Ann. Inst. H. Poincaré
Anal. Non Linéaire (Online first)
DOI 10.4171/AIHPC/73

Published by EMS Press

A quantitative strong parabolic maximum principle and application to a taxis-type migration–consumption model involving signal-dependent degenerate diffusion

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Abstract. The taxis-type migration–consumption model accounting for signal-dependent motilities, as given by $u_t = \Delta(u\phi(v))$, $v_t = \Delta v - uv$, is considered for suitably smooth functions $\phi: [0,\infty) \to \mathbb{R}$ which are such that $\phi > 0$ on $(0,\infty)$, but that in addition $\phi(0) = 0$ with $\phi'(0) > 0$. In order to appropriately cope with the diffusion degeneracies thereby included, this study separately examines the Neumann problem for the linear equation $V_t = \Delta V + \nabla \cdot (a(x,t)V) + b(x,t)V$ and establishes a statement on how pointwise positive lower bounds for nonnegative solutions depend on the supremum and the mass of the initial data, and on integrability features of a and b. This is thereafter used as a key tool in the derivation of a result on global existence of solutions to the equation above, smooth and classical for positive times, under the mere assumption that the suitably regular initial data be nonnegative in both components. Apart from that, these solutions are seen to stabilize toward some equilibrium, and as a qualitative effect genuinely due to degeneracy in diffusion, a criterion on initial smallness of the second component is identified as sufficient for this limit state to be spatially nonconstant.

1. Introduction

The primary subject of this study is the initial–boundary value problem

$$\begin{cases} u_t = \Delta(u\phi(v)), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ \ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with nonnegative initial data u_0 and v_0, and with a suitably regular nonnegative function ϕ on $[0,\infty)$. Parabolic systems of this form arise in the modeling of collective behavior in bacterial ensembles, as represented through their population densities $u = u(x,t)$, under the influence of certain signal

2020 Mathematics Subject Classification. Primary 35K55; Secondary 35B50, 35B40, 35K10, 92C17.
Keywords. Maximum principle, functional inequality, chemotaxis, pattern formation.

substances, described by their concentrations $v = v(x, t)$. Here the approach to describe migration by second-order operators of the form in the first equation of (1.1) accounts for recent advances in the modeling literature, addressing bacterial movement, especially in situations in which cell motility *as a whole* may be biased by chemical cues ([5, 24]).

Due to the resulting precise quantitative connection between diffusive and cross-diffusive contributions, (1.1) can be viewed as singling out a special subclass of Keller–Segel-type chemotaxis models with their commonly much less restricted interdependencies ([12]). The ambition to understand the implications of the particular structural properties going along with this type of link, and, more generally, to carve out possible peculiarities and characteristic features in population models including migration operators of this form, has stimulated considerable activity in the mathematical literature of the past few years. In this regard, the furthest-reaching insight seems to have been achieved for systems in which signal-dependent migration mechanisms of the form addressed in (1.1) are coupled to equations reflecting signal *production* through individuals, rather than consumption as in (1.1). Indeed, for various classes of the key ingredient ϕ, significant progress could be achieved for the corresponding initial–-boundary value problem associated with

$$\begin{cases} u_t = \Delta(u\phi(v)), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

both in the fully parabolic case when $\tau = 1$, and in the simplified parabolic–elliptic version obtained on letting $\tau = 0$. Besides basic results on global solvability (see [1, 4, 9, 11, 14, 15, 33, 34, 39]), several studies also include findings on asymptotic behavior, mainly identifying diffusion-dominated constellations in which solutions to either (1.2) or certain closely related variants stabilize toward homogeneous equilibria ([8, 13, 15, 25–28, 33, 41]). Beyond this, some interesting recent developments have provided rigorous evidence for a strong structure-supporting potential of such models, as already predicted by some numerical experiments ([4]), by revealing the occurrence of infinite-time blow-up phenomena ([10]).

Motility degeneracies at small signal densities. According to manifest positivity properties of v inherent in the production-determined signal evolution mechanism therein ([6]), considerations related to the behavior of ϕ close to the origin seem of secondary relevance in the context of (1.2); in particular, degeneracies due to either singular or vanishing asymptotics of ϕ near $v = 0$ appear to have no significant effects on corresponding solution theories, and thus have partially even been explicitly included in precedent analysis of (1.1) ([1, 9, 14, 39]).

