Ricci DeTurck Flow on Incomplete Manifolds

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Abstract. In this paper we construct a Ricci DeTurck flow on any incomplete Riemannian manifold with bounded curvature. The central property of the flow is that it stays uniformly equivalent to the initial incomplete Riemannian metric, and in that sense preserves any given initial singularity structure. Together with the corresponding result by Shi for complete manifolds [Shi89], this gives that any (complete or incomplete) manifold of bounded curvature can be evolved by the Ricci DeTurck flow for a short time.

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1 Introduction and statement of the main result

Consider an n-dimensional, smooth and possibly incomplete Riemannian manifold $(M, \widetilde{g})$. We denote the corresponding Riemannian curvature tensor by $\widetilde{\mathrm{Rm}}$ and its pointwise norm with respect to $\widetilde{g}$ by $|\widetilde{\mathrm{Rm}}|$. The Ricci DeTurck flow of $(M, \widetilde{g})$ is a smooth family $g(t), t \in [0,T]$, of Riemannian metrics on M, solving the initial value problem

$$\frac{\partial}{\partial t} g_{ij}(t) = -2\,\mathrm{Ric}_{ij}(t) + \nabla_i V_j(t) + \nabla_j V_i(t), \quad g(0) = \widetilde{g}. \tag{1.1}$$

where $V^i(t) = g(t)^{jk}(\Gamma^i_{jk}(g(t)) - \Gamma^i_{jk}(\widetilde{g}))$ is the DeTurck vector field defined[1] in terms of Christoffel symbols Γ^i_{jk} for $g(t)$ and $\widetilde{g}$; $(\mathrm{Ric}_{ij}(t))$ is the Ricci curvature tensor and ∇ the covariant derivative of $g(t)$. Our main theorem is then as follows.

[1] We employ the Einstein summation convention.

Theorem 1.1. *Assume $|\widetilde{\mathrm{Rm}}|^2 \leq k_0$ for some positive constant $k_0 > 0$. Then there exists $T(n, k_0) > 0$, depending only on n and k_0, such that the initial value problem (1.1) has a smooth solution $g(t)$ for $t \in [0, T(n, k_0)]$. Furthermore, for any $\delta > 0$ there exists $0 < T(n, k_0, \delta) \leq T(n, k_0)$ depending only on n, k_0 and δ, such that*

$$(1-\delta)\widetilde{g}(x) \leq g(x,t) \leq (1+\delta)\widetilde{g}(x), \tag{1.2}$$

for all $(x, t) \in M \times [0, T(n, k_0, \delta)]$. Moreover, if we assume that for all $m \geq 1$ there exists a constant $C_m > 0$, such that for all $x \in M$, $0 < \rho \leq 1$

$$|\widetilde{\nabla}^m \widetilde{\mathrm{Rm}}|(x) \leq \frac{C}{\rho^m}$$

whenever $B(x, \rho - r)$ is relatively compact for all $r > 0$, then there exist constants $C' > 0$, $C'_m > 0$, such that for all $x \in M$, $t \in [0, T]$, $0 < \rho \leq 1$

$$|\widetilde{\nabla}^m g|(x,t) \leq \frac{C'_m}{\rho^m}, \quad |\mathrm{Rm}|(x,t) \leq \frac{C'}{\rho^2}, \quad |\nabla^m \mathrm{Rm}|(x,t) \leq \frac{C'}{\rho^{m+2}}$$

whenever $B(x, \rho - r)$ is relatively compact for all $r > 0$.

Remark 1.2. *The condition that $B(x, \rho - r)$ is relatively compact in M for all $r > 0$ is an intrinsic way to express the distance of a point $x \in M$ to the singular strata of M. It means that this distance is larger or equal to ρ.*

We should point out that short-time existence and further properties of a Ricci DeTurck flow on incomplete manifolds has already been established in the special case of manifolds with conical or more generally wedge singularities in varying dimensions.

More specifically, Mazzeo, Rubinstein and Sesum [MRS15] as well as Yin [Yin10] discuss Ricci flow on surfaces with isolated conical singularities. In [BaVe14, BaVe16] the second named author jointly with Bahuaud discuss Yamabe flow on manifolds with wedge singularities. In [Ver21] the second named author introduces a Ricci DeTurck flow on manifolds with wedge singularities and discusses its short time existence and regularity. In [KrVe19a] the second named author, jointly with Kröncke discuss stability and convergence of the Ricci DeTurck flow on manifolds with isolated conical singularities near Ricci-flat metrics. In these references the flow stays uniformly equivalent to the initial metric and hence preserves the initial singularity. This list of references is not exhaustive.

Due to non-uniqueness of the flow in the singular setting, there exist solutions that are instantaneously complete, cf. Giesen and Topping [GiTo11], as well as solutions that smooth out the singularity, cf. Simon [Sim13].

The main novelty of the present paper is the assertion that such a Ricci DeTurck flow, preserving the initial singularity structure, exists on any *arbitrary* incomplete manifold of bounded curvature. This includes, but is not restricted to, orbifolds or more generally incomplete manifolds with isolated conical singularities, where the cone has bounded Riemannian curvature. In this setting we also establish explicit estimates for arbitrary higher derivatives of the metric and of the Riemann curvature tensor along the flow. We conjecture that this flow coincides with the Ricci DeTurck flow on wedge manifolds, introduced in a previous work by the second named author [Ver21].

Our paper is structured as follows. In §2 we review the argument of Shi [Shi89], which proves short time existence of Ricci DeTurck flow for complete manifolds of bounded curvature. We break down the argument to those points where completeness of the manifold is used. In §3 we establish estimates for quantities in balls which are compactly contained in an incomplete manifold. In the subsequent §4, §5 and §6 we establish a priori estimates for the first, second and higher derivatives of the metric along the flow. §5 and 6 also contain a priori estimates for the Riemann curvature tensor. In the final §7 we adapt the argument of §2 in order to establish the corresponding result for incomplete manifolds of bounded curvature as well.

Notation: Let us fix the notation for the discussion below. Let $g(t)$, $t \in [0, T]$ be a family of Riemannian metrics on an incomplete manifold M. We denote by ∇ and Γ the covariant derivative and the Christoffel symbols with respect to $g(t)$. Rm, Ric and R denote the Riemann curvature tensor, the Ricci tensor and the scalar curvature of $g(t)$, respectively.

Let $\widetilde{g}$ be the initial Riemannian metric on M. Quantities with respect to $\widetilde{g}$ are marked with an upper tilde. For example we write $\widetilde{\nabla}$ for the covariant derivative with respect to $\widetilde{g}$. There are the following exceptions to this rule: We denote by $B(x, r)$ the open ball with radius $r > 0$ and centre $x \in M$, and we write $B(A, r) := \{x \in M : d_{\widetilde{g}}(x, A) < r\}$ for the r-neighborhood of a given subset $A \subset M$, both with respect to the metric $\widetilde{g}$. The norm $|\cdot|$ will always be with respect to $\widetilde{g}$. We write $d_{\widetilde{g}}$ for the distance function induced by $\widetilde{g}$.

2 Review of Shi's local existence theorem

In this section we review results and proofs from Shi [Shi89] in the complete setting. Shi established the following short-time existence result for the Ricci DeTurck flow starting at complete manifolds with bounded curvature. Within this section, $(M, \widetilde{g})$ is always understood to be a complete n-dimensional Riemannian manifold of bounded curvature.

Theorem 2.1 ([Shi89], Theorems 4.3, 2.5). *Assume $|\widetilde{\mathrm{Rm}}|^2 \leq k_0$ for some positive constant $k_0 > 0$. Then there exists $T(n, k_0) > 0$ depending only on n and k_0, such that the initial value problem (1.1) has a smooth solution $g(t)$. Moreover, for any $\delta > 0$ there exists $0 < T(n, k_0, \delta) \leq T(n, k_0)$ depending only on n, k_0 and δ, such that*

$$(1 - \delta)\widetilde{g}(x) \leq g(x, t) \leq (1 + \delta)\widetilde{g}(x) \tag{2.1}$$

for all $(x,t) \in M \times [0, T(n,k_0,\delta)]$.

REMARK 2.2. *We emphasize that the lower bound on the injectivity radius does not enter in the definition of the time bounds* $T(n,k_0), T(n,k_0,\delta) > 0$. *Indeed, the local existence result still holds on complete manifolds without a positive lower bound on the injectivity radius. An obvious instance are manifolds with hyperbolic cusps, where Theorem 2.1 still holds despite the injectivity radius tending to zero at the cusp.*

The proof of this theorem is based on three main steps. The first is an a priori estimate for the Ricci DeTurck flow on a relatively compact domain $D \subset M$ with Dirichlet boundary conditions.

THEOREM 2.3 ([Shi89], Theorem 2.5). *Let* $D \subset M$ *be a relatively compact domain, whose boundary* ∂D *is an* $(n-1)$*-dimensional, smooth, compact submanifold. Let* $g(x,t)$, $t \in [0,T]$ *be a solution of the initial boundary value problem*

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij}(x,t) &= (-2\operatorname{Ric}_{ij} + \nabla_i V_j + \nabla_j V_i)(x,t), \quad (x,t) \in D \times [0,T], \\ g(x,t) &= \widetilde{g}(x), \quad (x,t) \in \partial D \times [0,T], \\ g(x,0) &= \widetilde{g}(x), \quad x \in D. \end{aligned} \tag{2.2}$$

where $V^i = g^{jk}(\Gamma^i_{jk} - \widetilde{\Gamma}^i_{jk})$ *is the DeTurck vector field. Then for any* $\delta > 0$ *there exists* $T(n,k_0,\delta) > 0$ *depending only on* n, k_0 *and* δ, *such that*

$$(1-\delta)\widetilde{g}(x) \leq g(x,t) \leq (1+\delta)\widetilde{g}(x) \tag{2.3}$$

for all $(x,t) \in M \times [0, \min\{T(n,k_0,\delta), T\}]$.

Proof outline. Shi controls the eigenvalues $\lambda_k(x,t)$ of $g(x,t)$ with respect to $\widetilde{g}(x)$ (i.e. the eigenvalues of $g(x,t)$ considered as a $(1,1)$-tensor using the metric $\widetilde{g}(x)$). Shi defines a function

$$\varphi(x,t) = \sum_{k=1}^{n} \lambda_k(x,t)^{-m},$$

where $m > 0$ is sufficiently large only depending on n and δ. Shi then shows that φ satisfies a differential inequality

$$\frac{\partial \varphi}{\partial t} \leq g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \varphi + 2mn\sqrt{k_0} \cdot \varphi^{1+1/m},$$

and applies the maximum principle to conclude $\varphi(x,t) \leq 2n$ for all $(x,t) \in D \times [0,T]$. This leads to the lower bound in (2.3). The upper bound in (2.3) is then obtained by a similar procedure applied to the function

$$F(x,t) = \left(1 - \frac{1}{2n} \sum_{k=1}^{n} \lambda_k(x,t)^{\widetilde{m}}\right)^{-1},$$

where $\widetilde{m} > 0$ is large enough and only depends on n and δ. ☐

The second step is the short-time existence of system (2.2).

Theorem 2.4 ([Shi89], Theorem 3.2). *Let $D \subset M$ be a relatively compact domain, whose boundary ∂D is an $(n-1)$-dimensional, smooth, compact submanifold. Then there exists $T(n,k_0) > 0$ only depending on n and k_0, such that the initial boundary value problem (2.2) admits a unique smooth solution $g(x,t)$, $(x,t) \in D \times [0, T(n,k_0)]$.*

The third step are interior estimates for the derivatives of the metric, only depending on $\widetilde{g}$ and not on any specified boundary conditions.

Lemma 2.5 ([Shi89], Lemma 4.1). *Fix $0 < \gamma, \delta, T < \infty$, and let $g(x,t)$ be a smooth solution of the initial value problem*

$$\frac{\partial}{\partial t} g_{ij}(x,t) = (-2\operatorname{Ric}_{ij} + \nabla_i V_j + \nabla_j V_i)(x,t), \quad (x,t) \in B(x_0, \gamma+\delta) \times [0,T],$$
$$g(x,0) = \widetilde{g}(x), \quad x \in B(x_0, \gamma+\delta),$$

where $V^i = g^{jk}(\Gamma^i_{jk} - \widetilde{\Gamma}^i_{jk})$ is the DeTurck vector field. Furthermore, assume that

$$(1-\varepsilon(n))\widetilde{g}(x) \le g(x,t) \le (1+\varepsilon(n))\widetilde{g}(x)$$

for $\varepsilon(n) > 0$ sufficiently small, only depending on n, and for all $(x,t) \in B(x_0, \gamma+\delta) \times [0,T]$. Then there exists a positive constant $c(n,\gamma,\delta,T,\widetilde{g}) > 0$, depending only on n, γ, δ, T and $\widetilde{g}$, such that

$$|\widetilde{\nabla} g(x,t)|^2 \le c(n,\gamma,\delta,T,\widetilde{g})$$

for all $(x,t) \in B(x_0, \gamma + \frac{\delta}{2}) \times [0,T]$.

Proof outline. Shi defines for any $(x,t) \in B(x_0,\gamma+\delta)\times[0,T]$ the function

$$\varphi(x,t) = a + \sum_{k=1}^{n} \lambda_k(x,t)^{m_0}, \tag{2.4}$$

where a, m_0 are carefully chosen positive constants only depending on n, and $\lambda_k(x,t)$ are the eigenvalues of $g(x,t)$ with respect to $\widetilde{g}(x)$. Shi then shows that the function

$$\psi(x,t) := |\widetilde{\nabla} g|^2 \varphi(x,t) \tag{2.5}$$

satisfies

$$\frac{\partial \psi}{\partial t} \le g^{\alpha\beta}\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \psi - \frac{1}{16}\psi^2 + c_0, \tag{2.6}$$

where $c_0 > 0$ is a constant only depending on n and $\widetilde{g}$. Then Shi takes a nonincreasing cutoff function $\eta \in C^\infty(\mathbb{R})$ such that $\eta \equiv 1$ on $(-\infty, 0]$, vanishing identically on $[1,\infty)$ as illustrated in Figure 1.
The crucial property of the function η is the control on its derivatives

$$|\eta''(x)| \le 8, \quad |\eta'(x)|^2 \le 16\eta(x), \quad \text{for any } x \in \mathbb{R}. \tag{2.7}$$

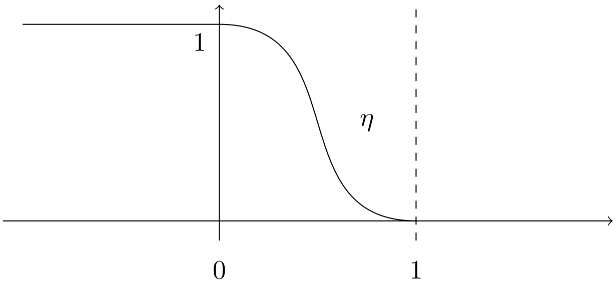

Figure 1: The cutoff function η.

One then defines a Lipschitz continuous bump function $\xi \in C(M)$ around any fixed $x_0 \in M$ by

$$\xi(x) := \eta\left(\frac{d_{\widetilde{g}}(x, x_0) - (\gamma + \delta/2)}{\delta/4}\right), \tag{2.8}$$

where $d_{\widetilde{g}}$ is the distance function with respect to the metric $\widetilde{g}$. Note that $d_{\widetilde{g}}(\cdot, x_0)$ is Lipschitz continuous but need not be smooth everywhere, and hence ξ need not be smooth everywhere. By construction, ξ has the following properties

$$\begin{aligned} \xi(x) &= 1, \quad x \in B(x_0, \gamma + \delta/2), \\ \xi(x) &= 0, \quad x \in M \setminus B(x_0, \gamma + 3\delta/4), \end{aligned} \tag{2.9}$$

which is illustrated in Figure 2.

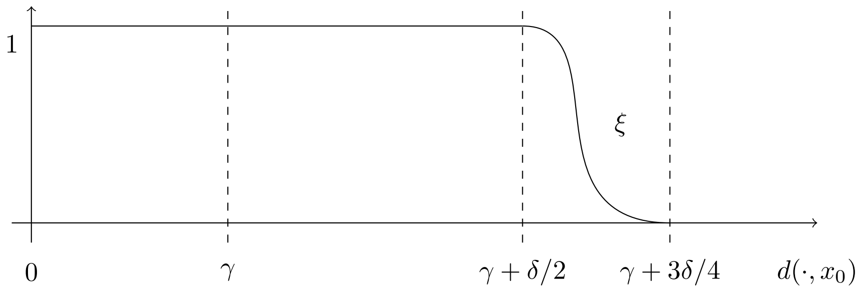

Figure 2: The bump function ξ.

Below in §4, starting with (4.28), we provide a careful argument differentiating between the case that ξ is smooth in a neighborhood of x and the case that ξ is not. The latter case is studied after (4.54) using a trick of Calabi. In case of smoothness, we have by (2.7) control on derivatives of ξ

$$|\widetilde{\nabla}\xi|^2(x) \leq \frac{16^2}{\delta^2}\xi(x), \quad x \in M. \tag{2.10}$$

Shi also proves an estimate

$$\widetilde{\nabla}\widetilde{\nabla}\xi(x) \geq -c_0(\gamma,\delta,k_0)\widetilde{g}(x), \quad x \in M, \tag{2.11}$$

where $c_0(\gamma,\delta,k_0) > 0$ is a constant only depending on γ, δ and k_0.

