})$ for some pp-pair ϕ/ψ for right A^{op}-modules, and hence $DU = (F_{D\psi/D\phi})$.

Let $\mathfrak{p} := P \cap R$. Since $I_{A^{\mathrm{op}}}(P)$ is $\mathfrak{p}$-local by (2.6), there is a ring homomorphism $R_{\mathfrak{p}} \to \operatorname{End}_{A^{\mathrm{op}}}(I_{A^{\mathrm{op}}}(P))$ given by scalar multiplication. Moreover, $E_R(R/\mathfrak{p}) \cong E_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}))$ is an injective cogenerator in $\operatorname{Mod} R_{\mathfrak{p}}$ ([ILL$^+$07, Lemma A.27]),

and by definition $T_A(P) = \mathrm{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/\mathfrak{p}))$. Thus, it follows from Lemma 8.3 that $F_{\phi/\psi}(I_{A^{\mathrm{op}}}(P)) \neq 0$ if and only if $F_{D\psi/D\phi}(T_A(P)) \neq 0$. Therefore $I_{A^{\mathrm{op}}}(P) \in U$ if and only if $T_A(P) \in DU$. $\qquad\square$

THEOREM 8.8. *Let A be a Noether R-algebra. Then the bijection $\mathrm{inj}_{A^{\mathrm{op}}} \xrightarrow{\sim} \mathrm{flcot}_A$ given by $I_{A^{\mathrm{op}}}(P) \mapsto T_A(P)$ is a homeomorphism.*

Proof. Lemma 8.7 implies that, for each open subset $U \subseteq \mathrm{Zg}_{A^{\mathrm{op}}}$, the bijection $\mathrm{inj}_{A^{\mathrm{op}}} \xrightarrow{\sim} \mathrm{flcot}_A$ restricts to a bijection $U \cap \mathrm{inj}_{A^{\mathrm{op}}} \xrightarrow{\sim} DU \cap \mathrm{flcot}_A$. Hence the result follows. $\qquad\square$

We can deduce from Lemma 8.7 that

$$D(\mathrm{inj}_{A^{\mathrm{op}}}) = \mathrm{flcot}_A. \tag{8.1}$$

for a Noether R-algebra A. Indeed, setting $U := (\mathrm{inj}_{A^{\mathrm{op}}})^{\mathrm{c}}$, we obtain $DU \cap \mathrm{flcot}_A = \emptyset$ from Lemma 8.7, and hence $D(\mathrm{inj}_{A^{\mathrm{op}}}) = (DU)^{\mathrm{c}} \supseteq \mathrm{flcot}_A$. On the other hand, setting $O := (\mathrm{flcot}_A)^{\mathrm{c}}$ and applying Lemma 8.7 to $DO \subseteq \mathrm{Zg}_{A^{\mathrm{op}}}$, we obtain $DO \cap \mathrm{inj}_{A^{\mathrm{op}}} = \emptyset$. This implies that $\mathrm{inj}_{A^{\mathrm{op}}} \subseteq (DO)^{\mathrm{c}} = D(\mathrm{flcot}_A)$, and hence $D(\mathrm{inj}_{A^{\mathrm{op}}}) \subseteq D^2(\mathrm{flcot}_A) = \mathrm{flcot}_A$. Therefore (8.1) holds.

In fact, (8.1) holds for an arbitrary left coherent ring A ([Her93, Theorem 9.3]). Moreover, Herzog proved that elementary duality "constitutes" a homeomorphism $\mathrm{inj}_{A^{\mathrm{op}}} \xrightarrow{\sim} \mathrm{flcot}_A$ for a class of rings A, including all left noetherian rings ([Her93, Corollary 9.6]). In the rest of this section, we prove that our homeomorphism in Theorem 8.8 coincides with Herzog's one when A is a Noether R-algebra.

Recall that a *generic point* of a topological space X is a point $x \in X$ whose closure is the whole space X.

DEFINITION 8.9. Let A be a ring. A point $N \in \mathrm{Zg}_A$ is called *reflexive* if its closure $\overline{\{N\}}$ has a unique generic point (which is necessarily N) and if the elementary dual $D\overline{\{N\}}$ of $\overline{\{N\}}$ also has a unique generic point. In this case, the generic point of $D\overline{\{N\}}$ is denoted by DN and called the *elementary dual* of N.

If $N \in \mathrm{Zg}_A$ is reflexive, then DN is also reflexive and $D^2N = N$ by definition. Thus we have a bijection between the reflexive points in Zg_A and those in $\mathrm{Zg}_{A^{\mathrm{op}}}$ given by $N \mapsto DN$. Herzog's homeomorphism $\mathrm{inj}_{A^{\mathrm{op}}} \xrightarrow{\sim} \mathrm{flcot}_A$ (for a left noetherian ring A) is realized as a restriction of this bijection based on the fact that all points of $\mathrm{inj}_{A^{\mathrm{op}}}$ and flcot_A are reflexive; see [Her93, the last paragraph of §4 and the paragraph preceding Corollary 9.6], where the definition of reflexivity (see [Her93, the paragraph preceding Theorem 4.10]) is stronger than ours following [Pre09, p. 271]. The dual of a reflexive point N in the former sense is actually DN defined as above; see [Pre09, Theorems 5.3.2 and 5.4.12].

Therefore, to see that Herzog's homeomorphism coincides with ours for a Noether algebra A, it is enough to show that each $T_A(P)$ is the elementary

dual of $I_{A^{\mathrm{op}}}(P)$; this will be done in Theorem 8.14. We also give an explicit proof for the reflexivity of points in $\mathrm{inj}_{A^{\mathrm{op}}}$ and flcot_A.
For this purpose, we describe the topology on inj_A in terms of prime ideals of A. The description is merely a paraphrase of known results.

DEFINITION 8.10. Let A be a Noether R-algebra. For a right A-module M, define the *support* of M to be

$$\mathrm{Supp}_A M := \{\, P \in \mathrm{Spec}\, A \mid \mathrm{Hom}_A(M, I_A(P)) \neq 0 \,\}.$$

This support coincides with the classical one in commutative algebra. Indeed, by (2.6) and Remark 2.13,

$$\mathrm{Hom}_A(M, I_A(P)) \cong \mathrm{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, I_A(P)), \qquad (8.2)$$

where $\mathfrak{p} := P \cap R$. If $A = R$, then $I_A(P) = E_R(R/\mathfrak{p})$ is an injective cogenerator in $\mathrm{Mod}\, R_{\mathfrak{p}}$. It should also be mentioned that Definition 8.10 just imitates the description of an open basis for inj_A given by Herzog and Krause; see Remark 8.12 below.
Let us state an auxiliary proposition. We say that a subset $\Phi \subseteq \mathrm{Spec}\, A$ is *specialization-closed* (resp. *generalization-closed*) if, for every pair $P \subseteq Q$ in $\mathrm{Spec}\, A$, $P \in \Phi$ implies $Q \in \Phi$ (resp. $Q \in \Phi$ implies $P \in \Phi$).

