On Class Number Relations and Intersections over Function Fields

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Abstract. The aim of this paper is to study class number relations over function fields and the intersections of Hirzebruch-Zagier type divisors on the Drinfeld-Stuhler modular surfaces. The main bridge is a particular "harmonic" theta series with nebentypus. Using the strong approximation theorem, the Fourier coefficients of this series are expressed in two ways; one comes from modified Hurwitz class numbers and another gives the intersection numbers in question.

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1 Introduction

1.1 Classical story

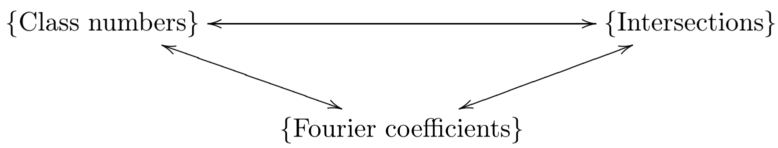
Given a negative integer d with $d \equiv 0$ or $1 \bmod 4$, let $h(d)$ be the proper ideal class number of the imaginary quadratic order $\mathcal{O}_d$ with discriminant d. Put $w(d) := \#(\mathcal{O}_d^\times)/2$. The classical Kronecker-Hurwitz class number relation says that for a non-square $n \in \mathbb{N}$,

$$\sum_{\substack{t \in \mathbb{Z} \\ t^2 < 4n}} \left(\sum_{\substack{d \in \mathbb{N} \\ d^2 \mid (t^2-4n)}} \frac{h\big((t^2-4n)/d^2\big)}{w\big((t^2-4n)/d^2\big)} \right) = \sum_{\substack{m \in \mathbb{N} \\ m \mid n}} \max(m, n/m). \tag{1.1}$$

One can derive the above identity via "modular polynomial", i.e. the defining equation of the graph of the Hecke correspondence T_n, $n \in \mathbb{N}$, on the modular curve X of full level (cf. [14]). In particular, the quantity in (1.1) is equal to the "finite part" of the intersection number of the divisors T_1 and T_n on the surface $X \times X$. Taking the "infinite part" (from cuspidal intersections) into account, the total intersection number of T_1 and T_n becomes

$$T_1 \cdot T_n = 2\sigma(n),$$

where $\sigma(n) := \sum_{m|n} m$ is precisely the n-th Fourier coefficient of the weight-two Eisenstein series (normalized so that the first Fourier coefficient equals to 1). This provides a very concrete example in the following connections:



In the celebrated work of Hirzebruch and Zagier [17], the whole theory on the ground of the Hilbert modular surfaces associated with real quadratic fields is well-established. More precisely, they express the intersections of certain special divisors in terms of Hurwitz class numbers, and show that the generating function associated with these intersection numbers is actually a particular Eisenstein series with nebentypus. The interpretations for the Fourier coefficients of Eisenstein series, which have been generalized to the "Kudla-Millson" theta integrals (cf. [22] and [24]) on the quotients of symmetric spaces for orthogonal and unitary groups, are viewed as *geometric Siegel-Weil formula* and have various applications (cf. [20], [6], [23], [21], and [9]). Moreover, connections with the class numbers make it possible to compute explicitly the intersections in question (cf. [17] and [9]).
The purpose of this paper is to attempt an exploration of this phenomenon in the function field setting, and to derive a Hirzebruch-Zagier style geometric interpretation for the class number relations in the world of positive characteristic.

1.2 Drinfeld-Stuhler modular curves

Let $A = \mathbb{F}_q[\theta]$, the polynomial ring with one variable θ over a finite field $\mathbb{F}_q$ with q elements, and let k be the field of fractions of A. Let k_∞ be the completion of k with respect to the "degree valuation" (cf. Section 2.1), and denote by $\mathbb{C}_\infty$ the completion of a chosen algebraic closure of k_∞. The *Drinfeld half plane* is $\mathfrak{H} := \mathbb{C}_\infty - k_\infty$, equipped with the Möbius left action of $\mathrm{GL}_2(k_\infty)$. Let B be a quaternion algebra over k which is split at ∞ (i.e. $B \otimes_k k_\infty \cong \mathrm{Mat}_2(k_\infty)$), and O_B be an Eichler A-order in B of type $(\mathfrak{n}^+, \mathfrak{n}^-)$ (cf. Section 2.3). Then the embedding $B^\times \hookrightarrow \mathrm{GL}_2(k_\infty)$ induces an action of $\Gamma(\mathfrak{n}^+, \mathfrak{n}^-) := O_B^\times$ on $\mathfrak{H}$. The quotient space

$$X(\mathfrak{n}^+, \mathfrak{n}^-) \ := \ \Gamma(\mathfrak{n}^+, \mathfrak{n}^-)\backslash\mathfrak{H}$$

is called the *Drinfeld-Stuhler modular curve for* $\Gamma(\mathfrak{n}^+,\mathfrak{n}^-)$. When $B = \mathrm{Mat}_2(k)$, the group $\Gamma(\mathfrak{n}^+,\mathfrak{n}^-)$ coincides (up to conjugations) with the congruence subgroup

$$\Gamma_0(\mathfrak{n}^+) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \;\middle|\; c \equiv 0 \bmod \mathfrak{n}^+ \right\},$$

and the compactification of $X(\mathfrak{n}^+,\mathfrak{n}^-)$ is the so-called *Drinfeld modular curve for* $\Gamma_0(\mathfrak{n}^+)$.

Remark 1.1. As in the classical case, the study of Drinfeld modular polynomials in [1], [2], and [19] give an analogue of the Kronecker-Hurwitz class number relation for "imaginary" quadratic A-orders (cf. [41] and [36]). Also, the connection with the intersections of the Hecke correspondence on the Drinfeld modular curves is derived in [41] when q is odd. Moverover, these intersection numbers appear in the Fourier expansion of the "improper" Eisenstein series on $\mathrm{GL}_2(k_\infty)$ which is introduced by Gekeler (cf. [10] and [11]). Thus, a parallel story for the Kronecker-Hurwitz case over rational function fields is developed. We may also expect to see these connections when the base field k is an arbitrary global function field.

1.3 Hirzebruch-Zagier-type divisors

From now on, we always assume that q is odd. Fix a monic square-free $\mathfrak{d} \in A$ with even degree. Then the quadratic field $F := k(\sqrt{\mathfrak{d}})$ is *real* over k, (i.e. the infinite place of k is split in F). The embedding $F \hookrightarrow F \otimes_k k_\infty \cong k_\infty \times k_\infty$ induces

$$\mathrm{GL}_2(F) \hookrightarrow \mathrm{GL}_2(k_\infty) \times \mathrm{GL}_2(k_\infty),$$

providing an action of $\mathrm{GL}_2(F)$ on $\mathfrak{H}_F := \mathfrak{H} \times \mathfrak{H}$. Let O_F be the integral closure of A in F. Given a monic $\mathfrak{n} \in A$, put

$$\Gamma_{0,F}(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_F) \;\middle|\; ad - bc \in \mathbb{F}_q^\times \text{ and } c \equiv 0 \bmod \mathfrak{n} \right\}.$$

The *Drinfeld-Stuhler modular surface for* $\Gamma_{0,F}(\mathfrak{n})$ is

$$\mathcal{S}_{0,F}(\mathfrak{n}) := \Gamma_{0,F}(\mathfrak{n}) \backslash \mathfrak{H}_F,$$

which is a coarse moduli scheme for the so-called *Frobenius-Hecke sheaves* (with additional "level-$\mathfrak{n}$ structure") introduced by Stuhler in [34].

We are interested in the intersections between the "Hirzebruch-Zagier-type divisors" on $\mathcal{S}_{0,F}(\mathfrak{n})$ which are defined as follows. For $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(F)$, we put

$$\bar{x} := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{and} \quad x' := \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

where for every $\alpha \in F$, α' is the conjugate of α under the action of the non-trivial element in $\mathrm{Gal}(F/k)$. Consider the involution $*$ on $\mathrm{Mat}_2(F)$ defined by

$$x^* := \begin{pmatrix} 0 & 1/\mathfrak{n} \\ 1 & 0 \end{pmatrix} \bar{x}' \begin{pmatrix} 0 & 1 \\ \mathfrak{n} & 0 \end{pmatrix} = \begin{pmatrix} a' & -\mathfrak{n}^{-1}c' \\ -\mathfrak{n}b' & d' \end{pmatrix}, \quad \forall x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(F).$$

Let Λ be the following A-lattice of rank 4:

$$\begin{aligned} \Lambda &:= \{x \in \mathrm{Mat}_2(O_F) \mid x^* = x\} \\ &= \left\{ \begin{pmatrix} a & \beta \\ -\mathfrak{n}\beta' & d \end{pmatrix} \;\middle|\; a, d \in A,\ \beta \in O_F \right\}. \end{aligned}$$

We have a left action of $\Gamma := \Gamma_{0,F}(\mathfrak{n})$ on Λ by

$$\gamma \star x := \gamma x \gamma^* \cdot (\det \gamma)^{-1}, \quad \gamma \in \Gamma_{0,F}(\mathfrak{n}) \text{ and } x \in \Lambda.$$

For each x in Λ with $\det x \neq 0$, let

$$B_x := \{b \in \mathrm{Mat}_2(F) \mid xb^* = \bar{b}x\} \quad \text{and} \quad \Gamma_x := B_x^\times \cap \Gamma.$$

From Lemma 3.11, we know that B_x is an *indefinite* quaternion algebra over k (i.e. unramified at the infinite place of k), whence the quotient $\mathcal{C}_x := \Gamma_x \backslash \mathfrak{H}$ becomes the Drinfeld-Stuhler modular curve for Γ_x. Put

$$S_x := \begin{pmatrix} 0 & 1 \\ \mathfrak{n} & 0 \end{pmatrix} \bar{x}.$$

The embedding from $\mathfrak{H}$ into $\mathfrak{H}_F$ defined by $(z \mapsto (z, S_x z))$ gives rise to a (rigid analytic) morphism $f_x : \mathcal{C}_x \to \mathcal{S}_{0,F}(\mathfrak{n})$, and we set

$$\mathcal{Z}_x := f_{x,*}(\mathcal{C}_x),$$

the push-forward divisor of f_x on $S_{0,F}(\mathfrak{n})$. For non-zero $a \in A$, the *Hirzebruch-Zagier divisor of discriminant* a is:

$$\mathcal{Z}(a) := \sum_{x \in \Gamma \backslash \Lambda_a} \mathcal{Z}_x, \quad \text{where } \Lambda_a := \{x \in \Lambda \mid \det(x) = a\}.$$

Notice that by Lemma 3.11 we may identify B_1 with the quaternion algebra

$$\left(\frac{\mathfrak{d}, \mathfrak{n}}{k}\right) := k + k\mathbf{i} + k\mathbf{j} + k\mathbf{ij} \quad \text{with } \mathbf{i}^2 = \mathfrak{d},\ \mathbf{j}^2 = \mathfrak{n}, \text{ and } \mathbf{ji} = -\mathbf{ij}.$$

In particular, suppose that $\mathfrak{n}$ is square-free and coprime to $\mathfrak{d}$. Write $\mathfrak{n} = \mathfrak{n}^+ \cdot \mathfrak{n}^-$ and $\mathfrak{d} = \mathfrak{d}^+ \cdot \mathfrak{d}^-$, where for each prime factor $\mathfrak{p}$ of $\mathfrak{n}^\pm$ (resp. $\mathfrak{d}^\pm$) we have the Legendre quadratic symbol $\left(\frac{\mathfrak{d}}{\mathfrak{p}}\right) = \pm 1$ (resp. $\left(\frac{\mathfrak{n}}{\mathfrak{p}}\right) = \pm 1$). Then B_1 is ramified precisely at the prime factors of $\mathfrak{d}^-\mathfrak{n}^-$ and $O_{B_1} := B_1 \cap \mathrm{Mat}_2(O_F)$ is an Eichler A-order of type $(\mathfrak{d}^+\mathfrak{n}^+, \mathfrak{d}^-\mathfrak{n}^-)$ in B_1. Hence $\mathcal{C}_1$ is actually the Drinfeld-Stuhler modular curve for $\Gamma(\mathfrak{d}^+\mathfrak{n}^+, \mathfrak{d}^-\mathfrak{n}^-)$. We pick $\mathcal{Z}_1$ as our "base" divisor on $\mathcal{S}_{0,F}(\mathfrak{n})$, and determine the intersection number of $\mathcal{Z}_1$ and $\mathcal{Z}(a)$ for non-zero $a \in A$ in the following theorem:

Theorem 1.2. *Given a square-free* $\mathfrak{n} \in A_+$ *coprime to* $\mathfrak{d}$, *suppose that* $\deg(\mathfrak{d}^-\mathfrak{n}^-) > 0$. *The intersection number of* $\mathcal{Z}_1$ *and* $\mathcal{Z}(a)$ *for non-zero* $a \in A$ *is equal to*

$$\mathcal{Z}_1 \cdot \mathcal{Z}(a) = 2 \cdot \sum_{\substack{t \in A \\ t^2-4a \preceq 0}} H^{\mathfrak{d}^+\mathfrak{n}^+,\mathfrak{d}^-\mathfrak{n}^-}(\mathfrak{d}(t^2-4a)).$$

Here for $d \in A$, *we write* $d \preceq 0$ *if* $d = 0$ *or* $k(\sqrt{d})$ *is an "imaginary" quadratic extension of* k *(i.e. the infinite place of* k *does not split in* $k(\sqrt{d})$*), and* $H^{\mathfrak{d}^+\mathfrak{n}^+,\mathfrak{d}^-\mathfrak{n}^-}(d)$ *is the modified Hurwitz class number in* Definition 2.4 *and Remark* 2.6.

We point out that when $a \in A$ is a square, the intersection number $\mathcal{Z}_1 \cdot \mathcal{Z}(a)$ includes the self-intersection $\mathcal{Z}_1 \cdot \mathcal{Z}_1$, which is defined to be an analogue of the "Euler characteristic" of $\mathcal{Z}_1$ in Definition 4.8.

To establish the equality in Theorem 1.2, the main bridge is the theta integral $I(\cdot;\varphi_\Lambda)$ associated with a particular chosen Schwartz function φ_Λ, see (3.5) and (3.7) in Section 3.1. Our strategy is briefly sketched as follows. Notice that using adelic language, we may express very naturally the a-th Fourier coefficient of $I(\cdot;\varphi_\Lambda)$ for a given non-zero $a \in A$ in terms of the modified Hurwitz class numbers (cf. Theorem 3.9). On the other hand, the strong approximation theorem (for the indefinite quaternion algebra ramified precisely at the prime factors of $\mathfrak{n}^-$) leads to an alternative expression of the a-th Fourier coefficient of $I(\cdot;\varphi_\Lambda)$ (cf. Theorem 3.14), which enables us to connect the Fourier coefficient with the intersection number $\mathcal{Z}_1 \cdot \mathcal{Z}(a)$ (Theorem 4.7 and Corollary 4.10). This completes the proof.

The theta integral $I(\cdot;\varphi_\Lambda)$ has nice invariant property and transformation law (cf. Proposition 3.7). In particular, the crucial choice of the "infinite component" $\varphi_{\Lambda,\infty}$ in (3.7) is a key ingredient in bridging two sides of the equality in Theorem 1.2. More precisely, as the place ∞ of k is non-archimedean, we apply the Eichler's theory of local optimal embeddings in Appendix A and B to ensure that our choice of $\varphi_{\Lambda,\infty}$ kills all the contributions of the K_x in Lemma 3.2 when K_x is a real quadratic field (cf. the equation (3.12)). This part is completely different from the classical case. Meanwhile, the choice of $\varphi_{\Lambda,\infty}$ provides as well the "harmonicity" of $I(\cdot;\varphi_\Lambda)$ (cf. Lemma 3.16). This allows us to extend $I(\cdot;\varphi_\Lambda)$ to a "Drinfeld-type" automorphic form on $\mathrm{GL}_2(k_\infty)$ (an analogue of weight-2 modular forms over function fields, see *Remark* 3.15 and [12]) with nebentypus character $\left(\frac{\cdot}{\mathfrak{d}}\right)$ for $\Gamma_0^{(1)}(\mathfrak{d}\mathfrak{n}) := \Gamma_0(\mathfrak{d}\mathfrak{n}) \cap \mathrm{SL}_2(A)$, cf. Proposition 3.17. In other words, we have the following theorem (cf. Theorem 3.18):

Theorem 1.3. *Under the assumptions in* Theorem 1.2, *there exists a Drinfeld-type automorphic form* ϑ_Λ *on* $\mathrm{GL}_2(k_\infty)$ *with nebentypus character* $\left(\frac{\cdot}{\mathfrak{d}}\right)$ *for the congruence subgroup* $\Gamma_0^{(1)}(\mathfrak{d}\mathfrak{n})$ *whose Fourier expansion is given as follows: for*

$(x, y) \in k_\infty \times k_\infty^\times$,

$$\vartheta_\Lambda \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = |y|_\infty \cdot \left[-\mathrm{vol}(\mathcal{Z}_1) + \sum_{\substack{0 \neq a \in A \\ \deg a + 2 \leq \mathrm{ord}_\infty(y)}} \big(\mathcal{Z}_1 \cdot \mathcal{Z}(a)\big) \psi_\infty(ax) \right].$$

Here:

- $|\cdot|_\infty$ *is the absolute value on* k_∞ *normalized so that* $|\theta|_\infty = q$*,*
- $\psi_\infty : k_\infty \to \mathbb{C}^\times$ *is a fixed additive character on* k_∞ *defined in* Section 2.1.1*,*
- $\mathrm{vol}(\mathcal{Z}_1) := -2H^{\mathfrak{d}^+\mathfrak{n}^+, \mathfrak{d}^-\mathfrak{n}^-}(0)$ *(cf. Remark* 2.6*).*

Remark 1.4.

(1) The theory of the geometric interpretation of the Fourier coefficients of automorphic forms as the corresponding intersection numbers are developed quite general over number fields (cf. [24] and [9]). One may expect a similar phenomenon occurs in the positive characteristic world, however, there are many technical issues needed to be carried out. Since this work is the first attempt to study the connection between class number relations and intersection numbers via this approach over the function field side, we include all the details for the sake of completeness.

(2) The technical assumption "$\deg(\mathfrak{d}^-\mathfrak{n}^-) > 0$" in Theorem 1.2 implies that B_1 is a division algebra, whence the Drinfeld-Stuhler modular curve $\mathcal{C}_1$ has no "cusps". Therefore there are no contributions of the "cuspidal intersections" to $\mathcal{Z}_1 \cdot \mathcal{Z}(a)$ in Theorem 1.2 and Theorem 1.3. When $\mathfrak{d}^-\mathfrak{n}^- = 1$, this argument would need to be adjusted by "regularizing the theta integral $I(\cdot; \varphi_\Lambda)$" as in [9], and the cuspidal intersections for $\mathcal{Z}_1 \cdot \mathcal{Z}(a)$ in a suitable "compactification of the surface $\mathcal{S}_{0,F}(\mathfrak{n})$" should be taken into account. However, due to a lack of studies in the literature for these two technical issues in the function field context, we make this assumption in Theorem 1.2 first. The general case will be explored in future work.

1.4 Content

The contents of this paper go as follows:

- (*Preliminaries.*) In Section 2.1, we set up basic notations used throughout this paper. The modified Hurwitz class number and the needed properties are reviewed in Section 2.2. The Tamagawa measures on the groups appearing in this paper are given in Section 2.1.1, 2.2, and 2.3, respectively. The definition of the Weil representation and theta series are recalled in Section 2.4.

- (*Fourier coefficients of theta series.*) In Section 3, we take a particular Schwatz function φ_Λ associated with the A-lattice Λ, and express the Fourier coefficients of the theta integral $I(\cdot;\varphi_\Lambda)$ explicitly in terms of the modified Hurwitz class numbers in Theorem 3.9. In Section 3.4, we show the harmonicity of $I(\cdot;\varphi_\Lambda)$ and extend it to a Drinfeld-type automorphic form ϑ_Λ on $\mathrm{GL}_2(k_\infty)$ in Proposition 3.17.

- (*Class number relations and intersections.*) In Section 4, we first introduce the Hirzebruch-Zagier-type divisors on the Drinfeld-Stuhler modular surfaces. Pulling back these divisors in the "fine coverings" of the surfaces, the projection formula in Proposition 4.4 enables us to interpret the intersection number $\mathcal{Z}_1 \cdot \mathcal{Z}_x$ as a a "double-coset summation" in Theorem 4.7 and Lemma 4.9. Together with the alternative expression of the Fourier coefficients of $I(\cdot;\varphi_\Lambda)$ in Theorem 3.14, we prove Theorem 1.2 and Theorem 1.3 in the end.

- (*Appendix: local optimal embeddings.*) The needed results in Eichler's theory of local optimal embeddings are recalled in Appendix A, and we express the technical local integrals used in Theorem 3.9 by the number of local optimal embeddings in Appendix B.

