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Small-time local stabilization of the two-dimensional incompressible Navier–Stokes equations

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Abstract. We construct explicit time-varying feedback laws that locally stabilize the two-dimensional internal controlled incompressible Navier–Stokes equations in arbitrarily small time. We also obtain quantitative rapid stabilization via stationary feedback laws, as well as quantitative null-controllability with explicit controls having $e^{C/T}$ costs.

1. Introduction

Let Ω be a bounded connected open set in $\mathbb{R}^2$ with smooth boundary. Let the controlled domain $\omega \subset \Omega$ be a nonempty open subset. We are interested in the stabilization and null-controllability of the incompressible Navier–Stokes system with internal control,

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = 1_\omega f & \text{in } \Omega, \\ \operatorname{div} y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the state $y(t, \cdot)$ and the control term $f(t, \cdot)$ belong to the space $\mathcal{H} \subset L^2(\Omega)^2$. In this paper we adapt the standard incompressible fluid mechanics framework,

$$\begin{aligned} \mathcal{H} &:= \{ y \in L^2(\Omega)^2 : \operatorname{div} y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \partial\Omega \} \\ & \qquad\qquad\qquad\qquad \text{with } \|y\|_{\mathcal{H}} := \|y\|_{L^2(\Omega)} \text{ for } y \in \mathcal{H}, \\ \mathcal{V} &:= \{ y \in H_0^1(\Omega)^2 \} \qquad\qquad \text{with } \|y\|_{\mathcal{V}} := \|\nabla y\|_{L^2(\Omega)} \text{ for } y \in \mathcal{V}, \\ \mathcal{V}_\sigma &:= \{ y \in H_0^1(\Omega)^2 : \operatorname{div} y = 0 \text{ in } \Omega \} \quad \text{with } \|y\|_{\mathcal{V}_\sigma} := \|\nabla y\|_{L^2(\Omega)} \text{ for } y \in \mathcal{V}_\sigma, \end{aligned}$$

satisfying $\mathcal{V}_\sigma \hookrightarrow \mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow L^2(\Omega)^2 \hookrightarrow \mathcal{V}' \hookrightarrow \mathcal{V}_\sigma'$. Here, in order to simplify the presentation, we have taken the viscosity coefficient as 1.

When dealing with stabilization problems, the control term f is regarded as a feedback control governed by some "feedback application" that depends on current states and time, $U(t; y)$:

$$f(t, x) := U(t; y(t, x)), \tag{1.2}$$

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where the application U is the so-called *time-varying feedback law*,

$$\begin{cases} U: \mathbb{R} \times \mathcal{H} \to \mathcal{H}, \\ \qquad (t; y) \mapsto U(t; y). \end{cases} \tag{1.3}$$

The *closed-loop system* associated to the preceding feedback law U is the evolution system (1.1)–(1.3). A *stationary* feedback law is such an application that only depends on $\mathcal{H}$. A T*-periodic* feedback law is a time-varying feedback law that is periodic with respect to time, i.e. $U(T + t; y) = U(t; y)$.

A *proper* feedback law U, roughly speaking, is some time-varying feedback law such that, for every $s \in \mathbb{R}$, and for every $y_0 \in \mathcal{H}$ as initial state at time s, i.e. $y(s, x) = y_0(x)$, the closed-loop system (1.1)–(1.3) admits a unique solution. For the closed-loop system with some proper feedback law we can define a "*flow*", $\Phi(t, s; y_0)$, as the state at time t of the solution of (1.1)–(1.3) with initial state $y(s, x) = y_0(x)$.

The controllability of Navier–Stokes equations has been extensively studied in the last decades. Let us mention, for example, [20, 29, 31, 33, 45, 49] for references on local controllability and [1, 19, 23, 24] for global controllability results. In order to study local controllability of nonlinear systems, it is classical to first study linearized control systems and expect nonlinear results via perturbation. In [30], based on this linearization approach and the global Carleman estimate method introduced by Fursikov–Imanuvilov [34], Fernández-Cara–Guerrero–Imanuvilov–Puel have proved the local exact controllability of the incompressible Navier–Stokes equations. This strategy also leads to the local exact controllability of the compressible Navier–Stokes equations [29]. We also refer to [20, 22] on local controllability with reduced control terms (i.e. the control terms have at least one vanishing component): for these systems one cannot directly obtain the desired controllability properties by considering the linearized systems around equilibrium points. Instead, one can construct some trajectory that starts from some equilibrium point and ends at the same point. In some circumstances it turns out that the linearized system around such a trajectory is indeed controllable, which further yields the local controllability of the nonlinear system: this is the so-called "return method". First introduced by Coron for asymptotic stabilization problems [16], this method has been successfully applied to a large class of nonlinear control systems, including incompressible Euler equations [17, 35], the one-dimensional isentropic Euler equations [36], and the Navier–Stokes equations [19, 23, 24]. We refer to [37] for a heuristic introduction on this method.

Concerning global controllability much less is known. One of the main difficulties is due to the fact that generally a linear result does not imply any global property. However, in some circumstances the "return method" together with scaling arguments yields global results; for instance, this is the case for the global controllability of Navier–Stokes equations on Riemannian manifolds [19]. Due to this main challenge, many global controllability problems still remain open, among which one of the most widely known problems is the "Lions' problem": whether we will be able to globally control Navier–Stokes equations via boundary control in arbitrarily small time. By far probably the best known contributions to this open problem are given by [23, 24].

The study of local exponential stabilization of Navier–Stokes equations is fruitful. Notably, based on the Riccati method, different types of local exponential stabilization results have been achieved: stabilization around equilibrium points or stabilization around other trajectories, using either internal feedback laws or boundary feedback laws. For example, we refer to [4,7] for local exponential stabilization with finite-dimensional internal feedback laws, to [5,32,48] for exponential stabilization with boundary feedback laws, and to [6, 11, 47] for stabilization around trajectories.

To the best of our knowledge, results on quantitative rapid stabilization or even finite-time stabilization of Navier–Stokes equations are extremely limited; we refer to [28] for a detailed review of these quantitative rapid stabilization problems. Recently, the author has introduced a method to stabilize the multidimensional heat equation in finite time [52], which is based on various techniques including *spectral inequalities*, *Lyapunov functionals*, and *piecewise feedback laws*. The main idea of the quantitative stabilization is on the construction of a Lyapunov functional such that spectral inequalities can be naturally adapted, leading to some quantitative estimates. Then this quantitative decay property, together with piecewise arguments, yields finite-time stabilization and small-time null-controllability:

$$\text{spectral inequality} + \text{Lyapunov functional} \Rightarrow \text{quantitative rapid stabilization},$$
$$\text{quantitative rapid stabilization} + \text{piecewise feedback law} \Rightarrow \text{finite-time stabilization},$$
$$\text{finite-time stabilization} \Rightarrow \text{constructive small-time null-controllability}.$$

Inspired by [52], in this paper we have proved the following theorems concerning quantitative rapid stabilization, constructive small-time null-controllability with cost estimates, and finite-time stabilization for the two-dimensional incompressible internal controlled Navier–Stokes equations.

Theorem 1.1 (Quantitative rapid stabilization). *There exists an effectively computable constant $C_2 > 0$ such that for any $\lambda > 0$ we can construct an explicit stationary feedback law $\mathcal{F}_\lambda\colon \mathcal{H} \to \mathcal{H}$, such that the closed-loop system (1.1)–(1.2) with the feedback law $U(t; y) := \mathcal{F}_\lambda y$ is locally exponentially stable:*

$$\|\Phi(t, s; y_0)\|_{\mathcal{H}} + \|\mathcal{F}_\lambda \Phi(t, s; y_0)\|_{\mathcal{H}} \leq 2C_2 e^{C_2\sqrt{\lambda}} e^{-\frac{\lambda}{4}(t-s)} \|y_0\|_{\mathcal{H}} \quad \forall s \in \mathbb{R},\ \forall t \in [s, +\infty),$$

for any $\|y_0\|_{\mathcal{H}} \leq C_2^{-1} e^{-C_2\sqrt{\lambda}}$.

Theorem 1.2 (Constructive null-controllability with cost estimates). *There exists an effectively computable constant $C_3 > 0$ such that, for any $T \in (0, 1)$ and for any $\|y_0\|_{\mathcal{H}} \leq e^{-\frac{C_3}{T}}$, we can find an explicit control $f|_{[0,T]}(t, x)$ satisfying*

$$\|f(t, x)\|_{L^\infty(0,T;\mathcal{H})} \leq e^{\frac{C_3}{T}} \|y_0\|_{\mathcal{H}},$$

such that the unique solution of the controlled system (1.1) with initial state $y(0, x) = y_0(x)$ and the control $f|_{[0,T]}$ verifies $y(T, x) = 0$.

Theorem 1.3 (Small-time local stabilization with explicit feedback laws). *For any $T > 0$, we find an effectively computable constant Λ_T and construct an explicit T-periodic proper feedback law U satisfying*

$$\|U(t; y)\|_{\mathcal{H}} \leq \min\{1, 2\|y\|_{\mathcal{H}}^{1/2}\} \quad \forall y \in \mathcal{H}, \ \forall t \in \mathbb{R},$$

that stabilizes system (1.1)–(1.3) *in finite time:*

(i) (2T stabilization) $\Phi(2T + t, t; y_0) = 0 \ \forall t \in \mathbb{R}, \forall \|y_0\|_{\mathcal{H}} \leq \Lambda_T$.

(ii) (Uniform stability) *For every $\delta > 0$ there exists an effectively computable $\eta > 0$ such that*

$$(\|(y_0\|_{\mathcal{H}} \leq \eta) \Rightarrow \big(\|\Phi(t, t'; y_0)\|_{\mathcal{H}} \leq \delta \ \forall t' \in \mathbb{R}, \ \forall t \in (t', +\infty)\big).$$

As we mentioned above, this stabilization method is a combination of *spectral inequalities*, *Lyapunov functionals*, and *piecewise feedback laws*. Here, for the reader's convenience, we briefly comment on the use of these tools in this work.