PROPOSITION 8.11. *Let A be a Noether R-algebra.*

(1) *For every short exact sequence $0 \to L \to M \to N \to 0$ of right A-modules,*

$$\mathrm{Supp}_A M = \mathrm{Supp}_A L \cup \mathrm{Supp}_A N.$$

(2) *For every $P \in \mathrm{Spec}\, A$, $\mathrm{Supp}_A(A/P) = \{\, Q \in \mathrm{Spec}\, A \mid P \subseteq Q \,\}$, which is the smallest specialization-closed subset of $\mathrm{Spec}\, A$ containing P.*

(3) *For every right A-module M, $\mathrm{Supp}_A M$ is specialization-closed.*

Proof. (1): Applying the exact functor $\mathrm{Hom}_A(-, I_A(P))$, for each $P \in \mathrm{Spec}\, A$, to the given short exact sequence, we obtain the result.
(2): Let $Q \in \mathrm{Spec}\, A$. First assume that $P \subseteq Q$. We have canonical morphisms $A/P \twoheadrightarrow A/Q \hookrightarrow E_A(A/Q)$. Since $E_A(A/Q)$ is a finite direct sum of copies of $I_A(Q)$ by (2.5), there exists a nonzero morphism $A/P \to I_A(Q)$. Thus $Q \in \mathrm{Supp}_A(A/P)$.
Conversely, assume that $Q \in \mathrm{Supp}_A(A/P)$. Then there exists a nonzero morphism $f \colon A/P \to E_A(A/Q)$. Since A/Q is an essential submodule of $E_A(A/Q)$, the intersection $\mathrm{Im}\, f \cap (A/Q)$ is nonzero. This means that A/Q has a nonzero submodule annihilated by P. Therefore $P \subseteq Q$ by the definition of prime ideals.
(3): Let $P \subseteq Q$ in $\mathrm{Spec}\, A$ and $P \in \mathrm{Supp}_A M$. Then there exists a nonzero morphism $g \colon M \to I_A(P)$. Let $N := \mathrm{Im}\, g$. Since $I_A(P)$ is $\mathfrak{p}$-local by (2.6), $N_{\mathfrak{p}}$ is a nonzero $A_{\mathfrak{p}}$-submodule of $I_A(P) = E_A(S_A(P))$, and hence $N_{\mathfrak{p}}$ contains $S_A(P)$

as an $A_{\mathfrak{p}}$-submodule. Thus, by [GN02, Lemma 2.5.1], there is a monomorphism from A/P to a finite direct sum of copies of N. Therefore $Q \in \operatorname{Supp}_A(A/P) \subseteq \operatorname{Supp}_A N \subseteq \operatorname{Supp}_A M$ by (1) and (2). $\qquad\square$

Remark 8.12. It is known (for any ring A) that there is a bijection from Zg_A to the set of isoclasses of indecomposable injective objects in $(\operatorname{mod} A^{\mathrm{op}}, \operatorname{Ab})$, the category of additive functors $\operatorname{mod} A^{\mathrm{op}} \to \operatorname{Ab}$, given by $M \mapsto M \otimes_A -$ ([Pre09, Corollary 12.1.9]). Extending this viewpoint, Herzog [Her97] and Krause [Kra97] studied the spectrum formed by isoclasses of indecomposable injective objects for an arbitrary locally coherent Grothendieck category. In particular, when A is a Noether R-algebra (or more generally, when A is a right coherent ring), their work provides another way to think of inj_A as a topological space, with open basis consisting of all subsets of the form

$$(M) := \{\, I \in \operatorname{inj}_A \mid \operatorname{Hom}_A(M, I) \neq 0 \,\}$$

for some finitely presented right A-module M; see [Her97, Corollary 3.5] or [Kra97, Corollary 4.6]. It follows from [Pre09, Theorem 5.1.11] and [Kra97, Corollary 4.3] that this topology coincides with the induced topology on inj_A as a (closed) subset of Zg_A.

Proposition 8.13. *Let A be a Noether R-algebra. There is an order-preserving bijection*

$$\{\, \text{specialization-closed subsets of } \operatorname{Spec} A \,\} \xrightarrow{\sim} \{\, \text{open subsets of } \operatorname{inj}_A \,\}$$

given by $\Phi \mapsto \{\, I_A(P) \mid P \in \Phi \,\}$.

Proof. We show that the bijection in Theorem 2.22 induces the desired bijection. By the above observation, inj_A (with topology induced from Zg_A) has an open basis consisting of all subsets of the form (M) for some finitely presented right A-module M. Furthermore, each subset $(M) \subseteq \operatorname{inj}_A$ corresponds to $\operatorname{Supp}_A M$ by the bijection in Theorem 2.22. So it suffices to show that a subset $\Phi \subseteq \operatorname{Spec} A$ is specialization-closed if and only if Φ is the union of subsets of the form $\operatorname{Supp}_A M$ for some $M \in \operatorname{mod} A$. The "if" part follows from Proposition 8.11(3). Conversely, if Φ is specialization-closed, then $\Phi = \bigcup_{P \in \Phi} \operatorname{Supp}_A(A/P)$ by Proposition 8.11(2). $\qquad\square$

By Theorem 8.8 and Proposition 8.13, we obtain an order-preserving bijection

$$\{\, \text{specialization-closed subsets of } \operatorname{Spec} A \,\} \xrightarrow{\sim} \{\, \text{open subsets of } \operatorname{flcot}_A \,\} \quad (8.3)$$

given by $\Phi \mapsto \{\, T_A(P) \mid P \in \Phi \,\}$.
The following is the main theorem in this section:

Theorem 8.14. *Let A be a Noether R-algebra. Then all points in $\operatorname{inj}_{A^{\mathrm{op}}}$ and flcot_A are reflexive. For each $P \in \operatorname{Spec} A$, the elementary dual of $I_{A^{\mathrm{op}}}(P)$ is $T_A(P)$.*

Proof. By Proposition 8.13 and (8.3), the generalization-closed subsets of $\operatorname{Spec} A$ bijectively correspond to the closed subsets of $\operatorname{inj}_{A^{\mathrm{op}}}$ and the closed subsets of flcot_A. Let $P \in \operatorname{Spec} A$, $I := I_{A^{\mathrm{op}}}(P) \in \operatorname{inj}_{A^{\mathrm{op}}}$, and $T := T_A(P) \in \operatorname{flcot}_A$. The generalization-closed subset $\Psi := \{\, Q \in \operatorname{Spec} A \mid Q \subseteq P \,\}$ corresponds to the closures $\overline{\{I\}} \subseteq \operatorname{inj}_{A^{\mathrm{op}}}$ and $\overline{\{T\}} \subseteq \operatorname{flcot}_A$, and none of the proper generalization-closed subsets of Ψ contains P. Hence I and T are the unique generic points of $\overline{\{I\}}$ and $\overline{\{T\}}$, respectively. Consequently, I and T are reflexive. It remains to show that $D\overline{\{I\}} = \overline{\{T\}}$. This follows from the next lemma. $\square$

LEMMA 8.15. *Let A be a Noether R-algebra. For every closed subset $C \subseteq \operatorname{inj}_{A^{\mathrm{op}}}$, we have*

$$DC = \{\, T_A(P) \in \operatorname{flcot}_A \mid P \in \operatorname{Spec} A,\ I_{A^{\mathrm{op}}}(P) \in C \,\}.$$

For every $P \in \operatorname{Spec} A$, it follows that $D\overline{\{I_{A^{\mathrm{op}}}(P)\}} = \overline{\{T_A(P)\}}$.

