

Existence and regularity of weak solutions for a fluid interacting with a non-linear shell in three dimensions

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Abstract. We study the unsteady incompressible Navier–Stokes equations in three dimensions interacting with a non-linear flexible shell of Koiter type. This leads to a coupled system of non-linear PDEs where the moving part of the boundary is an unknown of the problem. The known existence theory for weak solutions is extended to non-linear Koiter shell models. We introduce a priori estimates that reveal higher regularity of the shell displacement beyond energy estimates. These are essential for non-linear Koiter shell models, since such shell models are non-convex (with respect to terms of highest order). The estimates are obtained by introducing new analytical tools that allow dissipative effects of the fluid to be exploited for the (non-dissipative) solid. The regularity result depends on the geometric constitution alone and is independent of the approximation procedure; hence it holds for arbitrary weak solutions. The developed tools are further used to introduce a generalized Aubin–Lions-type compactness result suitable for fluid–structure interactions.

1. Introduction

Fluid–structure interactions (FSI) are everyday phenomena with many applications, for example in aeroelasticity [16] and biomedicine [4]. Mathematically, FSI problems are described by coupling fluid equations with elasticity equations. The analysis of FSI problems is challenging and attractive mainly due to the following properties. First, the resulting system of non-linear PDEs is of hyperbolic–parabolic type with the coupling taking place at the fluid–structure interface. Second, the fluid domain is an unknown of the problem, i.e. the resulting problem is a moving boundary problem. In this paper we study the coupling of the three-dimensional (3D) incompressible Navier–Stokes equations with the evolution of the non-linear Koiter shell equation. Our main result is that any finite-energy weak solution to the considered FSI problem satisfies an additional regularity property on its interval of existence (see Theorem 1.2). More precisely, we show that the elastic displacement belongs to the Bochner space $L_t^2(H_x^{2+s}) \cap H_t^1(H_x^s)$ for all $s < \frac{1}{2}$. Here H^s denotes the standard fractional Sobolev space (for a precise definition of a fractional Sobolev space, see Section 2.3). In particular, due to respective embedding theorems, the elastic displacement is Lipschitz continuous in the space variable for almost every moment of time. We use this result to show the existence of weak solutions to a fluid–non-linear

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Koiter shell interaction problem (see Theorem 1.1). Since the non-linear Koiter shell equations are quasi-linear with non-linear coefficients depending on the terms of leading order in the energy, the additional structure regularity estimate is crucial for the compactness argument in the construction of a weak solution. The main idea behind the regularity theorem is to use the fluid dissipation and the coupling conditions to prove the additional regularity estimate for the structure displacement. The realization of this idea is technically challenging. It includes the development of a comprehensive analysis to construct a solenoidal extension and smooth approximations for the time-changing domain with clear (local) dependence on the regularity of the boundary values and the boundary itself. The approach is quite general and thus seems suitable for further applications related to the analysis of variable geometries. In fact, the present result has already been applied; please see [6, 40, 43].

Fluid–structure interaction has been an increasingly active area of research in mathematics in the last 20 years. Due to the overwhelming number of contributions in the area, in this brief literature review we just mention analytic results that are most relevant for our work. Existence results for weak solutions for FSI problems where the incompressible Navier–Stokes equations are coupled with a lower-dimensional elasticity model (e.g. plate or shell laws) have been obtained in [10, 22, 25, 31, 34, 35]. The corresponding existence result for compressible fluid flow was proved in [7]. All mentioned results on the existence of weak solutions are valid up to the time of possible self-intersection of the domain. To our knowledge, the number of regularity estimates for long time solutions is rather limited. Recently, some significant results on strong solutions for large initial data and a two-dimensional fluid interacting with a one-dimensional solid have been obtained; see [23, 24]. For a three-dimensional fluid interacting with a three-dimensional elastic body, see [26, 27] for global results with small initial data and structural damping. The theory of local-in-time strong solutions for 3D–3D FSI problems is rather well developed; see recent results in [5, 29, 38] and references within. We wish to emphasize that in all these works the structure equations were linear. For an FSI problem with non-linear structure, the theory is far less developed. The existence of a weak solution to the FSI problem with a Koiter membrane energy that includes non-linearities of lower order and a leading-order linear regularizing term was proved in [36]. Short time or small data existence results in the context of strong solutions for various non-linear fluid–structure models have been obtained in [11, 12, 15, 39]. Finally, we wish to mention some results in the static case that can be found in [19, 21].

The role of fluid dissipation on the qualitative properties of the solution is one of the central questions in the area of fluid–structure interactions and related systems, and has been studied by many authors; see e.g. [3, 20, 44] and references within. We present here a new technique that allows dissipation features to be transferred from the fluid equation to the non-linear hyperbolic elastic displacement. We wish to point out that better regularity cannot be expected for a non-linear hyperbolic PDE with arbitrarily smooth right-hand sides and initial data. It is the coupling with a dissipative equation that allows this better regularity.

1.1. The coupled PDE

We first discuss the Koiter shell model (see e.g. [13, 28]) which describes the evolution of the elastic boundary of the fluid domain. Let $\Omega \subset \mathbb{R}^3$ be a domain such that its boundary $\Gamma = \partial\Omega$ is parameterized by a C^3 injective mapping $\varphi: \omega \rightarrow \mathbb{R}^3$, where $\omega \subset \mathbb{R}^2$. To simplify notation we assume in this paper that the boundary of Ω can be parameterized by a flat torus $\omega = \mathbb{R}^2/\mathbb{Z}^2$ which corresponds to the assumption of periodic boundary conditions for the structure displacement. We consider the periodic boundary conditions just to avoid unnecessary technical complications (see Remark 1.4). An example of such a domain is a cylindrical domain (see Figure 1) with periodic boundary condition for the fluid velocity on the inlet/outlet part of the boundary. Namely, in this setting we can identify the inlet and the outlet parts of the boundary, and thus the fluid domain is a three-dimensional torus, which fits our framework.

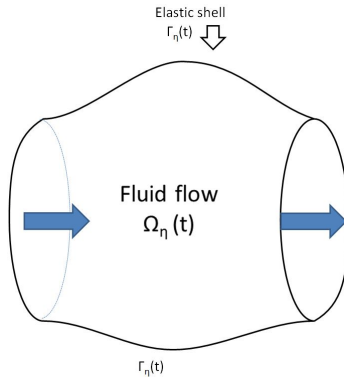


Figure 1. An example of the deformed cylindrical domain.

We denote tangential vectors at any point $\varphi(y)$ as

$$\mathbf{a}_\alpha(y) = \partial_\alpha \varphi(y), \quad \alpha = 1, 2, \quad y \in \omega.$$

The unit normal vector is given by

$$\mathbf{n}(y) = \frac{\mathbf{a}_1(y) \times \mathbf{a}_2(y)}{|\mathbf{a}_1(y) \times \mathbf{a}_2(y)|}.$$

The surface area element of $\partial\Omega$ is given by $dS = |\mathbf{a}_1(y) \times \mathbf{a}_2(y)| dy$. We assume that the domain deforms only in the normal direction and denote by $\eta(t, y)$ the magnitude of the displacement. This reflects the situation when the fluid pressure is the dominant force acting on the structure, in which case it is reasonable to assume that the shell is deforming in the normal direction. In this case the deformed boundary can be parameterized by the following coordinates:

$$\varphi_\eta(t, y) = \varphi(y) + \eta(t, y)\mathbf{n}(y), \quad t \in (0, T), \quad y \in \omega. \tag{1.1}$$

We wish to emphasize that this restriction is standard in the majority of mathematical works on the analysis of weak solutions, mainly due to severe technical difficulties associated with the analysis of the case where the full displacement is taken into account. The deformed boundary is denoted by $\Gamma_\eta(t) = \varphi_\eta(t, \omega)$. It is a well-known fact from differential geometry (see e.g. [30]) that there exist $\alpha(\Omega), \beta(\Omega) > 0$ such that for $\eta(y) \in (\alpha(\Omega), \beta(\Omega))$, $\varphi_\eta(t, \cdot)$ is a bijective parameterization of the surface $\Gamma_\eta(t)$ and it defines a domain $\Omega_\eta(t)$ in its interior such that $\partial\Omega_\eta(t) = \Gamma_\eta(t)$. Moreover, there exists a bijective transformation $\psi_\eta(t, \cdot): \Omega \rightarrow \Omega_\eta(t)$. For more details on the geometry see Section 2.1 and Definition 2.1.

We denote the moving domain in the following way:

$$(0, T) \times \Omega_\eta(t) := \bigcup_{t \in (0, T)} \{t\} \times \Omega_\eta(t).$$

The non-linear Koiter model is given in terms of the differences of the first and the second fundamental forms of $\Gamma_\eta(t)$ and Γ which represent membrane forces and bending forces respectively. These forces are summarized in its potential – the Koiter energy $\mathcal{E}_K(t, \eta)$. The definition of the potential is taken from [14, Section 4]. For a precise definition and the derivation of the energy for our coordinates, see (2.8) below. Let $\mathcal{L}_K \eta$ be the L^2 -gradient of the Koiter energy $\mathcal{E}_K(t, \eta)$, h be the (constant) thickness of the shell and ϱ_s the (constant) density of the shell. Then the respective momentum equation for the shell reads

$$\varrho_s h \partial_t^2 \eta + \mathcal{L}_K \eta = g, \tag{1.2}$$

where g is the momentum forces of the fluid acting on the shell.

The fluid flow is governed by the incompressible Navier–Stokes equations:

$$\varrho_f (\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)) \mathbf{u} = \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}, p) \quad \text{in } (0, T) \times \Omega_\eta(t), \tag{1.3}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } (0, T) \times \Omega_\eta(t), \tag{1.4}$$

where $\boldsymbol{\sigma}(\mathbf{u}, p) = -p\mathbb{I} + 2\mu \operatorname{sym} \nabla \mathbf{u}$ is the fluid stress tensor and ϱ_f the (constant) density of the fluid.

The fluid and the structure are coupled via kinematic and dynamic coupling conditions. We prescribe a no-slip kinematic coupling condition which means that the fluid and the structure velocities are equal on the elastic boundary:

$$\mathbf{u}(t, \varphi_\eta(t, y)) = \partial_t \eta(t, y) \mathbf{n}(y), \quad y \in \omega. \tag{1.5}$$

The dynamic boundary condition states that the total force in the normal direction on the boundary is zero:

$$g(t, y) = -\boldsymbol{\sigma}(\mathbf{u}, p)(t, \varphi_\eta(t, y)) \mathbf{n}(\eta(t, y)) \cdot \mathbf{n}(y), \quad y \in \omega, \tag{1.6}$$

where $\mathbf{n}(\eta(t, y)) = \partial_1 \varphi_\eta(t, y) \times \partial_2 \varphi_\eta(t, y)$ is defined as a weighted vector pointing in the direction of the outer normal to the deformed domain at point $\varphi_\eta(t, y)$; the weight is exactly the Jacobian of the change of variables from Eulerian to Lagrangian coordinates.

We may summarize and state the full fluid–structure interaction problem: find (\mathbf{u}, η) such that

$$\begin{aligned}
 \varrho_f(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) &= \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}, p) && \text{in } (0, T) \times \Omega_\eta(t), \\
 \operatorname{div} \mathbf{u} &= 0 && \text{in } (0, T) \times \Omega_\eta(t), \\
 \varrho_s h \partial_t^2 \eta + \mathcal{L}_K \eta &= -(\boldsymbol{\sigma}(\mathbf{u}, p) \circ \boldsymbol{\varphi}_\eta) \mathbf{n}(\eta) \cdot \mathbf{n} && \text{in } (0, T) \times \omega, \\
 \mathbf{u} \circ \boldsymbol{\varphi}_\eta &= \partial_t \eta \mathbf{n} && \text{in } (0, T) \times \omega, \\
 \mathbf{u}(0, \cdot) &= \mathbf{u}_0 && \text{in } \Omega_\eta(0), \\
 \eta(0) = \eta_0, \quad \partial_t \eta(0) &= \eta_1 && \text{in } \omega.
 \end{aligned} \tag{1.7}$$

The solution of the above coupled system formally satisfies the following energy equality:

$$\begin{aligned}
 \frac{d}{dt} \left(\frac{\varrho_f}{2} \|\mathbf{u}(t)\|_{L^2(\Omega_\eta(t))}^2 + \frac{h\varrho_s}{2} \|\partial_t \eta(t)\|_{L^2(\omega)}^2 + \mathcal{E}_K(t, \eta) \right) \\
 = -2\mu \int_{\Omega_\eta(t)} |\operatorname{sym} \nabla \mathbf{u}|^2.
 \end{aligned} \tag{1.8}$$

Due to the fact that the Koiter shell equation is non-linear – more precisely, since the curvature change is measured with respect to the deformed geometry – the H^2 -coercivity of the Koiter energy can become degenerate. This is quantified by the estimate shown in Lemma 4.3 below. At such a degenerate instant, the given existence and regularity proofs break down. This is a phenomenon purely due to the non-linearity of the Koiter shell equations. Indeed, in the case when the leading-order term of the elastic energy is quadratic (i.e. the equation is linear or semi-linear), this loss of coercivity is a priori excluded.

1.2. Main results

Let us now state the main theorems of the paper. The first main theorem is the existence of solutions to the non-linear Koiter shell model.

Theorem 1.1. *Assume that $\partial\Omega \in C^3$, $\eta_0 \in H^2(\omega)$, $\eta_1 \in L^2(\omega)$ and $\mathbf{u}_0 \in L^2(\Omega_{\eta_0})$, and η_0 is such that Γ_{η_0} has no self-intersection and $\gamma(\eta_0) \neq 0$. Moreover, we assume that the compatibility condition $\mathbf{u}_0|_{\Gamma_{\eta_0}} = \eta_1 \mathbf{n}$ is satisfied. Then there exists a weak solution (\mathbf{u}, η) on the time interval $(0, T)$ to (1.7) in the sense of Definition 2.3.*

Furthermore, one of the following is true: either $T = +\infty$, or the structure self-intersects, or $\gamma(\eta) \neq 0$, i.e. the H^2 -coercivity of the structure energy degenerates, where γ is defined in Definition 2.1 below.

The second main theorem says that all possible solutions in the natural existence class satisfy better structural regularity properties.

Theorem 1.2. *Let (\mathbf{u}, η) be a weak solution to (1.7) in the sense of Definition 2.3. Then the solution has the additional regularity properties $\eta \in L^2(0, T; H^{2+s}(\omega))$ and $\partial_t \eta \in L^2(0, T; H^s(\omega))$ for $s \in (0, \frac{1}{2})$. Moreover, it satisfies the regularity estimate*

$$\|\eta\|_{L^2(0,T;H^{2+s}(\omega))} + \|\partial_t \eta\|_{L^2(0,T;H^s(\omega))} \leq C_1$$

with C_1 depending on $\partial\Omega$, C_0 and the H^2 -coercivity size $\gamma(\eta)$.

Remark 1.3 (On the coercivity parameter $\gamma(\eta)$). Here we briefly discuss the dependence of the constant C_1 in Theorem 1.2 on the coercivity parameter $\gamma(\eta)$. The detailed analysis can be viewed in Section 4. First, one can use the energy inequality to deduce that η is continuous in space and time; see Lemma 4.2. The coercivity constant $\gamma(\eta)$ is explicitly calculated in Lemma 4.3, where it is shown that $\gamma(\eta)$ continuously depends on η . Consequently (depending on the initial state), coercivity in H^2 holds at least for some time interval. In particular, on this time interval, constant C_1 in Theorem 1.2 depends only on $\partial\Omega$ and C_0 . If the solution then did not reach a state of degeneracy ($\gamma(\eta) = 0$) or self-intersection, then it can be (iteratively) prolonged. In order to derive the estimate for the prolonged solution, one should revise the dependencies in Theorem 1.2 and this is the reason why in Theorem 1.2 the constant C_1 depends on $\gamma(\eta)$. Note, in particular, that in some cases $\gamma(\eta)$ can be explicitly computed; see Examples 1 and 2 in the next section, where very natural conditions for the exclusion of degeneracy are computed.

Remark 1.4. Since here $\partial\Omega$ is assumed to be a manifold without boundary, there is no boundary condition for η . We restrict ourselves to this case just for technical simplicity. The proof for the case where only a part of the boundary is elastic (see e.g. [7, 31]), with the appropriate Dirichlet boundary conditions, is completely analogous. In particular, the regularity estimate Theorem 1.2 is valid for the case of non-periodic shells and the proof can be adapted without any significant complications by using the zero extension of η to the whole boundary.

Remark 1.5. In the proof of Theorem 1.2 we use the no-slip coupling condition (1.5) to transfer the fluid regularity to the structure. Here the exact form of the structure equation (1.2) is not essential. Therefore the proof can be easily adapted to different structure models as long as we have that the corresponding structure energy is coercive in the H^2 norm. In particular, our result implies respective estimates for weak solutions to several FSI problems that have already been studied in the literature, and mentioned in the introduction, e.g. [10, 22, 31, 34, 36]. The geometric condition $\gamma(\eta) \neq 0$ is needed for H^2 -coercivity in the non-linear setting. Clearly, H^2 -coercivity is satisfied independently of $\gamma(\eta)$ when the leading-order term in the Koiter energy is quadratic. Hence, in the case when the structure equation is linear or semi-linear, the regularity result is valid until a self-intersection is approached.

1.3. Novelty and significance

The main novelty is the improved regularity of the elastic displacement. In particular, it allows the threshold between Lipschitz and non-Lipschitz domains to be overcome. This critical step has caused a significant amount of effort in previous works [10, 22, 31, 34, 36]. The regularity uses classical differential quotient techniques applied to the non-linear structure equation. The impact of the fluid on the structure, which is rather implicit in the framework of weak equations, is problematic. Here, newly developed extension operators are developed that are certainly of independent value (see Proposition 3.3). Of

critical technical difficulty are commutator estimates for the time-dependent extension of a difference quotient (see Lemma 3.5).

The power of the newly introduced method to gain higher regularity for the structure allows us to prove the existence of weak solutions for fluids interacting with non-linear Koiter shells. These more physical models have not been in reach for the theory of weak solutions that may exist for arbitrarily long times. The mathematical reason is that the respective energies are highly non-linear and non-convex. The extra regularity estimate, however, allows us to derive sequences that converge strongly to the solution with respect to the highest order of the operator. This is the reason why no *linearity or convexity assumptions* are needed anymore to pass to the limit within the non-linear stress tensor of the structure equation.

For previous results, the limit passage of the convective term in the Navier–Stokes equations was the main effort [10, 22, 31, 34, 36]. The limit passage usually relies on compactness results of Aubin–Lions type. The variable geometry make its application highly technical. In Section 5 we rewrite the celebrated result in a form that we believe to be suitable for coupled systems (see Theorem 5.1). Indeed, it can be applied to systems where the solution space depends on the solution itself. This section can be seen as the second main technical novelty.

The interval of existence is potentially arbitrarily large. The interval of existence is restricted to cases where the geometry degenerates. However, the minimal interval of existence depends on the reference geometry (which defined the shell model) and can be arbitrarily large for some commonly used models. We demonstrate this by providing explicit bounds for two popular reference geometries in Section 2.1: namely the cases where the reference geometry is a sphere or a cylinder.

The method seems very suitable to be adapted for further interaction problems. Possible future applications for FSI problems are in the fields of membrane energies, compressible fluids, tangential displacements, uniqueness issues and/or numerical analysis. In two space dimensions or in the regime of low Reynolds numbers, the method inherits great potential to further improve the regularity theory for the FSI problems [23]. However, due to the lack of a global regularity result for the fluid equations in three dimensions, we do not expect it is possible to further improve the regularity of the shell. In this sense our regularity result for the shell displacement can be viewed as optimal.

This is the point to mention that some applications of our results are already available. Firstly, in [40] the additional regularity for weak solution to *linear plates* is used in order to obtain *weak–strong uniqueness* results. In the case of three-dimensional fluids, the proof depends crucially on the extra regularity obtained here. Secondly, the results were used recently in [6], where Koiter shells are coupled to polymer fluids. Thirdly, we wish to mention work in progress on heat-conducting fluids [8], where the Navier–Stokes–Fourier system is considered. In order to obtain an energy equality, which is an essential part of the concept of weak solutions, compactness of the elastic solid energy is crucial. Hence, this result also depends sensitively on the regularity techniques presented here. Finally,

our methodology for the regularity estimate was also used in the proof of a weak–strong uniqueness result for FSI problems with compressible fluid [43].

