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On the sharp scattering threshold for the mass–energy double critical nonlinear Schrödinger equation via double track profile decomposition

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Abstract. The present paper is concerned with the large data scattering problem for the mass–energy double critical nonlinear Schrödinger equation $i\partial_t u + \Delta u \pm |u|^{\frac{4}{d}} u \pm |u|^{\frac{4}{d-2}} u = 0$ in $H^1(\mathbb{R}^d)$ with $d \geq 3$, referred to as DCNLS. In the defocusing–defocusing regime, Tao, Visan and Zhang showed that the unique solution of DCNLS is global and scattering in time for arbitrary initial data in $H^1(\mathbb{R}^d)$. This does not hold when at least one of the nonlinearities is focusing, due to the possible formation of blow-up and soliton solutions. However, precise thresholds for a solution of DCNLS being scattering were open in all the remaining regimes. Following the classical concentration compactness principle, we impose sharp scattering thresholds in terms of ground states for DCNLS in all the remaining regimes. The new challenge arises from the fact that the remainders of the standard L^2- or $\dot{H}^1$-profile decomposition fail to have asymptotically vanishing diagonal L^2- and $\dot{H}^1$-Strichartz norms simultaneously. To overcome this difficulty, we construct a double track profile decomposition which is capable of capturing the low-, medium- and high-frequency bubbles within a single profile decomposition and possesses remainders that are asymptotically small in both of the diagonal L^2- and $\dot{H}^1$-Strichartz spaces.

1. Introduction and main results

In this paper we study the large data scattering problem for the mass–energy double critical nonlinear Schrödinger equation (NLS)

$$i\partial_t u + \Delta u + \mu_1 |u|^{2_*-2} u + \mu_2 |u|^{2^*-2} u = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^d, \tag{DCNLS}$$

with $d \geq 3$, $\mu_1, \mu_2 \in \{\pm 1\}$, $2_* = 2 + \frac{4}{d}$ and $2^* = 2 + \frac{4}{d-2}$. Equation (DCNLS) is a special case of the NLS with combined nonlinearities

$$i\partial_t u + \Delta u + \mu_1 |u|^{p_1-2} u + \mu_2 |u|^{p_2-2} u = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^d, \tag{1.1}$$

with $\mu_1, \mu_2 \in \mathbb{R}$ and $p_1, p_2 \in (2, \infty)$. Equation (1.1) is a prototype model arising from numerous physical applications such as nonlinear optics and Bose–Einstein condensation.

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The signs μ_i can be tuned to be defocusing ($\mu_i < 0$) or focusing ($\mu_i > 0$), indicating the repulsivity or attractivity of the nonlinearity. For a comprehensive introduction to the physical background of (1.1), we refer to [3, 8, 34] and the references therein. Formally, (1.1) preserves

$$\begin{aligned} &\text{the mass} && \mathcal{M}(u) = \int_{\mathbb{R}^d} |u|^2\,dx, \\ &\text{the Hamiltonian} && \mathcal{H}(u) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2 - \frac{\mu_1}{p_1}|u|^{p_1} - \frac{\mu_2}{p_2}|u|^{p_2}\,dx, \\ &\text{the momentum} && \mathcal{P}(u) = \int_{\mathbb{R}^d} \operatorname{Im}(\bar{u}\nabla u)\,dx \end{aligned}$$

over time. It is also easy to check that any solution u of (1.1) is invariant under time and space translation. Direct calculation also shows that (1.1) remains invariant under the Galilean transformation

$$u(t,x) \mapsto e^{i\xi\cdot x}e^{-it|\xi|^2}u(t, x - 2\xi t)$$

for any $\xi \in \mathbb{R}^d$. Moreover, we say that a function P is a *soliton* solution of (1.1) if P solves the equation

$$-\Delta P + \omega P - \mu_1|P|^{p_1-2}P - \mu_2|P|^{p_2-2}P = 0 \tag{1.2}$$

for some $\omega \in \mathbb{R}$. One easily verifies that $u(t,x) := e^{i\omega t}P(x)$ is a solution of (1.1). As we will see later, the soliton solutions play a fundamental role in the study of (1.1), since they can be seen as the balance point between dispersive and nonlinear effects.

When $\mu_1 = 0$, (1.1) reduces to the NLS

$$i\partial_t u + \Delta u + \mu|u|^{p-2}u = 0 \tag{1.3}$$

with pure power-type nonlinearity, which has been extensively studied in the literature. In particular, a solution of (1.3) also exhibits the scaling invariance

$$u(t,x) \mapsto \lambda^{\frac{2}{p-2}}u(\lambda^2 t, \lambda x) \tag{1.4}$$

for any $\lambda > 0$, which distinguishes itself from (1.1) with combined nonlinearities. We also say that (1.3) is s_c-critical with $s_c = s_c(p) = \frac{d}{2} - \frac{2}{p-2}$. It is easy to verify that the $\dot{H}^{s_c}$-norm is invariant under the scaling (1.4). We are particularly interested in the cases $s_c = 0$ and $s_c = 1$: in order to guarantee one or more conservation laws, we demand the solution of the NLS is at least of class L^2 or $\dot{H}^1$. Moreover, we see that the mass and Hamiltonian are invariant under the 0- and 1-scalings respectively.

Concerning the Cauchy problem (1.3), Cazenave and Weissler [12, 13] showed that (1.3) with $p \in (2, 2^*]$ defined on some interval $I \ni t_0$ is locally well posed in $H^1(\mathbb{R}^d)$ on the maximal lifespan $I_{\max} \ni t_0$. In particular, if $p \in (2, 2^*)$ (namely the problem is energy-subcritical), then u blows up at finite time $t_{\sup} := \sup I_{\max}$ if and only if

$$\lim_{t\uparrow t_{\sup}} \|\nabla u(t)\|_2 = \infty. \tag{1.5}$$

A similar result holds for the negative time direction. Combining with the Gagliardo–Nirenberg inequality, it is immediate that (1.3) having defocusing energy-subcritical nonlinearity or mass-subcritical nonlinearity (regardless of the sign) is always globally well posed in $H^1(\mathbb{R}^d)$. However, this does not hold for focusing mass-supercritical and energy-subcritical (1.3): one can construct blow-up solutions using the celebrated virial identity due to Glassey [24] for initial data possessing negative virial. By a straightforward modification (see for instance [11]) the results from [12, 13] extend naturally to (1.1).

The blow-up criterion (1.5) does not carry over to the energy-critical case, since in this situation the well-posedness result also depends on the profile of the initial data. Using the so-called induction on energy method, Bourgain [7] was able to show that the defocusing energy-critical NLS is globally well posed and scattering[1] (we refer to Definition 1.12 below for a precise definition of a scattering solution) for any radial initial data in $\dot{H}^1(\mathbb{R}^d)$ in the case $d = 3$. Using the interaction Morawetz inequalities, the I-team [17] successfully removed the radial assumption in [7]. The result in [17] was later extended to arbitrary dimension $d \geq 4$ [41, 44] and the well-posedness and scattering problem for the defocusing energy-critical NLS was completely resolved.

Utilizing the Glassey's virial arguments one verifies that a solution of the focusing energy-critical NLS is not always globally well posed and scattering. On the other hand, appealing to standard contraction iteration we are able to show that the focusing energy-critical NLS is globally well posed and scattering for small initial data. It turns out that the strict threshold, under which the small data theory takes place, can be described by the Aubin–Talenti function

$$W(x) := \Big(1 + \frac{|x|^2}{d(d-2)}\Big)^{-\frac{d-2}{2}},$$

which solves the Lane–Emden equation

$$-\Delta W = W^{2^*-1}$$

and is an optimizer of the Sobolev inequality

$$\mathcal{S} := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^d)} \frac{\|u\|_{\dot{H}^1}^2}{\|u\|_{2^*}^2}.$$

Using the concentration compactness principle, Kenig and Merle [27] proved the following large data scattering result for the focusing energy-critical NLS:

Theorem 1.1 ([27]). *Let $d \in \{3, 4, 5\}$, $p = 2^*$ and $\mu = 1$. Also let u be a solution of* (1.3) *with $u(0) = u_0 \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^d)$, $\mathcal{H}^*(u_0) < \mathcal{H}^*(W)$ and $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, where*

$$\mathcal{H}^*(u) := \tfrac{1}{2}\|\nabla u\|_2^2 - \tfrac{1}{2^*}\|u\|_{2^*}^{2^*}. \tag{1.6}$$

Then u is global and scattering in time.

[1] For (1.3) with pure mass- or energy-critical nonlinearity, the scattering space is referred to $L^2(\mathbb{R}^d)$ or $\dot{H}^1(\mathbb{R}^d)$ respectively, while for (DCNLS) we consider scattering in $H^1(\mathbb{R}^d)$.

The result by Kenig and Merle was later extended by Killip and Visan [30] to arbitrary dimension $d \geq 5$, where the radial assumption was also removed. Very recently, Dodson [22] also removed the radial assumption in the case $d = 4$. The three-dimensional large data scattering problem for nonradial initial data in $\dot{H}^1(\mathbb{R}^3)$ still remains open.

Based on the methodologies developed for the energy-critical NLS, Dodson was able to prove similar global well-posedness and scattering results for the mass-critical NLS. For the defocusing case, Dodson [18, 20, 21] showed that a solution of the defocusing mass-critical NLS is always global and scattering in time for any initial data $u_0 \in L^2(\mathbb{R}^d)$ with $d \geq 1$. To formulate the corresponding result for the focusing case, we denote by Q the unique positive and radial solution of the stationary focusing mass-critical NLS

$$-\Delta Q + Q = Q^{2_*-1}.$$

For the existence and uniqueness of Q, we refer to [46] and [33] respectively. The following result is due to Dodson [19] concerning the focusing mass-critical NLS:

Theorem 1.2 ([19]). *Let $d \geq 1$, $p = 2_*$ and $\mu = 1$. Also let u be a solution of (1.3) with $u(0) = u_0 \in L^2(\mathbb{R}^d)$ and $\mathcal{M}(u_0) < \mathcal{M}(Q)$. Then u is global and scattering in time.*

In recent years, problems with combined nonlinearities (1.1) have been attracting much attention from the mathematical community. The mixed-type nature of (1.1) prevents it from being scale invariant, and several arguments for (1.3) fail to hold, which makes the analysis for (1.1) rather delicate and challenging. A systematic study of (1.1) was initiated by Tao, Visan and Zhang in their seminal paper [43]. In particular, based on the interaction Morawetz inequalities they showed that a solution of (1.1) with $\mu_1, \mu_2 < 0$ and $p_1 = 2_*$, $p_2 = 2^*$ (namely the defocusing–defocusing double critical regime) is always global and scattering in time for any initial data $u_0 \in H^1(\mathbb{R}^d)$.[2] As expected, this does not hold when at least one of the μ_i in (1.1) is negative. Using concentration compactness and perturbation arguments initiated by [25], Akahori, Ibrahim, Kikuchi and Nawa [1] were able to formulate a sharp scattering threshold for (1.1) in the case $d \geq 5$, $\mu_1, \mu_2 > 0$, $p_1 \in (2_*, 2^*)$ and $p_2 = 2^*$ (namely the focusing energy-critical NLS perturbed by a focusing mass-supercritical and energy-subcritical nonlinearity). The methodology of [1,25] has now become a golden rule for the study of large data scattering problems of the NLS with combined nonlinearities. In this direction, we refer to the representative papers [10, 14, 16, 28, 29, 36–39, 47] for large data scattering results of (1.1) in different regimes, where at least one of the nonlinearities possesses critical growth.

1.1. Main results

In this paper we study the most interesting and difficult case (DCNLS), where the mass- and energy-critical nonlinearities exist simultaneously in the equation. Roughly speaking,

[2] This was originally shown under the additional assumption that a solution of the defocusing mass-critical NLS is always global and scattering, which was later shown to be true by Dodson [18].

we cannot consider (DCNLS) as the energy-critical NLS perturbed by the mass-critical nonlinearity, nor vice versa, due to the endpoint critical nature of the potential terms. Nevertheless, it is quite natural to have the following heuristics on the long time dynamics of (DCNLS) based on the results for NLS with single mass- or energy-critical potentials:

- For the defocusing–defocusing case, we expect that both of the mass- and energy-critical nonlinear terms are harmless, and a solution of (DCNLS) should be global and scattering in time for arbitrary initial data u_0 from $H^1(\mathbb{R}^d)$.
- For the focusing–defocusing case, we expect that under the stabilization of the defocusing energy-critical potential, a solution of (DCNLS) should always be global. However, a bifurcation of scattering and soliton solutions might occur, which is determined by the mass of the initial data. In view of scaling, we conjecture that the threshold is given by $\mathcal{M}(Q)$.
- For the defocusing–focusing case, we expect that the scattering threshold should be uniquely determined by the Hamiltonian of the initial data. In view of scaling, we conjecture that the threshold is given by $\mathcal{H}^*(W)$.

We should discuss the focusing–focusing case separately, which is the most subtle one among the four regimes. One might expect that the restriction for the scattering threshold is coming from both the mass and the energy sides. In particular, a reasonable guess for the threshold would be

$$\mathcal{M}(u_0) < \mathcal{M}(Q) \wedge \mathcal{H}(u_0) < \mathcal{H}^*(W).$$

This is however not the case. As shown by the following result by Soave, the actual energy threshold is strictly less than $\mathcal{H}^*(W)$.

Theorem 1.3 ([42]). *Let $d \geq 3$ and $\mu_1 = \mu_2 = 1$. Define*

$$m_c := \inf_{u \in H^1(\mathbb{R}^d)} \big\{\mathcal{H}(u) : \mathcal{M}(u) = c,\ \mathcal{K}(u) = 0\big\}, \tag{1.7}$$

where $\mathcal{K}$ is defined by

$$\mathcal{K}(u) := \|\nabla u\|_2^2 - \frac{d}{d+2}\|u\|_{2_*}^{2_*} - \|u\|_{2^*}^{2^*}.$$

Then we have the following statements:

(i) (Existence of ground state). *For any $c \in (0, \mathcal{M}(Q))$, the variational problem (1.7) has a positive and radially symmetric minimizer P_c with $m_c = \mathcal{H}(P_c) \in (0, \mathcal{H}^*(W))$. Moreover, P_c is a solution of*

$$-\Delta P_c + \omega P_c = P_c^{2_*-1} + P_c^{2^*-1} \tag{1.8}$$

for some $\omega > 0$.

(ii) (Blow-up criterion). *Assume that* $u_0 \in H^1(\mathbb{R}^d)$ *satisfies*

$$\mathcal{M}(u_0) \in (0, \mathcal{M}(Q)) \wedge \mathcal{H}(u_0) < m_{\mathcal{M}(u_0)} \wedge \mathcal{K}(u_0) < 0.$$

Also assume that $|x|u_0 \in L^2(\mathbb{R}^d)$. *Then the solution* u *of* (DCNLS) *with* $u(0) = u_0$ *blows up in finite time.*

Remark 1.4. The quantity $\mathcal{K}(u)$ is referred to as the virial of u, which is closely related to the Glassey's virial identity and plays a fundamental role in the study of NLS. △

We make the intuitive heuristics into the following rigorous statements:

Conjecture 1.5. *Let* $d \geq 3$ *and consider* (DCNLS) *on some time interval* $I \ni 0$. *Let* u *be the unique solution of* (DCNLS) *with* $u(0) = u_0 \in H^1(\mathbb{R}^d)$. *We also define*

$$\mathcal{K}(u) := \|\nabla u\|_2^2 - \mu_1 \frac{d}{d+2}\|u\|_{2_*}^{2_*} - \mu_2 \|u\|_{2^*}^{2^*}.$$

Then we have the following statements:

(i) (Defocusing–defocusing regime). *Let* $\mu_1 = \mu_2 = -1$. *Then* u *is global and scattering in time.*

(ii) (Focusing–defocusing regime). *Let* $\mu_1 = 1$ *and* $\mu_2 = -1$. *Then* u *is a global solution. If additionally* $\mathcal{M}(u_0) < \mathcal{M}(Q)$, *then* u *is also scattering in time.*

(iii) (Defocusing–focusing regime). *Let* $\mu_1 = -1$ *and* $\mu_2 = 1$. *Assume that*

$$\mathcal{H}(u_0) < \mathcal{H}^*(W) \wedge \mathcal{K}(u_0) > 0.$$

Then u *is global and scattering in time.*

(iv) (Focusing–focusing regime). *Let* $\mu_1 = \mu_2 = 1$. *Assume that*

$$\mathcal{M}(u_0) < \mathcal{M}(Q) \wedge \mathcal{H}(u_0) < m_{\mathcal{M}(u_0)} \wedge \mathcal{K}(u_0) > 0,$$

where the quantity $m_{\mathcal{M}(u_0)}$ *is defined through* (1.7). *Then* u *is global and scattering in time.*

As mentioned previously, Conjecture 1.5 (i) has already been proved by Tao, Visan and Zhang [43]. The global well-posedness result in Conjecture 1.5 (ii) was shown by Zhang [48] and Tao, Visan and Zhang [43]. Moreover, Conjecture 1.5 (iii) was proved by Cheng, Miao and Zhao [16] in the case $d \leq 4$ and the author [35] in the case $d \geq 5$, both under the additional assumption that u_0 is radially symmetric.

In this paper we prove Conjecture 1.5 for initial data from $H^1(\mathbb{R}^d)$ which are not necessarily radial. Our main result is as follows:

Theorem 1.6. *We assume in the cases* $d = 3$, $\mu_1 = -1$, $\mu_2 = 1$ *and* $d = 3$, $\mu_1 = \mu_2 = 1$ *additionally that* u_0 *is radially symmetric. Then Conjecture* 1.5 *holds for any* $d \geq 3$.

Remark 1.7. The radial assumption by Theorem 1.6 is removable as long as Theorem 1.1 also holds for nonradial initial data from $\dot{H}^1(\mathbb{R}^3)$, which is widely believed to be true. △

The sharpness of the scattering threshold for the focusing–focusing (DCNLS) is already revealed by Theorem 1.3. The criticality of the threshold for the defocusing–focusing (DCNLS) is more subtle, since in general there exists no soliton solution for the corresponding stationary equation; see [42, Thm. 1.2]. Nevertheless, we have the following variational characterization of the scattering threshold:

Proposition 1.8. *Let $\mu_1 = -1$ and $\mu_2 = 1$. Let m_c be defined through (1.7). Then $m_c = \mathcal{H}^*(W)$ and (1.7) has no optimizer for any $c \in (0,\infty)$.*

The proof of Proposition 1.8 follows the same lines as [16, Prop. 1.2], but we will consider the variational problem on a manifold with prescribed mass, which complicates the arguments in several places. Moreover, it was shown in [16] that any solution of the defocusing–focusing (DCNLS) with initial data u_0 satisfying

$$|x|u_0 \in L^2(\mathbb{R}^d) \wedge \mathcal{H}(u_0) < \mathcal{H}^*(W) \wedge \mathcal{K}(u_0) < 0$$

must blow up in finite time. This gives a complete description of the criticality of the scattering threshold for the defocusing–focusing (DCNLS).

For the focusing–defocusing regime, it was shown by Zhang [48] and Tao, Visan and Zhang [43] that a solution of the focusing–defocusing (DCNLS) is always globally well posed, hence the blow-up solutions are ruled out. Using simple variational arguments we will show the existence of ground states at arbitrary mass level larger than $\mathcal{M}(Q)$.

Proposition 1.9. *Let $\mu_1 = 1$ and $\mu_2 = -1$. Define*

$$\gamma_c := \inf_{u \in H^1(\mathbb{R}^d)} \big\{\mathcal{H}(u) : \mathcal{M}(u) = c\big\}. \tag{1.9}$$

Then we have the following statements:

(i) *The mapping $c \mapsto \gamma_c$ is monotone decreasing on $(0,\infty)$, equal to zero on $(0,\mathcal{M}(Q)]$ and negative on $(\mathcal{M}(Q),\infty)$.*

(ii) *For all $c \in (0,\mathcal{M}(Q)]$, (1.9) has no minimizer.*

(iii) *For all $c \in (\mathcal{M}(Q),\infty)$, (1.9) has a positive and radially symmetric minimizer S_c. Consequently, S_c is a solution of*

$$-\Delta S_c + \omega S_c = S_c^{2_*-1} - S_c^{2^*-1} \tag{1.10}$$

with some $\omega \in (0, \frac{2}{d}(\frac{d}{d+2})^{\frac{d}{2}})$.

What can be said about the focusing–defocusing model in the borderline case $\mathcal{M}(u_0)=\mathcal{M}(Q)$ remains an interesting open problem. As suggested by the results in [9, 40], we conjecture that scattering also takes place in the critical mass case. We plan to tackle this problem in a forthcoming paper.

1.2. Road map for the large data scattering results

To prove Theorem 1.6, we utilize the standard concentration compactness arguments initiated by Kenig and Merle [27]. The idea can be briefly summarized as follows: by assuming that the scattering result below threshold does not hold, we may find a sequence of solutions $(u_n)_n$ of (DCNLS) which lie below the scattering threshold and have diverging space-time norms. We then apply a suitable linear profile decomposition to the initial data $(u_n(0))_n$ and define the nonlinear profiles as solutions of (DCNLS) with the linear profiles as their initial data. Under the inductive hypothesis we should deduce that there exists exactly one nonlinear profile, the so-called minimal blow-up solution, that must have infinite space-time norm and be equal to zero at the same time. This hence leads to a contradiction, which in turn proves the desired claim.

In view of the stability theory (Lemma 2.4), the main challenge will be to verify the smallness condition

$$\|\langle\nabla\rangle e\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(\mathbb{R})} \ll 1 \tag{1.11}$$

for an error term e associated to the nonlinear profiles (which is defined precisely through (4.90) given later). Loosely speaking, to achieve (1.11) we demand the remainders w_n^k given by the linear profile decomposition satisfy the asymptotic smallness condition

$$\lim_{k\to K^*}\lim_{n\to\infty}\|e^{it\Delta}w_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d}}\cap L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbb{R})} = 0. \tag{1.12}$$

However, this is impossible by applying solely the L^2- or $\dot{H}^1$-profile decomposition. To solve this problem, Cheng, Miao and Zhao [16] established a profile decomposition which was obtained by first applying the L^2-profile decomposition to the (radial) underlying sequence $(\langle\nabla\rangle\psi_n)_n$ and then undoing the transformation. The robustness of such a profile decomposition lies in the fact that the remainders satisfy the even stronger asymptotic smallness condition

$$\lim_{k\to K^*}\lim_{n\to\infty}\|\langle\nabla\rangle e^{it\Delta}w_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R})} = 0.$$

Equation (1.12) follows immediately from the Strichartz inequality and interpolation. Nevertheless, the radial assumption is essential, which guarantees that the Galilean boosts appearing in the L^2-profile decomposition are constantly equal to zero. Indeed, we may also apply the full L^2-profile decomposition to the possibly nonradial underlying sequence, by also taking the nonvanishing Galilean boosts into account. However, using this way the Galilean boosts are generally unbounded, and such unboundedness induces a very strong loss of compactness, which leads to the failure of decomposition of the Hamiltonian. Heuristically, the occurrence of the compactness defect is attributed to the fact that the profile decomposition in [16] can still be seen as a variant of the L^2-profile decomposition, and hence it is insufficiently sensitive to the high-frequency bubbles.

Our solution is based on a refinement of the classical profile decompositions. Notice that the profile decompositions are obtained by an iterative process. At each iterative step we will meet a bifurcation: either

(i) $\limsup_{n\to\infty} \|e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R})} \geq \limsup_{n\to\infty} \|e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbb{R})}$

or

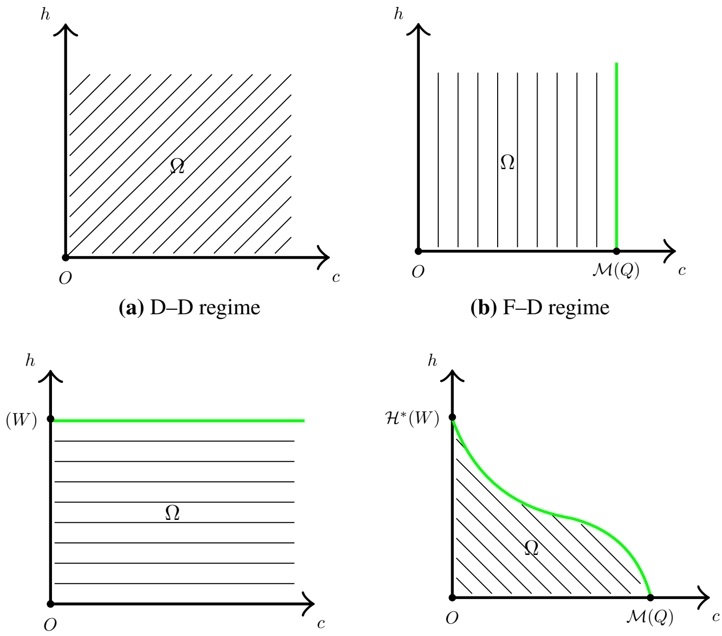
(ii) $\limsup_{n\to\infty} \|e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R})} < \limsup_{n\to\infty} \|e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbb{R})}$.

In the former case, we apply the L^2-decomposition to continue, while in the latter case we apply the $\dot{H}^1$-decomposition. Then (1.12) follows immediately from the construction of the profile decomposition. Moreover, since at each iterative step we are applying the profile decomposition to a bounded sequence in $H^1(\mathbb{R}^d)$, the resulting Galilean boosts are thus bounded. Using this additional property of the Galilean boosts we are able to show that the Hamiltonian of the bubbles are perfectly decoupled as desired. We refer to Lemma 3.6 for details.

On the other hand, we will build up the minimal blow-up solution using the mass–energy-indicator (MEI) functional $\mathcal{D}$. This was first introduced in [29] for studying the large data scattering problems for three-dimensional focusing–defocusing cubic–quintic NLS and further applied in [2, 36] for different models. The usage of the MEI functional is motivated by the fact that the underlying inductive scheme relies only on the mass and energy of the initial data and the scattering regime is immediately readable from the mass–energy diagram; see Figure 1. The idea can be described as follows: A mass–energy pair $(\mathcal{M}(u), \mathcal{H}(u))$ being admissible will imply $\mathcal{D}(u) \in (0, \infty)$. In order to escape the admissible region Ω, a function u must approach the boundary of Ω and one deduces that $\mathcal{D}(u) \to \infty$. We can therefore assume that the supremum $\mathcal{D}^*$ of $\mathcal{D}(u)$ running over all admissible u is finite, which leads to a contradiction and we conclude that $\mathcal{D}^* = \infty$, which will finish the desired proof. However, in the regime $\mu_2 = 1$, a mass–energy pair being admissible does not automatically imply the positivity of the virial $\mathcal{K}$. In particular, it is not trivial at first glance that the linear profiles have positive virial. We will appeal to the geometric properties of the MEI functional $\mathcal{D}$, combined with the variational arguments from [1], to overcome this difficulty.

Remark 1.10. By straightforward modification of the method developed in this paper, we are also able to give a new proof for the scattering result in the defocusing–defocusing regime using the concentration compactness principle. △

Outline of the paper. The paper is organized as follows: In Section 2 we establish the small data and stability theories for the (DCNLS). In Section 3 we construct the double track profile decomposition. Sections 4 to 6 are devoted to the proofs of Theorem 1.6 and Propositions 1.8 and 1.9. In the appendix we establish the endpoint values of the curve $c \mapsto m_c$ for the focusing–focusing (DCNLS).



(a) D–D regime **(b)** F–D regime

(c) D–F regime **(d)** F–F regime

Figure 1. An illustration of the admissible domains Ω in different regimes, where the shadow region is the intersection of Ω and $(0,\infty)^2$.

1.3. Notation and definitions

We use the notation $A \lesssim B$ whenever there exists some positive constant C such that $A \leq CB$. Similarly we define $A \gtrsim B$ and we will use $A \sim B$ when $A \lesssim B \lesssim A$. We denote by $\|\cdot\|_p$ the $L^p(\mathbb{R}^d)$-norm for $p \in [1,\infty]$. We similarly define the $H^1(\mathbb{R}^d)$-norm by $\|\cdot\|_{H^1}$. The following quantities will be used throughout the paper:

$$\mathcal{M}(u) := \|u\|_2^2, \tag{1.13}$$

$$\mathcal{H}(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu_1}{2_*}\|u\|_{2_*}^{2_*} - \frac{\mu_2}{2^*}\|u\|_{2^*}^{2^*}, \tag{1.14}$$

$$\mathcal{K}(u) := \|\nabla u\|_2^2 - \mu_1 \frac{d}{d+2}\|u\|_{2_*}^{2_*} - \mu_2\|u\|_{2^*}^{2^*}, \tag{1.15}$$

$$\mathcal{I}(u) := \mathcal{H}(u) - \frac{1}{2}\mathcal{K}(u) = \frac{\mu_2}{d}\|u\|_{2^*}^{2^*}. \tag{1.16}$$

We will also frequently use the scaling operator

$$T_\lambda u(x) := \lambda^{\frac{d}{2}} u(\lambda x). \tag{1.17}$$

One easily verifies that the L^2-norm is invariant under this scaling. Throughout the paper we denote by g_{ξ_0,x_0,λ_0} the L^2-symmetry transformation which is defined by

$$g_{\xi_0,x_0,\lambda_0} f(x) := \lambda_0^{-\frac{d}{2}} e^{i\xi_0\cdot x} f(\lambda_0^{-1}(x-x_0)) \tag{1.18}$$

for $(\xi_0, x_0, \lambda_0) \in \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty)$.

