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On Germs of Differentiable Functions in Two Variables

By

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§1. Introduction

In [4] we have shown some sufficient conditions for a germ (of a differentiable function in two variables) to be transformed into an analytic one or a polynomial through a change of coordinates. Here we shall refine the result above and find a necessary and sufficient condition.

We denote respectively by $\mathcal{O}$, $\mathcal{E}$ the rings of germs at 0 in $\mathbb{R}^2$ of real analytic and C^∞-functions, and by $\mathcal{F}$ the ring of formal power series in 2 indeterminates over $\mathbb{R}$. If A is one of the above rings, let $\mathfrak{m}(A)$ denote the maximal ideal of A. Let T_a denote the Taylor expansion at a. Elements f, g of $\mathcal{E}$ are called *equivalent* if there exists a local diffeomorphism (of class C^∞) τ of $\mathbb{R}^2$ around 0 such that $f\circ\tau=g$. If an element f of $\mathfrak{m}(\mathcal{E})$ can be factorized into the following form

$$f=\prod_{i=1}^{n} f_i^{\alpha_i}$$

such that $T_0 f_i(\neq 0)$ are prime and that any two of them are relatively prime in $\mathcal{F}$, then we call f *factorizable.*

Theorem 1. *Let f be in $\mathfrak{m}(\mathcal{E})$. Then f is equivalent to a non-zero element of $\mathcal{O}$ if and only if f is factorizable.*

Theorem 1 has the following corollary.

Corollary. *Any element of $\mathfrak{m}(\mathcal{E})$ is equivalent to some $f+g$ where f is in $\mathcal{O}$ and where g is in $\mathcal{E}$ and flat* (*i.e.* $T_0 g=0$).

The following is another result.

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Theorem 2. *Any element of $\mathcal{O}$ is equivalent to some polynomial one.*
We note that Theorem 2 is also valid in the complex case.

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§2. Proofs of Theorems

For the proof of Theorem 1 we need the following lemmas.

Lemma 1 (Shiota [4]). *Let f, g be in $\mathscr{E}$ and $a_i(x, t)$ $(i=1, 2)$ be germs at $0\times[0, 1]$ in $\mathbf{R}^2\times\mathbf{R}$ of C^∞-functions. If the conditions*

$$f(x)-g(x)=\sum_{i=1}^{2}a_i(x, t)\left(\frac{\partial f}{\partial x_i}t+\frac{\partial g}{\partial x_i}(1-t)\right),$$

$$a_i(0, t)=0$$

are satisfied, then f and g are equivalent.

Lemma 2 (Malgrange [3]). *Let $\tilde{\mathscr{F}}$ be the ring of germs at 0 in $\mathbf{R}^2$ of collections of formal power series at each point near 0 in $\mathbf{R}^2$. We may regard $\mathscr{E}$ as a subring of $\tilde{\mathscr{F}}$. Let $\mathfrak{p}$ be an ideal in $\mathcal{O}$. Then we have*

$$(\mathfrak{p}\tilde{\mathscr{F}})\cap\mathscr{E}=\mathfrak{p}\mathscr{E}.$$

Lemma 3 (Shiota [4]). *If an element f of $\mathfrak{m}(\mathscr{E})$ is factorizable, then there exists a germ $\phi\in\mathscr{E}-\mathfrak{m}(\mathscr{E})$ such that ϕf is equivalent to an element of $\mathcal{O}$.*

Lemma 4. *Let f be in $\mathfrak{m}(\mathscr{F})$ (resp. $\mathfrak{m}(\mathcal{O})$). Then, for some integer $k>0$, $f\mathfrak{m}^k(\mathscr{F})$ (resp. $f\mathfrak{m}^k(\mathcal{O})$) is contained in the ideal generated by $\frac{\partial f}{\partial x_i}$ in $\mathscr{F}$ (resp. $\mathcal{O}$).*