The auxiliary bump function ξ is used to define

$$F(x,t) := \xi(x)\psi(x,t), \quad (x,t) \in B(x_0,\gamma+\delta) \times [0,T].$$

By construction, it has the properties

$$\begin{aligned} &F(x,0) = 0, \quad x \in B(x_0,\gamma+\delta), \\ &F(x,t) = 0, \quad (x,t) \in M \backslash B(x_0,\gamma+3\delta/4) \times [0,T], \end{aligned} \tag{2.12}$$

In particular, F attains its maximum on $B(x_0,\gamma+3\delta/4) \times [0,T]$, i.e. there exists $(x_0,t_0) \in B(x_0,\gamma+3\delta/4) \times [0,T]$ such that

$$F(x_0,t_0) = \max\{\, F(x,t) \mid (x,t) \in B(x_0,\gamma+\delta) \times [0,T]\}.$$

Using the evolution inequality (2.6), especially the negative quadratic term $(-\frac{1}{16}\psi^2)$, as well as the properties (2.10) and (2.11) of the cutoff function ξ, Shi concludes by maximum principle arguments that

$$F(x_0,t_0) \leq c(n,\gamma,\delta,T,\widetilde{g}),$$

where $c(n,\gamma,\delta,T,\widetilde{g}) > 0$ is a constant only depending on $n,\gamma,\delta,T,\widetilde{g}$. Thus

$$\xi(x)\psi(x,t) = F(x,t) \leq F(x_0,t_0) \leq c(n,\gamma,\delta,T,\widetilde{g}), \tag{2.13}$$

for any $(x,t) \in B(x_0,\gamma+\delta) \times [0,T]$. Since $\xi \equiv 1$ on $B(x_0,\gamma+\delta/2)$, we conclude

$$|\widetilde{\nabla}g|^2\varphi(x,t) = \psi(x,t) \leq c(n,\gamma,\delta,T,\widetilde{g}), \tag{2.14}$$

for any $(x,t) \in B(x_0,\gamma+\delta/2) \times [0,T]$. Finally, since by definition $\varphi(x,t) \geq a$, the statement follows from

$$|\widetilde{\nabla}g|^2(x,t) \leq \frac{1}{a}c(n,\gamma,\delta,T,\widetilde{g}), \quad (x,t) \in B(x_0,\gamma+\delta/2) \times [0,T].$$

□

LEMMA 2.6 ([Shi89], Lemma 4.2). *Under the same assumptions as in Lemma 2.5, there exists a constant $c(n,m,\gamma,\delta,T,\widetilde{g}) > 0$ for any $m \geq 0$, depending only on n,m,γ,δ,T and $\widetilde{g}$, such that*

$$|\widetilde{\nabla}^m g(x,t)|^2 \leq c(n,m,\gamma,\delta,T,\widetilde{g}) \tag{2.15}$$

for all $(x,t) \in B(x_0,\gamma+\frac{\delta}{m+1}) \times [0,T]$.

Proof outline. Lemma 2.6 is proven by induction. Assuming that the statement holds for any integer $0 \leq m_0 < m$, Shi defines the function (cf. (2.5))

$$\Psi(x,t) = (a_0 + |\widetilde{\nabla}^{m-1} g(x,t)|^2)|\widetilde{\nabla}^m g(x,t)|^2$$

and proves that, if $a_0 > 0$, depending only on $m, n, \gamma, \delta, T, \widetilde{g}$, is chosen appropriately, then Ψ satisfies a differential inequality of the form (cf. (2.6))

$$\frac{\partial \Psi}{\partial t} \leq g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \Psi - c_1 \Psi^2 + c_0,$$

on $B(x_0, \gamma + \delta/m) \times [0,T]$, where $c_0, c_1 > 0$ only depend on m, n, γ, δ, T and $\widetilde{g}$. Then by the same steps as in the proof of Lemma 2.5, Shi obtains (cf. (2.14))

$$\Psi(x,t) \leq c_2(m,n,\gamma,\delta,T,\widetilde{g}), \quad \text{for } (x,t) \in B(\overline{U}, \delta/(m+1)) \times [0,T].$$

Hence, we conclude for all $(x,t) \in B(x_0, \delta/(m+1)) \times [0,T]$

$$|\widetilde{\nabla}^m g(x,t)|^2 \leq \frac{1}{a_0} \Psi(x,t) \leq \frac{1}{a_0} c_2(m,n,\gamma,\delta,T,\widetilde{g}),$$

which finishes the proof. □

Now Shi completes the proof of Theorem 2.1 as follows. Shi takes an exhaustion of the manifold M by relatively compact domains $D_k \subset M$, $k \in \mathbb{N}_0$, with $(n-1)$-dimensional, smooth, compact boundary ∂D_k, such that $B(x_0,k) \subset D_k$, for some fixed point $x_0 \in M$. By Theorem 2.4 and Theorem 2.3, there exists $T(n,k_0) > 0$ depending only on n and k_0 such that the system (cf. (2.2))

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij}(x,t) &= (-2\operatorname{Ric}_{ij} + \nabla_i V_j + \nabla_j V_i)(x,t), \quad (x,t) \in D_k \times [0,T], \\ g(x,t) &= \widetilde{g}(x), \quad (x,t) \in \partial D_k \times [0,T], \\ g(x,0) &= \widetilde{g}(x), \quad x \in D_k. \end{aligned} \tag{2.16}$$

has a unique smooth solution $g(k,x,t)$ on $D_k \times [0, T(n,k_0)]$ satisfying

$$(1 - \varepsilon(n))\widetilde{g}(x) \leq g(k,x,t) \leq (1+\varepsilon(n))\widetilde{g}(x) \tag{2.17}$$

for all $(x,t) \in D_k \times [0, T(n,k_0)]$. Here, $\varepsilon(n) > 0$ is a sufficiently small constant, depending only on n, introduced in Lemma 2.5. Now, for any $k \geq 2$, the solution $g(k,x,t)$ is defined on $B(x_0,1)$. By Lemma 2.6, we have for all $m \in \mathbb{N}_0$

$$|\widetilde{\nabla}^m g(k,x,t)|^2 \leq c(n,m,q,T(n,k_0),\widetilde{g}) \tag{2.18}$$

for all $(x,t) \in B(x_0,1) \times [0, T(n,k_0)]$ and all $k \geq 2$. Hence by Arzelà-Ascoli there exists a subsequence $(g(k_\ell,x,t))_{\ell \in \mathbb{N}_0}$, which converges on $B(x_0,1) \times [0, T(n,k_0)]$ in the C^∞ topology to a family of smooth metrics $g(x,t)$.

By the same argument a subsequence of this subsequence converges on $B(x_0,2) \times [0,T(n,k_0)]$. We iterate this argument and consider the diagonal sequence. Then, for every fixed $q \in \mathbb{N}$, the diagonal sequence converges to $g(x,t)$ on $B(x_0,q) \times [0, T(n,k_0)]$, and thus converges smoothly locally uniformly to $g(x,t)$. Thus $g(x,t)$ solves (1.1). The estimate (2.1) follows by restricting the solutions $g(k,x,t)$ to $0 \leq t \leq T(n,k_0,\delta)$, where $T(n,k_0,\delta)$ is from Theorem 2.3.

3 The geometry of incomplete manifolds

In the following sections we will establish estimates for quantities in balls which are compactly contained in an incomplete manifold. Since most theorems in the literature are stated for complete manifolds, in this section we give some background information on the geometry of incomplete manifolds inside relatively compact balls, and indicate how for example the proof of the Hessian comparison theorem can be modified to also hold in our setting.

Let (M, g) be a Riemannian manifold. Let $p \in M$. Let $\mathcal{D}_p \subset T_pM$ be the domain of the exponential map $\exp_p$. Following [Pet06] we refer to a shortest geodesic as a *segment*. Define the segment domain as

$$\operatorname{seg}(p) := \{v \in \mathcal{D}_p \mid \exp_{x_0}(tv) : [0,1] \to M \text{ is a segment}\}.$$

We define the segment "interior"

$$\operatorname{seg}^0(p) := \{tv \mid t \in [0,1), v \in \operatorname{seg}(p)\}.$$

Then, as in the case of complete manifolds, we have

1) $\exp_p : \operatorname{seg}^0(p) \to M$ is injective, and
2) $D\exp_p(v)$ is non-singular for all $v \in \operatorname{seg}^0(p)$

see [Pet06, Proposition 19, p.139 and Lemma 14, p.140].

Now assume that $\rho > 0$ such that $B(p, \rho - r) \subset\subset M$ for all $r > 0$. A direct consequence is

$$B(0, \rho) \subset D_p. \tag{3.1}$$

Also, by the same steps as in the proof of the Hopf-Rinow theorem in the complete case (see [Pet06, Theorem 16, p.137]), each $y \in B(p, \rho)$ can be joined to p by a segment. Thus, since

$$V_p := \exp_p(\operatorname{seg}(p)) = \{x \in M \mid \exists \text{ segment from } p \text{ to } x\},$$

it follows that

$$B(p, \rho) \subset V_p. \tag{3.2}$$

Furthermore, if $x \in V_p$ and $v \in \operatorname{seg}(p)$ with $\exp_p(v) = x$, then $d(p, x) = |v|$. Hence

$$\exp_p(B(0, \rho) \cap \operatorname{seg}(p)) = B(p, \rho) \cap V_p \overset{(3.2)}{=} B(p, \rho). \tag{3.3}$$

Now we can characterize the points in $\operatorname{seg}(p) \backslash \operatorname{seg}^0(p)$, which are inside $B(0, \rho)$, as in the complete case.

Lemma 3.1. *If $v \in (\operatorname{seg}(p) \backslash \operatorname{seg}^0(p)) \cap B(0, \rho)$, then*

1) $\exists w (\neq v) \in \operatorname{seg}(p) \cap B(0, \rho) : \exp_p(v) = \exp_p(w)$, or

2) $D\exp_p(v)$ is singular.

Proof. Analogous to the proof of the corresponding statement [Pet06, Lemma 15, p.141] in the complete case. □

Remark 3.2. *Note that* $\exp_p(\mathrm{seg}(p)\backslash\mathrm{seg}^0(p))$ *is the cut locus of* p *in* M.

This gives the following lemma.

Lemma 3.3. $\mathrm{seg}^0(p)\cap B(0,\rho)$ *is open.*

Proof. Analogous to the proof of the corresponding statement [Pet06, Proposition 20] in the complete case, using Lemma 3.1 instead of [Pet06, Lemma 15, p.141]. □

Letting $U_p := \exp_p(\mathrm{seg}^0(p))$, by the above

$$\exp_p : \mathrm{seg}^0(p)\cap B(0,\rho) \to U_p \cap B(p,\rho)$$

is a diffeomorphism. As

$$d(p,x) = |\exp_p^{-1}(x)|$$

for $x\in U_p\cap B(p,\rho)$, the distance function is smooth on $(U_p\cap B(p,\rho))\backslash\{p\}$. Also, by Lemma 3.1 the distance function is not smooth on $B(p,\rho)\backslash U_p$.

Now we can use the exponential map $\exp_p$ to define geodesic polar coordinates on $U_p\cap B(p,\rho)$. Then we can argue as in the complete case (cf. [CLN06, Proof of the Hessian Comparison Theorem 1.141, p.76]) to obtain the following version of the Hessian comparison theorem on incomplete manifolds.

Theorem 3.4. *Assume that the sectional curvatures satisfy* $\sec \geq K$ *on* $B(p,\rho)$. *Then*

$$\nabla_\alpha\nabla_\beta d_g(x,p) \leq \frac{1}{n-1} H_K(d_g(x,p))g_{\alpha\beta}(x)$$

at all points $x\in (U_p\cap B(p,\rho))\backslash\{p\}$. *Here*

$$H_K(r) := \begin{cases} (n-1)\sqrt{K}\cot(\sqrt{K}r), & \text{if } K>0,\\ \frac{n-1}{r}, & \text{if } K=0,\\ (n-1)\sqrt{|K|}\coth(\sqrt{|K|}r), & \text{if } K<0. \end{cases} \tag{3.4}$$

4 A priori estimates of ∇g along the flow

In this section we establish quantitative estimates for the first derivatives of the metric under Ricci DeTurck flow on singular manifolds. We assume bounded curvature at time $t=0$ and that the metrics $g(t)$ are uniformly equivalent and sufficiently close to the initial metric $\widetilde{g}$. As a byproduct we also obtain an estimate on the DeTurck vector field V. We continue in the setting of an n-dimensional, smooth and possibly incomplete Riemannian manifold $(M,\widetilde{g})$ and prove an analogue of Lemma 2.5.

Lemma 4.1. *Consider $x_0 \in M$ and fix any*[2] *finite $\gamma, \delta, T > 0$ with $\delta \le 1$. Let $g(x,t)$ be a smooth solution of the initial value problem*

$$\frac{\partial}{\partial t} g_{ij}(x,t) = (-2\operatorname{Ric}_{ij} + \nabla_i V_j + \nabla_j V_i)(x,t), \quad (x,t) \in B(x_0, \gamma+\delta) \times [0,T],$$
$$g(x,0) = \widetilde{g}(x), \quad x \in B(x_0, \gamma+\delta),$$

where $V^i = g^{jk}(\Gamma^i_{jk} - \widetilde{\Gamma}^i_{jk})$ is the DeTurck vector field. We assume that $B(x_0, \gamma+\delta-r)$ is relatively compact in M for all $r > 0$. Furthermore, we assume that for all $(x,t) \in B(x_0, \gamma+\delta) \times [0,T]$ we have the inequalities

$$(1-\varepsilon(n))\widetilde{g}(x) \le g(x,t) \le (1+\varepsilon(n))\widetilde{g}(x) \tag{4.1}$$

for $\varepsilon(n) > 0$ sufficiently small, only depending on n. Also assume that

$$|\widetilde{Rm}|^2 \le k_0$$

for some constant $k_0 > 0$. Then there exist constants $c(n), c(n,k_0) > 0$, only depending on the arguments in brackets, such that for all $(x,t) \in B(x_0, \gamma + \frac{\delta}{2}) \times [0,T]$

$$|\widetilde{\nabla} g|(x,t) \le \frac{c(n,k_0)}{\delta} + c(n)c_1, \quad \text{where } c_1 := \sup_{x \in B(x_0, \gamma + 3\delta/4)} |\widetilde{\nabla}\widetilde{\mathrm{Rm}}|(x). \tag{4.2}$$

Remark 4.2. *The restriction $\delta \le 1$ is for technical reasons to achieve a simpler expression for the right-hand side of (4.2). For our purposes this is sufficient as we are aiming at estimates on an incomplete manifold when we get closer and closer to the singularity. Also note that the estimates (4.2) are independent of γ, and only depend on the difference of radia of the smaller ball $B(x_0, \gamma + \frac{\delta}{2})$ and the larger ball $B(x_0, \gamma+\delta)$.*

We will prove the lemma below and first note its consequence − estimates on the first derivatives of the metric for Ricci DeTurck flow. More specifically, assuming additionally that $|\widetilde{\nabla}\widetilde{\mathrm{Rm}}| = \mathcal{O}(\rho^{-1})$, where $\rho > 0$ is the distance to the singularity, a natural condition in case $|\widetilde{\mathrm{Rm}}|$ is bounded, we obtain that $|\widetilde{\nabla} g| = \mathcal{O}(\rho^{-1})$ and $|V| = \mathcal{O}(\rho^{-1})$ uniformly in $t \in [0,T]$.

Corollary 4.3. *Let $(M, \widetilde{g})$ be a (possibly incomplete) smooth Riemannian manifold of dimension n. Fix $0 < T < \infty$ and let $g(x,t)$ be a smooth solution of*

$$\frac{\partial}{\partial t} g_{ij}(x,t) = (-2\operatorname{Ric}_{ij} + \nabla_i V_j + \nabla_j V_i)(x,t), \quad (x,t) \in M \times [0,T],$$
$$g(x,0) = \widetilde{g}(x), \quad x \in M,$$

[2] Below, in Corollary 4.3 we will set $\gamma = \delta > 0$ sufficiently small.

where V is the DeTurck vector field as above. Assume that for all $(x,t) \in M \times [0,T]$

$$(1-\varepsilon(n))\widetilde{g}(x) \leq g(x,t) \leq (1+\varepsilon(n))\widetilde{g}(x)$$

for $\varepsilon(n) > 0$ sufficiently small, only depending on n, and also assume that there exist constants $k_0, C > 0$, such that

$$|\widetilde{Rm}|^2 \leq k_0$$

and that for all $x \in M$, $0 < \rho \leq 1$

$$|\widetilde{\nabla}\widetilde{\mathrm{Rm}}|(x) \leq \frac{C}{\rho}$$

whenever $B(x, \rho - r)$ is relatively compact for all $r > 0$. Then there exists $C' > 0$ only depending on k_0, C and n such that for all $x \in M$, $t \in [0,T]$, $0 < \rho \leq 1$

$$|\widetilde{\nabla} g|(x,t) \leq \frac{C'}{\rho}, \quad |V|(x,t) \leq \frac{C'}{\rho}$$

whenever $B(x, \rho - r)$ is relatively compact for all $r > 0$.

Remark 4.4. *The (technical) condition $B(x, \rho - r)$ is relatively compact in M for all $r > 0$ is a way to express the distance of a point $x \in M$ to the singular strata of M intrinsically. It means that this distance is larger or equal to ρ.*

Proof of Corollary 4.3. Consider $x_0 \in M$ and $\rho \leq 1$ such that $B(x_0, \rho - r)$ is relatively compact in M for all $r > 0$. Then by Lemma 4.1 (choosing γ, δ in Lemma 4.1 as equal to $\rho/2$) we obtain

$$|\widetilde{\nabla} g|(x_0, t) \leq \frac{c(n, k_0)}{\rho} + c(n) c_1,$$

where the constant c_1 can be estimated as follows

$$c_1 = \sup_{x \in B(x_0, 7\rho/8)} |\widetilde{\nabla}\widetilde{\mathrm{Rm}}|(x) \leq \frac{8C}{\rho},$$

since for all $x \in B(x_0, 7\rho/8)$ we have that $B(x, \rho/8 - r)$ is relatively compact for all $r > 0$. This proves the estimate for $|\widetilde{\nabla} g|$. The estimate of the DeTurck vector field V follows from this and

$$V = g^{-1} * \widetilde{\nabla} g,$$

see [Shi89, p. 266, formula (32)]. ∎

We can now proceed with proof of Lemma 4.1.