We denote by Q the unique positive and radially symmetric solution of

$$-\Delta Q + Q = Q^{2_*-1}$$

and by C_{GN} the optimal L^2-critical Gagliardo–Nirenberg constant, i.e.

$$\mathrm{C}_{\mathrm{GN}} = \inf_{u\in H^1(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla u\|_2^2 \|u\|_2^{\frac{4}{d}}}{\|u\|_{2_*}^{2_*}}. \tag{1.19}$$

Using Pohozaev identities (see for instance [5]), the uniqueness of Q and scaling arguments one easily verifies that

$$\mathrm{C}_{\mathrm{GN}} = \frac{d}{d+2}(\mathcal{M}(Q))^{\frac{2}{d}}. \tag{1.20}$$

We also denote by $\mathcal{S}$ the optimal constant for the Sobolev inequality, i.e.

$$\mathcal{S} := \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}.$$

Here, the space $\mathcal{D}^{1,2}(\mathbb{R}^d)$ is defined by

$$\mathcal{D}^{1,2}(\mathbb{R}^d) := \big\{u \in L^{2^*}(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d)\big\}.$$

For an interval $I \subset \mathbb{R}$, the space $L_t^q L_x^r(I)$ is defined by

$$L_t^q L_x^r(I) := \big\{u\colon I\times\mathbb{R}^2 \to \mathbb{C} : \|u\|_{L_t^q L_x^r(I)} < \infty\big\},$$

where

$$\|u\|_{L_t^q L_x^r(I)}^q := \int_{\mathbb{R}} \|u\|_r^q \, dt.$$

The following spaces will be frequently used throughout the paper:

$$\begin{aligned} W_{2^*}(I) &:= L_{t,x}^{\frac{2(d+2)}{d-2}}(I),\\ W_{2_*}(I) &:= L_{t,x}^{\frac{2(d+2)}{d}}(I),\\ S(I) &:= L_t^\infty L_x^2(I) \cap L_t^2 L_x^{2^*}(I). \end{aligned}$$

A pair (q,r) is said to be L^2-admissible if $q,r \in [2,\infty]$, $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}$ and $(q,r,d) \neq (2,\infty,2)$. For any L^2-admissible pairs (q_1,r_1) and (q_2,r_2) we have the following Strichartz estimates: if u is a solution of

$$i\partial_t u + \Delta u = F(u)$$

in $I \subset \mathbb{R}$ with $t_0 \in I$ and $u(t_0) = u_0$, then

$$\|u\|_{L_t^q L_x^r(I)} \lesssim \|u_0\|_2 + \|F(u)\|_{L_t^{q_2'} L_x^{r_2'}(I)},$$

where (q_2', r_2') is the Hölder conjugate of (q_2, r_2). For a proof, we refer to [11, 26].

In this paper we use the following concepts for solution and scattering of (DCNLS):

Definiton 1.11 (Solution). A function $u: I \times \mathbb{R}^d \to \mathbb{C}$ is said to be a solution of (DCNLS) on the interval $I \subset \mathbb{R}$ if for any compact $J \subset I$, $u \in C(J; H^1(\mathbb{R}^d))$ and for all $t, t_0 \in I$,

$$u(t) = e^{i(t-t_0)\Delta} u(t_0) + i \int_{t_0}^{t} e^{i(t-s)\Delta} [\mu_1 |u|^{\frac{4}{d}} u + \mu_2 |u|^{\frac{4}{d-2}} u](s)\, ds.$$

Definiton 1.12 (Scattering). A global solution u of (DCNLS) is said to be forward-in-time scattering if there exists some $\phi_+ \in H^1(\mathbb{R}^d)$ such that

$$\lim_{t \to \infty} \|u(t) - e^{it\Delta}\phi_+\|_{H^1} = 0.$$

A backward-in-time scattering solution is similarly defined, and u is then called a scattering solution when it is both forward- and backward-in-time scattering.

We define the Fourier transformation of a function f by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x}\, dx.$$

For $s \in \mathbb{R}$, the multipliers $|\nabla|^s$ and $\langle \nabla \rangle^s$ are defined by the symbols

$$\begin{aligned} |\nabla|^s f(x) &= \mathcal{F}^{-1}\big(|\xi|^s \hat{f}(\xi)\big)(x), \\ \langle \nabla \rangle^s f(x) &= \mathcal{F}^{-1}\big((1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi)\big)(x). \end{aligned}$$

Let $\psi \in C_c^\infty(\mathbb{R}^2)$ be a fixed radial, nonnegative and radially decreasing function such that $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ for $|x| \ge \frac{11}{10}$. Then for $N > 0$, we define the Littlewood–Paley projectors by

$$\begin{aligned} P_{\le N} f(x) &= \mathcal{F}^{-1}\Big(\psi\Big(\frac{\xi}{N}\Big) \hat{f}(\xi)\Big)(x), \\ P_N f(x) &= \mathcal{F}^{-1}\Big(\Big(\psi\Big(\frac{\xi}{N}\Big) - \psi\Big(\frac{2\xi}{N}\Big)\Big) \hat{f}(\xi)\Big)(x), \\ P_{> N} f(x) &= \mathcal{F}^{-1}\Big(\Big(1 - \psi\Big(\frac{\xi}{N}\Big)\Big) \hat{f}(\xi)\Big)(x). \end{aligned}$$

We recall the following well-known Bernstein inequalities which will be frequently used throughout the paper: for all $s \ge 0$ and $1 \le p \le \infty$ we have

$$\begin{aligned} \|P_{> N} f\|_p &\lesssim N^{-s} \| |\nabla|^s P_{> N} f\|_p, \\ \| |\nabla|^s P_{\le N} f\|_p &\lesssim N^s \|P_{\le N} f\|_p. \end{aligned}$$

We also record the following useful elementary inequality, which can be proved by inductive applications of inequalities (1.7)–(1.9) in [44] over k: for $s \in \{0, 1\}$ and $u_1, \ldots, u_k : \mathbb{R}^d \to \mathbb{C}$ we have

$$\Big| |\nabla|^s \Big(\Big|\sum_{j=1}^k u_j\Big|^\alpha \Big(\sum_{j=1}^k u_j\Big) - \sum_{j=1}^k |u_j|^\alpha u_j \Big) \Big| \lesssim_{k,\alpha} \begin{cases} \sum_{j \neq j'} \big||\nabla|^s u_j\big| |u_{j'}|^\alpha & \text{if } 0 < \alpha \leq 1, \\ \sum_{j \neq j'} \big||\nabla|^s u_j\big| |u_{j'}| (|u_j| + |u_{j'}|)^{\alpha-1} & \text{if } \alpha > 1. \end{cases} \tag{1.21}$$

We end this section with the following useful local smoothing result:

Lemma 1.13 ([30]). *Given $\phi \in \dot{H}^1(\mathbb{R}^d)$ we have*

$$\|\nabla e^{it\Delta}\phi\|^3_{L^2_{t,x}([-T,T]\times\{|x|\leq R\})} \lesssim T^{\frac{2}{d+2}} R^{\frac{3d+2}{d+2}} \|e^{it\Delta}\phi\|_{W_{2^*}(\mathbb{R})} \|\nabla\phi\|_2^2. \tag{1.22}$$

2. Small data and stability theories

We record in this section the small data and stability theories for (DCNLS). The proof of the small data theory is standard; see for instance [11, 31]. We will therefore omit the details of the proof here.

Lemma 2.1 (Small data theory). *For any $A > 0$ there exists some $\beta > 0$ such that the following is true: Suppose that $t_0 \in I$ for some interval I. Suppose also that $u_0 \in H^1(\mathbb{R}^d)$ with*

$$\|u_0\|_{H^1} \leq A, \tag{2.1}$$

$$\|e^{i(t-t_0)\Delta} u_0\|_{W_{2_*} \cap W_{2^*}(I)} \leq \beta. \tag{2.2}$$

Then (DCNLS) has a unique solution $u \in C(I; H^1(\mathbb{R}^d))$ with $u(t_0) = u_0$ such that

$$\|\langle\nabla\rangle u\|_{S(I)} \lesssim \|u_0\|_{H^1}, \tag{2.3}$$

$$\|u\|_{W_{2_*} \cap W_{2^*}(I)} \leq 2\|e^{i(t-t_0)\Delta} u_0\|_{W_{2_*} \cap W_{2^*}(I)}. \tag{2.4}$$

By the uniqueness of the solution u we can extend I to some maximal open interval $I_{\max} = (T_{\min}, T_{\max})$. We have the following blow-up criterion: if $T_{\max} < \infty$, then

$$\|u\|_{W_{2_*} \cap W_{2^*}([T, T_{\max}))} = \infty$$

for any $T \in I_{\max}$. A similar result holds for $T_{\min} > -\infty$. Moreover, if

$$\|u\|_{W_{2_*} \cap W_{2^*}(I_{\max})} < \infty,$$

then $I_{\max} = \mathbb{R}$ and u scatters in time.

Remark 2.2. Using the Strichartz and Sobolev inequalities we infer that

$$\|e^{i(t-t_0)\Delta}u_0\|_{W_{2*}\cap W_{2^*}(I)} \lesssim \|u_0\|_{H^1}.$$

Thus Lemma 2.1 is applicable for all u_0 with sufficiently small H^1-norm. △

We will also need the following persistence of regularity result for (DCNLS).

Lemma 2.3 (Persistence of regularity for (DCNLS)). *Let u be a solution of (DCNLS) on some interval I with $t_0 \in I$ and $\|u\|_{W_{2*}\cap W_{2^*}(I)} < \infty$. Then*

$$\| \, |\nabla|^s u\|_{S(I)} \lesssim_{\|u\|_{W_{2*}\cap W_{2^*}(I)}} \| \, |\nabla|^s u(t_0)\|_2. \tag{2.5}$$

Proof. We divide I into m subintervals $I_1, I_2, \ldots, I_m$ with $I_j = [t_{j-1}, t_j]$ such that

$$\|u\|_{W_{2*}\cap W_{2^*}(I_j)} \leq \eta \ll 1$$

for some small η which is to be determined later. Then by Hölder and Strichartz we have

$$\| \, |\nabla|^s u\|_{S(I_j)} \lesssim \| \, |\nabla|^s u(t_j)\|_2 + (\eta^{\frac{4}{d}} + \eta^{\frac{4}{d-2}})\| \, |\nabla|^s u\|_{S(I_j)}.$$

Let $j = 1$. Choosing η sufficiently small (where the smallness depends only on the Strichartz constants and is uniform for all subintervals I_j) we have

$$\| \, |\nabla|^s u\|_{S(I_1)} \lesssim_{\|u\|_{W_{2*}\cap W_{2^*}(I)}} \| \, |\nabla|^s u(t_0)\|_2.$$

In particular,

$$\| \, |\nabla|^s u(t_1)\|_2 \lesssim_{\|u\|_{W_{2*}\cap W_{2^*}(I)}} \| \, |\nabla|^s u(t_0)\|_2.$$

Arguing inductively for all $j = 2, \ldots, m-1$ and summing the estimates on all subintervals yields the desired claim. ■

In the following we prove a suitable stability theory for (DCNLS). A similar stability result appeared first in [16] for the case $d \in \{3, 4\}$. For $d \geq 5$, we encounter the new difficulty that the gradient of the mass-critical nonlinearity is no longer Lipschitz. By appealing to fractional calculus the author was able to solve this issue and showed that the stability result from [16] continues to hold for all $d \geq 5$. We refer to [35] for details. In this paper we prove a stronger version of the stability result from [16, 35] under the enhanced condition (2.9).

Lemma 2.4 (Stability theory). *Let $d \geq 3$ and let $u \in C(I; H^1(\mathbb{R}^d))$ be a solution of (DCNLS) defined on some interval $I \ni t_0$. Also assume that $w \in C(I; H^1(\mathbb{R}^d))$ is an approximate solution of the perturbed NLS*

$$i\partial_t w + \Delta w + \mu_1 |w|^{\frac{4}{d}} w + \mu_2 |w|^{\frac{4}{d-2}} w + e = 0 \tag{2.6}$$

such that

$$\|u\|_{L_t^\infty H_x^1(I)} \leq B_1, \tag{2.7}$$

$$\|w\|_{W_{2*}\cap W_{2^*}(I)} \leq B_2 \tag{2.8}$$

for some $B_1, B_2 > 0$. *Then there exists some positive* $\beta_0 = \beta_0(B_1, B_2) \ll 1$ *with the following property: if*

$$\|u(t_0) - w(t_0)\|_{H^1} \le \beta, \tag{2.9}$$
$$\|\langle\nabla\rangle e\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I)} \le \beta \tag{2.10}$$

for some $0 < \beta < \beta_0$, *then*

$$\|\langle\nabla\rangle(u - w)\|_{S(I)} \lesssim_{B_1,B_2} \beta^{\kappa} \tag{2.11}$$

for some $\kappa \in (0, 1)$.

Proof. From the results given in [16,35] we already know that

$$\|u - w\|_{W_{2*}\cap W_{2^*}(I)} \lesssim_{B_1,B_2} \beta^{\kappa},$$
$$\|\langle\nabla\rangle u\|_{S(I)} + \|\langle\nabla\rangle w\|_{S(I)} \lesssim_{B_1,B_2} 1$$

for some $\kappa \in (0, 1)$. We divide I into $O(\frac{C(B_1,B_2)}{\delta})$ intervals $I_1, \ldots, I_m$ such that

$$\|u\|_{W_{2*}\cap W_{2^*}(I_j)} + \|w\|_{W_{2*}\cap W_{2^*}(I_j)} \le \delta$$

for all $j = 1, \ldots, m$, where $\delta > 0$ is some small number to be determined later. Let $I_1 = [t_0, t_1]$. Using Hölder and (1.21) we infer that

$$\begin{aligned}
&\| |\nabla|^s(|u|^{\frac{4}{d}}u - |w|^{\frac{4}{d}}w)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I_1)} \\
&\lesssim \begin{cases}
\|u - w\|_{W_{2*}(I_1)}(\|u\|^{\frac{4-d}{d}}_{W_{2*}(I_1)} + \|w\|^{\frac{4-d}{d}}_{W_{2*}(I_1)})\| |\nabla|^s w\|_{W_{2*}(I_1)} & \\
\quad + (\|v\|^{\frac{4}{d}}_{W_{2*}(I_1)} + \|w\|^{\frac{4}{d}}_{W_{2*}(I_1)})\| |\nabla|^s(u - w)\|_{W_{2*}(I_1)} & \text{if } d = 3, \\
(\|u\|^{\frac{4}{d}}_{W_{2*}(I_1)} + \|w\|^{\frac{4}{d}}_{W_{2*}(I_1)})\| |\nabla|^s(u - w)\|_{W_{2*}(I_1)} & \\
\quad + \|u - w\|^{\frac{4}{d}}_{W_{2*}(I_1)}(\| |\nabla|^s u\|_{W_{2*}(I_1)} + \| |\nabla|^s w\|_{W_{2*}(I_1)}) & \text{if } d \ge 4,
\end{cases}
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&\| |\nabla|^s(|u|^{\frac{4}{d-2}}u - |w|^{\frac{4}{d-2}}w)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I_1)} \\
&\lesssim \begin{cases}
\|u - w\|_{W_{2^*}(I_1)}(\|u\|^{\frac{6-d}{d-2}}_{W_{2^*}(I_1)} + \|w\|^{\frac{6-d}{d-2}}_{W_{2^*}(I_1)})\| |\nabla|^s w\|_{W_{2*}(I_1)} & \\
\quad + (\|u\|^{\frac{4}{d-2}}_{W_{2^*}(I_1)} + \|w\|^{\frac{4}{d-2}}_{W_{2^*}(I_1)})\| |\nabla|^s(u - w)\|_{W_{2*}(I_1)} & \text{if } d \le 5, \\
(\|u\|^{\frac{4}{d-2}}_{W_{2^*}(I_1)} + \|w\|^{\frac{4}{d-2}}_{W_{2^*}(I_1)})\| |\nabla|^s(u - w)\|_{W_{2*}(I_1)} & \\
\quad + \|u - w\|^{\frac{4}{d-2}}_{W_{2^*}(I_1)}(\| |\nabla|^s u\|_{W_{2*}(I_1)} + \| |\nabla|^s w\|_{W_{2*}(I_1)}) & \text{if } d \ge 6
\end{cases}
\end{aligned} \tag{2.13}$$

for $s \in \{0,1\}$. By Strichartz we also see that

$$\begin{aligned} &\| |\nabla|^s (u-w)\|_{S(I_1)} \\ &\quad\lesssim \| |\nabla|^s (u(t_0)-w(t_0))\|_{L^2} + \| |\nabla|^s (|u|^{\frac{4}{d}}u - |w|^{\frac{4}{d}}w)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I_1)} \\ &\qquad+ \| |\nabla|^s (|u|^{\frac{4}{d-2}}u - |w|^{\frac{4}{d-2}}w)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I_1)} + \| |\nabla|^s e\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I_1)}. \end{aligned} \tag{2.14}$$

Now we absorb the terms with $\| |\nabla|^s (u-w)\|_{W_{2*}(I_1)}$ on the r.h.s. of (2.14) to the l.h.s. (which is possible by choosing δ sufficiently small) to deduce that

$$\| |\nabla|^s (u-w)\|_{S(I_1)} \lesssim \beta^\kappa$$

for some (possibly smaller) $\kappa \in (0,1)$. In particular, we have

$$\|u(t_1)-w(t_1)\|_{H^1} \lesssim \beta^\kappa.$$

Therefore, we can proceed with the previous arguments for all $I_2,\dots,I_m$ to conclude that

$$\| |\nabla|^s (u-w)\|_{S(I_j)} \lesssim \beta^\kappa$$

for all $j=1,\dots,m$. The claim follows by summing the estimates on each subinterval. ■

3. Double track profile decomposition

In this section we construct the double track profile decomposition for a bounded sequence in $H^1(\mathbb{R}^d)$. We begin with the following inverse Strichartz inequality along the $\dot{H}^1$-track, which was originally proved in [29] in the case $d=3$ and can be extended to arbitrary dimension $d \geq 3$ straightforwardly by combining the results from [32].

Lemma 3.1 (Inverse Strichartz inequality, $\dot{H}^1$-track, [29]). *Let $d \geq 3$ and $(f_n)_n \subset H^1(\mathbb{R}^d)$. Suppose that*

$$\lim_{n\to\infty} \|f_n\|_{H^1} = A < \infty \quad \textit{and} \quad \lim_{n\to\infty} \|e^{it\Delta} f_n\|_{W_{2^*}(\mathbb{R})} = \varepsilon > 0. \tag{3.1}$$

Then up to a subsequence, there exist $\phi \in \dot{H}^1(\mathbb{R}^d)$ and $(t_n, x_n, \lambda_n)_n \subset \mathbb{R}\times\mathbb{R}^d\times(0,\infty)$ such that $\lambda_n \to \lambda_\infty \in [0,\infty)$, and if $\lambda_\infty > 0$, then $\phi \in H^1(\mathbb{R}^d)$. Moreover,

$$\lambda_n^{\frac{d}{2}-1}(e^{it_n\Delta} f_n)(\lambda_n x + x_n) \rightharpoonup \phi(x) \textit{ weakly in } \begin{cases} H^1(\mathbb{R}^d) & \textit{if } \lambda_\infty > 0, \\ \dot{H}^1(\mathbb{R}^d) & \textit{if } \lambda_\infty = 0. \end{cases} \tag{3.2}$$

Setting

$$\phi_n := \begin{cases} \lambda_n^{-\frac{d}{2}-1} e^{-it_n\Delta}\Big[\phi\Big(\frac{x-x_n}{\lambda_n}\Big)\Big] & \textit{if } \lambda_\infty > 0, \\ \lambda_n^{-\frac{d}{2}-1} e^{-it_n\Delta}\Big[(P_{>\lambda_n^\theta}\phi)\Big(\frac{x-x_n}{\lambda_n}\Big)\Big] & \textit{if } \lambda_\infty = 0 \end{cases} \tag{3.3}$$

for some fixed $\theta \in (0,1)$, *we have*

$$\lim_{n\to\infty}(\|f_n\|_{\dot H^1}^2 - \|f_n-\phi_n\|_{\dot H^1}^2) = \|\phi\|_{\dot H^1}^2 \gtrsim A^2\Big(\frac{\varepsilon}{A}\Big)^{\frac{d(d+2)}{4}}, \tag{3.4}$$

$$\lim_{n\to\infty}(\|f_n\|_{\dot H^1}^2 - \|f_n-\phi_n\|_{\dot H^1}^2 - \|\phi_n\|_{\dot H^1}^2) = 0, \tag{3.5}$$

$$\lim_{n\to\infty}(\|f_n\|_2^2 - \|f_n-\phi_n\|_2^2 - \|\phi_n\|_2^2) = 0. \tag{3.6}$$

Furthermore,

$$\lambda_n \equiv 1 \quad \textit{or} \quad \lambda_n \to 0, \tag{3.7}$$

$$t_n \equiv 0 \quad \textit{or} \quad \frac{t_n}{\lambda_n^2} \to \pm\infty \tag{3.8}$$

and

$$\|f_n\|_{2_*}^{2_*} = \|\phi_n\|_{2_*}^{2_*} + \|f_n-\phi_n\|_{2_*}^{2_*} + o_n(1), \tag{3.9}$$

$$\|f_n\|_{2^*}^{2^*} = \|\phi_n\|_{2^*}^{2^*} + \|f_n-\phi_n\|_{2^*}^{2^*} + o_n(1). \tag{3.10}$$

Next we establish the inverse Strichartz inequality along the L^2-track by using the arguments from the proof of Lemma 3.1 and from [15, 31]. For each $j \in \mathbb{Z}$, define $\mathcal{C}_j$ by

$$\mathcal{C}_j := \{\textstyle\prod_{i=1}^d [2^j k_i, 2^j(k_i+1)) \subset \mathbb{R}^d : k \in \mathbb{Z}^d\}$$

and $\mathcal{C} := \bigcup_{j\in\mathbb{Z}} \mathcal{C}_j$. Given $Q \in \mathcal{C}$ we define f_Q by $\hat f_Q := \chi_Q \hat f$, where χ_Q is the characteristic function of the cube Q. We have the following improved Strichartz estimate:

Lemma 3.2 (Improved Strichartz estimate, [31]). *Let* $d \geq 1$ *and* $q := \frac{2(d^2+3d+1)}{d^2}$. *Then*

$$\|e^{it\Delta} f\|_{W_{2_*}(\mathbb{R})} \lesssim \|f\|_2^{\frac{d+1}{d+2}} \Big(\sup_{Q\in\mathcal{C}} |Q|^{\frac{d+2}{dq}-\frac12} \|e^{it\Delta} f_Q\|_{L^q_{t,x}(\mathbb{R})}\Big)^{\frac{1}{d+2}}. \tag{3.11}$$

Utilizing Lemma 3.2 we give the following inverse Strichartz inequality along the L^2-track.

Lemma 3.3 (Inverse Strichartz inequality, L^2-track). *Let* $d \geq 3$ *and* $(f_n)_n \subset H^1(\mathbb{R}^d)$. *Suppose that*

$$\lim_{n\to\infty} \|f_n\|_{H^1} = A < \infty \quad \textit{and} \quad \lim_{n\to\infty} \|e^{it\Delta} f_n\|_{W_{2_*}(\mathbb{R})} = \varepsilon > 0. \tag{3.12}$$

Then up to a subsequence, there exist $\phi \in L^2(\mathbb{R}^d)$ *and* $(t_n, x_n, \xi_n, \lambda_n)_n \subset \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\times(0,\infty)$ *such that* $\limsup_{n\to\infty} |\xi_n| < \infty$ *and* $\lim_{n\to\infty}\lambda_n =: \lambda_\infty \in (0,\infty]$. *Moreover,*

$$\begin{aligned}
&\lambda_n^{\frac d2} e^{-i\xi_n\cdot(\lambda_n x + x_n)}(e^{it_n\Delta} f_n)(\lambda_n x + x_n)\\
&\quad \rightharpoonup \phi(x) \text{ weakly in } \begin{cases} H^1(\mathbb{R}^d) & \text{if } \limsup_{n\to\infty}|\lambda_n\xi_n| < \infty,\\ L^2(\mathbb{R}^d) & \text{if } |\lambda_n\xi_n| \to \infty. \end{cases}
\end{aligned} \tag{3.13}$$

Additionally, if $\limsup_{n\to\infty} |\lambda_n \xi_n| < \infty$*, then* $\xi_n \equiv 0$*. Setting*

$$\phi_n := \begin{cases} \lambda_n^{-\frac{d}{2}} e^{-it_n\Delta}\Big[\phi\Big(\frac{x-x_n}{\lambda_n}\Big)\Big] & \text{if } \lambda_\infty < \infty, \\ \lambda_n^{-\frac{d}{2}} e^{-it_n\Delta}\Big[e^{i\xi_n\cdot x}(P_{\leq\lambda_n^\theta}\phi)\Big(\frac{x-x_n}{\lambda_n}\Big)\Big] & \text{if } \lambda_\infty = \infty \end{cases} \tag{3.14}$$

for some fixed $\theta \in (0,1)$*, we have*

$$\lim_{n\to\infty}(\|f_n\|_2^2 - \|f_n - \phi_n\|_2^2) = \|\phi\|_2^2 \gtrsim A^2\Big(\frac{\varepsilon}{A}\Big)^{2(d+1)(d+2)}, \tag{3.15}$$

$$\lim_{n\to\infty}(\|f_n\|_{\dot H^1}^2 - \|f_n - \phi_n\|_{\dot H^1}^2 - \|\phi_n\|_{\dot H^1}^2) = 0, \tag{3.16}$$

$$\lim_{n\to\infty}(\|f_n\|_2^2 - \|f_n - \phi_n\|_2^2 - \|\phi_n\|_2^2) = 0. \tag{3.17}$$

Proof. For $R > 0$, denote by f^R the function such that $\mathcal{F}(f^R) = \chi_R \hat{f}$, where χ_R is the characteristic function of the ball $B_R(0)$. First we obtain

$$\sup_{n\in\mathbb{N}}\|f_n - f_n^R\|_2^2 = \sup_{n\in\mathbb{N}}\int_{|\xi|\geq R}|\hat{f}_n(\xi)|^2\,d\xi \leq R^{-2}\sup_{n\in\mathbb{N}}\|f_n\|_{\dot H^1}^2 \lesssim R^{-2}A^2 \to 0 \tag{3.18}$$

as $R\to\infty$. Combining with Strichartz, we infer that there exists some $K_1 > 0$ such that for all $R \geq K_1$ one has

$$\sup_{n\in\mathbb{N}}\|f_n^R\|_2 \lesssim A \quad \text{and} \quad \sup_{n\in\mathbb{N}}\|e^{it\Delta}f_n^R\|_{W_{2*}(\mathbb{R})} \gtrsim \varepsilon.$$

Applying Lemma 3.2 to $(f_n^R)_n$, we know that there exists $(Q_n)_n \subset \mathcal{C}$ such that

$$\varepsilon^{d+2}A^{-(d+1)} \lesssim \inf_{n\in\mathbb{N}}|Q_n|^{\frac{d+2}{dq}-\frac{1}{2}}\|e^{it\Delta}(f_n^R)_{Q_n}\|_{L^q_{t,x}(\mathbb{R})}. \tag{3.19}$$

Let λ_n^{-1} be the side length of Q_n. Also, denote by ξ_n the center of Q_n. Since $q \in (\frac{2(d+2)}{d}, \frac{2(d+2)}{d-2})$ for $d \geq 3$, Hölder and Strichartz yield

$$\sup_{n\in\mathbb{N}}\|e^{it\Delta}(f_n^R)_{Q_n}\|_{L^q_{t,x}(\mathbb{R})} \lesssim \sup_{n\in\mathbb{N}}\|f_n\|_{H^1} \lesssim A.$$

Combining with the fact that $\frac{d+2}{dq} - \frac{1}{2} < 0$, we deduce that $\sup_{n\in\mathbb{N}}|Q_n| \lesssim 1$. Since $(\mathcal{F}(f_n^R))_n$ are supported in $B_R(0)$, we may assume that $(Q_n)_n \subset B_{R'}(0)$ for some sufficiently large $R' = R'(R) > 0$. Therefore, $(\lambda_n)_n$ is bounded below and $(\xi_n)_n$ is bounded in $\mathbb{R}^d$. Hölder also gives

$$\begin{aligned} &|Q_n|^{\frac{d+2}{dq}-\frac{1}{2}}\|e^{it\Delta}(f_n^R)_{Q_n}\|_{L^q_{t,x}(\mathbb{R})} \\ &\quad\lesssim \lambda_n^{\frac{d}{2}-\frac{d+2}{q}}\|e^{it\Delta}(f_n^R)_{Q_n}\|_{W_{2*}(\mathbb{R})}^{\frac{d(d+2)}{d^2+3d+1}}\|e^{it\Delta}(f_n^R)_{Q_n}\|_{L^\infty_{t,x}(\mathbb{R})}^{\frac{d+1}{d^2+3d+1}} \\ &\quad\lesssim \lambda_n^{\frac{d}{2}-\frac{d+2}{q}}\varepsilon^{\frac{d(d+2)}{d^2+3d+1}}\|e^{it\Delta}(f_n^R)_{Q_n}\|_{L^\infty_{t,x}(\mathbb{R})}^{\frac{d+1}{d^2+3d+1}}. \end{aligned}$$

Combining with (3.19) we conclude that there exist $(t_n, x_n)_n \subset \mathbb{R} \times \mathbb{R}^d$ such that

$$\liminf_{n\to\infty} \lambda_n^{\frac{d}{2}} |[e^{it_n\Delta}(f_n^R)_{Q_n}](x_n)| \gtrsim \varepsilon^{(d+1)(d+2)} A^{-(d^2+3d+1)}. \tag{3.20}$$

Define

$$\begin{aligned} h_n(x) &:= \lambda_n^{\frac{d}{2}} e^{-i\xi_n(\lambda_n x + x_n)} (e^{it_n\Delta} f_n)(\lambda_n x + x_n), \\ h_n^R(x) &:= \lambda_n^{\frac{d}{2}} e^{-i\xi_n(\lambda_n x + x_n)} (e^{it_n\Delta} f_n^R)(\lambda_n x + x_n). \end{aligned}$$

It is easy to verify that $\|h_n\|_2 = \|f_n\|_2$. By the L^2-boundedness of $(f_n)_n$ we know that there exists some $\phi \in L^2(\mathbb{R}^d)$ such that $h_n \rightharpoonup \phi$ weakly in $L^2(\mathbb{R}^d)$. Arguing similarly, we also know that $(h_n^R)_n$ converges weakly to some $\phi^R \in L^2(\mathbb{R}^d)$. From the definition of ϕ and ϕ^R it follows that

$$\|\phi - \phi^R\|_2^2 = \lim_{n\to\infty} \langle h_n - h_n^R, \phi - \phi^R\rangle_{L^2} \le \Big(\limsup_{n\to\infty} \|h_n - h_n^R\|_2\Big) \|\phi - \phi^R\|_2.$$

Using (3.18) we then obtain

$$\phi^R \to \phi \quad \text{in } L^2(\mathbb{R}^d) \text{ as } R \to \infty. \tag{3.21}$$

Now define the function χ such that $\hat{\chi}$ is the characteristic function of the cube $[-\frac{1}{2}, \frac{1}{2})^d$. From (3.20), the weak convergence of h_n^R to ϕ^R in $L^2(\mathbb{R}^d)$ and change of variables it follows that

$$\langle \phi^R, \chi\rangle = \lim_{n\to\infty} \lambda_n^{\frac{d}{2}} |[e^{it_n\Delta}(f_n^R)_{Q_n}](x_n)| \gtrsim \varepsilon^{(d+1)(d+2)} A^{-(d^2+3d+1)}. \tag{3.22}$$

On the other hand, using Hölder we also have

$$|\langle \phi^R, \chi\rangle| \le \|\phi^R\|_2 \|\chi\|_2.$$

Thus

$$\|\phi^R\|_2^2 \ge C\varepsilon^{2(d+1)(d+2)} A^{-2(d^2+3d+1)} \tag{3.23}$$

for some $C = C(d) > 0$ which is uniform for all $R \ge K_1$. Now using (3.21) and (3.23) we finally deduce that

$$\begin{aligned} \|\phi\|_2^2 &\ge \|\phi^R\|_2^2 - \frac{C}{2} \varepsilon^{2(d+1)(d+2)} A^{-2(d^2+3d+1)} \\ &\ge \frac{C}{2} \varepsilon^{2(d+1)(d+2)} A^{-2(d^2+3d+1)} \end{aligned} \tag{3.24}$$

for sufficiently large R, which gives the lower bound of (3.15). From now on we fix R such that the lower bound of (3.15) is valid for this chosen R and let $(t_n, x_n, \xi_n, \lambda_n)_n$ be the corresponding symmetry parameters. Since $L^2(\mathbb{R}^d)$ is a Hilbert space, from the weak convergence of h_n to ϕ in $L^2(\mathbb{R}^d)$ we obtain

$$\lim_{n\to\infty} (\|h_n\|_2^2 - \|\phi\|_2^2 - \|h_n - \phi\|_2^2) = 2 \lim_{n\to\infty} \mathrm{Re}\langle \phi, h_n - \phi\rangle_{L^2} = 0.$$

Combining with the fact that

$$\|P_{\le\lambda_n^\theta}\phi-\phi\|_2\to 0\quad\text{as } n\to\infty$$

for $\lambda_n\to\infty$, we conclude the equalities of (3.15) and (3.17). Furthermore, in the case $\limsup_{n\to\infty}|\lambda_n\xi_n|<\infty$, using the boundedness of $(\lambda_n\xi_n)_n$ and chain rule, we also infer that $\|h_n\|_{H^1}\lesssim\|f_n\|_{H^1}$. By the H^1-boundedness of $(f_n)_n$ and uniqueness of weak convergence we deduce additionally that $\phi\in H^1(\mathbb{R}^d)$ and (3.13) follows.