Proof of Lemma 4. From the fact that $\mathscr{F}$ is faithfully flat over $\mathcal{O}$, it is enough to prove only for $\mathscr{F}$. As $\mathscr{F}$ is a unique factorization ring, there are prime elements $f_i\in\mathscr{F}$ such that

$$f=\prod_{i=1}^{n}f_i^{\alpha_i}$$

where f_i, f_j $(i\neq j)$ are relatively prime. Let $f'=\prod_{i=1}^{n}f_i^{\alpha_i-1}$. Then $\frac{\partial f}{\partial x_i}$ are divisible by f'. We can easily see that for each i at least one $\frac{\partial f}{\partial x_j}/f'$ is not divisible by f_i. Let $\mathfrak{p}$ be the ideal generated by $\frac{\partial f}{\partial x_i}/f'$. Let us show now that the height of $\mathfrak{p}$ is two. If its height is one, then $\mathfrak{p}$ is contained in a prime ideal $\mathfrak{q}$ of height one. According to a result in [5] that

$$f^m=\sum_{i=1}^{2} h_i \frac{\partial f}{\partial x_i} \text{ for some integer } m>0 \text{ and } h_i \in \mathscr{F},$$

$\mathfrak{q}$ contains f and hence also f_i for some i. The height of $\mathfrak{q}$ being one, we have $\mathfrak{q}=f_i\mathscr{F}$. This means that $\frac{\partial f}{\partial x_i}/f'$ are divisible by f_i; that is a contradiction. Hence the height of $\mathfrak{p}$ is two, and $\mathfrak{p}$ contains $\mathfrak{m}^k(\mathscr{F})$ for some integer $k>0$. Therefore the ideal generated by $\frac{\partial f}{\partial x_i}$ contains $f'\mathfrak{m}^k(\mathscr{F})$ and so $f\mathfrak{m}^k(\mathscr{F})$.

Proof of Theorem 1. For the necessity of the condition, see [4]. Let us prove its sufficiency. Suppose that an element f of $\mathfrak{m}(\mathscr{E})$ is factorizable. By Lemma 3 we may regard ϕf as in $\mathscr{O}$ for some germ $\phi\in\mathscr{E}-\mathfrak{m}(\mathscr{E})$. Here, for each integer $k>0$, we may choose a germ ϕ of the form $\phi=1+\phi_\circ$ with $\phi_\circ$ in $\mathfrak{m}^k(\mathscr{E})$. In fact we have

$$T_0\phi/\phi(0)=1+H_i+H_{i+1}+\cdots$$

for some integer $i>0$ where H_i are the homogeneous parts of degree i, then we see

$$\phi/\phi(0)-1\in\mathfrak{m}^i(\mathscr{E}),$$

$$(1-H_i)\phi/\phi(0)-1\in\mathfrak{m}^{i+1}(\mathscr{E}).$$

This gives the statement above.

Let $\phi f=g$, $\phi^{-1}=\psi$, $\psi=1+\psi_\circ$.

When the above k is sufficiently large, from Lemma 4 there exist elements b_i, $c_{i,j}$ of $\mathfrak{m}(\mathscr{F})$ $i, j=1, 2$ such that

$$T_0(\psi_\circ g)=\sum_{i=1}^{2} b_i T_0\left(\frac{\partial g}{\partial x_i}\right),$$

$$T_0\left(\frac{\partial\psi_\circ}{\partial x_i} g\right)=\sum_{j=1}^{2} c_{i,j} T_0\left(\frac{\partial g}{\partial x_j}\right).$$

If g has no multiple factors, Lemma 4 shows that the height of the ideal generated by $\frac{\partial g}{\partial x_i}$ in $\mathcal{O}$ is two, and that the set of critical points of g is $\{0\}$. It is shown that for each point $a\neq 0$ near 0 with $g(a)=0$, the germ of g at a is equivalent to x_1 where (x_1, x_2) is a local coordinates around a such that $a=(0, 0)$. We remark that exactly one factor of g vanishes at a. When g has multiple factors, by the remark above we easily see that for the same point a as the above, the germ of g at a is equivalent to x_1^{α} for some integer $\alpha>0$. Because of Lemma 4 we have for some integer $m>0$ and $h_i\in\mathcal{O}$