Outline of the paper. The next section first derives the Koiter energy with respect to our chosen coordinates, gives two explicit examples of Koiter energies with respective geometric restrictions on $\alpha(\Omega)$, $\beta(\Omega)$, $\gamma(\eta)$ and introduces the definition of a weak solution for fluid–structure interactions. Section 3 is the technical heart of the paper since there the solenoidal extension and approximation operators are introduced. In Section 4 we give the proof of the regularity result Theorem 1.2. Section 5 provides a new version of the Aubin–Lions compactness result that reveals the connection between the existence of suitable extensions and compactness results for fluid–structure interactions involving elastic shells. Finally, in Section 6 we show Theorem 1.1; the existence is shown by combining the extra regularity of the shell with the compactness theory.

2. Weak solutions

2.1. The elastic energy

Coordinates. Here we follow the strategy of [31, Section 2] by introducing the following coordinates attached to the reference geometry Ω , which are well defined in the tubular neighborhood of $\partial\Omega$ (see e.g. [30, Section 10] and Figure 2 for an illustration).

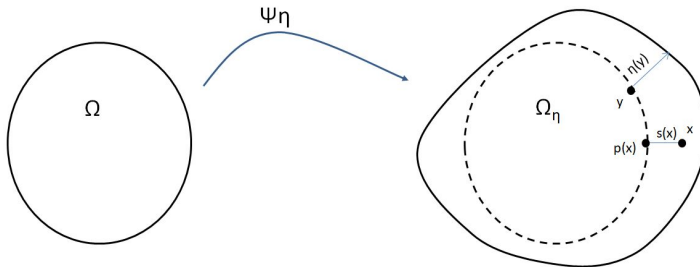


Figure 2. Cross section of a cylindrical domain, its deformation and corresponding coordinate system.

Definition 2.1. Let x be a point in the neighborhood of $\partial\Omega$. We define the distance parameter with respect to the reference point

$$y(x) = \arg \min_{y \in \omega} |x - \varphi(y)|, \quad s(x) = (x - y(x)) \cdot \mathbf{n}(x)$$

and the projection $\mathbf{p}(x) = \varphi(y(x))$.

We define the numbers $\alpha(\Omega)$, $\beta(\Omega)$ so that $(\alpha(\Omega), \beta(\Omega))$ is the largest open interval such that numbers $s(x)$, $\mathbf{p}(x)$, $y(x)$ are uniquely defined over $\{\varphi(y) + s\mathbf{n} : y \in \omega, s \in (\alpha(\Omega), \beta(\Omega))\}$.

For $\kappa > 0$ we introduce the indicator mapping $\sigma_\kappa \in C^\infty(\mathbb{R})$, such that

$$\sigma_\kappa(s) = 1 \text{ for } s \in (\alpha(\Omega) + \kappa, \infty), \quad \sigma_\kappa(s) = 0 \text{ for } s \leq \alpha(\Omega) + \frac{\kappa}{2} \quad \text{and } \sigma'_\kappa \geq 0.$$

We set

$$S_\kappa = \{\boldsymbol{\varphi}(y) + s\mathbf{n}(y) : (s, y) \in [\alpha(\Omega) + \kappa, \beta(\Omega) - \kappa] \times \omega\}.$$

Further

$$Q^\kappa = S_\kappa \cup \Omega \quad \text{and} \quad Q_\kappa = \Omega \setminus S_\kappa.$$

In particular, we have clear information on the support of the derivative:

$$\mathcal{A}_\kappa := \{\boldsymbol{\varphi}(y) + s\mathbf{n}(y) : (s, y) \in [\alpha(\Omega) + \kappa/2, \alpha(\Omega) + \kappa] \times \omega\} \supset \text{supp}(\sigma'_\kappa(s(\cdot))).$$

For $\eta(y) \in (\alpha(\Omega) + \kappa, \beta(\Omega) - \kappa)$, this allows us to introduce the mapping $\boldsymbol{\psi}_\eta(t, \cdot) : \Omega \rightarrow \Omega_{\eta(t)}$ by

$$z \mapsto (1 - \sigma_\kappa(s(z)))z + \sigma_\kappa(s(z))(\mathbf{p}(z) + (\eta(y(z)) + s(z))\mathbf{n}(y(z))),$$

and $\boldsymbol{\psi}_\eta^{-1}(t, \cdot) : \Omega_{\eta(t)} \rightarrow \Omega$ by

$$x \mapsto (1 - \sigma_\kappa(s(x)))x + \sigma_\kappa(s(x))(\mathbf{p}(x) + (s(x) - \eta(y(x)))\mathbf{n}(y(x))).$$

Moreover, we define

$$\boldsymbol{\Phi} : (\alpha(\Omega) + \kappa, \beta(\Omega) - \kappa) \times \omega \rightarrow S_\kappa, \quad \boldsymbol{\Phi}(s, y) = \boldsymbol{\varphi}(y) + \mathbf{n}(y)s.$$

The function's dependence on $\boldsymbol{\varphi}$ and κ is smooth and invertible. This implies that

$$\boldsymbol{\psi}_\eta \circ \boldsymbol{\Phi} : (\alpha(\Omega) + \kappa, 0) \times \omega \rightarrow \Omega_\eta, \quad \boldsymbol{\psi}_\eta \circ \boldsymbol{\Phi}(s, y) = \boldsymbol{\varphi}(y) + (s + \sigma_\kappa(s)\eta(y))\mathbf{n}(y).$$

Finally, we define the following geometric quantity depending on $\partial\Omega$ and η :

$$\gamma(\eta) = \frac{1}{|a_1 \times a_2|} (|a_1 \times a_2| + \eta(\mathbf{n} \cdot (\mathbf{a}_1 \times \partial_2 \mathbf{n} + \partial_1 \mathbf{n} \times \mathbf{a}_2)) + \eta^2 \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n})). \quad (2.1)$$

Remark 2.2. The numbers $\alpha(\Omega)$, $\beta(\Omega)$ do not have to be small. For example, if Ω is a ball or a cylinder with radius R , then $(\alpha(\Omega), \beta(\Omega)) = (-R, \infty)$. The geometric quantity $\gamma(\eta)$ is connected to the H^2 -coercivity of the non-linear structure energy and its meaning is clarified in Lemma 4.3 and Remark 4.4.

Derivation of the elastic energy. The non-linear Koiter model is given in terms of the differences of the first and the second fundamental forms of $\Gamma_\eta(t)$ and Γ . The tangent vectors to the deformed boundary are given by

$$\mathbf{a}_\alpha(\eta) = \partial_\alpha \boldsymbol{\varphi}_\eta = \mathbf{a}_\alpha + \partial_\alpha \eta \mathbf{n} + \eta \partial_\alpha \mathbf{n}, \quad \alpha = 1, 2. \quad (2.2)$$

Therefore, the components of the first fundamental form of the deformed configuration are given by

$$a_{\alpha\beta}(\eta) = \mathbf{a}_\alpha(\eta) \cdot \mathbf{a}_\beta(\eta) = a_{\alpha\beta} + \partial_\alpha \eta \partial_\beta \eta + \eta(\mathbf{a}_\alpha \cdot \partial_\beta \mathbf{n} + \mathbf{a}_\beta \cdot \partial_\alpha \mathbf{n}) + \eta^2 \partial_\alpha \mathbf{n} \cdot \partial_\beta \mathbf{n}. \quad (2.3)$$

We define the change of metric tensor $\mathbf{G}(\eta)$:

$$G_{\alpha\beta}(\eta) = a_{\alpha\beta}(\eta) - a_{\alpha\beta} = \partial_\alpha \eta \partial_\beta \eta + \eta(\mathbf{a}_\alpha \cdot \partial_\beta \mathbf{n} + \mathbf{a}_\beta \cdot \partial_\alpha \mathbf{n}) + \eta^2 \partial_\alpha \mathbf{n} \cdot \partial_\beta \mathbf{n}. \quad (2.4)$$

The normal vector to the deformed configuration is given by

$$\begin{aligned} \mathbf{n}(\eta) = \mathbf{a}_1(\eta) \times \mathbf{a}_2(\eta) &= |\mathbf{a}_1 \times \mathbf{a}_2| \mathbf{n} + \partial_2 \eta (\mathbf{a}_1 \times \mathbf{n} + \eta \partial_1 \mathbf{n} \times \mathbf{n}) \\ &\quad + \partial_1 \eta (\mathbf{n} \times \mathbf{a}_2 + \eta \mathbf{n} \times \partial_2 \mathbf{n}) + \eta (\mathbf{a}_1 \times \partial_2 \mathbf{n} + \partial_1 \mathbf{n} \times \mathbf{a}_2) \\ &\quad + \eta^2 (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}). \end{aligned} \quad (2.5)$$

Notice that $\mathbf{n}(\eta)$ is not a unit vector. We follow our reference literature [14] and use the following tensor \mathbf{R} (denoted by $\mathbf{R}^\#$ in [14, Section 4]) which is some non-normalized variant of the second fundamental form to measure the change of curvature:

$$R_{\alpha\beta}(\eta) = \frac{1}{|\mathbf{a}_1 \times \mathbf{a}_2|} \partial_\alpha \mathbf{a}_\beta(\eta) \cdot \mathbf{n}(\eta) - \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{n}, \quad \alpha, \beta = 1, 2. \quad (2.6)$$

Finally, we define the elasticity tensor in the classical way [14, Theorem 3.2]:

$$\mathbf{A}\mathbf{E} = \frac{4\lambda\mu}{\lambda + 2\mu} (\mathbf{A} : \mathbf{E})\mathbf{A} + 4\mu\mathbf{A}\mathbf{E}\mathbf{A}, \quad \mathbf{E} \in \text{Sym}(\mathbb{R}^{2 \times 2}). \quad (2.7)$$

Here \mathbf{A} is the contravariant metric tensor associated with $\partial\Omega$ (see e.g. [14, Section 2] for the precise definition of \mathbf{A}), and $\lambda > 0, \mu > 0$ are the Lamé constants. The Koiter energy of the shell is given by

$$\mathcal{E}_K(t, \eta) = \frac{h}{4} \int_\omega \mathcal{A}\mathbf{G}(\eta(t, \cdot)) : \mathbf{G}(\eta(t, \cdot)) \, dy + \frac{h^3}{48} \int_\omega \mathcal{A}\mathbf{R}(\eta(t, \cdot)) : \mathbf{R}(\eta(t, \cdot)) \, dy, \quad (2.8)$$

where h is the thickness of the shell. In order to simplify the notation we introduce the following forms connected to the membrane and bending effects in the variational formulation:

$$a_m(t, \eta, \xi) = \frac{h}{2} \int_\omega \mathcal{A}\mathbf{G}(\eta(t, \cdot)) : \mathbf{G}'(\eta(t, \cdot)) \xi \, dy, \quad (2.9)$$

$$a_b(t, \eta, \xi) = \frac{h^3}{24} \int_\omega \mathcal{A}\mathbf{R}(\eta(t, \cdot)) : \mathbf{R}'(\eta(t, \cdot)) \xi \, dy, \quad (2.10)$$

where \mathbf{G}' and \mathbf{R}' denote the Fréchet derivatives of \mathbf{G} and \mathbf{R} respectively. Therefore, the elastodynamics of the shell is given by the following variational formulation:

$$\begin{aligned} h\varrho_s \frac{d}{dt} \int_\omega \partial_t \eta(t, \cdot) \xi \, dy + a_m(t, \eta, \xi) + a_b(t, \eta, \xi) \\ = \int_\omega g \xi \, dy \quad \text{on } (0, T), \xi \in W^{2,p}(\omega), \end{aligned} \quad (2.11)$$

where ϱ_s is the structure density, g is the density of area force acting on the structure and $p > 2$. We denote the elasticity operator by \mathcal{L}_K which is formally given by

$$\langle \mathcal{L}_K \eta, \xi \rangle = a_m(t, \eta, \xi) + a_b(t, \eta, \xi), \quad \xi \in W^{2,p}(\omega). \tag{2.12}$$

Next we give some examples for which we can calculate our restrictive numbers $\alpha(\Omega)$, $\beta(\Omega)$ and $\gamma(\eta)$.

Example 1: Cylindrical Koiter shell. The parameterization of the reference cylinder is given by $\varphi(\theta, z) = (R \cos \theta, R \sin \theta, z)$, $(\theta, z) \in \omega = (0, 2\pi) \times (0, 1)$, where $R > 0$ is the radius of the cylinder. We compute

$$\mathbf{a}_1(\theta, z) = (-R \sin \theta, R \cos \theta, 0), \quad \mathbf{a}_2(\theta, z) = (0, 0, 1), \quad \mathbf{n}(\theta, z) = (\cos \theta, \sin \theta, 0).$$

The corresponding contravariant metric tensor is given by $A = \begin{pmatrix} 1/R^2 & 0 \\ 0 & 0 \end{pmatrix}$. The deformation of the cylindrical boundary is given by

$$\varphi_\eta(\theta, z) = (R \cos \theta + \eta(\theta, z) \cos \theta, R \sin \theta + \eta(\theta, z) \sin \theta, z).$$

Straightforward calculation yields

$$\begin{aligned} \mathbf{a}_1(\eta) &= \left(1 + \frac{1}{R}\right) \mathbf{a}_1 + \eta_\theta \mathbf{n}, & \mathbf{a}_2(\eta) &= \mathbf{a}_2 + \eta_z \mathbf{a}_3, \\ \mathbf{n}(\eta) &= (R + \eta) \mathbf{n} - \eta_z (R + \eta) \mathbf{a}_2 + \frac{\eta_\theta}{R} \mathbf{a}_1. \end{aligned}$$

Therefore, the change of metric tensor is given by

$$\mathbf{G}(\eta) = \begin{pmatrix} (R + \eta)^2 + \eta_\theta^2 - R^2 & \eta_\theta \eta_z \\ \eta_\theta \eta_z & 1 + \eta_z^2 \end{pmatrix}$$

and the change of curvature tensor by

$$\mathbf{R}(\eta) = \begin{pmatrix} \left(1 + \frac{\eta}{R}\right) \eta_{\theta\theta} - \frac{1}{R} (\eta + R)^2 - 2 \frac{\eta_\theta^2}{R} + R & \left(1 + \frac{\eta}{R}\right) \eta_{\theta z} - \frac{1}{R} \eta_\theta \eta_z \\ \left(1 + \frac{\eta}{R}\right) \eta_{\theta z} - \frac{1}{R} \eta_\theta \eta_z & \left(1 + \frac{\eta}{R}\right) \eta_{zz} \end{pmatrix}.$$

Here $(\alpha(\Omega), \beta(\Omega)) = (-R, \infty)$ and $\gamma(\eta) = 1 + \frac{\eta}{R}$.

Example 2: Spherical shell. Strictly speaking, the sphere does not fit in our framework since it does not have a global parameterization. However, this assumption was introduced just for technical simplicity and can be easily removed by working with local coordinates. In this example we consider an elastic sphere with holes around north and south poles. On these holes we prescribe the boundary condition for the fluid flow, e.g. inflow/outflow or Dirichlet. The shell is clamped on the boundary of the holes (see Figure 1 for illustration). More precisely, the parameterization is given by

$$\varphi(\theta, \phi) = (R(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)), \quad (\theta, \phi) \in w = (0, 2\pi) \times (a, \pi - a),$$

where $R > 0$ is the radius of the sphere, and $a > 0$ is the parameter determining the size of the holes. We compute the tangent and normal vectors to the reference configuration

$$\begin{aligned} \mathbf{a}_1 &= -R(\sin \theta \sin \phi, \cos \theta \sin \phi, 0), \\ \mathbf{a}_2 &= R(\cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \pi), \\ \mathbf{n} &= -(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi). \end{aligned}$$

The contravariant metric tensor is given by $A = \begin{pmatrix} 1/R^2 \sin^2 \phi & 0 \\ 0 & 1/R^2 \end{pmatrix}$, and the deformation of the cylindrical boundary by

$$\varphi_\eta(\theta, \phi) = (R - \eta(\theta, \phi))(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi).$$

We calculate the tangent and normal vectors to the deformed configuration:

$$\begin{aligned} \mathbf{a}_1(\eta) &= \left(1 - \frac{\eta}{R}\right)\mathbf{a}_1 + \eta_\theta \mathbf{n}, & \mathbf{a}_2(\eta) &= \left(1 - \frac{\eta}{R}\right)\mathbf{a}_2 + \eta_\phi \mathbf{n}, \\ \mathbf{n}(\eta) &= (R - \eta)^2 \sin \phi \mathbf{n} - \left(1 - \frac{\eta}{R}\right)\left(\frac{\eta_\theta}{\sin \phi} \mathbf{a}_1 + \eta_\phi \sin \phi \mathbf{a}_2\right). \end{aligned}$$

The change of the metric tensor is given by

$$\mathbf{G}(\eta) = \begin{pmatrix} \eta_\theta^2 + (\sin \phi)^2 \eta(\eta - 2R) & \eta_\theta \eta_\phi \\ \eta_\theta \eta_\phi & \eta_\phi^2 + (R - \eta)^2 - R^2 \end{pmatrix}.$$

Finally, the components of the change of curvature tensor are given by

$$\begin{aligned} R_{11}(\eta) &= \frac{1}{2R^2}(-2\eta^3 \sin^2 \phi + \eta^2(6R \sin^2 \phi + \eta_\phi \sin 2\phi + 2\eta_{\theta\theta}) \\ &\quad - 2\eta(3R^2 \sin^2 \phi + R\eta_\phi \sin 2\phi + 2\eta_\theta^2 + 2R\eta_{\theta\theta}) \\ &\quad + R(R\eta_\phi \sin 2\phi + 4\eta_\theta^2 + 2R\eta_{\theta\theta})), \\ R_{12}(\eta) &= R_{21}(\eta) \\ &= \frac{\eta - R}{R^2}(\eta_\theta(R \cot \phi - \eta \cot \phi - 2\eta_\phi) + \eta_{\theta\phi}(\eta - R)), \\ R_{22}(\eta) &= \frac{1}{R^2}(-\eta^3 + \eta^2(3R + \eta_{\phi\phi}) + R(2\eta_\phi^2 + R\eta_{\phi\phi}) - \eta(2\eta_\phi^2 + R(3R + 2\eta_{\phi\phi}))). \end{aligned}$$

The clamped boundary conditions are $\eta = \partial_\phi \eta = 0, \phi = a, \pi - a$. Since we will take finite differences of order less than 1, we can extend η by zero (over the poles) and complete all estimates related to the regularity. Here $(\alpha(\Omega), \beta(\Omega)) = (-\infty, R)$ and $\gamma(\eta) = \frac{(\eta - R)^2}{R^2}$.

2.2. Weak coupled solutions

We use here the standard notation of Bochner spaces related to Lebesgue and Sobolev spaces. We will use bold letters for vector-valued functions in three dimensions. Usually

we take $y \in \omega$ to be a two-dimensional variable and x a three-dimensional variable. In order to define weak solutions, let us first define the appropriate function spaces:

$$\begin{aligned}
 \mathcal{V}_\eta(t) &= \{\mathbf{u} \in H^1(\Omega_\eta(t)) : \operatorname{div} \mathbf{u} = 0\}, \\
 \mathcal{V}_F &= L^\infty(0, T; L^2(\Omega_\eta(t))) \cap L^2(0, T; V_\eta(t)), \\
 \mathcal{V}_K &= L^\infty(0, T; H^2(\omega)) \cap W^{1,\infty}(0, T; L^2(\omega)), \\
 \mathcal{V}_S &= \{(\mathbf{u}, \eta) \in \mathcal{V}_F \times \mathcal{V}_K : \mathbf{u}(t, \boldsymbol{\varphi}_\eta(t, \cdot)) = \partial_t \eta(t, \cdot) \mathbf{n}(\eta(t, \cdot))\}, \\
 \mathcal{V}_T &= \{(\mathbf{q}, \xi) \in \mathcal{V}_F \times \mathcal{V}_K : \mathbf{q}(t, \boldsymbol{\varphi}_\eta(t, \cdot)) = \xi(t, \cdot) \mathbf{n}(\eta(t, \cdot)), \\
 &\quad \partial_t \mathbf{q} \in L^2(0, T; L^2(\Omega_\eta(t)))\}.
 \end{aligned}
 \tag{2.13}$$

Here \mathcal{V}_S and \mathcal{V}_F are solution space and test space respectively. Even though for $\eta \in \mathcal{V}_K$, $\Omega_\eta(t)$ is not necessarily a Lipschitz domain, the traces used in definitions (2.13) and (3.3) are well defined; see [31, Corollary 2.9] (see also [10, 33]). We introduce the concept of a solution which we will consider here. Observe that from this point on we normalize all physical constants $\rho_s = \rho_f = h = \mu = \lambda = 1$ for notational simplicity since the proofs require just positivity of these constants. We emphasize that the restrictions on existence and regularity are only of a geometrical nature. This can be quantified by $\alpha(\Omega)$ and $\beta(\Omega)$ depending only on the reference geometry, and $\gamma(\eta)$ depending on the reference geometry and on the particular magnitude and direction of the displacement, but not on the above physical constants.