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2 Preliminaries

2.1 Basic settings

Let $\mathbb{F}_q$ be a finite field with q elements. Throughout this paper, we always assume q to be odd. Let $A := \mathbb{F}_q[\theta]$, the polynomial ring with one variable θ over $\mathbb{F}_q$, and $k := \mathbb{F}_q(\theta)$, the field of fractions of A. Let ∞ be the infinite place of k, i.e. the place corresponding to the "degree" valuation ord_∞ defined by

$$\mathrm{ord}_\infty\left(\frac{a}{b}\right) := \deg b - \deg a, \quad \forall a, b \in A \text{ with } b \neq 0.$$

The associated absolute value on k is normalized by $|\alpha|_\infty := q^{-\operatorname{ord}_\infty(\alpha)}$ for every $\alpha \in k$. Let k_∞ be the completion of k with respect to $|\cdot|_\infty$, which can be identified with the Laurent series field $\mathbb{F}_q(\!(\theta^{-1})\!)$. Put $\varpi := \theta^{-1}$, a fixed uniformizer at ∞, and $O_\infty := \mathbb{F}_q[\![\varpi]\!]$, the valuation ring in k_∞.
Let A_+ be the set of monic polynomials in A. By abuse of notations, we identify A_+ with the set of non-zero ideals of A. In particular, for $\mathfrak{a} \in A_+$ we put

$$\|\mathfrak{a}\| \;:=\; \#(A/\mathfrak{a}) \quad (= |\mathfrak{a}|_\infty).$$

Given a non-zero prime ideal $\mathfrak{p}$ of A, the normalized absolute value associated with $\mathfrak{p}$ is:

$$|\alpha|_\mathfrak{p} := \|\mathfrak{p}\|^{-\operatorname{ord}_\mathfrak{p}(\alpha)}, \quad \forall \alpha \in k.$$

Here $\operatorname{ord}_\mathfrak{p}(\alpha)$ is the order of α at $\mathfrak{p}$ for every $\alpha \in k$. The completion of k with respect to $|\cdot|_\mathfrak{p}$ is denoted by $k_\mathfrak{p}$, and put $O_\mathfrak{p}$ the valuation ring in $k_\mathfrak{p}$. We also refer the non-zero prime ideals of A to the finite places of k.
Let $k_\mathbb{A} := \prod'_v k_v$, the adele ring of k. The maximal compact subring of $k_\mathbb{A}$ is denoted by $O_\mathbb{A}$. The adelic norm $|\cdot|_\mathbb{A}$ on the idele group $k_\mathbb{A}^\times$ is:

$$|(\alpha_v)_v|_\mathbb{A} := \prod_v |\alpha_v|_v, \quad \forall (\alpha_v)_v \in k_\mathbb{A}^\times.$$

2.1.1 Additive character and Tamagawa measure

Let p be the characteristic of k and $\psi_\infty : k_\infty \to \mathbb{C}^\times$ be the additive character defined by: for $\sum_i a_i \varpi^i \in k_\infty$,

$$\psi_\infty\left(\sum_i a_i \varpi^i\right) := \exp\left(\frac{2\pi\sqrt{-1}}{p} \cdot \operatorname{Trace}_{\mathbb{F}_q/\mathbb{F}_p}(-a_1)\right).$$

The conductor of ψ_∞ is $\varpi^2 O_\infty$ and $\psi_\infty(A) = 1$. Since

$$k_\mathbb{A} = k + \left(k_\infty \times \prod_\mathfrak{p} O_\mathfrak{p}\right) \quad \text{and} \quad k \cap \left(\prod_\mathfrak{p} O_\mathfrak{p}\right) = A,$$

we may extend ψ_∞ uniquely to an additive character $\psi : k_\mathbb{A} \to \mathbb{C}^\times$ so that $\psi(\alpha) = 1$ for all $\alpha \in k + \big((\varpi^2 O_\infty) \times \prod_\mathfrak{p} O_\mathfrak{p}\big)$ and $\psi\big|_{k_\infty} = \psi_\infty$. Put $\psi_\mathfrak{p} := \psi\big|_{k_\mathfrak{p}}$ for each finite place $\mathfrak{p}$ of k, which is a non-trivial additive character on $k_\mathfrak{p}$ with trivial conductor.
For each place v of k, let dx_v be the "self-dual" Haar measure on k_v with respect to ψ_v, i.e.

$$\operatorname{vol}(O_\mathfrak{p}, dx_\mathfrak{p}) = 1 \text{ for each finite place } \mathfrak{p} \text{ of } k, \quad \text{and} \quad \operatorname{vol}(O_\infty, dx_\infty) = q.$$

Define the Haar measure $d^\times x_v$ on $k_v^\times$ by

$$d^\times x_\mathfrak{p} := \frac{\|\mathfrak{p}\|}{\|\mathfrak{p}\| - 1} \cdot \frac{dx_\mathfrak{p}}{|x_\mathfrak{p}|_\mathfrak{p}} \quad \text{and} \quad d^\times x_\infty := \frac{q}{q-1} \cdot \frac{dx_\infty}{|x_\infty|_\infty}.$$

The *Tamagawa measure* on $k_\mathbb{A}^\times$ (with respect to ψ) is $d^\times x = \prod_v d^\times x_v$.

2.2 Imaginary quadratic fields and class numbers

A quadratic field extension K/k is called *imaginary* if the infinite place of k does not split in K. Let $K_{\mathbb{A}} := K \otimes_k k_{\mathbb{A}}$ and $\mathrm{T}_{K/k} := K_{\mathbb{A}} \to k_{\mathbb{A}}$ be the trace map induced by the field trace map. Then the Tamagawa measure on $K_{\mathbb{A}}^\times$ (with respect to the additive character $\psi \circ \mathrm{T}_{K/k}$) and the one on $k_{\mathbb{A}}^\times$ induce a Haar measure $d^\times \alpha$ on the quotient group $K_{\mathbb{A}}^\times / k_{\mathbb{A}}^\times$. More precisely, let O_K (resp. O_{K_∞}) be the integral closure of A (resp. O_∞) in K (resp. $K_\infty := K \otimes_k k_\infty$). For each non-zero prime ideal $\mathfrak{p}$ of A, put $K_{\mathfrak{p}} := K \otimes_k k_{\mathfrak{p}}$ and $O_{K_{\mathfrak{p}}} := O_K \otimes_A O_{\mathfrak{p}}$. We normalize the Haar measure $d^\times \alpha_v$ on $K_v^\times / k_v^\times$ for each place of v by

$$\mathrm{vol}(O_{K_{\mathfrak{p}}}^\times / O_{\mathfrak{p}}^\times) = \|\mathfrak{p}\|^{-1+1/e_{\mathfrak{p}}(K/k)} \quad \text{and} \quad \mathrm{vol}(O_{K_\infty}^\times / O_\infty^\times) = q^{1/e_\infty(K/k)}. \tag{2.1}$$

Here $e_v(K/k)$ is the ramification index of the place v of k in K. Then $d^\times \alpha = \prod_v d^\times \alpha_v$.

Proposition 2.1. *Let K be an imaginary quadratic field over k, and O_K be the integral closure of A in K. Let $\Delta(O_K/A)$ be the discriminant ideal of O_K over A, $h(O_K)$ be the class number of O_K, and put $w(O_K) := \#(O_K^\times)/(q-1)$. We have*

$$\begin{aligned}\mathrm{vol}(K^\times \backslash K_{\mathbb{A}}^\times / k_{\mathbb{A}}^\times) &= \frac{h(O_K)}{w(O_K)} \cdot e_\infty(K/k) \cdot \prod_v \mathrm{vol}(O_{K_v}^\times / O_v^\times) \\ &= \frac{h(O_K)}{w(O_K)} \cdot e_\infty(K/k) \cdot q^{1/e_\infty(K/k)} \cdot \|\Delta(O_K/A)\|^{-1/2}.\end{aligned}$$

Proof. As A is a principal ideal domain, one gets

$$k_{\mathbb{A}}^\times = k^\times \cdot (k_\infty^\times \times \prod_{\mathfrak{p}} O_{\mathfrak{p}}^\times).$$

Thus the exact sequence

$$1 \longrightarrow \frac{O_K^\times}{\mathbb{F}_q^\times} \longrightarrow \frac{K_\infty^\times \times \prod_{\mathfrak{p}} O_{K_{\mathfrak{p}}}^\times}{k_\infty^\times \times \prod_{\mathfrak{p}} O_{\mathfrak{p}}^\times} \to \frac{K^\times \cdot (K_\infty^\times \times \prod_{\mathfrak{p}} O_{K_{\mathfrak{p}}}^\times)}{K^\times \cdot (k_\infty^\times \times \prod_{\mathfrak{p}} O_{\mathfrak{p}}^\times)} \longrightarrow 1$$

implies

$$\begin{aligned}\mathrm{vol}(K^\times \backslash K_{\mathbb{A}}^\times / k_{\mathbb{A}}^\times) &= \mathrm{vol}\Big(K^\times \backslash K_{\mathbb{A}}^\times / (k_\infty^\times \times \prod_{\mathfrak{p}} O_{\mathfrak{p}}^\times)\Big) \\ &= \frac{\#\big(K^\times \backslash K_{\mathbb{A}}^\times / (K_\infty^\times \times \prod_{\mathfrak{p}} O_{K_{\mathfrak{p}}}^\times)\big)}{\#(O_K^\times / \mathbb{F}_q^\times)} \cdot \mathrm{vol}\left(\frac{K_\infty^\times \times \prod_{\mathfrak{p}} O_{K_{\mathfrak{p}}}^\times}{k_\infty^\times \times \prod_{\mathfrak{p}} O_{\mathfrak{p}}^\times}\right).\end{aligned}$$

The result then follows from

$$\frac{\#\big(K^\times \backslash K_{\mathbb{A}}^\times / (K_\infty^\times \times \prod_{\mathfrak{p}} O_{K_{\mathfrak{p}}}^\times)\big)}{\#(O_K^\times / \mathbb{F}_q^\times)} = \frac{h(O_K)}{w(O_K)}$$

and

$$\operatorname{vol}\left(\frac{K_\infty^\times \times \prod_{\mathfrak{p}} O_{K_\mathfrak{p}}^\times}{k_\infty^\times \times \prod_{\mathfrak{p}} O_\mathfrak{p}^\times}\right) = \operatorname{vol}(K_\infty^\times/k_\infty^\times) \cdot \prod_{\mathfrak{p}} \operatorname{vol}(O_{K_\mathfrak{p}}^\times/O_\mathfrak{p}^\times)$$
$$= e_\infty(K/k) \cdot q^{1/e_\infty(K/k)} \cdot \|\Delta(O_K/A)\|^{-1/2}.$$

The last equality follows from (2.1) and

$$\operatorname{vol}(K_\infty^\times/k_\infty^\times) = e_\infty(K/k) \cdot \operatorname{vol}(O_{K\infty}^\times/O_\infty^\times), \quad \Delta(O_K/A) = \prod_{\substack{\text{prime } \mathfrak{p} \triangleleft A \\ \text{ramified in K}}} \mathfrak{p}.$$

□

Remark 2.2. Let $\varsigma_K : k^\times \backslash k_\mathbb{A}^\times \to \{\pm 1\}$ be the quadratic Hecke character associated with K/k, and let $L(s, \varsigma_K)$ be the L-function of ς_K. It is known that (cf. [4, Section 2.2], see also [31, Theorem 5.9])

$$L(1, \varsigma_K) = \frac{\#(O_K^\times)}{\#(\mathbb{F}_q^\times)} \cdot q \cdot \left(q^{(1-e_\infty(K/k))/2} \cdot \|\Delta(O_K/A)\|^{-1/2}\right) \cdot \frac{h(O_K)}{2/e_\infty(K/k)}.$$

The above proposition says in particular that

$$\operatorname{vol}(K^\times \backslash K_\mathbb{A}^\times / k_\mathbb{A}^\times) = 2 \cdot L(1, \varsigma_K).$$

Recall the following fact (cf. [28, section I (12.12) Theorem]):

Lemma 2.3. *For each A-order $\mathcal{O}$ in an imaginary quadratic extension K of k, let $h(\mathcal{O})$ be the proper ideal class number of $\mathcal{O}$ and $w(\mathcal{O}) := \#(\mathcal{O}^\times)/(q-1)$. Then*

$$\frac{h(\mathcal{O})}{w(\mathcal{O})} = \frac{h(O_K)}{w(O_K)} \cdot \prod_{\mathfrak{p}} \# \left(\frac{O_{K_\mathfrak{p}}^\times}{\mathcal{O}_\mathfrak{p}^\times}\right).$$

Here $\mathcal{O}_\mathfrak{p} := \mathcal{O} \otimes_A A_\mathfrak{p}$ for every non-zero prime ideal $\mathfrak{p}$ of A.

For $d \in A$, we write $d \prec 0$ if the quadratic extension $k(\sqrt{d})$ is imaginary over k. Given $d \in A$ with $d \prec 0$, denote by $\mathcal{O}_d := A[\sqrt{d}]$, $h(d) := h(\mathcal{O}_d)$, and $w(d) := w(\mathcal{O}_d)$.

Definition 2.4. For square-free $\mathfrak{n}^+, \mathfrak{n}^- \in A_+$ with $\gcd(\mathfrak{n}^+, \mathfrak{n}^-) = 1$, recall the following *modified Hurwitz class number*

$$H^{\mathfrak{n}^+,\mathfrak{n}^-}(d) := \sum_{\substack{\mathfrak{c} \in A_+ \\ \mathfrak{c}^2 \mid d}} \frac{h(d/\mathfrak{c}^2)}{w(d/\mathfrak{c}^2)} \cdot \prod_{\mathfrak{p} \mid \mathfrak{n}^+} \left(1 + \left\{\frac{d/\mathfrak{c}^2}{\mathfrak{p}}\right\}\right) \prod_{\mathfrak{p} \mid \mathfrak{n}^-} \left(1 - \left\{\frac{d/\mathfrak{c}^2}{\mathfrak{p}}\right\}\right).$$

Here

$$\left\{\frac{d}{\mathfrak{p}}\right\} := \begin{cases} 1, & \text{if either } \mathfrak{p} \text{ split in } k(\sqrt{d}) \text{ or } \mathfrak{p}^2 \mid d; \\ -1, & \text{if } \mathfrak{p} \text{ is inert in } k(\sqrt{d}) \text{ and } \operatorname{ord}_\mathfrak{p}(d) = 0; \\ 0, & \text{if } \operatorname{ord}_\mathfrak{p}(d) = 1. \end{cases}$$

Write

$$d = d_0 \cdot \prod_{\mathfrak{p}} \mathfrak{p}^{2c_{\mathfrak{p}}},$$

where $d_0 \in A$ is square-free (and $c_{\mathfrak{p}} = 0$ for almost all irreducible $\mathfrak{p} \in A_+$). For each irreducible $\mathfrak{p} \in A_+$ and integer $\ell_{\mathfrak{p}}$ with $0 \leq \ell_{\mathfrak{p}} \leq c_{\mathfrak{p}}$, put

$$e_{\mathfrak{p}}^{\mathfrak{n}^+,\mathfrak{n}^-}(\ell_{\mathfrak{p}}) := \begin{cases} 1 \pm \left\{\dfrac{d_0 \mathfrak{p}^{2\ell_{\mathfrak{p}}}}{\mathfrak{p}}\right\}, & \text{if } \mathfrak{p} \mid \mathfrak{n}^{\pm}; \\ 1, & \text{otherwise.} \end{cases} \tag{2.2}$$

We provide the following expression for the modified Hurwitz class numbers in later use:

Proposition 2.5. *Given $d \in A$ with $d \prec 0$, write $d = d_0 \prod_{\mathfrak{p}} \mathfrak{p}^{2c_{\mathfrak{p}}}$. Then*

$$H^{\mathfrak{n}^+,\mathfrak{n}^-}(d) = \frac{h(d_0)}{w(d_0)} \cdot \prod_{\mathfrak{p}} \left[\sum_{0 \leq \ell_{\mathfrak{p}} \leq c_{\mathfrak{p}}} \# \left(\frac{\mathcal{O}_{d_0,\mathfrak{p}}^{\times}}{\mathcal{O}_{d_0\mathfrak{p}^{2\ell_{\mathfrak{p}}},\mathfrak{p}}^{\times}} \right) \cdot e_{\mathfrak{p}}^{\mathfrak{n}^+,\mathfrak{n}^-}(\ell_{\mathfrak{p}}) \right].$$

Proof. For $\boldsymbol{\ell} = (\ell_{\mathfrak{p}})_{\mathfrak{p}} \in \prod_{\mathfrak{p}} \mathbb{Z}$ with $0 \leq \ell_{\mathfrak{p}} \leq c_{\mathfrak{p}}$, put

$$d_0(\boldsymbol{\ell}) := d_0 \prod_{\mathfrak{p}} \mathfrak{p}^{2\ell_{\mathfrak{p}}}.$$

Then

$$\left\{\frac{d_0(\boldsymbol{\ell})}{\mathfrak{p}}\right\} = \left\{\frac{d_0\mathfrak{p}^{2\ell_{\mathfrak{p}}}}{\mathfrak{p}}\right\} \quad \text{and} \quad \mathcal{O}_{d_0(\boldsymbol{\ell}),\mathfrak{p}} = \mathcal{O}_{d_0\mathfrak{p}^{2\ell_{\mathfrak{p}}},\mathfrak{p}}.$$

Therefore

$$\begin{aligned} H^{\mathfrak{n}^+,\mathfrak{n}^-}(d) &= \sum_{\substack{\boldsymbol{\ell} \in \prod_{\mathfrak{p}} \mathbb{Z} \\ 0 \leq \ell_{\mathfrak{p}} \leq c_{\mathfrak{p}}}} \frac{h(d(\boldsymbol{\ell}))}{w(d(\boldsymbol{\ell}))} \cdot \prod_{\mathfrak{p} \mid \mathfrak{n}^+} \left(1 + \left\{\frac{d(\boldsymbol{\ell})}{\mathfrak{p}}\right\}\right) \cdot \prod_{\mathfrak{p} \mid \mathfrak{n}^-} \left(1 - \left\{\frac{d(\boldsymbol{\ell})}{\mathfrak{p}}\right\}\right) \\ &= \frac{h(d_0)}{w(d_0)} \cdot \sum_{\substack{\boldsymbol{\ell} \in \prod_{\mathfrak{p}} \mathbb{Z} \\ 0 \leq \ell_{\mathfrak{p}} \leq c_{\mathfrak{p}}}} \left[\prod_{\mathfrak{p}} \# \left(\frac{\mathcal{O}_{d_0,\mathfrak{p}}^{\times}}{\mathcal{O}_{d_0\mathfrak{p}^{2\ell_{\mathfrak{p}}},\mathfrak{p}}^{\times}} \right) \cdot \prod_{\mathfrak{p}} e_{\mathfrak{p}}^{\mathfrak{n}^+,\mathfrak{n}^-}(\ell_{\mathfrak{p}}) \right] \\ &= \frac{h(d_0)}{w(d_0)} \cdot \prod_{\mathfrak{p}} \left[\sum_{0 \leq \ell_{\mathfrak{p}} \leq c_{\mathfrak{p}}} \# \left(\frac{\mathcal{O}_{d_0,\mathfrak{p}}^{\times}}{\mathcal{O}_{d_0\mathfrak{p}^{2\ell_{\mathfrak{p}}},\mathfrak{p}}^{\times}} \right) \cdot e_{\mathfrak{p}}^{\mathfrak{n}^+,\mathfrak{n}^-}(\ell_{\mathfrak{p}}) \right]. \end{aligned}$$

□

Remark 2.6. For convention, we put

$$H^{\mathfrak{n}^+,\mathfrak{n}^-}(0) := -\frac{1}{q^2-1} \cdot \prod_{\mathfrak{p} \mid \mathfrak{n}^+} (\|\mathfrak{p}\| + 1) \prod_{\mathfrak{p} \mid \mathfrak{n}^-} (\|\mathfrak{p}\| - 1).$$

This number is related to a volume quantity with respect to the "Tamagawa measure" on quaternion algebras in the next subsection.

2.3 Tamagawa measure on quaternion algebras

Let B be an *indefinite* quaternion algebra over k (i.e. $B_\infty := B \otimes_k k_\infty$ is not division). Put $B_{\mathbb{A}} := B \otimes_k k_{\mathbb{A}}$. Let $\mathrm{Tr} : B_{\mathbb{A}} \to k_{\mathbb{A}}$ be the reduced trace map. Choose a Haar measure $db = \prod_v db_v$ on $B_{\mathbb{A}}$ which is self-dual with respect to the additive character $\psi \circ \mathrm{Tr}$. More precisely, for each non-zero prime ideal $\mathfrak{p}$ of A, let $R_{\mathfrak{p}}$ be a maximal $O_{\mathfrak{p}}$-order in $B_{\mathfrak{p}} := B \otimes_k k_{\mathfrak{p}}$. Then

$$\mathrm{vol}(R_{\mathfrak{p}}, db_{\mathfrak{p}}) = \begin{cases} 1/\|\mathfrak{p}\|, & \text{if } B \text{ is ramified at } \mathfrak{p}; \\ 1, & \text{otherwise.} \end{cases}$$

Let O_{B_∞} be a maximal O_∞-order in B_∞. Then $\mathrm{vol}(O_{B_\infty}, db_\infty) = q^4$.
Let $\mathrm{Nr} : B_{\mathbb{A}}^\times \to k_{\mathbb{A}}^\times$ be the reduced norm map. For each non-zero prime ideal $\mathfrak{p}$ of A, we take the Haar measure $d^\times b_{\mathfrak{p}}$ on $B_{\mathfrak{p}}^\times$ defined by

$$d^\times b_{\mathfrak{p}} := \frac{\|\mathfrak{p}\|}{\|\mathfrak{p}\| - 1} \cdot \frac{db_{\mathfrak{p}}}{|\mathrm{Nr}(b_{\mathfrak{p}})|_{\mathfrak{p}}}.$$

In particular, the following lemma holds:

Lemma 2.7.

$$\mathrm{vol}(R_{\mathfrak{p}}^\times, d^\times b_{\mathfrak{p}}) = \left(1 - \frac{1}{\|\mathfrak{p}\|^2}\right) \cdot \begin{cases} 1/(\|\mathfrak{p}\| - 1), & \textit{if } B \textit{ is ramified at } \mathfrak{p}; \\ 1, & \textit{otherwise.} \end{cases}$$

Proof. Suppose $\mathfrak{p}$ is ramified in B, we may take $\tilde{\pi}_{\mathfrak{p}} \in R_{\mathfrak{p}}$ so that $\tilde{\pi}_{\mathfrak{p}}$ is a maximal two-sided ideal of $R_{\mathfrak{p}}$, and $R_{\mathfrak{p}}/\tilde{\pi}_{\mathfrak{p}} R_{\mathfrak{p}}$ is a quadratic field extension of $\mathbb{F}_{\mathfrak{p}}$. Hence

$$\begin{aligned} \mathrm{vol}(R_{\mathfrak{p}}^\times, d^\times b_{\mathfrak{p}}) &= (\|\mathfrak{p}\|^2 - 1) \cdot \mathrm{vol}(1 + \tilde{\pi}_{\mathfrak{p}} R_{\mathfrak{p}}, d^\times b_{\mathfrak{p}}) \\ &= (\|\mathfrak{p}\|^2 - 1) \cdot \frac{\|\mathfrak{p}\|}{\|\mathfrak{p}\| - 1} \cdot \mathrm{vol}(\tilde{\pi}_{\mathfrak{p}} R_{\mathfrak{p}}, db_{\mathfrak{p}}) \\ &= (\|\mathfrak{p}\|^2 - 1) \cdot \frac{\|\mathfrak{p}\|}{\|\mathfrak{p}\| - 1} \cdot \|\mathfrak{p}\|^{-3} \\ &= \left(1 - \frac{1}{\|\mathfrak{p}\|^2}\right) \cdot \frac{1}{\|\mathfrak{p}\| - 1}. \end{aligned}$$

