Additivity Violation of the Regularized Minimum Output Entropy

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Abstract. The problem of additivity of the Minimum Output Entropy is of fundamental importance in Quantum Information Theory (QIT). It was solved by Hastings [Has09] in the one-shot case by exhibiting a pair of random quantum channels. However, the initial motivation was arguably to understand regularized quantities, and there was so far no way to solve additivity questions in the regularized case. The purpose of this paper is to give a solution to this problem. Specifically, we exhibit a pair of quantum channels that unearths additivity violation of the regularized minimum output entropy. Unlike previously known results in the one-shot case, our construction is non-random, infinite-dimensional, and in the commuting-operator setup. The commuting-operator setup is equivalent to the tensor-product setup in the finite-dimensional case for this problem, but their difference in the infinite-dimensional setting has attracted substantial attention and legitimacy recently in QIT with the celebrated resolutions of Tsirelson's and Connes embedding problem [JNV+20]. Likewise, it is not clear that our approach works in the finite-dimensional setup. Our strategy of proof relies on developing a variant of the Haagerup inequality optimized for a product of free groups.

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1 Introduction

A crucial problem in quantum information theory is the problem of additivity of Minimum Output Entropy (MOE), which asks whether it is possible to find two quantum channels Φ_1, Φ_2 such that

$$H_{min}(\Phi_1 \otimes \Phi_2) < H_{min}(\Phi_1) + H_{min}(\Phi_2). \quad (1)$$

This problem was stated by King-Ruskai [KR01] as a natural question in the study of quantum channels. Shor proved in 2004 [Sho04] that a positive answer to the above question is equivalent to super-additivity of the Holevo information, i.e., there exist quantum channels Φ_1, Φ_2 such that

$$\chi(\Phi_1 \otimes \Phi_2) > \chi(\Phi_1) + \chi(\Phi_2). \tag{2}$$

Heuristically, the super-additivity of the Holevo information implies that entanglement inputs can be used to increase the transmission rate of classical information. We refer to Section 2.1 for the definitions of the MOE H_{min} and the Holevo information χ. This question attracted lots of attention, and Hastings eventually solved it in 2009 [Has09], with preliminary substantial contributions by Hayden, Winter, Werner, see in particular [HW08]. Subsequently, the mathematical aspects of the proof have been clarified in various directions by [ASW11, FKM10, BaH10, BCN16, Col18, CFZ15].

All previously known examples of additivity violation of MOE rely on subtle random constructions. In particular, to date, no deterministic construction of additivity violation has ever been given. For attempts and partial results in the direction of non-random techniques, we refer to [WH02, GHP10, BCLY20], etc.

Note that the above results do not imply anything about the problem of the additivity of the regularized MOE (see Definition 11 for details). Indeed, additivity violation is not known to pertain when the MOE is regularized. More precisely, the additivity question for the regularized MOE asks whether it is possible to find two quantum channels Φ_1, Φ_2 such that

$$\overline{H}_{min}(\Phi_1 \otimes \Phi_2) < \overline{H}_{min}(\Phi_1) + \overline{H}_{min}(\Phi_2). \tag{3}$$

where $\overline{H}_{min}$ stands for the regularized MOE. This question was raised in [Fuk14] and the affirmative answer to this implies superadditivity of classical capacity.

Very few results are known about regularized entropic quantities – see for example [Kin02] or [BCLY20] for partial results. In this paper, we focus on the additivity question of the regularized minimum output entropy, and the tensor product channel will be understood as a composition of two quantum channels whose systems of Kraus operators are commuting (see Section 2.2 for details).

In (quantum) information theory, one key paradigm is to allow repeated uses of a given quantum channel. To do this, we have to analyze a physical system by separated subsystems. Given quantum strategies for non-local games, there are two natural models to describe separated subsystems. One is the *tensor-product model*, and the other is *commuting-operator model*. This latter approach is the object of intense research, see for example [PT15, DP16, CLS17, Slo20, CCLP18], culminating with the recent resolution [JNV+20] in the negative of the celebrated Connes Embedding problem whose origin dates back to [Con76]. In our case, commuting systems of Kraus operators correspond to a commuting-operator model. We refer to Section 2.2 for details on this.

The main result of this paper is an explicit construction of a pair of quantum channels Φ_1 and Φ_2 which have commuting systems of Kraus operators and satisfy additivity violation of the regularized MOE. Specifically, our main result can be stated as follows:

Theorem 1.1. *There exist systems of operators* $\{E_i\}_{i=1}^m$ *and* $\{F_j\}_{j=1}^n$ *in* $B(H)$ *such that*

1. $E_iF_j = F_jE_i$ *for all* $1 \leq i \leq m$ *and* $1 \leq j \leq n$,
2. $\sum_{i=1}^m E_i^* E_i = Id_H = \sum_{j=1}^n F_j^* F_j$,
3. $\Phi_1, \Phi_2 : \mathcal{T}(H) \to \mathcal{T}(H)$ *are quantum channels given by*

$$\Phi_1(\rho) = \sum_{i=1}^m E_i \rho E_i^* \text{ and } \Phi_2(\rho) = \sum_{j=1}^n F_j \rho F_j^*, \tag{4}$$

4. $\overline{H}_{min}(\Phi_1 \circ \Phi_2) < \overline{H}_{min}(\Phi_1) + \overline{H}_{min}(\Phi_2)$.

Note that the above discussion for the regularized MOE makes sense since the given channels are generated by finitely many Kraus operators and given commuting systems will be chosen as an infinite-dimensional analog of i.i.d. Haar distributed unitary matrices, which will be explained in Section 2.2 and Theorem 4.1 in detail. One of the biggest benefits from this shift in perspective is that the regularized minimum output entropy becomes computable, whereas for random unitary channels, computing such regularized quantities still seems to remain totally out of reach at this point.

One of the key ingredients is to extend the *Haagerup inequality* [Haa79] to products of free groups (Proposition 3.2), which has numerous applications in operator algebras, non-commutative harmonic analysis and geometric group theory [Boż81, DCH85, Jol89, Laf00, Laf02].

This paper is organized as follows. After this introduction, Section 2 gathers some preliminaries about entropic quantities, quantum channels, and the infinite-dimensional framework. Section 3 contains the proof of a Haagerup-type inequality for products of free groups and estimates for the regularized Minimum Output Entropy of our main family of quantum channels. Section 4 explains how we can obtain additivity violation of the regularized MOE in the commuting operator setup, and Section 5 contains concluding remarks. A connection with the extended Haagerup inequality and operator space theory is given in the Appendix.