Definition 2.3 (Weak solution). We call $(\mathbf{u}, \eta) \in \mathcal{V}_S$ a weak solution of problem (1.7) if it satisfies the energy inequality (2.15) and for every $(\mathbf{q}, \xi) \in \mathcal{V}_T$ the following equality holds in $\mathcal{D}'(0, T)$:

$$\begin{aligned}
 &\frac{d}{dt} \int_{\Omega_\eta(t)} \mathbf{u} \cdot \mathbf{q} \, dx + \int_{\Omega_\eta(t)} (-\mathbf{u} \cdot \partial_t \mathbf{q} - \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{q} + 2 \operatorname{sym} \nabla \mathbf{u} : \operatorname{sym} \nabla \mathbf{q}) \, dx \\
 &\quad + \frac{d}{dt} \int_\omega \partial_t \eta \xi \, dy - \int_\omega \partial_t \eta \partial_t \xi + a_m(t, \eta, \xi) + a_b(t, \eta, \xi) \, dy = 0,
 \end{aligned}
 \tag{2.14}$$

Furthermore, the initial values $\eta_0, \eta_1, \mathbf{u}_0$ are attained in the respective weakly continuous sense.

By formally multiplying (1.7)₁ by \mathbf{u} and (1.7)₂ by $\partial_t \eta$, integrating over $\Omega_\eta(t)$ and ω respectively, integrating by parts and using the coupling conditions (1.7)₄, we obtain the following energy inequality (see e.g. [10, 34] for details of the computations related to the change of the domain and the convective term):

$$\begin{aligned}
 &\frac{1}{2} \|\mathbf{u}(t)\|_{L^2(\Omega_\eta(t))}^2 + \frac{1}{2} \|\partial_t \eta(t)\|_{L^2(\omega)}^2 + \mathcal{E}_K(t, \eta) + 2 \int_0^t \int_{(\Omega_\eta(t))} |\operatorname{sym} \nabla \mathbf{u}|^2 \, dx \, dt \\
 &\leq \frac{1}{2} \|\mathbf{u}_0\|_{L^2(\Omega_{\eta_0})}^2 + \frac{1}{2} \|\eta_1\|_{L^2(\omega)}^2 + \mathcal{E}_K(0, \eta_0) =: C_0.
 \end{aligned}
 \tag{2.15}$$

2.3. Fractional spaces

In this paper, we use the standard definitions of Bochner spaces related to Lebesgue and Sobolev spaces. In particular, we consider *fractional Sobolev spaces* and *Nikolskii spaces*. We recall their definitions here.

For $\alpha \in (0, 1)$ (the order of derivative) and $q \in [1, \infty)$ (the exponent of integrability) we say that $g \in W^{\alpha,q}(A)$, for a domain $A \subset \mathbb{R}^d$, if its norm

$$\|g\|_{W^{\alpha,q}(A)}^q := \left(\int_A \int_A \frac{|g(x) - g(y)|^q}{|x - y|^{n+\alpha q}} dx dy \right)^{\frac{1}{q}} + \left(\int_A |g(x)|^q dx \right)^{\frac{1}{q}}$$

is finite. Fractional Sobolev spaces can be extended to higher order. For $\alpha \in [k, k + 1)$ with $k \in \mathbb{N}$ it is said that $g \in W^{\alpha,q}(A)$ if all partial derivatives of order up to k are in $W^{\alpha-k,q}(A)$. In the particular case $q = 2$ we use the abbreviation

$$H^s(A) \equiv W^{s,2}(A) \quad \text{for } s \in [0, \infty).$$

We say that $g \in N^{\alpha,q}(A)$ if its norm

$$\|g\|_{N^{\alpha,q}(A)} := \sup_{i \in \{1, \dots, d\}} \sup_{h \neq 0} \left(\int_{A_h} \left| \frac{g(x + he_i) - g(x)}{|h|^{\alpha-1}h} \right|^q dx \right)^{\frac{1}{q}} + \left(\int_A |g(x)|^q dx \right)^{\frac{1}{q}},$$

where e_i is the i th unit vector and $A_h = \{x \in A : \text{dist}(x, \partial A) > h\}$ is finite. Nikolskii spaces are closely related to fractional Sobolev spaces $W^{\alpha,q}(A)$. Let us just mention that for $0 < \alpha < \beta < 1$ and a bounded domain A we have

$$W^{\beta,q}(A) \subset N^{\beta,q}(A) \subset W^{\alpha,q}(A).$$

Recall also that for fractional Sobolev spaces an embedding theorem is available for a Lipschitz domain $A \subset \mathbb{R}^n$. Therefore, for $g \in N^{\beta,q}(A)$ and $0 < \alpha < \beta < 1$ we have for $\alpha q < n$ that

$$\|g\|_{L^{\frac{nq}{n-\alpha q}}(A)} \leq c_1 \|g\|_{W^{\alpha,q}(A)} \leq c_2 \|g\|_{N^{\beta,q}(A)}, \tag{2.16}$$

and for $\alpha q > n$,

$$\|g\|_{C^{\alpha-\frac{n}{q}}(A)} \leq c_1 \|g\|_{W^{\alpha,q}(A)} \leq c_2 \|g\|_{N^{\beta,q}(A)}. \tag{2.17}$$

For the above estimates and more detailed study on the given function spaces we refer to [1, Chapter 7] and [42]. The Nikolskii spaces are very popular in the analysis of PDEs, since their definition via difference quotients is rather easy to handle. Namely, we introduce for $g \in L^1(\omega)$ and $h \neq 0$,

$$D_{h,e}^s(g)(x) := \frac{g(x + he) - g(x)}{|h|^{s-1}h} \quad \text{for any (unit) vector } e \in \mathbb{R}^2.$$

In the following, we will omit mentioning the direction e since it is never of relevance and write $D_h^s(q)(y) := D_{h,e}^s(q)(y)$ for an arbitrary direction e . At this point we just wish to mention that these expressions satisfy the summation-by-parts formula

$$\int_{\omega} D_{h,e}^s(g)(y)q(y) dy = - \int_{\omega} g(y)D_{-h,e}^s(q)(y) dy$$

for all periodic functions $g \in L^p(\omega)$ and $q \in L^{p'}(\omega)$ with $p \in [1, \infty]$.

For a vector field $\mathbf{g}: A \rightarrow \mathbb{R}^d$, we say that $\mathbf{g} \in W^{\alpha,p}(A)$ if $\mathbf{g}^i \in W^{\alpha,p}(A)$ for all $i \in \{1, \dots, d\}$. Finally we denote

$$W_{\text{div}}^{\alpha,p}(A) = \{\mathbf{g} \in W_{\text{div}}^{\alpha,p}(A) : \text{div}(\mathbf{g}) = 0 \text{ in a distributional sense}\}.$$

3. Solenoidal extensions and smooth approximations

In this section we construct a divergence-free extension operator from $(0, T) \times \partial\Omega$ to $(0, T) \times \Omega_\eta(t)$. The construction is based on the ideas of the construction in [31, Proposition 2.11]. In contrast to the approach there we will use the celebrated Bogovskii theorem in place of the steady Stokes operator. We use the following theorem that can be found in [18, Section 3.3] and in [17, Appendix 10.5].

Theorem 3.1. *Let the domain Ω be uniformly Lipschitz. There exists a linear operator $\text{Bog}: \widehat{C}_0^\infty(\Omega) \rightarrow C_0^\infty(\Omega)^d$, with the property $\text{div}(\text{Bog}(f)) = f$, which extends from $\widehat{W}_0^{k-1,p}(\Omega) \rightarrow W_0^{k,p}(\Omega)$ for $1 < p < \infty$ and $k \in \{0, 1, 2, \dots\}$, such that*

$$\|\text{Bog}(f)\|_{W^{k,p}(\Omega)} \leq C \|f\|_{\widehat{W}^{k-1,p}(\Omega)}, \quad k \in \mathbb{Z}, \tag{3.1}$$

where C is an absolute constant depending only on the Lipschitz constant. Here we use the notation $\widehat{C}_0^\infty(\Omega) = \{f \in C_0^\infty(\Omega) : \int_\Omega f \, dx = 0\}$, and for $l \geq 0$, $\widehat{W}_0^{l,p}(\Omega) = \{f \in W_0^{l,p}(\Omega) : \int_\Omega f \, dx = 0\}$, $\widehat{W}_0^{-l,p}(\Omega) = \{f \in \mathring{W}^{-l,p}(\Omega) : \langle f, 1 \rangle = 0\}$, where $\mathring{W}^{-l,p}(\Omega)$ is defined via the norm

$$\|f\|_{\mathring{W}^{-l,p}(\Omega)} = \sup_{\substack{\{\phi \in W^{l,p'}(\Omega) : \\ \|\phi\|_{W^{l,p'}(\Omega)} = 1\}}} \langle f, \phi \rangle.$$

Within this section we assume that $\eta: [0, T] \times \omega \rightarrow \mathbb{R}$ is such that there exist α_η, β_η such that

$$\alpha(\Omega) + \kappa \leq \alpha_\eta \leq \eta(t, y) \leq \beta_\eta \leq \beta(\Omega) - \kappa \quad \text{for all } (t, y) \in [0, T] \times \omega. \tag{3.2}$$

Moreover, in this section we use c or C as generic constants which may change their sizes in different instances. Since their dependence on the geometry is relevant for our arguments, it will always be given explicitly in the statements of the results.

The first step is to introduce a solenoidal extension operator. However, since all functions defined on the boundary do not necessarily allow for a solenoidal extension, we first need to construct a suitable corrector. We use the coordinates introduced in Definition 2.1. For ease of readability we define for a function $\xi: \omega \rightarrow \mathbb{R}$,

$$\tilde{\xi}: \partial\Omega \rightarrow \mathbb{R} \quad \text{with } \tilde{\xi}(\mathbf{p}(x)) = \tilde{\xi}(x) := \xi(y(x)).$$

In our solenoidal extension the Bogovskii theorem will be applied to

$$S_{\frac{\xi}{2}} \setminus S_\kappa =: \mathcal{A}_\kappa.$$

Observe that \mathcal{A}_κ is a C^2 domain that contains the support of the function $(\mathbf{p}, s) \mapsto \sigma'_\kappa(\mathbf{p} + s\tilde{\mathbf{n}}(\mathbf{p}))$ from Definition 2.1.

Next we introduce the following weighted mean value over that set. Let $\lambda \in L^\infty(\mathcal{A}_\kappa)$, $\lambda \geq 0$, and $\int_{\mathcal{A}_\kappa} \lambda(x) dx > 0$ be a given weight. Then

$$\langle \psi \rangle_\lambda := \frac{\int_{\mathcal{A}_\kappa} \psi(x)\lambda(x) dx}{\int_{\mathcal{A}_\kappa} \lambda(x) dx} \quad \text{for } \psi \in L^1(\mathcal{A}_\kappa).$$

We will denote

$$\lambda_\eta(t, x) := e^{(s(x)-\eta(t,y(x))) \operatorname{div}(\mathbf{n}(\mathbf{p}(x)))} \sigma'_\kappa(s(x)) \geq 0, \tag{3.3}$$

which has compact support in \mathcal{A}_κ and satisfies (uniformly in t)

$$c_1 \leq \|\lambda_\eta\|_{L^1(\mathcal{A}_\kappa)} \leq c_2 \|\lambda_\eta\|_{L^\infty(\mathcal{A}_\kappa)} \leq c_3$$

for some positive constants $c_1 \leq c_2 \leq c_3$ depending just on κ and the upper and lower bounds of η .

Corollary 3.2 (Corrector). *Let (3.2) be satisfied. Then the corrector map*

$$\mathcal{K}_\eta: L^1(\omega) \rightarrow \mathbb{R}, \quad \xi \mapsto \mathcal{K}_\eta(\xi) = \langle \tilde{\xi} \rangle_{\lambda_\eta} = \frac{\int_{\mathcal{A}_\kappa} \tilde{\xi}(\mathbf{p}(x))\lambda_\eta(t, x) dx}{\int_{\mathcal{A}_\kappa} \lambda_\eta(t, x) dx}$$

satisfies the following estimates for $q \in [1, \infty]$:

$$\|\mathcal{K}_\eta(\xi)\|_{L^q(0,T)} \leq C \|\xi\|_{L^q(0,T;L^1(\omega))}, \tag{3.4}$$

$$\|\partial_t \mathcal{K}_\eta(\xi)\|_{L^q(0,T)} \leq C (\|\partial_t \xi\|_{L^q(0,T;L^1(\omega))} + \|\xi \partial_t \eta\|_{L^q(0,T;L^1(\omega))}), \tag{3.5}$$

whenever the right-hand side is finite. Here C depends only on α_η, β_η and κ .

Proof. The estimates in $L^q(0, T)$ are immediate by the uniform bounds of λ_η and σ . In order to estimate the time derivative, we use the calculation

$$\begin{aligned} \partial_t \langle \xi(t) \rangle_{\lambda_\eta(t)} &= -\frac{1}{\|\lambda_\eta(t)\|_{L^1}^2} \int_{\mathcal{A}_\kappa} \partial_t \lambda_\eta(t) dx \int_{\mathcal{A}_\kappa} \tilde{\xi}(t)\lambda_\eta(t) dx \\ &\quad + \frac{1}{\|\lambda_\eta(t)\|_{L^1}} \int_{\mathcal{A}_\kappa} \partial_t \tilde{\xi}(t)\lambda_\eta(t) dx + \frac{1}{\|\lambda_\eta(t)\|_{L^1}} \int_{\mathcal{A}_\kappa} \tilde{\xi}(t)\partial_t \lambda_\eta(t) dx. \end{aligned}$$

The estimate now follows using $\partial_t \lambda_\eta = -\partial_t \eta \lambda_\eta$ and by the uniform bounds of λ_η and σ . ■

Proposition 3.3 (Solenoidal extension). *Let (3.2) be satisfied and $\eta \in L^\infty(0, T; W^{1,2}(\omega))$. Then there exists a linear solenoidal extension operator*

$$\operatorname{Test}_\eta: \{\xi \in L^1(0, T; W^{1,1}(\omega)) : \mathcal{K}_\eta(\xi) \equiv 0\} \rightarrow L^1(0, T; W^{1,1}(Q^{\frac{\kappa}{2}})),$$

such that $\operatorname{div} \operatorname{Test}_\eta(\xi - \mathcal{K}_\eta(\xi)) = 0$ for all $\xi \in L^1(0, T; W^{1,1}(\omega))$ and $(\operatorname{Test}_\eta(\xi - \mathcal{K}_\eta(\xi)), \xi - \mathcal{K}_\eta(\xi)) \in \mathcal{V}_T$ for $\xi \in \mathcal{V}_K$.

Moreover, $\operatorname{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))(t, x) = 0$ for $(t, x) \in (0, T) \times Q_{\frac{\kappa}{2}}$ and it satisfies the following estimates for $q \in [1, \infty]$, $p \in (1, \infty)$ and $l \in \mathbb{N}$:

$$\|\operatorname{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^q(0, T; L^p(Q_{\frac{\kappa}{2}}))} \leq C \|\xi\|_{L^q(0, T; L^p(\omega))}, \tag{3.6}$$

$$\begin{aligned} \|\operatorname{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^q(0, T; W^{1,p}(Q_{\frac{\kappa}{2}}))} \\ \leq C (\|\xi\|_{L^q(0, T; W^{1,p}(\omega))} + \|\xi \nabla \eta\|_{L^q(0, T; L^p(\omega))}), \end{aligned} \tag{3.7}$$

$$\begin{aligned} \|\partial_t \operatorname{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^q(0, T; L^p(Q_{\frac{\kappa}{2}}))} \\ \leq C (\|\partial_t \xi\|_{L^q(0, T; L^p(\omega))} + \|\xi \partial_t \eta\|_{L^q(0, T; L^p(\omega))}), \end{aligned} \tag{3.8}$$

$$\begin{aligned} \|\nabla^2 \operatorname{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^q(0, T; L^p(Q_{\frac{\kappa}{2}}))} \\ \leq C (\|\nabla^2 \xi\|_{L^q(0, T; L^p(\omega))} + \|\xi \nabla^2 \eta\|_{L^q(0, T; L^p(\omega))}) \\ + C (\|\nabla \xi\|_{L^q(0, T; L^p(\omega))} + \|\xi \nabla \eta\|_{L^q(0, T; L^p(\omega))}) \\ + \|\xi |\nabla \eta|^2\|_{L^q(0, T; L^p(\omega))}, \end{aligned}$$

$$\begin{aligned} \|\partial_t \nabla \operatorname{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^q(0, T; L^p(Q_{\frac{\kappa}{2}}))} \\ \leq C (\|\partial_t \nabla \xi\|_{L^q(0, T; L^p(\omega))} + \|\xi \partial_t \nabla \eta\|_{L^q(0, T; L^p(\omega))}) \\ + C (\|\partial_t \xi\|_{L^q(0, T; L^p(\omega))} + \|\nabla \xi\|_{L^q(0, T; L^p(\omega))} \\ + \|\xi \partial_t \eta\|_{L^q(0, T; L^p(\omega))}), \end{aligned}$$

$$\begin{aligned} \|\partial_n^l \operatorname{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^q(0, T; W^{1,p}(Q_{\frac{\kappa}{2}}))} \\ \leq C \|\operatorname{Test}_\eta(\xi)\|_{L^q(0, T; W^{1,p}(Q_{\frac{\kappa}{2}}))}, \end{aligned} \tag{3.9}$$

whenever the right-hand side is finite. Here C depends only on α_η , β_η and κ .

Proof. Construction. The construction relies exclusively on the reference geometry, namely on S_κ defined in Definition 2.1. Hence to keep the notation compact we will omit the dependence on the time variable t . Moreover, without loss of generality we assume that $\mathcal{K}_\eta(\xi) = 0$, since otherwise we replace ξ by $\xi - \mathcal{K}_\eta(\xi)$, for which we have

$$\begin{aligned} \mathcal{K}_\eta(\xi - \mathcal{K}_\eta(\xi)) &= \frac{\int_{\mathcal{A}_\kappa} (\tilde{\xi} - \mathcal{K}_\eta(\xi)) \lambda_\eta \, dx}{\int_{\mathcal{A}_\kappa} \lambda_\eta \, dx} \\ &= \frac{\int_{\mathcal{A}_\kappa} \tilde{\xi} \lambda_\eta \, dx}{\int_{\mathcal{A}_\kappa} \lambda_\eta \, dx} - \mathcal{K}_\eta(\xi) = 0. \end{aligned}$$

Hence, once the estimates are valid for ξ , such that $\mathcal{K}_\eta(\xi) = 0$, the estimates follow by Corollary 3.2, and also for the case $\mathcal{K}_\eta(\xi) \neq 0$.