When $\mathfrak{p}$ is unramified in B, we may identify $B_{\mathfrak{p}}$ with $\mathrm{Mat}_2(k_{\mathfrak{p}})$ and $R_{\mathfrak{p}}$ with $\mathrm{Mat}_2(O_{\mathfrak{p}})$. In particular, one has

$$R_{\mathfrak{p}}^\times / (1 + \mathfrak{p} R_{\mathfrak{p}}) \cong \mathrm{GL}_2(\mathbb{F}_{\mathfrak{p}}).$$

Therefore

$$\begin{aligned}\operatorname{vol}(R_{\mathfrak{p}}^{\times}, d^{\times}b_{\mathfrak{p}}) &= (\|\mathfrak{p}\|^2-1)(\|\mathfrak{p}\|^2-\|\mathfrak{p}\|)\cdot \operatorname{vol}(1+\mathfrak{p}R_{\mathfrak{p}}, d^{\times}b_{\mathfrak{p}})\\ &= (\|\mathfrak{p}\|^2-1)(\|\mathfrak{p}\|^2-\|\mathfrak{p}\|)\cdot \frac{\|\mathfrak{p}\|}{\|\mathfrak{p}\|-1}\cdot \operatorname{vol}(\mathfrak{p}R_{\mathfrak{p}}, db_{\mathfrak{p}})\\ &= (\|\mathfrak{p}\|^2-1)(\|\mathfrak{p}\|^2-\|\mathfrak{p}\|)\cdot \frac{\|\mathfrak{p}\|}{\|\mathfrak{p}\|-1}\cdot \|\mathfrak{p}\|^{-4}\\ &= 1-\frac{1}{\|\mathfrak{p}\|^2}.\end{aligned}$$

□

Similarly, put

$$d^{\times}b_{\infty} := \frac{q}{q-1}\cdot\frac{db_{\infty}}{|\operatorname{Nr}(b_{\infty})|_{\infty}}.$$

Then following the same argument in the above lemma, we get

$$\operatorname{vol}(O_{B_{\infty}}^{\times}, d^{\times}b_{\infty}) = q^4-q^2.$$

The *Tamagawa measure* $d^{\times}b$ on $B_{\mathbb{A}}^{\times}$ is the Haar measure satisfying that for every compact open subgroup $\mathcal{K}=\prod_v \mathcal{K}_v$ of $B_{\mathbb{A}}^{\times}$, one has

$$\operatorname{vol}(\mathcal{K}, d^{\times}b) = \prod_v \operatorname{vol}(\mathcal{K}_v, d^{\times}b_v).$$

Let $\mathfrak{n}^- \in A_+$ be the product of the primes at which B is ramified and $\mathfrak{n}^+ \in A_+$ be a square-free polynomial coprime to $\mathfrak{n}^-$. Let O_B be an Eichler A-order of type $(\mathfrak{n}^+,\mathfrak{n}^-)$ in B, i.e. O_B is an A-order in B satisfying that for each non-zero prime $\mathfrak{p}$ of A, $O_{B_{\mathfrak{p}}} := O_B\otimes_A O_{\mathfrak{p}}$ is the unique maximal $O_{\mathfrak{p}}$-order in $B_{\mathfrak{p}}$ if $\mathfrak{p}\mid\mathfrak{n}^-$; and

$$O_{B_{\mathfrak{p}}} \cong \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \operatorname{Mat}_2(O_{\mathfrak{p}}) \,\middle|\, c\equiv 0 \bmod \mathfrak{p}^{\operatorname{ord}_{\mathfrak{p}}(\mathfrak{n}^+)}\right\} \quad \text{if } \mathfrak{p}\nmid\mathfrak{n}^-.$$

Let $O_{B_{\mathbb{A}}} := \prod_v O_{B_v}$. Then:

LEMMA 2.8. *The Tamagawa measures on $B_{\mathbb{A}}^{\times}$ and $k_{\mathbb{A}}^{\times}$ induces a Haar measure on $B_{\mathbb{A}}^{\times}/k_{\mathbb{A}}^{\times}$ so that*

$$\operatorname{vol}(O_{B_{\mathbb{A}}}^{\times}/O_{\mathbb{A}}^{\times}) = \frac{(q-1)(q^2-1)}{\prod_{\mathfrak{p}\mid\mathfrak{n}^+}(\|\mathfrak{p}\|+1)\prod_{\mathfrak{p}\mid\mathfrak{n}^-}(\|\mathfrak{p}\|-1)} = -\frac{q-1}{H^{\mathfrak{n}^+,\mathfrak{n}^-}(0)}.$$

Proof. For each non-zero prime ideal $\mathfrak{p}$ of A, let $R_{\mathfrak{p}}$ be a maximal $O_{\mathfrak{p}}$-order containing $O_{B_{\mathfrak{p}}}$. As $\mathfrak{n}^+$ is square-free, we have

$$\#(R_{\mathfrak{p}}^{\times}/O_{B_{\mathfrak{p}}}^{\times}) = \begin{cases}\|\mathfrak{p}\|+1, & \text{if } \mathfrak{p}\mid\mathfrak{n}^+;\\ 1, & \text{otherwise.}\end{cases}$$

Thus

$$\mathrm{vol}(O_{B_{\mathfrak{p}}}^{\times}) = \left(1 - \frac{1}{\|\mathfrak{p}\|^2}\right) \cdot \begin{cases} 1/(\|\mathfrak{p}\|-1), & \text{if } \mathfrak{p} \mid \mathfrak{n}^-; \\ 1/(\|\mathfrak{p}\|+1), & \text{if } \mathfrak{p} \mid \mathfrak{n}^+; \\ 1, & \text{otherwise.} \end{cases}$$

Notice that

$$\prod_{\mathfrak{p}} \left(1 - \frac{1}{\|\mathfrak{p}\|^s}\right)^{-1} = \frac{1}{1-q^{1-s}}, \quad \mathrm{Re}(s) > 1.$$

Therefore we obtain

$$\prod_{\mathfrak{p}} \left(1 - \frac{1}{\|\mathfrak{p}\|^2}\right) = 1 - q^{1-2} = \frac{q-1}{q},$$

and

$$\begin{aligned} \mathrm{vol}(O_{B_{\mathbb{A}}}^{\times}/O_{\mathbb{A}}^{\times}) &= \frac{\mathrm{vol}(O_{B_\infty}^{\times})}{\mathrm{vol}(O_\infty^{\times})} \cdot \prod_{\mathfrak{p}} \frac{\mathrm{vol}(O_{B_{\mathfrak{p}}}^{\times})}{\mathrm{vol}(O_{\mathfrak{p}}^{\times})} \\ &= \frac{q^4-q^2}{q} \cdot \prod_{\mathfrak{p}} \left(1 - \frac{1}{\|\mathfrak{p}\|^2}\right) \prod_{\mathfrak{p}\mid\mathfrak{n}^+} \left(\frac{1}{\|\mathfrak{p}\|+1}\right) \prod_{\mathfrak{p}\mid\mathfrak{n}^-} \left(\frac{1}{\|\mathfrak{p}\|-1}\right) \\ &= \frac{(q-1)(q^2-1)}{\prod_{\mathfrak{p}\mid\mathfrak{n}^+}(\|\mathfrak{p}\|+1)\prod_{\mathfrak{p}\mid\mathfrak{n}^-}(\|\mathfrak{p}\|-1)} \\ &= -\frac{q-1}{H^{\mathfrak{n}^+,\mathfrak{n}^-}(0)}. \end{aligned}$$

The last equality follows directly from the definition of $H^{\mathfrak{n}^+,\mathfrak{n}^-}(0)$ in *Remark* 2.6. ☐

Remark 2.9. The Haar measure on $B_{\mathbb{A}}^{\times}/k_{\mathbb{A}}^{\times}$ induced by the Tamagawa measures on $B_{\mathbb{A}}^{\times}$ and on $k_{\mathbb{A}}^{\times}$ satisfies (cf. [40, Theorem 3.3.1])

$$\mathrm{vol}(B^{\times}\backslash B_{\mathbb{A}}^{\times}/k_{\mathbb{A}}^{\times}) = 2.$$

2.4 Weil representation and theta series

Let (V, Q_V) be a non-degenerat quadratic space over k, and suppose that $n := \dim_k(V)$ is even. For each place v of k, let $V(k_v) := V \otimes_k k_v$ and $S(V(k_v))$ be the space of Schwartz function on $V(k_v)$.

Definition 2.10. The *Weil representation* $\omega_{V,v}$ of $\mathrm{SL}_2(k_v) \times \mathrm{O}(V)(k_v)$ on $S(V(k_v))$, where $\mathrm{O}(V)$ is the orthogonal group of (V, Q_V), is given by (cf.

[13, Theorem 2.22]): for $\phi \in S(V(k_v))$,

(1) $\omega_{V,v}(h)\phi(x) = \phi(h^{-1}x), \quad h \in \mathrm{O}(V)(k_v);$

(2) $\omega_{V,v}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\phi(x) = \psi_v(uQ_V(x)) \cdot \phi(x), \quad u \in k_v;$

(3) $\omega_{V,v}\begin{pmatrix} a_v & 0 \\ 0 & a_v^{-1} \end{pmatrix}\phi(x) = |a_v|_v^{\frac{n}{2}} \cdot \chi_{V,v}(a_v) \cdot \phi(a_v x), \quad a_v \in k_v^\times;$

(4) $\omega_{V,v}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\phi(x) = \varepsilon_v(V) \cdot \widehat{\phi}(x).$

Here:

- $\chi_{V,v} := (\cdot, (-1)^{n/2} \det V)_v$ is the quadratic character associated with V, where $(\cdot,\cdot)_v$ is the Hilbert quadratic symbol,

  $$\det V := \det(\langle x_i, x_j\rangle_{1 \leq i,j \leq n}) \in k^\times/(k^\times)^2 \qquad ((k^\times)^2 := \{a^2 \mid a \in k^\times\})$$

  for any basis $\{x_1, ..., x_n\}$ of V and $\langle \cdot,\cdot\rangle_V$ is the bilinear form on V associated with Q_V;

- $\varepsilon_v(V)$ is the following *Weil index*:

  $$\varepsilon_v(V) := \int_{L_v} \psi_v(Q_V(x))\, dx,$$

  where L_v is a sufficiently large O_v-lattice in $V(k_v)$, and the Haar measure dx is self-dual with respect to the pairing

  $$(x, y) \mapsto \psi_v(\langle x, y\rangle_V), \quad \forall x, y \in V(k_v);$$

- $\widehat{\phi}(x)$ is the Fourier transform of ϕ (with respect to the self-dual Haar measure):

  $$\widehat{\phi}(x) := \int_{V(k_v)} \phi(y)\psi_v(\langle x, y\rangle_V)\, dy.$$

The (global) Weil representation of $\mathrm{SL}_2(k_{\mathbb{A}}) \times \mathrm{O}(V)(k_{\mathbb{A}})$ on the Schwartz space $S(V(k_{\mathbb{A}}))$, where $V(k_{\mathbb{A}}) := V \otimes_k k_{\mathbb{A}}$, is

$$\omega_V := \otimes_v \omega_{V,v}.$$

Remark 2.11. For each place v of k, one has that $\varepsilon_v(V)^2 = \chi_{V,v}(-1)$. Moreover, the Weil reciprocity says that (cf. [38, Proposition 5]):

$$\prod_v \varepsilon_v(V) = 1.$$

Given $\varphi \in S(V(k_{\mathbb{A}}))$, the *theta series associated with* φ is:

$$\Theta(g,h;\varphi) := \sum_{x \in V} \big(\omega_V(g,h)\varphi\big)(x), \quad \forall (g,h) \in \mathrm{SL}_2(k_{\mathbb{A}}) \times \mathrm{O}(V)(k_{\mathbb{A}}). \tag{2.3}$$

Then for every $\gamma \in \mathrm{SL}_2(k)$, $g \in \mathrm{SL}_2(k_{\mathbb{A}})$ $h \in \mathrm{O}(V)(k_{\mathbb{A}})$, and $\varphi \in S(V(k_{\mathbb{A}}))$ we have

$$\Theta(\gamma g, h; \varphi) = \Theta(g,h;\varphi).$$

Given $a \in k$ and $y \in k_{\mathbb{A}}^{\times}$, let:

$$\Theta^*(a,y;h;\varphi) := \int_{k\backslash k_{\mathbb{A}}} \Theta\left(\begin{pmatrix} y & uy^{-1} \\ 0 & y^{-1}\end{pmatrix}, h;\varphi\right)\psi(-au)\,du,$$

where the Haar measure du is normalized so that $\mathrm{vol}(k\backslash k_{\mathbb{A}}, du) = 1$. For $u \in k_{\mathbb{A}}$, one has the following Fourier expansion (cf. [39, p. 19])

$$\Theta\left(\begin{pmatrix} y & uy^{-1} \\ 0 & y^{-1}\end{pmatrix}, h;\varphi\right) = \sum_{a \in k} \Theta^*(a,y;h;\varphi)\psi(au).$$

We shall focus on particular quadratic spaces with degree 4 coming from quaternion algebras, and study the Fourier coefficients of the theta integrals associated with special Schwartz functions.

3 Theta series with nebentypus

Fix a square-free $\mathfrak{d} \in A_+$ with $\deg \mathfrak{d}$ even. Let $F = k(\sqrt{\mathfrak{d}})$. For each $\alpha \in F$, the Galois conjugate of α (over k) is denoted by α'. Given $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(F)$, put

$$\bar{x} := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{and} \quad x' := \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

Given $\mathfrak{n} \in A_+$, let $*$ be the involution on $\mathrm{Mat}_2(F)$ defined by: for $x \in \mathrm{Mat}_2(F)$,

$$x^* := \begin{pmatrix} 0 & 1/\mathfrak{n} \\ 1 & 0 \end{pmatrix} \bar{x}' \begin{pmatrix} 0 & 1 \\ \mathfrak{n} & 0 \end{pmatrix} = \begin{pmatrix} a' & -c'/\mathfrak{n} \\ -\mathfrak{n}b' & d' \end{pmatrix}.$$

Let

$$V := \{x \in \mathrm{Mat}_2(F) \mid x^* = x\} \quad \text{and} \quad Q_V := \det\big|_V.$$

Then (V, Q_V) is a quadratic space with degree 4 over k. In concrete terms, we have

$$V = \left\{\begin{pmatrix} a & \beta \\ -\mathfrak{n}\beta' & d \end{pmatrix} \,\middle|\, a, d \in k,\ \beta \in F\right\}. \tag{3.1}$$

In particular, take the following basis of V:

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -\mathfrak{n} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{\mathfrak{d}} \\ \mathfrak{n}\sqrt{\mathfrak{d}} & 0 \end{pmatrix}\right\},$$

one gets $\det V = 16\mathfrak{n}^2\mathfrak{d} = \mathfrak{d} \in k^\times/(k^\times)^2$. As $\dim_k(V) = 4$, for each place v of k one has that

$$\chi_{V,v}(a_v) = (a_v, \mathfrak{d})_v, \quad \forall a_v \in k_v^\times.$$

From now on, we make the following assumptions:

Assumption 3.1.

(1) The polynomial $\mathfrak{n} \in A_+$ is square-free and coprime to $\mathfrak{d}$.

(2) Write $\mathfrak{n} = \mathfrak{n}^+ \cdot \mathfrak{n}^-$ (resp. $\mathfrak{d} = \mathfrak{d}^+ \cdot \mathfrak{d}^-$), where each prime factor $\mathfrak{p}$ of $\mathfrak{n}^\pm$ (resp. $\mathfrak{d}^\pm$) satisfies that the Legendre quadratic symbol $\left(\frac{\mathfrak{d}}{\mathfrak{p}}\right) = \pm 1$ (resp. $\left(\frac{\mathfrak{n}}{\mathfrak{p}}\right) = \pm 1$). Then $\deg(\mathfrak{d}^-\mathfrak{n}^-) > 0$.

Let

$$B_1 := \{b \in \mathrm{Mat}_2(F) \mid b^* = \bar{b}\} = \left\{ \begin{pmatrix} \alpha & \beta \\ \mathfrak{n}\beta' & \alpha' \end{pmatrix} \;\middle|\; \alpha, \beta \in F \right\}. \tag{3.2}$$

We may identify B_1 with the quaternion algebra

$$\left(\frac{\mathfrak{d}, \mathfrak{n}}{k}\right) := k + k\mathbf{i} + k\mathbf{j} + k\mathbf{ij}, \quad \text{where } \mathbf{i}^2 = \mathfrak{d},\ \mathbf{j}^2 = \mathfrak{n}, \mathbf{ij} = -\mathbf{ji},$$

where $\mathbf{i}$ corresponds to $\begin{pmatrix} \sqrt{\mathfrak{d}} & 0 \\ 0 & -\sqrt{\mathfrak{d}} \end{pmatrix}$ and $\mathbf{j}$ corresponds to $\begin{pmatrix} 0 & 1 \\ \mathfrak{n} & 0 \end{pmatrix}$. Under Assumption 3.1, we observe that B_1 is the indefinite division quaternion algebra over k ramified precisely at prime factors of $\mathfrak{d}^-\mathfrak{n}^-$.

Consider the following left exact sequence

$$1 \longrightarrow k^\times \longrightarrow B_1^\times \longrightarrow \mathrm{SO}(V),$$

where the map from $B_1^\times$ into $\mathrm{SO}(V)$ is defined by

$$b \longmapsto h_b := (x \mapsto bxb^{-1}), \quad \forall b \in B_1^\times.$$

Given $\varphi \in S(V(k_\mathbb{A}))$, we are interested in the following theta integral:

$$I(g;\varphi) := \int_{B_1^\times \backslash B_{1,\mathbb{A}}^\times / k_\mathbb{A}^\times} \Theta(g, h_b; \varphi)\, d^\times b, \quad \forall g \in \mathrm{SL}_2(\mathbb{A}). \tag{3.3}$$

For $a \in k$ and $y \in k_\mathbb{A}^\times$, let (the a-th Fourier coefficient of I)

$$I^*(a, y; \varphi) := \int_{k\backslash k_\mathbb{A}} I\left(\begin{pmatrix} y & uy^{-1} \\ 0 & y^{-1} \end{pmatrix}; \varphi\right) \psi(-au)\, du.$$

Put

$$V_a := \{x \in V \mid Q_V(x) = a\}.$$

We obtain that:

Lemma 3.2. *For $a \in k$ and $y \in \mathbb{A}^\times$, we have*

$$I^*(a,y;\varphi) = |y|_{\mathbb{A}}^2 \cdot (y,\mathfrak{d})_{\mathbb{A}} \cdot \sum_{x \in B_1^\times \backslash V_a} \Big(\mathrm{vol}(K_x^\times \backslash K_{x,\mathbb{A}}^\times / k_{\mathbb{A}}^\times) \cdot \int_{K_{x,\mathbb{A}}^\times \backslash B_{1,\mathbb{A}}^\times} \varphi(yb^{-1}xb)\, d^\times b \Big).$$

Here $(y,\mathfrak{d})_{\mathbb{A}} := \prod_v (y_v,\mathfrak{d})_v$ when we write $y = (y_v)_v \in k_{\mathbb{A}}^\times$, K_x is the centralizer of x in B_1, and $K_{x,\mathbb{A}} = K_x \otimes_k k_{\mathbb{A}}$.

Proof. By definition, we get

$$\begin{aligned} & I^*(a,y;\varphi) \\ = & \int_{k\backslash k_{\mathbb{A}}} \left[\int_{B_1^\times \backslash B_{1,\mathbb{A}}^\times / k_{\mathbb{A}}^\times} \Theta\left(\begin{pmatrix} y & uy^{-1} \\ 0 & y^{-1} \end{pmatrix}, h_b; \varphi \right) d^\times b \right] \psi(-au)\, du \\ = & \int_{B_1^\times \backslash B_{1,\mathbb{A}}^\times / k_{\mathbb{A}}^\times} \left[\int_{k\backslash k_{\mathbb{A}}} \left(\sum_{x \in V} \left(\omega_V \begin{pmatrix} y & uy^{-1} \\ 0 & y^{-1} \end{pmatrix} \varphi \right)(b^{-1}xb) \right) \psi(-au) du \right] d^\times b. \end{aligned}$$

For $x \in V$ and $b \in B_{1,\mathbb{A}}^\times$, it is straightforward to check that

$$\left(\omega_V \begin{pmatrix} y & uy^{-1} \\ 0 & y^{-1} \end{pmatrix} \varphi \right)(b^{-1}xb) = \psi(uQ_V(x)) \cdot (y,\mathfrak{d})_{\mathbb{A}} \cdot |y|_{\mathbb{A}}^2 \cdot \varphi(y \cdot b^{-1}xb).$$

Since

$$\int_{k\backslash k_{\mathbb{A}}} \psi\big(uQ_V(x)\big) \cdot \psi(-au) du = \begin{cases} 1, & \text{if } Q_V(x) = a; \\ 0, & \text{otherwise,} \end{cases}$$

we have that

$$\begin{aligned} & \int_{k\backslash k_{\mathbb{A}}} \left(\sum_{x \in V} \left(\omega_V \begin{pmatrix} y & uy^{-1} \\ 0 & y^{-1} \end{pmatrix} \varphi \right)(b^{-1}xb) \right) \psi(-au)\, du \\ = & \ |y|_{\mathbb{A}}^2 \cdot (y,\mathfrak{d})_{\mathbb{A}} \cdot \sum_{x \in V_a} \varphi(y \cdot b^{-1}xb), \end{aligned}$$

where $V_a := \{x \in V \mid Q_V(x) = a\}$. Therefore,

$$I^*(a,y;\varphi) = |y|_{\mathbb{A}}^2 \cdot (y,\mathfrak{d})_{\mathbb{A}} \cdot \int_{B_1^\times \backslash B_{1,\mathbb{A}}^\times / k_{\mathbb{A}}^\times} \left(\sum_{x \in V_a} \varphi(y \cdot b^{-1}xb) \right) d^\times b. \tag{3.4}$$

Note that for $x \in V_a$, the stablizer of x in $B_1^\times$ under the conjugation is $K_x^\times$.