Proof. Since $\operatorname{inj}_{A^{\mathrm{op}}}$ is closed in $\operatorname{Zg}_{A^{\mathrm{op}}}$, the subset C is also closed in $\operatorname{Zg}_{A^{\mathrm{op}}}$ and $DC \subseteq D(\operatorname{inj}_{A^{\mathrm{op}}}) = \operatorname{flcot}_A$ by (8.1). Moreover, Lemma 8.7 implies that $I_{A^{\mathrm{op}}}(P) \in \operatorname{inj}_{A^{\mathrm{op}}} \setminus C$ if and only if $T_A(P) \in \operatorname{flcot}_A \setminus DC$ for each $P \in \operatorname{Spec} A$. In other words, $I_{A^{\mathrm{op}}}(P) \in C$ if and only if $T_A(P) \in DC$ for each $P \in \operatorname{Spec} A$. Thus we obtain the desired description of DC. The last statement of the lemma follows because Proposition 8.13 and (8.3) show that the closures of $I_{A^{\mathrm{op}}}(P)$ and $T_A(P)$ both correspond to the generalization closure of P. $\square$

EXAMPLE 8.16. Consider the algebra A in Example 6.3. For two prime ideals $P_i(\mathfrak{p})$ and $P_j(\mathfrak{q})$ of A, we have $P_i(\mathfrak{p}) \subseteq P_j(\mathfrak{q})$ if and only if $i = j$ and $\mathfrak{p} \subseteq \mathfrak{q}$. So we have an order-preserving bijection from $\operatorname{Spec} A$ to the disjoint union $\operatorname{Spec} R \amalg \operatorname{Spec} R$ given by $P_i(\mathfrak{p}) \mapsto (\mathfrak{p}$ in the ith $\operatorname{Spec} R)$. Every specialization-closed subset of $\operatorname{Spec} A$ is of the form $\Phi_1 \amalg \Phi_2$, where each Φ_i is a specialization-closed subset of the ith $\operatorname{Spec} R$. Hence, by (8.3), all open subsets of flcot_A are of the form

$$\{\, T_A(P_1(\mathfrak{p})) \mid \mathfrak{p} \in \Phi_1 \,\} \cup \{\, T_A(P_2(\mathfrak{p})) \mid \mathfrak{p} \in \Phi_2 \,\},$$

where Φ_1 and Φ_2 are specialization-closed subsets of $\operatorname{Spec} R$. The closure of each $T_A(P_i(\mathfrak{q}))$ in flcot_A is

$$\{\, T_A(P_i(\mathfrak{p})) \mid \mathfrak{q} \subseteq \mathfrak{p} \,\}.$$

Although (8.3) describes the induced topology on flcot_A explicitly, it is also possible to give an open basis for flcot_A in a similar way to Remark 8.12:

PROPOSITION 8.17. *The set of subsets of flcot_A of the form*

$$\{\, T_A(P) \in \operatorname{flcot}_A \mid T_A(P) \otimes_A M \neq 0 \,\}$$

for some finitely generated left A-module M is an open basis for flcot_A.

Proof. Recall that $I_{A^{\mathrm{op}}}(P) \cong \operatorname{Hom}_{\widehat{R_{\mathfrak{p}}}}(T_A(P), E_R(R/\mathfrak{p}))$, where $\mathfrak{p} := P \cap R$ (Proposition 4.13). Using this isomorphism and the tensor-hom adjunction, we obtain

$$\operatorname{Hom}_{A^{\mathrm{op}}}(M, I_{A^{\mathrm{op}}}(P)) \cong \operatorname{Hom}_{\widehat{R_{\mathfrak{p}}}}(T_A(P) \otimes_A M, E_R(R/\mathfrak{p}))$$

for every left A-module M. Since $E_R(R/\mathfrak{p}) \cong E_{\widehat{R_\mathfrak{p}}}(\kappa(\mathfrak{p}))$ is an injective cogenerator in $\operatorname{Mod} \widehat{R_\mathfrak{p}}$, we have

$$\operatorname{Supp}_{A^{\mathrm{op}}} M = \{\, P \in \operatorname{Spec} A \mid T_A(P) \otimes_A M \neq 0 \,\}.$$

Thus the desired conclusion follows from (8.3) and Proposition 8.11. $\qquad\qquad\square$

Appendix A   Ideal-adic completion

Let R be a commutative noetherian ring and A a Noether R-algebra. This appendix provides basic facts on $\mathfrak{a}$-adic completion of right A-modules, where $\mathfrak{a}$ is an ideal of R. All results here are generalizations or restatements of known results for R. Although the proofs resemble those for the commutative case, we provide a precise proof to each result for the reader's sake.

We denote by $\operatorname{Mod} A$ (resp. $\operatorname{mod} A$) the category of all (resp. finitely generated) right A-modules, and interpret $\operatorname{Mod} A^{\mathrm{op}}$ as the category of all left A-modules, where A^{op} is the opposite ring. The $\mathfrak{a}$-*adic completion functor* $\Lambda^{\mathfrak{a}} \colon \operatorname{Mod} A \to \operatorname{Mod} A$ is defined by

$$\Lambda^{\mathfrak{a}} := \varprojlim_{n \geq 1} (- \otimes_R R/\mathfrak{a}^n).$$

The functor $\Lambda^{\mathfrak{a}}$ is often written as $(-)_{\mathfrak{a}}^{\wedge}$. A right A-module M is called $\mathfrak{a}$-*complete* if the canonical morphism $M \to M_{\mathfrak{a}}^{\wedge}$ is an isomorphism.

We start with the following lemma, which follows from the Artin-Rees lemma over R and an intersection property of a flat right A-module.

Lemma A.1. *Let F be a flat right A-module and let $\mathfrak{a} \subseteq R$ be an ideal. Then the functor*

$$(F \otimes_A -)_{\mathfrak{a}}^{\wedge} \colon \operatorname{mod} A^{\mathrm{op}} \to \operatorname{Mod} R$$

is exact.

Proof. Let $0 \to L \to M \to N \to 0$ be an exact sequence of finitely generated left A-modules. This is sent by the functor $F \otimes_A -$ to an exact sequence of R-modules

$$0 \to F \otimes_A L \to F \otimes_A M \to F \otimes_A N \to 0.$$

We regard L (resp. $F \otimes_A L$) as a submodule of M (resp. $F \otimes_A M$). By [Mat89, Theorem 8.1], it is enough to see that the $\mathfrak{a}$-adic topology on $F \otimes_A L$ coincides with the topology induced from the $\mathfrak{a}$-adic topology on $F \otimes_A M$.

Let $n \geq 1$ be an integer. Since F is flat, the inclusion $\mathfrak{a}^n M \hookrightarrow M$ induces a canonical injection $F \otimes_A (\mathfrak{a}^n M) \hookrightarrow F \otimes_A M$, and

$$F \otimes_A (\mathfrak{a}^n M) = \mathfrak{a}^n (F \otimes_A M) \qquad\qquad\qquad (A.1)$$

as R-submodules of $F \otimes_A M$.