Proof of Lemma 4.1. Our strategy is a careful analysis of the proof of [Shi89, Lemma 4.1], which is written out here in Lemma 2.5, while making the dependencies of various constants explicit. For the convenience of the reader and to keep our argument here self-contained, we repeat the steps from [Shi89, Lemma 4.1] here.

In the following, $c(n)$ and $c(n,k_0)$ denote constants only depending on n and n,k_0, respectively. The constants may vary from estimate to estimate.

As in [Shi89, Proof of Lemma 4.1, p.247 (5)] we have

$$\begin{aligned}\frac{\partial}{\partial t}|\widetilde{\nabla} g|^2 =& g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla} g|^2 - 2g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla} g\cdot\widetilde{\nabla}_\beta\widetilde{\nabla} g\\ &+\widetilde{\mathrm{Rm}}*g^{-2}*g*\widetilde{\nabla} g*\widetilde{\nabla} g+g^{-1}*g*\widetilde{\nabla}\widetilde{\mathrm{Rm}}*\widetilde{\nabla} g\\ &+g^{-2}*\widetilde{\nabla} g*\widetilde{\nabla} g*\widetilde{\nabla}\widetilde{\nabla} g+g^{-3}*\widetilde{\nabla} g*\widetilde{\nabla} g*\widetilde{\nabla} g*\widetilde{\nabla} g.\end{aligned} \tag{4.3}$$

Here the product $A*B$ of two tensors A and B denotes a linear combination of terms which are obtained as follows: Starting from the tensor product $A\otimes B$, perform an arbitrary number of the following operations: taking contractions, raising, lowering or permuting indices. The important consequence in our case here is that it will always be possible to estimate

$$|A*B|\le c(n)|A|\cdot|B|,$$

where $c(n)$ depends on the specific form of the product. Since by assumption, the closure $\overline{B(x_0,\gamma+\frac{3}{4}\delta)}\subset M$ is compact, we conclude (c_1 is defined in (4.2))

$$|\widetilde{\nabla}\widetilde{\mathrm{Rm}}|\le c_1 \quad\text{on } B(x_0,\gamma+\frac{3}{4}\delta). \tag{4.4}$$

Furthermore, by (4.1) we have

$$\frac{1}{2}\widetilde{g}(x)\le g(x,t)\le 2\widetilde{g}(x) \qquad \text{on } B(x_0,\gamma+\delta). \tag{4.5}$$

Hence

$$\begin{aligned}\widetilde{\mathrm{Rm}}*g^{-2}*g*\widetilde{\nabla} g*\widetilde{\nabla} g&\le c(n,k_0)|\widetilde{\nabla} g|^2,\\ g^{-1}*g*\widetilde{\nabla}\widetilde{\mathrm{Rm}}*\widetilde{\nabla} g&\le c(n)c_1|\widetilde{\nabla} g|\end{aligned} \tag{4.6}$$

on $B(x_0,\gamma+3\delta/4)\times[0,T]$. Also, whenever we use the bound (4.4) on $\widetilde{\nabla}\widetilde{\mathrm{Rm}}$ it is understood that the estimate, which follows, holds on $B(x_0,\gamma+3\delta/4)\times[0,T]$. As in [Shi89, Proof of Lemma 4.1, p.247 (9)] we have

$$\begin{aligned}g^{-2}*\widetilde{g}*\widetilde{\nabla} g*\widetilde{\nabla}\widetilde{\nabla} g&\le 72n^5|\widetilde{\nabla} g|^2|\widetilde{\nabla}^2 g|,\\ g^{-3}*\widetilde{\nabla} g*\widetilde{\nabla} g*\widetilde{\nabla} g*\widetilde{\nabla} g&\le 160n^6|\widetilde{\nabla} g|^4.\end{aligned} \tag{4.7}$$

This gives

$$\begin{aligned} \frac{\partial}{\partial t}|\widetilde{\nabla} g|^2 \leq & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla} g|^2 - |\widetilde{\nabla}^2 g|^2 + c(n,k_0)|\widetilde{\nabla} g|^2 + c(n)c_1|\widetilde{\nabla} g| \\ & + 72n^5|\widetilde{\nabla} g|^2|\widetilde{\nabla}^2 g| + 160n^6|\widetilde{\nabla} g|^4. \end{aligned} \tag{4.8}$$

Estimating as in [Shi89, Proof of Lemma 4.1, p.247]

$$\begin{aligned} 72n^5|\widetilde{\nabla} g|^2|\widetilde{\nabla}^2 g| + 160n^6|\widetilde{\nabla} g|^4 &\leq \frac{1}{2}|\widetilde{\nabla}^2 g|^2 + 3200n^{10}|\widetilde{\nabla} g|^4, \\ c(n)c_1|\widetilde{\nabla} g| &\leq \frac{(c(n)c_1)^2}{2} + \frac{|\widetilde{\nabla} g|^2}{2}, \end{aligned} \tag{4.9}$$

we obtain from (4.8) after an appropriate change of constants $c(n,k_0)$ and $c(n)$

$$\begin{aligned} \frac{\partial}{\partial t}|\widetilde{\nabla} g|^2 \leq g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla} g|^2 - \frac{1}{2}|\widetilde{\nabla}^2 g|^2 + 3200n^{10}|\widetilde{\nabla} g|^4 \\ + c(n,k_0)|\widetilde{\nabla} g|^2 + c(n)c_1^2. \end{aligned} \tag{4.10}$$

As in [Shi89, Proof of Lemma 4.1, p.248], we fix a small constant $\varepsilon \equiv \varepsilon(n) := (256000n^{10})^{-1}$, such that the inequality (4.1) now reads as

$$1 - \varepsilon(n) \leq \lambda_k(x,t) \leq 1 + \varepsilon(n), \tag{4.11}$$

for any $k = 1, 2, \dots, n$, where $\lambda_k(x,t)$ refers to the eigenvalues of $g(x,t)$ with respect to $\widetilde{g}(x)$. Sometimes we use a rougher estimate $\frac{1}{2} \leq \lambda_k(x,t) \leq 2$ instead. We also set

$$m := 25600n^{10}, \qquad a := 6400n^{10} \tag{4.12}$$

and define (we simplify notation by writing $\lambda_k \equiv \lambda_k(x,t)$)

$$\varphi(x,t) := a + \sum_{k=1}^{n} \lambda_k^m, \qquad (x,t) \in B(x_0, \gamma+\delta) \times [0,T]. \tag{4.13}$$

Following [Shi89, Proof of Lemma 4.1, p.248 (16)] we obtain

$$\begin{aligned} \frac{\partial \varphi}{\partial t} =& m\lambda_k^{m-1} g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g_{kk} \\ & + m\lambda_k^{m-1} * (\widetilde{\mathrm{Rm}} * g^{-1} * g + g^{-2} * \widetilde{\nabla} g * \widetilde{\nabla} g). \end{aligned} \tag{4.14}$$

We now proceed as in Lemma 2.5 along the following steps.

Step 1: Derive an evolution inequality for $\psi := \varphi \cdot |\widetilde{\nabla} g|^2$ as in (2.6).

Step 2: Estimate $\widetilde{\nabla}\widetilde{\nabla}\xi$ from below as in (2.11).

Step 3: Estimate $\xi\psi$ from above as in (2.13) and conclude the proof.

STEP 1: DERIVE AN EVOLUTION INEQUALITY FOR $\psi := \varphi \cdot |\widetilde{\nabla} g|^2$ AS IN (2.6).

We estimate the individual terms on the right hand side of (4.14)

$$\begin{aligned} m\lambda_k^{m-1} * \widetilde{\mathrm{Rm}} * g^{-1} * g &\leq c(n, k_0), \\ m\lambda_k^{m-1} * g^{-2} * \widetilde{\nabla} g * \widetilde{\nabla} g &\leq 10n^3 m(1+\varepsilon)^{m-1} |\widetilde{\nabla} g|^2. \end{aligned} \tag{4.15}$$

As in [Shi89, Proof of Lemma 4.1, p.248] we have

$$\begin{aligned} g^{\alpha\beta}\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \varphi = {} & m\lambda_k^{m-1} g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta g_{kk} \\ & + m(\lambda_i^{m-2} + \lambda_i^{m-3}\lambda_j + \cdots + \lambda_j^{m-2}) \cdot g^{\alpha\beta} \widetilde{\nabla}_\alpha g \cdot \widetilde{\nabla}_\beta g \\ \geq {} & m\lambda_k^{m-1} g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta g_{kk} + \frac{m(m-1)}{2}(1-\varepsilon)^{m-2} |\widetilde{\nabla} g|^2. \end{aligned} \tag{4.16}$$

This yields

$$\begin{aligned} \frac{\partial \varphi}{\partial t} \leq {} & g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \varphi - \frac{m(m-1)}{2}(1-\varepsilon)^{m-2} |\widetilde{\nabla} g|^2 \\ & + c(n, k_0) + 10n^3 m(1+\varepsilon)^{m-1} |\widetilde{\nabla} g|^2. \end{aligned} \tag{4.17}$$

As in [Shi89, p.249 (20),(21),(22)], we easily check

$$\begin{aligned} & 10n^3 m(1+\varepsilon)^{m-1} \leq \frac{m^2}{16}, \\ & \frac{m(m-1)}{2}(1-\varepsilon)^{m-2} \geq \frac{m^2}{4}(1-\varepsilon)^{m-2} \geq \frac{3}{16} m^2, \end{aligned} \tag{4.18}$$

such that (4.17) reduces to

$$\frac{\partial \varphi}{\partial t} \leq g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \varphi + c(n, k_0) - \frac{m^2}{8} |\widetilde{\nabla} g|^2. \tag{4.19}$$

From (4.10) and (4.19) it follows that

$$\begin{aligned} \frac{\partial}{\partial t}(\varphi \cdot |\widetilde{\nabla} g|^2) \leq {} & g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta (\varphi \cdot |\widetilde{\nabla} g|^2) - 2g^{\alpha\beta} \widetilde{\nabla}_\alpha \varphi \widetilde{\nabla}_\beta |\widetilde{\nabla} g|^2 - \frac{\varphi}{2} |\widetilde{\nabla}^2 g|^2 \\ & + 3200 n^{10} \varphi |\widetilde{\nabla} g|^4 + c(n, k_0) \varphi |\widetilde{\nabla} g|^2 + c(n) c_1^2 \varphi \\ & + c(n, k_0) |\widetilde{\nabla} g|^2 - \frac{m^2}{8} |\widetilde{\nabla} g|^4. \end{aligned} \tag{4.20}$$

We estimate some of the terms on the right hand side of (4.20). As in [Shi89, Proof of Lemma 4.1, p.249 (26), p.250 (28)] we find for the fourth term on the right hand side of (4.20)

$$3200 n^{10} \varphi |\widetilde{\nabla} g|^4 \leq 3200 n^{10} (a + n(1+\varepsilon)^m) |\widetilde{\nabla} g|^4 \leq \frac{m^2}{16} |\widetilde{\nabla} g|^4. \tag{4.21}$$

The second term on the right hand side of (4.20) is estimated as follows.

$$
\begin{aligned}
-2g^{\alpha\beta}\widetilde{\nabla}_\alpha\varphi\widetilde{\nabla}_\beta|\widetilde{\nabla}g|^2 &= -2g^{\alpha\beta}\widetilde{\nabla}_\alpha\left(\sum_{k=1}^n \lambda_k^m\right)\cdot\widetilde{\nabla}_\beta|\widetilde{\nabla}g|^2 \\
&= -4g^{\alpha\beta}\cdot\left(m\,\lambda_k^{m-1}\cdot\widetilde{\nabla}_\alpha\lambda_k\right)\cdot\widetilde{\nabla}_\beta|\widetilde{\nabla}g|^2 \\
&\leq 8mn^5(1+\varepsilon)^{m-1}|\widetilde{\nabla}g|^2|\widetilde{\nabla}^2 g| \\
&\leq \sqrt{\phi}\,|\widetilde{\nabla}^2 g|\cdot\left(\frac{16mn^5|\widetilde{\nabla}g|^2}{\sqrt{\phi}}\right) \\
&\leq \frac{\varphi}{2}|\widetilde{\nabla}^2 g|^2 + \frac{128m^2n^{10}}{\varphi}|\widetilde{\nabla}g|^4.
\end{aligned}
\tag{4.22}
$$

Plugging these estimates back into (4.20) yields

$$
\begin{aligned}
\frac{\partial}{\partial t}(\varphi\cdot|\widetilde{\nabla}g|^2) \leq & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta(\varphi\cdot|\widetilde{\nabla}g|^2) + \frac{128m^2n^{10}}{\varphi}|\widetilde{\nabla}g|^4 - \frac{m^2}{16}|\widetilde{\nabla}g|^4 \\
&+ c(n,k_0)\varphi|\widetilde{\nabla}g|^2 + c(n)c_1^2.
\end{aligned}
\tag{4.23}
$$

Since $\varphi(x,t)\geq a$, with $a = 6400n^{10}$, we have

$$
\frac{128m^2n^{10}}{\varphi} \leq \frac{m^2}{32},
$$

such that (4.23) reduces to

$$
\begin{aligned}
\frac{\partial}{\partial t}(\varphi\cdot|\widetilde{\nabla}g|^2) \leq & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta(\varphi\cdot|\widetilde{\nabla}g|^2) - \frac{m^2}{32}|\widetilde{\nabla}g|^4 \\
&+ c(n,k_0)\varphi|\widetilde{\nabla}g|^2 + c(n)c_1^2.
\end{aligned}
\tag{4.24}
$$

Using (4.11) and the first estimate of (4.18) in the second inequality, we find

$$
\frac{m^2}{32}|\widetilde{\nabla}g|^4 \equiv \frac{m^2}{32\varphi}|\widetilde{\nabla}g|^4\varphi \geq \frac{m^2}{32(a+n(1+\varepsilon)^m)^2}|\widetilde{\nabla}g|^4\varphi^2 \geq \frac{1}{8}|\widetilde{\nabla}g|^4\varphi^2. \tag{4.25}
$$

Thus we obtain from (4.24), using the inequality $ab\leq\frac{1}{2}a^2+\frac{1}{2}b^2$ and adapting the constant $c(n,k_0)>0$ accordingly in the last estimate

$$
\begin{aligned}
\frac{\partial}{\partial t}(\varphi\cdot|\widetilde{\nabla}g|^2) &\leq g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta(\varphi\cdot|\widetilde{\nabla}g|^2) - \frac{1}{8}|\widetilde{\nabla}g|^4\varphi^2 \\
&\quad + c(n,k_0)\varphi|\widetilde{\nabla}g|^2 + c(n)c_1^2 \\
&\leq g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta(\varphi\cdot|\widetilde{\nabla}g|^2) - \frac{1}{16}|\widetilde{\nabla}g|^4\varphi^2 \\
&\quad + c(n,k_0) + c(n)c_1^2.
\end{aligned}
\tag{4.26}
$$

Defining $\psi(x,t) := (\varphi \cdot |\widetilde{\nabla} g|^2)(x,t)$ this inequality reads

$$\frac{\partial \psi}{\partial t} \le g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \psi - \frac{1}{16}\psi^2 + c(n,k_0) + c(n)c_1^2. \tag{4.27}$$

Step 2: Estimate $\widetilde{\nabla}\widetilde{\nabla}\xi$ from below as in (2.11).