In stark contrast to this, addressing the taxis–consumption problem (1.1) with its evident tendency toward enhancing small signal densities, seems to require a distinct focus on the behavior of $\phi(v)$ for small values of v, where especially migration-limiting mechanisms appear relevant in the modeling of bacterial motion in nutrient-poor environments ([16, 21]). In line with this, the present manuscript will be concerned with (1.1) under the assumption that ϕ be small near $v = 0$, while otherwise being fairly arbitrary for large

v and hence possibly retaining essential decay features at large signal densities that have underlain the modeling hypotheses in [5] and [24].

Already at the level of questions related to mere solvability, this degenerate framework seems to bring about noticeable challenges, especially in the application-relevant case when the initial signal distribution is small, or when v_0 even attains zeros. Adequately coping with such degeneracies in the course of an existence analysis for (1.1) will accordingly form our first objective, and our attempt to accomplish this will lead us to a more general problem from basic parabolic theory.

Quantifying positivity in a linear parabolic problem. Main results I. Specifically, a crucial step in our approach will, guided by the ambition to make efficient use of the dissipative action in the first subproblem of (1.1), consist in establishing appropriate lower bounds for the corresponding second components of solutions $(u_\varepsilon, v_\varepsilon)$ to suitably regularized variants of (1.1) (cf. (3.1)). Inter alia due to this approximation-based procedure, the necessity to thus simultaneously deal with whole solution *families*, instead of just one single object, seems to restrict accessibility to classical strong maximum principles.

A crucial part of our analysis will accordingly be concerned with the derivation of positive pointwise lower bounds for families of nonnegative solutions V to $V_t = \Delta V - U(x,t)V$ possibly attaining zeros initially, under adequate assumptions on U which are mild enough to allow for an application to $V := v_\varepsilon$ and $U := u_\varepsilon$ on the basis of a priori information on u_ε that can separately be obtained (cf. the further discussion near (1.12) below).

To address this in a context conveniently general, as an object of potentially independent interest we shall examine this question for the problem

$$\begin{cases} V_t = \Delta V + \nabla \cdot (a(x,t)V) + b(x,t)V, & x \in \Omega,\ t \in (0,T_0), \\ \dfrac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0,T_0), \\ V(x,0) = V_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

and our main result in this direction will indeed establish a quantitative link between basic properties of V_0, as well as a and b on the one hand, and positivity features of V on the other:

Proposition 1.1. *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and suppose that $p_1 \geq 2$, $p_2 \geq 1$, $q_1 > 2$, and $q_2 > 1$ are such that*

$$\frac{1}{q_1} + \frac{n}{2p_1} < \frac{1}{2} \quad \textit{and} \quad \frac{1}{q_2} + \frac{n}{2p_2} < 1. \tag{1.4}$$

Then, given any $L > 0$, $T > 0$, and $\tau \in (0,T)$, one can find $C(p_1, p_2, q_1, q_2, L, T, \tau) > 0$ with the property that whenever $T_0 \in (0,T]$ and $a \in C^{1,0}(\overline{\Omega} \times [0,T_0); \mathbb{R}^n)$, $b \in C^0(\overline{\Omega} \times [0,T_0))$, and $V \in C^0(\overline{\Omega} \times [0,T_0)) \cap C^{2,1}(\overline{\Omega} \times (0,T_0))$ are such that $a \cdot \nu = 0$

on $\partial\Omega \times (0, T_0)$*, that*

$$\int_0^{T_0} \|a(\cdot,t)\|_{L^{p_1}(\Omega)}^{q_1} \le L \quad \textit{and} \quad \int_0^{T_0} \|b(\cdot,t)\|_{L^{p_2}(\Omega)}^{q_2} \le L, \tag{1.5}$$

that

$$0 \le V_0 \le L \ \textit{ in } \Omega \quad \textit{and} \quad \int_\Omega V_0 \ge \frac{1}{L}, \tag{1.6}$$

and that (1.3) *holds, we have*

$$V(x,t) \ge C(p_1, p_2, q_1, q_2, L, T, \tau) \quad \textit{for all } x \in \Omega \textit{ and } t \in (\tau, T_0). \tag{1.7}$$

Here we note that the hypotheses in (1.5) and (1.4) cannot be substantially relaxed, not even in the simple case when $a \equiv 0$:

Proposition 1.2. *Let* $n \ge 1$ *and* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with smooth boundary, let* $p \ge 1$ *and* $q \ge 1$ *be such that*

$$\frac{1}{q} + \frac{n}{2p} > 1, \tag{1.8}$$

and let $T > 0$ *and* $x_0 \in \Omega$*. Then there exist* $L > 0$, $(b_k)_{k\in\mathbb{N}} \subset C^\infty(\bar\Omega \times [0,T])$ *and a positive function* $V_0 \in C^\infty(\bar\Omega)$ *such that*

$$\int_0^T \|b_k(\cdot,t)\|_{L^p(\Omega)}^q \, dt \le L \quad \textit{for all } k \in \mathbb{N} \tag{1.9}$$

and that (1.6) *holds, but that for each* $k \in \mathbb{N}$ *one can find* $V_k \in C^0(\bar\Omega \times [0,T]) \cap C^{2,1}(\bar\Omega \times (0,T))$ *such that*

$$\begin{cases} V_{kt} = \Delta V_k + b_k(x,t) V_k, & x \in \Omega,\ t \in (0,T), \\ \frac{\partial V_k}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0,T), \\ V_k(x,0) = V_0(x), & x \in \Omega, \end{cases} \tag{1.10}$$

and that

$$V_k(x_0,t) \to 0 \quad \textit{as } k \to \infty. \tag{1.11}$$

Global solvability and large-time behavior in (1.1)**. Main results II.** In line with the requirements expressed in (1.5) and (1.4), our application of Proposition 1.1 to approximate solutions of (1.1) needs to be preceded by the derivation of suitable integral bounds for the respective first components u_ε. When addressing this in the setting of a standard L^p testing procedure, we shall be forced to appropriately control ill-signed cross-diffusive contributions by means of correspondingly signal-weighted and hence weakened dissipation rates (cf. (3.17)). This will be achieved in the course of a further essential step in our analysis, to be accomplished in Lemma 3.6, by utilizing a functional inequality of the form

$$\int_\Omega \frac{\varphi^p}{\psi}|\nabla\psi|^2 \le \eta \int_\Omega \varphi^{p-2}\psi|\nabla\varphi|^2 + \eta\int_\Omega \varphi\psi + C(p)\cdot\Big(1+\frac{1}{\eta}\Big)\cdot\bigg\{\int_\Omega \varphi^p + \Big\{\int_\Omega \varphi\Big\}^{2p-1}\bigg\}\cdot\int_\Omega \frac{|\nabla\psi|^4}{\psi^3}, \tag{1.12}$$

to be deduced in Lemma 3.5 for smooth $\varphi \geq 0$ and $\psi > 0$ and any $p \geq 2$ and $\eta > 0$ with some $C(p) > 0$ in one- and two-dimensional domains. This will enable us to establish L^p bounds for u_ε in actually any L^p space with finite p, and a subsequent application of Proposition 1.1, as thereby facilitated, will thereupon provide accessibility to arguments from well-established parabolic regularity theories so as to finally yield $C^{2+\theta,1+\frac{\theta}{2}}$ estimates within the range where the said positivity result holds, that is, locally away from the temporal origin (Lemmas 4.3 and 4.5).

In consequence, this will enable us to establish the following result on global solvability of (1.1) by functions which are even smooth for all positive times, provided that ϕ and the initial data comply with mild assumptions which inter alia allow for large classes of merely nonnegative v_0:

Theorem 1.3. *Let $n \in \{1,2\}$ and $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary, assume that*

$$\phi \in C^1([0,\infty)) \cap C^3((0,\infty)) \quad \text{is such that } \phi(0)=0, \phi'(0)>0, \quad \text{and } \phi > 0 \text{ on } (0,\infty), \tag{1.13}$$

and suppose that

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative with } u_0 \not\equiv 0, \quad \text{and that} \\ v_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative with } v_0 \not\equiv 0 \text{ and } \sqrt{v_0} \in W^{1,2}(\Omega). \end{cases} \tag{1.14}$$

Then there exist functions

$$\begin{cases} u \in C^{2,1}(\bar{\Omega} \times (0,\infty)) \quad \text{and} \\ v \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \end{cases} \tag{1.15}$$

such that $u > 0$ and $v > 0$ in $\bar{\Omega} \times (0,\infty)$, and that (u,v) solves (1.1) in that in the classical pointwise sense we have $u_t = \Delta(u\phi(v))$ and $v_t = \Delta v - uv$ in $\Omega \times (0,\infty)$ and $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega \times (0,\infty)$, as well as $v(\cdot,0) = v_0$ in Ω, and that

$$u(\cdot,t) \rightharpoonup u_0 \quad \text{in } L^p(\Omega) \text{ for all } p \geq 1 \quad \text{as } t \searrow 0. \tag{1.16}$$