- For some given operator, the *spectral inequalities* are some quantitative estimates of the linear combinations of low-frequency eigenfunctions. Usually, these technical results are achieved upon elliptic-type operators using local Carleman estimates. Since the seminal works of Jerison–Lebeau–Robbiano [41, 42], this type of quantitative property has been extensively investigated in the literature; related works include but are not limited to [3, 9, 12–14]. In this paper, in particular we have used the spectral inequality on Stokes operators that is given by Chaves-Silva and Lebeau in [14, Theorem 3.1] (see Section 2.1 for more details).
- Generally speaking, the *Lyapunov functional* method aims to find artfully chosen energy and multipliers to characterize the variation of energy from a global point of view. They have been heavily adapted to the study of stabilities and stabilization of systems, including hyperbolic systems of conservation laws [8, 38, 53], wave equations [44], and parabolic equations [27], among others. The Lyapunov functional in this paper is highly inspired by the one introduced in [52]: the idea is to separate low frequency and high frequency with different weights. We refer to Section 3 for the precise choice of such Lyapunov functions in this work.
- Different from stationary feedback laws, a *piecewise (in time) feedback law* is some time-varying feedback law: sometimes we do not know how to stabilize a system with stationary feedback laws, and instead we construct time-varying feedback laws to stabilize the system. In [25], together with the backstepping method, an "infinite-piece" piecewise feedback law has been introduced by Coron–Nguyen to stabilize the one-dimensional heat equation in finite time. In Section 5 we have adapted similar piecewise feedback laws to achieve finite-time stabilization.

Finally, let us comment on the advantages of this stabilization method and the novelties of Theorems 1.1–1.3.

- The designed feedback laws that lead to quantitative stabilization are simple and explicit compared with some other stabilization techniques. For instance, in order to use the well-known Riccati method it is required to first solve some algebraic nonlinear Riccati equations. As we can see in Theorem 1.1, with the help of some precise feedback laws we are able to achieve quantitative rapid stabilization of Navier–Stokes equations.
  We believe that this explicit approach can be applied to various models. Indeed, recently, Alphonse–Martin [2] have successfully constructed this type of feedback law on quantitative rapid stabilization of a large class of diffusive equations from thick control supports.
- Quantitative rapid stabilization, together with piecewise continuous feedback laws, leads to finite-time stabilization of linear models. As a direct consequence, it also provides a constructive approach to obtain null-controllability results without using Lions' Hilbert uniqueness method [44]. Moreover, this approach also gives explicit control cost (which is even optimal in many cases).
  The *optimal cost* is a type of characterization of the cost leading to null-controllability, namely some uniform Ce^{C/T^α}-type upper bounds on the observability inequality constant for $T \in (0,1)$. Let us refer to, for example, the recent work of Beauchard–Pravda-Starov [10, Theorem 2.1], where the authors have provided an abstract characterization of such bounds for a large class of degenerate parabolic equations. We also refer to the work of Miller [46] and the references therein on this interesting topic.
- The closed-loop feedback stabilization is stable under perturbation in many circumstances; for example, let us mention [21, 39, 40, 48]. To be more precise, if a linearized system with some feedback law is exponentially stable, then we may expect the nonlinear system with the same feedback law to be locally exponentially stable. Thanks to this advantage of stabilization, the quantitative rapid stabilization and finite-time stabilization results on linear models can be directly generalized to several nonlinear models. For instance, to the best of our knowledge, Theorem 1.3 is the first finite-time stabilization result on Navier–Stokes equations.
  Similar to linear systems, this constructive process automatically leads to the local null-controllability of nonlinear models with explicit cost estimates. For example, in Theorem 1.2 we have provided a constructive proof of the small-time null-controllability of Navier–Stokes equations with $e^{C/T}$-type cost.

This paper is organized as follows: in Section 2 some preliminary results concerning well-posedness of Navier–Stokes equations, spectral inequalities of Stokes operators, as well as the related control problems are introduced; then Sections 3–5 are devoted to the proofs of Theorems 1.1–1.3.

2. Preliminary

2.1. Functional framework

We refer to the book by Chemin [15] for the functional analysis framework, standard energy estimates, and well-posedness results concerning incompressible Navier–Stokes equations, and the book by Coron [18] for an excellent introduction to the related control problems. In the text, if there is no confusion, sometimes we simply denote $L^2(\Omega)^2$ by $L^2(\Omega)$ or even L^2.

(1) Leray projection and spectral decomposition. According to Helmholtz decomposition, for any $u \in L^2(\Omega)^2$ there exist unique $v \in \mathcal{H}$ and $\nabla p \in L^2(\Omega)^2$ such that $u = v + \nabla p$, which defines the (orthogonal) Leray projection $\mathbb{P}$ on $L^2(\Omega)^2$:

$$\begin{cases} \mathbb{P}\colon L^2(\Omega)^2 \to \mathcal{H}, \\ \qquad\qquad u \mapsto \mathbb{P}u := u - \nabla p. \end{cases}$$

Notice that for any $f \in \mathcal{H}$,

$$\|\mathbb{P}(1_\omega f)\|_{\mathcal{H}} \le \|1_\omega f\|_{L^2(\Omega)^2} \le \|f\|_{L^2(\Omega)^2} = \|f\|_{\mathcal{H}},$$

which allows us to estimate the control term via $\|f\|_{L^2}$ (or equivalently $\|f\|_{\mathcal{H}}$).

Let $\{e_i\}_{i=1}^{\infty} \subset \mathcal{V}_\sigma$ be the orthonormal basis of $\mathcal{H}$ given by the eigenvectors of the Stokes operator

$$\begin{cases} -\Delta e_i + \nabla p_i = \tau_i e_i & \text{in } \Omega, \\ \operatorname{div} e_i = 0 & \text{in } \Omega, \\ e_i = 0 & \text{on } \partial\Omega, \end{cases}$$

with $0 < \tau_1 \le \tau_2 \le \tau_3 \le \cdots \le \tau_n \le \cdots$ and $\lim_{i\to\infty} \tau_i = +\infty$. Let $\mathcal{H}_N$ be the low-frequency subspace of $\mathcal{H}$, and $\mathbb{P}_N$ be its orthogonal projection,

$$\mathcal{H}_N := \operatorname{Vect}\{e_i\}_{i=1}^{N} \subset \mathcal{V}_\sigma.$$

In terms of the above eigenvectors, Leray projection can be extended to $\mathcal{V}'$,

$$\begin{cases} \mathbb{P}\colon \mathcal{V}' \to \mathcal{V}', \\ \qquad u \mapsto \mathbb{P}u := u - \nabla p, \end{cases}$$

where $p \in L^2_{\text{loc}}(\Omega)$, and ∇p belongs to $\mathcal{V}_\sigma^0$ as the polar space of $\mathcal{V}_\sigma$,

$$\mathcal{V}_\sigma^0 := \big\{ f \in \mathcal{V}' : \langle f, v\rangle_{\mathcal{V}'\times\mathcal{V}} = 0 \ \forall v \in \mathcal{V}_\sigma \big\}.$$

More precisely,

$$\mathbb{P}u := \sum_{i=1}^{\infty} \langle u, e_i\rangle_{\mathcal{V}'\times\mathcal{V}} e_i \in \mathcal{V}' \qquad \text{for } u \in \mathcal{V}',$$

$$\mathbb{P}_N u := \sum_{i=1}^{N} \langle u, e_i\rangle_{\mathcal{V}'\times\mathcal{V}} e_i \in \mathcal{V}_\sigma \qquad \text{for } u \in \mathcal{V}',$$

$$\mathbb{P}_N^{\perp} u := \sum_{i=N+1}^{\infty} \langle u, e_i \rangle_{\mathcal{V}' \times \mathcal{V}} e_i \in \mathcal{V}' \quad \text{for } u \in \mathcal{V}',$$
$$\mathbb{P} u := \sum_{i=1}^{\infty} (u, e_i)_{L^2(\Omega)^2} e_i \in \mathcal{H} \qquad \text{for } u \in L^2(\Omega)^2,$$

which satisfies

$$\langle u, v \rangle_{\mathcal{V}' \times \mathcal{V}} = \langle u, v \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} = \langle \mathbb{P} u, v \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} \quad \forall u \in \mathcal{V}',\ \forall v \in \mathcal{V}_\sigma.$$

Furthermore, the related $\mathcal{H}$-norm, $\mathcal{V}_\sigma$-norm, and $\mathcal{V}_\sigma'$-norm can be characterized by

$$\sum_{i=1}^{\infty} |(u, e_i)_{L^2(\Omega)^2}|^2 = \|\mathbb{P} u\|_{\mathcal{H}}^2 = \|\mathbb{P} u\|_{L^2}^2 \le \|u\|_{L^2}^2 \quad \text{for } u \in L^2(\Omega)^2,$$
$$\sum_{i=1}^{\infty} |(u, e_i)_{L^2(\Omega)^2}|^2 \tau_i = \|\mathbb{P} u\|_{\mathcal{V}_\sigma}^2 = \|\mathbb{P} u\|_{\mathcal{V}}^2 \le \|u\|_{\mathcal{V}}^2 \quad \text{for } u \in \mathcal{V},$$
$$\sum_{i=1}^{\infty} |\langle u, e_i \rangle_{\mathcal{V} \times \mathcal{V}'}|^2 \tau_i^{-1} = \|\mathbb{P} u\|_{\mathcal{V}_\sigma'}^2 = \|u\|_{\mathcal{V}_\sigma'}^2 \le \|u\|_{\mathcal{V}'}^2 \quad \text{for } u \in \mathcal{V}'.$$