First observe that for the coordinates $s(x)$, $\mathbf{p}(x)$ introduced in Definition 2.1 we find

$$\nabla s(x) = \partial_n s(x) \mathbf{n} \quad \text{and} \quad \nabla \mathbf{p}(x) = (\partial_{\tau_i(\mathbf{p}(x))} \mathbf{p}(x))_{i=1, \dots, d-1}$$

and (independent of $s(x)$)

$$\operatorname{div}(\mathbf{n}(\mathbf{p})) = \sum_{i=1}^{d-1} \partial_{\tau_i(\mathbf{p})} \mathbf{n}(\mathbf{p}) \cdot \tau_i(\mathbf{p}).$$

For $y \in \omega$ and $x \in S_\kappa$ we find, by the assumption on Ω , that $y = y(x)$ if and only if $\mathbf{p}(x) = \boldsymbol{\varphi}(y)$ and so (wherever well defined)

$$\partial_{\mathbf{n}(\mathbf{p})} y(x) = 0 \quad \text{and so} \quad \partial_{\mathbf{n}(\mathbf{p})} \xi(y(x)) \equiv 0.$$

Next we introduce the operator

$$\overline{\operatorname{Test}}_\eta(\xi)(x) := e^{(s(x)-\eta(y(x))) \operatorname{div}(\mathbf{n}(\mathbf{p}(x)))} \tilde{\xi}(\mathbf{p}(x)) \sigma_\kappa(s(x)) \mathbf{n}(\mathbf{p}(x)).$$

Observe that for $x \in \Omega_\eta \cap S_\kappa$ we find

$$\overline{\operatorname{Test}}_\eta(\xi)(x) = e^{(s(x)-\eta(y(x))) \operatorname{div}(\mathbf{n}(\mathbf{p}(x)))} \tilde{\xi}(\mathbf{p}(x)) \mathbf{n}(\mathbf{p}(x)).$$

In particular, for $x \in \partial\Omega_\eta$ we find $s(x) = \eta(y(x))$ and hence

$$\overline{\operatorname{Test}}_\eta(\xi)(x) = \mathbf{n}(\mathbf{p}(x)) \xi(y(x)), \quad x \in \partial\Omega_\eta.$$

Using that $\partial_{\mathbf{n}(\mathbf{p}(x))} f(x) = -\partial_s f(\mathbf{p}, s)$ we find for $x \in Q^{\frac{\kappa}{2}} \cap S_\kappa$,

$$\begin{aligned} \operatorname{div}(\overline{\operatorname{Test}}_\eta(\xi)(x)) &= \nabla \left(e^{(s(x)-\eta(y(x))) \operatorname{div}(\mathbf{n}(\mathbf{p}(x)))} \tilde{\xi}(\mathbf{p}(x)) \right) \cdot \mathbf{n}(\mathbf{p}(x)) \\ &\quad + e^{(s(x)-\eta(y(x))) \operatorname{div}(\mathbf{n}(\mathbf{p}(x)))} \tilde{\xi}(\mathbf{p}(x)) \operatorname{div}(\mathbf{n}(\mathbf{p}(x))) \\ &= -\partial_s \left(e^{(s(x)-\eta(y(x))) \operatorname{div}(\mathbf{n}(\mathbf{p}(x)))} \tilde{\xi}(\mathbf{p}(x)) \right) \\ &\quad + e^{(s(x)-\eta(y(x))) \operatorname{div}(\mathbf{n}(\mathbf{p}(x)))} \tilde{\xi}(\mathbf{p}(x)) \operatorname{div}(\mathbf{n}(\mathbf{p}(x))) \\ &= 0. \end{aligned}$$

On \mathcal{A}_κ we find (by the same calculations) that

$$\operatorname{div}(\overline{\operatorname{Test}}_\eta(\xi)(x)) = -e^{(s(x)-\eta(y(x))) \operatorname{div}(\mathbf{n}(\mathbf{p}(x)))} \tilde{\xi}(\mathbf{p}(x)) \sigma'_\kappa(s(x)),$$

which has compact support in \mathcal{A}_κ . Moreover,

$$\int_{\mathcal{A}_\kappa} \operatorname{div}(\overline{\operatorname{Test}}_\eta(\xi)(x)) \, dx = - \int_{\mathcal{A}_\kappa} \lambda_\eta(x) \tilde{\xi}(\mathbf{p}(x)) \, dx = 0.$$

Since \mathcal{A}_κ is by assumption a C^2 domain, we can apply the Bogovskiĭ operator, which we denote by $\operatorname{Bog}_\kappa$, on this domain. We define

$$\operatorname{Test}_\eta(\xi)(x) := \overline{\operatorname{Test}}_\eta(\xi)(x) - \operatorname{Bog}_\kappa(\operatorname{div}(\overline{\operatorname{Test}}_\eta(\xi)))(x).$$

Estimates. The estimates are quite standard, relying on the regularity of φ , namely on the C^2 -regularity of $\partial\Omega$. We give some details of the estimate in order to show a clear dependence on η .

We start with the estimates of the time derivative of $\overline{\text{Test}}_\eta(\xi)$. We calculate

$$\partial_t \text{Test}_\eta(\xi) = \partial_t \overline{\text{Test}}_\eta(\xi) - \text{Bog}_\kappa(\partial_t \text{div}(\overline{\text{Test}}_\eta(\xi))).$$

The Bogovskii operator is well defined due to the fact that (formally)

$$\begin{aligned} \int_{\Omega_{\eta(t)}} \partial_t \text{div}(\overline{\text{Test}}_\eta(\xi)) \, dx &= \int_{\mathcal{A}_\kappa} \partial_t \text{div}(\overline{\text{Test}}_\eta(\xi)) \, dx \\ &= \partial_t \left(\int_{\mathcal{A}_\kappa} \text{div}(\overline{\text{Test}}_\eta(\xi)) \, dx \right) = 0. \end{aligned}$$

We calculate further,

$$\begin{aligned} &\partial_t \overline{\text{Test}}_\eta(\xi)(t, x) \\ &= e^{(s(x)-\eta(t,y(x))) \text{div}(\mathbf{n}(\mathbf{p}(x)))} \partial_t \xi(t, y(x)) \mathbf{n}(\mathbf{p}(x)) \\ &\quad - \text{div}(\mathbf{n}(\mathbf{p}(x))) \partial_t \eta(t, y(x)) e^{(s(x)-\eta(t,y(x))) \text{div}(\mathbf{n}(\mathbf{p}(x)))} \xi(t, y(x)) \mathbf{n}(\mathbf{p}(x)), \end{aligned}$$

which implies the following pointwise estimate for $\partial_t \overline{\text{Test}}_\eta(\xi)$:

$$|\partial_t \overline{\text{Test}}_\eta(\xi)(t, x)| \leq c(|\partial_t \xi(t, \mathbf{p}(x))| + |\partial_t \eta(t, \mathbf{p}(x))| |\xi(t, \mathbf{p}(x))|), \tag{3.10}$$

where the constant only depends on $\kappa, \alpha_\eta, \beta_\eta$ and the C^2 -regularity of $\partial\Omega$. For the sake of better understanding we demonstrate that the assumption $\mathcal{K}_\eta(\xi) = 0$ is indeed without loss of generality. We estimate

$$\begin{aligned} &|\partial_t \overline{\text{Test}}_\eta(\xi - \mathcal{K}_\eta(\xi))(t, x)| \\ &\leq c(|\partial_t \xi(t, \mathbf{p}(x))| + |\partial_t \eta(t, \mathbf{p}(x))| (|\xi(t, \mathbf{p}(x))| + \|\xi(t)\|_{L^1}) \\ &\quad + \|\partial_t \eta(t)\xi(t)\|_{L^1}). \end{aligned} \tag{3.11}$$

In order to estimate the Bogovskii part we find by Theorem 3.1 (with a constant just depending on the Lipschitz constant of \mathcal{A}_κ) that

$$\begin{aligned} \|\text{Bog}_\kappa(\partial_t \text{div}(\overline{\text{Test}}_\eta(\xi - \mathcal{K}_\eta(\xi))))\|_{L^p(\Omega_\eta)} &= \|\text{Bog}_\kappa(\partial_t \text{div}(\overline{\text{Test}}_\eta(\xi - \mathcal{K}_\eta(\xi))))\|_{L^p(\mathcal{A}_\kappa)} \\ &\leq c \|\text{div}(\partial_t \overline{\text{Test}}_\eta(\xi - \mathcal{K}_\eta(\xi)))\|_{\widehat{W}^{-1,p}(\mathcal{A}_\kappa)} \\ &= c \|\partial_t \overline{\text{Test}}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^p(\mathcal{A}_\kappa)}, \end{aligned}$$

and so the estimate of $\partial_t \text{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))$ follows by (3.11).

The estimates of $\nabla \text{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))$, $\nabla^2 \text{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))$ and $\partial_t \nabla \text{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))$ are analogous and we skip the details here. Observe that due to the compact support of

$\operatorname{div}(\overline{\operatorname{Test}}_\eta(\xi - \mathcal{K}_\eta(\xi)))$, by the Gauss theorem we find that

$$\int_{\Omega_{\eta(t)}} \nabla^l \operatorname{div}(\overline{\operatorname{Test}}_\eta(\xi - \mathcal{K}_\eta \xi)) \, dx = 0 = \int_{\Omega_{\eta(t)}} \partial_t \nabla^l \operatorname{div}(\overline{\operatorname{Test}}_\eta(\xi - \mathcal{K}_\eta \xi)) \, dx;$$

hence $\operatorname{Bog}_\kappa$ is always well defined.

Clearly, the normal derivatives of the constructed function $\overline{\operatorname{Test}}_\eta(\xi - \mathcal{K}_\eta(\xi))$ depend on the estimates of the derivatives of σ_κ and not on the regularity of the derivatives of η . Since the Bogovskii theorem transfers the regularity to $\operatorname{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))$ with no further loss, (3.9) follows with according dependencies on the higher-order derivatives of σ_κ . ■

We include the following corollary that will be necessary for our compactness result (see Section 5).

Corollary 3.4 (Smooth solenoidal extension). *Let $a, r \in [2, \infty]$, $p, q \in (1, \infty)$ and $s \in [0, 1]$. Assume that $\eta \in L^r([0, T]; W^{2,a}(\omega)) \cap W^{1,r}([0, T]; L^a(\omega))$, such that $\alpha(\Omega) + \kappa \leq \alpha_\eta \leq \eta \leq \beta_\eta \leq \beta(\Omega) - \kappa$.*

Let $b \in W^{s,p}(\omega)$ and take $(b)_\delta$ as a smooth approximation of b in ω . Then $E_{\eta,\delta}(b) := \operatorname{Test}_\eta((b)_\delta - \mathcal{K}_\eta((b)_\delta))$ satisfies all the regularity of Proposition 3.3. In particular,

$$\|E_{\eta(t),\delta}(b) - \operatorname{Test}_\eta(b - \mathcal{K}_\eta(b))\|_{L^p(Q^{\frac{\kappa}{2}})} \leq c \|(b)_\delta - b\|_{L^p(\omega)}$$

and

$$\|\partial_t E_{\eta(t),\delta}(b)\|_{L^r(0,T;L^a(Q^{\frac{\kappa}{2}}))} \leq c \|(b)_\delta \partial_t \eta(t)\|_{L^r(0,T;L^a(\omega))}$$

uniformly in $t \in (0, T)$.

We include the following technical lemma, that will be necessary for the regularity result.

Lemma 3.5. *Let $p, \tilde{a} \in (1, \infty)$ such that $p' < \tilde{a} \leq \frac{dp'}{d-p'}$ if $p' < d$, and $p' < \tilde{a} < \infty$ otherwise, and let the assumptions of Proposition 3.3 be satisfied. Assume additionally that $\eta \in C^{0,\theta}(\omega) \cap W^{1,\frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}}(\omega)$ and $\mathbf{u} \in W^{1,p'}(\Omega_\eta)$. Then the above constructed test function satisfies*

$$\begin{aligned} & \left| \int_{\Omega_\eta} \mathbf{u} \cdot \operatorname{Test}_\eta(D_{h,e}^s \xi - \mathcal{K}_\eta(D_{h,e}^s \xi)) \, dx \right| \\ & \leq (h^{\theta-s} + \|D_{h,e}^s \eta\|_{W^{1,\frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}}(\omega)}) \|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)} \|\xi\|_{L^p(\omega)} \end{aligned} \tag{3.12}$$

and in the case $\partial_t \xi \in L^p(\omega)$,

$$\begin{aligned} & \left| \int_{\Omega_\eta} \mathbf{u} \cdot \partial_t \operatorname{Test}_\eta(D_{h,e}^s \xi - \mathcal{K}_\eta(D_{h,e}^s \xi)) \, dx \right| \\ & \leq c \left((h^{\theta-s} + \|D_{h,e}^s \eta\|_{W^{1,\frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}}(\omega)}) \|\partial_t \xi\|_{L^p(\omega)} \|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)} \right. \\ & \quad + \|D_{h,e}^s \xi(t)\|_{L^{\tilde{a}}(\omega)} \|\partial_t \eta\|_{L^{\tilde{a}}(\omega)} \|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)} \\ & \quad \left. + \|D_{h,e}^s \xi(t)\|_{L^1(\omega)} \|\partial_t \eta(t)\|_{L^1(\omega)} \|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)} \right). \end{aligned} \tag{3.13}$$

The constants depend only on $\alpha_\eta, \beta_\eta, \kappa$ and (linearly) on $\|\eta\|_{C^{1,\theta}(\omega)}$.

Proof. In the following we use the abbreviation $\delta_h f(y) = (f(y + e_i h) - f(y))$ for $i = 1, 2$. Moreover, since all estimates are done pointwise in time, we omit the dependence on t of η and Ω_η . First, since the support of $\text{Test}_\eta(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi))$ is $S_{\frac{\kappa}{2}}$, we can use the coordinates (\mathbf{p}, s) on the full support of $\text{Test}_\eta(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi))$. We will use the change of coordinates $\psi_\eta \circ \Phi: \omega \times (\alpha + \kappa/2, 0] \rightarrow \Omega_\eta$ in order to be able to do integration by parts. Hence

$$\begin{aligned} & \int_{\Omega_\eta} \mathbf{u} \cdot \text{Test}_\eta(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi)) \, dx \\ &= \int_{\alpha+\kappa/2}^0 \int_\omega (\mathbf{u} \cdot \text{Test}_\eta(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi))) \circ \psi_\eta \circ \Phi |\det(\nabla(\psi_\eta \circ \Phi))| \, dy \, ds. \end{aligned}$$

We will use the following abbreviations for the sake of a better overview:

$$\begin{aligned} \alpha &= \alpha(\Omega), \quad \tilde{\gamma}(s, y) := |\det(\nabla(\psi_\eta \circ \Phi)(s, y))|, \\ \tilde{\lambda}_\eta(s, y) &:= e^{(s-\eta(y) \operatorname{div}_x(\mathbf{n}(y)))} \sigma'_\kappa(s) \tilde{\gamma}(s, y). \end{aligned}$$

Hence, we calculate

$$\begin{aligned} \|\lambda_\eta\|_{L^1(\mathcal{A}_\kappa)} \mathcal{K}_\eta(\delta_h \xi) &= \int_{\alpha+\kappa/2}^{\alpha+\kappa} \int_\omega \lambda_\eta \circ \psi_\eta \circ \Phi \delta_h \xi |\det(\nabla(\psi_\eta \circ \Phi))| \, dy \, ds \\ &=: \int_{\alpha+\kappa/2}^{\alpha+\kappa} \int_\omega \delta_h \xi(y) \tilde{\lambda}_\eta(s, y) \, dy \, ds \\ &= \int_{\alpha+\kappa/2}^{\alpha+\kappa} \int_\omega \xi(y) \delta_h \tilde{\lambda}_\eta(s, y) \, dy \, ds, \end{aligned}$$

where we used the summation by parts formula for finite differences. Therefore,

$$\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi) = \delta_h(\xi - \mathcal{K}_\eta(\xi)) - \frac{\int_{\alpha+\kappa/2}^{\alpha+\kappa} \int_\omega \xi(y) \delta_h(\tilde{\lambda}_\eta) \, dy}{\|\lambda_\eta\|_{L^1(\mathcal{A}_\kappa)}}.$$

And so

$$\begin{aligned} & \overline{\text{Test}_\eta(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi))} \circ \psi_\eta \circ \Phi(s, y) \\ &= \delta_h(\overline{\text{Test}_\eta(\xi - \mathcal{K}_\eta(\xi))} \circ \psi_\eta \circ \Phi(s, y)) \\ &\quad - \delta_h(e^{(s-\eta(y) \operatorname{div}_x(\mathbf{n}(y)))} \sigma_\kappa(s) (\xi - \langle \xi \rangle_{\lambda_\eta}) \mathbf{n}(y)) \\ &\quad - e^{(s-\eta(y) \operatorname{div}_x(\mathbf{n}(y)))} \sigma_\kappa(s) (\xi - \langle \xi \rangle_{\lambda_\eta}) \delta_h(\mathbf{n}(y)) \\ &\quad - e^{(s-\eta(y) \operatorname{div}_x(\mathbf{n}(y)))} \sigma_\kappa(s) \frac{\int_{\alpha+\kappa/2}^{\alpha+\kappa} \int_\omega \xi(y) \delta_h(\tilde{\lambda}_\eta) \, dy \, ds}{\|\lambda_\eta\|_{L^1(\mathcal{A}_\kappa)}} \mathbf{n}(y) \\ &=: \delta_h(\overline{\text{Test}_\eta(\xi)} \circ \psi_\eta \circ \Phi(s, y)) + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3. \end{aligned}$$

Partial summation and Hölder’s inequality imply that

$$\begin{aligned} & \left| \int_{\Omega_\eta} \mathbf{u} \cdot \text{Test}_\eta(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi)) \, dx \right| \\ & \leq \left| \int_{\alpha+\kappa/2}^0 \int_\omega \mathbf{u} \circ \psi_\eta \circ \Phi \cdot \delta_h \overline{\text{Test}_\eta}(\xi - \mathcal{K}_\eta(\xi)) \circ \psi_\eta \circ \Phi \tilde{\gamma} \, dy \, ds \right| \\ & \quad + \left| \int_{\alpha+\kappa/2}^0 \int_\omega (\mathbf{u} \circ \psi_\eta \circ \Phi \cdot (\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3)) \tilde{\gamma} \, dy \, ds \right| \\ & \quad + \left| \int_{\mathcal{A}_\kappa} \mathbf{u} \cdot \text{Bog}_\kappa(\text{div}(\overline{\text{Test}_\eta}(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi)))) \, dx \right| \\ & = \text{(I)} + \text{(II)} + \text{(III)}. \end{aligned}$$

Recall, that $p' < \tilde{a} \leq \frac{dp'}{d-p'}$ (if $p < d$ and no upper bound otherwise). Observe that

$$|\delta_h \eta| \leq ch^\theta, \quad |\gamma| + |\nabla(\psi_\eta \circ \Phi)| \leq c(1 + |\nabla \eta|) \quad \text{and} \quad |\delta_h \tilde{\gamma}| \leq c|\delta_h \nabla \eta|$$

and

$$\begin{aligned} |\delta_h \mathbf{u} \circ \psi_\eta \circ \Phi| & \leq \left| \frac{\mathbf{u}(\psi_\eta \circ \Phi(x+h)) - \mathbf{u}(\psi_\eta \circ \Phi(x))}{\psi_\eta \circ \Phi(x+h) - \psi_\eta \circ \Phi(x)} \right| |\psi_\eta \circ \Phi(x+h) - \psi_\eta \circ \Phi(x)| \\ & \leq ch^\theta \int_{\psi_\eta \circ \Phi(x)}^{\psi_\eta \circ \Phi(x+h)} |\nabla u| \, ds. \end{aligned}$$

We estimate (I) using partial integration, the above inequality, Hölder’s inequality for $\frac{1}{p} + \frac{1}{\tilde{a}} + \frac{\tilde{a}p-p-\tilde{a}}{\tilde{a}p} = 1$ and Sobolev embedding:

$$\begin{aligned} \text{(I)} & = \left| \int_{\alpha+\kappa/2}^0 \int_\omega (\delta_h(\mathbf{u} \circ \psi_\eta \circ \Phi) \cdot \overline{\text{Test}_\eta}(\xi - \mathcal{K}_\eta(\xi)) \circ \psi_\eta \circ \Phi) \tilde{\gamma} \right. \\ & \quad \left. + (\mathbf{u} \circ \psi_\eta \circ \Phi) \cdot (\overline{\text{Test}_\eta}(\xi - \mathcal{K}_\eta(\xi)) \circ \psi_\eta \circ \Phi) \delta_h \tilde{\gamma} \, dy \, ds \right| \\ & \leq ch^\theta \|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)} \|\xi\|_{L^p(\omega)} + c\|\mathbf{u}\|_{L^{\tilde{a}}(\Omega_\eta)} \|\xi\|_{L^p(\omega)} \|\delta_h \eta\|_{W^{1, \frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}(\omega)}} \\ & \leq c(h^\theta + \|\delta_h \eta\|_{W^{1, \frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}(\omega)}}) \|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)} \|\xi\|_{L^p(\omega)}. \end{aligned} \tag{3.14}$$

We further estimate (II) by using, in a rather straightforward manner, the fact that $|\delta_h g| \leq h^\theta \|g\|_{\mathcal{C}}^{0,\theta}$:

$$\begin{aligned} |\mathcal{T}_1 \tilde{\gamma}| & \leq c(h + |\delta_h \eta|)(|\xi| \tilde{\gamma} + |\langle \xi \rangle_{\tilde{\lambda}_\eta} \tilde{\lambda}_\eta|) \leq ch^\theta (|\xi| \tilde{\gamma} + |\langle \xi \rangle_{\tilde{\lambda}_\eta} \tilde{\lambda}_\eta|), \\ |\mathcal{T}_2 \tilde{\gamma}| & \leq ch(|\xi| \tilde{\gamma} + |\langle \xi \rangle_{\tilde{\lambda}_\eta} \tilde{\lambda}_\eta|), \\ |\mathcal{T}_3 \tilde{\gamma}| & \leq c \frac{|\tilde{\lambda}_\eta|}{\|\tilde{\lambda}_\eta\|_{L^1(\mathcal{A}_\kappa)}} \|\xi\|_{L^p(\omega)} \|\delta_h \eta\|_{W^{1,p'}(\Omega)}. \end{aligned}$$