By the Artin-Rees lemma [Mat89, Theorem 8.5], there is an integer $c > 0$ such that

$$\mathfrak{a}^n L \subseteq (\mathfrak{a}^n M) \cap L \subseteq \mathfrak{a}^{n-c} L,$$

for every $n > c$. Application of $F \otimes_A -$ to this sequence yields

$$F \otimes_A (\mathfrak{a}^n L) \subseteq F \otimes_A ((\mathfrak{a}^n M) \cap L) \subseteq F \otimes_A (\mathfrak{a}^{n-c} L), \qquad \text{(A.2)}$$

where the middle term coincides with

$$(F \otimes_A (\mathfrak{a}^n M)) \cap (F \otimes_A L)$$

because the exact functor $F \otimes_A -$ preserves intersections of submodules. Hence, using (A.1), we can rewrite (A.2) as

$$\mathfrak{a}^n (F \otimes_A L) \subseteq (\mathfrak{a}^n (F \otimes_A M)) \cap (F \otimes_A L) \subseteq \mathfrak{a}^{n-c}(F \otimes_A L),$$

and this shows that the $\mathfrak{a}$-adic topology on $F \otimes_A L$ coincides with the topology induced from the $\mathfrak{a}$-adic topology on $F \otimes_A M$, as desired. $\qquad\square$

PROPOSITION A.2. *Let F be a flat right A-module and let $\mathfrak{a} \subseteq R$ be an ideal. Then there is a canonical isomorphism*

$$F_{\mathfrak{a}}^{\wedge} \otimes_A - \xrightarrow{\sim} (F \otimes_A -)_{\mathfrak{a}}^{\wedge}$$

of functors $\operatorname{mod} A^{\mathrm{op}} \to \operatorname{Mod} R$.

Proof. By Lemma A.1, the functor $(F \otimes_A -)_{\mathfrak{a}}^{\wedge}$ is right exact, so the Eilenberg-Watts theorem ([Wat60, Theorem 2]) gives a canonical isomorphism $(F \otimes_A A)_{\mathfrak{a}}^{\wedge} \otimes_A - \xrightarrow{\sim} (F \otimes_A -)_{\mathfrak{a}}^{\wedge}$. The desired isomorphism follows from the canonical isomorphism $F \otimes_A A \xrightarrow{\sim} F$ of right A-modules. $\qquad\square$

PROPOSITION A.3. *Let F be a flat right A-module and let $\mathfrak{a} \subseteq R$ be an ideal. Then $F_{\mathfrak{a}}^{\wedge}$ is a flat right A-module.*

Proof. By Lemma A.1 and Proposition A.2, the functor $F_{\mathfrak{a}}^{\wedge} \otimes_A -$ is exact on $\operatorname{mod} A^{\mathrm{op}}$. This implies that $F_{\mathfrak{a}}^{\wedge}$ is a flat right A-module (see [Ste75, Proposition I.10.6], for example). $\qquad\square$

In the case where $A = R$, Proposition A.3 was shown by Gruson and Raynaud [RG71, Part II, (2.4.2) and Proposition 2.4.3.1] when $\mathfrak{a} \subseteq R$ is a maximal ideal, and by Bartijn [Bar85, Chapter 1, Corollary 4.7] for arbitrary $\mathfrak{a}$. Another proof was given by Schenzel and Simon [SS18, Theorem 2.4.4]. See [Yek18, Theorem 1.6] for a certain generalization to non-noetherian commutative rings. Our proof of Proposition A.3 is essentially the same as Gabber and Ramero [GR03, Lemma 7.1.6] but the settings are different.

Schenzel and Simon [SS18, Theorem 2.4.4] also showed the flatness of $F_{\mathfrak{a}}^{\wedge}$ over $R_{\mathfrak{a}}^{\wedge}$. This will be generalized to Noether algebras in Proposition A.13.

The next two results are often used by experts implicitly.

LEMMA A.4. *Let $\mathfrak{a}, \mathfrak{b} \subseteq R$ be ideals such that $\mathfrak{a}^n \subseteq \mathfrak{b}$ for some $n > 0$ and let M be a right A-module. Then the canonical morphism $M \to M_{\mathfrak{a}}^{\wedge}$ induces an isomorphism $M \otimes_R (R/\mathfrak{b}) \xrightarrow{\sim} M_{\mathfrak{a}}^{\wedge} \otimes_R (R/\mathfrak{b})$.*

Proof. It suffices to prove this by regarding M as just an R-module, so the proof can be found in [Bar85, Chapter I, Theorem 3.1] or [Str90, Theorem 2.2.5], which deals with completion with respect to a finitely generated ideal of a (possibly non-noetherian) commutative ring. Note that, in [Str90, Theorem 2.2.5], a result like Proposition A.2 is implicitly used at the end of the proof. Another proof can be found in [SS18, Theorem 2.2.2]. $\square$

PROPOSITION A.5. *Let $\mathfrak{a}$ be an ideal of R. Denote by $\eta\colon \mathrm{id}_{\mathrm{Mod}\,A} \to \Lambda^{\mathfrak{a}}$ the canonical morphism of functors $\mathrm{Mod}\,A \to \mathrm{Mod}\,A$. For every right A-module M, the morphisms $\Lambda^{\mathfrak{a}}(\eta M)\colon \Lambda^{\mathfrak{a}}M \to \Lambda^{\mathfrak{a}}\Lambda^{\mathfrak{a}}M$ and $\eta(\Lambda^{\mathfrak{a}}M)\colon \Lambda^{\mathfrak{a}}M \to \Lambda^{\mathfrak{a}}\Lambda^{\mathfrak{a}}M$ are isomorphisms. In particular, $\Lambda^{\mathfrak{a}}M = M_{\mathfrak{a}}^{\wedge}$ is $\mathfrak{a}$-complete.*

Proof. Lemma A.4 applied to $\mathfrak{b} = \mathfrak{a}^n$ $(n \geq 1)$ yields the isomorphism $f_n\colon M \otimes_R (R/\mathfrak{a}^n) \xrightarrow{\sim} M_{\mathfrak{a}}^{\wedge} \otimes_R (R/\mathfrak{a}^n)$ induced from the completion map $M \to M_{\mathfrak{a}}^{\wedge}$. This implies that $\Lambda^{\mathfrak{a}}(\eta(M))$ is an isomorphism.

Applying $- \otimes_R R/\mathfrak{a}^n$ to the canonical map $M_{\mathfrak{a}}^{\wedge} \to M \otimes_R (R/\mathfrak{a}^n)$ appearing in the definition of the inverse limit, we obtain $g_n\colon M_{\mathfrak{a}}^{\wedge} \otimes_R (R/\mathfrak{a}^n) \to M \otimes_R (R/\mathfrak{a}^n)$. As mentioned in the proofs of [Bar85, Chapter I, Proposition 2.3] and [Str90, Theorem 2.2.5], it is easy to see that $g_n f_n$ is the identity map, so $g_n = f_n^{-1}$ is also an isomorphism. One can also check that the composition

$$M_{\mathfrak{a}}^{\wedge} \to \varprojlim_{n \geq 1} M_{\mathfrak{a}}^{\wedge} \otimes_R (R/\mathfrak{a}^n) \xrightarrow{\sim} \varprojlim_{n \geq 1} M \otimes_R (R/\mathfrak{a}^n) = M_{\mathfrak{a}}^{\wedge}$$

of $\eta(\Lambda^{\mathfrak{a}}M)$ and the isomorphism induced by $(g_n)_n$ is the identity map, so $\eta(\Lambda^{\mathfrak{a}}M)$ is also an isomorphism. $\square$

REMARK A.6. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. Then every $\mathfrak{b}$-complete right A-module M is $\mathfrak{a}$-complete. Indeed, the composition of the completion maps $M \to M_{\mathfrak{a}}^{\wedge}$ and $M_{\mathfrak{a}}^{\wedge} \to (M_{\mathfrak{a}}^{\wedge})_{\mathfrak{b}}^{\wedge}$ is an isomorphism since $(M_{\mathfrak{a}}^{\wedge})_{\mathfrak{b}}^{\wedge} \cong M_{\mathfrak{b}}^{\wedge}$ by Lemma A.4. Thus M is a direct summand of $M_{\mathfrak{a}}^{\wedge}$. This implies that M is $\mathfrak{a}$-complete by Proposition A.5.