Next we show that we may assume $\xi_n\equiv 0$ under the additional condition that $\limsup_{n\to\infty}|\lambda_n\xi_n|<\infty$. Define

$$\mathcal{T}_{a,b}u(x):=be^{ia\cdot x}u(x)$$

for $a\in\mathbb{R}^d$ and $b\in\mathbb{C}$ with $|b|=1$. Also let

$$\begin{aligned}(\lambda\xi)_\infty&:=\lim_{n\to\infty}\lambda_n\xi_n,\\ e^{i(\xi\cdot x)_\infty}&:=\lim_{n\to\infty}e^{i\xi_n\cdot x_n}.\end{aligned}$$

Notice that $e^{i(\xi\cdot x)_\infty}$ is well defined (up to a subsequence), since $(e^{i\xi_n\cdot x_n})_n$ is bounded. By the boundedness of $(\lambda_n\xi_n)_n$ we infer that $\mathcal{T}_{\lambda_n\xi_n,e^{i\xi_n\cdot x_n}}$ is an isometry on $L^2(\mathbb{R}^d)$ and converges strongly to $\mathcal{T}_{(\lambda\xi)_\infty,e^{i(\xi\cdot x)_\infty}}$ as operators on $H^1(\mathbb{R}^d)$. We may replace h_n by $\lambda_n^{\frac{d}{2}}(e^{it_n\Delta}f_n)(\lambda_n x+x_n)$ and ϕ by $\mathcal{T}_{(\lambda\xi)_\infty,e^{i(\xi\cdot x)_\infty}}\phi$ and (3.13), (3.15) and (3.16) carry over.

Finally, we prove (3.16). For the case $\lambda_\infty<\infty$ we additionally know that $\phi\in H^1(\mathbb{R}^d)$ and $\xi_n\equiv 0$. Using the fact that $\dot H^1$ is a Hilbert space and a change of variables we obtain

$$o_n(1)=\|h_n\|_{\dot H^1}-\|h_n-\phi\|_{\dot H^1}-\|\phi\|_{\dot H^1}=\lambda_n^2(\|f_n\|_{\dot H^1}-\|f_n-\phi_n\|_{\dot H^1}-\|\phi_n\|_{\dot H^1}).$$

Combining with the lower boundedness of $(\lambda_n)_n$, this implies that

$$\|f_n\|_{\dot H^1}-\|f_n-\phi_n\|_{\dot H^1}-\|\phi_n\|_{\dot H^1}=\lambda_n^{-2}o_n(1)=o_n(1),$$

which gives (3.16) in the case $\lambda_\infty<\infty$. Now assume $\lambda_\infty=\infty$. Using a change of variables and the chain rule we obtain

$$\begin{aligned}&\|f_n\|_{\dot H^1}^2-\|f_n-\phi_n\|_{\dot H^1}^2-\|\phi_n\|_{\dot H^1}^2\\ &\quad=|\xi_n|^2\big(\|h_n\|_2^2-\|h_n-P_{\le\lambda_n^\theta}\phi\|_2^2-\|P_{\le\lambda_n^\theta}\phi\|_2^2\big)\\ &\qquad+2\lambda_n^{-1}\operatorname{Re}\big(\langle i\xi_n(h_n-P_{\le\lambda_n^\theta}\phi),\nabla P_{\le\lambda_n^\theta}\phi\rangle+\langle i\xi_nP_{\le\lambda_n^\theta}\phi,\nabla(h_n-P_{\le\lambda_n^\theta}\phi)\rangle\big)\\ &\qquad+\lambda_n^{-2}\big(\|h_n\|_{\dot H^1}^2-\|h_n-P_{\le\lambda_n^\theta}\phi\|_{\dot H^1}^2-\|P_{\le\lambda_n^\theta}\phi\|_{\dot H^1}^2\big)\\ &\quad=:I_1+I_2+I_3.\end{aligned}\tag{3.25}$$

Using the boundedness of $(\xi_n)_n$ and (3.17) we already have $I_1\to 0$. For I_2, using Bernstein and the boundedness of $(\xi_n)_n$ in $\mathbb{R}^d$ and of $(h_n-P_{\le\lambda_n^\theta}\phi)$ in $L^2(\mathbb{R}^d)$ we see that

$$|I_2|\lesssim\lambda_n^{-1}\|h_n-P_{\le\lambda_n^\theta}\phi\|_2\|\nabla P_{\le\lambda_n^\theta}\phi\|_2\lesssim\lambda_n^{-(1-\theta)}\to 0.$$

Finally, I_3 can be similarly estimated using the Bernstein inequality; we omit the details here. Summing, we conclude (3.17). ∎

We show some further properties of the profile decomposition along the L^2-track.

Lemma 3.4. *In Lemma 3.3, we may always assume that*

$$\lambda_n \equiv 1 \quad or \quad \lambda_n \to \infty, \tag{3.26}$$

$$t_n \equiv 0 \quad or \quad \frac{t_n}{\lambda_n^2} \to \pm\infty. \tag{3.27}$$

Proof. If $\lambda_n \to \infty$, then there is nothing to prove. Otherwise assume that $\lambda_\infty < \infty$. By the boundedness of $(\xi_n)_n$ we also know that $\phi \in H^1(\mathbb{R}^d)$ and $(\lambda_n \xi_n)_n$ is bounded, thus $\xi_n \equiv 0$ and $h_n(x)$ reduces to $\lambda_n^{\frac{d}{2}}(e^{it_n\Delta} f_n)(\lambda_n x + x_n)$. Define

$$\mathcal{J}_\lambda f(x) := \lambda^{-\frac{d}{2}} f(\lambda^{-1}x).$$

Then $\mathcal{J}_{\lambda_n}$ and $\mathcal{J}_{\lambda_n}^{-1}$ converge strongly to $\mathcal{J}_{\lambda_\infty}$ and $\mathcal{J}_{\lambda_\infty}^{-1}$ respectively as operators in $H^1(\mathbb{R}^d)$. We may redefine $\lambda_n \equiv 1$ and replace ϕ by $\mathcal{J}_{\lambda_\infty}\phi$, and all the statements from Lemma 3.3 continue to hold.

We now prove (3.27). If $\frac{t_n}{\lambda_n^2} \to \pm\infty$, then we are done. Otherwise assume that $\frac{t_n}{\lambda_n^2} \to \tau_\infty \in \mathbb{R}$. Recall that for $(\xi_0, x_0, \lambda_0) \in \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty)$ the operator g_{ξ_0,x_0,λ_0} is defined by

$$g_{\xi_0,x_0,\lambda_0} f(x) = \lambda_0^{-\frac{d}{2}} e^{i\xi_0\cdot x} f(\lambda_0^{-1}(x - x_0)).$$

Then

$$f_n = e^{-it_n\Delta}[g_{\xi_n,x_n,\lambda_n} h_n](x)$$

and

$$\phi_n = \begin{cases} e^{-it_n\Delta}[g_{\xi_n,x_n,\lambda_n}\phi](x) & \text{if } \lambda_\infty < \infty, \\ e^{-it_n\Delta}[g_{\xi_n,x_n,\lambda_n} P_{\le\lambda_n^\theta}\phi](x) & \text{if } \lambda_\infty = \infty. \end{cases}$$

Using the invariance of the NLS flow under the Galilean transformation we infer that

$$e^{-it_n\Delta}[g_{\xi_n,x_n,\lambda_n} f](x) = g_{\xi_n,x_n-2t_n\xi_n,\lambda_n}[e^{it_n|\xi_n|^2} e^{-i\frac{t_n}{\lambda_n^2}\Delta} f](x). \tag{3.28}$$

Define $\beta := \lim_{n\to\infty} e^{it_n|\xi_n|^2}$. We can therefore redefine t_n as 0, x_n as $x_n - 2t_n\xi_n$ and ϕ as $\beta e^{-i\tau_\infty\Delta}\phi$. One easily checks that up to (3.16) in the case $\lambda_\infty = \infty$, the statements from Lemma 3.3 carry over, due to the strong continuity of the linear Schrödinger flow on $H^1(\mathbb{R}^d)$ and the fact that g is an isometry on $L^2(\mathbb{R}^d)$. To see (3.16) in the case $\lambda_\infty = \infty$, direct calculation results in

$$\begin{aligned} &\big\| g_{\xi_n,x_n-2t_n\xi_n,\lambda_n}[e^{it_n|\xi_n|^2} e^{-i\frac{t_n}{\lambda_n^2}\Delta} P_{\le\lambda_n^\theta}\phi] - g_{\xi_n,x_n-2t_n\xi_n,\lambda_n}[\beta e^{-i\tau_\infty\Delta} P_{\le\lambda_n^\theta}\phi]\big\|_{\dot H^1} \\ &\quad\lesssim |\xi_n| \, \big\| e^{it_n|\xi_n|^2} e^{-i\frac{t_n}{\lambda_n^2}\Delta} P_{\le\lambda_n^\theta}\phi - \beta e^{-i\tau_\infty\Delta} P_{\le\lambda_n^\theta}\phi\big\|_2 \\ &\qquad + \lambda_n^{-1} \big\| e^{it_n|\xi_n|^2} e^{-i\frac{t_n}{\lambda_n^2}\Delta} P_{\le\lambda_n^\theta}\phi - \beta e^{-i\tau_\infty\Delta} P_{\le\lambda_n^\theta}\phi\big\|_{\dot H^1} =: I_1 + I_2. \end{aligned} \tag{3.29}$$

By the boundedness of $(\xi_n)_n$ one easily verifies that $I_1 \to 0$. Using Bernstein we see that

$$|I_2| \lesssim \lambda_n^{-1} \|P_{\leq \lambda_n^\theta} \phi\|_2 \lesssim \lambda_n^{-(1-\theta)} \|\phi\|_2 \to 0. \tag{3.30}$$

This completes the desired proof. ∎

Using (3.28), redefining the parameters and taking Lemma 3.1 into account, we w.l.o.g. assume in the following that

$$\phi_n = \begin{cases} \lambda_n g_{0,x_n,\lambda_n}[e^{it_n\Delta} P_{>\lambda_n^\theta} \phi](x) & \text{if } \lambda_\infty = 0, \\ e^{it_n\Delta}\phi(x - x_n) & \text{if } \lambda_\infty = 1, \\ g_{\xi_n,x_n,\lambda_n}[e^{it_n\Delta} P_{\leq \lambda_n^\theta}\phi](x) & \text{if } \lambda_\infty = \infty. \end{cases}$$

Lemma 3.5. *Let $(f_n)_n$ and $(\phi_n)_n$ be the sequences from Lemma 3.3. Then*

$$\|f_n\|_{2_*}^{2_*} = \|\phi_n\|_{2_*}^{2_*} + \|f_n - \phi_n\|_{2_*}^{2_*} + o_n(1), \tag{3.31}$$

$$\|f_n\|_{2^*}^{2^*} = \|\phi_n\|_{2^*}^{2^*} + \|f_n - \phi_n\|_{2^*}^{2^*} + o_n(1). \tag{3.32}$$

Proof. Assume first that $\lambda_\infty = \infty$. Using Bernstein and Sobolev we infer that

$$\|\phi_n\|_{2^*} \lesssim \lambda_n^{-1} \|P_{\leq \lambda_n^\theta}\phi\|_{\dot{H}^1} \lesssim \lambda_n^{-(1-\theta)}\|\phi\|_2 \to 0.$$

Hence $\|\phi_n\|_{2^*} = o_n(1)$. Therefore, by the triangular inequality,

$$\big|\, \|f_n\|_{2^*} - \|f_n - \phi_n\|_{2^*} \big| \leq \|\phi_n\|_{2^*} \to 0$$

and (3.32) follows. Now suppose that $\lambda_\infty = 1$ and $t_n \to \pm\infty$. For $\beta > 0$ let $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\|\phi - \psi\|_{H^1} \leq \beta.$$

Define

$$\psi_n := e^{it_n\Delta}\psi(x - x_n).$$

Then by a dispersive estimate we deduce that

$$\|\psi_n\|_{2^*} \lesssim |t_n|^{-1} \|\psi\|_{(2^*)'} \to 0.$$

On the other hand, by Sobolev we have

$$\|\psi_n - \phi_n\|_{2^*} \lesssim \|\psi - \phi\|_{\dot{H}^1} \leq \beta.$$

Hence $\|\psi_n\|_{2^*} \lesssim \beta$ for all sufficiently large n. Therefore, by the triangular inequality,

$$\big|\, \|f_n\|_{2^*} - \|f_n - \psi_n\|_{2^*} \big| \lesssim \beta,$$

and (3.32) follows by taking β arbitrarily small. Now we assume $\lambda_\infty = 1$ and $t_n \equiv 0$. Then we additionally know that $\phi \in H^1(\mathbb{R}^d)$ and $h_n \rightharpoonup \phi$ in $H^1(\mathbb{R}^d)$. Using the Brezis–Lieb lemma we deduce that

$$\|h_n\|_{2^*}^{2^*} = \|\phi\|_{2^*}^{2^*} + \|h_n - \phi\|_{2^*}^{2^*} + o_n(1).$$

Undoing the transformation we obtain (3.32).

We now consider (3.31). When $\lambda_\infty = \infty$ or $\lambda_\infty = 1$ and $t_n \to \pm\infty$, then $\|\psi_n\|_{2^*} \to 0$, and by Hölder we will also have $\|\psi_n\|_{2_*} \to 0$, thus (3.31) follows. For the case $\lambda_\infty = 1$ and $t_n \equiv 0$, (3.31) follows again from the Brezis–Lieb lemma. This completes the desired proof. ∎

Before we finally establish the double track profile decomposition, we recall the operator g_{ξ_0,x_0,λ_0} defined by (1.18) and the quantities $\mathcal{H}$, $\mathcal{K}$, $\mathcal{I}$ defined by (1.14)–(1.16) which will be used to formulate the statement for the double track profile decomposition.

Having all the preliminaries we are in a position to state the double track profile decomposition.

Lemma 3.6 (Double track profile decomposition). *Let $(\psi_n)_n$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Then up to a subsequence, there exist nonzero linear profiles $(\phi^j)_j \subset \dot{H}^1(\mathbb{R}^d) \cup L^2(\mathbb{R}^d)$, remainders $(w_n^k)_{k,n} \subset H^1(\mathbb{R}^d)$, parameters $(t_n^j, x_n^j, \xi_n^j, \lambda_n^j)_{j,n} \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty)$ and $K^* \in \mathbb{N} \cup \{\infty\}$, such that we have the following statements:*

(i) *For any finite $1 \le j \le K^*$ the parameters satisfy*

$$\begin{aligned} 1 &\gtrsim_j \lim_{n\to\infty} |\xi_n^j|, \\ \lim_{n\to\infty} t_n^j &=: t_\infty^j \in \{0, \pm\infty\}, \\ \lim_{n\to\infty} \lambda_n^j &=: \lambda_\infty^j \in \{0, 1, \infty\}, \\ t_n^j &\equiv 0 \quad \text{if } t_\infty^j = 0, \\ \lambda_n^j &\equiv 1 \quad \text{if } \lambda_\infty^j = 1, \\ \xi_n^j &\equiv 0 \quad \text{if } \lambda_\infty^j \in \{0, 1\}. \end{aligned} \tag{3.33}$$

(ii) *For any finite $1 \le k \le K^*$ we have the decomposition*

$$\psi_n = \sum_{j=1}^{k} T_n^j P_n^j \phi^j + w_n^k. \tag{3.34}$$

Here, the operators T_n^j and P_n^j are defined by

$$T_n^j u(x) := \begin{cases} \lambda_n^j g_{0,x_n^j,\lambda_n^j}[e^{it_n^j\Delta}u](x) & \text{if } \lambda_\infty^j = 0, \\ [e^{it_n^j\Delta}u](x - x_n^j) & \text{if } \lambda_\infty^j = 1, \\ g_{\xi_n^j,x_n^j,\lambda_n^j}[e^{it_n^j\Delta}u](x) & \text{if } \lambda_\infty^j = \infty \end{cases} \tag{3.35}$$

and

$$P_n^j u := \begin{cases} P_{>(\lambda_n^j)^\theta} u & \text{if } \lambda_\infty^j = 0, \\ u & \text{if } \lambda_\infty^j = 1, \\ P_{\le(\lambda_n^j)^\theta} u & \text{if } \lambda_\infty^j = \infty \end{cases} \tag{3.36}$$

for some $\theta \in (0,1)$. *Moreover,*

$$\phi^j \in \begin{cases} \dot{H}^1(\mathbb{R}^d) & \text{if } \lambda_\infty^j = 0, \\ H^1(\mathbb{R}^d) & \text{if } \lambda_\infty^j = 1, \\ L^2(\mathbb{R}^d) & \text{if } \lambda_\infty^j = \infty. \end{cases} \tag{3.37}$$

(iii) *The remainders* $(w_n^k)_{k,n}$ *satisfy*

$$\lim_{k \to K^*} \lim_{n \to \infty} \|e^{it\Delta} w_n^k\|_{W_{2*} \cap W_{2^*}(\mathbb{R})} = 0. \tag{3.38}$$

(iv) *The parameters are orthogonal in the sense that*

$$\frac{\lambda_n^k}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^k} + \lambda_n^k|\xi_n^j - \xi_n^k| + \left|t_k\Big(\frac{\lambda_n^k}{\lambda_n^j}\Big)^2 - t_n^j\right| + \left|\frac{x_n^j - x_n^k - 2t_n^k(\lambda_n^k)^2(\xi_n^j - \xi_n^k)}{\lambda_n^k}\right| \to \infty \tag{3.39}$$

for any $j \neq k$.

(v) *For any finite* $1 \leq k \leq K^*$ *we have the energy decompositions*

$$\| |\nabla|^s \psi_n\|_2^2 = \sum_{j=1}^k \| |\nabla|^s T_n^j P_n^j \phi^j\|_2^2 + \| |\nabla|^s w_n^k\|_2^2 + o_n(1), \tag{3.40}$$

$$\mathcal{H}(\psi_n) = \sum_{j=1}^k \mathcal{H}(T_n^j P_n^j \phi^j) + \mathcal{H}(w_n^k) + o_n(1), \tag{3.41}$$

$$\mathcal{K}(\psi_n) = \sum_{j=1}^k \mathcal{K}(T_n^j P_n^j \phi^j) + \mathcal{K}(w_n^k) + o_n(1), \tag{3.42}$$

$$\mathcal{I}(\psi_n) = \sum_{j=1}^k \mathcal{I}(T_n^j P_n^j \phi^j) + \mathcal{I}(w_n^k) + o_n(1) \tag{3.43}$$

for $s \in \{0,1\}$ *and any finite* $1 \leq k \leq K^*$.

Proof. We construct the linear profiles iteratively and start with $k = 0$ and $w_n^0 := \psi_n$. We assume initially that the linear profile decomposition is given and its claimed properties are satisfied for some k. Define

$$\varepsilon_k := \lim_{n \to \infty} \|e^{it\Delta} w_n^k\|_{W_{2*} \cap W_{2^*}(\mathbb{R})}.$$

If $\varepsilon_k = 0$, then we stop and set $K^* = k$. Otherwise we have either

$$L^2\text{-track:} \quad \limsup_{n \to \infty} \|e^{it\Delta} w_n^k\|_{W_{2*}(\mathbb{R})} \geq \limsup_{n \to \infty} \|e^{it\Delta} w_n^k\|_{W_{2^*}(\mathbb{R})},$$

or

$$\dot{H}^1\text{-track:}\quad \limsup_{n\to\infty} \|e^{it\Delta} w_n^k\|_{W_{2_*}(\mathbb{R})} < \limsup_{n\to\infty} \|e^{it\Delta} w_n^k\|_{W_{2^*}(\mathbb{R})}. \tag{3.44}$$

For the first situation we apply Lemma 3.3 to w_n^k, while in the latter case we apply Lemma 3.1. In both cases we obtain the sequence

$$(\phi^{k+1}, w_n^{k+1}, t_n^{k+1}, x_n^{k+1}, \xi_n^{k+1}, \lambda_n^{k+1})_n.$$

We still need to check that (iii) and (iv) are satisfied for $k+1$. That the other items are also satisfied for $k+1$ follows directly from the construction of the linear profile decomposition. If $\varepsilon_k = 0$, then (iii) is automatic; otherwise we have $K^* = \infty$ and $\varepsilon_j > 0$ for all $j \in \mathbb{N} \cup \{0\}$. Let $S_1 \subset \mathbb{N}$ denote the set of indices such that for each $j \in S_1$, we apply the $\dot{H}^1$-profile decomposition at the $(j-1)$-step. Also define $S_2 := \mathbb{N} \setminus S_1$. Using (3.4), (3.15) and (3.40) we obtain

$$\begin{aligned}
&\sum_{j\in S_1} A_{j-1}^2 \Big(\frac{\varepsilon_{j-1}}{A_{j-1}}\Big)^{\frac{d(d+2)}{4}} + \sum_{j\in S_2} A_{j-1}^2 \Big(\frac{\varepsilon_{j-1}}{A_{j-1}}\Big)^{2(d+1)(d+2)} \\
&\quad\lesssim \sum_{j\in S_1} \|\phi^j\|_{\dot{H}^1}^2 + \sum_{j\in S_2} \|\phi^j\|_2^2 \\
&\quad= \sum_{j\in S_1} \lim_{n\to\infty} \|T_j P_n^j \phi^j\|_{\dot{H}^1}^2 + \sum_{j\in S_2} \lim_{n\to\infty} \|T_j P_n^j \phi^j\|_2^2 \\
&\quad\le \lim_{n\to\infty} \|\psi_n\|_{H^1}^2 = A_0^2,
\end{aligned} \tag{3.45}$$

where $A_j := \lim_{n\to\infty} \|w_n^j\|_{H^1}$. By (3.40) we know that $(A_j)_j$ is monotone decreasing, thus also bounded. Since $S_1 \cup S_2 = \mathbb{N}$, at least one of both is an infinite set. Suppose that $|S_1| = \infty$ and $|S_2| < \infty$. Then

$$\lim_{j\to\infty} A_j^2 \Big(\frac{\varepsilon_j}{A_j}\Big)^{\frac{d(d+2)}{4}} = 0.$$

Combining with the boundedness of $(A_j)_j$ we immediately conclude that $\varepsilon_j \to 0$. The same also holds for the cases $|S_2| = \infty$, $|S_1| < \infty$ and $|S_1| = |S_2| = \infty$, and the proof of (iii) is complete. Finally, we show (iv). Denote

$$g_n^j := \begin{cases} \lambda_n^j g_{0,x_n^j,\lambda_n^j} & \text{if } \lambda_\infty^j = 0, \\ g_{\xi_n^j,x_n^j,\lambda_n^j} & \text{if } \lambda_\infty^j \in \{1,\infty\}. \end{cases}$$

Assume that (iv) does not hold for some $j < k$. By construction of the profile decomposition we have

$$w_n^{k-1} = w_n^j - \sum_{l=j+1}^{k-1} g_n^l e^{-it_n^l\Delta} P_n^l \phi^l.$$

Then using the definition of ϕ^k we know that

$$\begin{aligned}\phi^k &= \operatorname*{w-lim}_{n\to\infty} e^{-it_n^k\Delta}[(g_n^k)^{-1}w_n^{k-1}]\\ &= \operatorname*{w-lim}_{n\to\infty} e^{-it_n^k\Delta}[(g_n^j)^{-1}w_n^j] - \sum_{l=j+1}^{k-1} \operatorname*{w-lim}_{n\to\infty} e^{-it_n^k\Delta}[(g_n^k)^{-1}P_n^l\phi^l],\end{aligned} \tag{3.46}$$

where the weak limits are taken in the $\dot{H}^1$- or L^2-topology, depending on the bifurcation (3.44). Our aim is to show that ϕ^k is zero, which leads to a contradiction and proves (iv). We first consider the case $\lambda_\infty^k = \infty$. Then the weak limit is taken w.r.t. the L^2-topology. Particularly, we must have $\lambda_\infty^j = \infty$, otherwise (iv) would be satisfied. For the first summand, we obtain

$$e^{-it_n^k\Delta}[(g_n^k)^{-1}w_n^j] = (e^{-it_n^k\Delta}(g_n^k)^{-1}g_n^j e^{it_n^j\Delta})[e^{-it_n^j\Delta}(g_n^j)^{-1}w_n^j].$$

Direct calculation yields

$$\begin{aligned}&e^{-it_n^k\Delta}(g_n^k)^{-1}g_n^j e^{it_n^j\Delta}\\ &\qquad= \beta_n^{j,k}\, g_{\lambda_n^k(\xi_n^j-\xi_n^k),\,\frac{x_n^j-x_n^k-2t_n^k(\lambda_n^k)^2(\xi_n^j-\xi_n^k)}{\lambda_n^k},\,\frac{\lambda_n^j}{\lambda_n^k}}\, e^{-i(t_n^k(\frac{\lambda_n^k}{\lambda_n^j})^2-t_n^j)\Delta}\end{aligned} \tag{3.47}$$

with $\beta_n^{j,k} = e^{i(\xi_n^j-\xi_n^k)x_n^k+t_n^k(\lambda_n^k)^2|\xi_n^j-\xi_n^k|^2}$. Therefore, the failure of (iv) results in the strong convergence of the adjoint of $e^{-it_n^k\Delta}(g_n^k)^{-1}g_n^j e^{it_n^j\Delta}$ in $L^2(\mathbb{R}^d)$. By construction of the profile decomposition we have

$$e^{-it_n^j\Delta}(g_n^j)^{-1}w_n^j \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}^d),$$

and we conclude that the first summand weakly converges to zero in $L^2(\mathbb{R}^d)$. Now we treat the single terms in the second summand. We can rewrite each single summand as

$$e^{-it_n^k\Delta}[(g_n^k)^{-1}P_n^l\phi^l] = (e^{-it_n^k\Delta}(g_n^k)^{-1}g_n^j e^{it_n^j\Delta})[e^{-it_n^j\Delta}(g_n^j)^{-1}P_n^l\phi^l].$$