2 Preliminaries

2.1 Minimum output entropy in the infinite-dimensional setting

Let $V : H_A \to H_B \otimes H_E$ be an isometry. Then partial traces on H_B and H_E define the following completely positive trace-preserving maps (aka *quan-*

tum channels)

$$\Phi : \mathcal{T}(H_A) \to \mathcal{T}(H_B),\ \rho \mapsto (\mathrm{id} \otimes \mathrm{tr})(V\rho V^*) \tag{5}$$
$$\Phi^c : \mathcal{T}(H_A) \to \mathcal{T}(H_E),\ \rho \mapsto (\mathrm{tr} \otimes \mathrm{id})(V\rho V^*) \tag{6}$$

where $\mathcal{T}(H)$ denotes the space of trace class operators on a Hilbert space H. The map Φ^c is called the *complementary channel* of Φ. The tensor product channels $\Phi^{\otimes k} : \mathcal{T}(H_A^{\otimes k}) \to \mathcal{T}(H_B^{\otimes k})$ are defined in the obvious way. A (quantum) state in H is a positive element of $\mathcal{T}(H)$ of trace 1, and for a state ρ, its Rényi entropy for $p \in (1, \infty)$ is defined as

$$H^p(\rho) = \frac{1}{1-p} \log\left(tr(\rho^p)\right). \tag{7}$$

Its limit as $p \to 1^+$ is called *the von Neumann entropy*, and if $\lambda_1(\rho) \geq \lambda_2(\rho) \geq \dots$ are the eigenvalues of ρ (counted with multiplicity), then the von Neumann entropy is

$$H(\rho) = -\sum_i \lambda_i(\rho) \log \lambda_i(\rho). \tag{8}$$

The *Holevo information* of a quantum channel is

$$\chi(\Phi) = \sup \left\{ H(\Phi(\sum_i \lambda_i \rho_i) - \sum_i \lambda_i H(\Phi(\rho_i)) \right\}, \tag{9}$$

where the supremum is taken over all probability distributions $(p_i)_i$ and all families of states $(\rho_i)_i$. It describes the amount of classical information that can be carried through a single use of a quantum channel. The ultimate transmission rate of classical information is described by

$$C(\Phi) = \lim_{k\to\infty} \frac{1}{k} \chi(\Phi^{\otimes k}), \tag{10}$$

which is called the *classical capacity*.

For a quantum channel Φ, *the Minimum Output Entropy* (MOE) and the *regularized MOE* are defined as

$$H_{min}(\Phi) = \inf_{\xi} H(\Phi(|\xi\rangle\langle\xi|)) \text{ and} \tag{11}$$

$$\overline{H}_{min}(\Phi) = \lim_{k\to\infty} \frac{1}{k} H_{min}(\Phi^{\otimes k}). \tag{12}$$

respectively, where the infimum runs over all unit vectors ξ of H_A. If Φ has finitely many Kraus operators $\{E_1, E_2, \cdots, E_N\}$ satisfying $\Phi(\rho) = \sum_{i=1}^N E_i \rho E_i^*$ (e.g. if H_E is finite dimensional), then

$$H_{min}(\Phi) + \chi(\Phi) \leq \log(N) \quad \text{and} \quad \overline{H}_{min}(\Phi) + C(\Phi) \leq \log(N). \tag{13}$$

Remark 2.1. *In Equation (11), taking the minimum over all states instead of pure states does not modify the quantity thanks to operator convexity of the function* $x\log(x)$*, see e.g. [Seg60, NU61].*

As in the finite dimensional setting, the following Schmidt decomposition theorem tells us that $\Phi(|\xi\rangle\langle\xi|)$ and $\Phi^c(|\xi\rangle\langle\xi|)$ have the same eigenvalues for each pure state $|\xi\rangle\langle\xi| \in \mathcal{T}(H)$.

Proposition 2.2. *Let* $V : H_A \to H_B \otimes H_E$ *be an isometry and* $\xi \in H_A$ *be a unit vector. For a quantum channel* $\Phi : \mathcal{T}(H_A) \to \mathcal{T}(H_B)$ *given by* $\Phi(\rho) = (id \otimes Tr_E)(V\rho V^*)$*, let us suppose that* $\Phi(|\xi\rangle\langle\xi|)$ *has the spectral decomposition* $\sum_i \lambda_i |e_i\rangle\langle e_i|$ *with* $\lambda_i > 0$*, where* $(e_i)_{i\in I}$ *is an orthonormal subset of* H_B*. Then there exists an orthonormal subset* $(f_i)_{i\in I}$ *of* H_E *satisfying*

$$V|\xi\rangle = \sum_i \sqrt{\lambda_i}|e_i\rangle \otimes |f_i\rangle \text{ and } \Phi^c(|\xi\rangle\langle\xi|) = \sum_i \lambda_i |f_i\rangle\langle f_i|. \tag{14}$$

In particular, $H(\Phi(|\xi\rangle\langle\xi|)) = H(\Phi^c(|\xi\rangle\langle\xi|))$ *for each unit vector* $\xi \in H_A$.

Proof. Since $(e_i)_i$ is an orthonormal basis of H_B, we can write $V|\xi\rangle$ as $\sum_i |e_i\rangle \otimes |\eta_i\rangle$ for a family $(\eta_i)_i \subseteq H_E$. Moreover, the given spectral decomposition of $\Phi(|\xi\rangle\langle\xi|)$ tells us that $\langle\eta_j|\eta_i\rangle = \lambda_i\delta_{i,j}$, which is equivalent to that $(f_i)_i := \left(\lambda_i^{-\frac{1}{2}}\eta_i\right)_i$ is an orthonormal subset of H_E. Then we have

$$V|\xi\rangle = \sum_i \sqrt{\lambda_i}|e_i\rangle \otimes |f_i\rangle \text{ and } \Phi^c(|\xi\rangle\langle\xi|) = \sum_i \lambda_i |f_i\rangle\langle f_i|. \tag{15}$$

□

2.2 Commuting systems of Kraus operators

Let H be a Hilbert space and $(a_{ij})_{(i,j)\in I\times J}$ be a family of bounded operators in $B(H)$ satisfying $\sum_{i\in I} a_{i,j}^* a_{i,j} = \mathrm{Id}_H$ for each $j \in J$. We assume that I is finite and J is arbitary. Let us define a family of quantum channels $(\Phi_j)_{j\in J}$ by

$$\Phi_j : \mathcal{T}(H) \to \mathcal{T}(H),\ X \mapsto \sum_{i\in I} a_{ij} X a_{ij}^*. \tag{16}$$

Their complementary channels are given by

$$\Phi_j^c : \mathcal{T}(H) \to M_{|I|}(\mathbb{C}),\ X \mapsto \sum_{i,i'\in I} \mathrm{tr}(a_{ij} X a_{i'j}^*)|i\rangle\langle i'|. \tag{17}$$

We say that $(\Phi_j)_{j\in J}$ is in the *commuting-operator setup* if the given channels Φ_j have commuting systems of Kraus operators in the sense that $a_{ij}a_{i'j'} = a_{i'j'}a_{ij}$ for any $i, i' \in I$ and $j, j' \in J$ such that $j \neq j'$.

An example of this is the *tensor-product setup*, but it is not the only example. A property is that Φ_j and $\Phi_{j'}$ commute and their products are again quantum channels. If J is finite and $\Phi_1, \cdots, \Phi_{|J|}$ are in the commuting-operator setup, then it is natural to ask whether the following additivity property holds when F is one of $H_{min}, \overline{H}_{min}, \chi, C$:

$$F\left(\prod_{j\in J}\Phi_j\right) = \sum_{j\in J} F(\Phi_j). \tag{18}$$

In particular, in the case $|J| = 2$, the product channel $\Phi_1 \circ \Phi_2$ is called a *local map* in the context of [CKLT20].