In particular, by recalling the definitions of $\mathbb{P}_N$, $\mathbb{P}_N^{\perp}$, and the $\|\cdot\|_{\mathcal{V}_\sigma}$-norm we have

$$\|\nabla u\|_{L^2(\Omega)}^2 = \|\nabla \mathbb{P}_N u\|_{L^2(\Omega)}^2 + \|\nabla \mathbb{P}_N^{\perp} u\|_{L^2(\Omega)}^2 \quad \text{for } u \in \mathcal{V}_\sigma.$$

(2) Spectral inequalities. For any $\lambda > 0$, we denote by $N(\lambda)$ the number of the eigenvalues that are smaller than or equal to λ, i.e. $\tau_{N(\lambda)} \le \lambda < \tau_{N(\lambda)+1}$, and define the symmetric matrix $J_{N(\lambda)}$:

$$J_{N(\lambda)} := \big((e_i, e_j)_{L^2(\omega)^2}\big)_{i,j=1}^{N(\lambda)}. \tag{2.1}$$

Proposition 2.1 (Spectral inequalities). *There exists an effectively computable constant $C_1 \ge 1$ which only depends on (Ω, ω) and is independent of $\lambda > 0$, such that, for any $\lambda > 0$ and for any $E_{N(\lambda)} = (a_1, a_2, \ldots, a_{N(\lambda)}) \in \mathbb{R}^{N(\lambda)}$ the following inequality holds:*

$$E_{N(\lambda)}^T J_{N(\lambda)} E_{N(\lambda)} \ge C_1^{-1} e^{-C_1 \sqrt{\lambda}} \|E_{N(\lambda)}\|_2^2.$$

Proof. This is a Lebeau–Robbiano-type spectral inequality on Stokes operators, which is proved by Chaves-Silva–Lebeau in [14, Theorem 3.1]: there exists some $C > 0$ such that

$$C e^{C\sqrt{\lambda}} \int_\omega \Big(\sum_{\tau_i \le \lambda} a_i e_i(x) \Big)^2 dx \ge \sum_{\tau_i \le \lambda} a_i^2.$$

Let us denote $\max\{1, C\}$ by C_1. By letting N represent N_λ, we get

$$E_N^T J_N E_N = \sum_{1 \le i,j \le N} a_i (e_i, e_j)_{L^2(\omega)^2} a_j = \Big\| \sum_{i=1}^{N} a_i e_i \Big\|_{L^2(\omega)^2}^2 \ge C_1^{-1} e^{-C_1 \sqrt{\lambda}} \|E_N\|_2^2.$$

Or equivalently, the preceding positive quadratic form can be expressed as

$$C_1 e^{C_1\sqrt{\lambda}} \int_\omega (\mathbb{P}_{N(\lambda)} y)^2 \, dx \geq \int_\Omega (\mathbb{P}_{N(\lambda)} y)^2 \, dx \quad \forall y \in \mathcal{H}. \qquad \blacksquare$$

(3) Nonlinear terms. Next we define the bilinear map $\mathcal{Q}$, as well as the trilinear functional $\mathcal{B}$:

$$\begin{cases} \mathcal{Q}\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}', \\ \qquad (u, v) \mapsto \operatorname{div}(u \otimes v), \end{cases}$$

$$\mathcal{B}(u, v, w) := \langle \mathcal{Q}(u, v), w \rangle_{\mathcal{V}' \times \mathcal{V}} \quad \forall u, v, w \in \mathcal{V}.$$

Proposition 2.2 (Nonlinear estimates). *There exists a constant c_0 such that for any u, v, and w in $\mathcal{V}$, we have the estimates*

$$\begin{aligned} \mathcal{B}(u, u, w) &= \langle (u \cdot \nabla) u, w \rangle_{\mathcal{V}' \times \mathcal{V}} && \text{if } u \in \mathcal{V}_\sigma, \\ \mathcal{B}(u, v, w) + \mathcal{B}(u, w, v) &= 0 && \text{if } u \in \mathcal{V}_\sigma, \\ |\mathcal{B}(u, v, w)| \leq c_0 \|u\|_{L^2}^{\frac{1}{2}} \|v\|_{L^2}^{\frac{1}{2}} & \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla v\|_{L^2}^{\frac{1}{2}} \|\nabla w\|_{L^2}. \end{aligned}$$

2.2. Open loop controlled (inhomogeneous) Navier–Stokes systems

The open loop controlled equation is indeed an inhomogeneous equation with a force term located in the controlled domain. A general inhomogeneous equation (without any restriction on force terms) is represented by

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = f(t, x), & (t, x) \in (t_1, t_2) \times \Omega, \\ \operatorname{div} y(t, x) = 0, & (t, x) \in (t_1, t_2) \times \Omega, \\ y(t, x) = 0, & (t, x) \in (t_1, t_2) \times \partial\Omega, \\ y(t_1, x) = y_0(x), & x \in \Omega, \end{cases} \tag{2.2}$$

where t_2 can be taken as $+\infty$. We are interested in Leray's weak solutions [43]: for any $y_0 \in \mathcal{H}$ and any $f \in L^2_{\text{loc}}(t_1, t_2; \mathcal{V}')$, the solution of equation (2.2) is some $y \in C([t_1, t_2]; \mathcal{H}) \cap L^2_{\text{loc}}(t_1, t_2; \mathcal{V}_\sigma)$ such that, for any test function ϕ in $C^1([t_1, t_2]; \mathcal{V}_\sigma)$, the vector field y satisfies the condition

$$\begin{aligned} (y(t), \phi(t))_{\mathcal{H}} = (y_0, \phi(0))_{\mathcal{H}} &+ \int_{t_1}^t \langle \Delta\phi(s) + \partial_t \phi(s), y(s) \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} \, ds \\ &+ \int_{t_1}^t (y(s) \otimes y(s), \nabla\phi(s))_{L^2(\Omega)^2} \, ds \\ &+ \int_{t_1}^t \langle f(s), \phi(s) \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} \, ds \end{aligned} \tag{2.3}$$

for every $t \in [t_1, t_2]$.

Theorem 2.3 (Leray theorem on well-posedness and stability of the solutions). *For any $y_0 \in \mathcal{H}$ and any $f \in L^2_{\text{loc}}(t_1, t_2; L^2(\Omega)^2)$, the Cauchy problem (2.2) admits a unique solution. This unique solution is also in $H^1_{\text{loc}}(t_1, t_2; \mathcal{V}'_\sigma)$. Moreover, there exists some constant C_0 independent of t_1 and t_2 such that this unique solution satisfies*

$$\frac{1}{2}\|y(t,x)\|^2_{\mathcal{H}} + \int_{t_1}^{t} \|\nabla y(s,x)\|^2_{L^2}\, ds = \frac{1}{2}\|y_0\|^2_{\mathcal{H}} + \int_{t_1}^{t} \langle f(s), y(s)\rangle_{\mathcal{V}'_\sigma \times \mathcal{V}_\sigma}\, ds, \tag{2.4}$$

$$\|y(t,x)\|^2_{\mathcal{H}} + \int_{t_1}^{t} \|\nabla y(s,x)\|^2_{L^2}\, ds \le \|y_0\|^2_{\mathcal{H}} + C_0 \int_{t_1}^{t} \|f(s)\|^2_{L^2}\, ds, \tag{2.5}$$

for any $t \in [t_1, t_2]$.

Furthermore, the Leray solutions are stable in the following sense. Let y (resp. z) be the Leray solution associated with y_0 (resp. z_0) in $\mathcal{H}$ and f (resp. g) in the space $L^2_{\text{loc}}(t_1, +\infty; L^2(\Omega)^2)$. Then for $w := y - z$ and for any $t \in (t_1, +\infty)$ we have

$$\|w(t)\|^2_{\mathcal{H}} + \int_{t_1}^{t} \|\nabla w(s,x)\|^2_{L^2}\, ds \le \left(\|w_0\|^2_{\mathcal{H}} + C_0 \int_{t_1}^{t} \|(f-g)(s)\|^2_{L^2}\, ds\right) \exp(C_s E^2(t)),$$
$$E(t) := \min\left\{\|y_0\|^2_{\mathcal{H}} + C_0 \textstyle\int_{t_1}^{t} \|f(s)\|^2_{L^2}\, ds,\ \|z_0\|^2_{\mathcal{H}} + C_0 \int_{t_1}^{t} \|g(s)\|^2_{L^2}\, ds\right\}.$$

Actually, Theorem 2.3 even holds for f in $L^2_{\text{loc}}(t_1, +\infty; \mathcal{V}')$, for which the related inequalities are governed by the $L^2(\mathcal{V}'_\sigma)$-norm of f and the constant C_0 can be taken as 1.

2.3. Time-varying feedback laws, closed-loop systems, and finite-time stabilization

In this section we recall the precise definition of time-varying feedback laws, as well as the related closed-loop solutions.

Definition 2.4 (Closed-loop systems). Let $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ be given such that $s_1 < s_2$. Let the time-varying feedback law on the interval $[s_1, s_2]$ be an application

$$\begin{cases} U\colon [s_1, s_2] \times \mathcal{H} \to \mathcal{H}, \\ \qquad\qquad (t; y) \mapsto U(t; y). \end{cases} \tag{2.6}$$

Let $t_1 \in [s_1, s_2]$, $t_2 \in (t_1, s_2]$, and $y_0 \in \mathcal{H}$. A solution on $[t_1, t_2]$ to the Cauchy problem associated to the closed-loop system (1.1)–(1.2) with (2.6) for initial data y_0 at time t_1 is some $y\colon [t_1, t_2] \to \mathcal{H}$ such that

$$t \in (t_1, t_2) \mapsto f(t,x) := 1_\omega U(t; y(t)) \in L^2(t_1, t_2; L^2(\Omega)^2),$$

where y is a Leray solution of (2.2), with initial data y_0 at time t_1 and the above force term $f(t,x)$.