By the previous arguments it suffices to show that

$$e^{-it_n^j\Delta}(g_n^j)^{-1}P_n^l\phi^l \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}^d).$$

Assume first $\lambda_\infty^l = 0$. In this case, we can in fact show that

$$e^{-it_n^j\Delta}(g_n^j)^{-1}P_n^l\phi^l \to 0 \quad \text{in } L^2(\mathbb{R}^d). \tag{3.48}$$

Indeed, using Bernstein we have

$$\|e^{-it_n^j\Delta}(g_n^j)^{-1}P_n^l\phi^l\|_2 = \lambda_n^l\|P_{>(\lambda_n^l)^\theta}\phi^l\|_2 \lesssim (\lambda_n^l)^{1-\theta}\|\phi^l\|_{\dot{H}^1} \to 0.$$

Next we consider the cases $\lambda^l_\infty \in \{1,\infty\}$. By the construction of the decomposition and the inductive hypothesis we know that $\phi^l \in L^2(\mathbb{R}^d)$ and (iv) is satisfied for the pair (j,l). Using the fact that

$$\|P_{\le (\lambda^l_n)^\theta}\phi^l - \phi^l\|_2 \to 0 \quad \text{when } \lambda^l_n \to \infty$$

and density arguments, it suffices to show that

$$I_n := e^{-it^j_n\Delta}(g^j_n)^{-1} g^l_n e^{it^l_n\Delta}\phi \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}^d)$$

for arbitrary $\phi \in C^\infty_c(\mathbb{R}^d)$. By (3.47) we obtain

$$I_n = \beta^{j,l}_n g_{\lambda^l_n(\xi^j_n-\xi^l_n),\frac{x^j_n-x^l_n-2t^l_n(\lambda^l_n)^2(\xi^j_n-\xi^l_n)}{\lambda^l_n},\frac{\lambda^j_n}{\lambda^l_n}} e^{-i(t^l_n(\frac{\lambda^l_n}{\lambda^j_n})^2-t^j_n)\Delta}\phi.$$

Assume first that $\lim_{n\to\infty}\frac{\lambda^j_n}{\lambda^l_n}+\frac{\lambda^l_n}{\lambda^j_n}=\infty$. Then for any $\psi \in C^\infty_c(\mathbb{R}^d)$ we have

$$|\langle I_n,\psi\rangle| \le \min\Big\{\Big(\frac{\lambda^l_n}{\lambda^j_n}\Big)^{\frac{d}{2}}\|\hat\phi\|_1\|\hat\psi\|_\infty, \Big(\frac{\lambda^j_n}{\lambda^l_n}\Big)^{\frac{d}{2}}\|\hat\psi\|_1\|\hat\phi\|_\infty\Big\} \to 0.$$

So we may assume that $\lim_{n\to\infty}\frac{\lambda^j_n}{\lambda^l_n}\in(0,\infty)$. Suppose now $t^l_n(\frac{\lambda^l_n}{\lambda^j_n})^2-t^j_n\to\pm\infty$. Then the weak convergence of I_n to zero in $L^2(\mathbb{R}^d)$ follows immediately from the dispersive estimate. Hence we may also assume that $\lim_{n\to\infty}t^l_n(\frac{\lambda^l_n}{\lambda^j_n})^2-t^j_n\in\mathbb{R}$. Finally, we are left with the options

$$|\lambda^l_n(\xi^j_n-\xi^l_n)|\to\infty \quad \text{or} \quad \Big|\frac{x^j_n-x^l_n-2t^l_n(\lambda^l_n)^2(\xi^j_n-\xi^l_n)}{\lambda^l_n}\Big|\to\infty.$$

For the latter case, we utilize the fact that the symmetry group composed by unbounded translations weakly converges to zero as operators in $L^2(\mathbb{R}^d)$ to deduce the claim. For the former case, we can use the same arguments as for the translation symmetry by considering the Fourier transformation of I_n in the frequency space. This completes the desired proof for the case $\lambda^k_n=\infty$.

It is still left to show the claim for the cases $\lambda^k_\infty\in\{0,1\}$. We only need to prove that for $\lambda^l_\infty=\infty$, we must have

$$e^{-it^j_n\Delta}(g^j_n)^{-1}g^l_n e^{it^l_n\Delta}P_{\le(\lambda^l_n)^\theta}\phi^l \to 0 \quad \text{in } \dot H^1(\mathbb{R}^d); \tag{3.49}$$

the other cases can be dealt similarly. Notice in this case that $e^{-it^j_n\Delta}(g^j_n)^{-1}$ is an isometry on $\dot H^1$. Using Bernstein, the boundedness of $(\xi^l_n)_n$ and chain rule we obtain

$$\begin{aligned}
&\|e^{-it^j_n\Delta}(g^j_n)^{-1}g^l_n e^{it^l_n\Delta}P_{\le(\lambda^l_n)^\theta}\phi^l\|_{\dot H^1}\\
&\lesssim (\lambda^l_n)^{-1}|\xi^l_n|\,\|P_{\le(\lambda^l_n)^\theta}\phi^l\|_2+(\lambda^l_n)^{-1}\|P_{\le(\lambda^l_n)^\theta}\phi^l\|_{\dot H^1}\\
&\lesssim (\lambda^l_n)^{-1}\|\phi^l\|_2+(\lambda^l_n)^{-(1-\theta)}\|\phi^l\|_2\to 0.
\end{aligned}$$

This finally completes the proof of (iv). ∎

4. Scattering threshold for the focusing–focusing (DCNLS)

Throughout this section we restrict ourselves to the focusing–focusing (DCNLS)

$$i\partial_t u + \Delta u + |u|^{2_*-2}u + |u|^{2^*-2}u = 0. \tag{4.1}$$

We also define the set $\mathcal{A}$ by

$$\mathcal{A} := \{u \in H^1(\mathbb{R}^d) : \mathcal{M}(u) < \mathcal{M}(Q),\ \mathcal{H}(u) < m_{\mathcal{M}(u)},\ \mathcal{K}(u) > 0\}.$$

4.1. Variational estimates and the MEI functional

We derive below some useful variational estimates which will be later used in Sections 4.3 and 4.4. Particularly, we give the precise construction of the MEI functional $\mathcal{D}$, which will help us to set up the inductive hypothesis given in Section 4.3.

Lemma 4.1. *Let $u \in H^1(\mathbb{R}^d) \setminus \{0\}$ with $\mathcal{M}(u) < \mathcal{M}(Q)$. Then there exists a unique $\lambda(u) > 0$ such that*

$$\mathcal{K}(T_\lambda u) \begin{cases} > 0 & \text{if } \lambda \in (0, \lambda(u)), \\ = 0 & \text{if } \lambda = \lambda(u), \\ < 0 & \text{if } \lambda \in (\lambda(u), \infty), \end{cases}$$

where the operator T_λ is defined by (1.17).

Proof. We first obtain

$$\mathcal{K}(T_\lambda u) = \lambda^2\Big(\|\nabla u\|_2^2 - \frac{d}{d+2}\|u\|_{2_*}^{2_*}\Big) - \lambda^{2^*}\|u\|_{2^*}^{2^*},$$
$$\frac{d}{d\lambda}\mathcal{K}(T_\lambda u) = 2\lambda\Big(\|\nabla u\|_2^2 - \frac{d}{d+2}\|u\|_{2_*}^{2_*}\Big) - 2^*\lambda^{2^*-1}\|u\|_{2^*}^{2^*},$$

with

$$\|\nabla u\|_2^2 - \frac{d}{d+2}\|u\|_{2_*}^{2_*} \geq \Big(1 - \Big(\frac{\mathcal{M}(u)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla u\|_2^2 > 0. \tag{4.2}$$

Since $2^* > 2$, $\frac{d}{d\lambda}\mathcal{K}(T_\lambda u)$ has a unique zero $\beta(u) \in (0,\infty)$ which is the global maximum of $\mathcal{K}(T_\lambda u)$. Also, $\mathcal{K}(T_\lambda u)$ is increasing on $(0,\beta(u))$ and decreasing on $(\beta(u),\infty)$. One easily verifies that $\mathcal{K}(T_\lambda u)$ is positive on $(0,\beta(u))$ and $\mathcal{K}(T_\lambda u) \to -\infty$ as $\lambda \to \infty$. Consequently, $\mathcal{K}(T_\lambda u)$ has a first and unique zero $\lambda(u) \in (\beta(u),\infty)$ and $\mathcal{K}(T_\lambda u)$ is positive on $(0,\lambda(u))$ and negative on $(\lambda(u),\infty)$. This completes the proof. ∎

Lemma 4.2. *Assume that $\mathcal{K}(u) \geq 0$. Then $\mathcal{H}(u) \geq 0$. If additionally $\mathcal{K}(u) > 0$, then also $\mathcal{H}(u) > 0$.*

Proof. We have

$$\mathcal{H}(u) \geq \mathcal{H}(u) - \frac{1}{2}\mathcal{K}(u) = \frac{1}{d}\|u\|_{2^*}^{2^*} \geq 0. \tag{4.3}$$

It is trivial that (4.3) becomes strict when $u \neq 0$, which is the case when $\mathcal{K}(u) > 0$. ∎

Lemma 4.3. *Let $u \in \mathcal{A}$. Suppose also that $\mathcal{M}(u) \leq (1-\delta)^{\frac{d}{2}}\mathcal{M}(Q)$ with some $\delta \in (0,1)$. Then*

$$\|u\|_{2^*}^{2^*} \leq \|\nabla u\|_2^2, \tag{4.4}$$

$$\frac{\delta}{d}\|\nabla u\|_2^2 \leq \mathcal{H}(u) \leq \frac{1}{2}\|\nabla u\|_2^2. \tag{4.5}$$

Proof. Inequality (4.4) follows immediately from the fact that $\mathcal{K}(u) \geq 0$ for $u \in \mathcal{A}$ and the nonpositivity of the nonlinear potentials. The first $\leq$ in (4.5) follows from

$$\begin{aligned}\mathcal{H}(u) &\geq \mathcal{H}(u) - \frac{1}{2^*}\mathcal{K}(u)\\ &= \frac{1}{d}\Big(\|\nabla u\|_2^2 - \frac{d}{d+2}\|u\|_{2_*}^{2_*}\Big)\\ &\geq \frac{1}{d}\Big(1 - \Big(\frac{\mathcal{M}(u)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla u\|_2^2 \geq \frac{\delta}{d}\|\nabla u\|_2^2,\end{aligned}$$

and the second $\leq$ follows immediately from the nonpositivity of the power potentials. ∎

Lemma 4.4. *The mapping $c \mapsto m_c$ is continuous and monotone decreasing on $(0, \mathcal{M}(Q))$, where m_c is defined by (1.7).*

Proof. The proof follows the arguments of [4], where we also need to take the mass constraint into account. We first show that the function f defined by

$$f(a,b) := \max_{t>0}\{at^2 - bt^{2^*}\}$$

is continuous on $(0,\infty)^2$. In fact, the global maximum can be calculated explicitly. Let

$$g(t,a,b) := at^2 - bt^{2^*}$$

and let $t^* \in (0,\infty)$ be such that $\partial_t g(t^*,a,b) = 0$. Then $t^* = (\frac{2a}{2^*b})^{\frac{d-2}{4}}$. Particularly, $\partial_t g(t,a,b)$ is positive on $(0,t^*)$ and negative on (t^*,∞). Thus

$$nf(a,b) = g(t^*,a,b) = \Big(\frac{2a}{2^*b}\Big)^{\frac{d-2}{2}}\frac{2a}{d},$$

and we conclude the continuity of f on $(0,\infty)^2$.

We now show the monotonicity of $c \mapsto m_c$. It suffices to show that for any $0 < c_1 < c_2 < \mathcal{M}(Q)$ and $\varepsilon > 0$ we have

$$m_{c_2} \leq m_{c_1} + \varepsilon.$$

Define the set $V(c)$ by

$$V(c) := \big\{u \in H^1(\mathbb{R}^d) : \mathcal{M}(u) = c,\ \mathcal{K}(u) = 0\big\}.$$

By the definition of m_{c_1} there exists some $u_1 \in V(c_1)$ such that

$$\mathcal{H}(u_1) \leq m_{c_1} + \frac{\varepsilon}{2}. \tag{4.6}$$

Let $\eta \in C_c^\infty(\mathbb{R}^d)$ be a cutoff function such that $\eta = 1$ for $|x| \leq 1$, $\eta = 0$ for $|x| \geq 2$ and $\eta \in [0,1]$ for $|x| \in (1,2)$. For $\delta > 0$, define

$$\tilde{u}_{1,\delta}(x) := \eta(\delta x) \cdot u_1(x).$$

Then $\tilde{u}_{1,\delta} \to u_1$ in $H^1(\mathbb{R}^d)$ as $\delta \to 0$. Therefore,

$$\begin{aligned} \|\nabla \tilde{u}_{1,\delta}\|_2 &\to \|\nabla u_1\|_2, \\ \|\tilde{u}_{1,\delta}\|_p &\to \|u_1\|_p \end{aligned}$$

for all $p \in [2, 2^*]$ as $\delta \to 0$. Using (4.2) we know that $\frac{1}{2}\|\nabla v\|_2^2 > \frac{1}{2_*}\|v\|_{2_*}^{2_*}$ for all $v \in H^1(\mathbb{R}^d)$ with $\mathcal{M}(v) < \mathcal{M}(Q)$. Since $c_1 \in (0, \mathcal{M}(Q))$, we infer that $\mathcal{M}(\tilde{u}_{1,\delta}) \in (0, \mathcal{M}(Q))$ for sufficiently small δ. Combining with the continuity of f we conclude that

$$\begin{aligned} \max_{t>0} \mathcal{H}(T_t \tilde{u}_{1,\delta}) &= \max_{t>0}\Big\{t^2\Big(\frac{1}{2}\|\nabla \tilde{u}_{1,\delta}\|_2^2 - \frac{1}{2_*}\|\tilde{u}_{1,\delta}\|_{2_*}^{2_*}\Big) - \frac{t^{2^*}}{2^*}\|\tilde{u}_{1,\delta}\|_{2^*}^{2^*}\Big\} \\ &\leq \max_{t>0}\Big\{t^2\Big(\frac{1}{2}\|\nabla u_1\|_2^2 - \frac{1}{2_*}\|u_1\|_{2_*}^{2_*}\Big) - \frac{t^{2^*}}{2^*}\|u_1\|_{2^*}^{2^*}\Big\} + \frac{\varepsilon}{4} \\ &= \max_{t>0} \mathcal{H}(T_t u_1) + \frac{\varepsilon}{4} \end{aligned} \tag{4.7}$$

for sufficiently small $\delta > 0$. Now let $v \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} v \subset B(0, 4\delta^{-1} + 1) \setminus B(0, 4\delta^{-1})$ and define

$$v_0 := \frac{(c_2 - \mathcal{M}(\tilde{u}_{1,\delta}))^{\frac{1}{2}}}{(\mathcal{M}(v))^{\frac{1}{2}}} v.$$

We have $\mathcal{M}(v_0) = c_2 - \mathcal{M}(\tilde{u}_{1,\delta})$. Define

$$w_\lambda := \tilde{u}_{1,\delta} + T_\lambda v_0$$

with some to be determined $\lambda > 0$. For sufficiently small δ the supports of $\tilde{u}_{1,\delta}$ and v_0 are disjoint, thus[3]

$$\|w_\lambda\|_p^p = \|\tilde{u}_{1,\delta}\|_p^p + \|T_\lambda v_0\|_p^p$$

for all $p \in [2, 2^*]$. Hence $\mathcal{M}(w_\lambda) = c_2$. Moreover, one easily verifies that

$$\begin{aligned} \|\nabla w_\lambda\|_2 &\to \|\nabla \tilde{u}_{1,\delta}\|_2, \\ \|w_\lambda\|_p &\to \|\tilde{u}_{1,\delta}\|_p \end{aligned}$$

[3] The order logic is as follows: we first fix δ such that $\tilde{u}_{1,\delta}$ and v_0 have disjoint supports. Then $\tilde{u}_{1,\delta}$ and $T_\lambda v_0$ have disjoint supports for any $\lambda \in (0,1)$.

for all $p \in (2, 2^*]$ as $\lambda \to 0$. Using the continuity of f once again we obtain

$$\max_{t>0} \mathcal{H}(T_t w_\lambda) \le \max_{t>0} \mathcal{H}(T_t \tilde{u}_{1,\delta}) + \frac{\varepsilon}{4}$$

for sufficiently small $\lambda > 0$. Finally, combining with (4.6) and (4.7) we infer that

$$\begin{aligned} m_{c_2} &\le \max_{t>0} \mathcal{H}(T_t w_\lambda) \le \max_{t>0} \mathcal{H}(T_t \tilde{u}_{1,\delta}) + \frac{\varepsilon}{4} \\ &\le \max_{t>0} \mathcal{H}(T_t u_1) + \frac{\varepsilon}{2} = \mathcal{H}(u_1) + \frac{\varepsilon}{2} \le m_{c_1} + \varepsilon, \end{aligned} \tag{4.8}$$

which implies the monotonicity of $c \mapsto m_c$ on $(0, \mathcal{M}(Q))$.

Finally, we show the continuity of the curve $c \mapsto m_c$. Since $c \mapsto m_c$ is nonincreasing, it suffices to show that for any $c \in (0, \mathcal{M}(Q))$ and any sequence $c_n \downarrow c$ we have

$$m_c \le \lim_{n\to\infty} m_{c_n}.$$

By the same reasoning we can also prove that $m_c \ge \lim_{n\to\infty} m_{c_n}$ for any sequence $c_n \uparrow c$ and the continuity follows. Let $\varepsilon > 0$ be an arbitrary positive number. By the definition of m_{c_n} we can find some $u_n \in V(c_n)$ such that

$$\mathcal{H}(u_n) \le m_{c_n} + \frac{\varepsilon}{2} \le m_c + \frac{\varepsilon}{2}. \tag{4.9}$$

We define $\tilde{u}_n = (c_n^{-1} c)^{\frac{1}{2}} \cdot u_n := \rho_n u_n$. Then $\mathcal{M}(\tilde{u}_n) = c$ and $\rho_n \uparrow 1$. Since $u_n \in V(c_n)$, we obtain

$$\begin{aligned} m_c + \frac{\varepsilon}{2} \ge m_{c_n} + \frac{\varepsilon}{2} \ge \mathcal{H}(u_n) &= \mathcal{H}(u_n) - \frac{1}{2^*}\mathcal{K}(u_n) \\ &= \frac{1}{d}\Big(\|\nabla u_n\|_2^2 - \frac{d}{d+2}\|u_n\|_{2_*}^{2_*}\Big) \\ &\ge \frac{1}{d}\Big(1 - \Big(\frac{\mathcal{M}(u_n)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla u_n\|_2^2 \\ &= \frac{1}{d}\Big(1 - \Big(\frac{c + o_n(1)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla u_n\|_2^2. \end{aligned} \tag{4.10}$$

Thus $(u_n)_n$ is bounded in $H^1(\mathbb{R}^d)$ and up to a subsequence we infer that there exist $A, B \ge 0$ such that

$$\|\nabla u_n\|_2^2 - \frac{d}{d+2}\|u_n\|_{2_*}^{2_*} = A + o_n(1), \quad \|u_n\|_{2^*}^{2^*} = B + o_n(1). \tag{4.11}$$

On the other hand, using $\mathcal{K}(u_n) = 0$ and the Sobolev inequality we see that

$$\begin{aligned} \frac{1}{d}\Big(1 - \Big(\frac{c + o_n(1)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla u_n\|_2^2 &\le \frac{1}{d}\Big(\|\nabla u_n\|_2^2 - \frac{d}{d+2}\|u_n\|_{2_*}^{2_*}\Big) \\ &= \frac{1}{d}\|u_n\|_{2^*}^{2^*} \\ &\le \frac{\mathcal{S}^{\frac{d}{2-d}}}{d}\|\nabla u_n\|_2^{2^*}. \end{aligned} \tag{4.12}$$

Hence $\liminf_{n\to\infty}\|\nabla u_n\|_2^2 > 0$, which combining with (4.12) also implies

$$A = \lim_{n\to\infty}\Big(\|\nabla u_n\|_2^2 - \frac{d}{d+2}\|u_n\|_{2_*}^{2_*}\Big) > 0, \quad B = \lim_{n\to\infty}\|u_n\|_{2^*}^{2^*} > 0.$$

Therefore, f is continuous at the point (A, B). Also using the fact that $\rho_n \uparrow 1$ we deduce that

$$\begin{aligned} m_c &\le \max_{t>0}\mathcal{H}(T_t\tilde{u}_n) = \max_{t>0}\Big\{\frac{t^2\rho_n^2}{2}\|\nabla u_n\|_2^2 - \frac{t^2\rho_n^{2_*}}{2_*}\|u_n\|_{2_*}^{2_*} - \frac{t^{2^*}\rho_n^{2^*}}{2^*}\|u_n\|_{2^*}^{2^*}\Big\} \\ &\le \max_{t>0}\Big\{t^2\frac{A}{2} - t^{2^*}\frac{B}{2^*}\Big\} + \frac{\varepsilon}{4} \\ &\le \max_{t>0}\Big\{\frac{t^2}{2}\|\nabla u_n\|_2^2 - \frac{t^2}{2_*}\|u_n\|_{2_*}^{2_*} - \frac{t^{2^*}}{2^*}\|u_n\|_{2^*}^{2^*}\Big\} + \frac{\varepsilon}{2} \\ &= \max_{t>0}\mathcal{H}(T_t u_n) + \frac{\varepsilon}{2} = \mathcal{H}(u_n) + \frac{\varepsilon}{2} \le m_{c_n} + \varepsilon \end{aligned} \tag{4.13}$$

by choosing n sufficiently large. The claim follows from the arbitrariness of ε. ■

The following lemma shows that the NLS flow leaves solutions starting from $\mathcal{A}$ invariant.

Lemma 4.5. *Let u be a solution of (4.1) with $u(0) \in \mathcal{A}$. Then $u(t) \in \mathcal{A}$ for all t in the maximal lifespan. Also assume $\mathcal{M}(u) = (1-\delta)^{\frac{d}{2}}\mathcal{M}(Q)$. Then*

$$\begin{aligned} &\inf_{t\in I_{\max}}\mathcal{K}(u(t)) \\ &\quad\ge \min\Big\{\frac{4\delta}{d}\mathcal{H}(u(0)), \Big(\Big(\frac{d}{\delta(d-2)}\Big)^{\frac{d-2}{4}} - 1\Big)^{-1}\Big(m_{\mathcal{M}(u(0))} - \mathcal{H}(u(0))\Big)\Big\}. \end{aligned} \tag{4.14}$$

Proof. By mass and energy conservation, to show the invariance of solutions starting from $\mathcal{A}$ under the NLS flow, we only need to show that $\mathcal{K}(u(t)) > 0$ for all $t \in I_{\max}$. Suppose that there exists some t in the maximal lifespan such that $\mathcal{K}(u(t)) \le 0$. By continuity of $u(t)$ there exists some $s \in (0, t]$ such that $\mathcal{K}(u(s)) = 0$. By conservation of mass we also know that $0 < \mathcal{M}(u(s)) < \mathcal{M}(Q)$. Using the definition of m_c we immediately obtain

$$m_{\mathcal{M}(u(s))} \le \mathcal{H}(u(s)) < m_{\mathcal{M}(u(0))} = m_{\mathcal{M}(u(s))},$$

which is a contradiction. We now show (4.14). Direct calculation yields

$$\frac{d^2}{d\lambda^2}\mathcal{H}(T_\lambda u(t)) = -\frac{1}{\lambda^2}\mathcal{K}(T_\lambda u(t)) + \frac{2}{\lambda^2}\Big(\mathcal{K}(T_\lambda u(t)) - \frac{2}{d-2}\|T_\lambda u(t)\|_{2^*}^{2^*}\Big). \tag{4.15}$$

If $\mathcal{K}(u(t)) - \frac{2}{d-2}\|u(t)\|_{2^*}^{2^*} \ge 0$, then using (4.2) we see that

$$\begin{aligned} \mathcal{K}(u(t)) &= \|\nabla u\|_2^2 - \frac{d}{d+2}\|u\|_{2_*}^{2_*} - \|u\|_{2^*}^{2^*} \\ &\ge \delta\|\nabla u\|_2^2 - \frac{d-2}{2}\mathcal{K}(u(t)), \end{aligned}$$

which combining with (4.5) implies

$$\mathcal{K}(u(t))\ geq \frac{2\delta}{d}\|\nabla u(t)\|_2^2 \geq \frac{4\delta}{d}\mathcal{H}(u(0)), \tag{4.16}$$

where for the last inequality we also used the conservation of energy. Suppose now that

$$\mathcal{K}(u(t)) - \frac{2}{d-2}\|u(t)\|_{2^*}^{2^*} < 0. \tag{4.17}$$

Then

$$\begin{aligned}\frac{2}{d-2}\|u(t)\|_{2^*}^{2^*} &> \|\nabla u(t)\|_2^2 - \frac{d}{d+2}\|u\|_{2_*}^{2_*} - \|u\|_{2^*}^{2^*}\\ &\geq \delta\|\nabla u(t)\|_2^2 - \|u(t)\|_{2^*}^{2^*}.\end{aligned}$$

Hence

$$\|u(t)\|_{2^*}^{2^*} > \frac{\delta(d-2)}{d}\|\nabla u(t)\|_2^2. \tag{4.18}$$

Since $\mathcal{K}(u(t)) > 0$, by Lemma 4.1 we know that there exists some $\lambda_* \in (1,\infty)$ such that

$$\mathcal{K}(T_\lambda u(t)) > 0 \quad \forall \lambda \in [1,\lambda_*) \tag{4.19}$$

and

$$0 = \mathcal{K}(T_{\lambda_*}u(t)) = \lambda_*^2\Big(\|\nabla u(t)\|_2^2 - \frac{d}{d+2}\|u(t)\|_{2_*}^{2_*}\Big) - \lambda_*^{2^*}\|u(t)\|_{2^*}^{2^*},$$

which gives

$$\|u(t)\|_{2^*}^{2^*} \leq \lambda_*^{2-2^*}\Big(\|\nabla u(t)\|_2^2 - \frac{d}{d+2}\|u(t)\|_{2_*}^{2_*}\Big) \leq \lambda_*^{2-2^*}\|\nabla u(t)\|_2^2. \tag{4.20}$$

Inequalities (4.18) and (4.20) then yield

$$\lambda_* \leq \Big(\frac{d}{\delta(d-2)}\Big)^{\frac{d-2}{4}}. \tag{4.21}$$

On the other hand, one easily checks that

$$\frac{d}{d\lambda}\Big(\frac{1}{\lambda^2}\Big(\mathcal{K}(T_\lambda u(t)) - \frac{2}{d-2}\|T_\lambda u(t)\|_{2^*}^{2^*}\Big)\Big) = -\frac{2(2^*-2)}{d-2}\lambda^{2^*-3}\|u(t)\|_{2^*}^{2^*} < 0. \tag{4.22}$$

Integrating (4.22) and using (4.17), we find that for $\lambda \geq 1$,

$$\frac{1}{\lambda^2}\Big(\mathcal{K}(T_\lambda u(t)) - \frac{2}{d-2}\|T_\lambda u(t)\|_{2^*}^{2^*}\Big) \leq 0. \tag{4.23}$$

Expressions (4.15), (4.19) and (4.23) then imply that $\frac{d^2}{d\lambda^2}\mathcal{H}(T_\lambda u(t)) \leq 0$ for all $\lambda \in [1,\lambda_*]$. Finally, combining with (4.21), the fact that $\mathcal{K}(T_{\lambda_*}u(t)) = 0$ and Taylor expansion we conclude that

$$\begin{aligned}\Big(\Big(\frac{d}{\delta(d-2)}\Big)^{\frac{d-2}{4}} - 1\Big)\mathcal{K}(u(t)) &\geq (\lambda_* - 1)\Big(\frac{d}{d\lambda}\mathcal{H}(T_\lambda u(t))\Big|_{\lambda=1}\Big)\\ &\geq \mathcal{H}(T_{\lambda_*}u(t)) - \mathcal{H}(u(t))\\ &\geq m_{\mathcal{M}(u(0))} - \mathcal{H}(u(0)).\end{aligned} \tag{4.24}$$

This together with (4.16) yields (4.14). ■

Lemma 4.6. *Let*

$$\widetilde{m}_c := \inf_{u \in H^1(\mathbb{R}^d)} \big\{ \mathcal{I}(u) : \mathcal{M}(u) = c,\ \mathcal{K}(u) \le 0 \big\}, \tag{4.25}$$

where $\mathcal{I}(u)$ is defined by (1.16). Then $m_c = \widetilde{m}_c$ for any $c \in (0, \mathcal{M}(Q))$.