Let us construct a non-trivial quantum channel within the commuting-operator setup from the view of abstract harmonic analysis and operator algebra. Let $\mathbb{F}_\infty$ be the free group whose generators are $g_1, g_2, \cdots$ and let us define unitary operators U_i and V_j on $\ell^2(\mathbb{F}_\infty)$ by

$$(U_i f)(x) = f(g_i^{-1}x) \text{ and } (V_j f)(x) = f(xg_j) \tag{19}$$

for any $f \in \ell^2(\mathbb{F}_\infty)$, $x \in \mathbb{F}_\infty$ and $i, j \in \mathbb{N}$. Since $U_iV_j = V_jU_i$ for all $i, j \in \mathbb{N}$, we have the following quantum channels that have commuting systems of Kraus operators:

$$\Phi_{N,l} : \mathcal{T}(\ell^2(\mathbb{F}_\infty)) \to \mathcal{T}(\ell^2(\mathbb{F}_\infty)),\ \rho \mapsto \frac{1}{N}\sum_{i=1}^N U_i\rho U_i^*, \tag{20}$$

$$\Phi_{N,r} : \mathcal{T}(\ell^2(\mathbb{F}_\infty)) \to \mathcal{T}(\ell^2(\mathbb{F}_\infty)),\ \rho \mapsto \frac{1}{N}\sum_{j=1}^N V_j\rho V_j^*. \tag{21}$$

Let $J \in B(\ell^2(\mathbb{F}_\infty))$ be a unitary given by

$$(Jf)(x) = f(x^{-1}) \tag{22}$$

for any $f \in \ell^2(\mathbb{F}_\infty)$ and $x \in \mathbb{F}_\infty$. Then, since $JU_iJ = V_i$ and $J^2 = \mathrm{Id}$, the above channels $\Phi_{N,l}$ and $\Phi_{N,r}$ are equivalent in the sense that

$$\Phi_{N,r}(\rho) = J\Phi_{N,l}(J\rho J)J \tag{23}$$

for any $\rho \in \mathcal{T}(\ell^2(\mathbb{F}_\infty))$. In particular, $H_{min}(\Phi_{N,l}^{\otimes k}) = H_{min}(\Phi_{N,r}^{\otimes k})$ for any $k \in \mathbb{N}$. Also, in order to express k-fold tensor product quantum channels $\Phi_{N,l}^{\otimes k}$, we will use the following notation

$$U_m = U_{m_1} \otimes U_{m_2} \otimes \cdots \otimes U_{m_k} \in B(\ell^2(\mathbb{F}_\infty^k)) \tag{24}$$

for any $m = (m_1, \cdots, m_k) \in I^k$, where $I = \{1, 2, \cdots, N\}$.

3 Generalized Haagerup inequality and regularized MOE

In this section, we prove that the *Haagerup inequality* extends naturally to r-products of free groups $\mathbb{F}_\infty^r$. Then we explain how this generalization allows proving lower bounds of the regularized minimum output entropies (MOE) for $\Phi_{N,l}$. All the results for $\Phi_{N,l}$ in this section are also true for $\Phi_{N,r}$ since they are unitarily equivalent to each other as noted in (23).

3.1 A generalized Haagerup inequality

For any element x in the free group $\mathbb{F}_\infty$, we call $|x|$ its reduced word length with respect to the canonical generators and their inverses. We consider products of free groups $\mathbb{F}_\infty^r$ for any $r \in \mathbb{N}$. Let us use the following notations $E_j = \{x \in \mathbb{F}_\infty : |x| = j\}$ for any $j \in \mathbb{N}_0$ and $E_m = E_{m_1} \times E_{m_2} \times \cdots \times E_{m_r} \subseteq \mathbb{F}_\infty^r$ for any $m = (m_1, \cdots, m_r) \in \mathbb{N}_0^r$.
We view $\mathbb{F}_\infty^r$ as an orthonormal basis that generates the Hilbert space $\ell^2(\mathbb{F}_\infty^r)$. As an algebraic vector space, $\mathbb{F}_\infty^r$ spans $\mathbb{C}[\mathbb{F}_\infty^r]$, on which we may define the convolution product

$$(f * g)(s) = \sum_{t,u \in \mathbb{F}_\infty^r : tu = s} f(t)g(u) \tag{25}$$

and the pointwise product $(f \cdot g)(s) = f(s)g(s)$. For $A \subset \mathbb{F}_\infty^r$, χ_A denotes the indicator function of A. First of all, we can generalize Lemma 1.3 of [Haa79] as follows:

Lemma 3.1. *Let $l, m, k \in \mathbb{N}_0^r$ and let f, g be supported on E_k and E_l respectively. Then*

$$\|(f * g) \cdot \chi_{E_m}\|_{\ell^2(\mathbb{F}_\infty^r)} \le \|f\|_{\ell^2(\mathbb{F}_\infty^r)} \cdot \|g\|_{\ell^2(\mathbb{F}_\infty^r)} \tag{26}$$

if $|k_j - l_j| \le m_j \le k_j + l_j$ and $k_j + l_j - m_j$ is even for all $1 \le j \le r$. Otherwise, we have $\|(f * g) \cdot \chi_{E_m}\|_{\ell^2(\mathbb{F}_\infty^r)} = 0$.

Proof. Since $E_{k_j} \cdot E_{l_j} \subseteq E_{k_j + l_j} \cup E_{k_j + l_j - 2} \cup \cdots \cup E_{|k_j - l_j|}$, we can see that $(f * g)\chi_{E_m} \neq 0$ should imply that m_j is one of $k_j + l_j$, $k_j + l_j - 2$, $\cdots$, $|k_j - l_j|$ for all $1 \le j \le r$. From now on, let us suppose that $|k_j - l_j| \le m_j \le k_j + l_j$ and $k_j + l_j - m_j$ is even for all $1 \le j \le r$. Also, it is enough to suppose that f, g are finitely supported since $(f, g) \mapsto (f * g) \cdot \chi_{E_m}$ is bilinear.
Let us use the induction argument with respect to $r \in \mathbb{N}$. The first case $r = 1$ follows from [Haa79, Lemma 1.3] and let us suppose that (26) holds true for $\mathbb{F}_\infty^r$. Under the notation $m = (m_0, m') \in \mathbb{N}_0^{r+1}$, we have

$$\|(f * g)\chi_{E_m}\|_{\ell^2(\mathbb{F}_\infty^{r+1})}^2 = \sum_{s \in E_m} \left| \sum_{t,u \in \mathbb{F}_\infty^{r+1} : tu = s} f(t)g(u) \right|^2 \tag{27}$$

$$= \sum_{s_0 \in E_{m_0}} \sum_{s' \in E_{m'}} \left| \sum_{t_0, u_0 \in \mathbb{F}_\infty : t_0 u_0 = s_0} \sum_{t', u' \in \mathbb{F}_\infty^r : t'u' = s'} f(t_0, t')g(u_0, u') \right|^2. \tag{28}$$

(Step 1) First of all, let us suppose that $m_0 = k_0 + l_0$. Note that, for each $s_0 \in E_{m_0}$, there is a unique choice of $t_0 \in E_{k_0}$ and $u_0 \in E_{l_0}$ satisfying $t_0 u_0 = s_0$. Thus, we have

$$\|(f * g)\chi_{E_m}\|^2_{\ell^2(\mathbb{F}^{r+1}_\infty)} = \sum_{s_0 \in E_{m_0}} \sum_{s' \in E_{m'}} \left| \sum_{t',u' \in \mathbb{F}^r_\infty : t'u' = s'} f(t_0, t') g(u_0, u') \right|^2 \quad (29)$$

Let us define functions $f_{t_0}(t') = f(t_0, t')$ and $g_{u_0}(u') = g(u_0, u')$ on $\mathbb{F}^r_\infty$. Then $\|(f * g)\chi_{E_m}\|_{\ell^2(\mathbb{F}^{r+1}_\infty)}$ is upper estimated by

$$\|(f * g)\chi_{E_m}\|^2_{\ell^2(\mathbb{F}^{r+1}_\infty)} = \sum_{s_0 \in E_{m_0}} \left\|(f_{t_0} * g_{u_0})\chi_{E_{m'}}\right\|^2_{\ell^2(\mathbb{F}^r_\infty)} \quad (30)$$

$$\leq \sum_{s_0 \in E_{m_0}} \|f_{t_0}\|^2_{\ell^2(\mathbb{F}^r_\infty)} \|g_{u_0}\|^2_{\ell^2(\mathbb{F}^r_\infty)} \quad (31)$$

$$\leq \sum_{t_0 \in E_{k_0}} \sum_{u_0 \in E_{l_0}} \|f_{t_0}\|^2_{\ell^2(\mathbb{F}^r_\infty)} \|g_{u_0}\|^2_{\ell^2(\mathbb{F}^r_\infty)} = \|f\|^2_{\ell^2(\mathbb{F}^{r+1}_\infty)} \|g\|^2_{\ell^2(\mathbb{F}^{r+1}_\infty)}. \quad (32)$$

Here, the first inequality comes from the induction hypothesis. Furthermore, the same idea applies whenever $m_j = k_j + l_j$ for some $0 \leq j \leq r$.