This implies

$$\begin{aligned} \sum_{i=1}^2 \int_{\alpha+\kappa/2}^0 \int_{\omega} |\mathcal{T}_i \tilde{\gamma}|^p dy ds &\leq ch^{\theta p} \|\xi\|_{L^p(\omega)}^p \left(1 + \frac{\|\tilde{\lambda}_\eta\|_{L^{p'}(\mathcal{A}_\kappa)}^p}{\|\tilde{\lambda}_\eta\|_{L^1(\mathcal{A}_\kappa)}^p}\right), \\ \int_{\alpha+\kappa/2}^0 \int_{\omega} |\mathcal{T}_3 \tilde{\gamma}|^p dy ds &\leq c \frac{\|\tilde{\lambda}_\eta\|_{L^p([\alpha+\kappa/2, 0] \times \omega)}^p}{\|\lambda_\eta\|_{L^1(\mathcal{A}_\kappa)}^p} \|\xi\|_{L^p(\omega)}^p \|\delta_h \eta\|_{W^{1,p'}(\omega)}^p. \end{aligned}$$

Hence, we find by Hölder’s and Poincaré’s inequalities that

$$\begin{aligned} \text{(II)} &\leq c \|\mathbf{u}\|_{L^{p'}(\Omega_\eta)} \sum_{i=1}^3 \|\mathcal{T}_i \tilde{\gamma}\|_{L^p((\alpha+\kappa/2, 0) \times \omega)} \\ &\leq c(h^\theta + \|\delta_h \eta\|_{W^{1,p'}(\omega)}) \|\xi\|_{L^p(\omega)} \|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)}. \end{aligned} \tag{3.15}$$

The estimates of (I) and (II) allow us to estimate the Bogovskiĭ term (III). This is possible since due to Theorem 3.1 and due to the compact support of σ' in \mathcal{A}_κ we find

$$\begin{aligned} \text{(III)} &:= \left\| \langle \mathbf{u}, \text{Bog}_\kappa(\text{div}(\overline{\text{Test}}_\eta(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi))) \rangle \right\| \\ &\leq \|\mathbf{u}\|_{W^{1,p'}(\mathcal{A}_\kappa)} \|\text{Bog}_\kappa(\text{div}(\overline{\text{Test}}_\eta(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi)))\|_{\dot{W}^{-1,p}(\mathcal{A}_\kappa)} \\ &\leq c \|\mathbf{u}\|_{W^{1,p'}(\mathcal{A}_\kappa)} \|\text{div}(\overline{\text{Test}}_\eta(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi)))\|_{\dot{W}^{-2,p}(\mathcal{A}_\kappa)} \\ &\leq c \|\mathbf{u}\|_{W^{1,p'}(\mathcal{A}_\kappa)} \|\overline{\text{Test}}_\eta(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi))\|_{\dot{W}^{-1,p}(\mathcal{A}_\kappa)}. \end{aligned}$$

Now take $q \in W_0^{1,p'}(\mathcal{A}_\kappa)$, with $\|q\|_{W^{1,p'}(\mathcal{A}_\kappa)} \leq 1$ arbitrary. From the calculations above, i.e. by replacing \mathbf{u} by q in (3.14) and (3.15) we find

$$\begin{aligned} &\langle \overline{\text{Test}}_\eta(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi)), q \rangle \\ &= \int_{\alpha+\kappa/2}^0 \int_{\omega} (\delta_h(\overline{\text{Test}}_\eta(\xi - \mathcal{K}_\eta(\xi))) \cdot q) \circ \psi_\eta \circ \Phi \tilde{\gamma} dy ds \\ &\quad + \sum_{i=1}^3 \int_{\alpha+\kappa/2}^{\alpha+\kappa} \int_{\omega} \mathcal{T}_i \cdot q \circ \psi_\eta \circ \Phi \tilde{\gamma} dy ds \\ &= \int_{\alpha+\kappa/2}^0 \int_{\omega} (\overline{\text{Test}}_\eta(\xi - \mathcal{K}_\eta(\delta_h \xi)) \circ \psi_\eta \circ \Phi \cdot \delta_h(q \circ \psi_\eta \circ \Phi \tilde{\gamma})) dy ds \\ &\quad + \sum_{i=1}^3 \int_{\alpha+\kappa/2}^{\alpha+\kappa} \int_{\omega} \mathcal{T}_i \cdot q \circ \psi_\eta \circ \Phi \tilde{\gamma} dy ds \\ &\leq c(h^\theta + \|\delta_h \eta\|_{W^{1, \frac{\tilde{a}p}{\tilde{a}p - \tilde{a} - p}}(\omega)}) \|\xi\|_{L^p(\omega)}. \end{aligned}$$

So

$$\begin{aligned} \text{(III)} &\leq c \|\mathbf{u}\|_{W^{1,p'}(\mathcal{A}_\kappa)} \|\overline{\text{Test}}_\eta(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi))\|_{\dot{W}^{-1,p}(\mathcal{A}_\kappa)} \\ &\leq c \|\mathbf{u}\|_{W^{1,p'}(\mathcal{A}_\kappa)} (h^\theta + \|\delta_h \eta\|_{W^{1, \frac{\tilde{a}p}{\tilde{a}p - \tilde{a} - p}}(\omega)}) \|\xi\|_{L^p(\omega)}. \end{aligned}$$

This finishes the proof of (3.12). For the time derivative we use the fact that

$$\begin{aligned} \partial_t(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi)) &= (\delta_h \partial_t \xi - \mathcal{K}_\eta(\delta_h \partial_t \xi)) - \frac{\langle \delta_h \tilde{\xi}(t) \rangle_{\lambda_\eta}}{\|\lambda_\eta(t)\|_{L^1}} \int_{\mathcal{A}_\kappa} \partial_t \lambda_\eta(t) dx \\ &\quad + \frac{1}{\|\lambda_\eta(t)\|_{L^1}} \int_{\mathcal{A}_\kappa} \delta_h \tilde{\xi}(t) \partial_t \lambda_\eta(t) dx \\ &=: (\delta_h \partial_t \xi - \mathcal{K}_\eta(\delta_h \partial_t \xi)) + K(t), \end{aligned} \tag{3.16}$$

and hence

$$\begin{aligned} &\partial_t \overline{\text{Test}_\eta}(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi))(t, x) \\ &= \sigma(s(x)) e^{(s(x) - \eta(t, y(x))) \operatorname{div}(\mathbf{n}(\mathbf{p}(x)))} (\delta_h(\partial_t \xi) - \mathcal{K}_\eta(\delta_h(\partial_t \xi)))(t, y(x)) \mathbf{n}(\mathbf{p}(x)) \\ &\quad - \sigma(s(x)) \operatorname{div}(\mathbf{n}(\mathbf{p}(x))) \partial_t \eta(t, y(x)) e^{(s(x) - \eta(t, y(x))) \operatorname{div}(\mathbf{n}(\mathbf{p}(x)))} \\ &\quad \times (\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi))(t, y(x)) \mathbf{n}(\mathbf{p}(x)) \\ &\quad - \sigma(s(x)) \operatorname{div}(\mathbf{n}(\mathbf{p}(x))) e^{(s(x) - \eta(t, y(x))) \operatorname{div}(\mathbf{n}(\mathbf{p}(x)))} K(t) \mathbf{n}(\mathbf{p}(x)) \\ &= (\text{A}) + (\text{B}) + (\text{C}). \end{aligned}$$

The estimate of (A) follows by (3.12). We proceed with the straightforward estimates

$$|(\text{B})| \leq c |\partial_t \eta(t, y(x))| (|\delta_h(\xi(t, y(x)))| + \|\delta_h(\xi(t))\|_{L^1(\mathcal{A}_\kappa)})$$

and

$$|(\text{C})| \leq \|\delta_h \xi(t)\|_{L^1(\omega)} \|\partial_t \eta(t)\|_{L^1(\omega)} + \|\delta_h \xi(t)\| |\partial_t \eta|_{L^1(\omega)}.$$

Hence we find, by (3.12), the estimates of (B) and (C), Hölder’s inequality and Sobolev embedding, that

$$\begin{aligned} &\left| \int_{\Omega_\eta} \mathbf{u} \cdot \partial_t \overline{\text{Test}_\eta}(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi)) dx \right| \\ &\leq (h^\theta + \|\delta_h \eta\|_{W^{1, \frac{\tilde{a}p}{\tilde{a}p - \tilde{a} - p}}(\omega)}) \|\partial_t \xi\|_{L^p(\omega)} \|\mathbf{u}\|_{W^{1, p'}(\Omega_\eta)} + c \|\mathbf{u}\|_{L^{\tilde{a}}(\Omega_\eta)} \|\delta_h \xi \partial_t \eta\|_{L^{\tilde{a}'(\omega)}} \\ &\leq c \left((h^\theta + \|\delta_h \eta\|_{W^{1, \frac{\tilde{a}p}{\tilde{a}p - \tilde{a} - p}}(\omega)}) \|\partial_t \xi\|_{L^p(\omega)} \right. \\ &\quad \left. + (\|\delta_h \xi\| |\partial_t \eta|_{L^{\tilde{a}}(\omega)} + \|\delta_h \xi(t)\|_{L^1(\omega)} \|\partial_t \eta(t)\|_{L^1(\omega)}) \|\mathbf{u}\|_{W^{1, p'}(\Omega_\eta)} \right). \end{aligned}$$

The Bogovskii part will be estimated once more in the form of negative norms using that

$$\begin{aligned} &\sup_{\|q\|_{W^{1, p'}(\mathcal{A}_\kappa)} \leq 1} \langle \partial_t \overline{\text{Test}_\eta}(\delta_h \xi - \mathcal{K}_\eta(\delta_h \xi)), q \rangle \\ &\leq c \left((h^\theta + \|\delta_h \eta\|_{W^{1, \frac{\tilde{a}p}{\tilde{a}p - \tilde{a} - p}}(\omega)}) \|\partial_t \xi\|_{L^p(\omega)} + \|\delta_h \xi(t)\|_{L^1(\omega)} \|\partial_t \eta(t)\|_{L^1(\omega)} \right. \\ &\quad \left. + \|\delta_h \xi(t)\| |\partial_t \eta|_{L^{\tilde{a}}(\omega)} \right), \end{aligned}$$

which finishes the proof. ■

4. The regularity result

4.1. Estimates for the structure

In this section we explore the consequences of the energy inequality (1.8).

Lemma 4.1 (Uniform Korn inequality). *For every $\mathbf{u} \in \mathcal{V}_F$ such that $\mathbf{u}(t, \varphi_\eta(t, \cdot)) = \xi \mathbf{n}$ the following Korn equality holds:*

$$\|\nabla \mathbf{u}\|_{L^2 \Omega_\eta(t)}^2 = 2 \|\text{sym } \nabla \mathbf{u}\|_{L^2 \Omega_\eta(t)}^2. \tag{4.1}$$

Proof. We follow the idea from [10, Lemma 6] and compute

$$\int_{\Omega_\eta(t)} |\text{sym } \nabla \mathbf{u}|^2 dx = \frac{1}{2} \left(\int_{\Omega_\eta(t)} |\nabla u|^2 dx + \int_{\Omega_\eta(t)} \nabla^T \mathbf{u} : \nabla \mathbf{u} dx \right).$$

Therefore it remains to show that the second term is zero:

$$\begin{aligned} \int_{\Omega_\eta(t)} \nabla^T \mathbf{u} : \nabla \mathbf{u} dx &= \sum_{i,j=1}^2 \int_{\Omega_\eta(t)} \partial_j u_i \partial_i u_j dx \\ &= - \sum_{i,j=1}^2 \int_{\Omega_\eta(t)} \partial_j \partial_i u_i u_j dx + \int_{\partial \Omega_\eta(t)} \partial_j u_i n_i u_j dS = \int_{\partial \Omega_\eta(t)} (\nabla \mathbf{u}) \mathbf{n} \cdot \mathbf{u} dS. \end{aligned}$$

Now using the no-slip condition (1.5) and the incompressibility condition we deduce that $\int_{\partial \Omega_\eta(t)} (\nabla \mathbf{u}) \mathbf{n} \cdot \mathbf{u} dS = 0$ (see [31, Lemma A.5]) and therefore Korn’s equality holds. ■

In the following we exploit the energy estimate (2.15). In particular, the number C_0 , which depends only on the initial conditions, always refers to this energy bound.

Lemma 4.2. *Let (\mathbf{u}, η) be such that energy inequality (2.15) is satisfied. Then $\eta \in L^\infty(0, T; W^{1,4}(\omega))$ and $\|\eta\|_{L_t^\infty W_x^{1,4}} \leq c C_0$, where c depends only on φ .*

Proof. The boundedness of $\|\eta\|_{L_t^\infty L_x^2}$ follows directly from the energy inequality (2.15). Now we use [13, Theorem 3.3-2] to conclude that by the definition of \mathcal{A} and (2.15),

$$\int_\omega |\mathbf{G}(\eta(t, \cdot))|^2 dy \leq c \int_\omega \mathcal{A} \mathbf{G}(\eta(t, \cdot)) : \mathbf{G}(\eta(t, \cdot)) dy \leq c C_0,$$

where the constant c just depends on the Lamé constants and the geometry of $\partial \Omega$. If $\partial_\alpha \mathbf{n} \neq 0$ we may use the bound for $G_{\alpha\alpha}(\eta)$ and (2.4) to get the bounds for $\|\partial_\alpha \eta(t)\|_{L^4(\omega)}$ and $\|\eta(t)\|_{L^4(\omega)}$ uniform in t . Using these bounds, (2.4) again and the bound for $G_{\beta\beta}(\eta)$ above for $\beta \neq \alpha$ we finish the proof.

If $\partial_1 \mathbf{n} = \partial_2 \mathbf{n} = 0$, we get the bound for $\|\nabla \eta\|_{L^4}$ directly from (2.4) and the boundedness of $\int_\omega |\mathbf{G}(\eta(t, \cdot))|^2$. However, since $\|\eta\|_{L_t^\infty L_x^2}$ is also bounded (using the bounds on $\partial_t \eta$ in (1.8)), the lemma follows also by the Poincaré inequality. ■

Lemma 4.3. *Let (\mathbf{u}, η) be such that the energy inequality (2.15) is satisfied. Then if $\gamma(\eta) \neq 0$ we have $\eta(t) \in H^2(\omega)$. Moreover,*

$$\sup_{t \in [0, T]} \int_{\omega} \gamma^2(\eta) |\nabla^2 \eta|^2 dy \leq c C_0,$$

where c depends only on φ .

Proof. We can again use [13, Theorem 3.3-2] and work with bounds on \mathbf{R} . From (2.2) we compute

$$\partial_{\beta} \mathbf{a}_{\alpha}(\eta) = \partial_{\alpha\beta}^2 \varphi + \partial_{\alpha\beta}^2 \eta \mathbf{n} + \partial_{\alpha} \eta \partial_{\beta} \mathbf{n} + \partial_{\beta} \eta \partial_{\alpha} \mathbf{n} + \eta \partial_{\alpha\beta}^2 \mathbf{n}, \quad \alpha, \beta = 1, 2. \tag{4.2}$$

Using (2.2), (2.5), (2.6) and the definition of γ from Definition 2.1 we have

$$\begin{aligned} R_{\alpha\beta}(\eta) &= \frac{1}{|\mathbf{a}_1 \times \mathbf{a}_2|} \partial_{\alpha\beta}^2 \eta (|a_1 \times a_2| + \eta(\mathbf{n} \cdot (\mathbf{a}_1 \times \partial_2 \mathbf{n} + \partial_1 \mathbf{n} \times \mathbf{a}_2)) \\ &\quad + \eta^2 \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n})) + P_0(\eta, \nabla \eta) \\ &=: \gamma(\eta) \partial_{\alpha\beta}^2 \eta + P_0(\eta, \nabla \eta), \end{aligned} \tag{4.3}$$

where P_0 is a polynomial of order 3 in η and $\nabla \eta$ such that all terms are at most quadratic in $\nabla \eta$, and the coefficients of P_0 depend on φ .

From Lemma 4.2 we gain, in particular by Sobolev embedding, that $\|\eta\|_{L_t^{\infty} L_x^{\infty}}$ and $\|\nabla \eta\|_{L_t^{\infty} L_x^4}$ are bounded by the energy. Therefore

$$\sup_{t \in [0, T]} \int_{\omega} \gamma^2(\eta) |\nabla^2 \eta|^2 dy \leq c (\|\mathbf{R}\|_{L_t^{\infty} L_x^2} + \|P_0(\eta, \nabla \eta)\|_{L_t^{\infty} L_x^2}) \leq c C_0. \quad \blacksquare$$

Remark 4.4. By definition we know that $\gamma(\eta) > 0$, as long as

$$\eta(\mathbf{n} \cdot (\mathbf{a}_1 \times \partial_2 \mathbf{n} + \partial_1 \mathbf{n} \times \mathbf{a}_2)) + \eta^2 \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}) > -\frac{1}{|\mathbf{a}_1 \times \mathbf{a}_2|}. \tag{4.4}$$

Therefore, since $\gamma(\eta_0) > 0$, it follows that there exists a c_2 (depending on φ only) such that if $\|\eta - \eta_0\|_{L_t^{\infty} L_x^{\infty}} \leq c_2$, then (4.4) is satisfied and hence $\gamma(\eta) > 0$. Finally, the energy estimate allows us to deduce directly (combining the $L_t^{\infty} W_x^{1,4}$ and the $W_t^{1,\infty} L_x^2$ estimate), that depending on the initial configuration there is a minimal time interval $(0, T)$ for which $\|\eta - \eta_0\|_{L_t^{\infty} L_x^{\infty}} \leq c_2$ is always satisfied.

Similarly to the previous lemma, let us write the form a_b defined by (2.9) as the sum of the bilinear form in second derivatives plus the remainder. We calculate the Fréchet derivative of \mathbf{R} :

$$R_{\alpha\beta}(\eta)\xi = \gamma(\eta) \partial_{\alpha\beta}^2 \xi + \gamma(\xi) \partial_{\alpha\beta}^2 \eta + P'_0(\eta, \nabla \eta)\xi.$$

Therefore, we have

$$\begin{aligned}
 a_b(t, \eta, \xi) &= \frac{h^3}{24} \int_w [\mathcal{A}(\gamma(\eta)\nabla^2\eta) : (\gamma(\eta)\nabla^2\xi) + \mathcal{A}(\gamma(\eta)\nabla^2\eta) : (\gamma(\xi)\nabla^2\eta) \\
 &\quad + (\mathcal{A}(\gamma(\eta)\nabla^2\eta) : P'_0(\eta, \nabla\eta)\xi + \mathcal{A}(P_0(\eta, \nabla\eta)) : (\gamma(\xi)\nabla^2\eta)) \\
 &\quad + \mathcal{A}(P_0(\eta, \nabla\eta)) : (\gamma(\eta)\nabla^2\xi) + \mathcal{A}(P_0(\eta, \nabla\eta)) : P'_0(\eta, \nabla\eta)\xi] dy \\
 &= a_b^1(\eta; \nabla^2\eta, \nabla^2\xi) + a_b^2(\eta, \nabla^2\eta; \xi) + a_b^3(\eta, \nabla\eta, \nabla^2\eta; \xi, \nabla\xi) \\
 &\quad + a_b^4(\eta, \nabla\eta; \nabla\xi, \nabla^2\xi) + a_b^5(\eta, \nabla\eta, \xi, \nabla\xi). \tag{4.5}
 \end{aligned}$$

We take $\xi = D_{-h}^s D_h^s \eta$, $0 < s < 1/2$, and obtain the following estimates.