Proof. Let $(u_n)_n$ be a minimizing sequence for the variational problem (4.25), i.e.

$$\lim_{n\to\infty} \mathcal{I}(u_n) = \widetilde{m}_c, \quad \mathcal{M}(u_n) = c, \quad \mathcal{K}(u_n) \le 0.$$

Using Lemma 4.1 we know that there exists some $\lambda_n \in (0, 1]$ such that $\mathcal{K}(T_{\lambda_n} u_n) = 0$. Thus

$$m_c \le \mathcal{H}(T_{\lambda_n} u_n) = \mathcal{I}(T_{\lambda_n} u_n) \le \mathcal{I}(u_n) = \widetilde{m}_c + o_n(1).$$

Sending $n \to \infty$ we infer that $m_c \le \widetilde{m}_c$. On the other hand,

$$\begin{aligned} \widetilde{m}_c &\le \inf_{u \in H^1(\mathbb{R}^d)} \big\{ \mathcal{I}(u) : \mathcal{M}(u) = c,\ \mathcal{K}(u) = 0 \big\} \\ &= \inf_{u \in H^1(\mathbb{R}^d)} \big\{ \mathcal{H}(u) : \mathcal{M}(u) = c,\ \mathcal{K}(u) = 0 \big\} = m_c. \end{aligned} \tag{4.26}$$

This completes the proof. ∎

We define the set Ω by its complement

$$\Omega^c := \big\{ (c, h) \in \mathbb{R}^2 : c \ge \mathcal{M}(Q) \big\} \cup \big\{ (c, h) \in \mathbb{R}^2 : c \in [0, \mathcal{M}(Q)),\ h \ge m_c \big\} \tag{4.27}$$

and the function $\mathcal{D} \colon \mathbb{R}^2 \to [0, \infty]$ by

$$\mathcal{D}(c, h) = \begin{cases} h + \dfrac{h + c}{\operatorname{dist}((c, h), \Omega^c)} & \text{if } (c, h) \in \Omega, \\ \infty & \text{otherwise.} \end{cases} \tag{4.28}$$

For $u \in H^1(\mathbb{R}^d)$ also define $\mathcal{D}(u) := \mathcal{D}(\mathcal{M}(u), \mathcal{H}(u))$.

Remark 4.7. Let $m_0 := \lim_{c \downarrow 0} m_c$ and $m_Q := \lim_{c \uparrow \mathcal{M}(Q)} m_c$. By modifying the arguments in [42, Thm. 1.2] and [45, Lem. 3.3] we are able to show that

$$m_0 = \mathcal{H}^*(W), \quad m_Q = 0.$$

Nevertheless, the precise values of m_0 and m_Q have no impact on the scattering result; all we need here is the monotonicity and continuity of the curve $c \mapsto m_c$. We therefore postpone the proof to the appendix. △

Lemma 4.8. *Assume $v \in H^1(\mathbb{R}^d)$ such that $\mathcal{K}(v) \ge 0$. Then we have the following statements:*

(i) $\mathcal{D}(v) = 0$ *if and only if* $v = 0$.

(ii) $0 < \mathcal{D}(v) < \infty$ *if and only if* $v \in \mathcal{A}$.

(iii) $\mathcal{D}$ *is conserved under the NLS flow* (4.1).

(iv) *Let* $u_1, u_2 \in \mathcal{A}$ *with* $\mathcal{M}(u_1) \le \mathcal{M}(u_2)$ *and* $\mathcal{H}(u_1) \le \mathcal{H}(u_2)$, *then* $\mathcal{D}(u_1) \le \mathcal{D}(u_2)$. *If in addition either* $\mathcal{M}(u_1) < \mathcal{M}(u_2)$ *or* $\mathcal{H}(u_1) < \mathcal{H}(u_2)$, *then* $\mathcal{D}(u_1) < \mathcal{D}(u_2)$.

(v) *Let* $\mathcal{D}_0 \in (0, \infty)$. *Then*

$$\|\nabla u\|_2^2 \sim_{\mathcal{D}_0} \mathcal{H}(u), \tag{4.29}$$

$$\|u\|_{H^1}^2 \sim_{\mathcal{D}_0} \mathcal{H}(u) + \mathcal{M}(u) \sim_{\mathcal{D}_0} \mathcal{D}(u) \tag{4.30}$$

uniformly for all $u \in \mathcal{A}$ *with* $\mathcal{D}(u) \le \mathcal{D}_0$.

(vi) *For all* $u \in \mathcal{A}$ *with* $\mathcal{D}(u) \le \mathcal{D}_0$ *for some* $\mathcal{D}_0 \in (0, \infty)$ *we have*

$$|\mathcal{H}(u) - m_{\mathcal{M}(u)}| \gtrsim 1. \tag{4.31}$$

Proof. (i) That $v = 0$ implies $\mathcal{D}(v) = 0$ is trivial. The other direction follows immediately from (4.5) and the definition of $\mathcal{D}$.

(ii) It is trivial that $v \in \mathcal{A}$ implies $\mathcal{D}(v) < \infty$. By Lemma 4.2 we also know that $\mathcal{H}(v) > 0$, which gives $\mathcal{D}(v) > 0$. Now let $0 < \mathcal{D}(v) < \infty$. Then $\mathcal{M}(v) \in (0, \mathcal{M}(Q))$. By definition of $\mathcal{D}$ and Lemma 4.2 we infer that $0 \le \mathcal{H}(v) < m_{\mathcal{M}(v)}$, which also yields $\mathcal{K}(v) > 0$ by the definition of $m_{\mathcal{M}(v)}$. Hence we conclude that $v \in \mathcal{A}$.

(iii) This follows immediately from the conservation of mass and energy of the NLS flow (4.1) and the definition of $\mathcal{D}$.

(iv) This follows directly from the fact that $c \mapsto m_c$ is monotone decreasing on $(0, \mathcal{M}(Q))$ and the definition of $\mathcal{D}$.

(v) Since $u \in \mathcal{A}$, we know that $\mathcal{M}(u) \in (0, \mathcal{M}(Q))$ and using Lemma 4.2 also $\mathcal{H}(u) \in [0, m_{\mathcal{M}(u)})$. Thus

$$\begin{aligned}\operatorname{dist}\big((\mathcal{M}(u), \mathcal{H}(u)), \Omega^c\big) &\le \operatorname{dist}\big((\mathcal{M}(u), \mathcal{H}(u)), (\mathcal{M}(Q), \mathcal{H}(u))\big) \\ &= \mathcal{M}(Q) - \mathcal{M}(u).\end{aligned}$$

Since $\mathcal{H}(u) \ge 0$, we have

$$\mathcal{D}(u) \ge \frac{\mathcal{M}(u)}{\mathcal{M}(Q) - \mathcal{M}(u)}, \tag{4.32}$$

which implies that

$$\frac{1}{1 + \mathcal{D}(u)} \le 1 - \frac{\mathcal{M}(u)}{\mathcal{M}(Q)}.$$

Since $1 - \alpha \le 1 - \alpha^{\frac{2}{d}}$ for $\alpha \in [0, 1]$, we deduce that

$$\frac{1}{1 + \mathcal{D}(u)} \le 1 - \Big(\frac{\mathcal{M}(u)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}.$$

Using $\mathcal{K}(u) > 0$ we have

$$\begin{aligned}\mathcal{D}(u) &\geq \mathcal{H}(u) > \mathcal{H}(u) - \frac{1}{2^*}\mathcal{K}(u)\\ &= \frac{1}{d}\Big(\|\nabla u\|_2^2 - \frac{d}{d+2}\|u\|_{2_*}^{2_*}\Big)\\ &\geq \frac{1}{d}\Big(1 - \Big(\frac{\mathcal{M}(u)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla u\|_2^2 \geq \frac{\|\nabla u\|_2^2}{d(1+\mathcal{D}(u))}. \end{aligned} \tag{4.33}$$

Therefore, $\|\nabla u\|_2^2 \lesssim_{\mathcal{D}_0} \mathcal{H}(u)$. Combining with (4.5) we conclude that

$$\|\nabla u\|_2^2 \sim_{\mathcal{D}_0} \mathcal{H}(u), \quad \|u\|_{H^1}^2 \sim_{\mathcal{D}_0} \mathcal{H}(u) + \mathcal{M}(u).$$

It remains to show $\mathcal{H}(u) + \mathcal{M}(u) \sim_{\mathcal{D}_0} \mathcal{D}(u)$. Using (4.32) and (4.33) we already know that

$$\mathcal{H}(u) + \mathcal{M}(u) \sim_{\mathcal{D}_0} \|u\|_{H^1}^2 \lesssim_{\mathcal{D}_0} \mathcal{D}(u).$$

To show $\mathcal{D}(u) \lesssim_{\mathcal{D}_0} \mathcal{H}(u) + \mathcal{M}(u)$ we discuss the following different cases: If $\mathcal{M}(u) \geq \frac{1}{2}\mathcal{M}(Q)$, then using the fact that $\mathcal{H}(u) \geq 0$ we have

$$\operatorname{dist}\big((\mathcal{M}(u), \mathcal{H}(u)), \Omega^c\big) \geq \frac{\mathcal{M}(u)}{\mathcal{D}_0} \geq \frac{\mathcal{M}(Q)}{2\mathcal{D}_0},$$

which implies

$$\mathcal{D}(u) \leq \frac{2\mathcal{D}_0}{\mathcal{M}(Q)}(\mathcal{M}(u) + \mathcal{H}(u)) + \mathcal{H}(u).$$

If $\mathcal{M}(u) < \frac{1}{2}\mathcal{M}(Q)$ and $\mathcal{H}(u) \geq \frac{1}{2}m_{\frac{1}{2}\mathcal{M}(Q)}$, then analogously we obtain

$$\mathcal{D}(u) \leq \frac{2\mathcal{D}_0}{m_{\frac{1}{2}\mathcal{M}(Q)}}(\mathcal{M}(u) + \mathcal{H}(u)) + \mathcal{H}(u).$$

If $\mathcal{M}(u) < \frac{1}{2}\mathcal{M}(Q)$ and $\mathcal{H}(u) < \frac{1}{2}m_{\frac{1}{2}\mathcal{M}(Q)}$, then

$$\operatorname{dist}\big((\mathcal{M}(u), \mathcal{H}(u)), \Omega^c\big) \geq \operatorname{dist}\big((\tfrac{1}{2}\mathcal{M}(Q), \tfrac{1}{2}m_{\frac{1}{2}\mathcal{M}(Q)}), \Omega^c\big) =: \alpha_0 > 0,$$

where the first inequality and the positivity of α_0 follow from the monotonicity of $c \mapsto m_c$. Therefore,

$$\mathcal{D}(u) \leq \frac{1}{\alpha_0}(\mathcal{M}(u) + \mathcal{H}(u)) + \mathcal{H}(u).$$

Summing, the proof of (v) is complete.

(vi) If this were not the case, then we could find a sequence $(u_n)_n \subset \mathcal{A}$ such that

$$|\mathcal{H}(u_n) - m_{\mathcal{M}(u_n)}| = o_n(1). \tag{4.34}$$

But then

$$\begin{aligned}\operatorname{dist}\big((\mathcal{M}(u_n),\mathcal{H}(u_n)),\Omega^c\big)&\leq \operatorname{dist}\big((\mathcal{M}(u_n),\mathcal{H}(u_n)),(\mathcal{M}(u_n),m_{\mathcal{M}(u_n)})\big)\\&=|m_{\mathcal{M}(u_n)}-\mathcal{H}(u_n)|=o_n(1).\end{aligned}$$

If $\mathcal{M}(u_n)\gtrsim 1$, then $\mathcal{D}(u_n)\gtrsim \frac{1}{o_n(1)}$, contradicting $\mathcal{D}(u_n)\leq\mathcal{D}_0$. If $\mathcal{M}(u_n)=o_n(1)$, then by (4.34) we know that $\mathcal{H}(u_n)\gtrsim 1$ and similarly we may again derive the contradiction $\mathcal{D}(u_n)\gtrsim\frac{1}{o_n(1)}$. This finishes the proof of (vi) and also the desired proof of Lemma 4.8. ■

4.2. Large- and small-scale approximations

In this section we show that the nonlinear profiles corresponding to low-frequency and high-frequency bubbles can be well approximated by the solutions of the mass- and energy-critical NLS respectively.

Lemma 4.9 (Large-scale approximation for $\lambda_\infty=\infty$). *Let u be the solution of the focusing mass-critical NLS*

$$i\partial_t u+\Delta u+|u|^{\frac{4}{d}}u=0, \tag{4.35}$$

with $u(0)=u_0\in H^1(\mathbb{R}^d)$ and $\mathcal{M}(u_0)<\mathcal{M}(Q)$. Then u is global and

$$\|u\|_{W_{2*}(\mathbb{R})}\leq C(\mathcal{M}(u_0)), \tag{4.36}$$

$$\|\,|\nabla|^s u\|_{S(\mathbb{R})}\lesssim_{\mathcal{M}(u_0)}\|\,|\nabla|^s u_0\|_2 \tag{4.37}$$

for $s\in\{0,1\}$. Moreover, we have the following large-scale approximation result for (4.35): Let $(\lambda_n)_n\subset(0,\infty)$ such that $\lambda_n\to\infty$, $(t_n)_n\subset\mathbb{R}$ such that either $t_n\equiv 0$ or $t_n\to\pm\infty$ and $(\xi_n)_n\subset\mathbb{R}^d$ such that $(\xi_n)_n$ is bounded. Define

$$\phi_n:=g_{\xi_n,x_n,\lambda_n}e^{it_n\Delta}P_{\leq\lambda_n^\theta}\phi \tag{4.38}$$

for some $\theta\in(0,1)$. Then for all sufficiently large n, the solution u_n of (4.1) with $u_n(0)=\phi_n$ is global and scattering in time with

$$\limsup_{n\to\infty}\|\langle\nabla\rangle u_n\|_{S(\mathbb{R})}\leq C(\mathcal{M}(\phi)), \tag{4.39}$$

$$\lim_{n\to\infty}\|u_n\|_{W_{2*}(\mathbb{R})}=0. \tag{4.40}$$

Furthermore, for every $\beta>0$ there exists $N_\beta\in\mathbb{N}$ and $\phi_\beta\in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ such that

$$\Big\|u_n-\lambda_n^{-\frac{d}{2}}e^{-it|\xi_n|^2}e^{i\xi_n\cdot x}\phi_\beta\Big(\frac{t}{\lambda_n^2}+t_n,\frac{x-x_n-2t\xi_n}{\lambda_n}\Big)\Big\|_{W_{2*}(\mathbb{R})}\leq\beta, \tag{4.41}$$

$$\Big\|\nabla u_n-i\xi_n\lambda_n^{-\frac{d}{2}}e^{-it|\xi_n|^2}e^{i\xi_n\cdot x}\phi_\beta\Big(\frac{t}{\lambda_n^2}+t_n,\frac{x-x_n-2t\xi_n}{\lambda_n}\Big)\Big\|_{W_{2*}(\mathbb{R})}\leq\beta \tag{4.42}$$

for all $n\geq N_\beta$.

Proof. Inequality (4.36) and the fact that u is global are proved in [19]; (4.37) can be proved similarly to Lemma 2.3 and we therefore omit the details here.

Next we prove the claims concerning the large-scale approximation. When $t_n \equiv 0$, we define w and w_n as the solutions of (4.35) with $w(0) = \phi$ and $w_n(0) = \phi_n$ respectively. When $t_n \to \pm\infty$, we define w and w_n as solutions of (4.35) which scatter to $e^{it\Delta}\phi$ and $e^{it\Delta}P_{\leq \lambda_n^\theta}\phi$ in $L^2(\mathbb{R}^d)$ as $t \to \pm\infty$ respectively. By (4.36) we know that w is global, scatters in time and

$$\|w\|_{S(\mathbb{R})} \leq C(\mathcal{M}(\phi)).$$

On the other hand, since

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{t\to\pm\infty}\|w_n(t) - w(t)\|_2 \\
&\quad \leq \lim_{n\to\infty}\lim_{t\to\pm\infty}\big(\|w_n(t) - e^{it\Delta}P_{\leq\lambda_n^\theta}\phi\|_2 + \|w(t) - e^{it\Delta}\phi\|_2 + \|\phi - P_{\leq\lambda_n^\theta}\phi\|_2\big) \\
&\quad = 0,
\end{aligned}$$

by the standard stability result for mass-critical NLS (see for instance [31]) we infer that w_n is global and scattering in time for all sufficiently large n and

$$\limsup_{n\to\infty}\|w_n\|_{W_{2*}(\mathbb{R})} \lesssim_{\mathcal{M}(\phi)} 1.$$

Using Bernstein, Strichartz and (4.37) we additionally have

$$\|w_n\|_{W_{2^*}(\mathbb{R})} \lesssim \|\nabla w_n\|_{S(\mathbb{R})} \lesssim_{\mathcal{M}(\phi)} \lambda_n^\theta.$$

We now define

$$\tilde{u}_n(t,x) := \lambda_n^{-\frac{d}{2}} e^{i\xi_n\cdot x} e^{-it|\xi_n|^2} w_n\Big(\frac{t}{\lambda_n^2} + t_n, \frac{x - x_n - 2t\xi_n}{\lambda_n}\Big). \tag{4.43}$$

Using the symmetry invariance for mass-critical NLS one easily verifies that $\tilde{u}_n$ is also a global and scattering solution of (4.35). In particular,

$$\|\langle\nabla\rangle\tilde{u}_n\|_{S(\mathbb{R})} \lesssim (1 + |\xi_n|)\|w_n\|_{S(\mathbb{R})} + \lambda_n^{-1}\|\nabla w_n\|_{S(\mathbb{R})} \lesssim 1 + \lambda_n^{-(1-\theta)} \to 1, \tag{4.44}$$

$$\|\tilde{u}_n\|_{W_{2^*}(\mathbb{R})} = \lambda_n^{-1}\|w_n\|_{W_{2^*}(\mathbb{R})} \lesssim \lambda_n^{-1}\|\nabla w_n\|_{S(\mathbb{R})} \lesssim \lambda_n^{-(1-\theta)} \to 0 \tag{4.45}$$

as $n \to \infty$. We next show that $\tilde{u}_n$ is asymptotically a good proxy of u_n using Lemma 2.4. Rewrite (4.35) for $\tilde{u}_n$ as

$$i\partial_t\tilde{u}_n + \Delta\tilde{u}_n + |\tilde{u}_n|^{\frac{4}{d}}\tilde{u}_n + |\tilde{u}_n|^{\frac{4}{d-2}}\tilde{u}_n + e = 0, \tag{4.46}$$

where $e = -|\tilde{u}_n|^{\frac{4}{d-2}}\tilde{u}_n$. Using (4.2), Sobolev and conservation of energy we obtain

$$\|\nabla u_n(t)\|_2^2 \lesssim \mathcal{H}(u_n(t)) + \tfrac{1}{2^*}\|u_n(t)\|_{2^*}^{2^*} \lesssim \mathcal{H}(\phi_n) + \|\nabla u_n(t)\|_2^{2^*}.$$

But using Bernstein we also see that

$$\|\nabla\phi_n\|_2 \lesssim \lambda_n^{-1}|\xi_n|\,\|\phi\|_2 + \lambda_n^{-(1-\theta)}\|\phi\|_2 \to 0,$$

which implies

$$\mathcal{H}(\phi_n) \lesssim \|\nabla \phi_n\|_2^2 \to 0.$$

By standard continuity arguments we conclude that $\limsup_{n\to\infty} \|u_n\|_{L_t^\infty \dot{H}_x^1(I)} < \infty$, and (2.7) is satisfied by combining with conservation of mass for sufficiently large n. It remains to show (2.10). Indeed, using Hölder we deduce that

$$\|\langle\nabla\rangle e\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \leq \|\tilde{u}_n\|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}} \|\langle\nabla\rangle\tilde{u}_n\|_{W_{2*}(\mathbb{R})}. \tag{4.47}$$

Then (2.10) follows from (4.44) and (4.45). Expressions (4.39) and (4.40) now follow from (2.11), (4.44), (4.45) and Strichartz. Finally, to show (4.41) and (4.42) we first choose $\phi_\beta \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ and sufficiently large n such that

$$\|w - \phi_\beta\|_{W_{2*}(\mathbb{R})} + \|w - w_n\|_{W_{2*}(\mathbb{R})} + \|\langle\nabla\rangle\tilde{u}_n - \langle\nabla\rangle u_n\|_{W_{2*}(\mathbb{R})} \lesssim \beta.$$

Using the chain rule and Bernstein we also deduce that

$$\|\nabla\tilde{u}_n - i\xi_n\tilde{u}_n\|_{W_{2*}(\mathbb{R})} = \lambda_n^{-1}\|\nabla w_n\|_{W_{2*}(\mathbb{R})} \lesssim \lambda_n^{-(1-\theta)} \to 0. \tag{4.48}$$

Then (4.41) and (4.42) follow from the triangular inequality and taking n sufficiently large. ∎

Analogously, we have the following small-scale analogue of Lemma 4.9, where the arguments from [19] are replaced by [22,27,30]. We therefore omit the proof.

Lemma 4.10 (Small-scale approximation for $\lambda_\infty = 0$). *Let u be the solution of the focusing energy-critical NLS*

$$i\partial_t u + \Delta u + |u|^{\frac{4}{d-2}}u = 0 \tag{4.49}$$

with $u(0) = u_0 \in H^1(\mathbb{R}^d)$, $\mathcal{H}^*(u_0) < \mathcal{H}^*(W)$ and $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$. Additionally assume that u_0 is radial when $d = 3$. Then u is global and

$$\|u\|_{W_{2^*}(\mathbb{R})} \leq C(\mathcal{H}^*(u_0)), \tag{4.50}$$

$$\|\,|\nabla|^s u\|_{S(\mathbb{R})} \lesssim_{\mathcal{H}^*(u_0)} \|\,|\nabla|^s u_0\|_2 \tag{4.51}$$

for $s \in \{0,1\}$. Moreover, we have the following small-scale approximation result for (4.49): Let $(\lambda_n)_n \subset (0,\infty)$ such that $\lambda_n \to 0$ and $(t_n)_n \subset \mathbb{R}$ such that either $t_n \equiv 0$ or $t_n \to \pm\infty$. Define

$$\phi_n := \lambda_n g_{0,x_n,\lambda_n} e^{it_n\Delta} P_{>\lambda_n^\theta}\phi$$

for some $\theta \in (0,1)$. Then for all sufficiently large n, the solution u_n of (4.1) with $u_n(0) = \phi_n$ is global and scattering in time with

$$\limsup_{n\to\infty} \|\langle\nabla\rangle u_n\|_{S(\mathbb{R})} \leq C(\mathcal{H}^*(\phi)), \tag{4.52}$$

$$\lim_{n\to\infty} \|u_n\|_{W_{2*}(\mathbb{R})} = 0. \tag{4.53}$$

Furthermore, for every $\beta > 0$ there exists $N_\beta \in \mathbb{N}$, $\phi_\beta \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ and $\psi_\beta \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d; \mathbb{C}^d)$ such that

$$\Big\| u_n - \lambda_n^{-\frac{d}{2}+1} \phi_\beta \Big(\frac{t}{\lambda_n^2} + t_n, \frac{x - x_n}{\lambda_n} \Big) \Big\|_{W_{2^*}(\mathbb{R})} \le \beta, \tag{4.54}$$

$$\Big\| \nabla u_n - \lambda_n^{-\frac{d}{2}} \psi_\beta \Big(\frac{t}{\lambda_n^2} + t_n, \frac{x - x_n}{\lambda_n} \Big) \Big\|_{W_{2_*}(\mathbb{R})} \le \beta \tag{4.55}$$

for all $n \ge N_\beta$.

4.3. Existence of the minimal blow-up solution

Having all the preliminaries we are ready to construct the minimal blow-up solution. Define

$$\tau(\mathcal{D}_0) := \sup\big\{ \|\psi\|_{W_{2_*} \cap W_{2^*}(I_{\max})} : \psi \text{ is solution of (4.1)}, \\ \psi(0) \in \mathcal{A},\ \mathcal{D}(\psi(0)) \le \mathcal{D}_0 \big\}$$

and

$$\mathcal{D}^* := \sup\big\{ \mathcal{D}_0 > 0 : \tau(\mathcal{D}_0) < \infty \big\}. \tag{4.56}$$

By Lemma 2.1, Remark 2.2 and Lemma 4.8 (v) we know that $\mathcal{D}^* > 0$ and $\tau(\mathcal{D}_0) < \infty$ for sufficiently small $\mathcal{D}_0$. We will therefore assume that $\mathcal{D}^* < \infty$ and aim to derive a contradiction, which will imply $\mathcal{D}^* = \infty$ and the whole proof will be complete in view of Lemma 4.8 (ii). By the inductive hypothesis we may find a sequence $(\psi_n)_n$ with $(\psi_n(0))_n \subset \mathcal{A}$ which are solutions of (4.1) with maximal lifespan $(I_n)_n$ such that

$$\lim_{n\to\infty} \|\psi_n\|_{W_{2_*} \cap W_{2^*}((\inf I_n, 0])} = \lim_{n\to\infty} \|\psi_n\|_{W_{2_*} \cap W_{2^*}([0, \sup I_n))} = \infty, \tag{4.57}$$

$$\lim_{n\to\infty} \mathcal{D}(\psi_n(0)) = \mathcal{D}^*. \tag{4.58}$$

Up to a subsequence we may also assume that

$$\big(\mathcal{M}(\psi_n(0)), \mathcal{H}(\psi_n(0)), \mathcal{I}(\psi_n(0)) \big) \to (\mathcal{M}_0, \mathcal{H}_0, \mathcal{I}_0) \quad \text{as } n \to \infty.$$

By continuity of $\mathcal{D}$ and finiteness of $\mathcal{D}^*$ we know that

$$\mathcal{D}^* = \mathcal{D}(\mathcal{M}_0, \mathcal{H}_0), \quad \mathcal{M}_0 \in (0, \mathcal{M}(Q)), \quad \mathcal{H}_0 \in [0, m_{\mathcal{M}_0}).$$

From Lemma 4.8 (v) it follows that $(\psi_n(0))_n$ is a bounded sequence in $H^1(\mathbb{R}^d)$ and Lemma 3.6 is applicable for $(\psi_n(0))_n$. More precisely, there exist nonzero linear profiles $(\phi^j)_j \subset \dot{H}^1(\mathbb{R}^d) \cup L^2(\mathbb{R}^d)$, remainders $(w_n^k)_{k,n} \subset H^1(\mathbb{R}^d)$, parameters $(t_n^j, x_n^j, \xi_n^j, \lambda_n^j)_{j,n} \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times (0, \infty)$ and $K^* \in \mathbb{N} \cup \{\infty\}$, such that we have the following statements:

(i) For any finite $1 \le j \le K^*$ the parameters satisfy

$$\begin{aligned} 1 &\gtrsim_j \lim_{n\to\infty} |\xi_n^j|, \\ \lim_{n\to\infty} t_n^j &=: t_\infty^j \in \{0, \pm\infty\}, \\ \lim_{n\to\infty} \lambda_n^j &=: \lambda_\infty^j \in \{0, 1, \infty\}, \\ t_n^j &\equiv 0 \quad \text{if } t_\infty^j = 0, \\ \lambda_n^j &\equiv 1 \quad \text{if } \lambda_\infty^j = 1, \\ \xi_n^j &\equiv 0 \quad \text{if } \lambda_\infty^j \in \{0, 1\}. \end{aligned} \tag{4.59}$$

(ii) For any finite $1 \le k \le K^*$ we have the decomposition

$$\psi_n(0) = \sum_{j=1}^{k} T_n^j P_n^j \phi^j + w_n^k. \tag{4.60}$$

Here, the operators T_n^j and P_n^j are defined by

$$T_n^j u(x) := \begin{cases} \lambda_n^j g_{0,x_n^j,\lambda_n^j}[e^{it_n^j\Delta}u](x) & \text{if } \lambda_\infty^j = 0, \\ [e^{it_n^j\Delta}u](x - x_n^j) & \text{if } \lambda_\infty^j = 1, \\ g_{\xi_n^j,x_n^j,\lambda_n^j}[e^{it_n^j\Delta}u](x) & \text{if } \lambda_\infty^j = \infty \end{cases} \tag{4.61}$$

and

$$P_n^j u := \begin{cases} P_{>(\lambda_n^j)^\theta} u & \text{if } \lambda_\infty^j = 0, \\ u & \text{if } \lambda_\infty^j = 1, \\ P_{\le(\lambda_n^j)^\theta} u & \text{if } \lambda_\infty^j = \infty \end{cases} \tag{4.62}$$

for some $\theta \in (0, 1)$. Moreover,

$$\phi^j \in \begin{cases} \dot{H}^1(\mathbb{R}^d) & \text{if } \lambda_\infty^j = 0, \\ H^1(\mathbb{R}^d) & \text{if } \lambda_\infty^j = 1, \\ L^2(\mathbb{R}^d) & \text{if } \lambda_\infty^j = \infty. \end{cases} \tag{4.63}$$

(iii) The remainders $(w_n^k)_{k,n}$ satisfy

$$\lim_{k\to K^*} \lim_{n\to\infty} \|e^{it\Delta} w_n^k\|_{W_{2*} \cap W_{2^*}(\mathbb{R})} = 0. \tag{4.64}$$

(iv) The parameters are orthogonal in the sense that

$$\begin{aligned} &\frac{\lambda_n^k}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^k} + \lambda_n^k|\xi_n^j - \xi_n^k| + \left| t_k \left(\frac{\lambda_n^k}{\lambda_n^j}\right)^2 - t_n^j \right| \\ &\quad + \left| \frac{x_n^j - x_n^k - 2t_n^k(\lambda_n^k)^2(\xi_n^j - \xi_n^k)}{\lambda_n^k} \right| \to \infty \end{aligned} \tag{4.65}$$

for any $j \neq k$.