Next, as in [Shi89, Proof of Lemma 4.1, p.251 (36),(37)] we take a cutoff function $\eta \in C^\infty(\mathbb{R})$ as in (2.7), illustrated in Figure 1. Then we define the cutoff function $\xi \in C(M)$

$$\xi(x) = \eta\left(\frac{d_{\widetilde{g}}(x,x_0) - (\gamma + \delta/2)}{\delta/4}\right), \tag{4.28}$$

where $d_{\widetilde{g}}$ is the distance function with respect to the metric $\widetilde{g}$. Note that $d_{\widetilde{g}}(\cdot, x_0)$ is Lipschitz continuous but need not be smooth everywhere, and hence ξ need not be smooth everywhere. From the properties of η we have

$$\begin{aligned} &\xi(x) = 1, \qquad x \in B(x_0, \gamma + \delta/2), \\ &\xi(x) = 0, \qquad x \in M \backslash B(x_0, \gamma + 3\delta/4), \\ &0 \le \xi(x) \le 1, \qquad x \in M. \end{aligned} \tag{4.29}$$

If $d_{\widetilde{g}}(\cdot, x_0)$ is smooth in a neighborhood of a point x, then we also have

$$\widetilde{\nabla}_\beta \xi(x) = \frac{4}{\delta}\eta'\left(\frac{d_{\widetilde{g}}(x,x_0) - (\gamma + \delta/2)}{\delta/4}\right)\widetilde{\nabla}_\beta d_{\widetilde{g}}(x,x_0) \tag{4.30}$$

$$\begin{aligned} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \xi(x) =& \frac{4}{\delta}\eta'\left(\frac{d_{\widetilde{g}}(x,x_0) - (\gamma + \delta/2)}{\delta/4}\right)\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta d_{\widetilde{g}}(x,x_0) \\ &+ \frac{16}{\delta^2}\eta''\left(\frac{d_{\widetilde{g}}(x,x_0) - (\gamma + \delta/2)}{\delta/4}\right)\widetilde{\nabla}_\alpha d_{\widetilde{g}}(x,x_0)\widetilde{\nabla}_\beta d_{\widetilde{g}}(x,x_0). \end{aligned} \tag{4.31}$$

Since $|\widetilde{\nabla} d_{\widetilde{g}}(x,x_0)| = 1$, it follows using $|\eta'|^2 \le 16\eta$ that

$$|\widetilde{\nabla}\xi(x)|^2 \le \frac{16}{\delta^2}(\eta')^2\left(\frac{d_{\widetilde{g}}(x,x_0) - (\gamma + \delta/2)}{\delta/4}\right) \le \frac{256}{\delta^2}\xi(x). \tag{4.32}$$

Furthermore, note that

$$\widetilde{\nabla}_\alpha d_{\widetilde{g}}(x,x_0)\widetilde{\nabla}_\beta d_{\widetilde{g}}(x,x_0) \le \widetilde{g}_{\alpha\beta}(x), \tag{4.33}$$

such that, using $|\eta''| \le 8$, we can estimate from below

$$\frac{16}{\delta^2}\eta''\left(\frac{d_{\widetilde{g}}(x,x_0) - (\gamma + \delta/2)}{\delta/4}\right)\widetilde{\nabla}_\alpha d_{\widetilde{g}}(x,x_0)\widetilde{\nabla}_\beta d_{\widetilde{g}}(x,x_0) \ge -\frac{128}{\delta^2}\widetilde{g}_{\alpha\beta}(x). \tag{4.34}$$

By assumption, $|\widetilde{\mathrm{Rm}}|^2 \leq k_0$ and thus the sectional curvature is in particular bounded from below $\sec \geq -\sqrt{k_0}$. From the Hessian comparison theorem Theorem 3.4, applied in a relatively compact ball, we conclude

$$\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta d_{\widetilde{g}}(x, x_0) \leq \sqrt[4]{k_0} \coth\left(\sqrt[4]{k_0} d_{\widetilde{g}}(x, x_0)\right) \widetilde{g}_{\alpha\beta}(x). \tag{4.35}$$

Using $0 \geq \eta'(s) \geq -4\,\eta^{1/2}(s) \geq -4$ for all $s \in \mathbb{R}$, it follows that

$$\begin{aligned} &\frac{4}{\delta}\eta'\left(\frac{d_{\widetilde{g}}(x, x_0) - (\gamma + \delta/2)}{\delta/4}\right) \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta d_{\widetilde{g}}(x, x_0) \\ &\geq -\frac{16}{\delta} \sqrt[4]{k_0} \coth\left(\sqrt[4]{k_0} d_{\widetilde{g}}(x, x_0)\right) \widetilde{g}_{\alpha\beta}(x). \end{aligned} \tag{4.36}$$

We now obtain from (4.31), combined with (4.34) and (4.36)

$$\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \xi(x) \geq -\left(\frac{128}{\delta^2} + \frac{16}{\delta} \sqrt[4]{k_0} \coth\left(\sqrt[4]{k_0} d_{\widetilde{g}}(x, x_0)\right)\right) \widetilde{g}_{\alpha\beta}(x). \tag{4.37}$$

Step 3: Estimate $\xi\psi$ from above as in (2.13) and conclude the proof.

Next we simplify notation by writing as in the proof of Lemma 2.5

$$F(x,t) := \xi(x)\psi(x,t), \qquad (x,t) \in B(x_0, \gamma + \delta) \times [0, T].$$

Since $|\widetilde{\nabla} g|^2(x, 0) = 0$, we have

$$F(x, 0) = 0, \qquad x \in B(x_0, \gamma + \delta). \tag{4.38}$$

Since $\xi(x) = 0$ for $x \in B(x_0, \gamma + \delta) \backslash B(x_0, \gamma + \frac{3}{4}\delta)$, it follows that

$$F(x,t) = 0, \qquad (x,t) \in B(x_0, \gamma + \delta) \backslash B(x_0, \gamma + \frac{3}{4}\delta) \times [0, T]. \tag{4.39}$$

Thus there exists a point $(y_0, t_0) \in B(x_0, \gamma + \frac{3}{4}\delta) \times [0, T]$ with $t_0 > 0$ such that

$$F(y_0, t_0) = \max\ \{F(x,t) \mid (x,t) \in B(x_0, \gamma + \delta) \times [0, T]\} \tag{4.40}$$

unless $F \equiv 0$ on $B(x_0, \gamma + \delta) \times [0, T]$.

In the following, as already alluded to in the proof of Lemma 2.5, we distinguish three cases, first case where $\xi \equiv 1$ in a neighborhood of y_0, second case where ξ is not identically 1, but smooth in a neighborhood of y_0, and third case, where ξ is not smooth and a trick needs to be applied.

Case 1. $y_0 \in B(x_0, \gamma + \delta/2)$

Then $\xi \equiv 1$ near y_0, such that $F = \psi$ near (y_0, t_0), and we have by (4.27)

$$0 \leq (\frac{\partial}{\partial t} - g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta)\psi(y_0, t_0) \leq -\frac{1}{16}\psi^2(y_0, t_0) + c(n, k_0) + c(n)c_1^2, \tag{4.41}$$

and thus we conclude

$$\frac{1}{16}F^2(y_0,t_0) = \frac{1}{16}\psi^2(y_0,t_0) \le c(n,k_0) + c(n)c_1^2. \tag{4.42}$$

This estimate is better than the one we will obtain in Case 2.

Case 2. $y_0 \notin B(x_0, \gamma+\delta/2)$ and y_0 is not in the cut locus of x_0

Then the distance function $d_{\widetilde{g}}(\cdot, x_0)$, and hence also ξ, is smooth in a neighborhood of y_0 and it follows that

$$\begin{aligned}
&0 \le \frac{\partial F}{\partial t}(y_0,t_0) = \xi(y_0)\frac{\partial \psi}{\partial t}(y_0,t_0),\\
&0 = \widetilde{\nabla}_\alpha F(y_0,t_0) = (\xi\widetilde{\nabla}_\alpha\psi + \psi\widetilde{\nabla}_\alpha\xi)(y_0,t_0),\\
&0 \ge g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta F(y_0,t_0) = (\xi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\psi + \psi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi\\
&\quad + 2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\psi)(y_0,t_0).
\end{aligned} \tag{4.43}$$

Using (4.27) in the final step, we obtain at the point (y_0,t_0)

$$\begin{aligned}
0 &\le \left(\frac{\partial F}{\partial t} - g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta F\right)(y_0,t_0)\\
&\le \xi(y_0)\left(\frac{\partial \psi}{\partial t} - g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta \psi\right)(y_0,t_0)\\
&- \left(\psi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi + 2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\psi\right)(y_0,t_0)\\
&\le \left(-\frac{1}{16}\xi\psi^2 - \psi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi - 2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\psi\right)(y_0,t_0)\\
&+ \xi(y_0)\Big(c(n,k_0) + c(n)c_1^2\Big).
\end{aligned}$$

Thus we conclude at the point (y_0,t_0)

$$\frac{1}{16}\xi\psi^2 \le -\psi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi - 2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\psi + \xi(c(n,k_0)+c(n)c_1^2). \tag{4.44}$$

From the second identity in (4.43) in the first step, and using (4.32) in the second estimate, we obtain at (y_0,t_0)

$$-2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\psi = \frac{2\psi}{\xi}g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\xi \le \frac{1024}{\delta^2}\psi, \tag{4.45}$$

Furthermore, we obtain from (4.37) at (y_0,t_0)

$$-\psi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi \le 2n\left(\frac{128}{\delta^2} + \frac{16}{\delta}\sqrt[4]{k_0}\coth\left(\sqrt[4]{k_0}\gamma(y_0,x_0)\right)\right)\psi. \tag{4.46}$$

We estimate the coth-term: Since $y_0 \notin B(x_0, \gamma+\delta/2)$ and coth is monotonically decreasing on the positive real axis, $\coth(\sqrt[4]{k_0}\gamma(y_0, x_0)) \le \coth(\sqrt[4]{k_0}\delta/2)$. Also, since $z \coth z \le 1 + Cz$ for $z > 0$ and since $\delta \le 1$

$$\sqrt[4]{k_0} \coth(\sqrt[4]{k_0}\delta/2) = \frac{2}{\delta} \sqrt[4]{k_0}\frac{\delta}{2} \coth(\sqrt[4]{k_0}\delta/2) \le \frac{2}{\delta}(1 + C\sqrt[4]{k_0}\frac{\delta}{2}) \le \frac{c(n, k_0)}{\delta}. \tag{4.47}$$

Thus we obtain at (y_0, t_0)

$$-\psi g^{\alpha\beta}\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \xi \le \frac{c(n, k_0)}{\delta^2}\psi. \tag{4.48}$$

Plugging (4.45) and (4.48) into (4.44), leads to

$$\frac{1}{16}\xi\psi^2 \le \frac{c(n, k_0)}{\delta^2}\psi + \xi(c(n, k_0) + c(n)c_1^2). \tag{4.49}$$

Multiplying this inequality with ξ and using $0 \le \xi \le 1$ we obtain

$$F(y_0, t_0)^2 \le \frac{c(n, k_0)}{\delta^2}F(y_0, t_0) + c(n, k_0) + c(n)c_1^2. \tag{4.50}$$

Thus

$$\begin{aligned} F(y_0, t_0) &\le \frac{c(n, k_0)}{\delta^2} + c(n, k_0) + c(n)c_1^2 \\ &\le \frac{c(n, k_0)}{\delta^2} + c(n)c_1^2, \end{aligned} \tag{4.51}$$

assuming $c(n, k_0) \ge 1$ if necessary. Thus

$$F(x, t) \le F(y_0, t_0) \le \frac{c(n, k_0)}{\delta^2} + c(n)c_1^2 \tag{4.52}$$

for all $(x, t) \in B(x_0, \gamma + \delta) \times [0, T]$. Since

$$\begin{aligned} &F(x, t) = \xi(x)\varphi(x, t)|\widetilde{\nabla} g|^2(x, t), \\ &\xi(x) = 1 \text{ for } x \in B(x_0, \gamma + \delta/2), \\ &\varphi(x, t) \ge a = 6400n^{10} \text{ for } (x, t) \in B(x_0, \gamma + \delta) \times [0, T], \end{aligned}$$

we obtain

$$\begin{aligned} |\widetilde{\nabla} g|^2(x, t) &\le \frac{1}{6400n^{10}} \left(\frac{c(n, k_0)}{\delta^2} + c(n)c_1^2 \right) \\ &= \frac{c(n, k_0)}{\delta^2} + c(n)c_1^2 \end{aligned} \tag{4.53}$$

for all $(x, t) \in B(x_0, \gamma + \delta/2) \times [0, T]$. Hence

$$|\widetilde{\nabla} g|(x, t) \le \frac{c(n, k_0)}{\delta} + c(n)c_1 \tag{4.54}$$

for all $(x,t) \in B(x_0, \gamma + \delta/2) \times [0,T]$.

Case 3. $y_0 \notin B(x_0, \gamma + \delta/2)$ and y_0 is in the cut locus of x_0

Then we apply Calabi's trick (see e. g. [CLN06, p.395]). Let $c : [0, d_{\widetilde{g}}(x_0, y_0)] \to M$ be a minimal geodesic from x_0 to y_0. Note that since $y_0 \in B(x_0, \gamma + \frac{3}{4}\delta)$ the assumption $B(x_0, \gamma + \delta - r) \subset\subset M$ for all $r > 0$ ensures that such a minimal geodesic exists. Fix $\varepsilon > 0$ sufficiently small and define

$$\xi_\varepsilon(x) := \eta\left(\frac{d_{\widetilde{g}}(x, c(\varepsilon)) + \varepsilon - (\gamma + \delta/2)}{\delta/4}\right), \quad F_\varepsilon(x,t) := \xi_\varepsilon(x)\psi(x,t).$$

Since $d_{\widetilde{g}}(x, x_0) \leq d_{\widetilde{g}}(x, c(\varepsilon)) + \varepsilon$ by the triangle inequality and since η is monotonically decreasing, we have

$$\xi_\varepsilon(x) \leq \xi(x)$$

for all $x \in M$. As $d_{\widetilde{g}}(y_0, x_0) = d_{\widetilde{g}}(y_0, c(\varepsilon)) + \varepsilon$, we have $\xi_\varepsilon(y_0) = \xi(y_0)$. Hence

$$\begin{aligned} &F_\varepsilon(x,t) \leq F(x,t) \qquad \forall x \in B(x_0, \gamma + \delta) \times [0,T], \\ &F_\varepsilon(y_0, t_0) = F(y_0, t_0), \end{aligned} \tag{4.55}$$

such that F_ε has a maximum at (y_0, t_0) as well.

The point now is that $d_{\widetilde{g}}(\cdot, c(\varepsilon))$ is smooth in a neighborhood of y_0. This can be seen as follows. First, $c(\varepsilon)$ is not conjugate to y_0 along c (more precisely running backwards from y_0 to $c(\varepsilon)$ along c), in the sense that there exists no non-trivial Jacobi field vanishing at y_0 and $c(\varepsilon)$. Hence y_0 is not conjugate to $c(\varepsilon)$ along c. Letting $v := (d_{\widetilde{g}}(x_0, y_0) - \varepsilon)\dot{c}(\varepsilon)$, such that $\exp_{c(\varepsilon)} v = y_0$, we thus have that $D\exp_{c(\varepsilon)}$ is non-singular at v.

We claim that $v \in \mathrm{seg}^0(c(\varepsilon))$. If $v \in \mathrm{seg}(c(\varepsilon)) \backslash \mathrm{seg}^0(c(\varepsilon))$, by Lemma 3.1 there exists $w\,(\neq v) \in \mathrm{seg}(c(\varepsilon))$ with $\exp_{c(\varepsilon)}(w) = \exp_{c(\varepsilon)}(v)$. Note that Lemma 3.1 can be applied since $|v| = d_{\widetilde{g}}(c(\varepsilon), y_0) < \gamma + \frac{3}{4}\delta$ and $B(c(\varepsilon), \gamma + \frac{3}{4}\delta) \subset B(x_0, \gamma + \frac{7}{8}\delta) \subset\subset M$. Hence $t \to \exp_{c(\varepsilon)}(tw)$ is another minimizing geodesic from $c(\varepsilon)$ to y_0. Following c from x_0 to $c(\varepsilon)$ and then this curve from $c(\varepsilon)$ to y_0 then gives a non-smooth minimizing curve from x_0 to y_0, which is a contradiction. Hence $v \in \mathrm{seg}^0(c(\varepsilon))$, such that $y_0 = \exp_{c(\varepsilon)}(v)$ is not in the cut locus of $c(\varepsilon)$.

Thus F_ε is smooth in a neighborhood of (y_0, t_0) and we can apply the same steps as in Case 2 to F_ε. Letting $\varepsilon \to 0$, we obtain (4.51), i.e.

$$F(y_0, t_0) \leq \frac{c(n, k_0)}{\delta^2} + c(n)c_1^2,$$

and we can finish the proof as in Case 2. □

5 A priori estimates of $\nabla^2 g$ along the flow

In this section we utilize the arguments in the proof of Lemma 4.1 to obtain a priori estimates of the second derivatives of g and the Riemann curvature tensor along the Ricci DeTurck flow.

Lemma 5.1. *Under the same assumptions as in Lemma 4.1, there exists a constant $c(n, k_0) > 0$ depending only on n and k_0, such that*

$$|\widetilde{\nabla}^2 g|(x,t) \leq c(n,k_0) \left(\frac{1}{\delta^2} + c_1^2 + \frac{c_2^{1/3}}{\delta^{2/3}} + c_2^{1/3} c_1^{2/3} \right) \tag{5.1}$$

for all $(x,t) \in B(x_0, \gamma + \delta/3) \times [0,T]$, where

$$c_1 = \sup_{x \in B(x_0, \gamma + 3\delta/4)} |\widetilde{\nabla} \widetilde{\mathrm{Rm}}|(x), \qquad c_2 = \sup_{x \in B(x_0, \gamma + 3\delta/4)} |\widetilde{\nabla}^2 \widetilde{\mathrm{Rm}}|(x).$$

We will prove this result below and first note an immediate consequence: Assuming additionally that $|\widetilde{\nabla}\widetilde{\mathrm{Rm}}| = \mathcal{O}(\rho^{-1})$ and $|\widetilde{\nabla}^2\widetilde{\mathrm{Rm}}| = \mathcal{O}(\rho^{-2})$, with $\rho > 0$ being the distance to the singularity, we obtain $|\widetilde{\nabla}^2 g| = \mathcal{O}(\rho^{-2})$ and $|\mathrm{Rm}| = \mathcal{O}(\rho^{-2})$ uniformly in $t \in [0,T]$.

Corollary 5.2. *Let $(M, \widetilde{g})$ be a (possibly incomplete) manifold. Fix $0 < T < \infty$ and let $g(x,t)$ be a smooth solution of the initial value problem*

$$\frac{\partial}{\partial t} g_{ij}(x,t) = (-2\,\mathrm{Ric}_{ij} + \nabla_i V_j + \nabla_j V_i)(x,t), \quad (x,t) \in M \times [0,T],$$
$$g(x,0) = \widetilde{g}(x), \quad x \in M,$$

where $V^i = g^{jk}(\Gamma^i_{jk} - \widetilde{\Gamma}^i_{jk})$ is the DeTurck vector field. We assume that

$$(1 - \varepsilon(n))\widetilde{g}(x) \leq g(x,t) \leq (1 + \varepsilon(n))\widetilde{g}(x)$$

for $\varepsilon(n) > 0$ sufficiently small, only depending on n, and for all $(x,t) \in M \times [0,T]$. Also assume that

$$|\widetilde{Rm}|^2 \leq k_0$$

for some constant $k_0 > 0$, and that there exists a constant $C > 0$ such that for all $x \in M$, $0 < \rho \leq 1$

$$|\widetilde{\nabla}\widetilde{\mathrm{Rm}}|(x) \leq \frac{C}{\rho}, \quad |\widetilde{\nabla}^2\widetilde{\mathrm{Rm}}|(x) \leq \frac{C}{\rho^2}$$

whenever $B(x, \rho - r)$ is relatively compact for all $r > 0$. Then there exists a constant $C' > 0$ only depending on k_0, C and n such that for all $x \in M$, $t \in [0,T]$, $0 < \rho \leq 1$

$$|\widetilde{\nabla}^2 g|(x,t) \leq \frac{C'}{\rho^2}, \ |\mathrm{Rm}|(x,t) \leq \frac{C'}{\rho^2}$$

whenever $B(x, \rho - r)$ is relatively compact for all $r > 0$.