Lemma 4.5. *Let $\eta \in H^2(\omega)$ such that $\gamma(\eta) \neq 0$. Then for every $h > 0$, $0 < s < 1/2$ the following inequality holds:*

$$a_b(t, \eta, D_{-h}^{-s} D_h^s \eta) \geq \|D_h^s \nabla^2 \eta\|_{L^2(\omega)} - C(\|\eta\|_{H^2(\omega)}).$$

Proof. Since all estimates in this lemma are uniform in t for simplicity of notation, we omit the t variable in this proof. First we use the fact that since $\omega \subset \mathbb{R}^2$, Sobolev embedding implies $\|D_h^s \eta\|_{L^\infty} \leq c\|\eta\|_{H^2(\omega)}$ and $\|D_{-h}^s D_h^s \eta\|_{L^\infty(\omega)} \leq c\|\eta\|_{H^2(\omega)}$. Due to Sobolev embedding the estimate is uniform in h for all $s \in (0, 1/2)$. This and the integration by parts formula for finite differences can be used to estimate a_b^1 :

$$\begin{aligned}
 a_b^1(\eta; \nabla^2\eta, \nabla^2 D_{-h}^s D_h^s \eta) &\geq C \int_\omega |D_h^s \nabla^2 \eta|^2 dy - C \|D_h^s \gamma(\eta)^2\|_{L^\infty} \|\nabla^2 \eta\|_{L^2} \|D_h^s \nabla^2 \eta\|_{L^2} \\
 &\geq \frac{C}{2} \|D_h^s \nabla^2 \eta\|_{L^2}^2 - C \|D_h^s \gamma(\eta)^2\|_{L^\infty}^2 \|\nabla^2 \eta\|_{L^2}^2 \\
 &\geq \frac{C}{2} \|D_h^s \nabla^2 \eta\|_{L^2}^2 - C(\|\eta\|_{H^2(\omega)}).
 \end{aligned}$$

Similarly, since $\|D_{-h}^s D_h^s \eta\|_{L^\infty(\omega)} \leq \|\eta\|_{H^2(\omega)}$ uniformly, we estimate

$$|a_b^2(\eta, \nabla^2\eta, D_{-h}^s D_h^s \eta)| \leq C(\|\eta\|_{H^2(\omega)}).$$

To estimate a_b^3 we first notice that $\|P_0(\eta, \nabla\eta)\|_{L^2} \leq C\|\eta\|_{L^\infty} \|\nabla\eta\|_{L^4}^2 \leq C(\|\eta\|_{H^2(\omega)})$. Moreover,

$$\|P'_0(\eta, \nabla\eta) D_h^s \eta\|_{L^2} \leq C\|\eta\|_{L^\infty} \|\nabla\eta\|_{L^4} \|\nabla D_h^s \eta\|_{L^4} \leq C(\|\eta\|_{H^2(\omega)}).$$

Now we can use integration by parts and Young’s inequality in the same way as in the estimate for a_b^1 to get

$$|a_b^3(\eta, \nabla\eta, \nabla^2\eta; D_{-h}^s D_h^s \eta, \nabla D_{-h}^s D_h^s \eta)| \leq \frac{C}{8} \|D_h^s \nabla^2 \eta\|_{L^2}^2 + C(\|\eta\|_{H^2(\omega)}).$$

The estimate for a_b^4 is done in an analogous way by integration by parts and using

$$\|D_h^s P_0(\eta, \nabla\eta)\|_{L^2} \leq \|\eta\|_{L^\infty} \|\eta\|_{W^{1,4}} \|\nabla D_h^s \eta\|_{L^4} \leq C(\|\eta\|_{H^2(\omega)}).$$

Hence,

$$|a_b^4(\eta, \nabla \eta; \nabla D_{-h}^s D_h^s \eta, \nabla^2 D_{-h}^s D_h^s \eta)| \leq \frac{C}{8} \|D_h^s \nabla^2 \eta\|_{L^2}^2 + C(\|\eta\|_{H^2(\omega)}).$$

Finally, the last term a_b^5 is a lower-order term and is easily estimated using the same inequalities:

$$|a_b^5(\eta, \nabla \eta, D_{-h}^s D_h^s \eta, \nabla D_{-h}^s D_h^s \eta)| \leq C(\|\eta\|_{H^2(\omega)}). \quad \blacksquare$$

4.2. Closing the estimates – Proof of Theorem 1.2

In this section we finish the proof of Theorem 1.2. Observe first that due to the Sobolev embedding theorem and due to the trace theorem [7, Lemma 2.4], we find for all $\theta \in (0, 1)$ and all $s \in (0, \frac{1}{2})$,

$$\begin{aligned} \|\eta\|_{L^\infty(0,T;C^{0,\theta}(\omega))} &\leq c\|\eta\|_{L^\infty(0,T;H^2(\omega))}, \\ \|\partial_t \eta\|_{L^2(0,T;H^s(\omega))} &\leq c\|\mathbf{u}\|_{L^2(0,T;H^1(\Omega_\eta))}. \end{aligned}$$

Assume that $s \in (0, \frac{1}{2})$ and take

$$(\text{Test}_\eta(D_{-h}^s D_h^s \eta - \mathcal{K}_\eta(D_{-h}^s D_h^s \eta)), D_{-h}^s D_h^s \eta - \mathcal{K}_\eta(D_{-h}^s D_h^s \eta))$$

as a test function in (2.14) and integrate from 0 to T . The test function is admissible by construction; see Proposition 3.3. The estimates of the forms a_m and a_b connected to the elastic energy follow directly by Lemma 4.5. Indeed, since $\mathcal{K}_\eta(D_{-h}^s D_h^s \eta)$ is constant in the space direction and hence does not change the estimate of the derivatives of η we find (using the uniform bounds on λ_η) that

$$\inf_\omega (\gamma^2(\eta)) a_b(t, \eta, (D_{-h}^s D_h^s \eta - \mathcal{K}_\eta(D_{-h}^s D_h^s \eta))) \geq \|D_h^s \nabla^2 \eta\|_{L^2(\omega)} - C(\|\eta\|_{H^2(\omega)}).$$

Hence we are left to estimate the term coming from the structure inertia. Using partial integration and Corollary 3.2, we find

$$\begin{aligned} &\int_0^T \left| \int_\omega \partial_t \eta \partial_t (D_{-h}^s D_h^s \eta - \mathcal{K}_\eta(D_{-h}^s D_h^s \eta)) \, dy \right| dt \\ &= \int_0^T \left| \int_\omega \partial_t \eta D_{-h}^s (\partial_t (D_h^s \eta - \mathcal{K}_\eta(D_h^s \eta))) \, dy \right| dt \\ &= \int_0^T \left| \int_\omega (D_h^s \partial_t \eta)^2 - D_h^s \partial_t \eta \partial_t \mathcal{K}_\eta(D_h^s \eta) \, dy \right| dt \\ &\leq c \|\partial_t \eta\|_{L^2(0,T;H^s(\omega))}^2 \\ &\quad + \int_0^T \|\partial_t \eta\|_{W^{1,s}(\omega)} (\|\partial_t \eta\|_{W^{1,s}(\omega)} + \|\partial_t \eta\|_{L^2(\omega)}) \|\nabla \eta\|_{L^2(\omega)} \, dt \\ &\leq c \|\partial_t \eta\|_{L^2(0,T;H^s(\omega))}^2 + cT \|\partial_t \eta\|_{L^\infty(0,T;L^2(\omega))} \|\nabla \eta\|_{L^\infty(0,T;L^2(\omega))}^2 \\ &\leq c \|\mathbf{u}\|_{L^2(0,T;H^1(\Omega_\eta(t)))}^2 + cT \|\partial_t \eta\|_{L^\infty(0,T;L^2(\omega))} \|\nabla \eta\|_{L^\infty(0,T;L^2(\omega))}^2 \\ &\leq cC_0^2. \end{aligned}$$

Here, in the last estimate, we used the trace theorem [7, Lemma 2.4] and the coupling condition (1.5). Notice that this term cannot be estimated in a purely hyperbolic problem and that here it is essential to use the coupling and the fluid dissipation.

Let us next prove the estimates related to the fluid part. From Proposition 3.3 and the energy inequality (2.15) we have the following estimate:

$$\begin{aligned} & \|\nabla \text{Test}_\eta (D_{-h}^s D_h^s \eta - \mathcal{K}_\eta(D_{-h}^s D_h^s \eta))\|_{L^\infty(0,T;L^2(\Omega_\eta(t)))} \\ & \leq C(\|D_{-h}^s D_h^s \eta\|_{L^\infty(0,T;H^1(\omega))} + \|(D_{-h}^s D_h^s \eta)\nabla \eta\|_{L^\infty(0,T;L^2(\omega))}) \\ & \leq C(\|\eta\|_{L^\infty(0,T;H^2(\omega))} + \|D_{-h}^s D_h^s \eta\|_{L^\infty(0,T;L^\infty(\omega))} \|\nabla \eta\|_{L^\infty(0,T;L^2(\omega))}) \\ & \leq C(\|\eta\|_{L^\infty(0,T;H^2(\omega))} + \|\eta\|_{L^\infty(0,T;H^2(\omega))}^2) \\ & \leq C(C_0 + C_0^2). \end{aligned}$$

This allows us to estimate the integrals:

$$\begin{aligned} & \left| \int_0^T \int_{\Omega_\eta(t)} (-\mathbf{u} \otimes \mathbf{u} : \nabla \text{Test}_\eta (D_{-h}^s D_h^s \eta - \mathcal{K}_\eta(D_{-h}^s D_h^s \eta)) \right. \\ & \quad \left. + \text{sym } \nabla \mathbf{u} : \text{sym } \nabla \text{Test}_\eta D_{-h}^s D_h^s \eta) dx \right| \\ & \leq \|\nabla \text{Test}_\eta D_{-h}^s D_h^s \eta\|_{L^\infty(0,T;L^2(\Omega_\eta(t)))} \\ & \quad \times (\|\nabla \mathbf{u}\|_{L^2(0,T;L^2(\Omega_\eta(t)))}^2 + \|\nabla \mathbf{u}\|_{L^2(0,T;L^2(\Omega_\eta(t)))}) \\ & \leq C(C_0 + C_0^2)^2. \end{aligned}$$

The most difficult estimate is the one involving the distributional time derivative of v . It can be estimated using Lemma 3.5; indeed, by defining $p = 2 = p'$ and $\tilde{a} = 6$ we get that $\frac{\tilde{a}p}{\tilde{a}p - \tilde{a} - p} = 3$. Hence, using the fact that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and

$$\frac{3}{2} - \frac{2}{3} = \frac{5}{6} < 1 = 2 - \frac{2}{2} \quad \text{and so} \quad W^{\frac{3}{2},3}(\omega) \subset W^{2,2}(\omega),$$

we find by Hölder’s inequality and Sobolev embedding that for every $\theta \in (0, 1)$ there is a constant c , such that

$$\begin{aligned} \text{(I)} & = \left| \int_0^T \int_{\Omega_\eta(t)} \mathbf{u} \cdot \partial_t \text{Test}_\eta (D_{-h}^s D_h^s \eta - \mathcal{K}_\eta(D_{-h}^s D_h^s \eta)) dx \right| \\ & \leq c(h^{\theta-s} + \|D_h^s \eta\|_{L^\infty(0,T;W^{1,3}(\omega))}) \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega_\eta(t)))} \|\partial_t D_h^s \eta\|_{L^2(0,T;L^2(\omega))} \\ & \quad + c\|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega_\eta(t)))} \|D_{-h}^s D_h^s \eta \partial_t \eta\|_{L^2(0,T;L^{6/5}(\omega))}. \end{aligned}$$

Hence, choosing $\theta = s$, we find

$$\begin{aligned} \text{(I)} & \leq c(1 + \|\eta\|_{L^\infty(0,T;W^{3/2,3}(\omega))}) \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega_\eta(t)))} \|\partial_t D_h^s \eta\|_{L^2(0,T;L^2(\omega))} \\ & \quad + c\|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega_\eta(t)))} \|\partial_t \eta\|_{L^\infty(0,T;L^2(\omega))} \|D_{-h}^s D_h^s \eta\|_{L^2(0,T;L^3(\omega))} \end{aligned}$$

$$\begin{aligned} &\leq c(1 + \|\eta\|_{L^\infty(0,T;W^{2,2}(\omega))})\|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega_\eta(t)))}\|\partial_t D_h^s \eta\|_{L^2(0,T;L^2(\omega))} \\ &\quad + c\|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega_\eta(t)))}\|\partial_t \eta\|_{L^\infty(0,T;L^2(\omega))}\|\eta\|_{L^2(0,T;W^{1,3}(\omega))} \\ &\leq cC_0(C_0 + C_0^2), \end{aligned}$$

and the estimate of the time-derivative term is complete. The result follows by combining the obtained estimates.

5. Compactness rewritten

We introduce the following version of the celebrated Aubin–Lions compactness lemma [2, 32]. The version below is tailored to be applicable for coupled systems of PDEs like the fluid–structure interaction which we are studying in this paper. The key point is to fully decouple the *compactness assumption* in space and the *compactness assumption* in time. We emphasize this fact by showing that under appropriate conditions the product of two weak convergent sequences decouples in the limit. It in some sense unifies ideas from time-space decoupling with compensated compactness approaches of div-curl type (see e.g. [9] for some further discussion on that matter).

In this context, the most difficult property to capture is the compactness assumption in time. Commonly it is given in the form of a uniform bound on the time derivative in a certain dual space or more precisely a uniform continuity assumption in time. What turns out to be the key observation is that it suffices only to extract the uniform continuity properties over a suitable approximation of its argument. In the theorem below, requirement (3) summarizes the time-compactness assumption. As can be seen, no function space appears. The assumption is that the pairing of the continuity in time for g_n is uniform with respect to a given suitable approximation of f_n .

This non-function-space-type requirement is necessary for the application in the context of fluid–structure interactions. Indeed, the weak time derivative of an approximate sequence $\partial_t \eta_\varepsilon, v_\varepsilon$ is defined merely over a non-linear coupled space that changes both with respect to time and with respect to the approximation parameter ε itself.

Theorem 5.1. *Let X, Z be two Banach spaces such that $X' \subset Z'$. Assume $f_n: (0, T) \rightarrow X$ and $g_n: (0, T) \rightarrow X'$. Moreover, assume the following:*

- (1) *Weak convergence: for some $s \in [1, \infty]$ we have that $f_n \overset{*}{\rightharpoonup} f$ in $L^s(X)$ and $g_n \overset{*}{\rightharpoonup} g$ in $L^{s'}(X')$.*
- (2) *The approximability condition is satisfied: for every $\delta \in (0, 1]$ there exists an $f_{n,\delta} \in L^s(0, T; X) \cap L^1(0, T; Z)$, such that for every $\varepsilon \in (0, 1)$ there exists a $\delta_\varepsilon \in (0, 1)$ (depending only on ε) such that*

$$\|f_n - f_{n,\delta}\|_{L^s(0,T;X)} \leq \varepsilon \quad \text{for all } \delta \in (0, \delta_\varepsilon]$$

and for every $\delta \in (0, 1]$ there is a $C(\delta)$ such that

$$\|f_{n,\delta}\|_{L^1(0,T;Z)} dt \leq C(\delta).$$

Moreover, we assume that for every δ there is a function f_δ and a subsequence such that $f_{n,\delta} \overset{*}{\rightharpoonup} f_\delta$ in $L^s(X)$.

- (3) Equicontinuity of g_n : for every $\varepsilon > 0$ and $\delta > 0$ there exist an $n_{\varepsilon,\delta}$ and a $\tau_{\varepsilon,\delta} > 0$, such that for all $n \geq n_{\varepsilon,\delta}$ and all $\tau \in (0, \tau_{\varepsilon,\delta}]$,

$$\int_0^{T-\tau} \left| \int_0^\tau \langle g_n(t) - g_n(t+s), f_{n,\delta}(t) \rangle_{X',X} ds \right| dt \leq \varepsilon.$$

- (4) The compactness assumption is satisfied: $X' \hookrightarrow Z'$. More precisely, every uniformly bounded sequence in X' has a strongly converging subsequence in Z' .

Then there is a subsequence, such that

$$\int_0^T \langle f_n, g_n \rangle_{X,X'} dt \rightarrow \int_0^T \langle f, g \rangle_{X,X'} dt.$$

Remark 5.2 (Modification for applications). With regard to our application it seems somehow natural to replace (3) by the following condition:

- (3') The equicontinuity of g_n . We require that there exist an $\alpha \in (0, 1]$ and a sequence A_n that is uniformly bounded in $L^1([0, T])$, such that for every $\delta > 0$ there exist a $C(\delta) > 0$ and an $n_\delta \in \mathbb{N}$ such that for $\tau > 0$ and a.e. $t \in [0, T - \tau]$,

$$\sup_{n \geq n_\delta} \left| \int_0^\tau \langle g_n(t) - g_n(t+s), f_{n,\delta}(t) \rangle_{X',X} ds \right| \leq C(\delta) \tau^\alpha (A_n(t) + 1).$$

Here (3') implies (3) by integration over $[0, T - \tau]$ and an appropriate choice of $\tau_{\delta,\varepsilon}$.

Remark 5.3 (Classic Aubin–Lions lemma). Let us explain how Theorem 5.1 relates to the classic Aubin–Lions lemma. The simplest case is when Z is a compact subspace of X , $f \in L^2(Z)$ and $\partial_t g_n \in L^2(Z')$. In this case one may take $f_{n,\delta} = f_n$ and find for $s < t$,

$$\begin{aligned} |\langle g_n(t) - g_n(s), f_m(t) \rangle| &= \left| \int_s^t \langle \partial_t g_n(\tau) \rangle f_m(t) d\tau \right| \leq \|f_m(t)\|_Z \int_s^t \|\partial_t g_n(\tau)\|_{Z'} d\tau \\ &\leq c |t - s|^{\frac{1}{2}} \|f_m(t)\|_Z, \end{aligned}$$

with $c = \|\partial_t g_n\|_{L^2(0,T;Z')}$. The classic Gelfand triple is then the particular case when X is a Hilbert space. Since then $Z \subset X \subset Z'$ implies the same argument as above.

The generalization to allow that Z is independent of the regularity of f_n is essentially some hidden interpolation result (also known as the Ehrling property). Here, classically, one can use convolution estimates to show that a mollifier in one space is uniformly close, while in the other (smaller space) they are merely bounded. One standard example is the periodic solutions over the torus Q and $X = H_{\text{per}}^a(Q)$ and $Z = H_{\text{per}}^c(Q)$, such that $f_n \in H_{\text{per}}^b(Q)$ uniformly with $a < b < c$. Then convolution with the standard mollifying kernel ψ_δ implies that $\|f - f * \psi_\delta\|_{H_{\text{per}}^r(Q)} \leq C \delta^{s-r} \|f\|_{H_{\text{per}}^s(Q)}$ which implies precisely the properties required in (2) above. This shows that condition (2) can be seen as a “spatial

compactness” condition in the Aubin–Lions lemma (or more generally in Simon’s compactness theorem [41]). Condition (3) could be viewed as a “temporal compactness”, i.e. as equicontinuity of time shifts in the weaker space Z' .

Remark 5.4 (Function spaces). Since we do not assume that Z is dense in X , an additional clarification of condition (4) is required. Let $\bar{Z}^{\|\cdot\|_X} = \bar{X} \cap \bar{Z}^{\|\cdot\|_X}$ be the closure of Z with respect to X and $(\bar{Z}^{\|\cdot\|_X})'$ its dual (with respect to X as pivot space). Then condition (4) has meaning in the sense of $(\bar{Z}^{\|\cdot\|_X})' \hookrightarrow \hookrightarrow Z'$.

Proof of Theorem 5.1. In this proof we will produce for every $\varepsilon > 0$ an $n_\varepsilon \in \mathbb{N}$, a $\tau_\varepsilon > 0$ with $\tau_\varepsilon \rightarrow 0$ for $\varepsilon \rightarrow 0$ and a subsequence of $\langle f_n, g_n \rangle_{X, X'}$ such that

$$\left| \int_0^{T-\tau_\varepsilon} \langle f_n, g_n \rangle_{X, X'} - \langle f_m, g_m \rangle_{X, X'} dt \right| \leq \varepsilon$$

for all $n, m \geq n_\varepsilon$. This then allows to construct the desired converging subsequence by taking a discrete sequence $\varepsilon_i \rightarrow 0$ and a respective diagonal argument.

Hence for $\varepsilon > 0$, we may choose δ_ε in such a way that, for all $\delta \in (0, \delta_\varepsilon]$,

$$\|f_n - f_{n, \delta}\|_{L^s(0, T; X)} \leq \varepsilon. \tag{5.1}$$

Next we fix $\tau_{\varepsilon, 0} > 0$ and $n_{\varepsilon, 0}$ such that, for all $\tau \in (0, \tau_{\varepsilon, 0}]$,

$$\sup_{n \geq n_{\varepsilon, 0}} \int_0^{T-\tau} \left| \int_0^\tau \langle g_n(t) - g_n(s), f_{n, \delta}(t) \rangle_{X', X} ds \right| dt \leq \varepsilon. \tag{5.2}$$

Fix $N \in \mathbb{N}$ such that $\tau_\varepsilon := \frac{T}{N} \leq \tau_{\varepsilon, 0}$. For $k \in \{0, \dots, N - 1\}$ and $n \in \mathbb{N}$ we define

$$g_n^k = \int_{k\tau}^{(k+1)\tau} g_n(s) ds.$$

This implies by Jensen’s inequality,

$$\|g_n^k\|_{X'} \leq \int_{k\tau}^{(k+1)\tau} \|g_n(s)\|_{X'} ds,$$

and so we define for the given τ_ε ,

$$g_n^{\tau_\varepsilon}(t) := g_n^k \quad \text{for } t \in [k\tau_\varepsilon, (k + 1)\tau_\varepsilon).$$

Since

$$\sup_{\substack{[k\tau_\varepsilon, (k+1)\tau_\varepsilon] \\ \subset [0, T]}} \sup_{n \in \mathbb{N}} \|g_n^k\|_{X'} \leq \frac{C}{\tau_\varepsilon},$$

we find by the compactness assumption, that we can find a subsequence for which there exists an $n_{\varepsilon, 1}$, such that

$$\sup_k \|g_n^k - g_m^k\|_{Z'} \leq \varepsilon_0 \quad \text{for all } n, m > n_{\varepsilon, 1}. \tag{5.3}$$

In particular, there exist g^{τ_ε} and a subsequence such that $g_n^{\tau_\varepsilon} \rightarrow g^{\tau_\varepsilon}$ strongly in $L^\infty(0, T; Z')$. Clearly, by the uniform bounds we find that $g^\tau \in L^{s'}(X')$.