(Step 2) Let us suppose that $m_j < k_j + l_j$ and put $q_j = \dfrac{k_j + l_j - m_j}{2}$ for all $0 \leq j \leq r$. Also, denote by $q = (q_0, \cdots, q_r)$ and define two functions F and G on $\mathbb{F}^{r+1}_\infty$ as follows:

$$F(x) = \begin{cases} \left(\sum_{v \in E_q} |f(xv)|^2\right)^{\frac{1}{2}} & \text{for any } x \in E_{k-q} \\ 0 & \text{otherwise} \end{cases} \quad (33)$$

$$G(y) = \begin{cases} \left(\sum_{v \in E_q} \left|g(v^{-1}y)\right|^2\right)^{\frac{1}{2}} & \text{for any } y \in E_{l-q} \\ 0 & \text{otherwise} \end{cases} \quad (34)$$

Note that F and G are supported on E_{k-q} and E_{l-q} respectively with

$$\|F\|^2_{\ell^2(\mathbb{F}^{r+1}_\infty)} = \sum_{x \in E_{k-q}} \sum_{v \in E_q} |f(xv)|^2 = \|f\|^2_{\ell^2(\mathbb{F}^{r+1}_\infty)} \text{ and} \quad (35)$$

$$\|G\|^2_{\ell^2(\mathbb{F}^{r+1}_\infty)} = \sum_{v \in E_q} \sum_{y \in E_{l-q}} |g(v^{-1}y)|^2 = \|g\|^2_{\ell^2(\mathbb{F}^{r+1}_\infty)}. \quad (36)$$

Then we can show that the convolution $F * G$ dominates $|f * g|$ on E_m. Indeed, for any $s \in E_m$, there exists a unique $(x, y) \in E_{k-q} \times E_{l-q}$ such that $s = xy$

and we have

$$|(f*g)(s)| = \left|\sum_{t,u\in\mathbb{F}_\infty^{r+1}:tu=s} f(t)g(u)\right| \tag{37}$$

$$= \left|\sum_{t\in E_k,u\in E_l:tu=s} f(t)g(u)\right| = \left|\sum_{v\in E_q} f(xv)g(v^{-1}y)\right| \tag{38}$$

$$\leq \left(\sum_{v\in E_q} |f(xv)|^2\right)^{\frac{1}{2}} \left(\sum_{v\in E_q} |g(v^{-1}y)|^2\right)^{\frac{1}{2}} = F(x)G(y) = (F*G)(s) \tag{39}$$

Finally, since F and G are supported on E_{k-q} and E_{l-q} respectively with $m=(k-q)+(l-q)$, we can apply the conclusion from (Step 1) to show

$$\|(f*g)\chi_{E_m}\|_{\ell^2(\mathbb{F}_\infty^{r+1})} \leq \|(F*G)\chi_{E_m}\|_{\ell^2(\mathbb{F}_\infty^{r+1})} \tag{40}$$

$$\leq \|F\|_{\ell^2(\mathbb{F}_\infty^{r+1})}\|G\|_{\ell^2(\mathbb{F}_\infty^{r+1})} = \|f\|_{\ell^2(\mathbb{F}_\infty^{r+1})}\|g\|_{\ell^2(\mathbb{F}_\infty^{r+1})}. \tag{41}$$

□

Then we can generalize the Haagerup inequality to products of free groups $\mathbb{F}_\infty^r$ as follows:

Proposition 3.2. *Let $n=(n_1,\cdots,n_r)\in\mathbb{N}_0^r$ and f be supported on $E_n\subseteq\mathbb{F}_\infty^r$. Then*

$$\|L_f\| \leq (n_1+1)\cdots(n_r+1)\|f\|_{\ell^2(\mathbb{F}_\infty^r)}, \tag{42}$$

where L_f is the convolution operator on $\ell^2(\mathbb{F}_\infty^r)$ given by $g\mapsto f*g$.

Proof. By density arguments, we may assume that f is finitely supported and it is enough to consider finitely supported functions to evaluate the norm of the associated convolution operator L_f. Let $g\in\ell^2(\mathbb{F}_\infty^r)$ be finitely supported and define $g_k=g\cdot\chi_{E_k}$ for each $k\in\mathbb{N}_0^r$. Then $g=\sum_{k\in\mathbb{N}_0^r} g\cdot\chi_{E_k}$ and we have

$$h := f*g = \sum_{k\in\mathbb{N}_0^r} f*g_k. \tag{43}$$

Then, by Lemma 3.1, we have the following estimate for $h_m = h\cdot\chi_{E_m}$ with $m=(m_1,\cdots,m_r)\in\mathbb{N}_0^r$ as follows:

$$\|h_m\|_{\ell^2(\mathbb{F}_\infty^r)} = \left\|\sum_{k\in\mathbb{N}_0^r}(f*g_k)\cdot\chi_{E_m}\right\|_{\ell^2(\mathbb{F}_\infty^r)} \tag{44}$$

$$\leq \sum_{k\in\mathbb{N}_0^r}\|(f*g_k)\cdot\chi_{E_m}\|_{\ell^2(\mathbb{F}_\infty^r)} \leq \sum_{\substack{k\in\mathbb{N}_0^r \\ n_j+k_j-m_j:even \\ |n_j-k_j|\leq m_j\leq n_j+k_j}} \|f\|_{\ell^2(\mathbb{F}_\infty^r)}\|g_k\|_{\ell^2(\mathbb{F}_\infty^r)} =: A_m \tag{45}$$

Writing $k_j = m_j + n_j - 2l_j$ for all $1 \leq j \leq r$, we obtain

$$A_m = \|f\|_{\ell^2(\mathbb{F}_\infty^r)} \sum_{\substack{l_1,\cdots,l_r \\ 0\leq l_j\leq \min\{m_j,n_j\}}} \|g_{m+n-2l}\|_{\ell^2(\mathbb{F}_\infty^r)} \tag{46}$$

$$\leq \|f\|_{\ell^2(\mathbb{F}_\infty^r)} \sqrt{(1+n_1)\cdots(1+n_r)} \left(\sum_{\substack{l_1,\cdots,l_r \\ 0\leq l_j\leq \min\{m_j,n_j\}}} \|g_{m+n-2l}\|^2_{\ell^2(\mathbb{F}_\infty^r)} \right)^{\frac{1}{2}} \tag{47}$$

by the Cauchy-Schwarz inequality. Therefore,

$$\|h\|^2_{\ell^2(\mathbb{F}_\infty^r)} = \sum_{m\in\mathbb{N}_0^r} \|h_m\|^2_{\ell^2(\mathbb{F}_\infty^r)} \leq \sum_{m\in\mathbb{N}_0^r} A_m^2 \tag{48}$$