Definition 2.5 (Proper feedback laws). Let $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ be given such that $s_1 < s_2$. A proper feedback law on $[s_1, s_2]$ is an application U of type (2.6) such that, for every

$t_1 \in [s_1, s_2]$, for every $t_2 \in (t_1, s_2]$, and for every $y_0 \in \mathcal{H}$, there exists a unique solution on $[t_1, t_2]$ to the Cauchy problem associated to the closed-loop system (1.1)–(1.2) with (2.6) for initial data y_0 at time t_1 according to Definition 2.4.

A proper feedback law is an application U of type (1.3) such that, for every $s_1 \in \mathbb{R}$ and for every $s_2 \in \mathbb{R}$ satisfying $s_1 < s_2$, the feedback law restricted to $[s_1, s_2] \times \mathcal{H}$ is a proper feedback law on $[s_1, s_2]$.

For a *proper* feedback law, one can define the *flow* $\Phi \colon \Lambda \times \mathcal{H} \to \mathcal{H}$ associated to this feedback law, with $\Lambda := \{(t, s); t > s\}$: $\Phi(t, s; y_0)$ is the value at time t of the solution y to the closed-loop system (1.1)–(1.3) which is equal to y_0 at time s.

Definition 2.6 (Finite-time local stabilization of Navier–Stokes equations). Let $T > 0$. A T-periodic proper feedback law U locally stabilizes the two-dimensional Navier–Stokes equations in finite time if, for some $\varepsilon > 0$, the flow Φ of the closed-loop system (1.1)–(1.3) verifies

(i) ($2T$ stabilization) $\Phi(2T + t, t; y_0) = 0 \ \forall t \in \mathbb{R}, \forall \|y_0\|_{\mathcal{H}} \leq \varepsilon$,

(ii) (Uniform stability) For every $\delta > 0$, there exists $\eta > 0$ such that

$$(\|y_0\|_{\mathcal{H}} \leq \eta) \Rightarrow \big(\|\Phi(t, t'; y_0)\|_{\mathcal{H}} \leq \delta \ \forall t' \in \mathbb{R}, \ \forall t \in (t', +\infty)\big).$$

2.4. Well-posedness of closed-loop systems

Finally, we present well-posedness results concerning closed-loop systems with stationary Lipschitz feedback laws. Concerning linear feedback laws one has the following well-posedness result.

Theorem 2.7. *Let $T > 0$. Let vector functions $\{\varphi_i\}_{i=1}^n \in \mathcal{H}$ and bounded linear operators $\{l_i\}_{i=1}^n \colon \mathcal{H} \to \mathbb{R}$ be given. For any $y_0 \in \mathcal{H}$, the Cauchy problem*

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = 1_\omega \left(\displaystyle\sum_{i=1}^n l_i(y) \varphi_i \right), & (t, x) \in (0, T) \times \Omega, \\ \operatorname{div} y = 0, & (t, x) \in (0, T) \times \Omega, \\ y(t, x) = 0, & (t, x) \in (0, T) \times \partial\Omega, \\ y(0, x) = y_0(x), & x \in \Omega, \end{cases}$$

admits a unique solution.

As we will consider finite-time stabilization problems we also introduce "cutoff"-type feedback laws. For any $r \in (0, 1/2]$ we find some smooth cutoff function $\chi_r \in C^\infty(\mathbb{R}^+; [0, 1])$ satisfying

$$\chi_r(x) = \begin{cases} 1 & \text{if } x \in [0, r], \\ 0 & \text{if } x \in [2r, +\infty), \end{cases} \tag{2.7}$$

and further define the related *Lipschitz* operator $\mathcal{K}_r\colon \mathcal{H} \to \mathcal{H}$ as

$$\mathcal{K}_r(y) := y \cdot \chi_r(\|y\|_{\mathcal{H}}) \quad \forall y \in \mathcal{H}, \tag{2.8}$$

satisfying, for some constant L_r depending on r,

$$\|\mathcal{K}_r(y)\|_{\mathcal{H}} \le \min\{1, \|y\|_{\mathcal{H}}\},$$
$$\|\mathcal{K}_r(y) - \mathcal{K}_r(z)\|_{\mathcal{H}} \le L_r\|y - z\|_{\mathcal{H}} \quad \forall y, z \in \mathcal{H}.$$

Theorem 2.8. *Let $T > 0$. Let $r \in (0, 1/2]$. Let vector functions $\{\varphi_i\}_{i=1}^n \in \mathcal{H}$ and bounded linear operators $\{l_i\}_{i=1}^n\colon \mathcal{H} \to \mathbb{R}$ be given. For any $y_0 \in \mathcal{H}$, the Cauchy problem*

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = 1_\omega \mathcal{K}_r\Big(\sum_{i=1}^n l_i(y)\varphi_i\Big), & (t,x) \in (0,T) \times \Omega, \\ \operatorname{div} y = 0, & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ y(0,x) = y_0(x), & x \in \Omega, \end{cases}$$

admits a unique solution.

Thanks to Theorems 2.7 and 2.8, both the closed-loop systems with linear feedback laws and the closed-loop systems with Lipschitz nonlinear feedback laws are well posed. We sketch the proofs of these theorems in the following.

Indeed, local (in time) existence and uniqueness of solutions are based on Leray's theorem, Theorem 2.3, concerning energy estimates and the stability of the solutions, and the Banach fixed point theorem. Let the Lipschitz constant of the feedback law be L. Let the $\mathcal{H}$-norm of the feedbacks of $y \in \mathcal{H}$ be bounded by $C\|y\|_{\mathcal{H}}$; i.e. for all $y \in \mathcal{H}$,

$$\Big\|\mathcal{K}_r\Big(\sum_{i=1}^n l_i(y)\varphi_i\Big)\Big\|_{\mathcal{H}} \le \Big\|\sum_{i=1}^n l_i(y)\varphi_i\Big\|_{\mathcal{H}} \le C\|y\|_{\mathcal{H}}. \tag{2.9}$$

We assume that $\|y_0\|_{\mathcal{H}} = M$. For some $\widetilde{T}$ small enough, to be fixed later, we consider the Banach space

$$\mathcal{X}_{\widetilde{T}} := C([0,\widetilde{T}]; \mathcal{H}) \cap L^2(0,\widetilde{T}; \mathcal{V}_\sigma),$$
$$\mathcal{X}_{\widetilde{T}}(2M) := \big\{y \in \mathcal{X}_{\widetilde{T}} : \|y\|^2_{\mathcal{X}_{\widetilde{T}}} = \|y\|^2_{C([0,\widetilde{T}];\mathcal{H})} + \|\nabla y\|^2_{L^2(0,\widetilde{T};L^2)} \le 4M^2\big\},$$

as well as the application

$$\begin{cases} \mathcal{S}\colon \mathcal{X}_{\widetilde{T}}(2M) \to \mathcal{X}_{\widetilde{T}}, \\ \qquad\qquad y \mapsto \mathcal{S}(y), \end{cases}$$

where $\mathcal{S}(y)$ is the solution of Cauchy problem (2.2) with the initial state y_0 and force (control) term $f = 1_\omega \mathcal{K}_r(\sum_{i=1}^n l_i(y)\varphi_i)$. One can check that, thanks to Theorem 2.3, the preceding application is a contraction map on $\mathcal{X}_{\widetilde{T}}(2M)$ for $\widetilde{T}$ sufficiently small, e.g.

$$\widetilde{T} \le \min\big\{(4C_0C^2)^{-1}, (4C_0L^2\exp(4C_sM^4))^{-1}\big\}$$

thus admits a fixed point $\tilde{y}$ which is the unique solution of the closed-loop system. Moreover, since $\tilde{y} = \mathcal{S}(\tilde{y})$ is the solution of the Cauchy problem (2.2) with control $f = 1_\omega \mathcal{K}_r(\sum_{i=1}^n l_i(\tilde{y})\varphi_i)$, thanks to Theorem 2.3, this solution also belongs to the space $H^1(0,T;\mathcal{V}'_\sigma)$.
In the end, some a priori estimates lead to global (in time) solutions. Indeed, suppose that the solution exists on the time interval $[0,T]$; then, thanks to the inequalities (2.5) and (2.9), we know that

$$\begin{aligned}\|y(t,x)\|^2_{\mathcal{H}} &\le \|y_0\|^2_{\mathcal{H}} + C_0 \int_0^t \left\| \sum_{i=1}^n l_i(y(s,x))\varphi_i \right\|^2_{\mathcal{H}} ds \\ &\le \|y_0\|^2_{\mathcal{H}} + C_0 C^2 \int_0^t \|y(s,x)\|^2_{\mathcal{H}}\, ds \quad \forall t \in (0,T).\end{aligned}$$

Therefore, by applying Grönwall's inequality we get

$$\|y(t,x)\|^2_{\mathcal{H}} \le e^{C_0C^2 t}\|y_0\|^2_{\mathcal{H}} \quad \forall t \in (0,T).$$

Since the value of C_0 and C are independent of $T \in (0,+\infty)$, we get the required a priori estimates. This estimate, together with the local (in time) existence of solutions, yields the global (in time) existence result.

We also emphasize the fact that the Lipschitz condition is crucial in order to guarantee the uniqueness. Otherwise, one may need to use some other compactness arguments to prove existence of solutions; see for example [26] for KdV equations.

2.5. On the choice of constants

In this section we fix the values of the constants that will be used later on. In particular, these constants also correspond to those having appeared in Theorems 1.1 and 1.2.