Hence

$$\begin{aligned} I^*(a,y;\varphi) &= |y|_{\mathbb{A}}^2 \cdot (y,\mathfrak{d})_{\mathbb{A}} \cdot \sum_{x \in B_1^\times \backslash V_a} \int_{K_x^\times \backslash B_{1,\mathbb{A}}^\times / k_{\mathbb{A}}^\times} \varphi(y \cdot b^{-1}xb)\, d^\times b \\ &= |y|_{\mathbb{A}}^2 \cdot (y,\mathfrak{d})_{\mathbb{A}} \cdot \sum_{x \in B_1^\times \backslash V_a} \Big(\mathrm{vol}(K_x^\times \backslash K_{x,\mathbb{A}}^\times / k_{\mathbb{A}}^\times) \\ &\qquad \cdot \int_{K_{x,\mathbb{A}}^\times \backslash B_{1,\mathbb{A}}^\times} \varphi(y \cdot b^{-1}xb)\, d^\times b \Big), \end{aligned}$$

and the proof is complete. □

Remark 3.3. Suppose φ is a pure-tensor, i.e. $\varphi = \otimes_v \varphi_v$, where $\varphi_v \in S(V(k_v))$. Then for $x \in V$, the following equality holds:

$$\int_{K_{x,\mathbb{A}}^\times \backslash B_{1,\mathbb{A}}^\times} \varphi(yb^{-1}xb)\, d^\times b = \prod_v \int_{K_{x,v}^\times \backslash B_{1,v}^\times} \varphi_v(y_v b_v^{-1} x b_v)\, d^\times b_v.$$

We shall choose a particular pure-tensor Schwartz function $\varphi_\Lambda = \otimes_v \varphi_{\Lambda,v} \in S(V(k_{\mathbb{A}}))$ so that the associated Fourier coefficients can be expressed in terms of modified Hurwitz class numbers.

3.1 Particular Schwartz function

Recall the definitions of V in (3.1) and B_1 in (3.2), and note that the trace map $\mathrm{Tr} : \mathrm{Mat}_2(F) \to F$ restricting to V gives a k-linear functional on V. For $x \in V$, put

$$x^\natural := \left(x - \frac{\mathrm{Tr}(x)}{2}\right) \cdot \sqrt{\mathfrak{d}} \in B_1^o,$$

where B_1^o is the space of of pure quaternions in B_1, i.e.

$$B_1^o = \{b \in B \mid \mathrm{Tr}(b) = 0\}.$$

Then the centralizer of x in B_1 is

$$K_x = \begin{cases} k(x^\natural), & \text{a quadratic field over } k \text{ if } x^\natural \neq 0; \\ B_1, & \text{otherwise.} \end{cases}$$

Lemma 3.4. *For $a \in k$, two elements x_1 and x_2 in V_a belong to the same orbit of $B_1^\times$ (under the conjugation action) if and only if* $\mathrm{Tr}(x_1) = \mathrm{Tr}(x_2)$.

Proof. It is clear that $\mathrm{Tr}(x_1) = \mathrm{Tr}(x_2)$ if x_1 and x_2 belong to the same orbit of $B_1^\times$. Conversely, suppose $\mathrm{Tr}(x_1) = \mathrm{Tr}(x_2)$. As

$$a = Q_V(x) = \frac{\mathrm{Tr}(x)^2}{4} - \frac{(x^\natural)^2}{\mathfrak{d}}, \quad \forall x \in V_a,$$

the condition $\mathrm{Tr}(x_1) = \mathrm{Tr}(x_2)$ says that $(x_1^\natural)^2 = (x_2^\natural)^2 \in k$. Thus there exists an isomorphism over k between two subfields $k(x_1^\natural)$ and $k(x_2^\natural)$ of B_1 sending $x_1^\natural$ to $x_2^\natural$. Extending this isomorphism to an inner automorphism of B_1, there exists $b \in B_1^\times$ so that $bx_1^\natural b^{-1} = x_2^\natural$. Therefore

$$bx_1b^{-1} = b\left(\frac{\mathrm{Tr}(x_1)}{2} + \frac{x_1^\natural}{\sqrt{\mathfrak{d}}}\right)b^{-1} = \frac{\mathrm{Tr}(x_2)}{2} + \frac{x_2^\natural}{\sqrt{\mathfrak{d}}} = x_2.$$

□

Take

$$\Lambda := \mathrm{Mat}_2(O_F) \cap V = \left\{ \begin{pmatrix} a & \beta \\ -\beta'\mathfrak{n} & d \end{pmatrix} \middle|\ a, d \in A,\ \beta \in O_F \right\}$$

and

$$O_{B_1} := \mathrm{Mat}_2(O_F) \cap B_1 = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta'\mathfrak{n} & \alpha' \end{pmatrix} \middle|\ \alpha, \beta \in O_F \right\}.$$

It is direct to check that O_{B_1} is an Eichler A-order in B_1 of type $(\mathfrak{d}^+\mathfrak{n}^+, \mathfrak{d}^-\mathfrak{n}^-)$ and $u^{-1}xu \in \Lambda$ for every $x \in \Lambda$ and $u \in O_{B_1}^\times$. For each non-zero prime ideal $\mathfrak{p}$ of A, put $\Lambda_\mathfrak{p} := \Lambda \otimes_A O_\mathfrak{p}$ and $\Lambda_\mathfrak{p}^\natural := O_{B_{1,\mathfrak{p}}}^o = \{b \in O_{B_{1,\mathfrak{p}}} \mid \mathrm{Tr}(b) = 0\}$. Then:

Lemma 3.5. *For $x_\mathfrak{p} \in V(k_\mathfrak{p})$ with $Q_V(x_\mathfrak{p}) \in O_\mathfrak{p}$, we have that*

$$x_\mathfrak{p} \in \Lambda_\mathfrak{p} \quad \textit{if and only if} \quad \mathrm{Tr}(x_\mathfrak{p}) \in O_\mathfrak{p} \textit{ and } x_\mathfrak{p}^\natural \in \Lambda_\mathfrak{p}^\natural.$$

Proof. It is straightforward to check that when $x_\mathfrak{p} \in \Lambda_\mathfrak{p}$, one has $\mathrm{Tr}(x_\mathfrak{p}) \in O_\mathfrak{p}$ and $x_\mathfrak{p}^\natural \in \Lambda_\mathfrak{p}^\natural$. Conversely, suppose $\mathrm{Tr}(x_\mathfrak{p}) \in O_\mathfrak{p}$ and $x_\mathfrak{p}^\natural \in \Lambda_\mathfrak{p}^\natural$. Write

$$x_\mathfrak{p}^\natural = \begin{pmatrix} a\sqrt{\mathfrak{d}} & \beta \\ \beta'\mathfrak{n} & -a\sqrt{\mathfrak{d}} \end{pmatrix} \quad \text{with } a \in O_\mathfrak{p} \text{ and } \beta \in O_{F,\mathfrak{p}} := O_F \otimes_A O_\mathfrak{p},$$

and $t = \mathrm{Tr}(x_\mathfrak{p}) \in O_\mathfrak{p}$. Then

$$Q_V(x_\mathfrak{p}) = \frac{t^2}{4} - \frac{(x_\mathfrak{p}^\natural)^2}{\mathfrak{d}} = \frac{t^2}{4} + \frac{a\mathfrak{d} + \mathrm{Nr}_{F/k}(\beta)\mathfrak{n}}{\mathfrak{d}} \in O_\mathfrak{p}.$$

Since $\mathfrak{n}$ is coprime to $\mathfrak{d}$, we obtain that $\mathrm{Nr}_{F/k}(\beta)/\mathfrak{d} \in O_\mathfrak{p}$ and so $\beta = \sqrt{\mathfrak{d}} \cdot \tilde{\beta}$ for some $\tilde{\beta} \in O_{F,\mathfrak{p}}$ (as $\mathfrak{d}$ is square-free). Therefore

$$x_\mathfrak{p} = \frac{t}{2} + \frac{x_\mathfrak{p}^\natural}{\sqrt{\mathfrak{d}}} = \begin{pmatrix} t/2 + a & \tilde{\beta} \\ -\tilde{\beta}\mathfrak{n} & t/2 - a \end{pmatrix} \in \Lambda_\mathfrak{p}.$$

□

We choose two special Schwartz functions for each place of k as follows. For each non-zero prime ideal $\mathfrak{p}$ of A we take

$$\varphi_{\Lambda,\mathfrak{p}} := \mathbf{1}_{\Lambda_\mathfrak{p}} \in S(V(k_\mathfrak{p})) \quad \text{and} \quad \varphi_\mathfrak{p}^\natural := \mathbf{1}_{\Lambda_\mathfrak{p}^\natural} \in S(B_{1,\mathfrak{p}}^o). \tag{3.5}$$

The above lemma says that for $x_\mathfrak{p} \in V(k_\mathfrak{p})$ with $Q_V(x_\mathfrak{p}) \in O_\mathfrak{p}$, we have

$$\varphi_{\Lambda,\mathfrak{p}}(x_\mathfrak{p}) = 1 \quad \text{if and only if} \quad \mathrm{Tr}(x) \in O_\mathfrak{p} \text{ and } \varphi_\mathfrak{p}^\natural(x_\mathfrak{p}^\natural) = 1. \tag{3.6}$$

As $\mathfrak{d}$ is monic with even degree, the field F is *real* over k, i.e. the infinite place ∞ of k splits in F. Fix an embedding $F \hookrightarrow k_\infty$, which induces a k_∞-algebra isomorphism

$$B_{1,\infty} = B_1 \otimes_k k_\infty \cong \mathrm{Mat}_2(k_\infty).$$

As the natural decomposition

$$V = k \oplus \frac{1}{\sqrt{\mathfrak{d}}} \cdot B_1^o \subset \mathrm{Mat}_2(F), \quad x = \frac{\mathrm{Tr}(x)}{2} + \frac{x^\natural}{\sqrt{\mathfrak{d}}}$$

induces an isomorphism $V(k_\infty) \cong B_{1,\infty}$ (as quadratic spaces over k_∞). Take

$$L_\infty := \varpi \cdot \mathrm{Mat}_2(O_\infty) \quad \text{and} \quad L'_\infty := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L_\infty \;\middle|\; c \in \varpi^2 O_\infty \right\}.$$

Via the identification $V(k_\infty) \cong B_{1,\infty} \cong \mathrm{Mat}_2(k_\infty)$, we may view L_∞ and L'_∞ as two O_∞-lattices in $V(k_\infty)$. Choose

$$\varphi_{\Lambda,\infty} := \mathbf{1}_{L_\infty} - \frac{q+1}{2} \cdot \mathbf{1}_{L'_\infty} \in S(V(k_\infty)) \tag{3.7}$$

and

$$\varphi_\infty^\natural := \mathbf{1}_{L_\infty^o} - \frac{q+1}{2} \cdot \mathbf{1}_{L'^{,o}_\infty} \in S(B_{1,\infty}^o),$$

where $L_\infty^o = L_\infty \cap B_{1,\infty}^o$ and $L'^{,o}_\infty = L'_\infty \cap B_{1,\infty}^o$. It is straightforward to check that:

Lemma 3.6. *For $x \in V(k_\infty)$, one has that*

$$\varphi_{\Lambda,\infty}(x) = \mathbf{1}_{\varpi O_\infty}(\mathrm{Tr}(x)) \cdot \varphi_\infty^\natural\left(\frac{x^\natural}{\sqrt{\mathfrak{d}}}\right).$$

Our particular Schwartz function $\varphi_\Lambda \in S(V(k_\mathbb{A}))$ is chosen to be:

$$\varphi_\Lambda := (\otimes_\mathfrak{p} \varphi_{\Lambda,\mathfrak{p}}) \otimes \varphi_{\Lambda,\infty} \in S(V(k_\mathbb{A})).$$

Let

$$\mathcal{K}_0^1(\mathfrak{d}\mathfrak{n}\infty) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(O_\mathbb{A}) \;\middle|\; c \equiv 0 \bmod \mathfrak{d}\mathfrak{n}\infty \right\}, \tag{3.8}$$

and let $\chi_{\mathfrak{d}} : \mathcal{K}_0^1(\mathfrak{d}\mathfrak{n}\infty) \to \{\pm 1\}$ be the quadratic character defined as follows: for each $\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{K}_0^1(\mathfrak{d}\mathfrak{n}\infty)$ with $d = (d_v)_v \in O_{\mathbb{A}}$ (and so $d_{\mathfrak{p}} \in O_{\mathfrak{p}}^\times$ for each prime factor $\mathfrak{p}$ of $\mathfrak{d}$),

$$\chi_{\mathfrak{d}}(\kappa) := \prod_{\mathfrak{p} \mid \mathfrak{d}} \left(\frac{d_{\mathfrak{p}}}{\mathfrak{p}}\right).$$

The following transformation law of the corresponding theta integral $I(\cdot;\varphi_\Lambda)$ holds:

Proposition 3.7. *Given $\gamma \in \mathrm{SL}_2(k)$, $g \in \mathrm{SL}_2(k_{\mathbb{A}})$ and $\kappa \in \mathcal{K}_0^1(\mathfrak{d}\mathfrak{n}\infty)$, we have*

$$I(\gamma g \kappa; \varphi_\Lambda) = \chi_{\mathfrak{d}}(\kappa) \cdot I(g; \varphi_\Lambda).$$

Proof. From the definition of the theta integral in (3.3) and the theta series in (2.3), we have that for $\gamma \in \mathrm{SL}_2(k)$, $g \in \mathrm{SL}_2(k_{\mathbb{A}})$ and $\kappa \in \mathcal{K}_0^1(\mathfrak{d}\mathfrak{n}\infty)$,

$$\begin{aligned} I(\gamma g\kappa;\varphi_\Lambda) &= \int_{B_1^\times \backslash B_{1,\mathbb{A}}^\times / k_{\mathbb{A}}^\times} \Theta(\gamma g \kappa, h_b; \varphi) d^\times b \\ &= \int_{B_1^\times \backslash B_{1,\mathbb{A}}^\times / k_{\mathbb{A}}^\times} \Theta\big(g, h_b; \omega_V(\kappa)\varphi\big) d^\times b \\ &= I\big(g; \omega_V(\kappa)\varphi_\Lambda\big). \end{aligned}$$

It suffices to show that $\omega_V(\kappa)\varphi_\Lambda = \chi_{\mathfrak{d}}(\kappa) \cdot \varphi_\Lambda$ for every $\kappa = (\kappa_v)_v \in \mathcal{K}_0^1(\mathfrak{d}\mathfrak{n}\infty)$. As φ_Λ is a pure-tensor, this can be checked "locally", i.e. for each place v of k, write $\kappa_v = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}$ and we need

$$\omega_{V,v}(\kappa_v)\varphi_{\Lambda,v} = \varphi_{\Lambda,v} \cdot \begin{cases} \left(\frac{d_{\mathfrak{p}}}{\mathfrak{p}}\right), & \text{if } v = \mathfrak{p} \mid \mathfrak{d}; \\ 1, & \text{otherwise.} \end{cases} \tag{3.9}$$

Given a place v of k, notice that $\mathrm{SL}_2(O_v)$ is generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & u_v \\ 0 & 1 \end{pmatrix}$, $u_v \in O_v$. Hence for $\begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \in \mathrm{SL}_2(O_v)$ with $c_v \equiv 0 \bmod v$, one has $d_v \in O_v^\times$ and

$$\begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} = \begin{pmatrix} 1 & b_v d_v^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_v^{-1} & 0 \\ 0 & d_v \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -d_v^{-1} c_v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Dealing with the case when $v \mid \mathfrak{d}\mathfrak{n}\infty$ and $v \nmid \mathfrak{d}\mathfrak{n}\infty$ separately, the equality (3.9) then follows from straightforward calculations. □

This transformation law implies in particular that for $a \in k$, $y \in k_{\mathbb{A}}^\times$, $\alpha \in k^\times$, $\varepsilon \in O_{\mathbb{A}}^\times$, and $u \in O_{\mathbb{A}}$, we have

$$(\varepsilon, \mathfrak{d})_{\mathbb{A}} \cdot I^*(\alpha^{-2}a, \alpha y \varepsilon; \varphi_\Lambda) = I^*(a, y; \varphi_\Lambda) = I^*(a, y; \varphi_\Lambda) \cdot \psi(a y^2 u). \tag{3.10}$$

Since $A = \mathbb{F}_q[\theta]$ is a principal ideal domain, one has that

$$k_{\mathbb{A}}^\times = k^\times \cdot (k_\infty^\times \times \prod_{\mathfrak{p}\neq\infty} O_{\mathfrak{p}}^\times).$$

From the first equality in (3.10), it suffices to consider $I^*(a,y;\varphi_\Lambda)$ for $y \in k_\infty^\times$. In this case, the second equality in (3.10) (when varying u in $O_{\mathbb{A}}$) implies that

$$I^*(a,y;\varphi_\Lambda) = 0 \quad \text{ unless } a \in A \text{ with } \deg a + 2 \leq 2\,\mathrm{ord}_\infty(y).$$

Next, we shall express $I^*(a,y;\varphi_\Lambda)$ in terms of the modified Hurwitz class numbers.

3.2 Fourier coefficients of $I(g;\varphi_\Lambda)$

Let $y \in k_\infty^\times$ and $a \in A$ with $\deg a + 2 \leq 2\,\mathrm{ord}_\infty(y)$. As $\mathfrak{d}$ is monic with even degree, one gets that $(y,\mathfrak{d})_\infty = 1$. By Lemma 3.2 and *Remark* 3.3, we have that

$$I^*(a,y;\varphi_\Lambda) = |y|_\infty^2 \cdot \sum_{x \in B_1^\times \backslash V_a} \Bigg[\mathrm{vol}(K_x^\times \backslash K_{x,\mathbb{A}}^\times / k_{\mathbb{A}}^\times) \tag{3.11}$$
$$\cdot \left(\prod_{\mathfrak{p}} \int_{K_{x,\mathfrak{p}}^\times \backslash B_{1,\mathfrak{p}}^\times} \varphi_{\Lambda,\mathfrak{p}}(b_{\mathfrak{p}}^{-1} x b_{\mathfrak{p}})\, d^\times b_{\mathfrak{p}}\right) \cdot \int_{K_{x,\infty}^\times \backslash B_{1,\infty}^\times} \varphi_{\Lambda,\infty}(y b_\infty^{-1} x b_\infty)\, d^\times b_\infty\Bigg].$$

For $x \in V_a$ with $\mathrm{Tr}(x) = t \in A$, one has that

$$(x^\natural)^2 = \mathfrak{d}\left(\frac{t^2}{4} - a\right).$$

Thus

$$K_x = \begin{cases} k(x^\natural) \cong k(\sqrt{\mathfrak{d}(t^2-4a)}), & \text{if } x \notin k; \\ B_1, & \text{otherwise.} \end{cases}$$

As the Eichler A-order O_{B_1} is of type $(\mathfrak{d}^+\mathfrak{n}^+, \mathfrak{d}^-\mathfrak{n}^-)$, applying Eichler's theory of local optimal embeddings in Appendix A and B we obtain that:

Proposition 3.8. *Given $a \in A$ and $y \in k_\infty^\times$ with $\deg a + 2 \leq 2\,\mathrm{ord}_\infty(y)$. Take $x \in \Lambda_a$ and put $t = \mathrm{Tr}(x) \in A$. We have that (recall Definition 2.4)*

$$\begin{aligned} &\mathrm{vol}(K_x^\times \backslash K_{x,\mathbb{A}}^\times / k_{\mathbb{A}}^\times) \cdot \left(\prod_{\mathfrak{p}} \int_{K_{x,\mathfrak{p}}^\times \backslash B_{1,\mathfrak{p}}^\times} \varphi_{\Lambda,\mathfrak{p}}(b_{\mathfrak{p}}^{-1} x b_{\mathfrak{p}})\, d^\times b_{\mathfrak{p}}\right) \\ &\qquad \cdot \int_{K_{x,\infty}^\times \backslash B_{1,\infty}^\times} \varphi_{\Lambda,\infty}(y b_\infty^{-1} x b_\infty)\, d^\times b_\infty. \\ = \ &\mathrm{vol}(O_{B_{1,\mathbb{A}}}^\times / O_{\mathbb{A}}^\times) \cdot \begin{cases} H^{\mathfrak{d}^+\mathfrak{n}^+, \mathfrak{d}^-\mathfrak{n}^-}(\mathfrak{d}(t^2-4a)), & \textit{if } t^2 - 4a \preceq 0; \\ 0, & \textit{otherwise.} \end{cases} \end{aligned}$$

Proof. Set

$$d := \mathfrak{d} \cdot \left(\frac{t^2}{4} - a\right) = (x^\natural)^2.$$

If $d = 0$, then $x \in k$ and $K_x = B_1$. Thus by by Lemma 2.8. we have

$$\begin{aligned} & \operatorname{vol}(K_x^\times \backslash K_{x,\mathbb{A}}^\times / k_{\mathbb{A}}^\times) \cdot \int_{K_{x,\mathbb{A}}^\times \backslash B_{1,\mathbb{A}}^\times} \varphi_\Lambda(b^{-1} x b)\, d^\times b \\ = \ & \operatorname{vol}(B_1^\times \backslash B_{1,\mathbb{A}}^\times / k_{\mathbb{A}}^\times) \cdot \varphi_\Lambda(0) \\ = \ & 2 \cdot \frac{1-q}{2} \\ = \ & \operatorname{vol}(O_{B_{1,\mathbb{A}}}^\times / O_{\mathbb{A}}^\times) \cdot H^{\mathfrak{d}^+ \mathfrak{n}^+, \mathfrak{d}^- \mathfrak{n}^-}(0). \end{aligned}$$

Now, suppose $d \neq 0$. From (3.6) and Lemma 3.6 one has that

$$\int_{K_{x,\mathfrak{p}}^\times \backslash B_{1,\mathfrak{p}}^\times} \varphi_{\Lambda,\mathfrak{p}}(b_\mathfrak{p}^{-1} x b_\mathfrak{p})\, d^\times b_\mathfrak{p} = \int_{K_{x,\mathfrak{p}}^\times \backslash B_{1,\mathfrak{p}}^\times} \varphi_\mathfrak{p}^\natural(b_\mathfrak{p}^{-1} x^\natural b_\mathfrak{p})\, d^\times b_\mathfrak{p}$$

and

$$\begin{aligned} & \int_{K_{x,\infty}^\times \backslash B_{1,\infty}^\times} \varphi_{\Lambda,\infty}(b_\infty^{-1} y x b_\infty)\, d^\times b_\infty \\ & \qquad = \mathbf{1}_{\varpi O_\infty}(yt) \cdot \int_{K_{x,\infty}^\times \backslash B_{1,\infty}^\times} \varphi_\infty^\natural\left(b_\infty^{-1} \cdot \frac{y x^\natural}{\sqrt{\mathfrak{d}}} \cdot b_\infty\right) d^\times b_\infty. \end{aligned}$$