Moreover, this solution has the property that for each $p \geq 1$ there exists $C(p) > 0$ fulfilling

$$\|u(\cdot,t)\|_{L^p(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C(p) \quad \text{for all } t > 0.$$

Next focusing on the qualitative behavior of the solutions gained above, we shall make use of the decay information contained in an inequality of the form

$$\int_0^\infty \int_\Omega uv \leq \int_\Omega v_0, \tag{1.17}$$

as constituting one of the most elementary features of the second equation in (1.1), to assert a bound in the style of an estimate in $BV([0,\infty);(W_N^{2,\infty}(\Omega))^\star)$ for u, where

$W_N^{2,\infty}(\Omega) := \{\varphi \in W^{2,\infty}(\Omega) \mid \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega\}$ (Lemma 5.1). Through interpolation, this will imply the essential part of the following result on large-time stabilization of each among the solutions obtained in Theorem 1.3:

Theorem 1.4. *Let $n \in \{1,2\}$ and $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary, and assume (1.13) as well as (1.14). Then there exists a nonnegative function $u_\infty \in \bigcap_{p\ge1} L^p(\Omega)$ such that $\int_\Omega u_\infty = \int_\Omega u_0$, and that as $t \to \infty$, the solution (u,v) of (1.1) from Theorem 1.3 satisfies*

$$u(\cdot,t) \rightharpoonup u_\infty \quad \text{in } L^p(\Omega) \quad \text{for all } p \ge 1 \tag{1.18}$$

and

$$v(\cdot,t) \overset{\star}{\rightharpoonup} 0 \quad \text{in } W^{1,\infty}(\Omega). \tag{1.19}$$

As a natural question related to the latter result, we finally address the problem of describing the limit functions u_∞ appearing in (1.18). To put this in perspective, let us recall that the literature has identified numerous situations in which, when accompanied by *nondegenerate* diffusion, taxis-type cross-diffusive interaction with absorptive signal evolution mechanisms as in (1.1) leads to asymptotic prevalence of spatial homogeneity: indeed, not only (1.1) with strictly positive ϕ ([22]), but also a considerable variety of chemotaxis–consumption systems, have been shown to have the common feature that for widely arbitrary initial data, corresponding solutions stabilize toward *constant* states in their first component (cf. [18, 31, 37] for a small selection of examples).

A noticeable difference to this type of behavior, and hence a qualitative effect genuinely due to the diffusion degeneracy in (1.1), will be revealed in our final result: by making appropriate use of the quantitative information contained in (1.17), we can derive a criterion, in its essence apparently reflecting quite well the nutrient-poor situation relevant to applications ([16, 21]), for the limit function in (1.18) to be *nonconstant*:

Theorem 1.5. *Let $n \in \{1,2\}$ and $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary, suppose that (1.13) holds, and let $u_0 \in W^{1,\infty}(\Omega)$ be nonnegative with $u_0 \not\equiv$ const. Then for all $K > 0$ there exists $\delta(K) > 0$ with the property that whenever $v_0 \in W^{1,\infty}(\Omega)$ is nonnegative with $\sqrt{v_0} \in W^{1,2}(\Omega)$ and such that*

$$0 < \|v_0\|_{L^\infty(\Omega)} \le K \quad \text{and} \quad \int_\Omega v_0 \le \delta(K),$$

the corresponding limit function obtained in Theorem 1.4 satisfies

$$u_\infty \not\equiv \text{const.}$$

2. A quantitative strong maximum principle. Proof of Proposition 1.1

Let us first turn our attention to the most essential among our tools, by namely focusing on the positivity property claimed in Proposition 1.1. Our reasoning in this direction will at its core be based on a comparison argument applied to the function $W := \ln \frac{C}{V}$ which, for suitably large C depending on the parameters in Proposition 1.1, given any V fulfilling (1.3) indeed satisfies an inhomogeneous linear parabolic inequality (cf. (2.17)). As an essential preparation for this, our derivation of some quantitative information on immediate smoothing of W into $L^1(\Omega)$ will rely on a Poincaré-type inequality, applicable here thanks to a short-time positive lower bound for $\int_\Omega V$ due to (1.5) (see (2.11)), which facilitates making appropriate use of a first-order superlinear absorptive contribution to the evolution of $\int_\Omega \ln \frac{\delta}{V}$ for suitably chosen $\delta > 0$ (see (2.13)).