- For any given $\lambda > 0$, we define

$$\gamma_\lambda := C_1 e^{C_1\sqrt{\lambda}}\lambda \quad \text{and} \quad \mu_\lambda := \frac{\gamma_\lambda^2}{\lambda^2} = C_1^2 e^{2C_1\sqrt{\lambda}} > 1. \tag{2.10}$$

- By recalling the definition of C_1 in Proposition 2.1, we further select some $C_2 \in [3C_1,+\infty)$ such that for all $\lambda > 0$,

$$(1+\lambda C_1)e^{C_1\sqrt{\lambda}},\ 8(1+\lambda)C_1^2 e^{2C_1\sqrt{\lambda}},\ 9c_0 C_1^3 e^{3C_1\sqrt{\lambda}} \le C_2 e^{C_2\sqrt{\lambda}}, \tag{2.11}$$

  and define

$$r_\lambda := (C_2 e^{C_2\sqrt{\lambda}})^{-1}. \tag{2.12}$$

- Then we choose some constant $Q > 0$ satisfying

$$C_1 e^{C_1 Q m},\ C_2 e^{C_2 Q m} \le e^{\frac{Q^2}{64}m} \quad \forall m \ge 1, \tag{2.13}$$

  and select

$$C_3 := \frac{Q^2}{32}. \tag{2.14}$$

3. Quantitative rapid stabilization

Inspired by the recent work [52] on the stabilization of the heat equations, we directly define the stationary feedback law

$$\mathcal{F}_\lambda y := -\gamma_\lambda \mathbb{P}_{N(\lambda)} y \quad \forall y \in \mathcal{H}, \tag{3.1}$$

and consider the closed-loop system

$$\begin{cases} y_t = \Delta y - (y\cdot\nabla)y - \nabla p - \gamma_\lambda 1_\omega \mathbb{P}_N y & \text{in } \Omega, \\ \operatorname{div} y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where, and from now on, we simply denote $N(\lambda)$ by N. Furthermore, the low frequency system satisfies

$$\frac{d}{dt}(\mathbb{P}_N y) = \mathbb{P}_N(\Delta y) - \mathbb{P}_N((y\cdot\nabla)y) - \gamma_\lambda \mathbb{P}_N(1_\omega \mathbb{P}_N y). \tag{3.3}$$

Because y lives in $\mathcal{H}$, we can decompose

$$y(t,x) = \mathbb{P}y(t,x) = \sum_{i=1}^{\infty} y_i(t) e_i,$$

$$\mathbb{P}(1_\omega e_i) = \sum_{j=1}^{\infty} (1_\omega e_i, e_j)_{L^2(\Omega)^2} e_j = \sum_{j=1}^{\infty} (e_i, e_j)_{L^2(\omega)^2} e_j,$$

which further implies

$$\mathbb{P}_N(1_\omega \mathbb{P}_N y) = \mathbb{P}_N\left(1_\omega \sum_{i=1}^{N} y_i(t) e_i\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} y_i(t)(e_i, e_j)_{L^2(\omega)^2} e_j.$$

Furthermore, for y in $\mathcal{V}_\sigma$ the nonlinear term $(y\cdot\nabla)y$ (which is equivalent to $\operatorname{div}(y\otimes y)$) belongs to the space $\mathcal{V}'$, thus

$$\mathbb{P}_N((y\cdot\nabla)y) = \sum_{i=1}^{N} \langle (y\cdot\nabla)y, e_i\rangle_{\mathcal{V}'\times\mathcal{V}} e_i;$$

moreover,

$$\mathbb{P}_N(\Delta y) = -\sum_{i=1}^{N} \tau_i y_i e_i.$$

By defining

$$X_N(t) := \begin{pmatrix} y_1(t) \\ y_2(t) \\ \cdots \\ y_N(t) \end{pmatrix}, \quad Y_N(t) := \begin{pmatrix} -\langle (y\cdot\nabla)y, e_1\rangle_{\mathcal{V}'\times\mathcal{V}}(t) \\ -\langle (y\cdot\nabla)y, e_2\rangle_{\mathcal{V}'\times\mathcal{V}}(t) \\ \cdots \\ -\langle (y\cdot\nabla)y, e_N\rangle_{\mathcal{V}'\times\mathcal{V}}(t) \end{pmatrix},$$

$$A_N := \begin{pmatrix} -\tau_1 & & & \\ & -\tau_2 & & \\ & & \dots & \\ & & & -\tau_N \end{pmatrix},$$

we know that the finite-dimensional system $X_N(t)$ satisfies the ordinary differential equation

$$\dot{X}_N(t) = A_N X_N(t) - \gamma_\lambda J_N X_N(t) + Y_N(t). \tag{3.4}$$

Let us consider the following Lyapunov functional on $\mathcal{H}$: for every $y \in \mathcal{H}$,

$$\begin{aligned} V(y) &:= \mu_\lambda (\mathbb{P}_N y, \mathbb{P}_N y)_{L^2(\Omega)^2} + (\mathbb{P}_N^\perp y, \mathbb{P}_N^\perp y)_{L^2(\Omega)^2} \\ &= \mu_\lambda \|X_N\|_2^2 + (\mathbb{P}_N^\perp y, \mathbb{P}_N^\perp y)_{L^2(\Omega)^2}, \end{aligned} \tag{3.5}$$

satisfying

$$\|y\|_{L^2(\Omega)^2}^2 \le V(y) \le \mu_\lambda \|y\|_{L^2(\Omega)^2}^2 \quad \forall y \in \mathcal{H},$$

where

$$\|X_N\|_2^2 := \sum_{i=1}^N y_i^2 = (\mathbb{P}_N y, \mathbb{P}_N y)_{L^2(\Omega)^2}.$$

Let $T > 0$. Concerning the variation of the value of the above Lyapunov function, at least when the solution is regular enough, for example $y \in C^1([0,T]; \mathcal{V}_\sigma') \cap C^0([0,T]; \mathcal{V}_\sigma)$ and (thus) $\mathbb{P}_N y \in C^1([0,T]; \mathcal{H}_N)$, one has

$$\begin{aligned} \frac{d}{dt} V(y(t)) &= \mu_\lambda \frac{d}{dt} \|X_N\|_2^2 + \frac{d}{dt} (\mathbb{P}_N^\perp y, \mathbb{P}_N^\perp y)_{L^2(\Omega)^2} \\ &= 2\mu_\lambda X_N^T \dot{X}_N + 2\Big\langle \mathbb{P}_N^\perp y, \frac{d}{dt}(\mathbb{P}_N^\perp y) \Big\rangle_{\mathcal{V}_\sigma \times \mathcal{V}_\sigma'} \\ &= 2\mu_\lambda X_N^T (A_N X_N - \gamma_\lambda J_N X_N + Y_N) + 2\Big\langle \mathbb{P}_N^\perp y, \frac{d}{dt} y \Big\rangle_{\mathcal{V}_\sigma \times \mathcal{V}_\sigma'}, \end{aligned}$$

where the value of $-X_N^T Y_N$ is given by

$$\begin{aligned} \langle \mathbb{P}_N((y \cdot \nabla) y), \mathbb{P}_N y \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} &= \langle \mathbb{P}((y \cdot \nabla) y), \mathbb{P}_N y \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} \\ &= \langle (y \cdot \nabla) y, \mathbb{P}_N y \rangle_{\mathcal{V}' \times \mathcal{V}} = \mathcal{B}(y, y, \mathbb{P}_N y). \end{aligned}$$

According to Theorems 2.3 and 2.7, the solution y indeed lives in the space $H^1(0,T; \mathcal{V}_\sigma') \cap C^0([0,T]; \mathcal{H}) \cap L^2(0,T; \mathcal{V}_\sigma)$. This further implies that

- Δy and $(y \cdot \nabla) y$ belong to $L^2(0,T; \mathcal{V}')$,
- the finite-dimensional projection Y_N is in $L^2(0,T; \mathbb{R}^N)$,
- $\frac{d}{dt}(X_N)$ is in $L^2(0,T; \mathbb{R}^N)$,
- X_N is in $C^0([0,T]; \mathbb{R}^N)$,

- $\frac{d}{dt}(\mathbb{P}_N y)$ is in $L^2(0,T;\mathcal{H}_N)$,
- $\frac{d}{dt}(\mathbb{P}_N^{\perp} y)$ and $\frac{d}{dt} y$ live in $L^2(0,T;\mathcal{V}_\sigma')$.

Consequently, the preceding equations hold in the distribution sense in $L^1(0,T)$.

Moreover, on the one hand, thanks to Propositions 2.1 and 2.2, as well as the choice of γ_λ and μ_λ, we have