$$\leq (1+n_1)\cdots(1+n_r)\, \|f\|^2_{\ell^2(\mathbb{F}_\infty^r)} \sum_{m\in\mathbb{N}_0^r} \sum_{\substack{l_1,\cdots,l_r \\ 0\leq l_j\leq \min\{m_j,n_j\}}} \|g_{m+n-2l}\|^2_{\ell^2(\mathbb{F}_\infty^r)} \tag{49}$$

$$= (1+n_1)\cdots(1+n_r)\, \|f\|^2_{\ell^2(\mathbb{F}_\infty^r)} \sum_{\substack{l_1,\cdots,l_r \\ 0\leq l_j\leq n_j}} \sum_{\substack{m_1,\cdots,m_r \\ l_j\leq m_j<\infty}} \|g_{m+n-2l}\|^2_{\ell^2(\mathbb{F}_\infty^r)} \tag{50}$$

$$= (1+n_1)\cdots(1+n_r)\, \|f\|^2_{\ell^2(\mathbb{F}_\infty^r)} \sum_{\substack{l_1,\cdots,l_r \\ 0\leq l_j\leq n_j}} \sum_{\substack{k_1,\cdots,k_r \\ n_j-l_j\leq k_j<\infty}} \|g_k\|^2_{\ell^2(\mathbb{F}_\infty^r)} \tag{51}$$

$$\leq (1+n_1)\cdots(1+n_r)\, \|f\|^2_{\ell^2(\mathbb{F}_\infty^r)} \sum_{\substack{l_1,\cdots,l_r \\ 0\leq l_j\leq n_j}} \sum_{k\in\mathbb{N}_0^r} \|g_k\|^2_{\ell^2(\mathbb{F}_\infty^r)} \tag{52}$$

$$= (1+n_1)^2\cdots(1+n_r)^2\, \|f\|^2_{\ell^2(\mathbb{F}_\infty^r)} \|g\|^2_{\ell^2(\mathbb{F}_\infty^r)}, \tag{53}$$

which gives us, since $h = L_f(g)$,

$$\|L_f\| \leq (1+n_1)\cdots(1+n_r)\, \|f\|_{\ell^2(\mathbb{F}_\infty^r)}. \tag{54}$$

□

3.2 A norm estimate

In this subsection, we investigate the operator norm of the following elements

$$\sum_{v,w\in E^k} a_{v,w}(U_v)^* U_w \text{ with } a_{vw} \in \mathbb{C}, \tag{55}$$

where $E = \{g_1, g_2, \cdots\}$ is the set of generators of $\mathbb{F}_\infty$ and $E^k = E \times \cdots \times E \subseteq E_{(1,\cdots,1)}$. This estimate will be needed to evaluate the regularized MOE in Section 3.3. Our estimate is as follows:

Theorem 3.3. *For any $a = (a_{vw})_{v,w\in E^k} \in M_{N^k}(\mathbb{C})$ such that $tr(a) = 0$, we have*

$$\left\| \sum_{v,w\in E^k} a_{vw}(U_v)^* U_w \right\| \leq N^{\frac{k}{2}} \sqrt{(1+9N^{-1})^k - 1}\, \|a\|_2. \tag{56}$$

Proof. Now we are summing over $(N^k)^2$ elements and we have to split the sum according to whether there are some cancellations to compute the reduced word of $v_i^{-1}w_i$. To do this, for any subset K of $\{1,2,\cdots,k\}$, we define

$$E_K = \left\{(v,w) \in E^k \times E^k : v_i = w_i \text{ if and only if } i \in K\right\}. \tag{57}$$

Note that $E^k \times E^k = \sqcup_{K\subset\{1,\cdots,k\}} E_K$ and $K = \{1,2,\cdots,k\}$ implies $E_K = \left\{(v,v) \in E^k \times E^k : v \in E^k\right\}$. Then, by the triangle inequality, we have

$$\left\|\sum_{v,w\in E^k} a_{vw}(U_v)^* U_w\right\| \le \sum_{K\subset\{1,\cdots,k\}} \left\|\sum_{(v,w)\in E_K} a_{vw}(U_v)^* U_w\right\| \tag{58}$$

$$= \sum_{s=0}^{k} \sum_{\substack{K\subset\{1,\cdots,k\}\\ |K|=s}} \left\|\sum_{(v,w)\in E_K} a_{vw}(U_v)^* U_w\right\|. \tag{59}$$

Note that $|K| = k$ implies $\sum_{(v,w)\in E_K} a_{vw}(U_v)^* U_w = \sum_{v\in E^k} a_{vv} = 0$. From now on, let us suppose that $|K| = s < k$. Then E_K can be identified with the set $\left\{(z,x,y) \in E^s \times E^{k-s} \times E^{k-s} : x_j \neq y_j \ \forall 1 \le j \le k-s\right\}$ for each K. Under this notation, (a_{vw}) can be written as $(a^{K,z}_{x,y})$ and we have

$$\sum_{\substack{K\subset\{1,\cdots,k\}\\ |K|=s}} \left\|\sum_{(v,w)\in E_K} a_{vw}(U_v)^* U_w\right\| \tag{60}$$

$$\le \sum_{\substack{K\subset\{1,\cdots,k\}\\ |K|=s}} \sum_{z\in E^s} \left\|\sum_{\substack{(x,y)\in E^{k-s}\times E^{k-s}\\ x_j\neq y_j}} a^{K,z}_{x,y}(U_x)^* U_y\right\|. \tag{61}$$

Moreover, Proposition 3.2 tells us that

$$\left\|\sum_{\substack{(x,y)\in E^{k-s}\times E^{k-s}\\ x_j\neq y_j}} a^{K,z}_{x,y}(U_x)^* U_y\right\| \le 3^{k-s} \left(\sum_{\substack{(x,y)\in E^{k-s}\times E^{k-s}\\ x_j\neq y_j}} |a^{K,z}_{x,y}|^2\right)^{\frac{1}{2}}, \tag{62}$$

which implies

$$\sum_{\substack{K\subset\{1,\cdots,k\}\\|K|=s}} \left\| \sum_{(v,w)\in E_K} a_{vw}(U_v)^* U_w \right\| \tag{63}$$

$$\leq \sum_{\substack{K\subset\{1,\cdots,k\}\\|K|=s}} \sum_{z\in E^s} 3^{k-s} \left(\sum_{\substack{(x,y)\in E^{k-s}\times E^{k-s}\\x_j\neq y_j}} |a^{K,z}_{x,y}|^2 \right)^{\frac{1}{2}}. \tag{64}$$

Then, since we are summing $N^s\binom{k}{s}$ elements, the Cauchy-Schwarz inequality tells us that the above is upper bounded by

$$3^{k-s} N^{\frac{s}{2}} \binom{k}{s}^{\frac{1}{2}} \left(\sum_{\substack{K\subset\{1,\cdots,k\}\\|K|=s}} \sum_{z\in E^s} \sum_{\substack{(x,y)\in E^{k-s}\times E^{k-s}\\x_j\neq y_j}} |a^{K,z}_{x,y}|^2 \right)^{\frac{1}{2}} \tag{65}$$

$$= 3^{k-s} N^{\frac{s}{2}} \binom{k}{s}^{\frac{1}{2}} \left(\sum_{\substack{K\subset\{1,\cdots,k\}\\|K|=s}} \sum_{(u,v)\in E_K} |a_{vw}|^2 \right)^{\frac{1}{2}}. \tag{66}$$