Let $\mathfrak{a}$ be an ideal of R. The functor $\Lambda^{\mathfrak{a}}\colon \mathrm{Mod}\,A \to \mathrm{Mod}\,A$ is not necessarily left exact or right exact (even if $A = R$; see [AM69, Chapter 10, Exercise 1], for example) so it is not isomorphic to $- \otimes_A A_{\mathfrak{a}}^{\wedge}$. However, Propositions A.7 and A.8 below show some basic properties of $\Lambda^{\mathfrak{a}}$.

PROPOSITION A.7. *The functor $\Lambda^{\mathfrak{a}}\colon \mathrm{Mod}\,A \to \mathrm{Mod}\,A$ preserves epimorphisms.*

Proof. The structure homomorphism $\varphi\colon R \to A$ induces the scalar restriction functor $\varphi_*\colon \mathrm{Mod}\,A \to \mathrm{Mod}\,R$. Since a morphism f in $\mathrm{Mod}\,A$ is an epimorphism if and only if $\varphi_*(f)$ is an epimorphism in $\mathrm{Mod}\,R$, it suffices to treat the case where $A = R$. Thus the result follows from [Mat89, Theorem 8.1(ii)] because the $\mathfrak{a}$-adic topology of a quotient module M/N coincides with the topology induced from the $\mathfrak{a}$-adic topology of M. $\square$

PROPOSITION A.8. *The functor $\Lambda^{\mathfrak{a}}\colon \mathrm{Mod}\,A \to \mathrm{Mod}\,A$ commutes with arbitrary direct products.*

Proof. The R-module $R/\mathfrak{a}^n R$ is finitely presented for each $n \geq 1$, so a standard argument shows that the functor $- \otimes_R R/\mathfrak{a}^n R \colon \operatorname{Mod} A \to \operatorname{Mod} A$ commutes with arbitrary direct products (see [EJ00, Theorem 3.2.22]). Hence the functor $\Lambda^{\mathfrak{a}} = \varprojlim_{n \geq 1}(- \otimes_R R/\mathfrak{a}^n R)$ commutes with arbitrary direct products. $\qquad\square$

Let $\mathfrak{a}$ be an ideal of R. The $\mathfrak{a}$-*torsion functor* $\Gamma_{\mathfrak{a}} \colon \operatorname{Mod} A \to \operatorname{Mod} A$ is defined by

$$\Gamma_{\mathfrak{a}} := \varinjlim_{n \geq 1} \operatorname{Hom}_R(R/\mathfrak{a}^n, -).$$

A right A-module M is called $\mathfrak{a}$-*torsion* if the canonical morphism $\Gamma_{\mathfrak{a}} M \to M$ is an isomorphism.

The functor $\Gamma_{\mathfrak{a}}$ from $\operatorname{Mod} A$ to its full subcategory consisting of all $\mathfrak{a}$-torsion modules is a right adjoint to the inclusion functor (Remark 2.23). A similar result holds for $\Lambda^{\mathfrak{a}}$:

PROPOSITION A.9. *The functor $\Lambda^{\mathfrak{a}}$ from $\operatorname{Mod} A$ to its full subcategory consisting of all $\mathfrak{a}$-complete modules is a left adjoint to the inclusion functor.*

Proof. This follows from Proposition A.5 and the general theory of categories; see [KS06, Proposition 4.1.3(iii)]. $\qquad\square$

Lemma A.10 and Proposition A.12 below are essentially stated in [ILL+07, Remark A.30(7) and (8)] for the case $A = R$.

LEMMA A.10. *Let M be an $\mathfrak{a}$-torsion right A-module. Then the canonical morphism $M \to M \otimes_A A_{\mathfrak{a}}^{\wedge}$ is an isomorphism of right A-modules.*

Proof. If N is a right A-module such that $\mathfrak{a}^n N = 0$ for some $n > 0$, then $N \cong N \otimes_R R/\mathfrak{a}^n$, so

$$N \otimes_A A_{\mathfrak{a}}^{\wedge} \cong (N \otimes_R R/\mathfrak{a}^n) \otimes_A A_{\mathfrak{a}}^{\wedge} \cong N \otimes_A (A_{\mathfrak{a}}^{\wedge} \otimes_R R/\mathfrak{a}^n) \cong N \otimes_A (A \otimes_R R/\mathfrak{a}^n) \cong N$$

as right A-modules, where the third isomorphism follows from Lemma A.4.
Now, if M is $\mathfrak{a}$-torsion, then M is canonically isomorphic to $\varinjlim_{n \geq 1} \operatorname{Hom}_R(R/\mathfrak{a}^n, M)$. The above argument shows that each $\operatorname{Hom}_R(R/\mathfrak{a}^n, M)$ satisfies the property in the statement. Since $- \otimes_A A_{\mathfrak{a}}^{\wedge}$ commutes with direct limits, so does M. $\qquad\square$

REMARK A.11. For a right A-module M, its $\mathfrak{a}$-adic completion $M_{\mathfrak{a}}^{\wedge}$ is naturally realized as a right A-submodule of $\prod_{n \geq 1} M/\mathfrak{a}^n M$. In particular, we may interpret $A_{\mathfrak{a}}^{\wedge}$ as a subring of $\prod_{n \geq 1} A/\mathfrak{a}^n A$. So the componentwise action defines a canonical right $A_{\mathfrak{a}}^{\wedge}$-module structure on $M_{\mathfrak{a}}^{\wedge}$. Moreover, taking the $\mathfrak{a}$-adic completion sends each A-homomorphism $M \to N$ to an $A_{\mathfrak{a}}^{\wedge}$-homomorphism $M_{\mathfrak{a}}^{\wedge} \to N_{\mathfrak{a}}^{\wedge}$, so we may regard $(-)_{\mathfrak{a}}^{\wedge}$ as a functor $\operatorname{Mod} A \to \operatorname{Mod} A_{\mathfrak{a}}^{\wedge}$.
For a finitely generated right A-module M, Proposition A.2 gives a canonical isomorphism $M \otimes_A A_{\mathfrak{a}}^{\wedge} \to M_{\mathfrak{a}}^{\wedge}$ of right A-modules. It is easily seen from the proof that this is an isomorphism of right $A_{\mathfrak{a}}^{\wedge}$-modules as well.

Note that a right $A_{\mathfrak{a}}^{\wedge}$-module is $\mathfrak{a}A_{\mathfrak{a}}^{\wedge}$-complete (resp. $\mathfrak{a}A_{\mathfrak{a}}^{\wedge}$-torsion) if and only if it is $\mathfrak{a}$-complete (resp. $\mathfrak{a}$-torsion) as a right A-module.