$$
\begin{aligned}
&2\mu_\lambda X_N^T \dot{X}_N \\
&\quad = 2\mu_\lambda X_N^T (A_N - \gamma_\lambda J_N) X_N + 2\mu_\lambda X_N^T Y_N \\
&\quad \le -2\mu_\lambda \gamma_\lambda (C_1 e^{C_1\sqrt{\lambda}})^{-1} \|X_N\|_2^2 - 2\mu_\lambda \|\nabla \mathbb{P}_N y\|_{L^2(\Omega)}^2 + 2\mu_\lambda |\mathcal{B}(y,y,\mathbb{P}_N y)| \\
&\quad \le -2\mu_\lambda \lambda \|X_N\|_2^2 - 2\mu_\lambda \|\nabla \mathbb{P}_N y\|_{L^2(\Omega)}^2 + 2\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)} \|\nabla \mathbb{P}_N y\|_{L^2(\Omega)} \\
&\quad \le -2\mu_\lambda \lambda \|X_N\|_2^2 - 2\mu_\lambda \|\nabla \mathbb{P}_N y\|_{L^2(\Omega)}^2 + 2\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}^2.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&2\Big\langle \mathbb{P}_N^{\perp} y, \frac{d}{dt} y \Big\rangle_{\mathcal{V}_\sigma \times \mathcal{V}_\sigma'} \\
&\quad = 2\langle \mathbb{P}_N^{\perp} y, \Delta y - (y\cdot\nabla) y - \gamma_\lambda 1_\omega \mathbb{P}_N y - \nabla p \rangle_{\mathcal{V}_\sigma \times \mathcal{V}_\sigma'} \\
&\quad = -2(\mathbb{P}_N^{\perp} y, y)_{\mathcal{V}_\sigma} - 2\gamma_\lambda (\mathbb{P}_N^{\perp} y, 1_\omega \mathbb{P}_N y)_{L^2(\Omega)^2} - 2\langle (y\cdot\nabla) y, \mathbb{P}_N^{\perp} y \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} \\
&\quad = -2\sum_{i=N+1}^{\infty} \tau_i y_i^2 - 2\gamma_\lambda (\mathbb{P}_N^{\perp} y, 1_\omega \mathbb{P}_N y)_{L^2(\Omega)^2} - 2\mathcal{B}(y,y,\mathbb{P}_N^{\perp} y) \\
&\quad \le -2\sum_{i=N+1}^{\infty} \tau_i y_i^2 + 2\gamma_\lambda \|\mathbb{P}_N^{\perp} y\|_{L^2(\Omega)} \|1_\omega \mathbb{P}_N y\|_{L^2(\Omega)} + 2\mathcal{B}(y,y,\mathbb{P}_N y) \\
&\quad \le -\frac{3}{2}\lambda \|\mathbb{P}_N^{\perp} y\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\nabla \mathbb{P}_N^{\perp} y\|_{L^2(\Omega)}^2 + \lambda \|\mathbb{P}_N^{\perp} y\|_{L^2(\Omega)}^2 + \frac{\gamma_\lambda^2}{\lambda}\|X_N\|_2^2 \\
&\qquad + 2c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}^2 \\
&\quad \le -\frac{1}{2}\lambda \|\mathbb{P}_N^{\perp} y\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\nabla \mathbb{P}_N^{\perp} y\|_{L^2(\Omega)}^2 + \mu_\lambda \lambda \|X_N\|_2^2 + 2c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Combining the preceding three inequalities, we further derive that

$$
\begin{aligned}
&\frac{d}{dt}\big(V(y(t))\big) \\
&\quad \le -2\mu_\lambda \lambda \|X_N\|_2^2 - 2\mu_\lambda \|\nabla \mathbb{P}_N y\|_{L^2(\Omega)}^2 + 2\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}^2 \\
&\qquad - \frac{1}{2}\lambda \|\mathbb{P}_N^{\perp} y\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\nabla \mathbb{P}_N^{\perp} y\|_{L^2(\Omega)}^2 + \mu_\lambda \lambda \|X_N\|_2^2 + 2c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}^2 \\
&\quad \le -\mu_\lambda \lambda \|X_N\|_2^2 - \frac{1}{2}\lambda \|\mathbb{P}_N^{\perp} y\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\nabla y\|_{L^2(\Omega)}^2 + 4\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}^2 \\
&\quad \le \Big(-\frac{\lambda}{2}\Big) V(y(t)) - \frac{1}{2}\|\nabla y\|_{L^2(\Omega)}^2 + 4\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}^2 \\
&\quad \le \Big(-\frac{\lambda}{2}\Big) V(y(t)) - \|\nabla y\|_{L^2(\Omega)}^2 \Big(\frac{1}{2} - 4\mu_\lambda c_0 V^{\frac{1}{2}}(y(t))\Big).
\end{aligned} \tag{3.6}
$$

Keep in mind that the continuous function $V(y(t)) \in W^{1,1}(0,T)$ and that inequality (3.6) holds almost everywhere, with both sides being $L^1(0,T)$. Inspired by the above formula, at first by ignoring the first term in the right-hand side of (3.6), we know that the value of $\min\{V(y(t)), (8\mu_\lambda c_0)^{-2}\}$ decreases with respect to time. Therefore, if $V(y(0)) < (8\mu_\lambda c_0)^{-2}$ then the value of $V(y(t))$ is always strictly smaller than $(8\mu_\lambda c_0)^{-2}$. As a consequence, in the preceding inequality one can next ignore the second term involving ∇y, which results in the fact that the Lyapunov functional $V(y(t))$ decays exponentially with decay rate $\lambda/2$.

More precisely, by the choice of r_λ from (2.11)–(2.12), for any initial data $y_0 \in \mathcal{H}$ satisfying $\|y_0\|_{L^2(\Omega)} \leq r_\lambda$, we have

$$V(y_0) \leq \mu_\lambda \|y_0\|^2_{L^2(\Omega)} \leq \mu_\lambda r_\lambda^2 \leq (9\mu_\lambda c_0)^{-2} < (8\mu_\lambda c_0)^{-2},$$

which, combined with the fact (due to inequality (3.6)) that for almost every t in $(0,T)$,

$$\frac{d}{dt}\big(\min\{V(y(t)), (8\mu_\lambda c_0)^{-2}\}\big) \leq 0,$$

implies that the continuous function $\min\{V(y(t)), (8\mu_\lambda c_0)^{-2}\}$ is always smaller than $(9\mu_\lambda c_0)^{-2}$. Hence $V(y(t))$ is always smaller than $(9\mu_\lambda c_0)^{-2}$, which, combined with inequality (3.6), yields that for almost every t in $(0,T)$,

$$\frac{d}{dt}(V(y(t))) \leq \Big(-\frac{\lambda}{2}\Big) V(y(t)).$$

Therefore, the continuous function $V(y(t))$ verifies

$$V(y(t)) \leq e^{-\frac{\lambda}{2}t} V(y(0)) \quad \forall t \in [0,T].$$

Consequently,

$$\begin{aligned}\|y(t)\|^2_{L^2(\Omega)} \leq V(y(t)) \leq e^{-\frac{\lambda}{2}t} V(y(0)) &\leq e^{-\frac{\lambda}{2}t} \mu_\lambda \|y(0)\|^2_{L^2(\Omega)} \\ &\leq C_1^2 e^{2C_1\sqrt{\lambda}} e^{-\frac{\lambda}{2}t} \|y(0)\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore, for any initial data $y_0 \in \mathcal{H}$ satisfying $\|y_0\|_{L^2(\Omega)} \leq r_\lambda$, the unique solution decays exponentially,

$$\begin{aligned}\|y(t)\|_{L^2(\Omega)} &\leq C_1 e^{C_1\sqrt{\lambda}} e^{-\frac{\lambda}{4}t} \|y(0)\|_{L^2(\Omega)} & \forall t \in [0,+\infty),\\ \|\mathcal{F}_\lambda y(t)\|_{L^2(\Omega)} \leq \gamma_\lambda \|y(t)\|_{L^2(\Omega)} &\leq \lambda C_1^2 e^{2C_1\sqrt{\lambda}} e^{-\frac{\lambda}{4}t} \|y(0)\|_{L^2(\Omega)} & \forall t \in [0,+\infty),\end{aligned}$$

which can be quantified in the following theorem.

Theorem 3.1 (Local stabilization with linear feedback laws). *For any $\lambda > 0$, for any $\|y_0\|_{\mathcal{H}} \leq r_\lambda$, and for any $s \in \mathbb{R}$, the Cauchy problem*

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = -\gamma_\lambda 1_\omega \mathcal{F}_\lambda y, & (t,x) \in [s,+\infty)\times\Omega,\\ \operatorname{div} y = 0, & (t,x) \in [s,+\infty)\times\Omega,\\ y(t,x) = 0, & (t,x) \in [s,+\infty)\times\partial\Omega,\\ y(s,x) = y_0(x), & x \in \Omega, \end{cases} \tag{3.7}$$

has a unique solution in $C^0([s,+\infty);\mathcal{H}) \cap L^2_{\text{loc}}(s,+\infty;\mathcal{V}_\sigma)$. *Moreover, this unique solution verifies*

$$\|y(t)\|_{\mathcal{H}} \le C_1 e^{C_1\sqrt{\lambda}} e^{-\frac{\lambda}{4}(t-s)} \|y_0\|_{\mathcal{H}} \quad \forall t \in [s,+\infty), \tag{3.8}$$

$$\|\mathcal{F}_\lambda y(t)\|_{\mathcal{H}} \le C_2 e^{C_2\sqrt{\lambda}} e^{-\frac{\lambda}{4}(t-s)} \|y_0\|_{\mathcal{H}} \quad \forall t \in [s,+\infty). \tag{3.9}$$

Nonlinear feedback laws. Actually, a similar result also holds for nonlinear feedback laws $\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y)$ provided by equations (2.7)–(2.8) and (3.1). From the preceding theorem we observe that for initial state $\|y_0\|_{\mathcal{H}} \le r_\lambda^2$, the unique solution $y(t)$ of the Cauchy problem (3.7) satisfies

$$\begin{aligned}\|\mathcal{F}_\lambda y(t)\|_{L^2(\Omega)} \le C_2 e^{C_2\sqrt{\lambda}} e^{-\frac{\lambda}{4}(t-s)} \|y_0\|_{L^2(\Omega)} &\le C_2 e^{C_2\sqrt{\lambda}} r_\lambda^2 \\ &= r_\lambda \quad \forall t \in [s,+\infty).\end{aligned} \tag{3.10}$$

Now we replace the linear feedback law $\mathcal{F}_\lambda$ by $\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y)$ (see equation (2.8)), which satisfies

$$\|\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y)\|_{L^2(\Omega)} \le \min\left\{1, \sqrt{2\|y\|_{L^2(\Omega)}}\right\}. \tag{3.11}$$

Indeed, if $\|\mathcal{F}_\lambda y\|_{L^2(\Omega)} \le 2r_\lambda$, then since the operator norm $\|\mathcal{F}_\lambda\| \le \gamma_\lambda \le r_\lambda^{-1}$, we have

$$\begin{aligned}\|\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y)\|_{L^2(\Omega)} \le \|\mathcal{F}_\lambda y\|_{L^2(\Omega)} \le \sqrt{2r_\lambda \|\mathcal{F}_\lambda y\|_{L^2(\Omega)}} &\le \sqrt{2r_\lambda \|\mathcal{F}_\lambda\| \|y\|_{L^2(\Omega)}} \\ &\le \sqrt{2\|y\|_{L^2(\Omega)}}.\end{aligned}$$

If $\|\mathcal{F}_\lambda y\|_{L^2(\Omega)} > 2r_\lambda$, then by the definition of $\mathcal{K}_{r_\lambda}$ we know that $\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y) = 0$, which completes the proof of condition (3.11).