Proof of Corollary 5.2. Let $x_0 \in M$ and $\rho \leq 1$ such that $B(x_0, \rho - r) \subset M$ relatively compact for all $r > 0$. Then by Lemma 5.1 (choosing γ, δ equal to $\rho/2$)

$$|\widetilde{\nabla}^2 g|(x,t) \leq c(n,k_0)\left(\frac{1}{\rho^2} + c_1^2 + \frac{c_2^{1/3}}{\rho^{2/3}} + c_2^{1/3}c_1^{2/3}\right) \tag{5.2}$$

with the constants estimated by

$$c_1 = \sup_{x\in B(x_0,7\rho/8)} |\widetilde{\nabla}\widetilde{\mathrm{Rm}}|(x) \leq \frac{8C}{\rho}, \quad c_2 = \sup_{x\in B(x_0,7\rho/8)} |\widetilde{\nabla}^2\widetilde{\mathrm{Rm}}|(x) \leq \frac{8\widehat{C}}{\rho^2},$$

since for all $x \in B(x_0, 7\rho/8)$ we have that $B(x, \rho/8 - r) \subset M$ relatively compact for all $r > 0$. The estimate of the Riemannian curvature tensor follows from this, Corollary 4.3 and

$$\mathrm{Rm} = \widetilde{\mathrm{Rm}} * \tilde{g}^{-1} * g + \widetilde{\nabla}^2 g + g^{-1} * \widetilde{\nabla} g * \widetilde{\nabla} g,$$

see [Shi89, p. 276, formula (83)]. □

Proof of Lemma 5.1. In the following all estimates and inequalities are supposed to hold on $B(x_0, \gamma + \delta/2) \times [0, T]$, when nothing else is mentioned. Differentiating the equation for the metric g from [Shi89, Lemma 2.1] m times we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{\nabla}^m g =& g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\widetilde{\nabla}^m g \\ &+ \sum_{\substack{0\leq k_1,k_2,\dots,k_{m+2}\leq m+1\\ k_1+k_2+\cdots+k_{m+2}\leq m+2}} \widetilde{\nabla}^{k_1} g * \widetilde{\nabla}^{k_2} g * \cdots * \widetilde{\nabla}^{k_{m+2}} g * P_{k_1k_2\dots k_{m+2}} \\ &+ \sum_{\substack{0\leq l_1,l_2,\dots,l_m,s\leq m\\ l_1+l_2+\cdots+l_m+s=m}} \widetilde{\nabla}^s\widetilde{\mathrm{Rm}} * \widetilde{\nabla}^{l_1} g * \widetilde{\nabla}^l_2 g * \cdots * \widetilde{\nabla}^{l_m} g * Q_{l_1l_2\dots l_m s},\end{aligned} \tag{5.3}$$

where $P_{k_1k_2\dots k_{m+2}}$ and $Q_{l_1l_2\dots l_m s}$ are polynomials of g, g^{-1}. Hence

$$\begin{aligned}&\frac{\partial}{\partial t}|\widetilde{\nabla}^m g|^2 = \\ &g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla}^m g|^2 - 2g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}^m g\cdot\widetilde{\nabla}_\beta\widetilde{\nabla}^m g \\ &+ \sum_{\substack{0\leq k_1,k_2,\dots,k_{m+2}\leq m+1\\ k_1+k_2+\cdots+k_{m+2}\leq m+2}} \widetilde{\nabla}^{k_1} g * \widetilde{\nabla}^{k_2} g * \cdots * \widetilde{\nabla}^{k_{m+2}} g * \widetilde{\nabla}^m g * P_{k_1k_2\dots k_{m+2}} \\ &+ \sum_{\substack{0\leq l_1,l_2,\dots,l_m,s\leq m\\ l_1+l_2+\cdots+l_m+s=m}} \widetilde{\nabla}^s\widetilde{\mathrm{Rm}} * \widetilde{\nabla}^{l_1} g * \widetilde{\nabla}^l_2 g * \cdots * \widetilde{\nabla}^{l_m} g * \widetilde{\nabla}^m g * Q_{l_1l_2\dots l_m s},\end{aligned} \tag{5.4}$$

For $m=2$ this gives, together with $2g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}^2 g\cdot\widetilde{\nabla}_\beta\widetilde{\nabla}^2 g \geq |\widetilde{\nabla}^3 g|^2$,

$$\begin{aligned}\frac{\partial}{\partial t}|\widetilde{\nabla}^2 g|^2 \leq & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla}^2 g|^2 - |\widetilde{\nabla}^3 g|^2 \\ & + c(n)|\widetilde{\nabla}^2 g|(|\widetilde{\nabla}^3 g||\widetilde{\nabla} g| + |\widetilde{\nabla}^2 g|^2 + |\widetilde{\nabla}^2 g||\widetilde{\nabla} g|^2 + |\widetilde{\nabla} g|^4) \\ & + c(n)|\widetilde{\nabla}^2 g|(|\widetilde{\nabla}^2\widetilde{\mathrm{Rm}}| + |\widetilde{\nabla}\widetilde{\mathrm{Rm}}||\widetilde{\nabla} g| + |\widetilde{\mathrm{Rm}}||\widetilde{\nabla}^2 g| + |\widetilde{\mathrm{Rm}}||\widetilde{\nabla} g|^2).\end{aligned} \tag{5.5}$$

It follows that

$$\begin{aligned}\frac{\partial}{\partial t}|\widetilde{\nabla}^2 g|^2 \leq & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla}^2 g|^2 - \frac{1}{2}|\widetilde{\nabla}^3 g|^2 \\ & + c(n)(|\widetilde{\nabla}^2 g|^2|\widetilde{\nabla} g|^2 + |\widetilde{\nabla}^2 g|^3 + |\widetilde{\nabla}^2 g||\widetilde{\nabla} g|^4 + c_2|\widetilde{\nabla}^2 g| \\ & + c_1|\widetilde{\nabla}^2 g||\widetilde{\nabla} g| + \sqrt{k_0}|\widetilde{\nabla}^2 g|^2 + \sqrt{k_0}|\widetilde{\nabla}^2 g||\widetilde{\nabla} g|^2),\end{aligned} \tag{5.6}$$

on $B(x_0,\gamma+3\delta/4)\times[0,T]$, where

$$c_1 = \sup_{x\in B(x_0,\gamma+3\delta/4)}|\widetilde{\nabla}\widetilde{\mathrm{Rm}}|(x), \qquad c_2 = \sup_{x\in B(x_0,\gamma+3\delta/4)}|\widetilde{\nabla}^2\widetilde{\mathrm{Rm}}|(x),$$

and where we used $|\widetilde{\nabla}^2 g||\widetilde{\nabla}^3 g||\widetilde{\nabla} g| \leq \frac{1}{2}|\widetilde{\nabla}^3 g|^2 + \frac{1}{2}|\widetilde{\nabla}^2 g|^2|\widetilde{\nabla} g|^2$. From (4.8) and (4.9) we have

$$\begin{aligned}\frac{\partial}{\partial t}|\widetilde{\nabla} g|^2 \leq & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla} g|^2 - \frac{1}{2}|\widetilde{\nabla}^2 g|^2 + c(n,k_0)|\widetilde{\nabla} g|^2 + c(n)c_1|\widetilde{\nabla} g| \\ & + 3200n^{10}|\widetilde{\nabla} g|^4.\end{aligned} \tag{5.7}$$

Now as in [Shi89, Proof of Lemma 4.2, p.256 (80)] let

$$\psi(x,t) = (a + |\widetilde{\nabla} g|^2)|\widetilde{\nabla}^2 g|^2, \tag{5.8}$$

where $a>0$ is a constant which is chosen later. Then

$$\begin{aligned}\left(\frac{\partial}{\partial t} - g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\right)\psi = & \left(\frac{\partial}{\partial t} - g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\right)(a + |\widetilde{\nabla} g|^2)\cdot|\widetilde{\nabla}^2 g|^2 \\ & + (a + |\widetilde{\nabla} g|^2)\left(\frac{\partial}{\partial t} - g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\right)|\widetilde{\nabla}^2 g|^2 \\ & - 2g^{\alpha\beta}\widetilde{\nabla}_\alpha|\widetilde{\nabla} g|^2\widetilde{\nabla}_\beta|\widetilde{\nabla}^2 g|^2.\end{aligned} \tag{5.9}$$

We proceed as before in Lemmas 2.5 and 4.1 along the following steps.

Step 1: Derive an evolution inequality for ψ.

Step 2: Estimate $\widetilde{\nabla}\widetilde{\nabla}\xi$ from below.

Step 3: Estimate $\xi\psi$ from above and conclude the proof.

Step 1: Derive an evolution inequality for ψ.

Together with (5.6) and (5.7) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\psi \leq & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\psi - \frac{1}{2}|\widetilde{\nabla}^2 g|^4 + c(n,k_0)|\widetilde{\nabla} g|^2|\widetilde{\nabla}^2 g|^2 + c(n)c_1|\widetilde{\nabla} g||\widetilde{\nabla}^2 g|^2 \\ & + c(n)|\widetilde{\nabla} g|^4|\widetilde{\nabla}^2 g|^2 - \frac{1}{2}(a+|\widetilde{\nabla} g|^2)|\widetilde{\nabla}^3 g|^2 \\ & + c(n,k_0)(a+|\widetilde{\nabla} g|^2)(|\widetilde{\nabla}^2 g|^2|\widetilde{\nabla} g|^2 + |\widetilde{\nabla}^2 g|^3 + |\widetilde{\nabla}^2 g||\widetilde{\nabla} g|^4 + c_2|\widetilde{\nabla}^2 g| \\ & + c_1|\widetilde{\nabla}^2 g||\widetilde{\nabla} g| + |\widetilde{\nabla}^2 g|^2 + |\widetilde{\nabla}^2 g||\widetilde{\nabla} g|^2) \\ & - 2g^{\alpha\beta}\widetilde{\nabla}_\alpha|\widetilde{\nabla} g|^2\widetilde{\nabla}_\beta|\widetilde{\nabla}^2 g|^2.\end{aligned} \tag{5.10}$$

We estimate the last term as

$$\begin{aligned}-2g^{\alpha\beta}\widetilde{\nabla}_\alpha|\widetilde{\nabla} g|^2\widetilde{\nabla}_\beta|\widetilde{\nabla}^2 g|^2 &\leq 16|\widetilde{\nabla} g||\widetilde{\nabla}^2 g|^2|\widetilde{\nabla}^3 g| \\ &\leq 16C_1|\widetilde{\nabla}^2 g|^2|\widetilde{\nabla}^3 g| \\ &\leq \frac{1}{2}a|\widetilde{\nabla}^3 g|^2 + \frac{1}{2a}\cdot 256C_1^2|\widetilde{\nabla}^2 g|^4 \\ &= \frac{1}{2}a|\widetilde{\nabla}^3 g|^2 + \frac{1}{4}|\widetilde{\nabla}^2 g|^4,\end{aligned} \tag{5.11}$$

where $C_1 := \frac{c(n,k_0)}{\delta} + c(n)c_1$ is the bound on $|\widetilde{\nabla} g|$ from (4.2) and we chose $a = 512C_1^2$. This gives

$$\begin{aligned}\frac{\partial}{\partial t}\psi \leq & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\psi - \frac{1}{4}|\widetilde{\nabla}^2 g|^4 + c(n,k_0)C_1^2|\widetilde{\nabla}^2 g|^2 + c(n)c_1C_1|\widetilde{\nabla}^2 g|^2 \\ & + c(n)C_1^4|\widetilde{\nabla}^2 g|^2 + c(n,k_0)C_1^2(C_1^2|\widetilde{\nabla}^2 g|^2 + |\widetilde{\nabla}^2 g|^3 + C_1^4|\widetilde{\nabla}^2 g| \\ & + c_2|\widetilde{\nabla}^2 g| + c_1C_1|\widetilde{\nabla}^2 g| + |\widetilde{\nabla}^2 g|^2 + C_1^2|\widetilde{\nabla}^2 g|).\end{aligned} \tag{5.12}$$

Now by definition of ψ we have

$$|\widetilde{\nabla}^2 g|^2 = \frac{\psi}{a+|\widetilde{\nabla} g|^2} \leq \frac{\psi}{a} = \frac{\psi}{512C_1^2} \tag{5.13}$$

and

$$|\widetilde{\nabla}^2 g|^2 = \frac{\psi}{a+|\widetilde{\nabla} g|^2} \geq \frac{\psi}{a+C_1^2} = \frac{\psi}{513C_1^2} \tag{5.14}$$

This yields

$$
\begin{aligned}
&\frac{\partial}{\partial t}\psi \leq \\
&g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\psi - \frac{1}{4}\frac{\psi^2}{513^2C_1^4} + c(n,k_0)(\psi + \frac{c_1}{C_1}\psi + C_1^2\psi) \\
&+ c(n,k_0)(C_1^2\psi + \frac{\psi^{3/2}}{C_1} + C_1^5\psi^{1/2} + c_2C_1\psi^{1/2} + c_1C_1^2\psi^{1/2} + \psi + C_1^3\psi^{1/2}) \\
&= g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\psi - \frac{1}{4}\frac{\psi^2}{513^2C_1^4} \\
&+ c(n,k_0)\left(\frac{\psi^{3/2}}{C_1} + (C_1^2 + \frac{c_1}{C_1} + 1)\psi + (C_1^5 + C_1^3 + c_1C_1^2 + c_2C_1)\psi^{1/2}\right) \\
&\leq g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\psi - \frac{1}{4}\frac{\psi^2}{513^2C_1^4} + c(n,k_0)\left(\frac{\psi^{3/2}}{C_1} + C_1^2\psi + (C_1^5 + c_2C_1)\psi^{1/2}\right),
\end{aligned}
\tag{5.15}
$$

where in the last step we used that $C_1 = \frac{c(n,k_0)}{\delta} + c(n)c_1 \geq c_1$ and that, assuming $c(n,k_0) \geq 1$, $C_1 \geq 1$.

Step 2: Estimate $\widetilde{\nabla}\widetilde{\nabla}\xi$ from below.

Now let $\eta \in C^\infty(\mathbb{R})$ be the cutoff function as before and define the cutoff function $\xi \in C(M)$ as

$$
\xi(x) = \eta\left(\frac{d_{\widetilde{g}}(x,x_0) - (\gamma + \delta/3)}{\delta/12}\right), \tag{5.16}
$$

where $d_{\widetilde{g}}$ denotes the distance function with respect to the metric $\widetilde{g}$. Then we have

$$
\begin{aligned}
&\xi(x) = 1, \qquad x \in B(x_0, \gamma + \delta/3), \\
&\xi(x) = 0, \qquad x \in M\backslash B(x_0, \gamma + 5\delta/12), \\
&0 \leq \xi(x) \leq 1, \qquad x \in M.
\end{aligned}
\tag{5.17}
$$

If $d_{\widetilde{g}}(\cdot, x_0)$ is smooth in a neighborhood of a point x, we have by a calculation analogous to (4.32)

$$
|\widetilde{\nabla}\xi(x)|^2 \leq \frac{2304}{\delta^2}\xi(x) \tag{5.18}
$$

and

$$
\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi(x) \geq -\left(\frac{1152}{\delta^2} + \frac{48}{\delta}\sqrt[4]{k_0}\coth\left(\sqrt[4]{k_0}\,d_{\widetilde{g}}(x,x_0)\right)\right)\widetilde{g}_{\alpha\beta}(x). \tag{5.19}
$$

Step 3: Estimate $\xi\psi$ from above and conclude the proof.

Let

$$F(x,t) := \xi(x)\psi(x,t), \qquad (x,t) \in B(x_0, \gamma+\delta) \times [0,T].$$

Since $|\widetilde{\nabla}^2 g|^2(x,0) = 0$, we have

$$F(x,0) = 0, \qquad x \in B(x_0, \gamma+\delta). \tag{5.20}$$

Since $\xi(x) = 0$ for $x \in B(x_0, \gamma+\delta)\backslash B(x_0, \gamma + \frac{5}{12}\delta)$, it follows that

$$F(x,t) = 0, \qquad (x,t) \in B(x_0,\gamma+\delta)\backslash B(x_0, \gamma + \frac{5}{12}\delta) \times [0,T]. \tag{5.21}$$

Thus there exists a point $(y_0, t_0) \in B(x_0, \gamma + \frac{5}{12}\delta) \times [0,T]$ with $t_0 > 0$ such that

$$F(y_0,t_0) = \max\ \{F(x,t) \mid (x,t) \in B(x_0,\gamma+\delta) \times [0,T]\} \tag{5.22}$$

unless $F \equiv 0$ on $B(x_0, \gamma+\delta) \times [0,T]$.

Next, as previously in Lemma 4.1, we distinguish three cases, first case where $\xi \equiv 1$ in a neighborhood of y_0, second case where ξ is not identically 1, but smooth in a neighborhood of y_0, and third case, where ξ is not smooth and a trick needs to be applied.