At this point τ_ε and δ_ε are fixed. Hence we may define

$$\varepsilon_0 := \frac{\varepsilon}{C(\delta_\varepsilon)},$$

where $C(\delta_\varepsilon)$ is defined via (2). Therefore, we find an $n_\varepsilon \in \mathbb{N}$, such that for all $n, m \geq n_\varepsilon$,

$$\left| \int_0^T \langle f_{n,\delta_\varepsilon}, g_n^{\tau_\varepsilon} - g_m^{\tau_\varepsilon} \rangle dt \right|_{X,X'} \leq \|f_{n,\delta_\varepsilon}\|_{L^1(0,T;Z)} \|g_n^{\tau_\varepsilon} - g_m^{\tau_\varepsilon}\|_{L^\infty(0,T;Z')} \leq \varepsilon. \tag{5.4}$$

Now all preparation has been made in order to estimate

$$\begin{aligned} & \left| \int_0^{T-\tau_\varepsilon} \langle f_n(t), g_n(t) \rangle_{X,X'} - \langle f_m(t), g_m(t) \rangle_{X,X'} dt \right| \\ & \leq \left| \int_0^{T-\tau_\varepsilon} \langle f_{n,\delta_\varepsilon}(t), g_n(t) \rangle_{X,X'} - \langle f_{m,\delta_\varepsilon}(t), g_m(t) \rangle_{X,X'} dt \right| \\ & \quad + \left| \int_0^{T-\tau_\varepsilon} \langle f_n(t) - f_{n,\delta_\varepsilon}(t), g_n(t) \rangle_{X,X'} - \langle f_m(t) - f_{m,\delta_\varepsilon}(t), g_m(t) \rangle_{X,X'} dt \right| \\ & \leq \left| \int_0^{T-\tau_\varepsilon} \langle f_{n,\delta_\varepsilon}(t), g_n(t) \rangle_{X,X'} - \langle f_{m,\delta_\varepsilon}(t), g_m(t) \rangle_{X,X'} dt \right| + 2\varepsilon. \end{aligned}$$

We estimate the right-hand side:

$$\begin{aligned} \text{(I)} & := \int_0^{T-\tau_\varepsilon} \langle f_{n,\delta_\varepsilon}(t), g_n(t) \rangle_{X,X'} - \langle f_{m,\delta_\varepsilon}(t), g_m(t) \rangle_{X,X'} dt \\ & = \int_0^{T-\tau_\varepsilon} \langle f_{n,\delta_\varepsilon}(t), g_n(t) - g_n^{\tau_\varepsilon}(t) \rangle_{X,X'} dt + \int_0^{T-\tau_\varepsilon} \langle f_{n,\delta_\varepsilon}(t), g_n^{\tau_\varepsilon}(t) - g_n^{\tau_\varepsilon}(t) \rangle_{X,X'} dt \\ & \quad + \int_0^{T-\tau_\varepsilon} \langle f_{m,\delta_\varepsilon}(t) - f_{n,\delta_\varepsilon}(t), g_m^{\tau_\varepsilon}(t) \rangle_{X,X'} dt \\ & \quad - \int_0^{T-\tau_\varepsilon} \langle f_{m,\delta_\varepsilon}(t), g_m(t) - g_n^{\tau_\varepsilon}(t) \rangle_{X,X'} dt \\ & = \sum_{k=0}^{N-2} \int_{k\tau_\varepsilon}^{(k+1)\tau_\varepsilon} \int_0^{\tau_\varepsilon} \langle f_{n,\delta_\varepsilon}(t), g_n(t) - g_n(s) \rangle_{X,X'} ds dt \\ & \quad + \sum_{k=0}^{N-2} \int_{k\tau_\varepsilon}^{(k+1)\tau_\varepsilon} \langle f_{n,\delta_\varepsilon}(t), g_n^k - g_m^k \rangle_{X,X'} dt \\ & \quad + \sum_{k=0}^{N-2} \int_{k\tau_\varepsilon}^{(k+1)\tau_\varepsilon} \langle f_{m,\delta_\varepsilon}(t) - f_{n,\delta_\varepsilon}(t), g_m^k \rangle_{X,X'} dt \\ & \quad - \sum_{k=0}^{N-2} \int_{k\tau_\varepsilon}^{(k+1)\tau_\varepsilon} \int_0^{\tau_\varepsilon} \langle f_{m,\delta_\varepsilon}(t), g_m(t) - g_m(s) \rangle_{X,X'} ds dt \\ & = \text{(II)} + \text{(III)} + \text{(IV)} + \text{(V)}. \end{aligned}$$

First observe that (II) and (V) can be estimated using the *equicontinuity condition*, namely (5.2). Term (III) is estimated using the *compactness condition*, namely (5.4). Finally, for (IV) we deviate and apply (5.4) a second and third time:

$$\begin{aligned}
 \text{(IV)} &= \int_0^T \langle f_{m,\delta_\varepsilon} - f_{n,\delta_\varepsilon}, g_m^{\tau_\varepsilon} \rangle_{X,X'} dt \\
 &= \int_0^T \langle f_{m,\delta_\varepsilon} - f_{n,\delta_\varepsilon}, g^{\tau_\varepsilon} \rangle_{X,X'} + \langle f_{m,\delta_\varepsilon} - f_{n,\delta_\varepsilon}, g_m^{\tau_\varepsilon} - g^{\tau_\varepsilon} \rangle_{X,X'} dt \\
 &\leq \left| \int_0^T \langle f_{m,\delta_\varepsilon} - f_{n,\delta_\varepsilon}, g^{\tau_\varepsilon} \rangle_{X,X'} \right| + 2\varepsilon.
 \end{aligned}$$

Now we take another subsequence of $f_{n,\delta_\varepsilon} \in L^s(X)$ that converges weakly*. Hence we may eventually increase n_ε one last time (depending on g^{τ_ε}) and find that for this subsequence and $n, m \geq n_\varepsilon$,

$$|\text{(IV)}| \leq 3\varepsilon.$$

This finishes the proof. ■

6. The existence result

6.1. The approximate system

In this section we construct approximate solutions $(\mathbf{u}^\varepsilon, \eta^\varepsilon) \in \mathcal{V}_S$, $\eta^\varepsilon \in L^\infty(0, T; H^3(\omega))$ which satisfy the following weak formulation:

$$\begin{aligned}
 &\frac{d}{dt} \int_{\Omega_\eta(t)} \mathbf{u}^\varepsilon \cdot \mathbf{q} dx + \int_{\Omega_\eta(t)} (-\mathbf{u}^\varepsilon \cdot \partial_t \mathbf{q} - \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon : \nabla \mathbf{q} + \text{sym} \nabla \mathbf{u}^\varepsilon : \text{sym} \nabla \mathbf{q}) dx \\
 &\quad + \frac{d}{dt} \int_\omega \partial_t \eta^\varepsilon \xi dy - \int_\omega \partial_t \eta^\varepsilon \partial_t \xi dy + a_m(t, \eta^\varepsilon, \xi) + a_b(t, \eta^\varepsilon, \xi) \\
 &\quad + \varepsilon \int_\omega \nabla_x^3 \eta^\varepsilon : \nabla_x^3 \xi dy = 0,
 \end{aligned} \tag{6.1}$$

where $\varepsilon > 0$ is a regularizing parameter, and with initial conditions $\eta_0, \eta_1, \mathbf{u}_0$. In this section we prove the following theorem:

Theorem 6.1. *There exists a $T > 0$ just depending on $\partial\Omega$ and the initial data, such that for every $\varepsilon \in (0, 1]$ there exists a weak solution $(\mathbf{u}^\varepsilon, \eta^\varepsilon)$ to the regularized problem (6.1). Moreover, the weak solution satisfies the following uniform in ε estimate:*

$$\|\mathbf{u}^\varepsilon\|_{\mathcal{V}_F} + \|\eta^\varepsilon\|_{\mathcal{V}_S} + \|\eta^\varepsilon\|_{L^2(0,T;N^{s,2}(\omega))} \leq C \tag{6.2}$$

for every (fixed) $s < \frac{5}{2}$, with C depending only on $\partial\Omega$ and the initial conditions.

The existence of regularized solutions can be proved following the ideas and techniques introduced in [36]. The problem solved in [36] is actually very similar to the

regularized system above since there the existence of a solution to an FSI problem with the structure being an elastic shell with non-linear Koiter membrane energy without bending energy, but with a (linear) regularization term of fourth order is shown. In order to be able to treat the non-linear bending energy in an analogous way we have to include a sixth-order regularization term. Another difference comes from the fact that in [36] a cylindrical geometry is considered. Nevertheless, the introduced existence scheme does not depend on the geometry of the problem and more general geometries can be handled by combining the existence proof with the estimates in this paper and in [31]. To avoid lengthy repetitions of the arguments analogous to [36] here, we summarize the main steps of the construction of a weak solution with emphasis on the differences coming from the non-linear bending term and the setting of more general geometries. The main steps of the construction are the following:

(1) **Arbitrary Lagrangian–Eulerian formulation.** We reformulate the problem in a fixed reference domain Ω using a suitable change of variables. This approach is popular in numerics and the change of variables is called an arbitrary Lagrangian–Eulerian (ALE) mapping. The formulation in the fixed reference domain is called the ALE formulation of the FSI problem. We use the mapping ψ_η (introduced in Definition 2.1) as an ALE mapping.

(2) **Construction of the approximate solutions.** We construct the approximate solutions using time discretizations and operator splitting methods. We use the Lie splitting strategy (also known as Marchuk–Yanenko splitting) to decouple the FSI problem.

(3) **Uniform estimates.** Let $\Delta t > 0$ be the time-discretization parameter. We show that the constructed approximate solutions satisfy uniform bounds with respect to Δt (and ε) in the energy function spaces. We identify weak and weak* limits.

(4) **Compactness.** We prove that the set of approximate solutions is compact in suitable norms. By using the compactness we prove that a limit of the sequence of approximate solutions is a weak solution to the regularized FSI problem. Here we use a generalization of the Aubin–Lions–Simon lemma from [37] for discrete-in-time solutions adapted to moving domain problems.

Since a solution is constructed by decoupling the problem, the largest difference from [36] is in the second step, where in the structure subproblem we also include the non-linear bending energy. However, we will show that the bending term can be discretized in an analogous way to the membrane term. Other steps are analogous to [36] using sixth-order regularization. For the convenience of the reader, we will describe the details of the time discretization of the structure subproblem with corresponding uniform estimates in the time-discretization parameter Δt . We conclude this chapter with a description of the compactness step. Generally, for more details of the procedure we refer the reader to [36].

In the rest of the subsection we fix the regularizing parameter ε and drop superscripts ε in $(\mathbf{u}^\varepsilon, \eta^\varepsilon)$ since there is no chance of confusion.

Construction of discrete approximations. The main problem in the construction of an approximate solution is how to discretize the Fréchet derivatives of \mathbf{G} and \mathbf{R} to obtain the discrete analogue of $\mathbf{R}'(\eta)\partial_t\eta = \partial_t\mathbf{R}(\eta)$. In [36] this was achieved by using the fact that the first fundamental form was a polynomial of order 2 of η and $\nabla\eta$, which was a consequence of the cylindrical geometry. Here we consider a more general geometry, so we need to develop a more general approach.

For a given end-time T , we fix Δt as the time step, such that $[0, T] = [0, N\Delta t]$ for some $N \in \mathbb{N}$. Now let $(\eta^n)_{n=1}^N$ be a given time-discrete solution and $\tilde{\eta}$ be the piecewise linear function in time such that $\tilde{\eta}(n\Delta t) = \eta^n$. Then we have

$$\mathbf{R}'(\tilde{\eta})\frac{\eta^{n+1} - \eta^n}{\Delta t} = \mathbf{R}'(\tilde{\eta})\partial_t\tilde{\eta} = \partial_t\mathbf{R}(\tilde{\eta}) \quad \text{on } [n\Delta t, (n+1)\Delta t].$$

Notice that the expression $\mathbf{R}'(\tilde{\eta})\partial_t\tilde{\eta}$ is a third-order polynomial in the t variable, so we can compute its integral $\int_{n\Delta t}^{(n+1)\Delta t}$ by using the Newton–Cotes formula. Hence, by defining $\bar{\eta}^{n+1} := \frac{\eta^{n+1} + \eta^n}{2}$ we find the approximations of $\mathbf{G}'(\eta)\xi$ and $\mathbf{R}'(\eta)\xi$ in the following way:

$$\mathbf{G}'(\eta^{n+1}, \eta^n)\xi := \frac{1}{3}(\mathbf{G}'(\eta^n) + 4\mathbf{G}'(\bar{\eta}^{n+1}) + \mathbf{G}^{n+1})\xi \tag{6.3}$$

and

$$\mathbf{R}'(\eta^{n+1}, \eta^n)\xi := \frac{1}{3}(\mathbf{R}'(\eta^n) + 4\mathbf{R}'(\bar{\eta}^{n+1}) + \mathbf{R}^{n+1})\xi. \tag{6.4}$$

By straightforward calculation it follows that

$$\mathbf{G}'(\eta^{n+\frac{1}{2}}, \eta^n)\frac{\eta^{n+\frac{1}{2}} - \eta^n}{\Delta t} = \Delta t \int_{n\Delta t}^{(n+1)\Delta t} \frac{d}{dt}\mathbf{G}(\tilde{\eta}) = \frac{1}{\Delta t}(\mathbf{G}(\eta^{n+\frac{1}{2}}) - \mathbf{G}(\eta^n)),$$

which is the correct substitute for “ $\partial_t\mathbf{G}(\eta) = \mathbf{G}'(\eta)\partial_t\eta$ ”. Analogously we find as the substitute for “ $\partial_t\mathbf{R}(\eta) = \mathbf{R}'(\eta)\partial_t\eta$ ”,

$$\mathbf{R}'(\eta^{n+\frac{1}{2}}, \eta^n)\frac{\eta^{n+\frac{1}{2}} - \eta^n}{\Delta t} = \frac{1}{\Delta t}(\mathbf{R}(\eta^{n+\frac{1}{2}}) - \mathbf{R}(\eta^n)).$$

These identities will be used to derive a semi-discrete uniform energy inequality. First we define the sequence of approximate solutions by solving the following problems.

Structure subproblem. Find $(v^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}) \in (H_0^2(\omega) \cap H^3(\omega))^2$ such that

$$\begin{aligned} \int_{\omega} \frac{\eta^{n+\frac{1}{2}} - \eta^n}{\Delta t} \phi \, dy &= \int_{\omega} v^{n+\frac{1}{2}} \phi \, dy, \\ \int_{\omega} \frac{v^{n+\frac{1}{2}} - v^n}{\Delta t} \psi \, dy + \frac{1}{2} \int_{\omega} \mathcal{A}\mathbf{G}(\eta^{n+\frac{1}{2}}) : \mathbf{G}'(\eta^{n+\frac{1}{2}}, \eta^n) \psi \, dy & \tag{6.5} \\ + \frac{1}{24} \int_{\omega} \mathcal{A}\mathbf{R}(\eta^{n+\frac{1}{2}}) : \mathbf{R}'(\eta^{n+\frac{1}{2}}, \eta^n) \psi \, dy + \varepsilon \int_{\omega} \nabla^3 \eta^{n+\frac{1}{2}} \nabla^3 \psi \, dy &= 0, \end{aligned}$$

for all $(\phi, \psi) \in L^2(\omega) \times (H_0^2(\omega) \cap H^3(\omega))$.

The existence of a solution to the above problem follows by Schaefer’s fixed point theorem, as was demonstrated in [36, Proposition 4]).

Fluid subproblem. The fluid problem stays the same as in [36] (which is the advantage of the operator splitting method). Since the domain deformation is calculated in the structure subproblem and does not change in the fluid subproblem, we set $\eta^{n+1} = \eta^{n+\frac{1}{2}}$, and define $(\mathbf{u}^{n+1}, v^{n+1}) \in \mathcal{V}_F^{\eta^n} \times L^2(\omega)$ by requiring that for all $(\mathbf{q}, \xi) \in \mathcal{V}_F^{\eta^n} \times L^2(\omega)$ such that $\mathbf{q}|_\Gamma = \xi \mathbf{n}$, the following weak formulation holds:

$$\begin{aligned} & \int_\Omega J^n \left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \cdot \mathbf{q} + \frac{1}{2} [(\mathbf{u}^n - \mathbf{w}^{n+\frac{1}{2}}) \cdot \nabla^{\eta^n}] \mathbf{u}^{n+1} \cdot \mathbf{q} \right. \\ & \quad \left. - \frac{1}{2} [(\mathbf{u}^n - \mathbf{w}^{n+\frac{1}{2}}) \cdot \nabla^{\eta^n}] \mathbf{q} \cdot \mathbf{u}^{n+1} \right) dx \\ & + \frac{1}{2} \int_\Omega \frac{J^{n+1} - J^n}{\Delta t} \mathbf{u}^{n+1} \cdot \mathbf{q} \, dx + 2 \int_\Omega J^n \mathbf{D}^{\eta^n}(\mathbf{u}) : \mathbf{D}^{\eta^n}(\mathbf{q}) \, dx \\ & + \int_\omega \frac{v^{n+1} - v^{n+\frac{1}{2}}}{\Delta t} \xi \, dy = 0 \end{aligned}$$

with $\nabla^{\eta^n} \cdot \mathbf{u}^{n+1} = 0$, $\mathbf{u}^{n+1}_\Gamma = v^{n+1} \mathbf{n}$. Here ∇^η is the transformed gradient, $\mathbf{w}^{n+1/2}$ is the ALE velocity (i.e. the time discretization of $\partial_t \psi_{\eta^n}$; see Definition 2.1) and $J^n = \det \nabla \psi_{\eta^n}$ is the Jacobian of the transformation from Ω_{η^n} to the reference configuration Ω . Please observe that the above system is a linear equation on a fixed domain and it is solvable as long as $J^n > 0$ by the Lax–Milgram lemma. One can see that no self-intersection implies $J^n > 0$.

Now we define the approximate solutions as a piecewise constant functions in time:

$$\begin{aligned} \mathbf{u}_{\Delta t}(t, \cdot) &= \mathbf{u}^n_{\Delta t}, & \eta_{\Delta t}(t, \cdot) &= \eta^n_{\Delta t}, & v_{\Delta t}(t, \cdot) &= v^n_{\Delta t}, \\ v^*_{\Delta t}(t, \cdot) &= v^{n-\frac{1}{2}}_{\Delta t} & \text{for } t &\in [n\Delta t, (n+1)\Delta t]. \end{aligned} \tag{6.6}$$

Uniform estimates in Δt . The following proposition gives us the uniform boundedness of the approximate solutions defined by (6.6). It is a consequence of [36, Lemma 8] combined with Lemmas 4.2 and 4.3.

Proposition 6.2. *Let $\Delta t > 0$. Then the approximate solutions defined by (6.6) satisfy the following estimate:*

$$\begin{aligned} & \|\mathbf{u}_{\Delta t}\|_{L_t^\infty L_x^2} + \|\mathbf{u}_{\Delta t}\|_{L_t^2 H_x^1} + \|\eta_{\Delta t}\|_{L_t^\infty H_x^2} \\ & + \|v_{\Delta t}\|_{L_t^\infty L_x^2} + \|v^*_{\Delta t}\|_{L_t^\infty L_x^2} + \sqrt{\varepsilon} \|\eta_{\Delta t}\|_{L_t^\infty H_x^3} \leq C, \end{aligned} \tag{6.7}$$

where C depends on the data only. Moreover, there exists a $T > 0$ independent of Δt such that no self-intersection is approached.

Proof. The proof can be directly adapted from [36, Lemma 8] combined with Lemmas 4.2 and 4.3. In particular, we find by the uniform $L_t^\infty H_x^2$ estimates of $\eta_{\Delta t}$ that $\|\eta_{\Delta t}\|_{L_t^\infty(L_x^\infty)}$ is uniformly bounded with constants just depending on $\partial\Omega$ and the initial condition. Moreover, since $v^*_{\Delta t}$ is bounded in $L_t^\infty L_x^2$ we can use the interpolation inequality for Sobolev spaces to show that there exists $T > 0$ such that $\eta_{\Delta t}$ satisfies (4.4) and $J_{\Delta t} > 0$

in $[0, T]$, uniformly in Δt (and ε); cf. [36, Proposition 9]. In particular, $\partial\Omega_{\eta_{\Delta t}}$ has no self-intersection on $[0, T]$. ■

Let us denote by \mathbf{u}, η, v and v^* the corresponding weak or weak* limits of $\Delta t \rightarrow 0$. From [36, Lemma 11] it follows that $v = v^*$.