Finally, we show that for $\|y_0\|_{\mathcal{H}} \le r_\lambda^2$ the solution of the closed-loop system with feedback law $\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y)$ also decays exponentially. Indeed, it suffices to prove that the solution y verifies

$$\mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y(t)) = \mathcal{F}_\lambda y(t) \quad \forall t \in [s,+\infty),$$

which, by recalling the definition of $\mathcal{K}_{r_\lambda}$ in (2.7)–(2.8), is true according to (3.10),

$$\|\mathcal{F}_\lambda y(t)\|_{L^2(\Omega)} \le C_2 e^{C_2\sqrt{\lambda}} \|y_0\|_{L^2(\Omega)} \le C_2 e^{C_2\sqrt{\lambda}} r_\lambda^2 = r_\lambda \quad \forall t \in [s,+\infty).$$

Theorem 3.2 (Local stabilization with nonlinear Lipschitz feedback laws). *For any* $\lambda > 0$, *for any* $\|y_0\|_{\mathcal{H}} \le r_\lambda^2$, *and for any* $s \in \mathbb{R}$ *the Cauchy problem*

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = -\gamma_\lambda 1_\omega \mathcal{K}_{r_\lambda}(\mathcal{F}_\lambda y), & (t,x) \in [s,+\infty)\times\Omega, \\ \operatorname{div} y = 0, & (t,x) \in [s,+\infty)\times\Omega, \\ y(t,x) = 0, & (t,x) \in [s,+\infty)\times\partial\Omega, \\ y(s,x) = y_0(x), & x \in \Omega, \end{cases} \tag{3.12}$$

has a unique solution in $C^0([s,+\infty);\mathcal{H})\cap L^2_{\mathrm{loc}}(s,+\infty;\mathcal{V}_\sigma)$. *Moreover, this unique solution verifies*

$$\|y(t)\|_{\mathcal{H}} \le C_1 e^{C_1\sqrt{\lambda}} e^{-\frac{\lambda}{4}(t-s)}\|y_0\|_{\mathcal{H}} \quad \forall t\in[s,+\infty),$$
$$\|\mathcal{F}_\lambda y(t)\|_{\mathcal{H}} \le C_2 e^{C_2\sqrt{\lambda}} e^{-\frac{\lambda}{4}(t-s)}\|y_0\|_{\mathcal{H}} \quad \forall t\in[s,+\infty).$$

4. Quantitative null-controllability with cost estimates

In this section we construct feedback laws (controls) that yield solutions that decay to zero in finite time.

Theorem 4.1. *There exists* $C_3>0$ *such that, for any* $T\in(0,1)$*, for any* $y_0\in\mathcal{H}$ *satisfying* $\|y_0\|_{\mathcal{H}}\le e^{-\frac{C_3}{T}}$ *we construct an explicit control* $f(t,x)$ *for the controlled system* (1.1) *such that the unique solution with initial data* $y(0,x)=y_0(x)$ *verifies* $y(T,x)=0$. *Moreover,*

$$\|f(t,x)\|_{L^\infty(0,T;\mathcal{H})}\le e^{\frac{C_3}{T}}\|y_0\|_{\mathcal{H}}.$$

Proof. For the ease of presentation, we only consider the case when $1/T=2^{n_0}$ with $n_0\in N^*$. The other cases can be treated via time transition, i.e. if $T\in(2^{-m-1},2^{-m})$ then we simply let the feedback law $U(t;y):=0$ on the time interval $[2^{-m-1},T]$. More precisely, we consider the following partition of $[0,T]$ and piecewise feedback laws:

$$T_n := 2^{-n_0}\Big(1-\frac{1}{2^n}\Big),\quad I_n:=[T_n,T_{n+1}),\quad \lambda_n:=Q^2 2^{2(n_0+n)}\quad \text{for any } n\ge 0; \tag{4.1}$$

$$\text{for any } n\ge 0 \text{ we consider the control (feedback law) as } \mathcal{F}_{\lambda_n} \text{ on interval } I_n, \tag{4.2}$$

where we recall that the exact value of Q is given in Section 2.5, equation (2.13).

Control design.

- Step 1. Let the constant $R_T>0$ be sufficiently small to be fixed later on. First, for $\|y_0\|_{\mathcal{H}}\le R_T$, on the interval I_0 we consider the closed-loop system (1.1)–(1.2) with feedback law $U:=\mathcal{F}_{\lambda_0}$ and initial data $y(0,x)=y_0(x)$. Assuming that $R_T\le r_{\lambda_0}$, then according to Theorem 3.1 the closed-loop system has a unique solution $\tilde{y}|_{\bar{I}_0}$ that decays exponentially with decay rate $\lambda_0/4$.
- Step 2. Next, we consider the closed-loop system with feedback law $\mathcal{F}_{\lambda_1}$ and $y(T_1,x):=\tilde{y}(T_1,x)$ on I_1. Again we assume $\|y(T_1)\|\le r_{\lambda_1}$ to find a unique solution $\tilde{y}|_{\bar{I}_1}$ that is exponentially stable.
- Step 3. By continuing this procedure on I_n and by always assuming $\|y(T_n)\|\le r_{\lambda_n}$, we find a stable solution $\tilde{y}|_{I_n}$.
- Step 4. We denote this constructed solution $\tilde{y}|_{[0,T)}\in C^0([0,T);\mathcal{H})$ by $y|_{[0,T)}$.
- Step 5. For some sufficiently small R_T we prove that $\|y(T_n)\|$ is indeed smaller than r_{λ_n} for every $n\in\mathbb{N}$, and show that the solution tends to zero as $y(T):=\lim_{t\to T^-}y(t)=0$.

Step 6. Eventually, thanks to Step 5, $y|_{[0,T]}$ is the unique solution of the Cauchy problem (2.2) with the control term f given by $f|_{I_n} := \mathcal{F}_{\lambda_n} y|_{I_n}$ for all $n \geq 0$, which satisfies $y(T) = 0$.

Step 7. We calculate precise cost estimates.

First we *assume* that for every I_n the value $\|y(T_n)\|_{L^2}$ is smaller than r_{λ_n}, which, together with Theorem 3.1, implies that the solution $y|_{I_n}$ verifies

$$\|y(t)\|_{L^2(\Omega)} \leq C_1 e^{C_1 Q 2^{n_0+n}} e^{-\frac{Q^2}{4} 2^{2(n_0+n)}(t-T_n)} \|y(T_n)\|_{L^2(\Omega)} \quad \forall t \in I_n, \tag{4.3}$$

$$\|\mathcal{F}_{\lambda_n} y(t)\|_{L^2(\Omega)} \leq C_2 e^{C_2 Q 2^{n_0+n}} e^{-\frac{Q^2}{4} 2^{2(n_0+n)}(t-T_n)} \|y(T_n)\|_{L^2(\Omega)} \quad \forall t \in I_n. \tag{4.4}$$

Consequently, for every $n \geq 1$ the value of $y(T_n)$ is dominated by

$$\begin{aligned}
\|y(T_n)\|_{L^2(\Omega)} &\leq \Big(\prod_{k=0}^{n-1} C_1 e^{C_1\sqrt{\lambda_k}} e^{-\frac{\lambda_k}{4} 2^{-(n_0+k+1)}}\Big) \|y_0\|_{L^2(\Omega)} \\
&= \Big(\prod_{k=0}^{n-1} C_1 e^{C_1 Q 2^{n_0+k}} e^{-\frac{Q^2}{8} 2^{n_0+k}}\Big) \|y_0\|_{L^2(\Omega)} \\
&\leq \Big(\prod_{k=0}^{n-1} e^{\frac{Q^2}{64} 2^{n_0+k}} e^{-\frac{Q^2}{8} 2^{n_0+k}}\Big) \|y_0\|_{L^2(\Omega)} \\
&= \Big(\prod_{k=0}^{n-1} e^{-\frac{7Q^2}{64} 2^{n_0+k}}\Big) \|y_0\|_{L^2(\Omega)} \\
&= \exp\Big(-\frac{7Q^2}{64} 2^{n_0}(2^n - 1)\Big) \|y_0\|_{L^2(\Omega)}.
\end{aligned} \tag{4.5}$$

Observe that the above inequality also holds for $n = 0$. Furthermore, for any $n \geq 1$ and for any $t \in I_n$, the control term is bounded by

$$\|\mathcal{F}_{\lambda_n} y(t)\|_{L^2(\Omega)} \leq C_2 e^{C_2 Q 2^{n_0+n}} \|y(T_n)\|_{L^2(\Omega)} \leq \exp\Big(-\frac{5Q^2}{64} 2^{n_0+n-1}\Big) \|y_0\|_{L^2(\Omega)}. \tag{4.6}$$

Clearly, the right-hand sides of inequalities (4.5) and (4.6) tend to 0 as n tends to ∞. Therefore, it suffices to prove the assumption $\|y(T_n)\|_{L^2} \leq r_{\lambda_n}$ to close the "bootstrap" and to conclude the null-controllability. By recalling the definitions of λ_n, r_{λ_n}, and Q we know that

$$e^{-\frac{Q^2}{64} 2^{n_0+n}} \leq (C_2 e^{C_2 Q 2^{n_0+n}})^{-1} = (C_2 e^{C_2\sqrt{\lambda_n}})^{-1} = r_{\lambda_n} \quad \forall n \in \mathbb{N}.$$