Case 1. $y_0 \in B(x_0, \gamma + \frac{5}{12}\delta)$

Then $\xi \equiv 1$ in a neighborhood of y_0, such that $F = \psi$ near (y_0, t_0), and we have

$$\begin{aligned} 0 &\le (\frac{\partial}{\partial t} - g^{\alpha\beta}\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta)\psi(y_0,t_0) && (5.23)\\ &\le -\frac{1}{4}\frac{\psi^2(y_0,t_0)}{513^2 C_1^4} + c(n,k_0)\left(\frac{\psi^{3/2}}{C_1} + C_1^2\psi + (C_1^5 + c_2 C_1)\psi^{1/2}\right)(y_0,t_0) && (5.24)\end{aligned}$$

and thus

$$\frac{1}{4}\frac{F^2(y_0,t_0)}{513^2 C_1^4} \le c(n,k_0)\left(\frac{F^{3/2}}{C_1} + C_1^2 F + (C_1^5 + c_2 C_1)F^{1/2}\right)(y_0,t_0) \tag{5.25}$$

which is a better estimate than the one below in Case 2, and thus Case 1 follows from Case 2.

Case 2. $y_0 \notin B(x_0, \gamma + \frac{5}{12}\delta)$ and y_0 is not in the cut locus of x_0

Then the distance function $d_{\widetilde{g}}(\cdot, x_0)$ is smooth in a neighborhood of y_0 and it follows that

$$\begin{aligned} &0 \le \frac{\partial F}{\partial t}(y_0,t_0) = \xi(y_0)\frac{\partial \psi}{\partial t}(y_0,t_0),\\ &0 = \widetilde{\nabla}_\alpha F(y_0,t_0) = (\xi \widetilde{\nabla}_\alpha \psi + \psi \widetilde{\nabla}_\alpha \xi)(y_0,t_0),\\ &0 \ge g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta F(y_0,t_0) = (\xi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta \psi + \psi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta \xi\\ &\quad + 2 g^{\alpha\beta}\widetilde{\nabla}_\alpha \xi \widetilde{\nabla}_\beta \psi)(y_0,t_0). \end{aligned} \tag{5.26}$$

Together with (5.15) we obtain at the point (y_0, t_0)

$$\begin{aligned}\frac{1}{4}\frac{\psi^2}{513^2 C_1^4}\xi &\leq \xi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\psi + \xi c(n,k_0)\left(\frac{\psi^{3/2}}{C_1} + C_1^2\psi + (C_1^5 + c_2 C_1)\psi^{1/2}\right)\\ &\leq -\psi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi - 2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\psi\\ &\quad + \xi c(n,k_0)\left(\frac{\psi^{3/2}}{C_1} + C_1^2\psi + (C_1^5 + c_2 C_1)\psi^{1/2}\right).\end{aligned} \tag{5.27}$$

From (6.29) we have at (y_0, t_0)

$$-2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\psi = \frac{2\psi}{\xi}g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\xi \leq \frac{9216}{\delta^2}\psi, \tag{5.28}$$

where the last inequality follows from (4.5). Furthermore, from (5.19) and an estimate analogous to (4.47) we obtain at (y_0, t_0)

$$-\psi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi \leq \frac{c(n,k_0)}{\delta^2}\psi. \tag{5.29}$$

This yields the following intermediate inequality

$$\frac{1}{4}\frac{\psi^2}{513^2 C_1^4}\xi \leq \frac{c(n,k_0)}{\delta^2}\psi + \xi c(n,k_0)\left(\frac{\psi^{3/2}}{C_1} + C_1^2\psi + (C_1^5 + c_2 C_1)\psi^{1/2}\right). \tag{5.30}$$

Multiplying this inequality with ξ, using $0 \leq \xi \leq 1$ and adjusting the constants $c(n,k_0) > 0$ appropriately, we obtain

$$\begin{aligned}\frac{F(y_0,t_0)^2}{C_1^4} &\leq c(n,k_0)\left(\frac{F(y_0,t_0)^{3/2}}{C_1} + C_1^2 F(y_0,t_0) + (C_1^5 + c_2 C_1)F(y_0,t_0)^{1/2}\right)\\ &\quad + \frac{c(n,k_0)}{\delta^2}F(y_0,t_0).\end{aligned} \tag{5.31}$$

Now we use the following elementary estimate: If $x \geq 0$ satisfies

$$x^2 \leq ax^{3/2} + bx + cx^{1/2} \tag{5.32}$$

with constants $a, b, c \geq 0$, then

$$x \leq \max\{a^2, b, c^{2/3}\}.$$

This reduces (6.35) to the following estimate

$$\begin{aligned}F(y_0,t_0) &\leq c(n,k_0)\left(C_1^6 + \frac{C_1^4}{\delta^2} + (C_1^9 + c_2 C_1^5)^{2/3}\right)\\ &\leq c(n,k_0)\left(C_1^6 + (C_1^9 + c_2 C_1^5)^{2/3}\right),\end{aligned} \tag{5.33}$$

since $\frac{1}{\delta} \leq C_1$. It follows that for all $(x,t) \in B(x_0, \gamma+\delta) \times [0,T]$

$$F(x,t) \leq F(y_0,t_0) \leq c(n,k_0)\left(C_1^6 + (C_1^9 + c_2 C_1^5)^{2/3}\right).$$

Since $F(x,t) = \xi(x)\psi(x,t)$ and $\xi(x) = 1$ for $x \in B(x_0, \gamma+\delta/3)$, we obtain

$$\psi(x,t) \leq c(n,k_0)\left(C_1^6 + (C_1^9 + c_2 C_1^5)^{2/3}\right) \qquad \forall (x,t) \in B(x_0, \gamma+\delta/3) \times [0,T].$$

As $\psi(x,t) = (a + |\widetilde{\nabla} g|^2)|\widetilde{\nabla}^2 g|^2$ and $a = 512 C_1^2$ we have

$$|\widetilde{\nabla}^2 g|^2(x,t) = \frac{\psi(x,t)}{(a + |\widetilde{\nabla} g|^2)(x,t)} \leq \frac{\psi(x,t)}{a} \leq c(n,k_0)\left(C_1^4 + (C_1^6 + c_2 C_1^2)^{2/3}\right)$$

for all $(x,t) \in B(x_0, \gamma+\delta/3) \times [0,T]$. Thus

$$\begin{aligned}
|\widetilde{\nabla}^2 g|(x,t) &\leq c(n,k_0)\sqrt{C_1^4 + (C_1^6 + c_2 C_1^2)^{2/3}} \\
&\leq c(n,k_0)(C_1^2 + (C_1^6 + c_2 C_1^2)^{1/3}) \\
&\leq c(n,k_0)(C_1^2 + c_2^{1/3} C_1^{2/3}) \\
&\leq c(n,k_0)\left(\left(\frac{1}{\delta} + c_1\right)^2 + c_2^{1/3}\left(\frac{1}{\delta} + c_1\right)^{2/3}\right) \\
&\leq c(n,k_0)\left(\frac{1}{\delta^2} + c_1^2 + \frac{c_2^{1/3}}{\delta^{2/3}} + c_2^{1/3} c_1^{2/3}\right)
\end{aligned} \tag{5.34}$$

for all $(x,t) \in B(x_0, \gamma+\delta/3) \times [0,T]$, where we used

$$\sqrt[3]{a+b} \leq \sqrt[3]{a} + \sqrt[3]{b}, \quad (a+b)^2 \leq 2a^2 + 2b^2$$

for real numbers $a, b \geq 0$.

Case 3. $y_0 \notin B(x_0, \gamma + \frac{5}{12}\delta)$ and y_0 is in the cut locus of x_0

Then we again apply Calabi's trick, see Case 3 in the Proof of Lemma 4.1. □

6 A priori estimates of $\nabla^m g$ along the flow

In this section prove we prove a priori estimates for all higher derivatives of g and the Riemann curvature tensor along the Ricci DeTurck flow. We treated the case of the second derivatives $\widetilde{\nabla}^2 g$ separately, since the evolution inequality (5.7) for $|\widetilde{\nabla} g|^2$ which goes into the estimate of the time-derivative of ψ (see the proof of Lemma 5.1 above) differs from the corresponding one (6.16) below that will be obtained for the higher derivatives $|\widetilde{\nabla}^{m-1} g|^2$.

Lemma 6.1. *Under the same assumptions as in Lemma 4.1 we set for* $k, s \in \mathbb{N}_0$

$$C_k := \sup_{x \in B(x_0, \gamma+\delta/(k+1))} |\widetilde{\nabla}^k g|, \quad c_s := \sup_{x \in B(x_0, \gamma+3\delta/4)} |\widetilde{\nabla}^s \widetilde{\mathrm{Rm}}|, \tag{6.1}$$

and define for any integer $p \geq 1$ *the following constants*

$$\begin{aligned} \mathcal{K}_p &:= \sum_{\substack{0 \leq k_1, \dots, k_{p+2} \leq p-1 \\ k_1 + \dots + k_{p+2} \leq p+2}} C_{k_1} \cdots C_{k_{p+2}}, \\ \mathcal{L}_p &:= \sum_{\substack{0 \leq l_1, \dots, l_p, s \leq p-1 \\ l_1 + \dots + l_p + s = p}} c_s C_{l_1} \cdots C_{l_p}. \end{aligned} \tag{6.2}$$

Then we find for $m \geq 3$ *and for all* $(x,t) \in B(x_0, \gamma + \frac{\delta}{m+1}) \times [0,T]$

$$|\widetilde{\nabla}^m g|^2(x,t) \leq \max\{A, B\} \tag{6.3}$$

where for some constants $c(n,m,k_0), c(n,m) > 0$

$$\begin{aligned} A &:= c(n,m,k_0) C_{m-1}^2 \left(\frac{1}{\delta^2} + C_1^2 + C_2 + \sqrt{k_0} + \frac{\mathcal{K}_{m-1} + \mathcal{L}_{m-1} + c_{m-1}}{C_{m-1}} \right), \\ B &:= c(n,m) \frac{1}{C_{m-1}^2} (C_{m-1}^5 (\mathcal{K}_m + \mathcal{L}_m + c_m))^{2/3}. \end{aligned} \tag{6.4}$$

We first prove a corollary of that result and later provide the proof of the lemma above. We point out that with more effort it would be possible to obtain an even more explicit bound of $|\widetilde{\nabla}^m g|^2$ analogous to the one in Lemma 5.1, but since our main interest is in the behaviour of the derivatives of the metric and the Riemann curvature tensor when approaching the singular strata, the bound above is sufficient for our purposes.

Corollary 6.2. *Let* $(M, \widetilde{g})$ *be a (possibly incomplete) manifold. Fix* $0 < T < \infty$ *and let* $g(x,t)$ *be a smooth solution of the initial value problem*

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij}(x,t) &= (-2\,\mathrm{Ric}_{ij} + \nabla_i V_j + \nabla_j V_i)(x,t), \quad (x,t) \in M \times [0,T], \\ g(x,0) &= \widetilde{g}(x), \quad x \in M, \end{aligned}$$

where $V^i = g^{jk}(\Gamma^i_{jk} - \widetilde{\Gamma}^i_{jk})$ *is the DeTurck vector field. We assume that*

$$(1-\varepsilon(n))\widetilde{g}(x) \leq g(x,t) \leq (1+\varepsilon(n))\widetilde{g}(x)$$

for $\varepsilon(n) > 0$ *sufficiently small, only depending on* n*, and for all* $(x,t) \in M \times [0,T]$*. Also assume that*

$$|\widetilde{Rm}|^2 \leq k_0$$

for some constant $k_0 > 0$*, and that for all* $m \geq 1$ *there exists a constant* $C_m > 0$*, such that for all* $x \in M$*,* $0 < \rho \leq 1$

$$|\widetilde{\nabla}^m \widetilde{\mathrm{Rm}}|(x) \leq \frac{C_m}{\rho^m}$$

whenever $B(x, \rho - r)$ *is relatively compact for all* $r > 0$*. Then there exists a constant* $C'_m > 0$ *only depending on* k_0*,* C_1*, ...,* C_m*,* m *and* n *such that for all* $x \in M$*,* $t \in [0, T]$*,* $0 < \rho \leq 1$

$$|\widetilde{\nabla}^m g|(x, t) \leq \frac{C'_m}{\rho^m}, \quad |\nabla^m \mathrm{Rm}|(x, t) \leq \frac{C'_m}{\rho^{m+2}}$$

whenever $B(x, \rho - r)$ *is relatively compact for all* $r > 0$*.*

Proof. We start with the estimates of the derivatives of the metric g. The cases $m = 1, 2$ have already been proven, so assume that $m \geq 3$. By induction, we can assume that there exists a constant $C' > 0$ such that for all $k = 1, \ldots, m-1$, $(x, t) \in M \times [0, T]$, $\rho \leq 1$, $r > 0$

$$|\widetilde{\nabla}^k g|(x, t) \leq \frac{C'}{\rho^k}$$

whenever $B(x, \rho - r) \subset M$ is relatively compact. Let $x_0 \in M$ and $\rho \leq 1$ such that $B(x_0, \rho - r) \subset M$ relatively compact for all $r > 0$. Then by Lemma 5.1 (choosing γ, δ equal to $\rho/2$)

$$|\widetilde{\nabla}^m g|^2(x, t) \leq \max\{A, B\} \tag{6.5}$$

for all $t \in [0, T]$. Recall the explicit form of A and B

$$\begin{aligned} A &= c(n, m, k_0) C_{m-1}^2 \left(\frac{1}{\delta^2} + C_1^2 + C_2 + \sqrt{k_0} + \frac{\mathcal{K}_{m-1} + \mathcal{L}_{m-1} + c_{m-1}}{C_{m-1}} \right), \\ B &= c(n, m) \frac{1}{C_{m-1}^2} (C_{m-1}^5 (\mathcal{K}_m + \mathcal{L}_m + c_m))^{2/3}. \end{aligned} \tag{6.6}$$

The individual constants can be estimated as follows:

$$C_k = \sup_{x \in B(x_0, \rho/2 + \rho/2/(k+1))} |\widetilde{\nabla}^k g| \leq \sup_{x \in B(x_0, 3\rho/4)} |\widetilde{\nabla}^k g| \leq \frac{4^k C'}{\rho^k} \tag{6.7}$$

for $k = 1, \ldots, m-1$, since for all $x \in B(x_0, 3\rho/4)$ we have that $B(x, \rho/4 - r) \subset M$ relatively compact for all $r > 0$,

$$c_s = \sup_{x \in B(x_0, \rho/2 + 3\rho/2/4)} |\widetilde{\nabla}^s \widetilde{\mathrm{Rm}}| = \sup_{x \in B(x_0, 7\rho/8)} |\widetilde{\nabla}^s \widetilde{\mathrm{Rm}}| \leq \frac{8^s C}{\rho^s} \tag{6.8}$$

for $s = 1, \dots, m$, since for all $x \in B(x_0, 7\rho/8)$ we have that $B(x, \rho/8 - r) \subset M$ relatively compact for all $r > 0$. Thus

$$\mathcal{K}_m \leq \frac{C}{\rho^{m+2}}, \quad \mathcal{L}_m \leq \frac{C}{\rho^m}, \quad \mathcal{K}_{m-1} \leq \frac{C}{\rho^{m+1}}, \quad \mathcal{L}_{m-1} \leq \frac{C}{\rho^{m-1}} \tag{6.9}$$

with the constant $C > 0$ only depending on m and the constants C', C from above. Plugging this in gives

$$|\widetilde{\nabla}^m g|^2(x_0, t) \leq \frac{C}{\rho^{2m}}$$

for all $t \in [0, T]$, with $C > 0$ only depending on m, n, k_0 and the constants C', C from above. This completes the proof for the derivatives of the metric.