Write $d = d_0 \prod_\mathfrak{p} \mathfrak{p}^{2c_\mathfrak{p}}$, where d_0 is square-free. Applying Corollary B.2 and B.3, we get

$$\begin{aligned} \int_{K_{x,\mathfrak{p}}^\times \backslash B_{1,\mathfrak{p}}^\times} \varphi_\mathfrak{p}^\natural(b_\mathfrak{p}^{-1} x^\natural b_\mathfrak{p})\, d^\times b_\mathfrak{p} \quad = \quad & \frac{\operatorname{vol}(O_{B_{1,\mathfrak{p}}}^\times / O_\mathfrak{p}^\times)}{\operatorname{vol}(\mathcal{O}_{d_0,\mathfrak{p}}^\times / O_\mathfrak{p}^\times)} \\ & \cdot \sum_{\ell_\mathfrak{p}=0}^{c_\mathfrak{p}} \# \left(\frac{\mathcal{O}_{d_0,\mathfrak{p}}^\times}{\mathcal{O}_{d_0 \mathfrak{p}^{2\ell_\mathfrak{p}},\mathfrak{p}}^\times}\right) \cdot e(\mathcal{O}_{d_0 \mathfrak{p}^{2\ell_\mathfrak{p}},\mathfrak{p}}, O_{B_{1,\mathfrak{p}}}), \end{aligned}$$

and

$$\begin{aligned} & \int_{K_{x,\infty}^\times \backslash B_{1,\infty}^\times} \varphi_\infty^\natural\left(b_\infty^{-1} \cdot \frac{y x^\natural}{\sqrt{\mathfrak{d}}} \cdot b_\infty\right) d^\times b_\infty \\ = \ & \frac{1}{e_\infty(K_x/k)} \cdot \frac{\operatorname{vol}(O_{B_{1,\infty}}^\times / O_\infty^\times)}{\operatorname{vol}(O_{K_{x,\infty}}^\times / O_\infty^\times)} \\ & \cdot \begin{cases} 1, & \text{if } k(\sqrt{d})/k \text{ is imaginary and } \operatorname{ord}_\infty(y^2(t^2-4a)) \geq 2; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Note that that condition $k(\sqrt{d})/k$ is imaginary is equivalent to $t^2-4a \prec 0$ and forces that $2\deg t \leq \deg a$. Since our assumption $\deg a + 2 \leq 2\operatorname{ord}_\infty(y)$ guarantees that $yt \in \varpi O_\infty$ and $\operatorname{ord}_\infty(y^2(t^2-4a)) \geq 2$, we have

$$\int_{K_{x,\infty}^\times \backslash B_{1,\infty}^\times} \varphi_{\Lambda,\infty}(yb_\infty^{-1}xb_\infty)\, d^\times b_\infty \;=\; \frac{1}{e_\infty(K_x/k)} \cdot \frac{\operatorname{vol}(O_{B_{1,\infty}}^\times/O_\infty^\times)}{\operatorname{vol}(O_{K_{x,\infty}}^\times/O_\infty^\times)} \cdot \begin{cases} 1, & \text{if } t^2-4a \prec 0; \\ 0, & \text{otherwise.}\end{cases}$$

Finally, recall by Proposition 2.1 that

$$\operatorname{vol}(K_x^\times \backslash K_{x,\mathbb{A}}^\times/k_\mathbb{A}^\times) = \frac{h(d_0)}{w(d_0)} \cdot e_\infty(K_x/k) \cdot \operatorname{vol}(O_{K_{x,\infty}}^\times/O_\infty^\times) \cdot \prod_{\mathfrak{p}} \operatorname{vol}(\mathcal{O}_{d_0,\mathfrak{p}}^\times/O_\mathfrak{p}^\times)$$

and notice that for each non-zero prime ideal $\mathfrak{p}$ of A we have

$$e(\mathcal{O}_{d_0\mathfrak{p}^{2\ell_\mathfrak{p}},\mathfrak{p}}, O_{B_\mathfrak{p}}) \;=\; e_\mathfrak{p}^{\mathfrak{d}^+\mathfrak{n}^+,\mathfrak{d}^-\mathfrak{n}^-}(\ell_\mathfrak{p})$$

by Lemma A.1 and A.2 and (2.2). Hence when $t^2-4a \prec 0$, we conclude that

$$\begin{aligned} &\operatorname{vol}(K_x^\times \backslash K_{x,\mathbb{A}}^\times/k_\mathbb{A}^\times) \cdot \left(\prod_\mathfrak{p} \int_{K_{x,\mathfrak{p}}^\times \backslash B_{1,\mathfrak{p}}^\times} \varphi_{\Lambda,\mathfrak{p}}(b_\mathfrak{p}^{-1}xb_\mathfrak{p})\, d^\times b_\mathfrak{p}\right) \\ &\qquad\qquad \cdot \int_{K_{x,\infty}^\times \backslash B_{1,\infty}^\times} \varphi_{\Lambda,\infty}(yb_\infty^{-1}xb_\infty)\, d^\times b_\infty. \\ = \;& \operatorname{vol}(O_{B_{1,\mathbb{A}}}^\times/O_\mathbb{A}^\times) \cdot \frac{h(d_0)}{w(d_0)} \cdot \prod_\mathfrak{p} \left[\sum_{\ell_\mathfrak{p}=0}^{c_\mathfrak{p}} \#\left(\frac{\mathcal{O}_{d_0,\mathfrak{p}}^\times}{\mathcal{O}_{d_0\mathfrak{p}^{2\ell_\mathfrak{p}},\mathfrak{p}}^\times}\right) \cdot e_\mathfrak{p}^{\mathfrak{d}^+\mathfrak{n}^+,\mathfrak{d}^-\mathfrak{n}^-}(\ell_\mathfrak{p})\right] \\ = \;& \operatorname{vol}(O_{B_{1,\mathbb{A}}}^\times/O_\mathbb{A}^\times) \cdot H^{\mathfrak{d}^+\mathfrak{n}^+,\mathfrak{d}^-\mathfrak{n}^-}(\mathfrak{d}(t^2-4a)), \end{aligned}$$

where the last equality follows from Proposition 2.5. □

Notice that two elements $x_1, x_2 \in V_a$ belong to the same $B_1^\times$-orbit if and only if $\operatorname{Tr}(x_1) = \operatorname{Tr}(x_2)$. From the equation (3.11) and Proposition 3.8, we conclude that:

Theorem 3.9. *Given $a \in A$ and $y \in k_\infty^\times$ with $\deg a + 2 \leq 2\operatorname{ord}_\infty(y)$, the following equality holds:*

$$I^*(a,y;\varphi_\Lambda) = \operatorname{vol}(O_{B_{1,\mathbb{A}}}^\times/O_\mathbb{A}^\times) \cdot |y|_\infty^2 \cdot \sum_{\substack{t\in A \\ t^2 \preceq 4a}} H^{\mathfrak{d}^+\mathfrak{n}^+,\mathfrak{d}^-\mathfrak{n}^-}(\mathfrak{d}(t^2-4a)).$$

3.3 Alternative expression of the Fourier coefficients

For $y \in k_\infty^\times$ and $a \in A$ with $\deg a + 2 \leq 2\,\mathrm{ord}_\infty(y)$, from the equation (3.4) one may express $I^*(a,y;\varphi_\Lambda)$ as

$$I^*(a,y;\varphi_\Lambda) = |y|_\infty^2 \cdot \int_{B_1^\times \backslash B_{1,\mathbb{A}}^\times / k_\mathbb{A}^\times} \left(\sum_{x \in V_a} \varphi_\Lambda(y b^{-1} x b) \right) d^\times b.$$

Let $\widehat{A} := \prod_\mathfrak{p} O_\mathfrak{p}$ and $O_{\widehat{B}_1} := O_{B_1} \otimes_A \widehat{A}$. From the strong approximation theorem one has the following bijection:

$$O_{B_1}^\times \backslash B_{1,\infty}^\times / k_\infty^\times \longleftrightarrow B_1^\times \backslash B_{1,\mathbb{A}}^\times / k_\mathbb{A}^\times O_{\widehat{B}_1}^\times. \tag{3.12}$$

Let $\Lambda_a := \Lambda \cap V_a$. The above bijection leads to

$$\begin{aligned} I^*(a,y;\varphi_\Lambda) \quad = \quad & |y|_\infty^2 \mathrm{vol}(O_{\widehat{B}_1}^\times / \widehat{A}^\times) \qquad\qquad (3.13) \\ & \cdot \int_{O_{B_1}^\times \backslash B_{1,\infty}^\times / k_\infty^\times} \left(\sum_{x \in \Lambda_a} \varphi_{\Lambda,\infty}(y b_\infty^{-1} x b_\infty) \right) d^\times b_\infty. \end{aligned}$$

Let

$$\Gamma = \Gamma_{0,F}(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_F) \;\middle|\; ad - bc \in \mathbb{F}_q^\times,\ c \equiv 0 \bmod \mathfrak{n} \right\}.$$

Define an action $\star$ of Γ on Λ by:

$$\gamma \star x := \gamma x \gamma^* \cdot \det(\gamma)^{-1}, \quad \forall \gamma \in \Gamma,\ x \in \Lambda.$$

Then for a non-zero $a \in A$, Λ_a is invariant under the action of Γ. Moreover, given $x \in \Lambda_a$, let

$$B_x := \{ b \in \mathrm{Mat}_2(F) \mid x b^* = \bar{b} x \}.$$

The stablizer of x via the action $\star$ in Γ coincides with $\Gamma_x := B_x^\times \cap \Gamma$, whence

$$\Lambda_a = \coprod_{x \in \Gamma \backslash \Lambda_a} (\Gamma/\Gamma_x) \star x,$$

and

$$(\Gamma/\Gamma_x) \star x = \coprod_{\gamma \in O_{B_1}^\times \backslash \Gamma / \Gamma_x} \Big(O_{B_1}^\times / (O_{B_1}^\times \cap \Gamma_{\gamma \star x}) \Big) \star (\gamma \star x).$$

Therefore we may rewrite (3.13) as follows:

Lemma 3.10. *For $a \in A$ and $y \in k_\infty^\times$ with* $\deg a + 2 \leq 2\,\mathrm{ord}_\infty(y)$,

$$\begin{aligned} I^*(a,y;\varphi_\Lambda) = & |y|_\infty^2 \cdot \mathrm{vol}(O_{\widehat{B}_1}^\times / \widehat{A}^\times) \\ & \cdot \sum_{x \in \Gamma \backslash \Lambda_a} \sum_{\gamma \in O_{B_1}^\times \backslash \Gamma / \Gamma_x} \int_{(O_{B_1}^\times \cap \Gamma_{\gamma \star x}) \backslash B_{1,\infty}^\times / k_\infty^\times} \varphi_{\Lambda,\infty}(y b_\infty^{-1} (\gamma \star x) b_\infty)\, d^\times b_\infty. \end{aligned}$$

To determine the integral inside the above summation, we need the following lemmas:

Lemma 3.11. *Given a non-zero $a \in A$ and $x \in \Lambda_a$, B_x is a quaternion algebra over k which is isomorphic to:*

$$\left(\frac{\mathfrak{d}, a\mathfrak{n}}{k}\right) := k + k\mathbf{i} + k\mathbf{j} + k\mathbf{ij}, \qquad \text{where } \mathbf{i}^2 = \mathfrak{d},\ \mathbf{j}^2 = a\mathfrak{n},\ \text{and } \mathbf{ji} = -\mathbf{ij}.$$

Proof. Write $x = \begin{pmatrix} d_1 & \beta \\ -\mathfrak{n}\beta' & d_2 \end{pmatrix}$ where $d_1,\ d_2 \in A$ and $\beta \in O_F$ with $d_1 d_2 + \mathfrak{n}\beta\beta' = a$. Take

$$U := \begin{cases} \begin{pmatrix} 1 & 0 \\ \mathfrak{n}\beta' & d_1 \end{pmatrix}, & \text{if } d_1 \neq 0; \\ \begin{pmatrix} d_2 & -\beta \\ 0 & a \end{pmatrix}, & \text{if } d_1 = 0 \text{ and } d_2 \neq 0\ ; \\ \begin{pmatrix} -1 & \beta \\ \mathfrak{n}\beta & \mathfrak{n}\beta^2 \end{pmatrix}, & \text{if } d_1 = d_2 = 0. \end{cases}$$

Then

$$x_U := UxU^* = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \cdot \begin{cases} d_1, & \text{if } d_1 \neq 0; \\ ad_2, & \text{if } d_1 = 0 \text{ and } d_2 \neq 0; \\ 2a, & \text{if } d_1 = d_2 = 0. \end{cases}$$

It is straightforward to check that $B_x = U^{-1} B_{x_U} U$ and

$$B_{x_U} = \left\{ \begin{pmatrix} \alpha & \beta \\ a\mathfrak{n}\beta' & \alpha' \end{pmatrix} \ \middle|\ \alpha, \beta \in F \right\}.$$

Thus

$$B_x \cong B_{x_U} \cong \left(\frac{\mathfrak{d}, a\mathfrak{n}}{k}\right).$$

□

Remark 3.12. Observe that $B_x = B_1$ if and only if $x \in k^\times$. In this case, a is a square in A, and $\Gamma_x = O_{B_1}^\times$.

Lemma 3.13. *Let $a \in A$ and $y \in k_\infty^\times$ with $a \neq 0$ and $\deg a + 2 \leq 2\,\mathrm{ord}_\infty(y)$. Take $x \in V_a$.*

(1) If $B_x = B_1$, then $\Gamma_x = \Gamma_1$ and

$$\int_{(\Gamma_1 \cap \Gamma_x)\backslash B_{1,\infty}^\times / k_\infty^\times} \varphi_{\Lambda,\infty}(y b_\infty^{-1} x b_\infty)\, d^\times b_\infty = \frac{1-q}{2} \cdot \mathrm{vol}(\Gamma_1 \backslash B_{1,\infty}^\times / k_\infty^\times).$$

(2) If $B_x \neq B_1$, then $B_x \cap B_1 = K_x$, and

$$\begin{aligned}&\int_{(\Gamma_1\cap\Gamma_x)\backslash B_{1,\infty}^\times/k_\infty^\times} \varphi_{\Lambda,\infty}(yb_\infty^{-1}xb_\infty)\, d^\times b_\infty \\ = \quad & \operatorname{vol}(O_{B_{1,\infty}}^\times/O_\infty^\times)\cdot \begin{cases} \dfrac{q-1}{\#(\Gamma_1\cap\Gamma_x)}, & \text{if } K_x/k \text{ is imaginary;} \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Proof. When $B_x = B_1$, we get $x \in k^\times$ with $x^2 = a$. Thus the condition $\deg a + 2 \leq 2\operatorname{ord}_\infty(y)$ implies

$$\varphi_{\Lambda,\infty}(yb_\infty^{-1}xb_\infty) = 1 - \frac{q+1}{2} = \frac{1-q}{2}, \quad \forall b_\infty \in B_{1,\infty}^\times.$$

Hence the assertion (1) holds.

For (2), the integral vanishes unless K_x/k is imaginary. In this case, $\Gamma_1 \cap \Gamma_x$ is a finite subgroup of $K_x^\times$, and

$$\begin{aligned}&\int_{(\Gamma_1\cap\Gamma_x)\backslash B_{1,\infty}^\times/k_\infty^\times} \varphi_{\Lambda,\infty}(yb_\infty^{-1}xb_\infty)\, d^\times b_\infty \\ = \quad & \frac{\operatorname{vol}(K_{x,\infty}^\times/k_\infty^\times)}{\#(\Gamma_1\cap\Gamma_x)}\cdot \int_{K_{x,\infty}^\times\backslash B_{1,\infty}^\times} \varphi_\infty^\natural\big((y\sqrt{\mathfrak{d}})b_\infty^{-1}x^\natural b_\infty\big)\, d^\times b_\infty \\ = \quad & \operatorname{vol}(O_{B_{1,\infty}}^\times/O_\infty^\times)\cdot\frac{q-1}{\#(\Gamma_1\cap\Gamma_x)}.\end{aligned}$$

The last equality follows from Corollary B.3. □

The bijection (3.12) implies that

$$\operatorname{vol}(O_{\widehat{B}_1}^\times/\widehat{A}^\times)\cdot\operatorname{vol}(\Gamma_1\backslash B_{1,\infty}^\times/k_\infty^\times) = \operatorname{vol}(B_1^\times\backslash B_{1,\mathbb{A}}^\times/k_{\mathbb{A}}^\times) = 2.$$

Hence by Lemma 2.8 we get

$$\frac{1-q}{2}\cdot\frac{\operatorname{vol}(\Gamma_1\backslash B_{1,\infty}^\times/k_\infty^\times)}{\operatorname{vol}(O_{B_{1,\infty}}^\times/O_\infty^\times)} = \frac{1-q}{2}\cdot\frac{2}{\operatorname{vol}(O_{B_{1,\mathbb{A}}}^\times/O_{\mathbb{A}}^\times)} = H^{\mathfrak{d}^+\mathfrak{n}^+,\mathfrak{d}^-\mathfrak{n}^-}(0). \tag{3.14}$$

For non-zero $x \in \Lambda$, put

$$\iota(x) := \begin{cases} H^{\mathfrak{d}^+\mathfrak{n}^+,\mathfrak{d}^-\mathfrak{n}^-}(0), & \text{if } B_x = B_1; \\ \dfrac{q-1}{\#(\Gamma_1\cap\Gamma_x)}, & \text{if } K_x/k \text{ is imaginary;} \\ 0, & \text{otherwise.}\end{cases} \tag{3.15}$$

Define

$$\mathcal{I}(x) := \sum_{\gamma\in\Gamma_1\backslash\Gamma/\Gamma_x} \iota(\gamma\star x). \tag{3.16}$$

From Lemma 3.10, Lemma 3.13, (3.14), (3.15) and (3.16), we then obtain:

Theorem 3.14. *Given $a \in A$ and $y \in k_\infty^\times$ with $a \neq 0$ and $2\operatorname{ord}_\infty(y)+2 \geq \deg a$, we have:*

$$I^*(a,y;\varphi_\Lambda) = \operatorname{vol}(O_{B_{1,\mathbb{A}}^\times}/O_{\mathbb{A}}^\times)\cdot |y|_\infty^2 \cdot \sum_{x\in\Gamma\backslash\Lambda_a} \mathcal{I}(x).$$

In Section 4, the above theorem enables us to connect the Fourier coefficients of the theta integral $I(g;\varphi_\Lambda)$ with the intersection numbers of the "Hirzebruch-Zagier-type divisors" on the "Drinfeld-Stuhler modular surfaces".

3.4 Extension of $I(g;\varphi_\Lambda)$

Let

$$\mathcal{K}_\infty := \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{GL}_2(O_\infty) \;\middle|\; c \equiv 0 \bmod \varpi\right\}$$

and

$$\Gamma_0(\mathfrak{d}\mathfrak{n}) := \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{GL}_2(A) \;\middle|\; c \equiv 0 \bmod \mathfrak{d}\mathfrak{n}\right\}.$$

Put $\mathcal{K}_\infty^1 := \mathcal{K}_\infty \cap \mathrm{SL}_2(k_\infty)$ and $\Gamma_0^1(\mathfrak{d}\mathfrak{n}) := \Gamma_0(\mathfrak{d}\mathfrak{n})\cap\mathrm{SL}_2(A)$ and recall (3.8). From the strong approximation theorem, the natural embedding $\mathrm{SL}_2(k_\infty) \hookrightarrow \mathrm{SL}_2(k_\mathbb{A})$ induces the following bijection

$$\Gamma_0^1(\mathfrak{d}\mathfrak{n})\backslash \mathrm{SL}_2(k_\infty)/\mathcal{K}_\infty^1 \longleftrightarrow \mathrm{SL}_2(k)\backslash \mathrm{SL}_2(k_\mathbb{A})/\mathcal{K}_0^1(\mathfrak{d}\mathfrak{n}\infty).$$

This allows us to view $I(g;\varphi_\Lambda)$ as a function on $\mathrm{SL}_2(k_\infty)/\mathcal{K}_\infty^1$ satisfying

$$I(\gamma g_\infty;\varphi_\Lambda) = \chi_\mathfrak{d}(\gamma) I(g_\infty;\varphi_\Lambda), \quad \forall g_\infty \in \mathrm{SL}_2(k_\infty) \text{ and } \gamma \in \Gamma_0^1(\mathfrak{d}\mathfrak{n}).$$

We shall extend $I(\cdot;\varphi_\Lambda)$ to a function ϑ_Λ on $\mathrm{GL}_2(k_\infty)/k_\infty^\times\mathcal{K}_\infty$ which is "Drinfeld-type", i.e. the following *harmonic property* holds: for $g_\infty \in \mathrm{GL}_2(k_\infty)$ we have

$$\vartheta_\Lambda(g_\infty) + \vartheta_\Lambda\left(g_\infty\begin{pmatrix} 0 & 1 \\ \varpi & 0\end{pmatrix}\right) = 0 = \sum_{\kappa\in\mathrm{GL}_2(O_\infty)/\mathcal{K}_\infty} \vartheta_\Lambda(g_\infty\kappa).$$

Remark 3.15. Let f be a Drinfeld-type automorphic form on $\mathrm{GL}_2(k_\infty)/k_\infty^\times\mathcal{K}_\infty$. The harmonicity of f implies that f is invariant by the "Iwahori" Hecke operator at ∞, i.e. for $g_\infty \in \mathrm{GL}_2(k_\infty)$,

$$\sum_{\epsilon\in\mathbb{F}_q} f\left(g_\infty\begin{pmatrix} \pi_\infty & \epsilon \\ 0 & 1\end{pmatrix}\right) = f(g_\infty).$$

Viewed as analogue to classical weight-two modular forms, Drinfeld-type automorphic forms are objects of great interest in the study of function field arithmetic. We refer the readers to [12], [3], [4], and [37]) for further discussions.