To summarize, we have

$$\left\| \sum_{v,w\in E^k} a_{vw}(U_v)^* U_w \right\| \leq \sum_{s=0}^{k-1} \sum_{\substack{K\subset\{1,\cdots,k\}\\|K|=s}} \left\| \sum_{(v,w)\in E_K} a_{vw}(U_v)^* U_w \right\| \tag{67}$$

$$\leq \sum_{s=0}^{k-1} 3^{k-s} N^{\frac{s}{2}} \binom{k}{s}^{\frac{1}{2}} \left(\sum_{\substack{K\subset\{1,\cdots,k\}\\|K|=s}} \sum_{(u,v)\in E_K} |a_{vw}|^2 \right)^{\frac{1}{2}} \tag{68}$$

and, applying the Cauchy-Schwartz inequality once more, the above is upper bounded by

$$\left(\sum_{s=0}^{k-1} 9^{k-s} N^s \binom{k}{s} \right)^{\frac{1}{2}} \left(\sum_{K\subset\{1,\cdots,k\}} \sum_{(u,v)\in E_K} |a_{vw}|^2 \right)^{\frac{1}{2}} \tag{69}$$

$$= \sqrt{(N+9)^k - N^k}\, \|a\|_2 = N^{\frac{k}{2}} \sqrt{(1+9N^{-1})^k - 1}\, \|a\|_2 \,. \tag{70}$$

Here, the first equality is thanks to the binomial expansion $(N+9)^k = \sum_{s=0}^{k} 9^{k-s} N^s \binom{k}{s}$. □

3.3 The regularized minimum output entropy of Φ_N

Recall that the quantum channel $\Phi_{N,l} : \mathcal{T}(\ell^2(\mathbb{F}_\infty)) \to \mathcal{T}(\ell^2(\mathbb{F}_\infty))$ is given by

$$\Phi_{N,l}(\rho) = \frac{1}{N}\sum_{i=1}^{N} U_i \rho U_i \tag{71}$$

where U_i is a unitary acting on $\ell^2(\mathbb{F}_\infty)$ by $(U_i f)(x) = f(g_i^{-1}x)$ and $g_1, g_2, \cdots$ are the generators of $\mathbb{F}_\infty$. Let us simply write $\Phi_N = \Phi_{N,l}$ in this section. Theorem 3.3 enables us to show that for any density operator S

$$\left\| (\Phi_N^c)^{\otimes k}(S) - \frac{1}{N^k}\mathrm{Id}_N^{\otimes k} \right\|_2 \tag{72}$$

is sufficiently small with respect to the Hilbert-Schmidt norm. This generalizes [Col18, Theorem 3.1]. Specifically, we prove the following theorem:

Theorem 3.4. *Let Φ_N be a quantum channel of the form (71). For each $k \in \mathbb{N}$, we have*

$$\sup_S \left\| (\Phi_N^c)^{\otimes k}(S) - \frac{1}{N^k} Id_N^{\otimes k} \right\|_2 \leq \frac{\sqrt{(1+9N^{-1})^k - 1}}{N^{\frac{k}{2}}}, \tag{73}$$

where S runs over all density operators in $\mathcal{T}(\ell^2(\mathbb{F}_\infty^k))$.

Proof. Let $X = (x_{i,j})_{i,j\in I^k} = (\Phi_N^c)^{\otimes k}(S) - \frac{1}{N^k}\mathrm{Id}_N^{\otimes k}$. Since $\mathrm{tr}(X) = 0$, we have

$$\mathrm{tr}(X^2) = \mathrm{tr}((\Phi_N^c)^{\otimes k}(S)X) - \frac{1}{N^k}\mathrm{tr}(X) \tag{74}$$

$$= \mathrm{tr}([(\Phi_N^c)^{\otimes k}(S)] \cdot X) - 0 = \mathrm{tr}(S \cdot [\left((\Phi_N^c)^{\otimes k}\right)^* (X)]) \tag{75}$$

where $\left((\Phi_N^c)^{\otimes k}\right)^*$ denotes the adjoint map of $(\Phi_N^c)^{\otimes k}$. Since

$$(\Phi_N^c)^{\otimes k}(A) = \frac{1}{N^k}\sum_{i,j\in I^k} \mathrm{tr}(U_i A U_j^*)|i\rangle\langle j| \tag{76}$$

for any $A \in \mathcal{T}(\ell^2(\mathbb{F}_\infty))$ where $I = \{1, 2, \cdots, N\}$, the above $\left((\Phi_N^c)^{\otimes k}\right)^*(X)$ is given by

$$\left((\Phi_N^c)^{\otimes k}\right)^*(X) = \frac{1}{N^k}\sum_{i,j\in I^k} x_{j,i} U_j^* U_i = \frac{1}{N^k}\sum_{\substack{i,j\in I^k\\ i\neq j}} x_{j,i} U_j^* U_i. \tag{77}$$

Here, $\mathrm{tr}(X) = 0$ is used for the second equality. Thus, we have

$$\mathrm{tr}(X^2) = \mathrm{tr}(S((\Phi_N^c)^{\otimes k})^*(X)) \leq \left\|((\Phi_N^c)^{\otimes k})^*(X)\right\| = \frac{1}{N^k}\left\| \sum_{\substack{i,j\in I^k\\ i\neq j}} x_{i,j} U_i^* U_j \right\| \tag{78}$$

and, according to Theorem 3.3, we get

$$\|X\|_2^2 = \mathrm{tr}(X^2) \le \frac{N^{\frac{k}{2}}\sqrt{(1+9N^{-1})^k - 1}\,\|X\|_2}{N^k}, \tag{79}$$

as claimed. □

This allows us to estimate the regularized minimum output entropies of Φ_N as follows:

Theorem 3.5. *Let Φ_N be a quantum channel of the form (71). For any $k \in \mathbb{N}$ we have*

$$H_{min}(\Phi_N^{\otimes k}) \ge k\log(N) - 2\log(1 + \sqrt{(1+9N^{-1})^k - 1}). \tag{80}$$

In particular, we have the following estimate for the regularized MOE

$$\overline{H}_{min}(\Phi_N) \ge \log(N) - \log\left(1 + \frac{9}{N}\right) \ge \log(N) - \frac{9}{N}. \tag{81}$$

Proof. Thanks to the fact that the von Neumann entropy is greater than or equal to the Rényi entropy of order $\alpha = 2$ [MLDS+13], we have

$$H((\Phi_N^c)^{\otimes k}(S)) \ge \frac{\alpha}{1-\alpha}\log\left(\left\|(\Phi_N^c)^{\otimes k}(S)\right\|_\alpha\right) \tag{82}$$

$$\ge -2\log\left(N^{-\frac{k}{2}}(1 + \sqrt{(1+9N^{-1})^k - 1})\right) \tag{83}$$

$$= k\log(N) - 2\log\left(1 + \sqrt{(1+9N^{-1})^k - 1}\right) \tag{84}$$

for any density operator $S \in \mathcal{T}(\ell^2(\mathbb{F}_\infty^k))$ by Theorem 3.4. In particular, we have

$$H_{min}(\Phi_N^{\otimes k}) \ge k\log(N) - 2\log(1 + \sqrt{(1+9N^{-1})^k - 1}) \tag{85}$$

and the last conclusion follow from the following computation with L'Hôpital's rule:

$$\lim_{k\to\infty} \frac{\log(1 + \sqrt{(1+9N^{-1})^k - 1})}{k} \tag{86}$$

$$= \lim_{k\to\infty} \frac{\frac{(1+\frac{9}{N})^k \log(1+\frac{9}{N})}{2\sqrt{(1+\frac{9}{N})^k - 1}}}{1 + \sqrt{(1+\frac{9}{N})^k - 1}} = \frac{1}{2}\log\left(1 + \frac{9}{N}\right). \tag{87}$$