To estimate the derivatives of the curvature tensor, we start by the following general identities for any (say $(1,2)$-tensor) A

$$\nabla_l A^i_{jk} = \frac{\partial}{\partial x^l} A^i_{jk} + A^m_{jk}\Gamma^i_{ml} - A^i_{mk}\Gamma^m_{jl} - A^i_{jm}\Gamma^m_{kl},$$

$$\widetilde{\nabla}_l A^i_{jk} = \frac{\partial}{\partial x^l} A^i_{jk} + A^m_{jk}\widetilde{\Gamma}^i_{ml} - A^i_{mk}\widetilde{\Gamma}^m_{jl} - A^i_{jm}\widetilde{\Gamma}^m_{kl}.$$

Thus ∇ and $\widetilde{\nabla}$, acting on $(1,2)$-tensors, differ by the following expression

$$\nabla_l A^i_{jk} = \widetilde{\nabla}_l A^i_{jk} + A^m_{jk}(\Gamma^i_{ml} - \widetilde{\Gamma}^i_{ml}) - A^i_{mk}(\Gamma^m_{jl} - \widetilde{\Gamma}^m_{jl}) - A^i_{jm}(\Gamma^m_{kl} - \widetilde{\Gamma}^m_{kl}). \tag{6.10}$$

In normal coordinates at a point $p \in M$ with respect to the metric $\widetilde{g}$ we have $\widetilde{\Gamma}^k_{ij} = 0$ and $\frac{\partial}{\partial x^i} g_{jk} = \widetilde{\nabla}_i g_{jk}$ at the point p, such that

$$\Gamma^k_{ij} - \widetilde{\Gamma}^k_{ij} = \frac{1}{2} g^{km}(\widetilde{\nabla}_j g_{im} + \widetilde{\nabla}_i g_{jm} - \widetilde{\nabla}_m g_{ij})$$

at p. But since this is an identity of tensors, it actually holds for all points in any coordinate system. Using the $*$-notation we can write this shorter as

$$\Gamma - \widetilde{\Gamma} = g^{-1} * \widetilde{\nabla} g.$$

Hence (6.10) takes the form

$$\nabla A = \widetilde{\nabla} A + A * g^{-1} * \widetilde{\nabla} g.$$

By induction, together with the product rule

$$\widetilde{\nabla}(A * B) = \widetilde{\nabla} A * B + A * \widetilde{\nabla} B,$$

and the covariant derivative of the inverse metric tensor given by

$$\widetilde{\nabla}(g^{-1}) = g^{-1} * g^{-1} * \widetilde{\nabla} g,$$

we obtain for all $k \geq 1$

$$\nabla^k A = \sum_{\substack{0 \leq k_1, \ldots, k_r \leq k \\ k_1 + \cdots + k_r = k}} \widetilde{\nabla}^{k_1} A * \widetilde{\nabla}^{k_2} g * \cdots * \widetilde{\nabla}^{k_r} g * P_{k_1 \ldots k_r}, \tag{6.11}$$

where $P_{k_1 \ldots k_r}$ is a polynomial in g^{-1}. Now from the identity for the Riemann curvature tensor

$$\mathrm{Rm} = \widetilde{\mathrm{Rm}} * \widetilde{g}^{-1} * g + \widetilde{\nabla}^2 g + g^{-1} * \widetilde{\nabla} g * \widetilde{\nabla} g,$$

see [Shi89, p. 276, formula (83)], we obtain by induction for all $k \geq 1$

$$\begin{aligned} \widetilde{\nabla}^k \mathrm{Rm} = & \sum_{\substack{0 \leq s, k_1, \ldots, k_r \leq k \\ s + k_1 + \cdots + k_r = k}} \widetilde{\nabla}^s \widetilde{\mathrm{Rm}} * \widetilde{\nabla}^{k_1} g * \cdots * \widetilde{\nabla}^{k_r} g * Q_{s k_1 \ldots k_r} \\ & + \sum_{\substack{0 \leq l_1, \ldots, l_s \leq k+2 \\ l_1 + \cdots + l_s = k+2}} \widetilde{\nabla}^{l_1} g * \cdots * \widetilde{\nabla}^{l_s} g * R_{l_1 \ldots l_s}, \end{aligned} \tag{6.12}$$

where Q, R are polynomials in g, g^{-1} and $\widetilde{g}^{-1}$. Plugging (6.12) into (6.11) gives

$$\begin{aligned} \nabla^k \mathrm{Rm} = & \sum_{\substack{0 \leq s, k_1, \ldots, k_r \leq k \\ s + k_1 + \cdots + k_r = k}} \widetilde{\nabla}^s \widetilde{\mathrm{Rm}} * \widetilde{\nabla}^{k_1} g * \cdots * \widetilde{\nabla}^{k_r} g * S_{s k_1 \ldots k_r} \\ & + \sum_{\substack{0 \leq l_1, \ldots, l_s \leq k+2 \\ l_1 + \cdots + l_s = k+2}} \widetilde{\nabla}^{l_1} g * \cdots * \widetilde{\nabla}^{l_s} g * T_{l_1 \ldots l_s}, \end{aligned} \tag{6.13}$$

where S, T are polynomials in g, g^{-1} and $\widetilde{g}^{-1}$, and thus

$$\begin{aligned} & |\nabla^k \mathrm{Rm}| \leq C(n,k) \times \\ & \times \left(\sum_{\substack{0 \leq s, k_1, \ldots, k_r \leq k \\ s + k_1 + \cdots + k_r = k}} |\widetilde{\nabla}^s \widetilde{\mathrm{Rm}}| |\widetilde{\nabla}^{k_1} g| \cdots |\widetilde{\nabla}^{k_r} g| + \sum_{\substack{0 \leq l_1, \ldots, l_s \leq k+2 \\ l_1 + \cdots + l_s = k+2}} |\widetilde{\nabla}^{l_1} g| \cdots |\widetilde{\nabla}^{l_s} g| \right). \end{aligned}$$

Now the claim follows from the estimates of the derivatives of g. □

Proof of Lemma 6.1. Let $m \geq 2$. From (5.4) and since

$$2 g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}^m g \cdot \widetilde{\nabla}_\beta \widetilde{\nabla}^m g \geq |\widetilde{\nabla}^{m+1} g|^2,$$

we have the following differential inequality

$$\begin{aligned}\frac{\partial}{\partial t}|\widetilde{\nabla}^m g|^2 \leq\ & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla}^m g|^2 - |\widetilde{\nabla}^{m+1}g|^2 \\ & + c(n)|\widetilde{\nabla}^m g| \sum_{\substack{0\leq k_1,\dots,k_{m+2}\leq m+1 \\ k_1+\dots+k_{m+2}\leq m+2}} |\widetilde{\nabla}^{k_1}g|\cdots|\widetilde{\nabla}^{k_{m+2}}g| \\ & + c(n)|\widetilde{\nabla}^m g| \sum_{\substack{0\leq l_1,\dots,l_m,s\leq m \\ l_1+\dots+l_m+s=m}} |\widetilde{\nabla}^s\widetilde{\mathrm{Rm}}||\widetilde{\nabla}^{l_1}g|\cdots|\widetilde{\nabla}^{l_m}g| \\ \leq\ & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla}^m g|^2 - |\widetilde{\nabla}^{m+1}|^2 \\ & + c(n,m)|\widetilde{\nabla}^m g|\cdot[|\widetilde{\nabla}^{m+1}g||\widetilde{\nabla}g| + |\widetilde{\nabla}^m g||\widetilde{\nabla}^2 g| + |\widetilde{\nabla}^m g||\widetilde{\nabla}g|^2 \\ & + \sum_{\substack{0\leq k_1,\dots,k_{m+2}\leq m-1 \\ k_1+\dots+k_{m+2}\leq m+2}} C_{k_1}\cdots C_{k_{m+2}}] \\ & + c(n,m)|\widetilde{\nabla}^m g|\cdot[|\widetilde{\mathrm{Rm}}||\widetilde{\nabla}^m g| + |\widetilde{\nabla}^m\widetilde{\mathrm{Rm}}| \\ & + \sum_{\substack{0\leq l_1,\dots,l_m,s\leq m-1 \\ l_1+\dots+l_m+s=m}} c_s C_{l_1}\cdots C_{l_m}]\end{aligned} \tag{6.14}$$

on $B(x_0,\gamma+\delta/m)\times[0,T]$, where we have set as before

$$c_s := \sup_{x\in B(x_0,\gamma+3\delta/4)}|\widetilde{\nabla}^s\widetilde{\mathrm{Rm}}|, \qquad C_k := \sup_{x\in B(x_0,\gamma+\delta/(k+1))}|\widetilde{\nabla}^k g|.$$

The following estimates also hold on $B(x_0,\gamma+\delta/m)\times[0,T]$, when nothing else is mentioned. With the abbreviations

$$\mathcal{K}_m := \sum_{\substack{0\leq k_1,\dots,k_{m+2}\leq m-1 \\ k_1+\dots+k_{m+2}\leq m+2}} C_{k_1}\cdots C_{k_{m+2}},\ \mathcal{L}_m := \sum_{\substack{0\leq l_1,\dots,l_m,s\leq m-1 \\ l_1+\dots+l_m+s=m}} c_s C_{l_1}\cdots C_{l_m}$$

and using $|\widetilde{\nabla}^m g||\widetilde{\nabla}^{m+1}g||\widetilde{\nabla}g| \leq \frac{1}{2}|\widetilde{\nabla}^{m+1}g|^2 + \frac{1}{2}|\widetilde{\nabla}^m g|^2|\widetilde{\nabla}g|^2$ we obtain

$$\begin{aligned}\frac{\partial}{\partial t}|\widetilde{\nabla}^m g|^2 \leq\ & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla}^m g|^2 - \frac{1}{2}|\widetilde{\nabla}^{m+1}|^2 \\ & + c(n,m)\cdot[|\widetilde{\nabla}g|^2|\widetilde{\nabla}^m g|^2 + |\widetilde{\nabla}^2 g||\widetilde{\nabla}^m g|^2 + \mathcal{K}_m\cdot|\widetilde{\nabla}^m g| \\ & + \sqrt{k_0}|\widetilde{\nabla}^m g|^2 + c_m|\widetilde{\nabla}^m g| + \mathcal{L}_m\cdot|\widetilde{\nabla}^m g|] \\ \leq\ & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla}^m g|^2 - \frac{1}{2}|\widetilde{\nabla}^{m+1}|^2 \\ & + c(n,m)\cdot\left(|\widetilde{\nabla}^m g|^2(C_1^2 + C_2 + \sqrt{k_0})\right) \\ & + c(n,m)\cdot\left(|\widetilde{\nabla}^m g|(\mathcal{K}_m + \mathcal{L}_m + c_m)\right).\end{aligned} \tag{6.15}$$

Assume from now on that $m \geq 3$. Then we can replace m by $m-1$ and obtain

$$\begin{aligned}\frac{\partial}{\partial t}|\widetilde{\nabla}^{m-1}g|^2 \leq g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla}^m g|^2 - \frac{1}{2}|\widetilde{\nabla}^m|^2 \\ + c(n,m-1)\cdot\Big(C_{m-1}^2(C_1^2+C_2+\sqrt{k_0}) \\ + C_{m-1}(\mathcal{K}_{m-1}+\mathcal{L}_{m-1}+c_{m-1})\Big)\,.\end{aligned} \tag{6.16}$$

We define similar to (5.8)

$$\psi(x,t) = (a+|\widetilde{\nabla}^{m-1}g|^2)|\widetilde{\nabla}^m g|^2,$$

where $a > 0$ is a constant to be chosen later. Exactly as before in we proceed in the following three steps:

Step 1: Derive an evolution inequality for ψ.

Step 2: Estimate $\widetilde{\nabla}\widetilde{\nabla}\xi$ from below.

Step 3: Estimate $\xi\psi$ from above and conclude the proof.

Step 1: Derive an evolution inequality for ψ.

From (6.15) and (6.16) we obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\right)\psi \\ &= \left(\frac{\partial}{\partial t} - g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\right)(a+|\widetilde{\nabla}^{m-1}g|^2)\cdot|\widetilde{\nabla}^m g|^2 \\ &\quad + (a+|\widetilde{\nabla}^{m-1}g|^2)\left(\frac{\partial}{\partial t} - g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\right)|\widetilde{\nabla}^m g|^2 \\ &\quad - 2g^{\alpha\beta}\widetilde{\nabla}_\alpha|\widetilde{\nabla}^{m-1}g|^2\widetilde{\nabla}_\beta|\widetilde{\nabla}^m g|^2 \\ &\leq -\frac{1}{2}|\widetilde{\nabla}^m g|^4 + c(n,m-1)|\widetilde{\nabla}^m g|^2\cdot \\ &\quad \cdot[C_{m-1}^2(C_1^2+C_2+\sqrt{k_0}) + C_{m-1}(\mathcal{K}_{m-1}+\mathcal{L}_{m-1}+c_{m-1})] \\ &\quad + (a+|\widetilde{\nabla}^{m-1}g|^2)(-\frac{1}{2}|\widetilde{\nabla}^{m+1}g|^2 + c(n,m)[|\widetilde{\nabla}^m g|^2(C_1^2+C_2+\sqrt{k_0}) \\ &\quad + |\widetilde{\nabla}^m g|(\mathcal{K}_m+\mathcal{L}_m+c_m)]) \\ &\quad - 2g^{\alpha\beta}\widetilde{\nabla}_\alpha|\widetilde{\nabla}^{m-1}g|^2\widetilde{\nabla}_\beta|\widetilde{\nabla}^m g|^2.\end{aligned} \tag{6.17}$$

We estimate the last term on the right-hand side

$$\begin{aligned}
& - 2g^{\alpha\beta}\widetilde{\nabla}_\alpha|\widetilde{\nabla}^{m-1}g|^2\widetilde{\nabla}_\beta|\widetilde{\nabla}^m g|^2 \\
& \leq 16|\widetilde{\nabla}^{m-1}g||\widetilde{\nabla}^m g|^2|\widetilde{\nabla}^{m+1}g| \\
& \leq 16C_{m-1}|\widetilde{\nabla}^m g|^2|\widetilde{\nabla}^{m+1}g| \\
& \leq \frac{1}{2}a|\widetilde{\nabla}^{m+1}g|^2 + \frac{1}{2a}\cdot 256C_{m-1}^2|\widetilde{\nabla}^m g|^4.
\end{aligned} \tag{6.18}$$

Now choosing $a := 512C_{m-1}^2$ yields

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} - g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\right)\psi \\
& \leq -\frac{1}{4}|\widetilde{\nabla}^m g|^4 + c(n, m-1)|\widetilde{\nabla}^m g|^2\cdot \\
& \quad \cdot [C_{m-1}^2(C_1^2 + C_2 + \sqrt{k_0}) + C_{m-1}(\mathcal{K}_{m-1} + \mathcal{L}_{m-1} + c_{m-1})] \\
& \quad + c(n,m)C_{m-1}^2[|\widetilde{\nabla}^m g|^2(C_1^2 + C_2 + \sqrt{k_0}) + |\widetilde{\nabla}^m g|(\mathcal{K}_m + \mathcal{L}_m + c_m)].
\end{aligned} \tag{6.19}$$

Since

$$|\widetilde{\nabla}^m g|^2 = \frac{\psi}{a + |\widetilde{\nabla}^{m-1}g|^2} \leq \frac{\psi}{a} = \frac{\psi}{512C_{m-1}^2}$$

and

$$|\widetilde{\nabla}^m g|^2 = \frac{\psi}{a + |\widetilde{\nabla}^{m-1}g|^2} \geq \frac{\psi}{(512+1)C_{m-1}^2}$$

it follows that

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} - g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\right)\psi \\
& \leq -\frac{1}{4}\frac{\psi^2}{513^2C_{m-1}^4} + c(n,m-1)[C_1^2 + C_2 + \sqrt{k_0} + \frac{\mathcal{K}_{m-1} + \mathcal{L}_{m-1} + c_{m-1}}{C_{m-1}}]\psi \\
& \quad + c(n,m)(C_1^2 + C_2 + \sqrt{k_0})\psi \\
& \quad + c(n,m)C_{m-1}(\mathcal{K}_m + \mathcal{L}_m + c_m)\sqrt{\psi}.
\end{aligned} \tag{6.20}$$

Step 2: Estimate $\widetilde{\nabla}\widetilde{\nabla}\xi$ from below.

Let $\eta \in C^\infty(\mathbb{R})$ be the cutoff function as before and define $\xi \in C(M)$ to be the cutoff function

$$\xi(x) = \eta\left(\frac{d_{\widetilde{g}}(x, x_0) - \left(\gamma + \frac{\delta}{m+1}\right)}{\delta\cdot\left(\frac{1}{2}\left(\frac{1}{m+1} + \frac{1}{m}\right) - \frac{1}{m+1}\right)}\right), \tag{6.21}$$

where $d_{\widetilde{g}}$ denotes the distance function with respect to the metric $\widetilde{g}$. Then

$$\begin{aligned} &\xi(x) = 1, && x \in B(x_0, \gamma + \frac{\delta}{m+1}), \\ &\xi(x) = 0, && x \in B(x_0, \gamma+\delta)\backslash B(x_0, \gamma + \delta \cdot \frac{1}{2}\left(\frac{1}{m+1} + \frac{1}{m}\right)), \\ &0 \le \xi(x) \le 1, && x \in M. \end{aligned} \tag{6.22}$$

If $d_{\widetilde{g}}(\cdot, x_0)$ is smooth in a neighborhood of a point x, we obtain by estimates analogous to (4.32)

$$|\widetilde{\nabla}\xi(x)|^2 \le \frac{c(m)}{\delta^2}\xi(x) \tag{6.23}$$

and

$$\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \xi(x) \ge -\left(\frac{c(m)}{\delta^2} + \frac{c(m)}{\delta}\sqrt[4]{k_0}\coth\left(\sqrt[4]{k_0} d_{\widetilde{g}}(x,x_0)\right)\right)\widetilde{g}_{\alpha\beta}(x). \tag{6.24}$$

STEP 3: ESTIMATE $\xi\psi$ FROM ABOVE AND CONCLUDE THE PROOF.

Let

$$F(x,t) = \xi(x)\psi(x,t), \qquad (x,t) \in B(x_0, \gamma+\delta) \times [0,T].$$

Since $|\widetilde{\nabla}^m g|^2(x,0) = 0$, we have

$$F(x,0) = 0, \quad x \in B(x_0, \gamma+\delta). \tag{6.25}$$

Since $\xi(x) = 0$ for $x \in B(x_0, \gamma+\delta)\backslash B(x_0, \gamma + \delta \cdot \frac{1}{2}\left(\frac{1}{m+1} + \frac{1}{m}\right))$, it follows that

$$F(x,t) = 0, \quad (x,t) \in B(x_0,\gamma+\delta)\backslash B\left(x_0, \gamma+\delta\cdot\frac{1}{2}\left(\frac{1}{m+1}+\frac{1}{m}\right)\right) \times [0,T].$$

Thus there exists a point $(y_0, t_0) \in B(x_0, \gamma + \delta \cdot \frac{1}{2}\left(\frac{1}{m+1} + \frac{1}{m}\right)) \times [0,T]$ with $t_0 > 0$ such that

$$F(y_0,t_0) = \max\ \{F(x,t) \mid (x,t) \in B(x_0,\gamma+\delta) \times [0,T]\} \tag{6.26}$$

unless $F \equiv 0$ on $B(x_0, \gamma+\delta) \times [0,T]$.

Now as before in Lemma 4.1, we distinguish three cases, first case where $\xi \equiv 1$ in a neighborhood of y_0, second case where ξ is not identically 1, but smooth in a neighborhood of y_0, and third case, where ξ is not smooth and a trick needs to be applied.