PROPOSITION A.12. *The functor* $- \otimes_A A_{\mathfrak{a}}^{\wedge} \colon \operatorname{Mod} A \to \operatorname{Mod} A_{\mathfrak{a}}^{\wedge}$ *induces an equivalence from the full subcategory of $\mathfrak{a}$-torsion right A-modules to the full subcategory of $\mathfrak{a}$-torsion right $A_{\mathfrak{a}}^{\wedge}$-modules. Its quasi-inverse is given by the scalar restriction functor.*

Proof. If M is an $\mathfrak{a}$-torsion right A-module, then we have the canonical isomorphism $M \xrightarrow{\sim} M \otimes_A A_{\mathfrak{a}}^{\wedge}$ of right A-modules by Lemma A.10, and this means that the composition of $- \otimes_A A_{\mathfrak{a}}^{\wedge} \colon \operatorname{Mod} A \to \operatorname{Mod} A_{\mathfrak{a}}^{\wedge}$ and the scalar restriction functor $\operatorname{Mod} A_{\mathfrak{a}}^{\wedge} \to \operatorname{Mod} A$ induces an autoequivalence on the full subcategory of $\mathfrak{a}$-torsion right A-modules.

Let N be an $\mathfrak{a}$-torsion right $A_{\mathfrak{a}}^{\wedge}$-module. We only need to check that the canonical morphism $N \otimes_A A_{\mathfrak{a}}^{\wedge} \to N$ of right $A_{\mathfrak{a}}^{\wedge}$-modules is an isomorphism. This also follows from Lemma A.10 because the composition of the canonical maps $N \xrightarrow{\sim} N \otimes_A A_{\mathfrak{a}}^{\wedge} \to N$ is the identity map. □

PROPOSITION A.13. *For every flat right A-module F, its $\mathfrak{a}$-adic completion $F_{\mathfrak{a}}^{\wedge}$ is a flat right $A_{\mathfrak{a}}^{\wedge}$-module.*

Proof. If L is an $\mathfrak{a}$-torsion left $A_{\mathfrak{a}}^{\wedge}$-module, then $L \cong A_{\mathfrak{a}}^{\wedge} \otimes_A L$ as left $A_{\mathfrak{a}}^{\wedge}$-modules by Proposition A.12, so

$$- \otimes_{A_{\mathfrak{a}}^{\wedge}} L \cong - \otimes_{A_{\mathfrak{a}}^{\wedge}} (A_{\mathfrak{a}}^{\wedge} \otimes_A L) \cong - \otimes_A L \qquad (A.3)$$

as functors $\operatorname{Mod} A_{\mathfrak{a}}^{\wedge} \to \operatorname{Mod} R_{\mathfrak{a}}^{\wedge}$. Similarly to the proof of Lemma A.1, we show that the functor

$$(F_{\mathfrak{a}}^{\wedge} \otimes_{A_{\mathfrak{a}}^{\wedge}} -)_{\mathfrak{a}}^{\wedge} \colon \operatorname{mod} A_{\mathfrak{a}}^{\wedge \operatorname{op}} \to \operatorname{Mod} R_{\mathfrak{a}}^{\wedge}$$

is exact. By Lemma A.4 and (A.3),

$$(F_{\mathfrak{a}}^{\wedge} \otimes_{A_{\mathfrak{a}}^{\wedge}} -)_{\mathfrak{a}}^{\wedge} = \varprojlim_{n \geq 1} (F_{\mathfrak{a}}^{\wedge} \otimes_{A_{\mathfrak{a}}^{\wedge}} -) \otimes_R (R/\mathfrak{a}^n)$$

$$\cong \varprojlim_{n \geq 1} (F \otimes_A -) \otimes_R (R/\mathfrak{a}^n) = (F \otimes_A -)_{\mathfrak{a}}^{\wedge}$$

as functors $\operatorname{mod} A_{\mathfrak{a}}^{\wedge \operatorname{op}} \to \operatorname{Mod} R_{\mathfrak{a}}^{\wedge}$. The exactness of the functor $(F \otimes_A -)_{\mathfrak{a}}^{\wedge}$ can be shown in the same way as Lemma A.1, using the Artin-Rees lemma for finitely generated left $R_{\mathfrak{a}}^{\wedge}$-modules. Hence the functor $(F_{\mathfrak{a}}^{\wedge} \otimes_{A_{\mathfrak{a}}^{\wedge}} -)_{\mathfrak{a}}^{\wedge}$ is also exact. In the same way as in the proofs of Propositions A.2 and A.3, we have $F_{\mathfrak{a}}^{\wedge} \otimes_{A_{\mathfrak{a}}^{\wedge}} - \cong (F_{\mathfrak{a}}^{\wedge} \otimes_{A_{\mathfrak{a}}^{\wedge}} -)_{\mathfrak{a}}^{\wedge}$, so $F_{\mathfrak{a}}^{\wedge}$ is a flat right $A_{\mathfrak{a}}^{\wedge}$-module. □

The following result is analogous to Proposition A.12.

PROPOSITION A.14. *The functor* $(-)_{\mathfrak{a}}^{\wedge} \colon \operatorname{Mod} A \to \operatorname{Mod} A_{\mathfrak{a}}^{\wedge}$ *induces an equivalence from the full subcategory of $\mathfrak{a}$-complete right A-modules to the full subcategory of $\mathfrak{a}$-complete right $A_{\mathfrak{a}}^{\wedge}$-modules. Its quasi-inverse is given by the scalar restriction functor.*

Proof. If M is an $\mathfrak{a}$-complete right A-module, then by definition we have the canonical isomorphism $M \xrightarrow{\sim} M_{\mathfrak{a}}^{\wedge}$ of right A-modules, and this means that the composition of $\Lambda^{\mathfrak{a}} \colon \operatorname{Mod} A \to \operatorname{Mod} A_{\mathfrak{a}}^{\wedge}$ and the scalar restriction functor $\operatorname{Mod} A_{\mathfrak{a}}^{\wedge} \to \operatorname{Mod} A$ induces an autoequivalence on the full subcategory of $\mathfrak{a}$-complete right A-modules.