Let $w_\infty := \begin{pmatrix} 0 & 1 \\ \varpi & 0\end{pmatrix}$. We first prove that:

Lemma 3.16. *Given $g_\infty \in \mathrm{SL}_2(k_\infty)$, the following equality holds:*

$$\sum_{\kappa \in \mathrm{SL}_2(O_\infty)/\mathcal{K}_\infty^1} I(g_\infty \kappa; \varphi_\Lambda) = 0 = \sum_{\kappa \in \mathrm{SL}_2(O_\infty)/\mathcal{K}_\infty^1} I(g_\infty w_\infty^{-1} \kappa w_\infty; \varphi_\Lambda).$$

Proof. Notice that $\widehat{\mathbf{1}}_{L_\infty} = \mathbf{1}_{L_\infty}$ and $\widehat{\mathbf{1}}_{L'_\infty} = q^{-1} \cdot \mathbf{1}_{\widetilde{L}_\infty}$, where

$$\widetilde{L}_\infty := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(O_\infty) \;\middle|\; a, c, d \in \varpi O_\infty \right\}.$$

Thus $\omega_{V,\infty}(\kappa)\mathbf{1}_{L_\infty} = \mathbf{1}_{L_\infty}$ for every $\kappa \in \mathrm{SL}_2(O_\infty)$, and

$$\begin{aligned} \sum_{\kappa \in \mathrm{SL}_2(O_\infty)/\mathcal{K}_\infty^1} \omega_{V,\infty}(\kappa)\mathbf{1}_{L'_\infty} &= \mathbf{1}_{L_\infty \cup w_\infty L_\infty w_\infty^{-1}} + \mathbf{1}_{L_\infty \cap w_\infty L_\infty w_\infty^{-1}} \\ &= \mathbf{1}_{L_\infty} + \mathbf{1}_{w_\infty L_\infty w_\infty^{-1}}. \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{\kappa \in \mathrm{SL}_2(O_\infty)/\mathcal{K}_\infty^1} I(g_\infty \kappa; \varphi_\Lambda) \\ &= (q+1) \cdot I(g_\infty; \otimes_{\mathfrak{p}} \varphi_{\Lambda,\mathfrak{p}} \otimes \mathbf{1}_{L_\infty}) \\ &\qquad\qquad - \frac{q+1}{2} \cdot I\Big(g_\infty; \otimes_{\mathfrak{p}} \varphi_{\Lambda,\mathfrak{p}} \otimes \Big(\mathbf{1}_{L_\infty} + \mathbf{1}_{w_\infty L_\infty w_\infty^{-1}}\Big)\Big) \\ &= (q+1) \cdot I(g_\infty; \otimes_{\mathfrak{p}} \varphi_{\Lambda,\mathfrak{p}} \otimes \mathbf{1}_{L_\infty}) - \frac{q+1}{2} \cdot 2 \cdot I(g_\infty; \otimes_{\mathfrak{p}} \varphi_{\Lambda,\mathfrak{p}} \otimes \mathbf{1}_{L_\infty}) \\ &= 0. \end{aligned}$$

Similarly, let

$$L''_\infty := \begin{pmatrix} \varpi O_\infty & O_\infty \\ \varpi O_\infty & O_\infty \end{pmatrix}.$$

Then

$$\sum_{\kappa \in \mathrm{SL}_2(O_\infty)/\mathcal{K}_\infty^1} \omega_{V,\infty}\left(w_\infty \kappa w_\infty^{-1}\right) \mathbf{1}_{L_\infty} = q \cdot \sum_{\kappa \in \mathrm{SL}_2(O_\infty)/\mathcal{K}_\infty^1} \mathbf{1}_{\kappa L''_\infty \kappa^{-1}}$$

and

$$\sum_{\kappa \in \mathrm{SL}_2(O_\infty)/\mathcal{K}_\infty^1} \omega_{V,\infty}(w_\infty \kappa w_\infty^{-1})\mathbf{1}_{L'_\infty} = q \cdot \Big(\mathbf{1}_{L''_\infty} + \mathbf{1}_{w_\infty L''_\infty w_\infty^{-1}}\Big).$$

Therefore

$$\begin{aligned}
&\sum_{\kappa\in \mathrm{SL}_2(O_\infty)/\mathcal{K}_\infty^1} I\left(g_\infty w_\infty \kappa w_\infty^{-1};\varphi_\Lambda\right)\\
&= \sum_{\kappa\in \mathrm{SL}_2(O_\infty)/\mathcal{K}_\infty^1} q\cdot I\left(g_\infty;\otimes_{\mathfrak{p}}\varphi_{\Lambda,\mathfrak{p}}\otimes \mathbf{1}_{\kappa L''_\infty \kappa^{-1}}\right)\\
&\qquad\qquad - \frac{q+1}{2}\cdot q\cdot I\Big(g_\infty;\otimes_{\mathfrak{p}}\varphi_{\Lambda,\mathfrak{p}}\otimes \left(\mathbf{1}_{L''_\infty}+\mathbf{1}_{w_\infty L''_\infty w_\infty^{-1}}\right)\Big)\\
&= (q+1)\cdot q\cdot I\left(g_\infty;\otimes_{\mathfrak{p}}\varphi_{\Lambda,\mathfrak{p}}\otimes\mathbf{1}_{L''_\infty}\right) - \frac{q+1}{2}\cdot q\cdot 2\cdot I\left(g_\infty;\otimes_{\mathfrak{p}}\varphi_{\Lambda,\mathfrak{p}}\otimes\mathbf{1}_{L''_\infty}\right)\\
&= 0.
\end{aligned}$$

□

Let

$$\mathrm{GL}_2^+(k_\infty) := \{g\in \mathrm{GL}_2(k_\infty) \mid \mathrm{ord}_\infty(\det g)\equiv 0 \bmod 2\}.$$

The natural inclusion $\mathrm{SL}_2(k_\infty)\hookrightarrow \mathrm{GL}_2(k_\infty)$ gives a bijection

$$\mathrm{SL}_2(k_\infty)/\mathcal{K}_\infty^1 \longleftrightarrow \mathrm{GL}_2^+(k_\infty)/k_\infty^\times\mathcal{K}_\infty.$$

Thus $I(\cdot;\varphi_\Lambda)$ can be viewed as a function on $\mathrm{GL}_2^+(k_\infty)/k_\infty^\times\mathcal{K}_\infty$. For g_∞ in $\mathrm{GL}_2(k_\infty)$, define $\vartheta_\Lambda(g_\infty)$ by:

$$\vartheta_\Lambda(g_\infty) := \frac{2}{\mathrm{vol}(O_{B_{\mathbb{A}}}^\times/O_{\mathbb{A}}^\times)}\cdot\begin{cases} I(g_\infty;\varphi_\Lambda), & \text{if } g_\infty\in \mathrm{GL}_2^+(k_\infty);\\ -I(g_\infty w_\infty;\varphi_\Lambda), & \text{otherwise.}\end{cases} \tag{3.17}$$

The above lemma implies immediately that:

Proposition 3.17. *The function ϑ_Λ on $\mathrm{GL}_2(k_\infty)/k_\infty^\times\mathcal{K}_\infty$ satisfies the harmonic property, i.e. for $g_\infty\in\mathrm{GL}_2(k_\infty)$,*

$$\vartheta_\Lambda(g_\infty)+\vartheta_\Lambda(g_\infty w_\infty) = 0 = \sum_{\kappa\in \mathrm{GL}_2(O_\infty)/\mathcal{K}_\infty}\vartheta_\Lambda(g_\infty\kappa).$$

Moreover, for $\gamma\in\Gamma_0^{(1)}(\mathfrak{d}\mathfrak{n})$ we have

$$\vartheta_\Lambda(\gamma g_\infty) = \chi_{\mathfrak{d}}(\gamma)\cdot\vartheta_\Lambda(g_\infty),\quad \forall g_\infty\in\mathrm{GL}_2(k_\infty).$$

Here for $\gamma=\begin{pmatrix} a & b\\ \mathfrak{d}\mathfrak{n}c & d\end{pmatrix}\in\Gamma_0^{(1)}(\mathfrak{d}\mathfrak{n})$, $\chi_{\mathfrak{d}}(\gamma)$ is equal to the Legendre quadratic symbol $\left(\frac{d}{\mathfrak{d}}\right)$.

Proof. The second assertion follows directly from Proposition 3.7. To show the harmonicity of ϑ_Λ, by definition we get immediately that

$$\vartheta_\Lambda(g_\infty)+\vartheta_\Lambda(g_\infty w_\infty) = 0.$$

Moreover, suppose $g_\infty \in \mathrm{GL}_2^+(k_\infty)$. Then by Lemma 3.16, one has

$$\sum_{\kappa \in \mathrm{GL}_2(O_\infty)/\mathcal{K}_\infty} \vartheta_\Lambda(g_\infty \kappa) = \frac{2}{\mathrm{vol}(O_{B_\mathbb{A}}^\times / O_\mathbb{A}^\times)} \cdot \sum_{\kappa \in \mathrm{SL}_2(O_\infty)/\mathcal{K}_\infty^1} I(g_\infty \kappa; \varphi_\Lambda) = 0.$$

When $g_\infty \notin \mathrm{GL}_2^+(k_\infty)$, by Lemma 3.16 again we get

$$\begin{aligned} &\sum_{\kappa \in \mathrm{GL}_2(O_\infty)/\mathcal{K}_\infty} \vartheta_\Lambda(g_\infty \kappa) \\ &= \frac{-2}{\mathrm{vol}(O_{B_\mathbb{A}}^\times / O_\mathbb{A}^\times)} \cdot \sum_{\kappa \in \mathrm{SL}_2(O_\infty)/\mathcal{K}_\infty^1} I\big((g_\infty w_\infty) w_\infty^{-1} \kappa w_\infty; \varphi_\Lambda\big) \\ &= 0. \end{aligned}$$

Therefore the proof is complete. ☐

In conclusion:

Theorem 3.18. *We extend $I(\cdot; \varphi_\Lambda)$ to a Drinfeld-type automorphic form ϑ_Λ on $\mathrm{GL}_2(k_\infty)$ for the congruence subgroup $\Gamma_0^1(\mathfrak{d}\mathfrak{n})$ with nebentypus $\chi_\mathfrak{d}$, whose Fourier expansion is: for $(x, y) \in k_\infty \times k_\infty^\times$,*

$$\vartheta_\Lambda \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = |y|_\infty \cdot \sum_{\substack{a \in A \\ \deg a + 2 \leq \mathrm{ord}_\infty(y)}} \left(2 \cdot \sum_{\substack{t \in A \\ t^2 \preceq 4a}} H^{\mathfrak{d}^+\mathfrak{n}^+, \mathfrak{d}^-\mathfrak{n}^-}\big(\mathfrak{d}(t^2 - 4a)\big) \right) \cdot \psi_\infty(ax).$$

Remark 3.19. The above construction of ϑ_Λ gives us a way to produce Drinfeld-type automorphic forms on $\mathrm{GL}_2(k_\infty)$ with non-trivial nebentypus, which is different from the theta series given in [32], [29], [3], or [4].

4 Intersections of the Hirzebruch-Zagier-type divisors

4.1 Drinfeld-Stuhler modular curves

Let $\mathbb{C}_\infty$ be the completion of a chosen algebraic closure of k_∞. The *Drinfeld half plane* is

$$\mathfrak{H} := \mathbb{C}_\infty - k_\infty,$$

which is equipped with the Möbius action of $\mathrm{GL}_2(k_\infty)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k_\infty),\ z \in \mathfrak{H}.$$

We recall the analytic construction of Drinfeld-Stuhler modular curves as follows. Let B be an indefinite quaternion algebra over k, and $\mathfrak{n}^- \in A_+$ be the product of the primes at which B is ramified. Take a square-free $\mathfrak{n}^+ \in A_+$ coprime to $\mathfrak{n}^-$, and let O_B be an Eichler A-order in B of type $(\mathfrak{n}^+, \mathfrak{n}^-)$. Fix an isomorphism $B \otimes_k k_\infty \cong \mathrm{Mat}_2(k_\infty)$, which embeds $\Gamma(\mathfrak{n}^+, \mathfrak{n}^-) := O_B^\times$ into $\mathrm{GL}_2(k_\infty)$ as a discrete subgroup. This induces an action of $O_B^\times$ on the Drinfeld half plane $\mathfrak{H}$. Let

$$X(\mathfrak{n}^+, \mathfrak{n}^-) := \Gamma(\mathfrak{n}^+, \mathfrak{n}^-)\backslash\mathfrak{H},$$

which is a rigid analytic space (compact if B is division). From the moduli interpretation of $X(\mathfrak{n}^+, \mathfrak{n}^-)$ (which parametrizes the "$\mathscr{B}$-elliptic sheaves with additional level-$\mathfrak{n}^+$ structure", cf. [25] and [30]), we may identify $X(\mathfrak{n}^+, \mathfrak{n}^-)$ (rigidly analytically) with the $\mathbb{C}_\infty$-valued points of a smooth curve (projective if B is division) over $\mathbb{C}_\infty$, called the *Drinfeld-Stuhler modular curve for* $\Gamma(\mathfrak{n}^+, \mathfrak{n}^-)$. For our purpose, we shall only use the analytic description of $X(\mathfrak{n}^+, \mathfrak{n}^-)$.
Notice that when $B = \mathrm{Mat}_2(k)$, every Eichler A-order O_B of type $(\mathfrak{n}^+, 1)$ is equal (up to conjugation) to

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(A) \;\middle|\; c \equiv 0 \bmod \mathfrak{n}^+ \right\},$$

and so $\Gamma(\mathfrak{n}^+, 1)$ coincides with the congruence subgroup

$$\Gamma_0(\mathfrak{n}^+) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \;\middle|\; c \equiv 0 \bmod \mathfrak{n}^+ \right\}.$$

The "compactification"

$$X_0(\mathfrak{n}^+) := \Gamma_0(\mathfrak{n})\backslash\Big(\mathfrak{H} \cup \mathbb{P}^1(k)\Big)$$

is called the *Drinfeld modular curve for* $\Gamma_0(\mathfrak{n}^+)$ (cf. [12]).

4.2 Drinfeld-Stuhler modular surface

Let $\mathfrak{d} \in A_+$ be square-free with $\deg \mathfrak{d}$ even and $F = k(\sqrt{\mathfrak{d}})$. Identifying $F_\infty := F \otimes_k k_\infty \cong k_\infty \times k_\infty$, we denote the image of $\alpha \in F$ in k_∞^2 by (α, α') (α' is the Galois conjugate of α over k). Let $\mathfrak{H}_F := \mathfrak{H} \times \mathfrak{H}$, equipped with the Möbius action of $\mathrm{GL}_2(k_\infty)^2$. The above embedding $F \hookrightarrow k_\infty \times k_\infty$ gives $\mathrm{GL}_2(F) \hookrightarrow \mathrm{GL}_2(k_\infty)^2$, which induces an action of $\mathrm{GL}_2(F)$ on $\mathfrak{H}_F$. In concrete terms, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F)$ and $\vec{z} = (z_1, z_2) \in \mathfrak{H}_F$, we have

$$g \cdot \vec{z} = \left(\frac{az_1 + b}{cz_1 + d}, \frac{a' z_2 + b'}{c' z_2 + d'} \right).$$

For $\mathfrak{n} \in A_+$, recall that

$$\Gamma_{0,F}(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_F) \;\middle|\; ad - bc \in \mathbb{F}_q^\times,\ c \equiv 0 \bmod \mathfrak{n} \right\}.$$

The *Drinfeld-Stuhler modular surface for* $\Gamma_{0,F}(\mathfrak{n})$ is

$$\mathcal{S}_{0,F}(\mathfrak{n}) := \Gamma_{0,F}(\mathfrak{n})\backslash\mathfrak{H}_F.$$

From the work of Stuhler [34], $\mathcal{S}_{0,F}(\mathfrak{n})$ is a moduli space of the so-called "Frobenius-Hecke sheaves" (an analogue of the Hilbert-Blumenthal abelian surfaces in the classical case) with additional "level-$\mathfrak{n}$ structure". This provides the algebraic structure of the surface $\mathcal{S}_{0,F}(\mathfrak{n})$. For our purpose, we only consider $\mathcal{S}_{0,F}(\mathfrak{n})$ as a rigid analytic space, and study the intersections of the "Hirzebruch-Zagier-type" divisors on $\mathcal{S}_{0,F}(\mathfrak{n})$.

4.3 Hirzebruch-Zagier-type divisors

Recall in Section 3 that

$$V = \{x \in \mathrm{Mat}_2(F) \mid x^* = x\} \quad \text{and} \quad \Lambda = V \cap \mathrm{Mat}_2(O_F).$$

Given $x \in \Lambda$ with $\det(x) \neq 0$, let $\mathcal{C}_x := \Gamma_x \backslash \mathfrak{H}$, the Drinfeld-Stuhler modular curve for Γ_x (where Γ_x is the stabilizer of x in $\Gamma = \Gamma_{0,F}(\mathfrak{n})$ via the action $\star$). Put

$$S_x := \begin{pmatrix} 0 & 1 \\ \mathfrak{n} & 0 \end{pmatrix} \bar{x}.$$

The closed immersion $\mathfrak{H} \to \mathfrak{H}_F$ defined by $(z \mapsto (z, S_x z))$ induces a (rigid analytic) proper morphism $f_x : \mathcal{C}_x \to \mathcal{S}_{0,F}(\mathfrak{n})$. We put $X_x := f_x(\mathcal{C}_x)$ and $\mathcal{Z}_x := f_{x,*}(\mathcal{C}_x)$, the pushforward divisor of $\mathcal{C}_x$ under f_x on $\mathcal{S}_{0,F}(\mathfrak{n})$. Let

$$\widehat{\Gamma}_x := \{\gamma \in \Gamma_{0,F}(\mathfrak{n}) \mid \gamma \star x = \pm x\}.$$

Then $[\widehat{\Gamma}_x : \Gamma_x] = 1$ or 2, and:

Lemma 4.1. *For* $x \in \Lambda$ *with* $\deg(x) \neq 0$, *one has*

$$\mathcal{Z}_x = [\widehat{\Gamma}_x : \Gamma_x] \cdot X_x.$$

Proof. We need to show that the proper morphism $f_x : \mathcal{C}_x \to X_x$ has degree equal to $[\widehat{\Gamma}_x : \Gamma_x]$.
Let $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathcal{C}_x$ be two points with $f_x(\boldsymbol{z}_1) = f_x(\boldsymbol{z}_2) \in X_x$. Take representatives $\vec{z}_1 = (z_1, S_x z_1)$ and $\vec{z}_2 = (z_2, S_x z_2)$ of $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ on $\mathfrak{H}_F$, respectively. There exists $\gamma \in \Gamma$ so that

$$\vec{z}_1 = \gamma \cdot \vec{z}_2, \quad \text{i.e. } (z_1, S_x z_1) = (\gamma z_2, \gamma' S_x z_2).$$

Thus

$$z_1 = \gamma z_2 = \gamma((\gamma' S_x)^{-1} S_x) z_1 = \big((\gamma \star x)\bar{x}\big) z_1.$$

When z_1 is in "general position", e.g. the stabilizer of z_1 in $\mathrm{GL}_2(F)$ is $F^\times$, one has $(\gamma \star x)\bar{x} \in F^\times$. Taking the determinant of $(\gamma \star x)\bar{x}$, we obtain $\gamma \star x = \pm x$, which says that $\gamma \in \widehat{\Gamma}_x$. Therefore the result holds. $\square$

Suppose now that $\mathfrak{n}$ satisfies Assumption 3.1. We shall study the number of intersections of $\mathcal{Z}_1$ and $\mathcal{Z}_x$ by lifting to a "fine covering" of $\mathcal{S}_{0,F}(\mathfrak{n})$. More precisely, for $\mathfrak{m} \in A_+$, we let

$$\Gamma_F(\mathfrak{m}) := \left\{ \gamma \in \mathrm{GL}_2(O_F) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{m} \right\}.$$

Choose $\mathfrak{m}$ so that $\mathfrak{n}^2$ divides $\mathfrak{m}$. Then

$$\Gamma_F(\mathfrak{m})^* := \{\gamma^* \mid \gamma \in \Gamma_F(\mathfrak{m})\} \subset \Gamma_F(\mathfrak{n}).$$

Consider the finite morphism

$$\pi : \Gamma_F(\mathfrak{m})\backslash \mathfrak{H}_F =: \mathcal{S}_F(\mathfrak{m}) \twoheadrightarrow \mathcal{S}_{0,F}(\mathfrak{n}).$$

For $x \in \Lambda$ with $\det(x) \neq 0$, let $\mathfrak{H}_x := \big\{(z, S_x z) \mid z \in \mathfrak{H}\big\} \subset \mathfrak{H}_F$. Observe that $\gamma\mathfrak{H}_x = \mathfrak{H}_{\gamma\star x}$ for all $\gamma \in \Gamma_{0,F}(\mathfrak{n})$. Let $\widetilde{X}_x$ be the image of $\mathfrak{H}_x$ in $\mathcal{S}_F(\mathfrak{m})$ under the canonical map from $\mathfrak{H}_F$ onto $\mathcal{S}_F(\mathfrak{m})$. Let

$$\Gamma_x(\mathfrak{m}) := \Gamma_x \cap \Gamma_F(\mathfrak{m}) \quad \text{and} \quad \widetilde{\mathcal{C}}_x := \Gamma_x(\mathfrak{m})\backslash \mathfrak{H}.$$

We have:

Lemma 4.2. *Assume $\mathfrak{n}^2 \det(x)$ divides $\mathfrak{m}$. Then the identification between $\mathfrak{H} \cong \mathfrak{H}_x$ induces an isomorphism $\tilde{f}_x : \widetilde{\mathcal{C}}_x \cong \widetilde{X}_x$.*

Proof. Notice that the defining equation of $\mathfrak{H}_x$ in $\mathfrak{H}_F$ makes it smooth everywhere. As each point in $\mathfrak{H}_F$ has trivial stabilizer in $\Gamma_F(\mathfrak{m})$, we may identify a sufficiently small admissible open neighborhood of a given point in $\mathfrak{H}_F$ with the corresponding affinoid subdomains in $\mathcal{S}_F(\mathfrak{m})$. This assures the smoothness of $\widetilde{X}_x$. Therefore it suffices to show that the morphism from $\tilde{f}_x : \widetilde{\mathcal{C}}_x \to \widetilde{X}_x$ is a bijection.