(v) For any finite $1 \le k \le K^*$ we have the energy decompositions

$$\| |\nabla|^s \psi_n\|_2^2 = \sum_{j=1}^k \| |\nabla|^s T_n^j P_n^j \phi^j \|_2^2 + \| |\nabla|^s w_n^k\|_2^2 + o_n(1), \tag{4.66}$$

$$\mathcal{H}(\psi_n) = \sum_{j=1}^k \mathcal{H}(T_n^j P_n^j \phi^j) + \mathcal{H}(w_n^k) + o_n(1), \tag{4.67}$$

$$\mathcal{K}(\psi_n) = \sum_{j=1}^k \mathcal{K}(T_n^j P_n^j \phi^j) + \mathcal{K}(w_n^k) + o_n(1), \tag{4.68}$$

$$\mathcal{I}(\psi_n) = \sum_{j=1}^k \mathcal{I}(T_n^j P_n^j \phi^j) + \mathcal{I}(w_n^k) + o_n(1) \tag{4.69}$$

for $s \in \{0, 1\}$ and any finite $1 \le k \le K^*$.

We define the nonlinear profiles as follows: For $\lambda_\infty^k \in \{0, \infty\}$, we define v_n^k as the solution of (4.1) with $v_n^k(0) = T_n^k P_n^k \phi^k$. For $\lambda_\infty^k = 1$ and $t_\infty^k = 0$, we define v^k as the solution of (4.1) with $v^k(0) = \phi^k$. For $\lambda_\infty^k = 1$ and $t_\infty^k \to \pm\infty$, we define v^k as the solution of (4.1) that scatters forward (backward) to $e^{it\Delta}\phi^k$ in $H^1(\mathbb{R}^d)$. In both cases for $\lambda_\infty^k = 1$ we define

$$v_n^k := v^k(t + t_n^k, x - x_n^k).$$

Then v_n^k is also a solution of (4.1). In all cases we have for each finite $1 \le k \le K^*$,

$$\lim_{n\to\infty} \|v_n^k(0) - T_n^k P_n^k \phi^k\|_{H^1} = 0. \tag{4.70}$$

In the following, we establish a Palais–Smale-type lemma which is essential for the construction of the minimal blow-up solution.

Lemma 4.11 (Palais–Smale condition). *Let $(\psi_n)_n$ be a sequence of solutions of (4.1) with maximal lifespan I_n, $\psi_n \in \mathcal{A}$ and $\lim_{n\to\infty} \mathcal{D}(u_n) = \mathcal{D}^*$. Also assume that there exists a sequence $(t_n)_n \subset \prod_n I_n$ such that*

$$\lim_{n\to\infty} \|\psi_n\|_{W_{2*}\cap W_{2^*}((\inf I_n, t_n])} = \lim_{n\to\infty} \|\psi_n\|_{W_{2*}\cap W_{2^*}([t_n, \sup I_n)} = \infty. \tag{4.71}$$

Then up to a subsequence, there exists a sequence $(x_n)_n \subset \mathbb{R}^d$ such that $(\psi_n(t_n, \cdot + x_n))_n$ strongly converges in $H^1(\mathbb{R}^d)$.

Proof. By time-translation invariance we may assume that $t_n \equiv 0$. Let $(v_n^j)_{j,n}$ be the nonlinear profiles corresponding to the linear profile decomposition of $(\psi_n(0))_n$. Define

$$\Psi_n^k := \sum_{j=1}^k v_n^j + e^{it\Delta} w_n^k.$$

We will show that there exists exactly one nontrivial bad linear profile, relying on which the desired claim follows. We divide the remainder of the proof into three steps.

Step 1: Decomposition of energies and large-/small-scale proxies. In the first step we show that the low- and high-frequency bubbles asymptotically meet the preconditions of Lemmas 4.9 and 4.10 respectively. We first show that

$$\mathcal{H}(T_n^j P_n^j \phi^j) > 0, \tag{4.72}$$
$$\mathcal{K}(T_n^j P_n^j \phi^j) > 0 \tag{4.73}$$

for any finite $1 \le j \le K^*$ and all sufficiently large $n = n(j) \in \mathbb{N}$. Since $\phi^j \ne 0$ we know that $T_n^j P_n^j \phi^j \ne 0$ for sufficiently large n. Suppose now that (4.73) does not hold. Up to a subsequence we may assume that $\mathcal{K}(T_n^j P_n^j \phi^j) \le 0$ for all sufficiently large n. By the nonnegativity of $\mathcal{I}$, (4.69) and (4.31) we know that there exists some sufficiently small $\delta > 0$ depending on $\mathcal{D}^*$ and some sufficiently large N_1 such that for all $n > N_1$ we have

$$\begin{aligned} \widetilde{m}_{\mathcal{M}(T_n^j P_n^j \phi^j)} &\le \mathcal{I}(T_n^j P_n^j \phi^j) \le \mathcal{I}(\psi_n(0)) + \delta \\ &\le \mathcal{H}(\psi_n(0)) + \delta \le m_{\mathcal{M}(\psi_n(0))} - 2\delta, \end{aligned} \tag{4.74}$$

where $\widetilde{m}$ is the quantity defined by Lemma 4.6. By continuity of $c \mapsto m_c$ we also know that for sufficiently large n we have

$$m_{\mathcal{M}(\psi_n(0))} - 2\delta \le m_{\mathcal{M}_0} - \delta. \tag{4.75}$$

Using (4.66) we deduce that for any $\varepsilon > 0$ there exists some large N_2 such that for all $n > N_2$ we have

$$\mathcal{M}(T_n^j P_n^j \phi^j) \le \mathcal{M}_0 + \varepsilon.$$

From the continuity and monotonicity of $c \mapsto m_c$ and Lemma 4.6, we may choose some sufficiently small ε to see that

$$\widetilde{m}_{\mathcal{M}(T_n^j P_n^j \phi^j)} = m_{\mathcal{M}(T_n^j P_n^j \phi^j)} \ge m_{\mathcal{M}_0 + \varepsilon} \ge m_{\mathcal{M}_0} - \frac{\delta}{2}. \tag{4.76}$$

Now (4.74), (4.75) and (4.76) yield a contradiction. Thus (4.73) holds, which combining with Lemma 4.2 also yields (4.72). Similarly, for each $1 \le k \le K^*$ we deduce

$$\mathcal{H}(w_n^k) > 0, \tag{4.77}$$
$$\mathcal{K}(w_n^k) > 0 \tag{4.78}$$

for sufficiently large n. Now using (4.66)–(4.69) we have for any $1 \le k \le K^*$,

$$\mathcal{M}_0 = \mathcal{M}(\psi_n(0)) + o_n(1) = \sum_{j=1}^{k} \mathcal{M}(S_n^j \phi^j) + \mathcal{M}(w_n^k) + o_n(1), \tag{4.79}$$

$$\mathcal{H}_0 = \mathcal{H}(\psi_n(0)) + o_n(1) = \sum_{j=1}^{k} \mathcal{H}(S_n^j \phi^j) + \mathcal{H}(w_n^k) + o_n(1), \tag{4.80}$$

$$\mathcal{I}_0 = \mathcal{H}(\psi_n(0)) + o_n(1) = \sum_{j=1}^{k} \mathcal{I}(S_n^j \phi^j) + \mathcal{I}(w_n^k) + o_n(1). \tag{4.81}$$

From (4.79) it is immediate that Lemma 4.9 is applicable for solutions with initial data $T_n^j P_n^j \phi^j$ for all sufficiently large n in the case $\lambda_\infty^j = \infty$. We will show that Lemma 4.10 is applicable for solutions with initial data $T_n^j P_n^j \phi^j$ for all sufficiently large n in the case $\lambda_\infty^j = 0$. From Theorem 1.3, and Lemmas 4.6 and 4.8 we know that there exists some $\varepsilon > 0$ such that

$$\mathcal{M}(u_0) \leq \mathcal{M}(Q) - 2\varepsilon, \quad \mathcal{H}_0 \leq \mathcal{H}^*(W) - 2\varepsilon, \quad \mathcal{I}_0 \leq \mathcal{H}^*(W) - 2\varepsilon, \tag{4.82}$$

where $\mathcal{H}^*(W)$ is the quantity defined by (1.6). Since $\|T_n^j P_n^j \phi^j\|_2 \to 0$, by interpolation we have

$$\mathcal{H}(T_n^j P_n^j \phi^j) - \mathcal{H}^*(T_n^j P_n^j \phi^j) \to 0,$$

which implies

$$\mathcal{H}^*(T_n^j P_n^j \phi^j) \leq \mathcal{H}_0 + \varepsilon \leq \mathcal{H}^*(W) - \varepsilon$$

for all sufficiently large n. Similarly,

$$\begin{aligned}
\|T_n^j P_n^j \phi^j\|_{\dot{H}^1} &= 2\mathcal{H}^*(T_n^j P_n^j \phi^j) + \frac{d-2}{d}\mathcal{I}(T_n^j P_n^j \phi^j) \\
&\leq 2(\mathcal{H}_0 + \varepsilon) + \frac{d-2}{d}(\mathcal{I}_0 + \varepsilon) \\
&\leq 2(\mathcal{H}^*(W) - \varepsilon) + \frac{d-2}{d}(\mathcal{H}^*(W) - \varepsilon) = \|W\|_{\dot{H}^1} - \Big(3 - \frac{2}{d}\Big)\varepsilon
\end{aligned}$$

for all sufficiently large n. This completes the proof of Step 1.

Step 2: Existence of at least one bad profile. First we claim that there exists some $1 \leq J \leq K^*$ such that for all $j \geq J+1$ and all sufficiently large n, v_n^j is global and

$$\sup_{J+1 \leq j \leq K^*} \lim_{n \to \infty} \|v_n^j\|_{W_{2_*} \cap W_{2^*}(\mathbb{R})} \lesssim 1. \tag{4.83}$$

Indeed, using (4.66) we infer that

$$\lim_{k \to K^*} \lim_{n \to \infty} \sum_{j=1}^{k} \|T_n^j P_n^j \phi^j\|_{H^1}^2 < \infty. \tag{4.84}$$

Then (4.83) follows from Lemma 2.1. In the same manner, by Lemma 2.1 we infer that

$$\sup_{J+1 \leq k \leq K^*} \lim_{n \to \infty} \sum_{j=J+1}^{k} \|\langle\nabla\rangle v_n^j\|_{S(\mathbb{R})}^2 \lesssim 1. \tag{4.85}$$

We now claim that there exists some $1 \leq J_0 \leq J$ such that

$$\limsup_{n \to \infty} \|v_n^{J_0}\|_{W_{2_*} \cap W_{2^*}(\mathbb{R})} = \infty. \tag{4.86}$$

We argue by contradiction and assume that

$$\limsup_{n\to\infty} \|v_n^j\|_{W_{2*}\cap W_{2^*}(\mathbb{R})} < \infty \quad \forall 1 \le j \le J. \tag{4.87}$$

Combining with (4.85), Lemma 2.3 and (4.93) (to be independently proved in Step 2a below) we deduce

$$\sup_{1\le k\le K^*} \lim_{n\to\infty} \Big\| \sum_{j=1}^{k} \langle\nabla\rangle v_n^j \Big\|_{S(\mathbb{R})} \lesssim 1. \tag{4.88}$$

Therefore, using (4.66), (4.70) and Strichartz we confirm that conditions (2.7)–(2.9) are satisfied for sufficiently large k and n, where we set $u = \psi_n$ and $w = \Psi_n^k$ therein. Once we can show that (2.10) is satisfied, we may apply Lemma 2.4 to obtain the contradiction

$$\limsup_{n\to\infty} \|\psi_n\|_{W_{2*}\cap W_{2^*}(\mathbb{R})} < \infty. \tag{4.89}$$

It is readily seen that

$$\begin{aligned}
e &= i\partial_t \Psi_n^k + \Delta\Psi_n^k + |\Psi_n^k|^{\frac{4}{d}}\Psi_n^k + |\Psi_n^k|^{\frac{4}{d-2}}\Psi_n^k \\
&= \Big(\sum_{j=1}^{k} (i\partial_t v_n^j + \Delta v_n^j) + \Big|\sum_{j=1}^{k} v_n^j\Big|^{\frac{4}{d}} \sum_{j=1}^{k} v_n^j + \Big|\sum_{j=1}^{k} v_n^j\Big|^{\frac{4}{d-2}} \sum_{j=1}^{k} v_n^j \Big) \\
&\quad + \big(|\Psi_n^k|^{\frac{4}{d}}\Psi_n^k - |\Psi_n^k - e^{it\Delta}w_n^k|^{\frac{4}{d}}(\Psi_n^k - e^{it\Delta}w_n^k) \big) \\
&\quad + \big(|\Psi_n^k|^{\frac{4}{d-2}}\Psi_n^k - |\Psi_n^k - e^{it\Delta}w_n^k|^{\frac{4}{d-2}}(\Psi_n^k - e^{it\Delta}w_n^k) \big) \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{4.90}$$

In the following we show the asymptotic smallness of I_1, I_2, I_3.

Step 2a: Smallness of I_1. We first show

$$\lim_{k\to K^*} \lim_{n\to\infty} \|\langle\nabla\rangle I_1\|_{L_{t,x}^{\frac{2(d+2)}{d+2}}} = 0. \tag{4.91}$$

Since v_n^j solves (4.1), we can rewrite I_1 as

$$\begin{aligned}
I_1 &= \sum_{j=1}^{k} \big(-|v_n^j|^{\frac{4}{d}} v_n^j - |v_n^j|^{\frac{4}{d-2}} v_n^j\big) + \Big|\sum_{j=1}^{k} v_n^j\Big|^{\frac{4}{d}} \sum_{j=1}^{k} v_n^j - \Big|\sum_{j=1}^{k} v_n^j\Big|^{\frac{4}{d-2}} \sum_{j=1}^{k} v_n^j \\
&= -\Big(\sum_{j=1}^{k} |v_n^j|^{\frac{4}{d}} v_n^j - \Big|\sum_{j=1}^{k} v_n^j\Big|^{\frac{4}{d}} \sum_{j=1}^{k} v_n^j \Big) \\
&\quad - \Big(\sum_{j=1}^{k} |v_n^j|^{\frac{4}{d-2}} v_n^j - \Big|\sum_{j=1}^{k} v_n^j\Big|^{\frac{4}{d-2}} \sum_{j=1}^{k} v_n^j \Big).
\end{aligned}$$

By Hölder and (1.21) we obtain for $s \in \{0,1\}$ that

$$
\begin{aligned}
&\| |\nabla|^s I_1 \|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\
&\lesssim_k \begin{cases}
\sum_{j \neq j'} \Big(\|v_n^j |\nabla|^s v_n^{j'}\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} (\|v_n^j\|_{W_{2*}(\mathbb{R})}^{\frac{4}{d}-1} + \|v_n^{j'}\|_{W_{2*}(\mathbb{R})}^{\frac{4}{d}-1}) \\
\quad + \|v_n^j |\nabla|^s v_n^{j'}\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})} (\|v_n^j\|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}-1} + \|v_n^{j'}\|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}-1}) \Big) & \text{if } d = 3, \\
\sum_{j \neq j'} \Big(\|v_n^j |\nabla|^s v_n^{j'}\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})}^{\frac{4}{d}} \| |\nabla|^s v_n^{j'}\|_{W_{2*}(\mathbb{R})}^{1-\frac{4}{d}} \\
\quad + \|v_n^j |\nabla|^s v_n^{j'}\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})} (\|v_n^j\|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}-1} + \|v_n^{j'}\|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}-1}) \Big) & \text{if } d \in \{4,5\}, \\
\sum_{j \neq j'} \Big(\|v_n^j |\nabla|^s v_n^{j'}\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})}^{\frac{4}{d}} \| |\nabla|^s v_n^{j'}\|_{W_{2*}(\mathbb{R})}^{1-\frac{4}{d}} \\
\quad + \|v_n^j |\nabla|^s v_n^{j'}\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})}^{\frac{4}{d-2}} \| |\nabla|^s v_n^{j'}\|_{W_{2^*}(\mathbb{R})}^{1-\frac{4}{d-2}} \Big) & \text{if } d \geq 6.
\end{cases}
\end{aligned}
\tag{4.92}
$$

In view of (4.83) and (4.87) and for the purpose of closing the proof of (4.88), we only need to show that for any fixed $1 \leq i, j \leq K^*$ with $i \neq j$ and any $s \in \{0,1\}$,

$$
\lim_{n \to \infty} \Big(\|v_n^i |\nabla|^s v_n^j\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} + \|v_n^i |\nabla|^s v_n^j\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})} + \|\nabla v_n^i \nabla v_n^j\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} \Big) = 0. \tag{4.93}
$$

First consider the term $\|v_n^i v_n^j\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R}^d)}$. Notice that it suffices to consider the case $\lambda_\infty^i, \lambda_\infty^j \in \{1, \infty\}$. Indeed, using (4.53) (which is applicable due to Step 1) and Hölder we already conclude that

$$
\|v_n^i v_n^j\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} \lesssim \|v_n^i\|_{W_{2*}(\mathbb{R})} \|v_n^j\|_{W_{2*}(\mathbb{R})} \to 0 \tag{4.94}
$$

when λ_∞^i or λ_∞^j is equal to zero. Next we claim that for any $\beta > 0$ there exists some $\psi_\beta^i, \psi_\beta^j \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ such that

$$
\Big\| v_n^i - (\lambda_n^i)^{-\frac{d}{2}} e^{-it|\xi_n^i|^2} e^{i\xi_n^i \cdot x} \psi_\beta^i \Big(\frac{t}{(\lambda_n^i)^2} + t_n^i, \frac{x - x_n^i - 2t\xi_n^i}{\lambda_n^i} \Big) \Big\|_{W_{2*}(\mathbb{R})} \leq \beta, \tag{4.95}
$$

$$
\Big\| v_n^j - (\lambda_n^j)^{-\frac{d}{2}} e^{-it|\xi_n^j|^2} e^{i\xi_n^j \cdot x} \psi_\beta^j \Big(\frac{t}{(\lambda_n^j)^2} + t_n^j, \frac{x - x_n^j - 2t\xi_n^j}{\lambda_n^j} \Big) \Big\|_{W_{2*}(\mathbb{R})} \leq \beta. \tag{4.96}
$$

Indeed, for $\lambda_\infty^i, \lambda_\infty^j = \infty$, this follows already from (4.41), while for $\lambda_\infty^i, \lambda_\infty^j = 1$ we choose some $\psi_\beta^i, \psi_\beta^j \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ such that

$$
\|v^i - \psi_\beta^i\|_{W_{2*}(\mathbb{R})} \leq \beta, \quad \|v^j - \psi_\beta^j\|_{W_{2*}(\mathbb{R})} \leq \beta \tag{4.97}
$$

and the claim follows. Define

$$\Lambda_n(\psi_\beta^i) := (\lambda_n^i)^{-\frac{d}{2}} \psi_\beta^i\Big(\frac{t}{(\lambda_n^i)^2} + t_n^i, \frac{x - x_n^i - 2t\xi_n^i}{\lambda_n^i}\Big).$$

Using Hölder we infer that

$$\|v_n^i v_n^j\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R}^d)} \lesssim \beta + \|\Lambda_n(\psi_\beta^i)\Lambda_n(\psi_\beta^j)\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R}^d)}.$$

Since β can be chosen arbitrarily small, it suffices to show

$$\lim_{n\to\infty} \|\Lambda_n(\psi_\beta^i)\Lambda_n(\psi_\beta^j)\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R}^d)} = 0. \tag{4.98}$$

Assume that $\frac{\lambda_n^i}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^i} \to \infty$. By symmetry we may w.l.o.g. assume that $\frac{\lambda_n^i}{\lambda_n^j} \to 0$. Using a change of variables we obtain

$$\begin{aligned}
&\|\Lambda_n(\psi_\beta^i)\Lambda_n(\psi_\beta^j)\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R}^d)} \\
&\quad = \Big(\frac{\lambda_n^i}{\lambda_n^j}\Big)^{\frac{d}{2}} \Big\| \psi_\beta^i(t,x)\psi_\beta^j\Big(\Big(\frac{\lambda_n^i}{\lambda_n^j}\Big)^2 t - \Big(\Big(\frac{\lambda_n^i}{\lambda_n^j}\Big)^2 t_n^i - t_n^j\Big), \\
&\qquad\qquad \Big(\frac{\lambda_n^i}{\lambda_n^j}\Big)x + 2\Big(\frac{\lambda_n^i}{\lambda_n^j}\Big)\lambda_n^i(\xi_n^i - \xi_n^j)t \\
&\qquad\qquad + \frac{x_n^i - x_n^j - 2t_n^i(\lambda_n^i)^2(\xi_n^i - \xi_n^j)}{\lambda_n^j}\Big)\Big\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R}^d)} \\
&\quad \lesssim \Big(\frac{\lambda_n^i}{\lambda_n^j}\Big)^{\frac{d}{2}} \|\psi_\beta^i\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R}^d)} \|\psi_\beta^j\|_{L_{t,x}^\infty(\mathbb{R}^d)} \to 0.
\end{aligned} \tag{4.99}$$

Suppose therefore $\frac{\lambda_n^i}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^i} \to \lambda_0 \in (0,\infty)$. If $(\frac{\lambda_n^i}{\lambda_n^j})^2 t_n^i - t_n^j \to \pm\infty$, then by (4.99) the supports of the integrands become disjoint in the temporal direction.

We may therefore further assume that $(\frac{\lambda_n^i}{\lambda_n^j})^2 t_n^i - t_n^j \to t_0 \in \mathbb{R}$.

- If $|\frac{x_n^i - x_n^j - 2t_n^i(\lambda_n^i)^2(\xi_n^i - \xi_n^j)}{\lambda_n^j}| \to \infty$ and $\xi_n^i = \xi_n^j$ for infinitely many n, then the supports of the integrands become disjoint in the spatial direction.
- If $|\frac{x_n^i - x_n^j - 2t_n^i(\lambda_n^i)^2(\xi_n^i - \xi_n^j)}{\lambda_n^j}| \to \infty$ and $\xi_n^i \neq \xi_n^j$ for infinitely many n, then we apply the change of temporal variable $t \mapsto \frac{t}{\lambda_n^i|\xi_n^i - \xi_n^j|}$ to see the decoupling of the supports of the integrands in the spatial direction.
- Finally, if $\frac{x_n^i - x_n^j - 2t_n^i(\lambda_n^i)^2(\xi_n^i - \xi_n^j)}{\lambda_n^j} \to x_0 \in \mathbb{R}^d$, then by (4.65) we must have that $\lambda_n^i|\xi_n^i - \xi_n^j| \to \infty$. Hence for all $t \neq 0$ the integrand converges pointwise to zero. Using the dominated convergence theorem (setting $\|\psi_\beta^j\|_{L_{t,x}^\infty(\mathbb{R})}\psi_\beta^i$ as the majorant) we finally conclude (4.98).

We now consider the remaining terms:

- For $\|v_n^i \nabla v_n^j\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})}$, arguing similarly to (4.94) and using (4.53) we know that $\lambda_\infty^i \in \{1, \infty\}$. For ∇v_n^j, we use (4.42) or (4.55) as a proxy for ∇v_n^j, depending on the value of λ_∞^j.
- For $\|v_n^i v_n^j\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})}$, we first obtain

$$\|v_n^i v_n^j\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})} \leq \min\Big\{\|v_n^i\|_{W_{2^*}(\mathbb{R})}^{\frac{d-2}{d-1}} \|v_n^j\|_{W_{2_*}(\mathbb{R})}^{\frac{1}{d-1}}, \|v_n^j\|_{W_{2^*}(\mathbb{R})}^{\frac{d-2}{d-1}} \|v_n^i\|_{W_{2_*}(\mathbb{R})}^{\frac{1}{d-1}}\Big\}.$$

  Therefore, using (4.40) and (4.53) we can reduce the analysis to the case $\lambda_\infty^i, \lambda_\infty^j = 1$.
- For $\big\|v_n^i \nabla v_n^j\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})}$ we can reduce our analysis to the case $\lambda_\infty^i \in \{0, 1\}$ and use (4.42) or (4.55) as proxy for ∇v_n^j and (4.54) for v_n^i.
- For $\|\nabla v_n^i \nabla v_n^j\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})}$ we use (4.42) or (4.55) as proxies for both ∇v_n^i and ∇v_n^j.

Combining also with the boundedness of $(\xi_n^j)_n$, we can proceed as before to conclude (4.93). We omit the details of the repeating arguments. This completes the proof of Step 2a.

Step 2b: Smallness of I_2 and I_3. We establish in this substep the asymptotic smallness of I_2 and I_3. Using Hölder and (1.21) we obtain the following:

- For $d = 3$,

$$\begin{aligned}
&\| |\nabla|^s (I_2 + I_3)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\
&\lesssim_k \|\Psi_n^k |\nabla|^s e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} (\|\Psi_n^k\|_{W_{2_*}(\mathbb{R})}^{\frac{4}{d}-1} + \|e^{it\Delta} w_n^k\|_{W_{2_*}(\mathbb{R})}^{\frac{4}{d}-1}) \\
&\quad + \| |\nabla|^s \Psi_n^k e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} (\|\Psi_n^k\|_{W_{2_*}(\mathbb{R})}^{\frac{4}{d}-1} + \|e^{it\Delta} w_n^k\|_{W_{2_*}(\mathbb{R})}^{\frac{4}{d}-1}) \\
&\quad + \|\Psi_n^k |\nabla|^s e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})} (\|\Psi_n^k\|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}-1} + \|e^{it\Delta} w_n^k\|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}-1}) \\
&\quad + \| |\nabla|^s \Psi_n^k e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})} (\|\Psi_n^k\|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}-1} + \|e^{it\Delta} w_n^k\|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}-1}) \\
&\quad + \|e^{it\Delta} w_n^k\|_{W_{2_*}(\mathbb{R})}^{\frac{4}{d}} \| |\nabla|^s e^{it\Delta} w_n^k\|_{W_{2_*}(\mathbb{R})} \\
&\quad + \|e^{it\Delta} w_n^k\|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}} \| |\nabla|^s e^{it\Delta} w_n^k\|_{W_{2_*}(\mathbb{R})}.
\end{aligned}$$

- For $d \in \{4, 5\}$,

$$\begin{aligned}
&\| |\nabla|^s (I_2 + I_3)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\
&\lesssim_k \|\Psi_n^k |\nabla|^s e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})}^{\frac{4}{d}} \| |\nabla|^s e^{it\Delta} w_n^k\|_{W_{2_*}(\mathbb{R})}^{1-\frac{4}{d}}
\end{aligned}$$

$$+\, \| \, |\nabla|^s \Psi_n^k e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})}^{\frac{4}{d}} \| \, |\nabla|^s \Psi_n^k \|_{W_{2*}(\mathbb{R})}^{1-\frac{4}{d}}$$
$$+\, \| \Psi_n^k |\nabla|^s e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})} (\| \Psi_n^k \|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}-1} + \| e^{it\Delta} w_n^k \|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}-1})$$
$$+\, \| \, |\nabla|^s \Psi_n^k e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})} (\| \Psi_n^k \|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}-1} + \| e^{it\Delta} w_n^k \|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}-1})$$
$$+\, \| e^{it\Delta} w_n^k \|_{W_{2*}(\mathbb{R})}^{\frac{4}{d}} \| \, |\nabla|^s e^{it\Delta} w_n^k \|_{W_{2*}(\mathbb{R})}$$
$$+\, \| e^{it\Delta} w_n^k \|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}} \| \, |\nabla|^s e^{it\Delta} w_n^k \|_{W_{2*}(\mathbb{R})}.$$

- For $d \geq 6$,

$$\begin{aligned}
\| \, |\nabla|^s (I_2 + I_3) \|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \lesssim_k\ & \| \Psi_n^k |\nabla|^s e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})}^{\frac{4}{d}} \| \, |\nabla|^s e^{it\Delta} w_n^k \|_{W_{2*}(\mathbb{R})}^{1-\frac{4}{d}} \\
& + \| \, |\nabla|^s \Psi_n^k e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})}^{\frac{4}{d}} \| \, |\nabla|^s \Psi_n^k \|_{W_{2*}(\mathbb{R})}^{1-\frac{4}{d}} \\
& + \| \Psi_n^k |\nabla|^s e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})}^{\frac{4}{d-2}} \| \, |\nabla|^s e^{it\Delta} w_n^k \|_{W_{2*}(\mathbb{R})}^{1-\frac{4}{d-2}} \\
& + \| \, |\nabla|^s \Psi_n^k e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})}^{\frac{4}{d-2}} \| \, |\nabla|^s \Psi_n^k \|_{W_{2*}(\mathbb{R})}^{1-\frac{4}{d-2}} \\
& + \| e^{it\Delta} w_n^k \|_{W_{2*}(\mathbb{R})}^{\frac{4}{d}} \| \, |\nabla|^s e^{it\Delta} w_n^k \|_{W_{2*}(\mathbb{R})} \\
& + \| e^{it\Delta} w_n^k \|_{W_{2^*}(\mathbb{R})}^{\frac{4}{d-2}} \| \, |\nabla|^s e^{it\Delta} w_n^k \|_{W_{2*}(\mathbb{R})}.
\end{aligned}$$

In view of (4.64), (4.66), Strichartz and (4.88) it suffices to show that for $s \in \{0, 1\}$,

$$\begin{aligned}
\lim_{k\to K^*} \lim_{n\to\infty} \Big(& \| \Psi_n^k |\nabla|^s e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} + \| \, |\nabla|^s \Psi_n^k e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} \\
& + \| \Psi_n^k |\nabla|^s e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})} + \| \, |\nabla|^s \Psi_n^k e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})} \Big) = 0. \quad (4.100)
\end{aligned}$$