Through this type of design, our strategy is able to cope with the mild regularity requirements in Proposition 1.1, and thereby, unlike alternative approaches based on lower estimates for Green's functions ([3]), especially remains applicable throughout the essentially optimal parameter range described by (1.4); in particular, for our subsequently performed analysis of (1.1) it will be of crucial importance that our argument in Proposition 1.1 is robust enough to make do without requiring L^∞ bounds for b.

Proof of Proposition 1.1. We abbreviate $\Pi := (p_1, q_1, p_2, q_2)$ and first recall known regularization features of the Neumann heat semigroup $(e^{t\Delta})_{t\geq 0}$ on Ω ([7,35]) to fix positive constants $c_1(\Pi, T)$, $c_2(\Pi, T)$, $c_3(\Pi, T)$, and $c_4(T) > 0$ such that for any $t \in (0, T)$,

$$\|e^{t\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \leq c_1(\Pi, T)t^{-\frac{1}{2}-\frac{n}{2p_1}}\|\varphi\|_{L^{p_1}(\Omega)} \quad \text{for all } \varphi \in C^1(\bar{\Omega};\mathbb{R}^n) \text{ fulfilling } \varphi\cdot\nu|_{\partial\Omega} = 0 \tag{2.1}$$

and

$$\|e^{t\Delta}\varphi\|_{L^\infty(\Omega)} \leq c_2(\Pi, T)t^{-\frac{n}{p_1}}\|\varphi\|_{L^{\frac{p_1}{2}}(\Omega)} \quad \text{for all } \varphi \in C^0(\bar{\Omega}) \tag{2.2}$$

and

$$\|e^{t\Delta}\varphi\|_{L^\infty(\Omega)} \leq c_3(\Pi, T)t^{-\frac{n}{2p_2}}\|\varphi\|_{L^{p_2}(\Omega)} \quad \text{for all } \varphi \in C^0(\bar{\Omega}), \tag{2.3}$$

as well as

$$\|e^{t\Delta}\varphi\|_{L^\infty(\Omega)} \leq c_4(T)t^{-\frac{n}{2}}\|\varphi\|_{L^1(\Omega)} \quad \text{for all } \varphi \in C^0(\bar{\Omega}). \tag{2.4}$$

Using that

$$\Big(\frac{1}{2} + \frac{n}{2p_1}\Big)\cdot\frac{q_1}{q_1-1} < 1 \quad \text{and} \quad \frac{n}{2p_2}\cdot\frac{q_2}{q_2-1} < 1 \tag{2.5}$$

according to (1.4), we can thereafter rely on Beppo Levi's theorem to fix $\mu(\Pi, L, T) > 0$ suitably large such that

$$c_1(\Pi, T)L^{\frac{1}{q_1}}\cdot\bigg\{\int_0^T \sigma^{-(\frac{1}{2}+\frac{n}{2p_1})\cdot\frac{q_1}{q_1-1}} e^{-\frac{\mu(\Pi,L,T)q_1}{q_1-1}\cdot\sigma}\,d\sigma\bigg\}^{\frac{q_1-1}{q_1}} \leq \frac{1}{4} \tag{2.6}$$

and

$$c_3(\Pi, T)L^{\frac{1}{q_2}}\cdot\bigg\{\int_0^T \sigma^{-\frac{n}{2p_2}\cdot\frac{q_2}{q_2-1}} e^{-\frac{\mu(\Pi,L,T)q_2}{q_2-1}\cdot\sigma}\,d\sigma\bigg\}^{\frac{q_2-1}{q_2}} \leq \frac{1}{4}. \tag{2.7}$$

We moreover employ a consequence of a Poincaré-type inequality ([20, Lemma 8.4 and appendix], [32, Lemma 4.3]) to choose $c_5(\Pi, L, T) > 0$ in such a way that whenever $\delta > 0$,

$$\frac{1}{2}\int_\Omega \frac{|\nabla\varphi|^2}{\varphi^2} \ge c_5(\Pi, L, T)\cdot\left\{\int_\Omega \ln\frac{\delta}{\varphi}\right\}_+^2 \quad \begin{array}{l}\text{for all } \varphi \in C^1(\bar\Omega)\\ \text{such that } \varphi > 0 \text{ in } \bar\Omega\\ \text{and } |\{\varphi > \delta\}| \ge \frac{1}{4Lc_6(\Pi,L,T)},\end{array} \tag{2.8}$$