The surjectivity of $\tilde{f}_x$ comes directly from the definition. On the other hand, let $\tilde{\boldsymbol{z}}_1$ and $\tilde{\boldsymbol{z}}_2$ be two points on $\widetilde{\mathcal{C}}_x$ so that $\tilde{f}_x(\tilde{\boldsymbol{z}}_1) = \tilde{f}_x(\tilde{\boldsymbol{z}}_2)$. Take representatives $\vec{z}_1 = (z_1, S_x z_1)$ and $\vec{z}_2 = (z_2, S_x z_2)$ of $\tilde{\boldsymbol{z}}_1$ and $\tilde{\boldsymbol{z}}_2$ on $\mathfrak{H}_x$, respectively. Then there exists $\gamma \in \Gamma_F(\mathfrak{m})$ so that $\vec{z}_1 = \gamma \cdot \vec{z}_2$, i.e.

$$(z_1, S_x z_1) = (\gamma z_2, \gamma' S_x z_2).$$

Thus

$$z_1 = \gamma z_2 = \big(\gamma(\gamma' S_x)^{-1} S_x\big) z_1 = \big((\gamma \star x)\bar{x}\big) z_1.$$

Since $\mathfrak{n}^2 \det(x)$ divides $\mathfrak{m}$, we obtain that

$$(\gamma \star x)x^{-1} = \gamma(x\gamma^* x^{-1}) \cdot \det(\gamma)^{-1} \in \Gamma_F(\mathfrak{n}), \quad \text{i.e. } (\gamma\star x)x^{-1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{n}.$$

As it fixes $z_1 \in \mathfrak{H}$, we obtain that

$$(\gamma \star x)x^{-1} = 1, \quad \text{i.e.} \quad \gamma \star x = x.$$

Hence $\gamma \in \Gamma_x \cap \Gamma_F(\mathfrak{m}) = \Gamma_x(\mathfrak{m})$. In other words, the morphism $\tilde{f}_x$ is bijective, and the proof is complete. ∎

4.4 Formula of intersections

Let $x \in \Lambda$ with $\det(x) \neq 0$, and $\mathfrak{m} \in A_+$ with $\mathfrak{n}^2 \det(x) \mid \mathfrak{m}$. We first verify the transversality of the intersections of $\widetilde{X}_1$ and $\widetilde{X}_x$ on $\mathcal{S}_F(\mathfrak{m})$.

Lemma 4.3. *Suppose $\widetilde{X}_1 \neq \widetilde{X}_x$. Then $\widetilde{X}_1$ and $\widetilde{X}_x$ intersect transversally.*

Proof. It suffices to check that the preimages of $\widetilde{X}_1$ and $\widetilde{X}_x$ in $\mathfrak{H}_F$ intersect transversally. Since $\gamma \mathfrak{H}_x = \mathfrak{H}_{\gamma \star x}$ for every $\gamma \in \Gamma$, it is reduced to show the transversality of the intersection of $\mathfrak{H}_1$ and $\mathfrak{H}_x$ when $x \notin A$.

Suppose $\vec{z} = (z, S_1 z) = (z, S_x z) \in \mathfrak{H}_1 \cap \mathfrak{H}_x$. Write $x = \begin{pmatrix} d_1 & \beta \\ -\mathfrak{n}\beta' & d_2 \end{pmatrix}$ with $d_1, d_2 \in A$, $\beta \in O_F$, and put $a := \det(x) = d_1 d_2 + \mathfrak{n}\beta\beta' \neq 0$. Then $\bar{x} z = z$, i.e.

$$\frac{d_2 z - \beta}{\mathfrak{n}\beta' z + d_1} = z.$$

Thus $\mathfrak{n}\beta' z^2 + (d_1 - d_2) z + \beta = 0$. Multiplying β on both sides we get

$$(a - d_1 d_2) z^2 + (d_1 - d_2)\beta z + \beta^2 = 0. \tag{4.1}$$

On the other hand, the tangent vectors of $\vec{z}$ along $\mathfrak{H}_1$ and $\mathfrak{H}_x$, respectively, are

$$\left(1, -\frac{1}{\mathfrak{n} z^2}\right) \quad \text{and} \quad \left(1, \frac{-a}{\mathfrak{n}(d_2 z - \beta)^2}\right).$$

If these two vectors coincide, we get $az^2 = (d_2 z - \beta)^2$, which says that

$$(a - d_2^2) z^2 + 2 d_2 \beta z - \beta^2 = 0. \tag{4.2}$$

Suppose $\beta \neq 0$. As $z \in \mathfrak{H}$, comparing equations (4.1) and (4.2) we get

$$a - d_1 d_2 = -(a - d_2^2) \quad \text{and} \quad d_1 - d_2 = -2 d_2,$$

which imply $a = 0$ and cause a contradiction. Hence $\beta = 0$ and $a = d_1 d_2 = d_2^2$. Since $a \neq 0$, we have $d_1 = d_2$. Therefore, $x \in A$ and $\mathfrak{H}_x = \mathfrak{H}_1$ also cause a contradiction. As $\mathfrak{H}_x \neq \mathfrak{H}_1$, the two tangent vectors much be different, i.e. the intersection of $\mathfrak{H}_x$ and $\mathfrak{H}_1$ at $\vec{z}$ must be transversal. ☐

Let $\widetilde{\mathcal{Z}}_x$ be the prime divisor associated with $\widetilde{X}_x$ on $\mathcal{S}_F(\mathfrak{m})$. We get

$$\pi_*(\widetilde{\mathcal{Z}}_x) = [\Gamma_x : \Gamma_x(\mathfrak{m}) \cdot \mathbb{F}_q^\times] \cdot \mathcal{Z}_x.$$

From the above lemmas, the intersection number of $\mathcal{Z}_1$ and $\mathcal{Z}_x$ is determined in the following:

Proposition 4.4. *Given $x \in \Lambda$ with $\det(x) \neq 0$, suppose $\mathcal{Z}_1 \neq \mathcal{Z}_x$. Choose $\mathfrak{m} \in A_+$ so that $\mathfrak{n}^2 \det(x) \mid \mathfrak{m}$. The intersection number of $\mathcal{Z}_1$ and $\mathcal{Z}_x$ is equal to*

$$\mathcal{Z}_1 \cdot \mathcal{Z}_x = \frac{q-1}{[\Gamma_1 : \Gamma_1(\mathfrak{m})] \cdot [\Gamma_x : \Gamma_x(\mathfrak{m})]} \cdot \sum_{\gamma \in \Gamma / \Gamma_F(\mathfrak{m})} \widetilde{\mathcal{Z}}_1 \cdot \gamma \widetilde{\mathcal{Z}}_x.$$

Proof. Observe that $\mathcal{Z}_1$ is a $\mathbb{Q}$-Cartier divisor on $\mathcal{S}_{0,F}(\mathfrak{n})$. Thus the result is a rigid-analytic version of the projection formula (cf. [26, Remark 2.13 in Chapter 9]) for the intersection of $\mathcal{Z}_1$ and $\mathcal{Z}_x = \pi_*(\widetilde{\mathcal{Z}}_x)$. We include the argument here for completeness.

Let $\Gamma = \Gamma_{0,F}(\mathfrak{n})$. Given $x \in \Lambda$ with $\det(x) \neq 0$, the normalization of the irreducible curve X_x is isomorphic to $\widehat{\Gamma}_x \backslash \mathfrak{H}$. Notice that for each $\boldsymbol{z} \in X_1 \cap X_x$, take $\tilde{\boldsymbol{z}}_1$ and $\tilde{\boldsymbol{z}}_x$ be two lifts of $\boldsymbol{z}$ in $\widetilde{X}_1$ and $\widetilde{X}_x$, respectively. The intersection multiplicity of X_1 and X_x at $\boldsymbol{z}$ is actually equal to

$$m_{\boldsymbol{z}}(X_1, X_x) = \frac{(q-1)\cdot \#\operatorname{Stab}_\Gamma(\tilde{\boldsymbol{z}}_1)}{\#\operatorname{Stab}_{\widehat{\Gamma}_1}(\tilde{\boldsymbol{z}}_1)\cdot \#\operatorname{Stab}_{\widehat{\Gamma}_x}(\tilde{\boldsymbol{z}}_x)}.$$

Indeed, let $\pi_x := \pi\big|_{\tilde{X}_x} : \tilde{X}_x \to X_x$. As X_1 is a $\mathbb{Q}$-Cartier divisor on $\mathcal{S}_{0,F}(\mathfrak{n})$, we have the following equality (between $\mathbb{Q}$-divisors on X_x):

$$X_1\big|_{X_x} = \frac{1}{\deg \pi_x}\cdot \pi_{x,*}\big(\pi_x^*(X_1\big|_{X_x})\big) \quad \in \operatorname{Div}_{\mathbb{Q}}(X_x) := \mathbb{Q}\otimes_{\mathbb{Z}} \operatorname{Div}(X_x).$$

Let $i_{\boldsymbol{z}}(D)$ (resp. $i_{\tilde{\boldsymbol{z}}_x}(\tilde{D})$) be the multiplicity of a $\mathbb{Q}$-divisor D on X_x (resp. $\tilde{D}$ on $\tilde{X}_x$) at $\boldsymbol{z}$ (resp. $\tilde{\boldsymbol{z}}_x$). Take $\gamma \in \Gamma$ so that $\gamma\tilde{\boldsymbol{z}}_x = \tilde{\boldsymbol{z}}_1$. We then obtain that

$$\begin{aligned} m_{\boldsymbol{z}}(X_1, X_x) = i_{\boldsymbol{z}}(X_1\big|_{X_x}) &= \frac{q-1}{\#\operatorname{Stab}_{\widehat{\Gamma}_x}(\tilde{\boldsymbol{z}}_x)}\cdot i_{\tilde{\boldsymbol{z}}_x}\big(\pi_x^*(X_1\big|_{X_x})\big) \\ &= \frac{q-1}{\#\operatorname{Stab}_{\widehat{\Gamma}_x}(\tilde{\boldsymbol{z}}_x)}\cdot i_{\tilde{\boldsymbol{z}}_x}\big(\pi^*(X_1)\big|_{\tilde{X}_x}\big) \\ &= \frac{q-1}{\#\operatorname{Stab}_{\widehat{\Gamma}_x}(\tilde{\boldsymbol{z}}_x)}\cdot i_{\tilde{\boldsymbol{z}}_1}\big(\pi^*(X_1)\big|_{\gamma\tilde{X}_x}\big) \\ &= \frac{q-1}{\#\operatorname{Stab}_{\widehat{\Gamma}_x}(\tilde{\boldsymbol{z}}_x)}\cdot \frac{\#\operatorname{Stab}_\Gamma(\tilde{\boldsymbol{z}}_1)}{\#\operatorname{Stab}_{\widehat{\Gamma}_1}(\tilde{\boldsymbol{z}}_1)}. \end{aligned}$$

Now, consider the disjoint union

$$\Phi := \coprod_{\gamma \in \Gamma/\widehat{\Gamma}_x\cdot\Gamma_F(\mathfrak{m})} \widetilde{X}_1 \cap \gamma\widetilde{X}_x$$

which maps surjectively to $X_1 \cap X_x$ via the finite morphism π on each component (we denote this surjection $\Phi \twoheadrightarrow X_1 \cap X_x$ by $\tilde{\pi}$). For each point $\boldsymbol{z} \in X_1 \cap X_x$, the pre-image of $\boldsymbol{z}$ in Φ has cardinality equal to

$$[\widehat{\Gamma}_1 : \operatorname{Stab}_{\widehat{\Gamma}_1}(\tilde{\boldsymbol{z}}_1)\cdot\Gamma_1(\mathfrak{m})]\cdot \frac{\#\operatorname{Stab}_\Gamma(\tilde{\boldsymbol{z}}_1)}{\#\operatorname{Stab}_{\widehat{\Gamma}_x}(\tilde{\boldsymbol{z}}_x)}.$$

Thus the cardinality of Φ can be expressed as

$$\begin{aligned}
&\sum_{\gamma\in\Gamma/\widehat{\Gamma}_x\cdot\Gamma_F(\mathfrak{m})} \#(\widetilde{X}_1\cap\gamma\widetilde{X}_x)\\
=&\sum_{\boldsymbol{z}\in X_1\cap X_x}\#\big(\tilde{\pi}^{-1}(\boldsymbol{z})\big)\\
=&\sum_{\boldsymbol{z}\in X_1\cap X_x}[\widehat{\Gamma}_1:\operatorname{Stab}_{\widehat{\Gamma}_1}(\tilde{\boldsymbol{z}}_1)\cdot\Gamma_1(\mathfrak{m})]\cdot\frac{\#\operatorname{Stab}_\Gamma(\tilde{\boldsymbol{z}}_1)}{\#\operatorname{Stab}_{\widehat{\Gamma}_x}(\tilde{\boldsymbol{z}}_x)}\\
=&[\widehat{\Gamma}_1:\Gamma_1(\mathfrak{m})\mathbb{F}_q^\times]\cdot\sum_{\boldsymbol{z}\in X_1\cap X_x}\frac{(q-1)\cdot\#\operatorname{Stab}_\Gamma(\tilde{\boldsymbol{z}}_1)}{\#\operatorname{Stab}_{\widehat{\Gamma}_1}(\tilde{\boldsymbol{z}}_1)\cdot\#\operatorname{Stab}_{\widehat{\Gamma}_x}(\tilde{\boldsymbol{z}}_x)}\\
=&[\widehat{\Gamma}_1:\Gamma_1(\mathfrak{m})\mathbb{F}_q^\times]\cdot\sum_{\boldsymbol{z}\in X_1\cap X_x}m_{\boldsymbol{z}}(X_1,X_x).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{Z}_1\cdot\mathcal{Z}_x &= [\widehat{\Gamma}_1:\Gamma_1]\cdot[\widehat{\Gamma}_x:\Gamma_x]\cdot\sum_{\boldsymbol{z}\in X_1\cap X_x}m_{\boldsymbol{z}}(X_1,X_x)\\
&=\frac{[\widehat{\Gamma}_1:\Gamma_1]\cdot[\widehat{\Gamma}_x:\Gamma_x]}{[\widehat{\Gamma}_1:\Gamma_1(\mathfrak{m})\mathbb{F}_q^\times]}\cdot\sum_{\gamma\in\Gamma/\widehat{\Gamma}_x\cdot\Gamma_F(\mathfrak{m})}\#(\widetilde{X}_1\cap\gamma\widetilde{X}_x)\\
&=\frac{q-1}{[\Gamma_1:\Gamma_1(\mathfrak{m})]\cdot[\Gamma_x:\Gamma_x(\mathfrak{m})]}\cdot\sum_{\gamma\in\Gamma/\Gamma_F(\mathfrak{m})}\widetilde{\mathcal{Z}}_1\cdot\gamma\widetilde{\mathcal{Z}}_x,
\end{aligned}$$

where the last equality holds as $\widetilde{X}_1$ and $\gamma\widetilde{X}_x(=\widetilde{X}_{\gamma\star x})$ intersect transversally for every $\gamma\in\Gamma$. □

Proposition 4.5. *Let* $x\in\Lambda$ *with* $\det(x)\neq 0$, *and* $\mathfrak{m}\in A_+$ *so that* $\mathfrak{n}^2\det(x)$ *divides* $\mathfrak{m}$. *Given* $\gamma\in\Gamma$, *suppose* $\widetilde{\mathcal{Z}}_1\neq\gamma\widetilde{\mathcal{Z}}_x$. *We have*

$$\widetilde{\mathcal{Z}}_1\cdot\gamma\widetilde{\mathcal{Z}}_x=\sum_{\gamma_0\in\Gamma_1(\mathfrak{m})\backslash\Gamma_F(\mathfrak{m})/\Gamma_{\gamma\star x}(\mathfrak{m})}\#(\mathfrak{H}_1\cap\gamma_0\gamma\mathfrak{H}_x).$$

Proof. It suffices to show that the union

$$\bigcup_{\gamma_0\in\Gamma_1(\mathfrak{m})\backslash\Gamma_F(\mathfrak{m})/\Gamma_{\gamma\star x}(\mathfrak{m})}\mathfrak{H}_1\cap\gamma_0\gamma\mathfrak{H}_x\ (\subset\mathfrak{H}_F)$$

is disjoint and in bijection with the intersection points of $\widetilde{X}_1$ and $\gamma\widetilde{X}_x$ under the canonical map $\mathfrak{H}_F\to\mathcal{S}_F(\mathfrak{m})$.

The surjectivity is straightforward. On the other hand, given $\gamma_1,\gamma_2\in\Gamma_F(\mathfrak{m})$ and $\vec{z}_i\in\mathfrak{H}_1\cap\gamma_i\gamma\mathfrak{H}_x$ for $i=1,2$, write

$$\vec{z}_i=(z_i,S_1z_i)=(\gamma_i\gamma w_i,\gamma_i'\gamma'S_xw_i)\quad\text{ with } z_i,w_i\in\mathfrak{H}\text{ for } i=1,2.$$

Suppose the image of $\vec{z}_1$ and $\vec{z}_2$ coincides in $\mathcal{S}_F(\mathfrak{m})$, i.e. there exists $\gamma_0 \in \Gamma_F(\mathfrak{m})$ so that $(z_1, S_1 z_1) = (\gamma_0 z_2, \gamma_0' S_1 z_2)$. Then $\deg \gamma_0 = 1$ and

$$S_1 z_1 = \gamma_0' S_1 z_2 = \gamma_0' S_1 \gamma_0^{-1} z_1,$$

which says $(\gamma_0 \gamma_0^*) z_1 = z_1$. From our choice of $\mathfrak{m}$, we get $\gamma_0 \gamma_0^* \in \Gamma_F(\mathfrak{n} \det(x))$ which fixes z_1. This implies $\gamma_0 \gamma_0^* = 1$. As $\det \gamma_0 = 1$, we have that $\gamma_0^* = \bar{\gamma}_0$, whence

$$\gamma_0 \in \Gamma_1 \cap \Gamma_F(\mathfrak{m}) = \Gamma_1(\mathfrak{m}).$$

Moreover, let

$$\gamma_3 = \gamma^{-1} \gamma_1^{-1} \gamma_0 \gamma_2 \gamma \in \Gamma_F(\mathfrak{m}) \quad \text{(as } \Gamma_F(\mathfrak{m}) \text{ is normal in } \Gamma).$$

We get $\det \gamma_3 = 1$ and $(w_1, S_x w_1) = (\gamma_3 w_2, \gamma_3' S_x w_2)$, which says

$$(\gamma_3 \cdot \bar{x}^{-1} \gamma_3^* \bar{x}) w_1 = w_1.$$

Similarly, from our choice of $\mathfrak{m}$ we get $\gamma_3 \cdot \bar{x}^{-1} \gamma_3^* \bar{x} \in \Gamma_F(\mathfrak{n})$ and fixes w_1. Thus

$$\gamma_3 \cdot \bar{x}^{-1} \gamma_3^* \bar{x} = 1, \qquad \text{which shows that } x \gamma_3^* = \bar{\gamma}_3 x \text{ (as } \det \gamma_3 = 1).$$

Therefore $\gamma_3 \in \Gamma_x \cap \Gamma_F(\mathfrak{m}) = \Gamma_x(\mathfrak{m})$. In conclusion, we have

$$\gamma_1 \cdot (\gamma \gamma_3 \gamma^{-1}) = \gamma_0 \gamma_2,$$

i.e. γ_1 and γ_2 represents the same double cosets in $\Gamma_1(\mathfrak{m}) \backslash \Gamma_F(\mathfrak{m}) / \Gamma_{\gamma \star x}(\mathfrak{m})$. This assures the injectivity and completes the proof. □

LEMMA 4.6. *Given* $x \in \Lambda$ *with* $\det(x) \neq 0$. *For* $\gamma \in \Gamma$ *with* $\gamma \mathfrak{H}_x \neq \mathfrak{H}_1$ *one has*

$$\mathfrak{H}_1 \cap \gamma \mathfrak{H}_x = \{\vec{z} = (z, S_1 z) \mid (\gamma \star x) \cdot z = z\}.$$

Consequently, put

$$\tilde{\imath}(x) := \begin{cases} 1 & \text{if } K_x/k \text{ is an imaginary quadratic field extension;} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\#(\mathfrak{H}_1 \cap \gamma \mathfrak{H}_x) = 2 \cdot \tilde{\imath}(\gamma \star x).$$

Proof. Given $\vec{z} \in \mathfrak{H}_1 \cap \gamma \mathfrak{H}_x$, write $\vec{z} = (z, S_1 z) = (\gamma w, \gamma' S_x w)$ for $z, w \in \mathfrak{H}$. We get

$$\gamma' S_x \gamma^{-1} z = \gamma' S_x w = S_1 z.$$

Thus

$$\begin{aligned} z &= \gamma S_x^{-1} \gamma'^{-1} S_1 z = \gamma \bar{x}^{-1} (S_1^{-1} \gamma'^{-1} S_1) \cdot z \\ &= \gamma x (S_1^{-1} \bar{\gamma}' S_1) \cdot z = (\gamma \star x) \cdot z. \end{aligned}$$

Conversely, given $z \in \mathfrak{H}$ so that $(\gamma \star x) \cdot z = z$, we obtain $\gamma' S_x \gamma^{-1} z = S_1 z$. Let $w = \gamma^{-1} z$. Then

$$\vec{z} := (z, S_1 z) = (\gamma w, \gamma' S_x w) \in \mathfrak{H}_1 \cap \gamma \mathfrak{H}_x.$$

This shows the first equality. Note that from the assumption that $\gamma \mathfrak{H}_x \neq \mathfrak{H}_1$, the element $\gamma \star x \notin k^\times$. Write $\gamma \star x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in F \hookrightarrow k_\infty$. Observe that $(\gamma \star x) \cdot z = z$ if and only if the column vector $(z, 1)^t$ is an eigen-vector of $\gamma \star x$ with respect to the eigen-value $cz + d$. This implies that $K_x \cong k(z)$ is an imaginary quadratic field over k, whence the second equality holds. □

4.5 Geometric interpretation of the Fourier coefficients of ϑ_Λ

For non-zero $x \in \Lambda$, recall the number $\iota(x)$ defined in (3.15). Given $\gamma \in \Gamma$ with $\gamma \mathfrak{H}_x \neq \mathfrak{H}_1$ (which implies $B_{\gamma \star x} \neq B_1$), observe that for $\gamma_1 \in \Gamma_1$, and $\gamma_x \in \Gamma_x$ one has

$$\tilde{\iota}((\gamma_1 \gamma \gamma_x) \star x) = \tilde{\iota}(\gamma \star x) = \#(\Gamma_1 \cap \Gamma_{\gamma \star x}) \cdot \iota(\gamma \star x)/(q-1).$$

We are now able to express the intersection number $\mathcal{Z}_1 \cdot \mathcal{Z}_x$ as follows:

Theorem 4.7. *Given $x \in \Lambda$ with $\det x \neq 0$. Suppose $\mathcal{Z}_1 \neq \mathcal{Z}_x$, or equivalently, $\gamma \star x \notin k$ for every $\gamma \in \Gamma$. Then*

$$\mathcal{Z}_1 \cdot \mathcal{Z}_x = 2 \cdot \sum_{\gamma \in \Gamma_1 \backslash \Gamma / \Gamma_x} \iota(\gamma \star x).$$

Proof. From Proposition 4.5 and Lemma 4.6 we have

$$\begin{aligned} \mathcal{Z}_1 \cdot \mathcal{Z}_x &= \frac{q-1}{[\Gamma_1 : \Gamma_1(\mathfrak{m})] \cdot [\Gamma_x : \Gamma_x(\mathfrak{m})]} \cdot \sum_{\gamma \in \Gamma / \Gamma_F(\mathfrak{m})} \widetilde{\mathcal{Z}}_1 \cdot \gamma \widetilde{\mathcal{Z}}_x \\ &= \frac{q-1}{[\Gamma_1 : \Gamma_1(\mathfrak{m})] \cdot [\Gamma_x : \Gamma_x(\mathfrak{m})]} \cdot \sum_{\gamma \in \Gamma_1(\mathfrak{m}) \backslash \Gamma / \Gamma_x(\mathfrak{m})} 2 \cdot \tilde{\iota}(\gamma \star x) \\ &= \sum_{\gamma \in \Gamma_1 \backslash \Gamma / \Gamma_x} \frac{q-1}{\#(\Gamma_1 \cap \Gamma_{\gamma \star x})} \cdot 2 \cdot \tilde{\iota}(\gamma \star x) \\ &= 2 \cdot \sum_{\gamma \in \Gamma_1 \backslash \Gamma / \Gamma_x} \iota(\gamma \star x). \end{aligned}$$

□

We now define the self-intersection number of $\mathcal{Z}_1$ (following [17, p. 84]). First, put

$$\mathrm{vol}(X_1) := -\frac{2}{[\widehat{\Gamma}_1 : \Gamma_1]} \cdot H^{\mathfrak{d}^+\mathfrak{n}^+, \mathfrak{d}^-\mathfrak{n}^-}(0) \tag{4.3}$$

and

$$\mathrm{vol}(\mathcal{Z}_1) \;:=\; [\widehat{\Gamma}_1 : \Gamma_1] \cdot \mathrm{vol}(X_1) \;=\; -2H^{\mathfrak{d}^+\mathfrak{n}^+,\mathfrak{d}^-\mathfrak{n}^-}(0). \tag{4.4}$$

For each point $\boldsymbol{z} \in X_1$, take a lift $\tilde{\boldsymbol{z}} \in \mathfrak{H}_1$, and let

$$r_{\boldsymbol{z}} := \frac{\#\big(\mathrm{Stab}_\Gamma(\tilde{\boldsymbol{z}})\big)}{\#\big(\mathrm{Stab}_{\widehat{\Gamma}_1}(\tilde{\boldsymbol{z}})\big)}.$$

We set the following “Plücker-type” number:

$$\mu_{\boldsymbol{z}}(X_1) := \frac{q-1}{\#(\mathrm{Stab}_\Gamma(\tilde{\boldsymbol{z}}))} \cdot \big(r_{\boldsymbol{z}}(r_{\boldsymbol{z}}-1)\big).$$

Definition 4.8. The self-intersection number of $\mathcal{Z}_1$ is then defined to be:

$$\mathcal{Z}_1 \cdot \mathcal{Z}_1 := [\widehat{\Gamma}_1 : \Gamma_1]^2 \cdot \left(-\mathrm{vol}(X_1) + \sum_{\boldsymbol{z} \in X_1} \mu_{\boldsymbol{z}}(X_1)\right).$$

Lemma 4.9. *We may express the self-intersection number of $\mathcal{Z}_1$ as follows:*

$$\mathcal{Z}_1 \cdot \mathcal{Z}_1 = 2 \cdot \sum_{\gamma \in \Gamma_1 \backslash \Gamma / \Gamma_1} \iota(\gamma \star 1).$$

Proof. Given $\gamma \in \Gamma$, notice that $\gamma \star 1 \in k$ if and only if $\gamma \in \widehat{\Gamma}_1$. As Γ_1 is normal in $\widehat{\Gamma}_1$, one has

$$\begin{aligned} 2 \cdot \sum_{\gamma \in \Gamma_1 \backslash \Gamma / \Gamma_1} \iota(\gamma \star 1) &= 2 \cdot \sum_{\gamma_1 \in \widehat{\Gamma}_1 / \Gamma_1} \iota(\gamma \star 1) + 2 \cdot \sum_{\substack{\gamma \in \Gamma_1 \backslash \Gamma / \Gamma_1 \\ \gamma \notin \widehat{\Gamma}_1}} \iota(\gamma \star 1) \\ &= [\widehat{\Gamma}_1 : \Gamma_1] \cdot \big(2H^{\mathfrak{d}^+\mathfrak{n}^+,\mathfrak{d}^-\mathfrak{n}^-}(0)\big) + 2 \cdot \sum_{\substack{\gamma \in \Gamma_1 \backslash \Gamma / \Gamma_1 \\ \gamma \notin \widehat{\Gamma}_1}} \iota(\gamma \star 1). \end{aligned}$$

Because of (4.3), the result holds if we show

$$[\widehat{\Gamma}_1 : \Gamma_1]^2 \cdot \sum_{\boldsymbol{z} \in X_1} \mu_{\boldsymbol{z}}(X_1) \;=\; 2 \cdot \sum_{\substack{\gamma \in \Gamma_1 \backslash \Gamma / \Gamma_1 \\ \gamma \notin \widehat{\Gamma}_1}} \iota(\gamma \star 1). \tag{4.5}$$

Take $\mathfrak{m} \in A_+$ with $\mathfrak{n}^2 \mid \mathfrak{m}$. Adapting the argument in the proof of Proposition 4.4 (which we omit the details), we get

$$[\widehat{\Gamma}_1 : \Gamma_1]^2 \cdot \sum_{\boldsymbol{z} \in X_1} \mu_{\boldsymbol{z}}(X_1) = \frac{q-1}{[\Gamma_1 : \Gamma_1(\mathfrak{m})]^2} \cdot \sum_{\substack{\gamma \in \Gamma / \Gamma_F(\mathfrak{m}) \\ \gamma \notin \widehat{\Gamma}_1}} \widetilde{\mathcal{Z}}_1 \cdot \gamma \widetilde{\mathcal{Z}}_1.$$

From Proposition 4.5 and Lemma 4.6, we have

$$\begin{aligned}
\sum_{\substack{\gamma \in \Gamma/\Gamma_F(\mathfrak{m}) \\ \gamma \notin \widehat{\Gamma}_1}} \widetilde{\mathcal{Z}}_1 \cdot \gamma \widetilde{\mathcal{Z}}_1 &= \sum_{\substack{\gamma \in \Gamma_1(\mathfrak{m}) \backslash \Gamma/\Gamma_1(\mathfrak{m}) \\ \gamma \notin \widehat{\Gamma}_1}} 2 \cdot \tilde{\iota}(\gamma \star 1) \\
&= \frac{[\Gamma_1 : \Gamma_1(\mathfrak{m})]^2}{q-1} \cdot \sum_{\substack{\gamma \in \Gamma_1 \backslash \Gamma/\Gamma_1 \\ \gamma \notin \widehat{\Gamma}_1}} \frac{q-1}{\#(\Gamma_1 \cap \Gamma_{\gamma \star 1})} \cdot 2 \cdot \tilde{\iota}(\gamma \star 1) \\
&= \frac{[\Gamma_1 : \Gamma_1(\mathfrak{m})]^2}{q-1} \cdot 2 \cdot \sum_{\substack{\gamma \in \Gamma_1 \backslash \Gamma/\Gamma_1 \\ \gamma \notin \widehat{\Gamma}_1}} \iota(\gamma \star 1).
\end{aligned}$$

Therefore the equality (4.5) follows and the proof is complete. □

For non-zero $a \in A$, consider the following *Hirzebruch-Zagier-type divisor*

$$\mathcal{Z}(a) := \sum_{\Gamma \backslash \Lambda_a} \mathcal{Z}_x.$$

From (3.16), Theorem 3.14, Theorem 4.7, Lemma 4.9, Theorem 3.9 and (4.4), we finally arrive at:

Corollary 4.10. *Given non-zero $a \in A$ and $y \in k_\infty^\times$ with $\deg a \leq 2\,\mathrm{ord}_\infty(y) + 2$, we have*

$$\mathrm{vol}(O_{B_{1,\mathbb{A}}^\times}/O_{\mathbb{A}})^{-1} \cdot I^*(a, y; \varphi_\Lambda) = \frac{|y|_\infty^2}{2} \cdot (\mathcal{Z}_1 \cdot \mathcal{Z}(a)),$$

and

$$\mathrm{vol}(O_{B_{1,\mathbb{A}}^\times}/O_{\mathbb{A}})^{-1} \cdot I^*(0, y; \varphi_\Lambda) = -\frac{|y|_\infty^2}{2} \cdot \mathrm{vol}(\mathcal{Z}_1).$$

Remark 4.11. From Theorem 3.18, we may express the Fourier expansion of the Drinfeld-type automorphic form ϑ_Λ defined in (3.17) in terms of the corresponding intersection numbers: for $(x, y) \in k_\infty^\times \times k_\infty$,

$$\vartheta_\Lambda \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = |y|_\infty \cdot \left[-\mathrm{vol}(\mathcal{Z}_1) + \sum_{\substack{0 \neq a \in A \\ \deg a + 2 \leq \mathrm{ord}_\infty(y)}} \big(\mathcal{Z}_1 \cdot \mathcal{Z}(a)\big) \cdot \psi_\infty(ax) \right].$$

Therefore in our case, ϑ_Λ plays the same role as Gekeler's improper Eisenstein series in the Kronecker-Hurwitz class number relation over function fields discussed in *Remark* 1.1.

A Local optimal embeddings

Here we recall the needed properties of local optimal embeddings from a quadratic order into a hereditary order of a quaternion algebra over a local field. Further details are referred to [35, Chapter 2, Section 3] and [3, Chapter 5, Section 1.1].

Let $(L, |\cdot|_L)$ be a non-archimedean local field, and O_L be the ring of integers in L. Given a separable quadratic algebra E over L and a quaternion algebra $\mathcal{D}$ over L together with a fixed embedding $\iota : E \hookrightarrow \mathcal{D}$, it is known that every embedding from E into $\mathcal{D}$ must be conjugates of ι by elements of $\mathcal{D}^\times$. Let $\mathcal{O}$ be an O_L-order in E and $O_{\mathcal{D}}$ a maximal O_L-order in $\mathcal{D}$. Put

$$\mathcal{E}(\mathcal{O}, O_{\mathcal{D}}) := \{b \in \mathcal{D}^\times \mid b^{-1}Eb \cap O_{\mathcal{D}} = b^{-1}\mathcal{O}b\},$$

where we identify E as a subalgebra of $\mathcal{D}$ via ι. For $\alpha \in E$, $b \in \mathcal{E}(\mathcal{O}, O_{\mathcal{D}})$, and $\kappa \in O_{\mathcal{D}}^\times$, one has

$$\alpha \cdot b \cdot \kappa \in \mathcal{E}(\mathcal{O}, O_{\mathcal{D}}).$$

Moreover, the following result holds (cf. [35, Chapter 2, Theorem 3.1 and 3.2]):

Lemma A.1. *(1) Let O_E be the maximal O_L-order in E. Then*

$$e(O_E, O_{\mathcal{D}}) := \#\left(E^\times \backslash \mathcal{E}(O_E, O_{\mathcal{D}})/O_{\mathcal{D}}^\times\right) = \begin{cases} 2, & \text{if } \mathcal{D} \text{ is division and } E/L \text{ is inert;} \\ 0, & \text{if } \mathcal{D} \text{ is division and } E/L \text{ is split;} \\ 1, & \text{otherwise.} \end{cases}$$

(2) If $\mathcal{O} \subsetneq O_E$, then

$$e(\mathcal{O}, O_{\mathcal{D}}) := \#\left(E^\times \backslash \mathcal{E}(\mathcal{O}, O_{\mathcal{D}})/O_{\mathcal{D}}^\times\right) = \begin{cases} 0, & \text{if } \mathcal{D} \text{ is division;} \\ 1, & \text{otherwise.} \end{cases}$$

Suppose $\mathcal{D}$ is not division (i.e. $\mathcal{D} \cong \mathrm{Mat}_2(L)$). Let $O'_{\mathcal{D}}$ be a hereditary O_L-order in $O_{\mathcal{D}}$. Put

$$\mathcal{E}(\mathcal{O}, O'_{\mathcal{D}}) := \{b \in \mathcal{D}^\times \mid b^{-1}Eb \cap O'_{\mathcal{D}} = b^{-1}\mathcal{O}b\}.$$

Then for $\alpha \in E$, $b \in \mathcal{E}(\mathcal{O}, O'_{\mathcal{D}})$, and $\kappa' \in (O'_{\mathcal{D}})^\times$, one has

$$\alpha \cdot b \cdot \kappa' \in \mathcal{E}(\mathcal{O}, O'_{\mathcal{D}}).$$

Moreover (cf. [35, Chapter 2, Theorem 3.2]):

Lemma A.2. *(1)*

$$e(O_E, O'_{\mathcal{D}}) := \#(E^\times \backslash \mathcal{E}(O_E, O'_{\mathcal{D}})/(O'_{\mathcal{D}})^\times) = \begin{cases} 0, & \text{if } E/L \text{ is inert;} \\ 1, & \text{if } E/L \text{ is ramified;} \\ 2, & \text{if } E/L \text{ is split.} \end{cases}$$

(2) If $\mathcal{O} \subsetneq O_E$, then

$$e(\mathcal{O}, O'_{\mathcal{D}}) := \#(E^\times \backslash \mathcal{E}(\mathcal{O}, O'_{\mathcal{D}})/(O'_{\mathcal{D}})^\times) = 2.$$

B Special local integrals

Given $c \in \mathbb{Z}_{\geq 0}$, put $\mathcal{O}(c) := O_L + \pi_L^c O_E$, where $\pi_L \in O_L$ is a uniformizer in L. For $x \in E \backslash L$, we can find a unique $c_x \in \mathbb{Z}_{\geq 0}$ if $x \in O_E$ so that $O_L[x] = \mathcal{O}(c_x)$; and put $c_x := -1$ if $x \notin O_E$. Let $\mathcal{D}^o$ be the space of pure quaternions in $\mathcal{D}$, i.e.

$$\mathcal{D}^o := \{b \in \mathcal{D} \mid \operatorname{Tr}(b) = 0\}.$$

Put $O_{\mathcal{D}}^o := O_{\mathcal{D}} \cap \mathcal{D}^o$ and $O_{\mathcal{D}}'^{,o} := O'_{\mathcal{D}} \cap \mathcal{D}^o$. We observe that:

Lemma B.1. *Given $x \in E\backslash L$ with $\operatorname{Tr}(x) = 0$, one has*

$$\mathbf{1}_{O_{\mathcal{D}}^o}(b^{-1}xb) = \sum_{\ell=0}^{c_x} \mathbf{1}_{\mathcal{E}(\mathcal{O}(\ell), O_{\mathcal{D}})}(b),$$

Moreover, if $\mathcal{D}$ is not division, then

$$\mathbf{1}_{O_{\mathcal{D}}'^{,o}}(b^{-1}xb) = \sum_{\ell=0}^{c_x} \mathbf{1}_{\mathcal{E}(\mathcal{O}(\ell), O'_{\mathcal{D}})}(b).$$

Proof. Notice that $\mathcal{E}\left(\mathcal{O}(\ell), \mathcal{O}_{\mathcal{D}}\right)$ and $\mathcal{E}\left(\mathcal{O}(\ell'), \mathcal{O}_{\mathcal{D}}\right)$ are disjoint if $\ell \neq \ell'$. Thus for $b \in \mathcal{D}^\times$ one has

$$\sum_{\ell=0}^{c_x} \mathbf{1}_{\mathcal{E}(\mathcal{O}(\ell), \mathcal{O}_{\mathcal{D}})}(b) = 0 \text{ or } 1.$$

Suppose the value is 1, i.e. $b \in \mathcal{E}(\mathcal{O}(\ell_0), \mathcal{O}_{\mathcal{D}})$ for some $0 \leq \ell_0 \leq c_x$. Then

$$x \in \mathcal{O}[x] = \mathcal{O}(c_x) \subset \mathcal{O}(\ell_0) \subset E \cap b\mathcal{O}_{\mathcal{D}}b^{-1} \subset b\mathcal{O}_{\mathcal{D}}b^{-1}.$$

Since $\operatorname{Tr}(x) = 0$, we get $b^{-1}xb \in \mathcal{O}_{\mathcal{D}}^\circ$, i.e. $\mathbf{1}_{\mathcal{O}_{\mathcal{D}}^\circ}(b^{-1}xb) = 1$.
Conversely, let $b \in \mathcal{D}^\times$ with $\mathbf{1}_{\mathcal{O}_{\mathcal{D}}^\circ}(b^{-1}xb) = 1$. Then $x \in b\mathcal{O}_{\mathcal{D}}^\circ b^{-1}$, which implies $\mathcal{O}(c_x) \subset E \cap b\mathcal{O}_{\mathcal{D}}b^{-1}$. Thus there exists ℓ_0 with $0 \leq \ell_0 \leq c_x$ such that

$$E \cap b\mathcal{O}_{\mathcal{D}}b^{-1} = \mathcal{O}(\ell_0).$$

which means that $b \in \mathcal{E}(\mathcal{O}(\ell_0), \mathcal{O}_{\mathcal{D}})$. Therefore

$$\sum_{\ell=0}^{c_x} \mathbf{1}_{\mathcal{E}(\mathcal{O}(\ell), \mathcal{O}_{\mathcal{D}})}(b) = \mathbf{1}_{\mathcal{E}(\mathcal{O}(\ell_0), \mathcal{O}_{\mathcal{D}})}(b) = 1.$$

□

Suppose Haar measures of $\mathcal{D}^\times$ and $E^\times$ are chosen, respectively. The above lemma leads to:

Corollary B.2. *For $x \in E\backslash L$ with $\mathrm{Tr}(x)=0$, one has*

$$\int_{E^\times\backslash \mathcal{D}^\times} \mathbf{1}_{O_{\mathcal{D}}^{\circ}}(b^{-1}xb)\, d^\times b \;=\; \frac{\mathrm{vol}(O_{\mathcal{D}}^\times)}{\mathrm{vol}(O_E^\times)} \cdot \sum_{\ell=0}^{c_x} \#\left(\frac{O_E^\times}{\mathcal{O}(\ell)^\times}\right) \cdot e(\mathcal{O}(\ell), O_{\mathcal{D}}).$$

Suppose $\mathcal{D}$ is not division, then

$$\int_{E^\times\backslash \mathcal{D}^\times} \mathbf{1}_{O_{\mathcal{D}}^{\prime,\circ}}(b^{-1}xb)\, d^\times b \;=\; \frac{\mathrm{vol}(O_{\mathcal{D}}^{\prime\times})}{\mathrm{vol}(O_E^\times)} \cdot \sum_{\ell=0}^{c_x} \#\left(\frac{O_E^\times}{\mathcal{O}(\ell)^\times}\right) \cdot e(\mathcal{O}(\ell), O_{\mathcal{D}}').$$

Proof. Given $0 \le \ell \le c_x$, one has

$$\begin{aligned}\mathrm{vol}(E^\times\backslash \mathcal{E}(\mathcal{O}(\ell),\mathcal{O}_{\mathcal{D}})) &= \sum_{b\in E^\times\backslash \mathcal{E}(\mathcal{O}(\ell),\mathcal{O}_D)/\mathcal{O}_{\mathcal{D}}^\times} \frac{\mathrm{vol}(\mathcal{O}_{\mathcal{D}}^\times)}{\mathrm{vol}(E^\times \cap b\mathcal{O}_{\mathcal{D}}^\times b^{-1})} \\ &= \frac{\mathrm{vol}(\mathcal{O}_{\mathcal{D}}^\times)}{\mathrm{vol}\left(\mathcal{O}_E^\times\right)} \cdot \#\left(\frac{\mathcal{O}_E^\times}{\mathcal{O}(\ell)^\times}\right) \cdot e\left(\mathcal{O}(\ell),\mathcal{O}_{\mathcal{D}}\right).\end{aligned}$$

Thus

$$\begin{aligned}\int_{E^\times\backslash \mathcal{D}^\times} \mathbf{1}_{\mathcal{O}_{\mathcal{D}}^{\circ}}(b^{-1}xb)\, d^\times b &= \sum_{\ell=0}^{c_x} \int_{E^\times\backslash \mathcal{D}^\times} \mathbf{1}_{\mathcal{E}(\mathcal{O}(\ell),\mathcal{O}_{\mathcal{D}})}(b)\, d^\times b \\ &= \frac{\mathrm{vol}(\mathcal{O}_{\mathcal{D}}^\times)}{\mathrm{vol}(\mathcal{O}_E^\times)} \cdot \sum_{\ell=0}^{c_x} \#\left(\frac{\mathcal{O}_E^\times}{\mathcal{O}(\ell)^\times}\right) \cdot e\left(\mathcal{O}(\ell),\mathcal{O}_{\mathcal{D}}\right).\end{aligned}$$

□

Let q_L be the cardinality of the residue field of L. Since

$$\mathrm{vol}(O_{\mathcal{D}}^{\prime\times}) = \frac{1}{q_L+1}\cdot \mathrm{vol}(O_{\mathcal{D}}^\times),$$

combining Lemma A.1, Lemma A.2, and Corollary B.2 we obtain:

Corollary B.3. *Suppose $\mathcal{D}$ is not division. Then for $x \in O_E\backslash O_L$ with $\mathrm{Tr}(x)=0$, one has*

$$\begin{aligned}&\int_{E^\times\backslash \mathcal{D}^\times} \left(\mathbf{1}_{O_{\mathcal{D}}^{\circ}}(b^{-1}xb) - \frac{q_L+1}{2}\cdot \mathbf{1}_{O_{\mathcal{D}}^{\prime,\circ}}(b^{-1}xb)\right) d^\times b \\ &= \begin{cases} \dfrac{1}{e(E/L)}\cdot \dfrac{\mathrm{vol}(O_{\mathcal{D}}^\times)}{\mathrm{vol}(O_E^\times)}, & \text{if } E \text{ is a field;} \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Here $e(E/L)$ is the ramification index of E/L.

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