Hence, it suffices to find some $R_T > 0$ such that

$$R_T \exp\Big(-\frac{7Q^2}{64} 2^{n_0}(2^n - 1)\Big) \leq e^{-\frac{Q^2}{64} 2^{n_0+n}} \leq r_{\lambda_n} \quad \forall n \in \mathbb{N}. \tag{4.7}$$

Thus one can take

$$R_T := e^{-\frac{Q^2}{32}2^{n_0}} = e^{-\frac{Q^2}{32T}} = e^{-\frac{C_3}{T}}, \quad \text{where } C_3 = \frac{Q^2}{32}. \tag{4.8}$$

It only remains to estimate the controlling cost. Thanks to (4.6) we know that

$$\|f(t)\|_{L^2(\Omega)} \le \|y_0\|_{L^2(\Omega)} \quad \forall t \in [T_1, T].$$

As for $t \in [0, T_1)$ and the control $f|_{I_0}(t)$, we have

$$\|\mathcal{F}_{\lambda_0} y(t)\|_{L^2(\Omega)} \le C_2 e^{C_2 Q 2^{n_0}} \|y_0\|_{L^2(\Omega)} \le e^{\frac{Q^2}{64}2^{n_0}} \|y_0\|_{L^2(\Omega)} \le e^{\frac{C_3}{T}} \|y_0\|_{L^2(\Omega)}.$$

In conclusion, for any $\|y_0\|_{\mathcal{H}} \le e^{-\frac{C_3}{T}}$, the constructed solution $y(t,x)$ and control $f(t,x)$ satisfy

$$\begin{aligned} &\|y(t,\cdot)\|_{L^2(\Omega)} \text{ and } \|f(t,\cdot)\|_{L^2(\Omega)} \to 0^+ \quad \text{as } t \to T^-, \\ &\|y(t,\cdot)\|_{L^2(\Omega)} \text{ and } \|f(t,\cdot)\|_{L^2(\Omega)} \le e^{\frac{C_3}{T}} \|y_0\|_{L^2(\Omega)} \quad \forall t \in [0,T]. \end{aligned}$$

■

Remark 4.2. If we replace the linear feedback laws $\{\mathcal{F}_{\lambda_n} y\}_{n=1}^{\infty}$ by $\{\mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n} y)\}_{n=1}^{\infty}$ on interval I_n, then a similar result holds. Indeed, according to Theorem 3.2 it suffices to find some initial state such that for every $n \in \mathbb{N}$ the value of $\|y(T_n)\|$ is smaller than $r_{\lambda_n}^2$. More precisely, instead of taking some $R_T > 0$ that verifies (4.7), one only needs to find $\widetilde{R}_T := e^{-\frac{Q^2}{16T}} = e^{-\frac{2C_3}{T}}$ satisfying

$$\widetilde{R}_T \exp\Big(-\frac{7Q^2}{64} 2^{n_0}(2^n - 1)\Big) \le e^{-\frac{Q^2}{32}2^{n_0+n}} \le r_{\lambda_n}^2 \quad \forall n \in \mathbb{N}$$

to guarantee that for every $n \in \mathbb{N}$ we have $\|y(T_n)\|_{L^2} \le r_{\lambda_n}^2$.

5. Small-time local stabilization

As in the preceding section, we focus only on the case when $T = 1/2^{n_0}$ with n_0 an integer. We also adapt the same construction of T_n and λ_n given by (4.1) in Section 4.

Theorem 5.1 (Small-time local stabilization of Navier–Stokes equations). *Let $T = 1/2^{n_0}$ with $n_0 \in \mathbb{N}^*$. The following T-periodic feedback law $U(t;y)\colon \mathbb{R} \times \mathcal{H} \to \mathcal{H}$ satisfying (3.11),*

$$U|_{[0,T)\times\mathcal{H}}(t;y) := \mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n} y) \quad \forall y \in \mathcal{H},\ \forall t \in I_n,\ \forall n \in \mathbb{N}, \tag{5.1}$$

is a proper feedback law for system (1.1)–(1.2). Moreover, for some effectively computable constant Λ_T this feedback law stabilizes system (1.1)–(1.2) in finite time:

(i) (2T stabilization) $\Phi(2T+t,t;y_0) = 0\ \forall t \in \mathbb{R},\ \forall \|y_0\|_{\mathcal{H}} \le \Lambda_T$.

(ii) (Uniform stability) *For every* $\delta > 0$, *there exists an effectively computable* $\eta > 0$ *such that*

$$(\|(y_0\|_{\mathcal{H}} \leq \eta) \Rightarrow \big(\|\Phi(t,t';y_0)\|_{\mathcal{H}} \leq \delta \ \forall t' \in \mathbb{R},\ \forall t \in (t',+\infty)\big).$$

Proof. We mimic the proof of the finite-time stabilization of the heat equations [25, 52], as relatively standard; see also [28, 50, 51] for similar results. The proof is followed by five steps:

Step 1. The feedback law U is a proper feedback law, i.e. for any $y_0 \in \mathcal{H}$ and for any initial time $s \in \mathbb{R}$ there exists a unique global (in time) solution.

Step 2. Null-controllability: $\Phi(T,0;y_0) = 0$ for any y_0 satisfying $\|y_0\|_{\mathcal{H}} \leq \widetilde{R}_T = e^{-\frac{2C_3}{T}}$. Moreover,

$$\|\Phi(t,0;y_0)\|_{\mathcal{H}} \leq e^{\frac{C_3}{T}} \|y_0\|_{\mathcal{H}} \quad \forall \|y_0\|_{\mathcal{H}} \leq e^{-\frac{2C_3}{T}},\ \forall t \in [0,T]. \tag{5.2}$$

Step 3. For any $\tilde{\eta} > 0$, there exists some $\varepsilon(\tilde{\eta}) \in (0,\tilde{\eta})$ such that

$$\|\Phi(t,s;y_0)\|_{\mathcal{H}} \leq \tilde{\eta} \quad \forall \|y_0\|_{\mathcal{H}} \leq \varepsilon(\tilde{\eta}),\ \forall s \in [0,T),\ \forall t \in [s,T]. \tag{5.3}$$

Step 4. $2T$ stabilization: $\Phi(2T,s;y_0) = 0$ for any $s \in [0,T)$, for any y_0 satisfying $\|y_0\|_{\mathcal{H}} \leq \varepsilon(e^{-\frac{2C_3}{T}}) =: \Lambda_T$.

Step 5. Uniform stability as a direct consequence of Steps 2–4.

Step 1. It suffices to prove that for any $s \in [0,T)$, the closed-loop system has a unique solution on $[s,T]$. Indeed, thanks to Theorem 2.8 there exists a unique solution on I_n for any I_n that intersects with $[s,T)$. Hence we find a unique solution y in $C^0([s,T);\mathcal{H}) \cap L^2_{\mathrm{loc}}(s,T;\mathcal{V}_\sigma)$. Observe that the control (provided by the related feedback law) is smaller than 1, i.e. $\|f(t,x)\|_{L^2(s,T;\mathcal{H})} \leq \sqrt{T}$. Theorem 2.3 implies that the solution y is indeed in $C^0([s,T];\mathcal{H}) \cap L^2(s,T;\mathcal{V}_\sigma)$. Finally, thanks to Theorem 2.3 again, the unique solution y never blows up,

$$\|y(t,x)\|^2_{\mathcal{H}} + \|\nabla y(t,x)\|^2_{L^2(s,t;L^2)} \leq \|y_0\|^2_{\mathcal{H}} + C_0(t-s) \quad \forall t \in (s,+\infty).$$

Step 2. This step is a consequence of Theorem 4.1 and Remark 4.2.

Step 3. Thanks to the fact that $\|f(t,x)\|_{\mathcal{H}} \leq 1$ and Theorem 2.3, there exists $\widetilde{T} \in (0,T)$ such that

$$\|\Phi(t,s;y_0)\|_{\mathcal{H}} \leq \tilde{\eta} \quad \forall \|y_0\|_{\mathcal{H}} \leq \tilde{\eta}/2,\ \forall s \in [\widetilde{T},T),\ \forall t \in [s,T].$$

Observe that the feedback law U on $[0,\widetilde{T})$ is composed of finitely many stationary feedback laws on intervals $\{I_n\}$, while, thanks to Theorem 3.2, on each of these intervals I_n the system is locally exponentially stable. Consequently, there exists some $\varepsilon = \varepsilon(\tilde{\eta}) \in (0,\tilde{\eta}/2)$ such that

$$\|\Phi(t,s;y_0)\|_{\mathcal{H}} \leq \tilde{\eta}/2 \quad \forall \|y_0\|_{\mathcal{H}} \leq \varepsilon,\ \forall s \in [0,\widetilde{T}),\ \forall t \in [s,\widetilde{T}].$$

Step 4 is a trivial combination of Steps 2 and 3 by taking $\varepsilon(e^{-\frac{2C_3}{T}})$.

Step 5 follows directly from Steps 2–4. For instance, for $\delta > 0$, we can take

$$\delta_1 := \min\{\varepsilon(\delta), e^{-\frac{2C_3}{T}}\} < \delta \quad \text{and} \quad \eta := \varepsilon(\delta_1) < \delta_1 < \delta.$$

Indeed, for $s \in [0, T)$, and for $\|y_0\|_{\mathcal{H}} \le \eta$, thanks to the choice of η as $\varepsilon(\delta_1)$, Step 3 yields

$$\|y(t)\|_{\mathcal{H}} = \|\Phi(t, s; y_0)\|_{\mathcal{H}} \le \delta_1 \quad \forall t \in [s, T].$$

Next, thanks to the choice of δ_1, Steps 3 and 2 lead to

$$\|\Phi(t, s; y_0)\|_{\mathcal{H}} = \|\Phi(t, T; y(T))\|_{\mathcal{H}} \le \delta \quad \forall t \in [T, 2T],$$
$$\Phi(2T, s; y_0) = \Phi(2T, T; y(T)) = 0. \qquad \blacksquare$$

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