For $\| \, |\nabla|^s \Psi_n^k e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})}$ and $\| \, |\nabla|^s \Psi_n^k e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})}$, using Hölder, (4.88), Strichartz, (4.66) and (4.64) we have

$$\begin{aligned}
& \lim_{k\to K^*} \lim_{n\to\infty} (\| \, |\nabla|^s \Psi_n^k e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} + \| \, |\nabla|^s \Psi_n^k e^{it\Delta} w_n^k \|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})}) \\
& \quad \lesssim \lim_{k\to K^*} \lim_{n\to\infty} \Big(\| \, |\nabla|^s \Big(\sum_{j=1}^k v_n^k \Big) \|_{W_{2*}(\mathbb{R})} \| e^{it\Delta} w_n^k \|_{W_{2*}(\mathbb{R})} \\
& \qquad + \| \, |\nabla|^s \Big(\sum_{j=1}^k v_n^k \Big) \|_{W_{2*}(\mathbb{R})}^{\frac{1}{d-2}} \| e^{it\Delta} w_n^k \|_{W_{2^*}(\mathbb{R})}^{\frac{d-2}{d-1}}
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \| |\nabla|^s e^{it\Delta} w_n^k \|_{W_{2*}(\mathbb{R})} \| e^{it\Delta} w_n^k \|_{W_{2*}(\mathbb{R})} \\
&\qquad + \| |\nabla|^s e^{it\Delta} w_n^k \|^{\frac{1}{d-2}}_{W_{2*}(\mathbb{R})} \| e^{it\Delta} w_n^k \|^{\frac{d-2}{d-1}}_{W_{2^*}(\mathbb{R})} \Big) \\
&\lesssim \lim_{k\to K^*} \lim_{n\to\infty} \big((1 + \|w_n^k\|_{H^1} + \|w_n^k\|^{\frac{1}{d-2}}_{H^1}) (\|e^{it\Delta} w_n^k\|_{W_{2*}(\mathbb{R})} + \|e^{it\Delta} w_n^k\|^{\frac{d-2}{d-1}}_{W_{2^*}(\mathbb{R})}) \big) \\
&= 0. \qquad (4.101)
\end{aligned}
$$

It is left to estimate $\|\Psi_n^k \nabla e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})}$ and $\|\Psi_n^k \nabla e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})}$. By (4.85), Hölder, Strichartz and (4.66) we know that for each $\eta > 0$ there exists some $1 \leq J' = J'(\eta) \leq K^*$ such that

$$
\begin{aligned}
\sup_{J'\leq k\leq K^*} \lim_{n\to\infty} \Bigg(&\Big\| \Big(\sum_{j=J'}^{k} v_n^j \Big) \nabla e^{it\Delta} w_n^k \Big\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} \\
&+ \Big\| \Big(\sum_{j=J'}^{k} v_n^j \Big) \nabla e^{it\Delta} w_n^k \Big\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})} \Bigg) \lesssim \eta. \qquad (4.102)
\end{aligned}
$$

Hence, it suffices to show that

$$
\lim_{k\to K^*} \lim_{n\to\infty} (\|v_n^j \nabla e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} + \|v_n^j \nabla e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})}) = 0 \qquad (4.103)
$$

for any $1 \leq j < J'$. For $\|v_n^j \nabla e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})}$, using (4.53) we may further assume that $\lambda_\infty^j \in \{1, \infty\}$. For $\beta > 0$, let $\phi_\beta \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ be given according to (4.41). Let $T, R > 0$ such that $\operatorname{supp} \phi_\beta \subset [-T, T] \times \{|x| \leq R\}$. Then using Hölder we infer that

$$
\|v_n^j \nabla e^{it\Delta} w_n^k\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} \lesssim \beta \|\nabla e^{it\Delta} w_n^k\|_{W_{2*}(\mathbb{R})} + \Lambda, \qquad (4.104)
$$

where

$$
\begin{aligned}
\Lambda := &\Big\| \phi_\beta(t,x) \Big((\lambda_n^j)^{\frac{d}{2}} [e^{it\Delta} \nabla w_n^k] \big((\lambda_n^j)^2 t - (\lambda_n^j)^2 t_n^j, \\
&\qquad \lambda_n^j x + 2\xi_n^j (\lambda_n^j)^2 t + x_n^j - 2\xi_n^j (\lambda_n^j)^2 t_n^j \big) \Big) \Big\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} \\
= &\Big\| \phi_\beta(t,x) G_n^j \big([e^{it\Delta} \nabla w_n^k](t, x + 2\xi_n^j t) \big) \Big\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} \qquad (4.105)
\end{aligned}
$$

and

$$
G_n^j u(t,x) := (\lambda_n^j)^{\frac{d}{2}} u((\lambda_n^j)^2 (t - t_n^j), \lambda_n^j x + x_n^j).
$$

By the arbitrariness of β it suffices to show the asymptotic smallness of Λ. Using the invariance of the NLS flow under Galilean transformation we know that

$$
\begin{aligned}
&[e^{it\Delta} \nabla w_n^k](t, x + 2\xi_n^j t) \\
&\quad = e^{i\xi_n^j \cdot x} e^{it|\xi_n^j|^2} \big[e^{it\Delta} [e^{-i\xi_n^j \cdot x} \nabla w_n^k] \big](t,x)
\end{aligned}
$$

$$
\begin{aligned}
&= e^{i\xi_n^j\cdot x}e^{it|\xi_n^j|^2}\big[e^{it\Delta}[\nabla(e^{-i\xi_n^j\cdot x}w_n^k)]\big](t,x)\\
&\quad + i\xi_n^j e^{i\xi_n^j\cdot x}e^{it|\xi_n^j|^2}\big[e^{it\Delta}[e^{-i\xi_n^j\cdot x}w_n^k]\big](t,x)\\
&= e^{i\xi_n^j\cdot x}e^{it|\xi_n^j|^2}\big[\nabla[e^{it\Delta}[e^{-i\xi_n^j\cdot x}w_n^k]]\big](t,x) + i\xi_n^j[e^{it\Delta}w_n^k](t,x+2\xi_n^j t)\\
&=: e^{i\xi_n^j\cdot x}e^{it|\xi_n^j|^2}\Lambda_1 + \Lambda_2.
\end{aligned}\tag{4.106}
$$

Using Hölder, (4.64) and the boundedness of $(\xi_n^j)_n$ we infer that

$$
\begin{aligned}
\|\phi_\beta G_n^j(\Lambda_2)\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} &\lesssim \|\phi_\beta\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R})}\|\Lambda_2\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R})}\\
&= |\xi_n^j|\,\|\phi_\beta\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R})}\|e^{it\Delta}w_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R})} = o_n(1).
\end{aligned}\tag{4.107}
$$

Finally, using Hölder, a change of variables, (1.22) and the boundedness of $(\xi_n^j)_n$ we obtain

$$
\begin{aligned}
\|\phi_\beta G_n^j(e^{i\xi_n^j\cdot x}e^{it|\xi_n^j|^2}\Lambda_2)\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})} &\le C(T,R)\|G_n^j(\Lambda_2)\|_{L^2_{t,x}([-T,T]\times\{|x|\le R\})}\\
&\le C(T,R)\|e^{it\Delta}w_n^k\|^{\frac13}_{W_{2^*}(\mathbb{R})}\|e^{-i\xi_n^j\cdot x}w_n^k\|^{\frac23}_{\dot H^1}\\
&\le C(T,R,\sup_n|\xi_n^j|)\|e^{it\Delta}w_n^k\|^{\frac13}_{W_{2^*}(\mathbb{R})}\|w_n^k\|^{\frac23}_{\dot H^1}.
\end{aligned}\tag{4.108}
$$

The claim then follows by invoking (4.64) and (4.66). For $d\ge 4$, $\|v_n^j\nabla e^{it\Delta}w_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R})}$ can be estimated similarly to $\|v_n^j\nabla e^{it\Delta}w_n^k\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R})}$. In this case we can further assume that $\lambda_\infty^j\in\{0,1\}$ and $\xi_n^j\equiv 0$ (which also holds for $d=3$) and the proof is in fact much easier; we therefore omit the details here. For $d=3$, we notice that $\frac{d+2}{d-1}>2$ and hence we should use the interpolation estimate

$$
\|\phi_\beta\nabla\widetilde{w}_n^k\|_{L_{t,x}^{\frac52}(\mathbb{R})} \lesssim C(T,R)\|\nabla\widetilde{w}_n^k\|^{\frac12}_{W_{2_*}}\|\nabla\widetilde{w}_n^k\|^{\frac12}_{L^2_{t,x}([-T,T]\times\{|x|\le R\})}\tag{4.109}
$$

in order to apply the local smoothing estimate (1.22), where ϕ_β is deduced from (4.54) and $\widetilde{w}_n^k := \lambda_n^j G_n^j w_n^k$. This completes the proof of Step 2b and thus also the desired proof of Step 2.

Step 3: Reduction to one bad profile and conclusion. From Step 2 we conclude that there exists some $1\le J_1\le K^*$ such that

$$
\limsup_{n\to\infty}\|v_n^j\|_{W_{2_*}\cap W_{2^*}(\mathbb{R})}=\infty\quad \forall 1\le j\le J_1,\tag{4.110}
$$

$$
\limsup_{n\to\infty}\|v_n^j\|_{W_{2_*}\cap W_{2^*}(\mathbb{R})}<\infty\quad \forall J_1+1\le j\le K^*.\tag{4.111}
$$

Let $j \in \{1, \dots, J_1\}$. If $\lambda_\infty^j = \infty$, then by Lemma 3.6 we know that $v_n^j(0)$ takes the form (4.38). Consequently, using Lemma 4.9 we infer that

$$\limsup_{n\to\infty} \|v_n^j\|_{W_{2_*}\cap W_{2^*}(\mathbb{R})} < \infty, \tag{4.112}$$

which contradicts (4.110). In the same manner, we exclude the case $\lambda_\infty^j = 0$ using Lemma 4.10 and by Lemma 3.6 we conclude $\lambda_n^j \equiv \lambda_\infty^j = 1$. If $J_1 > 1$, then using (4.79), (4.80) and Lemma 4.8 (iv) we know that $\limsup_{n\to\infty} \mathcal{D}^*(v_n^j) < \mathcal{D}^*$ for any $1 \leq j \leq J_1$. By the inductive hypothesis we arrive at the contradiction (4.112) again and we deduce $J_1 = 1$. Note also that from Lemma 3.6 we know that in the case $\lambda_\infty^1 = 1$ one has $\xi_n^1 \equiv 0$. Therefore, by applying the linear profile expansion (4.60) at step $k = 1$ we obtain

$$\psi_n(0, x) = e^{it_n^1\Delta}\phi^1(x - x_n^1) + w_n^1(x).$$

In particular, by Lemma 3.6 we know that $\phi^1 \in H^1(\mathbb{R}^d)$. Similarly, we must have $\mathcal{M}(w_n^1) = o_n(1)$ and $\mathcal{H}(w_n^1) = o_n(1)$, otherwise we deduce the contradiction (4.89) again using Lemma 2.4. Combining with Lemma 4.8 (v) we conclude that $\|w_n^1\|_{H^1} = o_n(1)$. Finally, we exclude the case $t_n^1 \to \pm\infty$. We only consider the case $t_n^1 \to \infty$; the case $t_n^1 \to -\infty$ can be dealt with similarly. Indeed, using Strichartz we obtain

$$\|e^{it\Delta}\psi_n(0)\|_{W_{2_*}\cap W_{2^*}([0,\infty))} \lesssim \|e^{it\Delta}\phi^1\|_{W_{2_*}\cap W_{2^*}([t_n,\infty))} + \|w_n^1\|_{H^1} \to 0 \tag{4.113}$$

and using Lemma 2.1 we deduce the contradiction (4.89) again. This completes the desired proof. ∎

Lemma 4.12 (Existence of the minimal blow-up solution). *Suppose that $\mathcal{D}^* < \infty$. Then there exists a global solution u_c of (4.1) such that $\mathcal{D}(u_c) = \mathcal{D}^*$ and*

$$\|u_c\|_{W_{2_*}\cap W_{2^*}((-\infty,0])} = \|u_c\|_{W_{2_*}\cap W_{2^*}([0,\infty))} = \infty. \tag{4.114}$$

Moreover, u_c is almost periodic in $H^1(\mathbb{R}^d)$ modulo translations, i.e. the set $\{u(t) : t \in \mathbb{R}\}$ is precompact in $H^1(\mathbb{R}^d)$ modulo translations.

Proof. As discussed at the beginning of this section, under the assumption $\mathcal{D}^* < \infty$ one can find a sequence such that (4.57) and (4.58) hold. We apply Lemma 4.11 to the sequence $(\psi_n(0))_n$ to infer that $(\psi_n(0))_n$ (up to modifying time and space translation) is precompact in $H^1(\mathbb{R}^d)$. We denote its strong H^1-limit by ψ. Let u_c be the solution of (4.1) with $u_c(0) = \psi$. Then $\mathcal{D}(u_c(t)) = \mathcal{D}(\psi) = \mathcal{D}^*$ for all t in the maximal lifespan $I_{\max}$ of u_c (recall that $\mathcal{D}$ is a conserved quantity by Lemma 4.8).

We first show that u_c is a global solution. We only show that $s_0 := \sup I_{\max} = \infty$; the negative direction can be similarly proved. If this does not hold, then by Lemma 2.1 there exists a sequence $(s_n)_n \subset \mathbb{R}$ with $s_n \to s_0$ such that

$$\limsup_{n\to\infty} \|u_c\|_{W_{2_*}\cap W_{2^*}([s_n,s_0))} = \infty.$$

Define $\psi_n(t) := u_c(t + s_n)$. Then (4.71) is satisfied with $t_n \equiv 0$. We then apply Lemma 4.11 to the sequence $(\psi_n(0))_n$ to infer that there exists some $\varphi \in H^1(\mathbb{R}^d)$ such that, up to modifying the space translation, $u_c(s_n)$ strongly converges to φ in $H^1(\mathbb{R}^d)$. But then using Strichartz we obtain

$$\|e^{it\Delta}u_c(s_n)\|_{W_{2_*}\cap W_{2^*}([s_n,s_0))} = \|e^{it\Delta}\varphi\|_{W_{2_*}\cap W_{2^*}([s_n,s_0))} + o_n(1) = o_n(1).$$

By Lemma 2.1 we can extend u_c beyond s_0, which contradicts the maximality of s_0. Now by (4.57) and Lemma 2.4 it is necessary that

$$\|u_c\|_{W_{2_*}\cap W_{2^*}((-\infty,0])} = \|u_c\|_{W_{2_*}\cap W_{2^*}([0,\infty))} = \infty. \tag{4.115}$$

We finally show that the orbit $\{u_c(t) : t \in \mathbb{R}\}$ is precompact in $H^1(\mathbb{R}^d)$ modulo translations. Let $(\tau_n)_n \subset \mathbb{R}$ be an arbitrary time sequence. Then (4.115) implies

$$\|u_c\|_{W_{2_*}\cap W_{2^*}((-\infty,\tau_n])} = \|u_c\|_{W_{2_*}\cap W_{2^*}([\tau_n,\infty))} = \infty.$$

The claim follows by applying Lemma 4.11 to $(u_c(\tau_n))_n$. ∎

4.4. Extinction of the minimal blow-up solution

The following lemma is an immediate consequence of the fact that u_c is almost periodic in $H^1(\mathbb{R}^d)$ and conservation of momentum. The proof is standard; we refer to [23,29] for details of the proof.

Lemma 4.13. *Let u_c be the minimal blow-up solution given by Lemma 4.12. Then there exists some function $x\colon \mathbb{R} \to \mathbb{R}^d$ such that we have the following statements:*

(i) *For each $\varepsilon > 0$, there exists $R > 0$ so that*

$$\int_{|x+x(t)|\geq R} |\nabla u_c(t)|^2 + |u_c(t)|^2 + |u_c|^{2_*} + |u_c|^{2^*}\,dx \leq \varepsilon \quad \forall t \in \mathbb{R}. \tag{4.116}$$

(ii) *The center function $x(t)$ obeys the decay condition $x(t) = o(t)$ as $|t| \to \infty$.*

Proof of Theorem 1.6 for the focusing–focusing regime. We will show the contradiction that the minimal blow-up solution u_c given by Lemma 4.12 is equal to zero, which will finally imply Theorem 1.6 for the focusing–focusing case. Let χ be a smooth radial cutoff function satisfying

$$\chi = \begin{cases} |x|^2 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

Also define the local virial function

$$z_R(t) := \int R^2\chi\Big(\frac{x}{R}\Big)|u_c(t,x)|^2\,dx.$$

Direct calculation yields

$$\partial_t z_R(t) = 2\,\mathrm{Im} \int R\nabla\chi\Big(\frac{x}{R}\Big)\cdot\nabla u_c(t)\bar{u}_c(t)\,dx, \tag{4.117}$$

$$\begin{aligned}\partial_{tt} z_R(t) = 4\int \partial^2_{jk}\chi\Big(\frac{x}{R}\Big)\partial_j u_c\partial_k\bar{u}_c - \frac{1}{R^2}\int \Delta^2\chi\Big(\frac{x}{R}\Big)|u_c|^2 \\ - \frac{4}{d+2}\int \Delta\chi\Big(\frac{x}{R}\Big)|u_c|^{2_*}\,dx - \frac{4}{d}\int \Delta\chi\Big(\frac{x}{R}\Big)|u_c|^{2^*}\,dx.\end{aligned} \tag{4.118}$$

We then obtain

$$\partial_{tt} z_R(t) = 8\mathcal{K}(u_c) + A_R(u_c(t)), \tag{4.119}$$

where

$$\begin{aligned}A_R(u_c(t)) = 4\int \Big(\partial_{jj}\chi\Big(\frac{x}{R}\Big) - 2\Big)|\partial_j u_c|^2 + 4\sum_{j\neq k}\int_{R\le|x|\le 2R}\partial_{jk}\chi\Big(\frac{x}{R}\Big)\partial_j u\partial_k\bar{u}_c \\ - \frac{1}{R^2}\int \Delta^2\chi\Big(\frac{x}{R}\Big)|u_c|^2 - \frac{4}{d+2}\int\Big(\Delta\chi\Big(\frac{x}{R}\Big) - 2d\Big)|u_c|^{2_*}\,dx \\ - \frac{4}{d}\int\Big(\Delta\chi\Big(\frac{x}{R}\Big) - 2d\Big)|u_c|^{2^*}\,dx.\end{aligned}$$

We thus infer the estimate

$$|A_R(u(t))| \le C_1\int_{|x|\ge R}|\nabla u(t)|^2 + \frac{1}{R^2}|u(t)|^2 + |u|^{2_*} + |u|^{2^*}$$

for some $C_1 > 0$. Assume that $\mathcal{M}(u_c) = (1-\delta)^{\frac{d}{2}}\mathcal{M}(Q)$ for some $\delta \in (0,1)$. Using (4.14) we deduce that

$$\begin{aligned}\mathcal{K}(u_c(t)) &\ge \min\Big\{\frac{4\delta}{d}\mathcal{H}(u(0)), \Big(\Big(\frac{d}{\delta(d-2)}\Big)^{\frac{d-2}{4}} - 1\Big)^{-1}(m_{\mathcal{M}(u(0))} - \mathcal{H}(u(0)))\Big\} \\ &=: \frac{\eta_1}{4}\end{aligned} \tag{4.120}$$

for all $t \in \mathbb{R}$. From Lemma 4.13 it follows that there exists some $R_0 \ge 1$ such that

$$\int_{|x+x(t)|}|\nabla u_c|^2 + |u_c|^2 + |u|^{2_*} + |u|^{2^*}\,dx \le \frac{\eta}{C_1}.$$

Thus for any $R \ge R_0 + \sup_{t\in[t_0,t_1]}|x(t)|$ with some to be determined $t_0, t_1 \in [0,\infty)$, we have

$$\partial_{tt} z_R(t) \ge \eta_1 \tag{4.121}$$

for all $t \in [t_0, t_1]$. By Lemma 4.13 we know that for any $\eta_2 > 0$ there exists some $t_0 \gg 1$ such that $|x(t)| \le \eta_2 t$ for all $t \ge t_0$. Now set $R = R_0 + \eta_2 t_1$. Integrating (4.121) over $[t_0, t_1]$ yields

$$\partial_t z_R(t_1) - \partial_t z_R(t_0) \ge \eta_1(t_1 - t_0). \tag{4.122}$$

Using (4.117), Cauchy–Schwarz and Lemma 4.8 we have

$$|\partial_t z_R(t)| \leq C_2 \mathcal{D}^* R = C_2 \mathcal{D}^* (R_0 + \eta_2 t_1) \tag{4.123}$$

for some $C_2 = C_2(\mathcal{D}^*) > 0$. Expressions (4.122) and (4.123) give us

$$2C_2 \mathcal{D}^* (R_0 + \eta_2 t_1) \geq \eta_1 (t_1 - t_0).$$

Setting $\eta_2 = \frac{\eta_1}{4C_2 \mathcal{D}^*}$, dividing both sides by t_1 and then sending t_1 to infinity we obtain $\frac{1}{2}\eta_1 \geq \eta_1$, which implies $\eta_1 = 0$ and consequently $\mathcal{H}_0 = \mathcal{H}(u_c) = 0$. From Lemma 4.8 we know that $\nabla u_c = 0$, hence $u_c = 0$. This completes the proof. ∎

5. Scattering threshold for the defocusing–focusing (DCNLS)

In this section we prove Theorem 1.6 for the defocusing–focusing model and Proposition 1.8. Throughout the section, we assume that (DCNLS) reduces to

$$i\partial_t u + \Delta u - |u|^{\frac{4}{d}} u + |u|^{\frac{4}{d-2}} u = 0. \tag{5.1}$$

We also define the set $\mathcal{A}$ by

$$\mathcal{A} := \big\{u \in H^1(\mathbb{R}^d) : \mathcal{H}(u) < \mathcal{H}^*(W),\ \mathcal{K}(u) > 0\big\}.$$

5.1. Variational formulation for m_c

This subsection is devoted to the proof of Proposition 1.8. We first record some auxiliary variational tools for (5.1) which are similar to those given in Section 4.1.

Lemma 5.1. *The following statements hold true:*

(i) *Let $u \in H^1(\mathbb{R}^d) \setminus \{0\}$. Then there exists a unique $\lambda(u) > 0$ such that*

$$\mathcal{K}(T_\lambda u) \begin{cases} > 0 & \text{if } \lambda \in (0, \lambda(u)), \\ = 0 & \text{if } \lambda = \lambda(u), \\ < 0 & \text{if } \lambda \in (\lambda(u), \infty), \end{cases}$$

where the operator T_λ is defined by (1.17).

(ii) *The mapping $c \mapsto m_c$ is continuous and monotone decreasing on $(0, \infty)$.*

(iii) *Let*

$$\tilde{m}_c := \inf_{u \in H^1(\mathbb{R}^d)} \big\{\mathcal{I}(u) : \mathcal{M}(u) = c,\ \mathcal{K}(u) \leq 0\big\},$$

where $\mathcal{I}(u)$ is defined by (1.16). Then $m_c = \tilde{m}_c$ for any $c \in (0, \infty)$.

Proof. This is a straightforward modification of Lemmas 4.1, 4.4 and 4.6; we therefore omit the details here. ∎

Lemma 5.2. *Let* $\mathcal{K}^c(u) := \|\nabla u\|_2^2 - \|u\|_{2^*}^{2^*}$ *and*

$$\widehat{m}_c := \inf_{u\in H^1(\mathbb{R}^d)} \big\{\mathcal{I}(u) : \mathcal{M}(u) = c,\ \mathcal{K}^c(u) \le 0\big\}. \tag{5.2}$$

Then $m_c = \widehat{m}_c$ *for any* $c \in (0,\infty)$.

Proof. If $\mathcal{M}(u) = c$ and $\mathcal{K}(u) = 0$, then it is clear that $\mathcal{K}^c(u) < 0$ and $\mathcal{H}(u) = \mathcal{I}(u)$, which implies $m_c \ge \widehat{m}_c$. For the converse direction, in view of Lemma 5.1 (iii) it suffices to show $\widetilde{m}_c \le \widehat{m}_c$. By Lemma 5.1 (ii) we can further define $\widetilde{m}_c$ by

$$\widetilde{m}_c = \inf_{u\in H^1(\mathbb{R}^d)} \big\{\mathcal{I}(u) : \mathcal{M}(u) \in (0,c],\ \mathcal{K}(u) \le 0\big\}. \tag{5.3}$$

Assume that $u \in H^1(\mathbb{R}^d)$ with $\mathcal{M}(u) = c$ and $\mathcal{K}^c(u) \le 0$. Then

$$\frac{d}{dt}\mathcal{K}^c(T_t u)\Big|_{t=1} = 2\mathcal{K}^c(u) - \frac{4}{d-2}\|u\|_{2^*}^{2^*} < 0. \tag{5.4}$$

Hence there exists some sufficiently small $\delta > 0$ such that $\mathcal{K}^c(T_t u) < 0$ for all $t \in (1, 1+\delta)$. In particular,

$$\mathcal{I}(T_t u) \to \mathcal{I}(u), \quad \mathcal{K}^c(T_t u) \to \mathcal{K}^c(u) \quad \text{as } t \downarrow 1.$$

We now define

$$U_\lambda u(x) := \lambda^{\frac{d-2}{2}} u(\lambda x).$$

Then $\mathcal{K}^c(U_\lambda u) = \mathcal{K}^c(u)$ and $\mathcal{I}(U_\lambda u) = \mathcal{I}(u)$ for any $\lambda > 0$. Moreover,

$$\mathcal{K}(U_\lambda u) = \mathcal{K}^c(u) + \frac{2\lambda^{-\frac{4}{d}}}{d+2}\|u\|_{2_*}^{2_*} \to \mathcal{K}^c(u), \tag{5.5}$$

$$\mathcal{M}(U_\lambda u) = \lambda^{-2}\mathcal{M}(u) \to 0 \tag{5.6}$$

as $\lambda \to \infty$. Let $\varepsilon > 0$ be an arbitrary positive number. We can then find some $t > 1$ sufficiently close to 1 such that

$$|\mathcal{I}(T_t u) - \mathcal{I}(u)| \le \varepsilon.$$

Moreover, we can further find some sufficiently large $\lambda = \lambda(t)$ such that $\mathcal{K}(U_\lambda T_t u) < 0$. Then by (5.3) and (5.6) we infer that

$$\mathcal{I}(u) \ge \mathcal{I}(U_\lambda T_t u) - \varepsilon \ge \widetilde{m}_c - \varepsilon.$$

The claim follows from the arbitrariness of u and ε. ∎

Proof of Proposition 1.8. Let $c > 0$ and let $u_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ with $\|u_\varepsilon - W\|_{\dot{H}^1} \le \varepsilon$ for some given $\varepsilon > 0$. We define

$$v_\varepsilon := \sqrt{\frac{c}{\mathcal{M}(u_\varepsilon)}}\, u_\varepsilon.$$

Then $\mathcal{M}(v_\varepsilon) = c$. Let $t_\varepsilon \in (0,\infty)$ such that $\mathcal{K}^c(T_{t_\varepsilon} v_\varepsilon) = 0$. Direct calculation yields

$$t_\varepsilon = \Big(\frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_{2^*}^{2^*}}\Big)^{\frac{d-2}{4}}. \tag{5.7}$$

By Lemma 5.2 we have

$$m_c \le \mathcal{I}(T_{t_\varepsilon} v_\varepsilon) = \frac{1}{d}\Big(\frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_{2^*}^2}\Big)^{\frac{d}{2}} = \frac{1}{d}\Big(\frac{\|\nabla u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2^*}^2}\Big)^{\frac{d}{2}}. \tag{5.8}$$

Taking $\varepsilon \to 0$ we immediately conclude that $m_c \le \frac{1}{d} \cdot (\frac{\|\nabla W\|_2^2}{\|W\|_{2^*}^2})^{\frac{d}{2}} = \mathcal{H}^*(W)$. On the other hand, one easily verifies that

$$\mathcal{K}^c(u) \le 0 \Rightarrow \mathcal{I}(u) \ge \Big(\frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}\Big)^{\frac{d}{2}}.$$

But by the Sobolev inequality we always have $(\frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2})^{\frac{d}{2}} \ge \mathcal{S}^{\frac{d}{2}} = d\mathcal{H}^*(W)$. Hence $m_c = \mathcal{H}^*(W)$. By [42, Thm. 1.2], any optimizer P of m_c must satisfy $\mathcal{H}(P) > \mathcal{H}^*(W)$, which is a contradiction. This completes the proof of Proposition 1.8. ■

5.2. Scattering for the defocusing–focusing (DCNLS)

In this subsection we establish similar variational estimates as those given in Section 4.1. The scattering result then follows from the variational estimates by using the arguments given in Sections 4.3 and 4.4 verbatim.