<u>CASE 1. $y_0 \in B(x_0, \gamma + \delta \cdot \frac{1}{2}\left(\frac{1}{m+1} + \frac{1}{m}\right))$</u>

Then $\xi \equiv 1$ in a neighborhood of y_0, such that $F = \psi$ near (y_0, t_0), so that

$$\begin{aligned}
0 &\leq \left(\frac{\partial}{\partial t} - g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\right)\psi \\
&\leq -\frac{1}{4}\frac{\psi^2}{513^2 C_{m-1}^4} + c(n, m-1)[C_1^2 + C_2 + \sqrt{k_0} + \frac{\mathcal{K}_{m-1} + \mathcal{L}_{m-1} + c_{m-1}}{C_{m-1}}]\psi \\
&\quad + c(n, m)(C_1^2 + C_2 + \sqrt{k_0})\psi \\
&\quad + c(n, m)C_{m-1}(\mathcal{K}_m + \mathcal{L}_m + c_m)\sqrt{\psi}
\end{aligned} \tag{6.27}$$

and hence

$$\begin{aligned}
&\frac{1}{4}\frac{F^2(y_0, t_0)}{513^2 C_{m-1}^4} \leq \\
&c(n, m-1)[C_1^2 + C_2 + \sqrt{k_0} + \frac{\mathcal{K}_{m-1} + \mathcal{L}_{m-1} + c_{m-1}}{C_{m-1}}]F(y_0, t_0) \\
&+ c(n, m)(C_1^2 + C_2 + \sqrt{k_0})F(y_0, t_0) \\
&+ c(n, m)C_{m-1}(\mathcal{K}_m + \mathcal{L}_m + c_m)\sqrt{F(y_0, t_0)},
\end{aligned} \tag{6.28}$$

which again is a better estimate than the one below in Case 2, and hence Case 1 follows from Case 2.

Case 2. $y_0 \notin B(x_0, \gamma + \frac{\delta}{2}\left(\frac{1}{m+1} + \frac{1}{m}\right))$ and y_0 is not in cut locus of x_0

Then the distance function $d_{\widetilde{g}}(\cdot, x_0)$ is smooth in a neighborhood of y_0 and we have

$$\begin{aligned}
&0 \leq \frac{\partial F}{\partial t}(y_0, t_0) = \xi(y_0)\frac{\partial \psi}{\partial t}(y_0, t_0), \\
&0 = \widetilde{\nabla}_\alpha F(y_0, t_0) = (\xi\widetilde{\nabla}_\alpha\psi + \psi\widetilde{\nabla}_\alpha\xi)(y_0, t_0), \\
&0 \geq g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta F(y_0, t_0) = (\xi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\psi + \psi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi \\
&\qquad + 2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\psi)(y_0, t_0).
\end{aligned} \tag{6.29}$$

Together with (6.20) we obtain at the point (y_0, t_0)

$$\begin{aligned}
\frac{1}{4}\frac{\psi^2}{513^2 C_{m-1}^4}\xi &\leq \xi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\psi + \xi \cdot \mathcal{A} \\
&\leq -\psi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi - 2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\psi + \xi \cdot \mathcal{A}
\end{aligned} \tag{6.30}$$

with

$$\begin{aligned}
\mathcal{A} := &\, c(n, m-1)[C_1^2 + C_2 + \sqrt{k_0} + \frac{\mathcal{K}_{m-1} + \mathcal{L}_{m-1} + c_{m-1}}{C_{m-1}}]\psi \\
&+ c(n, m)(C_1^2 + C_2 + \sqrt{k_0})\psi \\
&+ c(n, m)C_{m-1}(\mathcal{K}_m + \mathcal{L}_m + c_m)\sqrt{\psi}.
\end{aligned} \tag{6.31}$$

Using (6.29), (6.23) and (4.5) we have at (y_0, t_0)

$$-2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\psi = \frac{2\psi}{\xi}g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi\widetilde{\nabla}_\beta\xi \leq \frac{c(m)}{\delta^2}\psi. \tag{6.32}$$

Also (6.24) and an estimate analogous to (4.47) yields at (y_0, t_0)

$$-\psi g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi \leq \frac{c(n,m,k_0)}{\delta^2}\psi. \tag{6.33}$$

Plugging (6.32) and (6.33) into (6.30) leads to

$$\frac{1}{4}\frac{\psi^2}{513^2 C_{m-1}^4}\xi \leq \frac{c(n,m,k_0)}{\delta^2}\psi + \xi\cdot\mathcal{A}. \tag{6.34}$$

Multiplying by ξ while using $0 \leq \xi \leq 1$ and adjusting the constants $c(n, m, k_0) > 0$ we obtain

$$\begin{aligned}\frac{F(y_0,t_0)^2}{C_{m-1}^4} \leq{}& c(n,m-1)[C_1^2 + C_2 + \sqrt{k_0} + \frac{\mathcal{K}_{m-1}+\mathcal{L}_{m-1}+c_{m-1}}{C_{m-1}}]F(y_0,t_0)\\ &+ c(n,m)(C_1^2 + C_2 + \sqrt{k_0})F(y_0,t_0)\\ &+ c(n,m)C_{m-1}(\mathcal{K}_m + \mathcal{L}_m + c_m)\sqrt{F(y_0,t_0)}\\ &+ \frac{c(n,m,k_0)}{\delta^2}F(y_0,t_0).\end{aligned} \tag{6.35}$$

Now we use the elementary estimate (5.32) with $a = 0$, namely: If $x \geq 0$ satisfies

$$x^2 \leq bx + cx^{1/2}$$

with constants $b, c \geq 0$, then

$$x \leq \max\{b, c^{2/3}\}.$$

This reduces (6.35) to

$$\begin{aligned}&F(y_0,t_0) \leq\\ &\max\{c(n,m,k_0)C_{m-1}^4\left(\frac{1}{\delta^2} + C_1^2 + C_2 + \sqrt{k_0} + \frac{\mathcal{K}_{m-1}+\mathcal{L}_{m-1}+c_{m-1}}{C_{m-1}}\right),\\ &\qquad c(n,m)(C_{m-1}^5(\mathcal{K}_m + \mathcal{L}_m + c_m))^{2/3}\}.\end{aligned} \tag{6.36}$$

Hence for all $(x, t) \in B(x_0, \gamma + \delta) \times [0, T]$

$$\begin{aligned}&F(x,t) \leq F(y_0,t_0) \leq\\ &\max\{c(n,m,k_0)C_{m-1}^4\left(\frac{1}{\delta^2} + C_1^2 + C_2 + \sqrt{k_0} + \frac{\mathcal{K}_{m-1}+\mathcal{L}_{m-1}+c_{m-1}}{C_{m-1}}\right),\\ &\qquad c(n,m)(C_{m-1}^5(\mathcal{K}_m + \mathcal{L}_m + c_m))^{2/3}\}.\end{aligned} \tag{6.37}$$

As $F(x,t) = \xi(x)\psi(x,t)$ and $\xi(x) = 1$ for $x \in B(x_0, \gamma + \frac{\delta}{m+1})$, we conclude

$$\begin{aligned} &\psi(x,t) \leq \\ &\max\{c(n,m,k_0)C_{m-1}^4 \left(\frac{1}{\delta^2} + C_1^2 + C_2 + \sqrt{k_0} + \frac{\mathcal{K}_{m-1} + \mathcal{L}_{m-1} + c_{m-1}}{C_{m-1}} \right), \\ &\qquad c(n,m)(C_{m-1}^5(\mathcal{K}_m + \mathcal{L}_m + c_m))^{2/3}\} \end{aligned} \tag{6.38}$$

for all $(x,t) \in B(x_0, \gamma + \frac{\delta}{m+1}) \times [0,T]$. Now since $\psi(x,t) = (a + |\widetilde{\nabla}^{m-1} g|^2)|\widetilde{\nabla}^m g|^2$ and $a = 512 C_{m-1}^2$ we finally obtain

$$\begin{aligned} |\widetilde{\nabla}^m g|^2(x,t) &= \frac{\psi(x,t)}{(a + |\widetilde{\nabla} g|^2)(x,t)} \leq \frac{\psi(x,t)}{a} \\ &\leq \max\{c(n,m,k_0)C_{m-1}^2 \left(\frac{1}{\delta^2} + C_1^2 + C_2 + \sqrt{k_0} + \frac{\mathcal{K}_{m-1} + \mathcal{L}_{m-1} + c_{m-1}}{C_{m-1}} \right), \\ &\qquad c(n,m)\frac{1}{C_{m-1}^2}(C_{m-1}^5(\mathcal{K}_m + \mathcal{L}_m + c_m))^{2/3}\} \end{aligned} \tag{6.39}$$

for all $(x,t) \in B(x_0, \gamma + \frac{\delta}{m+1}) \times [0,T]$.

Case 3. $y_0 \notin B(x_0, \gamma + \frac{\delta}{2}\left(\frac{1}{m+1} + \frac{1}{m}\right))$ and y_0 is in the cut locus of x_0

Here we again apply Calabi's trick, see Case 3 in the Proof of Lemma 4.1. □

7 Proof of the main existence and regularity result

In this section we describe the necessary modifications of the proofs in Section 2 for the case of incomplete manifolds. Despite incompleteness, we still continue under the assumption of bounded geometry $|\widetilde{\mathrm{Rm}}|^2 \leq k$ for some positive constant $k > 0$.

7.1 Validity of Theorems 2.3 and 2.4 in the incomplete case

We start by observing that, due to the relative compactness of the domain D, Theorem 2.3 and Theorem 2.4 are also valid in case the initial manifold $(M, \widetilde{g})$ is incomplete. Indeed, we can follow the same steps as in the proof of Theorem 2.3 outlined above. To give just one example from the proof of Theorem 2.4, the compactness of the closure of the domain D still gives a positive lower bound for the injectivity radius on $\overline{D}$, which is needed for the estimate (7) in Lemma 3.1 in [Shi89].

Notice that the injectivity radius on $\overline{D}$ can become small in the case of incomplete manifolds when D is close to the singularity, but that smallness of the injectivity radius can also happen in the complete case, e.g. on manifolds with hyperbolic cusps. At this point we emphasize once again. cf. Remark 2.2, that the lower bound for the injectivity radius on $\overline{D}$ does not enter the existence times $T(n,k_0)$ and $T(n,k_0,\delta)$. This is a crucial point, as we will later take an exhaustion of the manifold M by such domains which then get closer and closer to the singularity.

7.2 Extension of Lemmas 2.5 and 2.6 to the incomplete case

Next we formulate and prove interior estimates for the derivatives of the metric, corresponding to Lemma 2.5 and Lemma 2.6. Let $U \subset M$ be open and relatively compact, such that ∂U is an $(n-1)$-dimensional, smooth, compact submanifold. Choose $\delta > 0$ small enough, that $\overline{B(\overline{U},\delta)} \subset M$ is compact and that the function $d_{\widetilde{g}}(\cdot,\overline{U}) : M \to \mathbb{R}$ giving the distance to $\overline{U}$ is smooth on $B(\overline{U},\delta)\backslash\overline{U}$. The latter is possible, since $d_{\widetilde{g}}(x,\overline{U}) = d_{\widetilde{g}}(x,\partial U)$ for all $x \in M\backslash\overline{U}$, and $d_{\widetilde{g}}(\cdot,\partial U)$ is smooth in a neighborhood of ∂U by [Foo84, Theorem 1 and Remark (1)]. The following result is an extension of Lemmas 2.5 and 2.6 to the incomplete case.

Lemma 7.1. *Fix U,δ as above, and a finite $T > 0$. Let $g(x,t)$ be a smooth solution of the initial value problem*

$$\frac{\partial}{\partial t} g_{ij}(x,t) = (-2\operatorname{Ric}_{ij} + \nabla_i V_j + \nabla_j V_i)(x,t), \quad (x,t) \in B(\overline{U},\delta)\times[0,T],$$
$$g(x,0) = \widetilde{g}(x), \quad x \in B(\overline{U},\delta),$$

where $V^i = g^{jk}(\Gamma^i_{jk} - \widetilde{\Gamma}^i_{jk})$ is the DeTurck vector field. Furthermore, assume that

$$(1-\varepsilon(n))\widetilde{g}(x) \le g(x,t) \le (1+\varepsilon(n))\widetilde{g}(x)$$

for $\varepsilon(n) > 0$ sufficiently small only depending on n and for all $(x,t) \in B(\overline{U},\delta)\times [0,T]$. Then for all $m \in \mathbb{N}_0$ there exists $c(n,m,U,\delta,\widetilde{g}) > 0$ depending only on n,m,U,δ and $\widetilde{g}$, such that for all $(x,t) \in B(\overline{U},\frac{\delta}{m+1})\times[0,T]$

$$|\widetilde{\nabla}^m g(x,t)|^2 \le c(n,m,U,\delta,\widetilde{g}).$$

Proof. By compactness of $\overline{B(\overline{U},\delta/(m+1))}$ we can cover it by finitely many balls $B(x_i,\delta/(m+1)+\delta/4)$, $i=1,\dots,N$ with $x_i \in \overline{U}$ and where $N = N(U,m,\widetilde{g})$. Now by the derivative estimates for the metric Lemma 4.1, Lemma 5.1, Lemma 6.1 we obtain

$$|\widetilde{\nabla}^m g|^2(x,t) \le c_i(n,m,\delta,\widetilde{g}), \qquad (x,t) \in B(x_i,\frac{\delta}{m+1}+\frac{\delta}{4})\times[0,T]$$

for $i=1,\dots,N$, implying the desired estimate. □

7.3 Proof of Theorem 1.1 by exhaustion

Now we can finish the proof of Theorem 1.1. Let $\{\overline{U_k}\}_{k\in\mathbb{N}}$, be an exhaustion of M by n-dimensional, smooth, compact manifolds with boundary, i.e. $U_k \subset M$ is open, $\overline{U_k} \subset M$ is compact, ∂U_k is an $(n-1)$-dimensional, smooth, compact submanifold, $U_k \subset U_{k+1}$ for all $k \in \mathbb{N}$ and $\bigcup_{k\in\mathbb{N}} U_k = M$. By Theorem 2.3 and Theorem 2.4 there exists $T(n, k_0) > 0$, such that the system (cf. (2.16))

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij}(x,t) = (-2\operatorname{Ric}_{ij} + \nabla_i V_j + \nabla_j V_i)(x,t), &\quad (x,t) \in U_k \times [0, T(n,k_0)], \\ g(x,t) = \widetilde{g}(x), &\quad (x,t) \in \partial U_k \times [0, T(n,k_0)], \\ g(x,0) = \widetilde{g}(x), &\quad x \in U_k. \end{aligned} \tag{7.1}$$

where $V^i = g^{jk}(\Gamma^i_{jk} - \widetilde{\Gamma}^i_{jk})$ is the DeTurck vector field, has a unique smooth solution $g(k,x,t)$ on $0 \le t \le T(n,k_0)$ for all $k \in \mathbb{N}$, satisfying the estimate

$$(1 - \varepsilon(n))\widetilde{g}(x) \le g(k,x,t) \le (1 + \varepsilon(n))\widetilde{g}(x) \tag{7.2}$$

for all $(x,t) \in \overline{U_k} \times [0, T(n,k_0)]$ and for $\varepsilon(n) > 0$ sufficiently small only depending on n. Note that $\overline{U_k}$ from the exhaustion above need not be connected, but since it is compact, it has at most finitely many connected components, so that Theorem 2.3 and Theorem 2.4 can be applied to each component.

Choose $\delta_k > 0$ sufficiently small, such that the closure of $B(\overline{U_k}, \delta_k) \subset M$ is compact and such that the function $d_{\widetilde{g}}(\cdot, \overline{U_k}) : M \to \mathbb{R}$ is smooth on $B(\overline{U}_k, \delta_k)\backslash\overline{U_k}$. By compactness of the closure of $B(\overline{U_1}, \delta_1)$, there exists $N \in \mathbb{N}$, such that the solution $g(k,x,t)$ is defined on $B(\overline{U_1}, \delta_1)$ for all $k \ge N$. By Lemma 7.1

$$|\widetilde{\nabla}^m g(k,x,t)|^2 \le c(n, m, U_k, \delta_k, \widetilde{g}),$$

for all $k \ge N$, $m \in \mathbb{N}_0$, and $(x,t) \in \overline{U_1} \times [0, T(n,k_0)]$. Then by Arzelà-Ascoli there exists a subsequence $(g(k_\ell, x, t))_{\ell\in\mathbb{N}}$, which converges on $\overline{U_1} \times [0,T]$ in the C^∞ topology to a family of C^∞ metrics $g(x,t)$.

Similarly a subsequence of the subsequence converges on $\overline{U_2} \times [0,T]$, etc. Now the diagonal sequence converges on every $\overline{U_k} \times [0,T]$ to $g(x,t)$. As the sequence (U_k) eventually contains any given compact subset of M, the diagonal sequence converges smoothly locally uniformly to $g(x,t)$. Then $g(x,t)$ solves (1.1). The estimate (1.2) follows by restricting the solutions $g(k,x,t)$ to $t \in [0, T(n,k_0,\delta)]$, where $T(n,k_0,\delta)$ is from Theorem 2.3.

8 Open problems and future research directions

We intend to discuss the following questions in the subsequent publications.

1. Does the Ricci DeTurck flow, presented here, and the flow constructed by the second author in [Ver21], coincide in the setting of incomplete manifolds of bounded geometry with wedge singularities?

2. Can we extend the tensor maximum principle to the incomplete setting?

3. Is there a way to define a flow of arbitrary incomplete manifolds without assuming bounded curvature, for instance imposing bounded Ricci curvature only?

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