$$g^m=\sum_{i=1}^{2} h_i\frac{\partial g}{\partial x_i}.$$

This shows that at least one first partial derivative does not vanish at each point a near 0 with $g(a)\neq 0$. From these, $T_a g$ is contained in the ideal generated by $T_a\frac{\partial g}{\partial x_i}$ in $\mathscr{F}$ except when $a=0$. By Lemma 2 there exist elements $\beta_i, \gamma_{i,j}$ of $\mathfrak{m}(\mathscr{E})$ $i, j=1, 2$ such that

$$\psi_{\circ} g=\sum_{i=1}^{2}\beta_i\frac{\partial g}{\partial x_i},$$

$$\frac{\partial\psi_{\circ}}{\partial x_i}g=\sum_{j=1}^{2}\gamma_{i,j}\frac{\partial g}{\partial x_j}.$$

We can prove in the same way as in the proof of Proposition 1 in [4] that f, g satisfy the condition in Lemma 1; hence f is equivalent to g.

Proof of Theorem 2. Let f be an element of $\mathcal{O}$. Since $\mathcal{O}$ is a unique factorization ring, there are prime elements $f_i\in\mathcal{O}$ and positive integers α_i such that

$$f=\prod_{i=1}^{n} f_i^{\alpha_i}$$

where f_i, f_j $(i\neq j)$ are relatively prime. Let

$$g=\prod_{i=1}^{n} f_i,$$

and let $\bar{f}_i$ be polynomials of degree k which have the same partial differential coefficients at 0 up to k order as f_i where k is sufficiently large. We know the following fact (Tougeron [5]). Let h be an element of $\mathfrak{m}(\mathcal{O})$ without multiple factors. Then there is an integer $k>0$ such that h is equivalent to any element of $\mathcal{O}$ which has the same partial differential coefficients up to k order as h. This shows that g is equivalent to $\prod_{i=1}^{n} \bar{f}_1$. Hence there exists a local diffeomorphism τ of $\mathbb{R}^2$ around 0 such that $g\circ\tau = \prod_{i=1}^{n} \bar{f}_i$. By the uniqueness of the factorization, there exist invertible elements ψ_i of $\mathcal{O}$ such that $\psi_i(f_i\circ\tau) = \bar{f}_{i'}$ where i' is not always i. Let

$$\psi = \prod_{i=1}^{n} \psi_i^{\alpha_i}.$$

Then we see that $\psi(f\circ\tau)$ is a polynomial element, this result corresponds to our Lemma 3 in the differential case. It is enough for the remainder of the proof to proceed in the same way as in the proof of Theorem 1.

§3. An application

Let us recall the following concept introduced by Kuo [2].

Definition. An element $f\in\mathcal{F}$ is called *C^0-sufficient in C^r* if for any two C^r-realizations g, g' of f (i.e. g, g' are C^r-functions defined near 0 and the natural images of f, g, g' in $\mathcal{F}/\mathfrak{m}^{r+1}(\mathcal{F})$ coincide) there exists a local homeomorphism τ around 0 such that $g\circ\tau = g'$ near 0.

An element $f\in\mathcal{F}$ is simply called *C^0-sufficient* if f is C^0-sufficient in C^r for some r.

Theorem 3 (Kuo [2]). *An element $f\in\mathcal{F}$ is not C^0-sufficient if and only if there is a C^∞-realization g of f which satisfies the condition that g is divisible by an element of the form h^2 where h has zeros arbitrary close to 0. If f is in $\mathcal{O}$ (resp. a germ of a polynomial), we may choose g, h in $\mathcal{O}$ (resp. in the domain of germs of polynomials).*

Proof. Theorem 7-2 in [2] treats only the polynomial case but it proof is also valid in the analytic case. For the differentiable case, the necessity of the condition is obvious from our Corollary in §1, and its sufficiency follows from the converse of the Kuiper-Kuo Theorem in [1] (i.e. if g is a C^r-realization of a C^0-sufficient element in C^r then $|\text{grad}\, g(x)| \geqq c|x|^{r-1}$ in a neighbourhood of 0, with some $c>0$).

References

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