where

$$c_6(\Pi, L, T) := 2Le^{\mu(\Pi,L,T)T}. \tag{2.9}$$

We now suppose that $T_0 \in (0, T]$ and that a, V_0, and V have the listed properties, and begin our derivation of (1.7) by relying on a Duhamel representation associated with (1.3) to see that thanks to the maximum principle, the identity $a\cdot\nu|_{\partial\Omega\times(0,T_0)} = 0$, (2.1), (2.3), and the Hölder inequality, the continuous function y given by $y(t) := e^{-\mu(\Pi,L,T)t}\|V(\cdot,t)\|_{L^\infty(\Omega)}$, $t \in [0, T_0)$, satisfies

$$\begin{aligned}
y(t) &= e^{-\mu(\Pi,L,T)t}\bigg\| e^{t\Delta}V_0 + \int_0^t e^{(t-s)\Delta}\nabla\cdot\{a(\cdot,s)V(\cdot,s)\}\,ds \\
&\qquad + \int_0^t e^{(t-s)\Delta}\{b(\cdot,s)V(\cdot,s)\}\,ds\bigg\|_{L^\infty(\Omega)} \\
&\le e^{-\mu(\Pi,L,T)t}\|V_0\|_{L^\infty(\Omega)} \\
&\quad + c_1(\Pi,T)e^{-\mu(\Pi,L,T)t}\int_0^t (t-s)^{-\frac12-\frac{n}{2p_1}}\|a(\cdot,s)V(\cdot,s)\|_{L^{p_1}(\Omega)}\,ds \\
&\quad + c_3(\Pi,T)e^{-\mu(\Pi,L,T)t}\int_0^t (t-s)^{-\frac{n}{2p_2}}\|b(\cdot,s)V(\cdot,s)\|_{L^{p_2}(\Omega)}\,ds \\
&\le e^{-\mu(\Pi,L,T)t}\|V_0\|_{L^\infty(\Omega)} \\
&\quad + c_1(\Pi,T)e^{-\mu(\Pi,L,T)t}\int_0^t (t-s)^{-\frac12-\frac{n}{2p_1}}\|a(\cdot,s)\|_{L^{p_1}(\Omega)}\|V(\cdot,s)\|_{L^\infty(\Omega)}\,ds \\
&\quad + c_3(\Pi,T)e^{-\mu(\Pi,L,T)t}\int_0^t (t-s)^{-\frac{n}{2p_2}}\|b(\cdot,s)\|_{L^{p_2}(\Omega)}\|V(\cdot,s)\|_{L^\infty(\Omega)}\,ds \\
&\le e^{-\mu(\Pi,L,T)t}\|V_0\|_{L^\infty(\Omega)} \\
&\quad + c_1(\Pi,T)\cdot\left\{\int_0^t \|a(\cdot,s)\|_{L^{p_1}(\Omega)}^{q_1}\,ds\right\}^{\frac{1}{q_1}} \\
&\qquad \times\left\{\int_0^t (t-s)^{-(\frac12+\frac{n}{2p_1})\cdot\frac{q_1}{q_1-1}}e^{-\frac{\mu(\Pi,L,T)q_1}{q_1-1}\cdot(t-s)}\,ds\right\}^{\frac{q_1-1}{q_1}}\cdot\|y\|_{L^\infty((0,t))} \\
&\quad + c_3(\Pi,T)\cdot\left\{\int_0^t \|b(\cdot,s)\|_{L^{p_2}(\Omega)}^{q_2}\,ds\right\}^{\frac{1}{q_2}} \\
&\qquad \times\left\{\int_0^t (t-s)^{-\frac{n}{2p_2}\cdot\frac{q_2}{q_2-1}}e^{-\frac{\mu(\Pi,L,T)q_2}{q_2-1}\cdot(t-s)}\,ds\right\}^{\frac{q_2-1}{q_2}}\cdot\|y\|_{L^\infty((0,t))}
\end{aligned}$$

for all $t \in (0, T_0)$. Therefore, (1.6) and (1.5) together with (2.6) and (2.7) ensure that