□

4 Additivity violation of the regularized MOE

In this section, we choose two copies of Φ_N as $\Phi_{N,l}$ and $\Phi_{N,r}$. Indeed, these two quantum channels $\Phi_{N,l}$ and $\Phi_{N,r}$ are unitarily equivalent as explained in Section 2.2 and are in the commuting-operator setup.
Then we can obtain the following additivity violation of the regularized MOE by generalizing the Winter-Holevo-Hayden-Werner trick for $\Phi_{N,l}\circ\Phi_{N,r}$:

Theorem 4.1. *For quantum channels $\Phi_{N,l}$ and $\Phi_{N,r}$ of the form (20) and (21) respectively, the regularized MOE is not additive: For any $N > e^{18}$, we have*

$$\overline{H}_{min}(\Phi_{N,l}\circ\Phi_{N,r}) < \overline{H}_{min}(\Phi_{N,l}) + \overline{H}_{min}(\Phi_{N,r}). \tag{88}$$

Proof. Note that under notations from Subsection 2.2,

$$(\Phi_{N,l}\circ\Phi_{N,r})(\rho) = \frac{1}{N^2}\sum_{i,j=1}^{N} U_iV_j\rho V_j^* U_i^*. \tag{89}$$

Since $|e\rangle\langle e|$ is an invariant for U_iV_i, we have

$$(\Phi_{N,l}\circ\Phi_{N,r})(|e\rangle\langle e|) = \frac{1}{N}|e\rangle\langle e| + \frac{1}{N^2}\sum_{\substack{1\le i,j\le N:\\ i\neq j}} |g_ig_j^{-1}\rangle\langle g_ig_j^{-1}| \tag{90}$$

where $g_1, g_2, \cdots$ are the generators of $\mathbb{F}_\infty$. This implies

$$\overline{H}_{min}(\Phi_{N,l}\circ\Phi_{N,r}) \le H_{min}(\Phi_{N,l}\circ\Phi_{N,r}) \le H((\Phi_{N,l}\circ\Phi_{N,r})(|e\rangle\langle e|)) \tag{91}$$

$$= \frac{\log(N)}{N} + (N^2 - N)\cdot\frac{\log(N^2)}{N^2} = 2\log(N) - \frac{\log(N)}{N}. \tag{92}$$

Moreover, $\Phi_{N,l}$ and $\Phi_{N,r}$ are copies of Φ_N, so that we have

$$2\log(N) - \frac{\log(N)}{N} < 2\log(N) - \frac{18}{N} \le \overline{H}_{min}(\Phi_{N,l}) + \overline{H}_{min}(\Phi_{N,r}) \tag{93}$$

by Theorem 3.5 if $N > e^{18}$ ($\Leftrightarrow -\frac{\log(N)}{N} < -\frac{18}{N}$). $\square$

5 Concluding remarks

(1) Various versions of C^*-tensor products can be used to obtain commuting systems of operators. For example, let A, B be unital C^*-algebras and take families of operators $(E_i)_{i=1}^m \subseteq A$ and $(F_j)_{j=1}^n \subseteq B$. Also, suppose that $A\otimes_{max} B \subseteq B(K)$. Then

$$\{E_i\otimes 1_B\}_{i=1}^m \text{ and } \{1_A\otimes F_j\}_{j=1}^n \tag{94}$$

give us commuting systems of operators in $B(K)$. Moreover, if we suppose that $C_r^*(\mathbb{F}_\infty)\otimes_{max} C_r^*(\mathbb{F}_\infty) \subseteq B(K)$ where $C_r^*(\mathbb{F}_\infty)$ is the reduced group C^*-algebra

of the free group $\mathbb{F}_\infty$, then commuting systems $\{U_i \otimes \mathrm{Id}\}_i$ and $\{\mathrm{Id} \otimes U_j\}_j$ give another example of additivity violation in the commuting-operator setup.
(2) Since Haagerup type inequalities exist for other groups (e.g., hyperbolic groups [dlH88]) or certain reduced free products of C^*-algebras [Boż91], it is natural to expect that similar results should hold and will yield other examples of additivity violation phenomena.
(3) It is worthwhile to compare the main results of this paper and the cases of random unitary channels. On the side of random unitary channels, the regularized MOE is unknown, whereas our Theorem 3.5 gives us a strong estimate for the regularized MOE of Φ_N.
(4) One might wonder if we can evaluate the classical capacity of Φ_N^c whose output space is finite-dimensional. Thanks to Theorem 3.5 and a standard argument, the classical capacity of Φ_N^c is upper bounded by

$$C(\Phi_N^c) \leq \log(N) - \overline{H}_{min}(\Phi_N^c) \leq \frac{9}{N}. \tag{95}$$

However, unlike in the tensor-product setup [Fuk14, Theorem 6.1], it is not clear whether additivity violation of the regularized MOE implies super-additivity of the classical capacity within the commuting-operator framework, so the question of the additivity of the classical capacity remains open.

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Appendix

In this section, we present two alternative proofs of Proposition 3.2 using operator space theory, while the proof in Section 3 is more elementary for readers. See [Pis03] for more details and the basics of operator space theory.
For the free group $\mathbb{F}_\infty$ with generators $g_1, g_2, \cdots$, we denote by $U_x = U_{s_1}^{t_1} U_{s_2}^{t_2} \cdots U_{s_k}^{t_k}$ if $x = g_{s_1}^{t_1} g_{s_2}^{t_2} \cdots g_{s_k}^{t_k}$ in a reduced word. Recall that $E_n = \{x \in \mathbb{F}_\infty : |x| = n\}$ has a natural embedding into $E_k \times E_{n-k}$ for any $0 \leq k \leq n$,

which we denote by $x = (x_{k,1}, x_{k,2})$, and the operator space structure of span $\{U_x : x \in E_n\}$ was analyzed in [Buc99]. More precisely, the following operator-valued matrix $\sum_{x\in E_n} A_x \otimes U_x \in B(H) \otimes_{min} C^*_r(\mathbb{F}_\infty)$ is identified with

$$\sum_{x\in E_n} A_x \otimes |x_{k,1}\rangle\langle x_{k,2}| \in B(H) \otimes_{min} M_{|E_k|,|E_{n-k}|}(\mathbb{C}) \tag{A.1}$$

and [Buc99] states that

$$\left\| \sum_{x\in E_n} A_x \otimes U_x \right\|_{B(H)\otimes_{min} C^*_r(\mathbb{F}_\infty)} \leq (1+n) \cdot \max_{0\leq k\leq n} \left\| \sum_{x\in E_n} A_x \otimes |x_{k,1}\rangle\langle x_{k,2}| \right\|. \tag{A.2}$$

The above inequality can be interpreted as follows in view of operator space theory. A natural operator space structure on span $\{U_x : |x| = n\}$ is given by

$$\left\| \sum_{x\in E_n} A_x \otimes U_x \right\|_{B(H)\otimes_{min} D_n} = \max_{0\leq k\leq n} \left\| \sum_{x\in E_n} A_x \otimes |x_{k,1}\rangle\langle x_{k,2}| \right\|, \tag{A.3}$$

where the associated operator space D_n is understood as a subspace of $\bigoplus_{k=0}^{n} C^k \otimes_{min} R^{n-k}$. Then (A.2) implies that the completely bounded norm of the formal identity from D_n into $C^*_r(\mathbb{F}_\infty)$ is less than or equal to $1+n$. From now on, let us explain how we reach the conclusion of Proposition 3.2 via two different ways using operator space techniques.