Compactness for $\Delta t \rightarrow 0$. First, we prove the strong convergence of the sequence $\eta_{\Delta t}$. This is a consequence of the uniform boundedness of the discrete time derivatives $\|\frac{\eta^{n+1}-\eta^n}{\Delta t}\|_{L^2(\omega)}$ and the boundedness of $\eta_{\Delta t}$ in $L^\infty(0, T; H^3(\omega))$. By using the classical Arzelà–Ascoli theorem for piecewise affine interpolation we get, as in [34, Lemma 3], that

$$\eta_{\Delta t} \rightarrow \eta \quad \text{in } L^\infty(0, T; H^s(\omega)) \text{ for } s \in (0, 3).$$

This is enough to pass to the limit in the terms connected to the elastic energy. In order to pass to the limit in the convective term and the terms connected to the moving boundary we need strong L^2 convergence of $(\mathbf{u}_{\Delta t}, v_{\Delta t})$. This is the most delicate part of the existence proof, where one has to use the uniform convergence of $\eta_{\Delta t}$ and the fact that the fluid dissipates higher frequencies of the structure velocities.

In the current case this follows by a version of the Aubin–Lions–Simon lemma adapted for problems with a moving boundary [37, Theorem 3.1. and Section 4.2]. Hence, by passing to the limit we find a $T > 0$ such that for every fixed $\varepsilon > 0$ there exists a weak solution to (6.1).

6.2. Proof of Theorem 1.1

In this subsection we first collect the necessary a priori estimates (which essentially follow from the regularity theorem) and then pass to the limit with $\varepsilon \rightarrow 0$. Here the establishment of the non-linearity in the convective term is (as usually) the most delicate part.

Uniform estimates in ε . We use the test function

$$(\mathbf{q}, \xi) = (\text{Test}_\eta(D_{-h}^s D_h^s \eta^\varepsilon - \mathcal{K}_\eta(D_{-h}^s D_h^s \eta^\varepsilon)), D_{-h}^s D_h^s \eta^\varepsilon - \mathcal{K}_\eta(D_{-h}^s D_h^s \eta^\varepsilon))$$

in (6.1) in an analogous way to the proof of Theorem 1.2. In combination with the energy estimates we obtain the following uniform regularity estimate for all (fixed) $s < \frac{1}{2}$:

$$\begin{aligned} &\|\eta^\varepsilon\|_{L^\infty(0, T; (H^2 \cap \sqrt{\varepsilon} H^3)(\omega))} + \|\partial_t \eta^\varepsilon\|_{L^\infty(0, T; L^2(\omega))} \\ &\quad + \|\eta^\varepsilon\|_{L^2(0, T; N^{2+s, 2}(\omega))} + \|\partial_t \eta^\varepsilon\|_{L^2(0, T; N^{s, 2}(\omega))} \leq C. \end{aligned} \tag{6.8}$$

Passing with $\varepsilon \rightarrow 0$. From the (classic) Aubin–Lions lemma we obtain

$$\eta^\varepsilon \rightarrow \eta \quad \text{in } L^2(0, T; H^s(\omega)) \cap L^\infty(0, T; H^{s-1/2}(\omega)) \text{ for } s < 5/2. \tag{6.9}$$

In particular, $\eta^\varepsilon \rightarrow \eta$ in $L^2(0, T; H^2(\omega)) \cap L^\infty(0, T; L^\infty(\omega))$, which is enough to pass to the limit in the elastic terms; see (4.5) for the highest-order terms. The existence result is completed once we can show that (for a subsequence) $(\partial_t \eta^\varepsilon, \mathbf{u}^\varepsilon) \rightarrow (\partial_t \eta, \mathbf{u})$, since this allows us to establish all non-linearities in the limit equation and the existence is complete.

The proof of the L^2 convergence of the velocities is known to be the most delicate part of the construction of weak solutions in the framework of FSI in the incompressible regime; see [22, 31, 37]. Here we present a more universal approach based on the reformulation of the Aubin–Lions lemma (Theorem 5.1) combined with the extension operator presented in Corollary 3.4.

Lemma 6.3. *There exists a strongly converging subsequence $\varepsilon^n \rightarrow 0$ such that*

$$(\partial_t \eta^{\varepsilon^n}, \mathbf{u}^{\varepsilon^n}) \rightarrow (\eta, \mathbf{u})$$

in $L_t^2 L_x^2$.

Proof. The strong convergence follows from Theorem 5.1. Actually we will apply the theorem twice: first for the boundary compactness and then for the interior compactness. We denote by $\mathbf{n}_{\eta^{\varepsilon^n}}$ the unit outer normal of $\Omega_{\eta^{\varepsilon^n}}$. From $\operatorname{div} \operatorname{Test}_{\eta^{\varepsilon^n}} (\partial_t \eta^{\varepsilon^n} - \mathcal{K}_{\eta^{\varepsilon^n}} (\partial_t \eta^{\varepsilon^n})) = 0$ we conclude that

$$\mathcal{K}_{\eta^{\varepsilon^n}} (\partial_t \eta^{\varepsilon^n}) \int_{\partial \Omega_{\eta^{\varepsilon^n}}} \mathbf{n}(y(x)) \cdot \mathbf{n}_{\eta^{\varepsilon^n}}(x) \, dx = \int_{\partial \Omega_{\eta^{\varepsilon^n}}} \mathbf{u}^{\varepsilon^n} \cdot \mathbf{n}_{\eta^{\varepsilon^n}}(x) \, dx = 0,$$

since $\operatorname{div} \mathbf{u}^{\varepsilon^n} = 0$ and $\mathbf{u}^{\varepsilon^n}(x) = \partial_t \eta^{\varepsilon^n}(y(x)) \mathbf{n}(\mathbf{p}(x))$ for $x \in \partial \Omega_{\eta^{\varepsilon^n}}$. Since $\eta^{\varepsilon^n}(\mathbf{p}(x))$ is a graph over the surfaces Ω (see Figure 2) which has a well-defined tangent plane almost everywhere, we conclude that $\mathbf{n}(y(x)) \cdot \mathbf{n}_{\eta^{\varepsilon^n}}(x) > 0$ a.e.; this implies that $\mathcal{K}_{\eta^{\varepsilon^n}} (\partial_t \eta^{\varepsilon^n}) = 0$. And consequently $\operatorname{Test}_{\eta^{\varepsilon^n}} (\partial_t \eta^{\varepsilon^n})$ is well defined:

$$\begin{aligned} \int_0^T \|\partial_t \eta^{\varepsilon^n}\|_{L^2(\omega)}^2 + \|\mathbf{u}^{\varepsilon^n}\|_{L^2(\Omega_{\eta^{\varepsilon^n}})}^2 \, dt &= \int_0^T \langle \partial_t \eta^{\varepsilon^n}, \partial_t \eta^{\varepsilon^n} \rangle + \langle \mathbf{u}^{\varepsilon^n}, \operatorname{Test}_{\eta^{\varepsilon^n}} (\partial_t \eta^{\varepsilon^n}) \rangle \, dt \\ &\quad + \int_0^T \langle \mathbf{u}^{\varepsilon^n}, \mathbf{u}^{\varepsilon^n} - \operatorname{Test}_{\eta^{\varepsilon^n}} (\partial_t \eta^{\varepsilon^n}) \rangle \, dt \\ &=: (I_n) + (II_n). \end{aligned}$$

For these two terms we will show the convergence separately using Theorem 5.1. The convergence implies that

$$\|\partial_t \eta^{\varepsilon^n}\|_{L_t^2 L_x^2} + \|\mathbf{u}^{\varepsilon^n} \chi_{\Omega_{\eta^{\varepsilon^n}}}\|_{L_t^2 L_x^2} \rightarrow \|\partial_t \eta\|_{L_t^2 L_x^2} + \|\mathbf{u} \chi_{\Omega_\eta}\|_{L_t^2 L_x^2},$$

which implies, by the uniform convexity of L^2 , the strong convergence $(\partial_t \eta^\varepsilon, \mathbf{u}^\varepsilon) \rightarrow (\eta, \mathbf{u})$ and the lemma is proved.

For the first term (I_n) , we define

$$g_n = (\partial_t \eta^{\varepsilon^n}, \mathbf{u}^{\varepsilon^n} \chi_{\Omega_{\eta^{\varepsilon^n}}}) \quad \text{and} \quad f_n = (\partial_t \eta^{\varepsilon^n}, \operatorname{Test}_{\eta^{\varepsilon^n}} (\partial_t \eta^{\varepsilon^n})),$$

and apply Theorem 5.1 using the spaces $X = L^2(\omega) \times H^{-s}(Q^K)$ and consequently $X' = L^2(\omega) \times H^s(Q^K)$. The space $Z = H^{s_0}(\omega) \times H^{s_0}(Q^K)$ for $0 < s < s_0 < \frac{1}{4}$. Also, with respect to time, we work in the setting of Hilbert spaces, which means that all Lebesgue exponents are 2. Further, we recall the smooth extension $E_{\eta^{\varepsilon^n}(t), \delta}$ introduced in Corollary 3.4 and denote $(\partial_t \eta^{\varepsilon^n})_\delta := \partial_t \eta^{\varepsilon^n} * \psi_\delta$. This allows us to define

$$f_{n,\delta} := ((\partial_t \eta^{\varepsilon^n})_\delta, E_{\eta^{\varepsilon^n}, \delta} (\partial_t \eta^{\varepsilon^n})).$$

Next let us check the assumptions of Theorem 5.1 and Remark 5.2. First observe that [31, Proposition 2.28] and (6.8) imply that g_n is uniformly bounded in $L_t^2(H_x^s)$ (for $s \leq \frac{1}{4}$). Hence, assumptions (1) follow in a rather straightforward manner by weak compactness in Hilbert spaces. Next, (2) follows by Corollary 3.4 and the standard estimates for mollifiers:

$$\|f_n - f_{n,\delta}\|_{L^2(Q^\kappa)} \leq c \|\partial_t \eta^{\varepsilon_n} * \psi_\delta - \partial_t \eta^{\varepsilon_n}\|_{L^2(\omega)} \leq C \delta^s \|\partial_t \eta^{\varepsilon_n}\|_{H^s(\omega)}.$$

Hence we are left to check (3'). As usual for equicontinuity in time, this is a consequence of the weak formulation of problem (6.1). For $\sigma \in [t, t + \tau]$ (using the solenoidality and the matching of the extension) we have

$$\begin{aligned} & | \langle g_n(t) - g_n(\sigma), f_{n,\delta}(t) \rangle | \\ &= \left| \int_{Q^\kappa} (\mathbf{u}^{\varepsilon_n}(t) \chi_{\Omega_{\eta^{\varepsilon_n}(t)}} - \mathbf{u}^{\varepsilon_n}(\sigma) \chi_{\Omega_{\eta^{\varepsilon_n}(\sigma)}}) \cdot E_{\eta^{\varepsilon_n}(t),\delta}(\partial_t \eta^{\varepsilon_n}(t)) \, dx \right. \\ &\quad \left. + \int_\omega (\partial_t \eta^{\varepsilon_n}(t) - \partial_t \eta^{\varepsilon_n}(\sigma)) (\partial_t \eta^{\varepsilon_n}(t))_\delta \, dy \right| \\ &= \left| \int_{Q^\kappa} \mathbf{u}^{\varepsilon_n}(t) \chi_{\Omega_{\eta^{\varepsilon_n}(t)}} \cdot E_{\eta^{\varepsilon_n}(t),\delta}(\partial_t \eta^{\varepsilon_n}(t)) - \mathbf{u}^{\varepsilon_n}(\sigma) \chi_{\Omega_{\eta^{\varepsilon_n}(\sigma)}} \cdot E_{\eta^{\varepsilon_n}(\sigma),\delta}(\partial_t \eta^{\varepsilon_n}(t)) \, dx \right. \\ &\quad \left. + \int_\omega (\partial_t \eta^{\varepsilon_n}(t) - \partial_t \eta^{\varepsilon_n}(\sigma)) (\partial_t \eta^{\varepsilon_n}(t))_\delta \, dy \right| \\ &\quad + \left| \int_{Q^\kappa} \mathbf{u}^{\varepsilon_n}(\sigma) \chi_{\Omega_{\eta^{\varepsilon_n}(\sigma)}} \cdot (E_{\eta^{\varepsilon_n}(\sigma),\delta}(\partial_t \eta^{\varepsilon_n}(t)) - E_{\eta^{\varepsilon_n}(t),\delta}(\partial_t \eta^{\varepsilon_n}(t))) \, dx \right| \\ &=: (A_1) + (A_2). \end{aligned}$$

First observe that by Corollary 3.4,

$$\begin{aligned} (A_2) &\leq \int |\mathbf{u}^{\varepsilon_n}(\sigma)| \chi_{\Omega_{\eta^{\varepsilon_n}(\sigma)}} \cdot \int_t^\sigma |\partial_s E_{\eta^{\varepsilon_n}(s),\delta}(\partial_t \eta^{\varepsilon_n}(t))| \, ds \, dx \\ &\leq c \|\mathbf{u}^{\varepsilon_n}(\sigma)\|_{L^2(\Omega_{\eta^{\varepsilon_n}(\sigma)})} \|\partial_t \eta^{\varepsilon_n}(\partial_t \eta^{\varepsilon_n})_\delta\|_{L_t^\infty(L_x^2)} |\sigma - t|^{\frac{1}{2}} \leq C(\delta) \tau^{\frac{1}{2}}, \end{aligned}$$

where we used the uniform $L_t^\infty L_x^2$ estimates of $\mathbf{u}^{\varepsilon_n}$ and $\partial_t \eta^{\varepsilon_n}$ multiple times. Second, by the weak formulation (6.1), we find

$$\begin{aligned} (A_1) &= \left| \int_\sigma^t \int_{\Omega_\eta} -\mathbf{u}^{\varepsilon_n}(s) \cdot \partial_s E_{\eta^{\varepsilon_n}(s),\delta}(\partial_t \eta^{\varepsilon_n}(t)) \right. \\ &\quad \left. + (\text{sym } \nabla \mathbf{u}^{\varepsilon_n}(s) - \mathbf{u}^{\varepsilon_n}(s) \otimes \mathbf{u}^{\varepsilon_n}(s)) : \nabla E_{\eta^{\varepsilon_n}(s),\delta}(\partial_t \eta^{\varepsilon_n}(t)) \, dx \right. \\ &\quad \left. + \int_\omega a_m(t, \eta^{\varepsilon_n}, (\partial_t \eta^{\varepsilon_n}(t))_\delta) + a_b(t, \eta^{\varepsilon_n}, (\partial_t \eta^{\varepsilon_n})_\delta) \right. \\ &\quad \left. + \varepsilon_n \nabla^3 \eta^{\varepsilon_n}(s) : \nabla^3 (\partial_t \eta^{\varepsilon_n}(t))_\delta \, dy \, ds \right| \\ &\leq C(\delta) (\|\mathbf{u}^{\varepsilon_n}\|_{L^2(t,t+\tau;W^{1,2+1})(\Omega_{\eta^{\varepsilon_n}})} (\|\partial_t \eta^{\varepsilon_n}\|_{L_t^\infty(L_x^2)} + \|\eta^{\varepsilon_n}\|_{L_t^\infty(H_x^2 + \varepsilon H_x^3)}) \\ &\quad + \|\eta^{\varepsilon_n}\|_{L^2(t,t+\tau;W^{1,\infty}(\omega))} \|\mathbf{u}^{\varepsilon_n}\|_{L^\infty(0,T;L^2(\Omega_{\eta^{\varepsilon_n}}))}^2) |t - \sigma|^{\frac{1}{2}} \\ &\leq C(\delta) \tau^{\frac{1}{2}}. \end{aligned}$$

This implies (3'), namely

$$\left| \int_t^{t+\tau} \langle g_n(t) - g_n(\sigma), f_{n,\delta}(t) \rangle d\sigma \right| \leq C(\delta)\tau^{\frac{1}{2}}.$$

This finishes the proof of the convergence of the (I_n) term. For the second term (II_n) , we again apply Theorem 5.1. Here we set

$$g_n = \mathbf{u}^{\varepsilon_n} \chi_{\Omega_{\eta^{\varepsilon_n}}} \quad \text{and} \quad f_n = (\mathbf{u}^{\varepsilon_n} - \text{Test}_{\eta^{\varepsilon_n}}(\partial_t \eta^{\varepsilon_n})) \chi_{\Omega_{\eta^{\varepsilon_n}}}.$$

We apply Theorem 5.1 with the spaces $X = H^{-s}(Q^\kappa)$ and consequently $X' = H^s(Q^\kappa)$ for some $s \in (0, \frac{1}{4})$. Further, we define $Z = L^2(Q^\kappa)$. Please observe that we may extend all involved quantities by zero to be functions over Q^κ . Finally, we again set all Lebesgue exponents to 2. Similarly to the first term, again the main effort is the construction of the right mollification. Indeed, assumptions (1) and (4) follow by standard compactness arguments. In particular, for assumption (1) it has been shown in [31, Proposition 2.28] that g_n is uniformly bounded in $H^s(Q^\kappa)$ (if $s \leq \frac{1}{4}$). For (2) we use the fact that f_n has a zero trace on $\partial\Omega_{\eta^{\varepsilon_n}}(t)$. First, let $\delta > 0$ be given. We take n_δ large enough and $\tau_\delta > 0$ small enough, such that

$$\sup_{n \geq n_\delta} \sup_{\tau \in (t-\tau_\delta, t+\tau_\delta) \cap [0, T]} \|\eta(t, x) - \eta^{\varepsilon_n}(\tau, x)\|_\infty \leq \delta. \tag{6.10}$$

Second, we fix $0 < s_0 < s$ and $\varepsilon > 0$. By [31, Lemma A.13] there exists a σ_ε and a sequence $\tilde{f}_{n,\delta}$, such that $\text{supp}(\tilde{f}_{n,\delta}(t)) \subset \Omega_{\eta^{\varepsilon_n}}(t)_{-3\delta}$ for all $3\delta \leq \sigma_\varepsilon$, that is divergence-free and $\|f - \tilde{f}_{n,\delta}\|_{(H^{-s_0}(Q^\kappa))} \leq \varepsilon \|f\|_{L^2(Q^\kappa)}$. We mollify this solenoidal function to define

$$f_{n,\delta} = \tilde{f}_{n,\delta} * \psi_\delta \quad \text{where } \psi \text{ is the standard mollifier in space.}$$

Then this definition implies (3') by a standard mollification estimate. Indeed, choosing

$$\begin{aligned} \|f_n - f_{n,\delta}\|_{H^{-s}(Q^\kappa)} &\leq \|\tilde{f}_{n,\delta} - f_{n,\delta}\|_{H^{-s}(Q^\kappa)} + \|\tilde{f}_{n,\delta} - f_n\|_{H^{-s_0}(Q^\kappa)} \\ &\leq c\delta^{s-s_0} \|\tilde{f}_{n,\delta}\|_{H^{-s_0}(Q^\kappa)} + \varepsilon \|f_n\|_{L^2(Q^\kappa)} \\ &\leq c\varepsilon \|f_n\|_{L^2(Q^\kappa)} \end{aligned}$$

for δ small enough, depending on $s - s_0$. By the properties of the mollification and (6.10), observe that $\text{supp}(f_{n,\delta}) \subset \Omega_{\eta^{\varepsilon_n}}(t)_{-2\delta} \subset \Omega_{\eta^{\varepsilon_m}}(\tau)$, which implies that $f_{n,\delta}(t)$ can be used as a test function on the fluid equation alone over the interval $[t, t + \tau]$ for $t \in [0, T - \tau]$. Hence (3') follows exactly along the lines of the above estimates using the weak formulation (6.1). Notice that here we will work just with the fluid equation since the traces of the test function are zero at the moving interface. ■

End of the proof of Theorem 1.1. The a priori estimates and the above compactness arguments guarantee that for given initial conditions there is a minimal time interval $T > 0$ for which a weak solution exists (see Remark 4.4). Once the solution is established we can

repeat the argument (by using $\eta(T)$, $\partial_t \eta(T)$, $\mathbf{u}(T)$ as initial conditions) until either a self-intersection is approached or a degeneracy of the H^2 -coercivity is violated (namely if $\gamma(\eta(t, x)) \rightarrow 0$ for some $t \rightarrow T$).

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