$$
\begin{aligned}
y(t) &\le L + c_1(\Pi, T) L^{\frac{1}{q_1}} \cdot \left\{ \int_0^{T_0} \sigma^{-(\frac{1}{2}+\frac{n}{2p_1})\cdot\frac{q_1}{q_1-1}} e^{-\frac{\mu(\Pi,L,T)q_1}{q_1-1}\cdot\sigma} d\sigma \right\}^{\frac{q_1-1}{q_1}} \cdot \|y\|_{L^\infty((0,t))} \\
&\quad + c_3(\Pi, T) L^{\frac{1}{q_2}} \cdot \left\{ \int_0^{T_0} \sigma^{-\frac{n}{2p_2}\cdot\frac{q_2}{q_2-1}} e^{-\frac{\mu(\Pi,L,T)q_2}{q_2-1}\cdot\sigma} d\sigma \right\}^{\frac{q_2-1}{q_2}} \cdot \|y\|_{L^\infty((0,t))} \\
&\le L + \frac{1}{4}\|y\|_{L^\infty((0,t))} + \frac{1}{4}\|y\|_{L^\infty((0,t))} \quad \text{for all } t \in (0, T_0),
\end{aligned}
$$

from which it follows that, in line with (2.9),

$$
\|V(\cdot,t)\|_{L^\infty(\Omega)} \le c_6(\Pi, L, T) \quad \text{for all } t \in (0, T_0). \tag{2.10}
$$

Again, since $a \cdot \nu = 0$ on $\partial\Omega \times (0, T_0)$, in view of the Hölder inequality this especially ensures that

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega V &= \int_\Omega b(x,t)V \\
&\ge -c_6(\Pi, L, T)|\Omega|^{\frac{p_2-1}{p_2}} \|b(\cdot,t)\|_{L^{p_2}(\Omega)} \quad \text{for all } t \in (0, T_0)
\end{aligned}
$$

and that hence, by (1.6) and (1.5),

$$
\begin{aligned}
\int_\Omega V(\cdot,t) &\ge \int_\Omega V_0 - c_6(\Pi, L, T)|\Omega|^{\frac{p_2-1}{p_2}} \int_0^t \|b(\cdot,s)\|_{L^{p_2}(\Omega)} ds \\
&\ge \frac{1}{L} - c_6(\Pi, L, T)|\Omega|^{\frac{p_2-1}{p_2}} \cdot \left\{ \int_0^t \|b(\cdot,s)\|_{L^{p_2}(\Omega)}^{q_2} ds \right\}^{\frac{1}{q_2}} \cdot t^{\frac{q_2-1}{q_2}} \\
&\ge \frac{1}{L} - c_6(\Pi, L, T)|\Omega|^{\frac{p_2-1}{p_2}} L^{\frac{1}{q_2}} t^{\frac{q_2-1}{q_2}} \\
&\ge \frac{1}{2L} \quad \text{for all } t \in (0, \hat{t}_1),
\end{aligned} \tag{2.11}
$$

where

$$
\begin{aligned}
\hat{t}_1 := \min\{t_1, T_0\} \quad \text{with } t_1 &\equiv t_1(\Pi, L, T) \\
&:= \left\{ 2c_6(\Pi, L, T)|\Omega|^{\frac{p_2-1}{p_2}} L^{\frac{q_2+1}{q_2}} \right\}^{-\frac{q_2}{q_2-1}}.
\end{aligned} \tag{2.12}
$$

Combining this with (2.10), we see that for $\delta(L) := \frac{1}{4|\Omega|L}$ we have

$$
\begin{aligned}
\frac{1}{2L} &\le \int_{\{V(\cdot,t)\le\delta(L)\}} V(\cdot,t) + \int_{\{V(\cdot,t)>\delta(L)\}} V(\cdot,t) \\
&\le \delta(L)|\Omega| + c_6(\Pi, L, T) \cdot |\{V(\cdot,t) > \delta(L)\}| \\
&= \frac{1}{4L} + c_6(\Pi, L, T) \cdot |\{V(\cdot,t) > \delta(L)\}| \quad \text{for all } t \in (0, \hat{t}_1)
\end{aligned}
$$

and thus

$$|\{V(\cdot,t)>\delta(L)\}| \ge \frac{1}{4Lc_6(\Pi,L,T)} \quad \text{for all } t\in(0,\hat t_1).$$

We may therefore draw on (2.8) to find that in the identity

$$\begin{aligned}\frac{d}{dt}\int_\Omega \ln\frac{\delta(L)}{V} &= -\int_\Omega \frac{V_t}{V}\\ &= -\int_\Omega \frac{|\nabla V|^2}{V^2} - \int_\Omega a\cdot\frac{\nabla V}{V} - \int_\Omega b,\end{aligned} \tag{2.13}$$

valid throughout $(0,T_0)$ since clearly V is positive on $\bar\Omega\times(0,T_0)$ by (1.6) and the classical strong maximum principle, and again since $a\cdot\nu=0$ on $\partial\Omega\times(0,T_0)$, we can estimate

$$\frac12\int_\Omega\frac{|\nabla V|^2}{V^2} \ge c_5(\Pi,L,T)\cdot\left\{\int_\Omega \ln\frac{\delta(L)}{V}\right\}_+^2 \quad \text{for all } t\in(0,\hat t_1).$$