Lemma 5.3. *The following statements hold true:*

(i) *Assume that $\mathcal{K}(u) \ge 0$. Then $\mathcal{H}(u) \ge 0$. If additionally $\mathcal{K}(u) > 0$, then $\mathcal{H}(u) > 0$ also.*

(ii) *Let $u \in \mathcal{A}$. Then*

$$\|u\|_{2^*}^{2^*} \le \|\nabla u\|_2^2 + \frac{d}{d+2}\|u\|_{2_*}^{2_*}, \tag{5.9}$$

$$\frac{1}{d}\Big(\|\nabla u\|_2^2 + \frac{d}{d+2}\|u\|_{2_*}^{2_*}\Big) \le \mathcal{H}(u) \le \frac{1}{2}\Big(\|\nabla u\|_2^2 + \frac{d}{d+2}\|u\|_{2_*}^{2_*}\Big). \tag{5.10}$$

(iii) *Let u be a solution of (5.1) with $u(0) \in \mathcal{A}$. Then $u(t) \in \mathcal{A}$ for all t in the maximal lifespan. Moreover, we have*

$$\inf_{t\in I_{\max}} \mathcal{K}(u(t)) \ge \min\Big\{\frac{4}{d}\mathcal{H}(u(0)), \Big(\Big(\frac{d}{d-2}\Big)^{\frac{d-2}{4}} - 1\Big)^{-1}\big(\mathcal{H}^*(W) - \mathcal{H}(u(0))\big)\Big\}. \tag{5.11}$$

Proof. This is a straightforward modification of Lemmas 4.2, 4.3 and 4.5; we therefore omit the details here. ■

We now define the MEI functional for (5.1). Let $\Omega := \mathbb{R}^2 \setminus ([0,\infty) \times [\mathcal{H}^*(W),\infty))$ and let the MEI functional $\mathcal{D}$ be given by (4.28). One has the following analogue of Lemma 4.8.

Lemma 5.4. *Assume $v \in H^1(\mathbb{R}^d)$ such that $\mathcal{K}(v) \geq 0$. Then we have the following statements:*

(i) *$\mathcal{D}(v) = 0$ if and only if $v = 0$.*

(ii) *$0 < \mathcal{D}(v) < \infty$ if and only if $v \in \mathcal{A}$.*

(iii) *$\mathcal{D}$ is conserved under the NLS flow (5.1).*

(iv) *Let $u_1, u_2 \in \mathcal{A}$ with $\mathcal{M}(u_1) \leq \mathcal{M}(u_2)$ and $\mathcal{H}(u_1) \leq \mathcal{H}(u_2)$. Then $\mathcal{D}(u_1) \leq \mathcal{D}(u_2)$. If in addition either $\mathcal{M}(u_1) < \mathcal{M}(u_2)$ or $\mathcal{H}(u_1) < \mathcal{H}(u_2)$, then $\mathcal{D}(u_1) < \mathcal{D}(u_2)$.*

(v) *Let $\mathcal{D}_0 \in (0,\infty)$. Then*

$$\|\nabla u\|_2^2 \sim_{\mathcal{D}_0} \mathcal{H}(u), \tag{5.12}$$

$$\|u\|_{H^1}^2 \sim_{\mathcal{D}_0} \mathcal{H}(u) + \mathcal{M}(u) \sim_{\mathcal{D}_0} \mathcal{D}(u) \tag{5.13}$$

uniformly for all $u \in \mathcal{A}$ with $\mathcal{D}(u) \leq \mathcal{D}_0$.

(vi) *For all $u \in \mathcal{A}$ with $\mathcal{D}(u) \leq \mathcal{D}_0$ for some $\mathcal{D}_0 \in (0,\infty)$ we have*

$$|\mathcal{H}(u) - \mathcal{H}^*(W)| \gtrsim 1. \tag{5.14}$$

Proof. (i) to (iv) can be proved similarly to those from Lemma 4.8; we omit the details here.

Next we verify (v). Let $u \in \mathcal{A}$ with $\mathcal{D}(u) \leq \mathcal{D}_0$. Using (5.10) we already have $\|\nabla u\|_2^2 \leq d\,\mathcal{H}(u)$. On the other hand, by the definition of $\mathcal{D}$ it is readily seen that

$$\mathcal{D}_0 \geq \mathcal{D}(u) = \mathcal{H}(u) + \frac{\mathcal{H}(u) + \mathcal{M}(u)}{\mathcal{H}^*(W) - \mathcal{H}(u)} \geq \frac{\mathcal{M}(u)}{\mathcal{H}^*(W)}, \tag{5.15}$$

which implies $\mathcal{M}(u) \leq \mathcal{D}_0 \mathcal{H}^*(W)$. Using Gagliardo–Nirenberg we infer that

$$\frac{d}{d+2}\|u\|_{2_*}^{2_*} \leq \Big(\frac{\mathcal{M}(u)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}} \|\nabla u\|_2^2 \leq \Big(\frac{\mathcal{D}_0 \mathcal{H}^*(W)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}} \|\nabla u\|_2^2. \tag{5.16}$$

Applying (5.10) one more time we conclude that

$$\mathcal{H}(u) \leq \frac{1}{2}\Big(\|\nabla u\|_2^2 + \frac{d}{d+2}\|u\|_{2_*}^{2_*}\Big) \leq \frac{1}{2}\Big(1 + \Big(\frac{\mathcal{D}_0 \mathcal{H}^*(W)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla u\|_2^2 \tag{5.17}$$

and (5.12) and the first equivalence of (5.13) follow. From (5.15) it also follows that $\mathcal{H}(u) + \mathcal{M}(u) \lesssim_{\mathcal{D}_0} \mathcal{D}(u)$. To prove the inverse direction, we first obtain

$$\mathcal{D}_0 \geq \mathcal{D}(u) = \mathcal{H}(u) + \frac{\mathcal{H}(u) + \mathcal{M}(u)}{\mathcal{H}^*(W) - \mathcal{H}(u)} \geq \frac{\mathcal{H}(u)}{\mathcal{H}^*(W) - \mathcal{H}(u)},$$

which implies $\mathcal{H}(u) \le (1+\mathcal{D}_0)^{-1}\mathcal{D}_0\mathcal{H}^*(W)$. Then

$$\begin{aligned}\mathcal{D}(u) &= \mathcal{H}(u) + \frac{\mathcal{H}(u)+\mathcal{M}(u)}{\mathcal{H}^*(W)-\mathcal{H}(u)}\\ &\le \mathcal{H}(u) + \frac{\mathcal{H}(u)+\mathcal{M}(u)}{(1-(1+\mathcal{D}_0)^{-1}\mathcal{D}_0)\mathcal{H}^*(W)}\\ &= \mathcal{H}(u) + \frac{(1+\mathcal{D}_0)(\mathcal{H}(u)+\mathcal{M}(u))}{\mathcal{H}^*(W)},\end{aligned}$$

which finishes the proof of (v). For (vi), if this were not the case, then we could find a sequence $(u_n)_n \subset \mathcal{A}$ such that

$$\mathcal{H}^*(W) - \mathcal{H}(u_n) = o_n(1). \tag{5.18}$$

Then (5.18) implies $\mathcal{H}(u_n) \gtrsim 1$ and therefore

$$\mathcal{D}(u_n) \ge \frac{\mathcal{H}(u_n)}{\mathcal{H}^*(W)-\mathcal{H}(u_n)} \to \infty,$$

which is a contradiction to $\mathcal{D}(u_n) \le \mathcal{D}_0$. This completes the proof of (vi) and also the desired proof of Lemma 5.4. ∎

Proof of Theorem 1.6 for the defocusing–focusing regime. The proof is almost identical to the one for the focusing–focusing regime; one only needs to replace the results from [19] applied in Lemma 4.9 by those from [18, 20, 21], the arguments from Lemma 4.8 by those from Lemma 5.4 and (4.120) by (5.11). We therefore omit the details here. ∎

6. Scattering threshold, existence and nonexistence of ground states for the focusing–defocusing (DCNLS)

In this section we prove Theorem 1.6 for the focusing–defocusing model and Proposition 1.9. Throughout the section, we assume that (DCNLS) reduces to

$$i\partial_t u + \Delta u + |u|^{\frac{4}{d}}u - |u|^{\frac{4}{d-2}}u = 0. \tag{6.1}$$

The corresponding stationary equation reads

$$-\Delta u + \omega u - |u|^{\frac{4}{d}}u + |u|^{\frac{4}{d-2}}u = 0. \tag{6.2}$$

We also define the set $\mathcal{A}$ by

$$\mathcal{A} := \{u \in H^1(\mathbb{R}^d) : 0 < \mathcal{M}(u) < \mathcal{M}(Q)\}.$$

6.1. Monotonicity formulae and nonexistence of minimizers for $c \le \mathcal{M}(Q)$

Lemma 6.1. *Suppose that u is a solution of (6.2). Then*

$$0 = \|\nabla u\|_2^2 + \omega\|u\|_2^2 - \|u\|_{2_*}^{2_*} + \|u\|_{2^*}^{2^*}, \tag{6.3}$$

$$0 = \|\nabla u\|_2^2 + \frac{d}{d-2}\omega\|u\|_2^2 - \frac{d^2}{d^2-4}\|u\|_{2_*}^{2_*} + \|u\|_{2^*}^{2^*}, \tag{6.4}$$

and

$$\omega\|u\|_2^2 = \frac{2}{d+2}\|u\|_{2_*}^{2_*}. \tag{6.5}$$

Moreover, if $u \neq 0$, then $\omega \in (0, \frac{2}{d}(\frac{d}{d+2})^{\frac{d}{2}})$.

Proof. Equation (6.3) follows from multiplying (6.2) by $\bar{u}$ and then integrating by parts. Equation (6.4) is the Pohozaev inequality; see for instance [6]. Equation (6.5) follows immediately from (6.3) and (6.4). That $\omega > 0$ for $u \neq 0$ follows directly from (6.5). To see $\omega < \frac{2}{d}(\frac{d}{d+2})^{\frac{d}{2}}$, one can easily check this by using the fact that the polynomial

$$t^{\frac{4}{d-2}} - \frac{d^2}{d^2-4}t^{\frac{4}{d}} + \frac{d}{d-2}\omega$$

is nonnegative for $\omega \ge \frac{2}{d}(\frac{d}{d+2})^{\frac{d}{2}}$. ■

Lemma 6.2. *The mapping $c \to \gamma_c$ is nonpositive on $(0,\infty)$ and equal to zero on $(0, \mathcal{M}(Q)]$. Consequently, γ_c has no minimizer for any $c \in (0, \mathcal{M}(Q)]$.*

Proof. First we obtain

$$\mathcal{H}(T_\lambda u) = \frac{\lambda^2}{2}\Big(\|\nabla u\|_2^2 - \frac{d}{d+2}\|u\|_{2_*}^{2_*}\Big) + \frac{\lambda^{2^*}}{2^*}\|u\|_{2^*}^{2^*}.$$

By sending $\lambda \to 0$ we see that $\gamma_c \le 0$. On the other hand, using (4.2) we infer that

$$\mathcal{H}(u) \ge \frac{1}{2}\Big(1 - \Big(\frac{\mathcal{M}(u)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla u\|_2^2 + \frac{\lambda^{2^*}}{2^*}\|u\|_{2^*}^{2^*} \ge 0$$

for $\mathcal{M}(u) \in (0, \mathcal{M}(Q)]$. In particular, since $(1 - (\frac{\mathcal{M}(u)}{\mathcal{M}(Q)})^{\frac{2}{d}})$ is nonnegative for $\mathcal{M}(u) \in (0, \mathcal{M}(Q)]$, we deduce that $\mathcal{H}(u) = 0$ is only possible when $u = 0$, which is a contradiction since $\mathcal{M}(u) > 0$. Thus there is no minimizer for γ_c when $c \in (0, \mathcal{M}(Q)]$. ■

Lemma 6.3. *The mapping $c \mapsto \gamma_c$ is monotone decreasing and $\gamma_c > -\infty$ on $(0,\infty)$. Moreover, γ_c is negative on $(\mathcal{M}(Q),\infty)$.*

Proof. We define the scaling operator U_λ by

$$U_\lambda u(x) := \lambda^{\frac{d-2}{2}}u(\lambda x).$$

Then

$$\mathcal{H}(U_\lambda u) = \mathcal{H}(u) + \frac{1}{2_*}(1 - \lambda^{-\frac{4}{d}})\|u\|_{2_*}^{2_*},$$
$$\mathcal{M}(U_\lambda u) = \lambda^{-2}\mathcal{M}(u).$$

For $u \neq 0$ we see that $\mathcal{H}(U_\lambda u) \to -\infty$ and $\mathcal{M}(U_\lambda u) \to \infty$ as $\lambda \to 0$, which implies that $\gamma_c < 0$ for large c. Next we show the monotonicity of $c \mapsto \gamma_c$. Let $0 < c_1 < c_2 < \infty$. By definition of γ_{c_1} there exists a sequence $(u_n)_n \subset H^1(\mathbb{R}^d)$ satisfying

$$\mathcal{M}(u_n) = c_1,$$
$$\mathcal{H}(u_n) = \gamma_{c_1} + o_n(1).$$

Let $\lambda_* := \sqrt{\frac{c_1}{c_2}} < 1$. Then $\mathcal{M}(U_{\lambda_*}u_n) = c_2$ and we conclude that

$$\gamma_{c_1} = \mathcal{H}(u_n) + o_n(1) \geq \mathcal{H}(U_{\lambda_*}u_n) + o_n(1) \geq \gamma_{c_2} + o_n(1).$$

Sending $n \to \infty$ follows the monotonicity. To see that γ_c is negative on $(\mathcal{M}(Q), \infty)$, we define $S = tQ$ for some to be determined $t \in (1, \infty)$. Using Pohozaev we infer that

$$\|\nabla Q\|_2^2 = \frac{d}{d+2}\|Q\|_{2_*}^{2_*},$$

which yields

$$\mathcal{H}(T_\lambda S) = -\frac{\lambda^2}{2_*}(t^{2_*} - t^2)\|Q\|_{2_*}^{2_*} + \frac{\lambda^{2^*}}{2^*}t^{2^*}\|Q\|_{2^*}^{2^*}.$$

By direct calculation we also see that

$$0 < \lambda < \Big(\frac{2^*(t^{2_*} - t^2)\|Q\|_{2_*}^{2_*}}{2_* t^{2^*}\|Q\|_{2^*}^{2^*}}\Big)^{\frac{d-2}{4}} \Rightarrow \mathcal{H}(T_\lambda S) < 0.$$

This shows that $\gamma_c < 0$ on $(\mathcal{M}(Q), \infty)$. Finally, we show that γ_c is bounded below. By theHölder inequality we obtain

$$\|u\|_{2_*}^{2_*} \leq (\mathcal{M}(u))^{\frac{2}{d}}\|u\|_{2^*}^2.$$

Then for $u \in H^1(\mathbb{R}^d)$ with $\mathcal{M}(u) = c$ we have

$$\mathcal{H}(u) \geq -\frac{c^{\frac{2}{d}}}{2_*}\|u\|_{2^*}^2 + \frac{1}{2^*}\|u\|_{2^*}^{2^*}. \tag{6.6}$$

But the function $t \mapsto -\frac{c^{\frac{2}{d}}}{2_*}t^2 + \frac{1}{2^*}t^{2^*}$ is bounded below on $[0, \infty)$. This completes the proof. ∎

6.2. Existence of minimizers of γ_c for $c > \mathcal{M}(Q)$

Lemma 6.4. *For each $c > \mathcal{M}(Q)$, the variational problem γ_c has a minimizer which is positive and radially symmetric.*

Proof. Let $(u_n)_n \subset H^1(\mathbb{R}^d)$ be a minimizing sequence, i.e.

$$\mathcal{M}(u_n) = c,$$
$$\mathcal{H}(u_n) = \gamma_c + o_n(1).$$

Since $\mathcal{H}$ is stable under the Steiner symmetrization, we may further assume that u_n is radially symmetric. Using (6.6) we infer that

$$\gamma_c + o_n(1) \geq -\frac{c^{\frac{2}{d}}}{2_*}\|u_n\|_{2^*}^2 + \frac{1}{2^*}\|u_n\|_{2^*}^{2^*},$$

thus $(\|u_n\|_{2^*})_n$ is a bounded sequence. Hence

$$\frac{1}{2}\|\nabla u_n\|_2^2 \leq \gamma_c + o_n(1) + \frac{c^{\frac{2}{d}}}{2_*}\|u_n\|_{2^*}^2 \lesssim 1,$$

and therefore $(u_n)_n$ is a bounded sequence in $H^1(\mathbb{R}^d)$. Up to a subsequence $(u_n)_n$ converges to some radially symmetric $u \in H^1(\mathbb{R}^d)$ weakly in $H^1(\mathbb{R}^d)$ and $\mathcal{M}(u) \leq c$. By weak lower semicontinuity of norms and the Strauss compact embedding for radial functions we know that

$$\mathcal{H}(u) \leq \gamma_c < 0,$$

and therefore $u \neq 0$. Suppose that $\mathcal{M}(u) < c$. Then $\mathcal{M}(U_\lambda u) = \lambda^{-2}\mathcal{M}(u) < c$ for λ in a neighborhood of 1 and

$$\mathcal{H}(U_\lambda u) = \mathcal{H}(u) + \frac{1}{2_*}(1 - \lambda^{-\frac{4}{d}})\|u\|_{2_*}^{2_*} = \gamma_c + \frac{1}{2_*}(1 - \lambda^{-\frac{4}{d}})\|u\|_{2_*}^{2_*} < \gamma_c$$

for $\lambda < 1$ sufficiently close to 1. This contradicts the monotonicity of $c \mapsto \gamma_c$, thus $\mathcal{M}(u) = c$. By the Lagrange multiplier theorem we know that any minimizer of γ_c is automatically a solution of (6.2) and thus the positivity of u follows from the strong maximum principle. The proof is then complete. ∎

Proof of Proposition 1.9. This follows immediately from Lemmas 6.1–6.4. ∎

6.3. Scattering for the focusing–defocusing (DCNLS)

Lemma 6.5. *Let u be a solution of* (6.1) *with $u(0) \in \mathcal{A}$. Then $u(t) \in \mathcal{A}$ for all $t \in \mathbb{R}$. Also assume $\mathcal{M}(u) = (1-\delta)^{\frac{d}{2}}\mathcal{M}(Q)$. Then*

$$\inf_{t \in I_{\max}} \mathcal{K}(u(t)) \geq 2\mathcal{H}(u(0)). \tag{6.7}$$

Proof. That $u(t) \in \mathcal{A}$ for all $t \in \mathbb{R}$ follows immediately from the conservation of mass. Moreover, (6.7) follows from

$$\mathcal{K}(u(t)) = 2\mathcal{H}(u(t)) + \frac{2}{d}\|u\|_{2^*}^{2^*} \geq 2\mathcal{H}(u(0)),$$

where we also used the conservation of energy. ■

We now define the MEI functional for (6.1). Let $\Omega := (-\infty, \mathcal{M}(Q)) \times \mathbb{R}$ and let the MEI functional $\mathcal{D}$ be given by (4.28). One has the following analogue of Lemma 4.8.

Lemma 6.6. *Assume $u \in H^1(\mathbb{R}^d)$. Then we have the following statements:*

(i) $\mathcal{D}(u) = 0$ *if and only if* $u = 0$.

(ii) $0 < \mathcal{D}(u) < \infty$ *if and only if* $u \in \mathcal{A}$.

(iii) $\mathcal{D}$ *is conserved under the NLS flow* (6.1).

(iv) *Let $u_1, u_2 \in \mathcal{A}$ with $\mathcal{M}(u_1) \leq \mathcal{M}(u_2)$ and $\mathcal{H}(u_1) \leq \mathcal{H}(u_2)$. Then $\mathcal{D}(u_1) \leq \mathcal{D}(u_2)$. If additionally either $\mathcal{M}(u_1) < \mathcal{M}(u_2)$ or $\mathcal{H}(u_1) < \mathcal{H}(u_2)$, then $\mathcal{D}(u_1) < \mathcal{D}(u_2)$.*

(v) *Let $\mathcal{D}_0 \in (0, \infty)$. Then*

$$\|\nabla u\|_2^2 \sim_{\mathcal{D}_0} \mathcal{H}(u), \tag{6.8}$$
$$\|u\|_{H^1}^2 \sim_{\mathcal{D}_0} \mathcal{H}(u) + \mathcal{M}(u) \sim_{\mathcal{D}_0} \mathcal{D}(u) \tag{6.9}$$

uniformly for all $u \in \mathcal{A}$ with $\mathcal{D}(u) \leq \mathcal{D}_0$.

Remark 6.7. Due to the positivity of the defocusing energy-critical potential we do not need to impose the additional condition $\mathcal{K}(u) \geq 0$. △

Proof of Lemma 6.6. Items (i) to (iv) are trivial. We still need to verify (v). Let $u \in \mathcal{A}$ with $\mathcal{D}(u) \leq \mathcal{D}_0$. It is readily seen that

$$\mathcal{D}_0 \geq \mathcal{D}(u) = \mathcal{H}(u) + \frac{\mathcal{H}(u) + \mathcal{M}(u)}{\mathcal{M}(Q) - \mathcal{M}(u)} \geq \frac{\mathcal{M}(u)}{\mathcal{M}(Q) - \mathcal{M}(u)}, \tag{6.10}$$

which implies $\mathcal{M}(u) \leq (1 + \mathcal{D}_0)^{-1}\mathcal{D}_0\mathcal{M}(Q)$. Hence

$$\begin{aligned}\mathcal{H}(u) &\geq \frac{1}{2}\Big(1 - \Big(\frac{\mathcal{M}(u)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla u\|_2^2 + \frac{1}{2^*}\|u\|_{2^*}^{2^*} \\ &\geq \frac{1}{2}\big(1 - ((1 + \mathcal{D}_0)^{-1}\mathcal{D}_0)^{\frac{2}{d}}\big)\|\nabla u\|_2^2. \end{aligned} \tag{6.11}$$

Similarly, we obtain

$$\mathcal{D}_0 \geq \mathcal{H}(u) \geq \frac{1}{2}\big(1 - ((1 + \mathcal{D}_0)^{-1}\mathcal{D}_0)^{\frac{2}{d}}\big)\|\nabla u\|_2^2,$$

which implies

$$\|\nabla u\|_2^2 \le \frac{2\mathcal{D}_0}{1-\big((1+\mathcal{D}_0)^{-1}\mathcal{D}_0\big)^{\frac{2}{d}}}.$$

Using the Sobolev inequality and (6.11) we obtain

$$\begin{aligned}\mathcal{H}(u) &\le \frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2^*}\|u\|_{2^*}^{2^*}\le \frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2^*}\|u\|_{2^*}^{2^*}\\ &\le \frac{1}{2}\|\nabla u\|_2^2+\frac{\mathcal{S}^{-\frac{d}{d-2}}}{2^*}\bigg(\frac{2\mathcal{D}_0}{1-\big((1+\mathcal{D}_0)^{-1}\mathcal{D}_0\big)^{\frac{2}{d}}}\bigg)^{\frac{2}{d-2}}\|\nabla u\|_2^2. \end{aligned}\tag{6.12}$$

Expression (6.8) and the first equivalence of (6.9) now follow from (6.11) and (6.12). From (6.10) it also follows that $\mathcal{H}(u)+\mathcal{M}(u)\lesssim_{\mathcal{D}_0}\mathcal{D}(u)$. That $\mathcal{D}(u)\lesssim_{\mathcal{D}_0}\mathcal{H}(u)+\mathcal{M}(u)$ follows immediately from

$$\begin{aligned}\mathcal{D}(u)&=\mathcal{H}(u)+\frac{\mathcal{H}(u)+\mathcal{M}(u)}{\mathcal{M}(Q)-\mathcal{M}(u)}\le \mathcal{H}(u)+\frac{\mathcal{H}(u)+\mathcal{M}(u)}{(1-(1+\mathcal{D}_0)^{-1}\mathcal{D}_0)\mathcal{M}(Q)}\\ &=\mathcal{H}(u)+\frac{(1+\mathcal{D}_0)(\mathcal{H}(u)+\mathcal{M}(u))}{\mathcal{M}(Q)}.\end{aligned}$$

■

Proof of Theorem 1.6 for the focusing–defocusing regime. The proof is almost identical to the one for the focusing–focusing regime; one only needs to replace the results from [22,27,30] applied in Lemma 4.10 by those from [17,41,44], the arguments from Lemma 4.8 by those from Lemma 6.6 and (4.120) by (6.7). We therefore omit the details here. ■

A. Endpoint values of the curve $c \mapsto m_c$ for the focusing–focusing (DCNLS)

Proposition A.1. *Let $\mu_1=\mu_2=1$ and m_c be defined through (1.7). Let*

$$m_0 := \lim_{c\downarrow 0} m_c,\quad m_Q := \lim_{c\uparrow \mathcal{M}(Q)} m_c.$$

Then $m_0=\mathcal{H}^*(W)$ and $m_Q=0$.

Proof. By Theorem 1.3 we already know that $m_0\le \mathcal{H}^*(W)$. For $c\in(0,\mathcal{M}(Q))$, let P_c be an optimizer of the variational problem m_c, whose existence is guaranteed by Theorem 1.3. We first show $m_0=\mathcal{H}^*(W)$ and let $c\downarrow 0$. Then by $\mathcal{K}(P_c)=0$ and (4.2) we obtain

$$\begin{aligned}m_c=\mathcal{H}(P_c)&=\mathcal{H}(P_c)-\frac{1}{2^*}\mathcal{K}(P_c)\\ &=\frac{1}{d}\Big(\|\nabla P_c\|_2^2-\frac{d}{d+2}\|P_c\|_{2_*}^{2_*}\Big)\\ &\ge \frac{1}{d}\Big(1-\Big(\frac{\mathcal{M}(P_c)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla P_c\|_2^2\\ &=\frac{1}{d}\Big(1-\Big(\frac{o_c(1)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla P_c\|_2^2.\end{aligned}\tag{A.1}$$

Hence $(P_c)_{c\downarrow 0}$ is bounded in $H^1(\mathbb{R}^d)$. On the other hand, using $\mathcal{K}(P_c)=0$ and the Sobolev inequality we infer that

$$\frac{1}{d}\Big(1-\Big(\frac{o_c(1)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla P_c\|_2^2 \le \frac{1}{d}\Big(\|\nabla P_c\|_2^2-\frac{d}{d+2}\|P_c\|_{2_*}^{2_*}\Big)$$
$$=\frac{1}{d}\|P_c\|_{2^*}^{2^*} \le \frac{\mathcal{S}^{\frac{d}{2-d}}}{d}\|\nabla P_c\|_2^{2^*},$$

which implies that (up to a subsequence) $l := \lim_{c\downarrow 0}\|\nabla P_c\|_2^2 > 0$. But then, by the Gagliardo–Nirenberg inequality and $\mathcal{K}(P_c)=0$, we obtain

$$\|\nabla P_c\|_2^{2^*}\mathcal{S}^{\frac{d}{2-d}} \ge \|P_c\|_{2^*}^{2^*} = \|\nabla P_c\|_2^2-\frac{d}{d+2}\|P_c\|_{2_*}^{2_*} \ge \Big(1-\Big(\frac{o_c(1)}{\mathcal{M}(Q)}\Big)^{\frac{2}{d}}\Big)\|\nabla P_c\|_2^2 \to l.$$

Therefore, $l^{2^*}\mathcal{S}^{\frac{d}{2-d}} \ge l$. Since $l\neq 0$, we infer that $l \ge \mathcal{S}^{\frac{d}{2}}$. But then (A.1) implies $m_0 \ge \frac{\mathcal{S}^{\frac{d}{2}}}{d} = \mathcal{H}^*(W)$, which completes the proof.

Next we show $m_Q = 0$. Let $(u_n)_n$ be a minimizing sequence for (1.19). By rescaling we may assume that $\mathcal{M}(u_n)=\delta\mathcal{M}(Q)$ and $\|u_n\|_{2_*}=1$ for a fixed $\delta\in(0,1)$, which will be sent to 1 later. Then combining with (1.20) we obtain $\|\nabla u_n\|_2^2=\frac{d}{d+2}\delta^{-\frac{2}{d}}+o_n(1)$. We then conclude that

$$\mathcal{K}(T_\lambda u_n)=\frac{d\lambda^2}{d+2}\big(\delta^{-\frac{2}{d}}-1+o_n(1)\big)-\lambda^{2^*}\|u_n\|_{2^*}^{2^*}.$$

By setting

$$\lambda_{n,\delta}=\Big(\frac{d}{(d+2)\|u_n\|_{2^*}^{2^*}}\Big(\delta^{-\frac{2}{d}}-1+o_n(1)\Big)\Big)^{\frac{d-2}{4}}$$

we see that $\mathcal{K}(T_{\lambda_{n,\delta}}u_n)=0$. By Hölder we deduce that

$$\|u_n\|_{2^*}^{2^*} \ge \mathcal{M}(u_n)^{-\frac{2}{d-2}}\|u_n\|_{2_*}^{\frac{2(d+2)}{d-2}}=(\delta\mathcal{M}(Q))^{-\frac{2}{d-2}}.$$

We now choose $N=N(\delta)\in\mathbb{N}$ such that $|o_n(1)|\le \delta^{-\frac{2}{d}}-1$ for all $n>N$. Summing and using the definition of m_c we finally conclude that

$$m_{\delta M(Q)} \le \sup_{n>N}\mathcal{H}(T_{\lambda_{n,\delta}}u_n)=\sup_{n>N}\Big(\mathcal{H}(T_{\lambda_{n,\delta}}u_n)-\frac{1}{2}\mathcal{K}(T_{\lambda_{n,\delta}}u_n)\Big)$$
$$=\sup_{n>N}\frac{1}{2^*}\|T_{\lambda_{n,\delta}}u_n\|_{2^*}^{2^*}=\sup_{n>N}\frac{\lambda_{n,\delta}^{2^*}}{2^*}\|u_n\|_{2^*}^{2^*}$$
$$\le \frac{2^{\frac{d}{2}}}{2^*}\Big(\frac{d}{d+2}\Big)^{\frac{d}{2}}(\delta^{-\frac{2}{d}}-1)\delta\mathcal{M}(Q)\to 0$$

as $\delta\to 1$. This proves $m_Q=0$. ∎

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