Let N be an $\mathfrak{a}$-complete right $A_{\mathfrak{a}}^{\wedge}$-module. Then, by definition, we have an isomorphism $N \to N_{\mathfrak{a}}^{\wedge}$ of right A-modules. We show that this is an isomorphism of right $A_{\mathfrak{a}}^{\wedge}$-module, where the $A_{\mathfrak{a}}^{\wedge}$-module structure on $N_{\mathfrak{a}}^{\wedge}$ is the one defined in Remark A.11. The embedding $f \colon N \hookrightarrow \prod_{n \geq 1} N/\mathfrak{a}^n N$ induced by the projections $N \twoheadrightarrow N/\mathfrak{a}^n N$ is an $A_{\mathfrak{a}}^{\wedge}$-homomorphism if we regard $\prod_{n \geq 1} N/\mathfrak{a}^n N$ as the product of right $A_{\mathfrak{a}}^{\wedge}$-modules $N/\mathfrak{a}^n N$. On the other hand, we observed in Remark A.11 that the natural embedding $N_{\mathfrak{a}}^{\wedge} \hookrightarrow \prod_{n \geq 1} N/\mathfrak{a}^n N$ is an $A_{\mathfrak{a}}^{\wedge}$-homomorphism, but here $A_{\mathfrak{a}}^{\wedge}$ acts on the product componentwise. Since these embeddings are identified via the isomorphism $N \to N_{\mathfrak{a}}^{\wedge}$, it suffices to prove that those two $A_{\mathfrak{a}}^{\wedge}$-module structures on $\prod_{n \geq 1} N/\mathfrak{a}^n N$ coincides. In other words, it suffices to prove that, for each $n \geq 1$, the $A_{\mathfrak{a}}^{\wedge}$-module structure on $N/\mathfrak{a}^n N$ induced from that of N is the same as the $A_{\mathfrak{a}}^{\wedge}$-module structure on $N/\mathfrak{a}^n N$ obtained from the $A/\mathfrak{a}^n A$-module structure of $N/\mathfrak{a}^n N$ via the canonical map $A_{\mathfrak{a}}^{\wedge} \to A/\mathfrak{a}^n A$. The former structure factors through the $A_{\mathfrak{a}}^{\wedge}/\mathfrak{a}^n A_{\mathfrak{a}}^{\wedge}$-structure on $N/\mathfrak{a}^n N$. As we observed in the proof of Proposition A.5, the map $A_{\mathfrak{a}}^{\wedge} \to A/\mathfrak{a}^n A$ induces an isomorphism $A_{\mathfrak{a}}^{\wedge}/\mathfrak{a}^n A_{\mathfrak{a}}^{\wedge} \xrightarrow{\sim} A/\mathfrak{a}^n A$. So we have a commutative diagram

$$
\begin{array}{ccc}
A & \longrightarrow\!\!\!\!\!\to & A/\mathfrak{a}^n A \\
\downarrow & \nearrow & \uparrow\wr \\
A_{\mathfrak{a}}^{\wedge} & \longrightarrow\!\!\!\!\!\to & A_{\mathfrak{a}}^{\wedge}/\mathfrak{a}^n A_{\mathfrak{a}}^{\wedge},
\end{array}
$$

where all maps are canonical ones. This means that each $A_{\mathfrak{a}}^{\wedge}$-module structures on $N/\mathfrak{a}^n N$ is determined by the induced A-module structure on $N/\mathfrak{a}^n N$ via the canonical map $A \to A_{\mathfrak{a}}^{\wedge}$. Since the induced A-module structures are the same, so are the $A_{\mathfrak{a}}^{\wedge}$-module structures. This completes the proof. $\square$

The following fact is shown in [SS18, Proposition 2.1.15(a)] for $\mathfrak{a}$-torsion R-modules.

PROPOSITION A.15. *Every $\mathfrak{a}$-torsion (resp. $\mathfrak{a}$-complete) right A-module has a unique right $A_{\mathfrak{a}}^{\wedge}$-module structure that is compatible with the right A-module structure via the canonical map $A \to A_{\mathfrak{a}}^{\wedge}$.*

Proof. This follows from Proposition A.12 (resp. Proposition A.14). Indeed, such a structure exists since every $\mathfrak{a}$-torsion (resp. $\mathfrak{a}$-complete) right A-module M belongs to the essential image of the scalar restriction functor $\operatorname{Mod} A_{\mathfrak{a}}^{\wedge} \to \operatorname{Mod} A$. If N_1 and N_2 are right $A_{\mathfrak{a}}^{\wedge}$-modules that are equal to M as right A-modules, then they are $\mathfrak{a}$-torsion (resp. $\mathfrak{a}$-complete), and the scalar restriction functor gives a bijection $\operatorname{Hom}_{A_{\mathfrak{a}}^{\wedge}}(N_1, N_2) \to \operatorname{Hom}_A(M, M)$. Therefore the identity map $M \to M$ gives the equality of N_1 and N_2 as right $A_{\mathfrak{a}}^{\wedge}$-modules. $\square$

Assume that an ideal $\mathfrak{a} \subseteq R$ is contained by the Jacobson radical of R (which is by definition the intersection of all maximal ideals of R). Then the ring homomorphism $R \to R_{\mathfrak{a}}^{\wedge}$ is faithfully flat ([Mat89, Theorem 8.14]), and hence the induced map $\operatorname{Spec} R_{\mathfrak{a}}^{\wedge} \to \operatorname{Spec} R$ is surjective by [Mat89, Theorem 7.3(i)]. It is natural to ask whether this holds for a Noether R-algebra A. Under the same assumption on $\mathfrak{a} \subseteq R$, it follows that the canonical ring homomorphism $A \to A_{\mathfrak{a}}^{\wedge}$ is a pure monomorphism in $\operatorname{Mod} A$ (since $R \to R_{\mathfrak{a}}^{\wedge}$ is a pure monomorphism by [Mat89, Theorem 7.5(i)] and $A_{\mathfrak{a}}^{\wedge} = A \otimes_R R_{\mathfrak{a}}^{\wedge}$). Thus the following proposition gives an affirmative answer to the question:

PROPOSITION A.16. *Let R be a commutative ring and let A and B be rings. Let $\varphi \colon R \to A$ and $f \colon A \to B$ be ring homomorphisms such that $\varphi(R)$ and $f(\varphi(R))$ are contained in the centers of A and B, respectively (that is, f is an R-algebra homomorphism). Assume that A is finitely generated as an R-module, B is a centralizing extension of $f(A)$, and f is a pure monomorphism in $\operatorname{Mod} A$. Then the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.*

Proof. Let $P \in \operatorname{Spec} A$ and $\mathfrak{p} := P \cap R$, which belongs to $\operatorname{Spec} R$ by Remark 2.17. Since B is a centralizing extension of $f(A)$, $BP = PB$ is a (two-sided) ideal of B. We have a commutative diagram

$$
\begin{array}{ccc}
A & \overset{f}{\hookrightarrow} & B \\
\downarrow & & \downarrow \\
A_{\mathfrak{p}}/P_{\mathfrak{p}} & \hookrightarrow & B_{\mathfrak{p}}/PB_{\mathfrak{p}}
\end{array}
$$

of canonical ring homomorphisms, where the second horizontal map is injective since it can be identified with $f \otimes_A (A_{\mathfrak{p}}/P_{\mathfrak{p}})$ and f is a pure monomorphism in $\operatorname{Mod} A$. By Remarks 2.16 and 2.17, the diagram induces the following commutative diagram of maps:

$$
\begin{array}{ccc}
\operatorname{Spec} A & \longleftarrow & \operatorname{Spec} B \\
\uparrow & & \uparrow \\
\operatorname{Spec}(A_{\mathfrak{p}}/P_{\mathfrak{p}}) & \longleftarrow & \operatorname{Spec}(B_{\mathfrak{p}}/PB_{\mathfrak{p}}).
\end{array}
$$

Since the ring $B_{\mathfrak{p}}/PB_{\mathfrak{p}}$ is nonzero, it has at least one maximal (hence prime) ideal Q. By Proposition 2.15 along with Remark 2.16, $\operatorname{Spec}(A_{\mathfrak{p}}/P_{\mathfrak{p}}) = \operatorname{Spec}((A/P) \otimes_R \kappa(\mathfrak{p}))$ consists of only the zero ideal, and it is sent to $P \in \operatorname{Spec} A$ by the left vertical map in the diagram. By the commutativity of the last diagram, the image of Q in $\operatorname{Spec} B$ is sent to P by the map $\operatorname{Spec} B \to \operatorname{Spec} A$. $\square$

Let $f \colon A \to A_{\mathfrak{a}}^{\wedge}$ be the canonical ring homomorphism. Then the induced map $\operatorname{Spec} A_{\mathfrak{a}}^{\wedge} \to \operatorname{Spec} A$ given by $Q \mapsto f^{-1}(Q)$ is surjective by Proposition A.16. The next proposition shows that, when R is local and $\mathfrak{a}$ is its maximal ideal, the correspondence of maximal ideals can be understood well. The completion

functor with respect to the maximal ideal of R is written as $\widehat{(-)}$. Recall that, for each maximal ideal P of A, $S_A(P)$ is the corresponding simple right A-module and $I_A(P)$ is the injective envelope of $S_A(P)$; see section 2.4.