(Proof 1) One natural way is to consider tensor products of the above formal identities at the level of operator spaces. Then

$$\mathrm{id} \otimes \cdots \otimes \mathrm{id} : D_{n_1} \otimes_{min} \cdots \otimes_{min} D_{n_r} \to E_{n_1} \otimes_{min} \cdots \otimes_{min} E_{n_r} \tag{A.4}$$

has the completely bounded norm less than or equal to $(1+n_1)\cdots(1+n_r)$, which implies

$$\left\| \sum_{x\in E_{n_1}\times\cdots\times E_{n_r}} f(x) U_{x_1} \otimes \cdots \otimes U_{x_r} \right\|_{C^*_r(\mathbb{F}_\infty)\otimes_{min}\cdots\otimes_{min} C^*_r(\mathbb{F}_\infty)} \tag{A.5}$$

$$\leq (1+n_1)\cdots(1+n_r) \cdot \left\| \sum_{x\in E_{n_1}\times\cdots\times E_{n_r}} f(x) U_{x_1} \otimes \cdots \otimes U_{x_r} \right\|_{D_{n_1}\otimes_{min}\cdots\otimes_{min} D_{n_r}} \tag{A.6}$$

for any $n = (n_1, \cdots, n_r) \in \mathbb{N}_0^r$ and finitely supported $f : E_{n_1} \times \cdots \times E_{n_r} \to \mathbb{C}$. Hence, the only remaining part is to check whether

$$\left\| \sum_{x\in E_{n_1}\times\cdots\times E_{n_r}} f(x) U_{x_1} \otimes \cdots \otimes U_{x_r} \right\|_{D_{n_1}\otimes_{min}\cdots\otimes_{min} D_{n_r}} \tag{A.7}$$

is less than or equal to $\left(\sum_{x\in E_{n_1}\times\cdots\times E_{n_r}} |f(x)|^2\right)^{\frac{1}{2}}$. Indeed, it follows from basic operator space theory since

$$\bigoplus_{0\leq k\leq n} C^{k_1}\otimes_{min} R^{n_1-k_1}\otimes_{min}\cdots\otimes_{min} C^{k_r}\otimes_{min} R^{n_r-k_r} \tag{A.8}$$

$$\cong \bigoplus_{0\leq k\leq n} C^{k_1+\cdots+k_r}\otimes_{min} R^{n_1+\cdots+n_r-(k_1+\cdots+k_r)} \tag{A.9}$$

where $0\leq k\leq n$ means $0\leq k_j\leq n_j$ for all $j=1,2,\cdots,r$ and

$$\left\|\sum_{x\in E_{n_1}\times\cdots\times E_{n_r}} f(x)|x_{k_1,1}\cdots x_{k_r,1}\rangle\langle x_{k_1,2}\cdots x_{k_r,2}|\right\| \tag{A.10}$$

$$\leq \left\|\sum_{x\in E_{n_1}\times\cdots\times E_{n_r}} f(x)|x_{k_1,1}\cdots x_{k_r,1}\rangle\langle x_{k_1,2}\cdots x_{k_r,2}|\right\|_{HS}. \tag{A.11}$$

Then the right hand side is given by $\left(\sum_{x\in E_{n_1}\times\cdots\times E_{n_r}} |f(x)|^2\right)^{\frac{1}{2}}$.

(Proof 2) Let us elaborate on an operator-valued version of Proposition 3.2. More precisely, let us prove the following inequality

$$\left\|\sum_{x\in E_{n_1}\times\cdots\times E_{n_r}} f(x)\otimes U_{x_1}\otimes\cdots\otimes U_{x_r}\right\|_{M_l(\mathbb{C})\otimes_{min}C^*_r(\mathbb{F}_\infty)\otimes_{min}\cdots\otimes_{min}C^*_r(\mathbb{F}_\infty)} \tag{A.12}$$

$$\leq (1+n_1)\cdots(1+n_r)\cdot\left(\sum_{x\in E_{n_1}\times\cdots\times E_{n_r}} \|f(x)\|_{HS}^2\right)^{\frac{1}{2}} \tag{A.13}$$

for any finitely supported matrix-valued functions $f: E_{n_1}\times\cdots\times E_{n_r}\to M_l(\mathbb{C})$ by induction. If $r=1$, then a direct application of [Buc99] tells us that

$$\left\|\sum_{x\in E_n} f(x)\otimes U_x\right\|_{M_l(\mathbb{C})\otimes_{min}C^*_r(\mathbb{F}_\infty)} \leq (1+n)\cdot\max_{0\leq k\leq n}\left\|\sum_{x\in E_n} f(x)\otimes|x_{k,1}\rangle\langle x_{k,2}|\right\| \tag{A.14}$$

$$\leq (1+n)\cdot\left\|\sum_{x\in E_n} f(x)\otimes|x_{k,1}\rangle\langle x_{k,2}|\right\|_{HS} \tag{A.15}$$

$$= (1+n)\cdot\left(\sum_{x\in E_n}\|f(x)\|_{HS}^2\right)^{\frac{1}{2}}. \tag{A.16}$$

Now, let us suppose that the above inequality extends naturally to $E_n = E_{(n_1,\cdots,n_r)}$. Then for any finitely supported $f : E_{(n_1,\cdots,n_{r+1})} \to M_l(\mathbb{C})$ we have

$$\left\| \sum_{x\in E_n} \sum_{y\in E_{n_{r+1}}} f(x,y) \otimes U_{x_1} \otimes \cdots \otimes U_{x_r} \otimes U_y \right\| \tag{A.17}$$

$$= \left\| \sum_{y\in E_{n_{r+1}}} \left(\sum_{x\in E_n} f(x,y) \otimes U_{x_1} \otimes \cdots \otimes U_{x_r} \right) \otimes U_y \right\| \tag{A.18}$$

$$\leq (1+n_{r+1}) \max_{0\leq k\leq n_{r+1}} \left\| \sum_{y\in E_{n_{r+1}}} \left(\sum_{x\in E_n} f(x,y) \otimes U_{x_1} \otimes \cdots \otimes U_{x_r} \right) \otimes |y_{k,1}\rangle\langle y_{k,2}| \right\| \tag{A.19}$$

$$= (1+n_{r+1}) \max_{0\leq k\leq n_{r+1}} \left\| \sum_{x\in E_n} \left(\sum_{y\in E_{n_{r+1}}} f(x,y) \otimes |y_{k,1}\rangle\langle y_{k,2}| \right) \otimes U_{x_1} \otimes \cdots \otimes U_{x_r} \right\|. \tag{A.20}$$

Here, $\sum_{y\in E_{n_{r+1}}} f(x,y) \otimes |y_{k,1}\rangle\langle y_{k,2}|$ can be understood as a function

$$F_k : E_n \to M_l(\mathbb{C}) \otimes M_{|E_k|,|E_{n_{r+1}-k}|}(\mathbb{C}) \tag{A.21}$$

given by $F_k(x) = \sum_{y\in E_{n_{r+1}}} f(x,y) \otimes |y_{k,1}\rangle\langle y_{k,2}|$. Then, from the inductive assumption, we have

$$\left\| \sum_{x\in E_n} \sum_{y\in E_{n_{r+1}}} f(x,y) \otimes U_{x_1} \otimes \cdots \otimes U_{x_r} \otimes U_y \right\| \tag{A.22}$$

$$\leq (1+n_1)\cdots(1+n_{r+1}) \max_{0\leq k\leq n_{r+1}} \left(\sum_{x\in E_n} \|F_k(x)\|_{HS}^2 \right)^{\frac{1}{2}} \tag{A.23}$$

$$= (1+n_1)\cdots(1+n_{r+1}) \left(\sum_{x\in E_n} \sum_{y\in E_{n_{r+1}}} \|f(x,y)\|_{HS}^2 \right)^{\frac{1}{2}}. \tag{A.24}$$

Hence, we can prove an operator-valued extension of Proposition 3.2.

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