PROPOSITION A.17. *Assume that (R, m, k) is a commutative noetherian local ring, and let $f : A \to \widehat{A}$ be the canonical ring homomorphism.*

(1) *Let $I \subseteq \widehat{A}$ be an ideal. Then $I \in \operatorname{Max} \widehat{A}$ if and only if $f^{-1}(I) \in \operatorname{Max} A$. If this is the case, then $I = \widehat{f^{-1}(I)}$, and $f : A \to \widehat{A}$ induces an isomorphism $A/f^{-1}(I) \xrightarrow{\sim} \widehat{A}/I$ of rings.*

(2) *The canonical surjection $\operatorname{Spec} \widehat{A} \to \operatorname{Spec} A$ restricts to a bijection $\operatorname{Max} \widehat{A} \xrightarrow{\sim} \operatorname{Max} A$ between the sets of maximal ideals, and $\operatorname{Max} \widehat{A} = \{ \widehat{P} \mid P \in \operatorname{Max} A \}$.*

(3) *For every $P \in \operatorname{Max} A$, we have isomorphisms $S_A(P) \cong S_{\widehat{A}}(\widehat{P})$ and $I_A(P) \cong I_{\widehat{A}}(\widehat{P})$ in $\operatorname{Mod} \widehat{A}$ (and also in $\operatorname{Mod} A$).*

Proof. (1): Let $J := f^{-1}(I)$. We have a commutative diagram

$$
\begin{array}{ccc}
R/(J \cap R) & \longhookrightarrow & \widehat{R}/(I \cap \widehat{R}) \\
\downarrow & & \downarrow \\
A/J & \xrightarrow{\overline{f}} & \widehat{A}/I,
\end{array}
$$

in which all maps are canonical ones. If $J \in \operatorname{Max} A$, then $J \cap R \in \operatorname{Max} R$ by Lemma 2.12, and hence $J \cap R = m$ and $R/(J \cap R) = k$. This means that A/J is a finite-dimensional k-algebra. In particular, A/J is of finite length as an R-module, so it is m-complete. On the other hand, $\widehat{A}/I$ is also m-complete as it is finitely generated $\widehat{R}$-module (see the third paragraph of section 2.5). Thus $\overline{f}$ is canonically identified with its completion $\Lambda^m \overline{f}$. By Lemma A.1, $\Lambda^m(A/J) \cong \widehat{A}/\widehat{J}$, so $\Lambda^m \overline{f}$ is the ring homomorphism $\widehat{A}/\widehat{J} \to \widehat{A}/I$, which is surjective. It then follows that $\overline{f} = \Lambda^m \overline{f}$ is an isomorphism and $\widehat{J} = I$. The isomorphism $\overline{f} : A/J \xrightarrow{\sim} \widehat{A}/I$ of rings implies that I is a maximal ideal of $\widehat{A}$ since J is maximal. This proves the "if" part of the first claim and the second claim of (1).
Conversely, if $I \in \operatorname{Max} \widehat{A}$, then $I \cap \widehat{R} = \widehat{m} \in \operatorname{Max} \widehat{R}$ by Lemma 2.12. Since the preimage of $I \cap \widehat{R}$ by the canonical map $R \to \widehat{R}$ is $J \cap R$ (see the commutative diagram above), we have $J \cap R = m \in \operatorname{Max} R$. Thus $J \in \operatorname{Max} A$ by Lemma 2.12 again.
(2): It follows from the first claim of (1) that the canonical surjection $\operatorname{Spec} \widehat{A} \to \operatorname{Spec} A$ restricts to a surjection $\operatorname{Max} \widehat{A} \to \operatorname{Max} A$, and this must be injective by the second claim of (1).
(3): For every $P \in \operatorname{Max} A$, A/P is a finite direct sum of copies of $S_A(P)$ as a right A-module; see (2.2). By Proposition A.12, this decomposition can be

regarded as that of right $\widehat{A}$-modules, and $S_A(P)$ is a simple $\widehat{A}$-module. Since $A/P \cong \widehat{A}/\widehat{P}$ by (1) and (2), and $\widehat{A}/\widehat{P}$ is a finite direct sum of copies of $S_{\widehat{A}}(\widehat{P})$ as a right $\widehat{A}$-module, we obtain an isomorphism $S_A(P) \cong S_{\widehat{A}}(\widehat{P})$ of right $\widehat{A}$-modules.

By this isomorphism, the injective envelope $E_{\widehat{A}}(S_A(P))$ of $S_A(P)$ coincides with the injective envelope $I_{\widehat{A}}(\widehat{P}) = E_{\widehat{A}}(S_{\widehat{A}}(\widehat{P}))$ of $S_{\widehat{A}}(\widehat{P})$ in $\mathrm{Mod}\,\widehat{A}$. As $I_{\widehat{A}}(\widehat{P})$ is $\widehat{\mathfrak{m}}$-torsion (Remark 2.26), it is $\mathfrak{m}$-torsion, so the essential extension $S_A(P) \hookrightarrow E_{\widehat{A}}(S_A(P)) \cong I_{\widehat{A}}(\widehat{P})$ in $\mathrm{Mod}\,\widehat{A}$ is also an essential extension in $\mathrm{Mod}\,A$ by Proposition A.12. Moreover, $E_{\widehat{A}}(S_A(P))$ is injective as a right A-module, because $\widehat{A}$ is a flat left A-module by Proposition A.3 and

$$\mathrm{Hom}_A(-, E_{\widehat{A}}(S_A(P))) \cong \mathrm{Hom}_A(-, \mathrm{Hom}_{\widehat{A}}(\widehat{A}, E_{\widehat{A}}(S_A(P))))$$
$$\cong \mathrm{Hom}_{\widehat{A}}(- \otimes_A \widehat{A}, E_{\widehat{A}}(S_A(P)))$$

by the tensor-hom adjunction. Therefore $S_A(P) \hookrightarrow E_{\widehat{A}}(S_A(P))$ is an injective envelope in $\mathrm{Mod}\,A$ as well, and hence $I_A(P) = E_A(S_A(P)) \cong E_{\widehat{A}}(S_A(P)) = I_{\widehat{A}}(\widehat{P})$ in $\mathrm{Mod}\,A$. This is also an isomorphism in $\mathrm{Mod}\,\widehat{A}$ by Proposition A.12. $\qquad\square$

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