As moreover

$$\begin{aligned}-\int_\Omega a\cdot\frac{\nabla V}{V} &\le \frac12\int_\Omega\frac{|\nabla V|^2}{V^2}+\frac12\int_\Omega|a|^2\\ &\le \frac12\int_\Omega\frac{|\nabla V|^2}{V^2}+\frac12|\Omega|^{\frac{p_1-2}{p_1}}\|a(\cdot,t)\|_{L^{p_1}(\Omega)}^2 \quad \text{for all } t\in(0,T_0)\end{aligned}$$

and

$$-\int_\Omega b \le |\Omega|^{\frac{p_2-1}{p_2}}\|b(\cdot,t)\|_{L^{p_2}(\Omega)} \quad \text{for all } t\in(0,T_0)$$

by the Hölder inequality, this implies that if we let $c_7(\Pi):=\max\{\frac12|\Omega|^{\frac{p_1-2}{p_1}},|\Omega|^{\frac{p_2-1}{p_2}}\}$, then $z(t):=\int_\Omega\ln\frac{\delta(L)}{V(\cdot,t)}$, $t\in(0,\hat t_1)$, has the property that

$$\begin{aligned}z'(t) \le -c_5(\Pi,L,T)z_+^2(t)&+c_7(\Pi)\|a(\cdot,t)\|_{L^{p_1}(\Omega)}^2\\ &+c_7(\Pi)\|b(\cdot,t)\|_{L^{p_2}(\Omega)} \quad \text{for all } t\in(0,\hat t_1).\end{aligned}$$

By means of an ODE comparison argument, this can be seen to entail that with

$$h(t):=c_7(\Pi)\int_0^t\|a(\cdot,s)\|_{L^{p_1}(\Omega)}^2\,ds+c_7(\Pi)\int_0^t\|b(\cdot,s)\|_{L^{p_2}(\Omega)}\,ds, \quad t\in(0,T_0), \tag{2.14}$$

we have

$$z(t)\le\frac{1}{c_5(\Pi,L,T)t}+h(t) \quad \text{for all } t\in(0,\hat t_1), \tag{2.15}$$

because for each $\eta\in(0,\hat t_1)$,

$$\bar z(t):=\frac{1}{c_5(\Pi,L,T)\cdot(t-\eta)}+h(t), \quad t>\eta,$$

satisfies

$$\begin{aligned}
&\bar{z}'(t) + c_5(\Pi, L, T)\bar{z}_+^2(t) - c_7(\Pi)\|a(\cdot,t)\|_{L^{p_1}(\Omega)}^2 - c_7(\Pi)\|b(\cdot,t)\|_{L^{p_2}(\Omega)} \\
&\quad = \Big\{-\frac{1}{c_5(\Pi, L, T)\cdot(t-\eta)^2} + h'(t)\Big\} \\
&\qquad + c_5(\Pi, L, T)\cdot\Big\{\frac{1}{c_5(\Pi, L, T)\cdot(t-\eta)} + h(t)\Big\}^2 \\
&\qquad - c_7(\Pi)\|a(\cdot,t)\|_{L^{p_1}(\Omega)}^2 - c_7(\Pi)\|b(\cdot,t)\|_{L^{p_2}(\Omega)} \\
&\quad = \frac{2h(t)}{t-\eta} + c_5(\Pi, L, T)h^2(t) \\
&\quad \ge 0 \quad \text{for all } t \in (\eta, \hat{t}_1)
\end{aligned}$$

according to (2.14). In order to make this applicable to accomplishing the final step of our argument, we note that once more due to the Hölder inequality, (1.5) ensures that

$$\begin{aligned}
h(t) &\le c_7(\Pi)t^{\frac{q_1-2}{q_1}}\cdot\Big\{\int_0^t \|a(\cdot,s)\|_{L^{p_1}(\Omega)}^{q_1}\,ds\Big\}^{\frac{2}{q_1}} \\
&\qquad + c_7(\Pi)t^{\frac{q_2-1}{q_2}}\cdot\Big\{\int_0^t \|b(\cdot,s)\|_{L^{p_2}(\Omega)}^{q_2}\,ds\Big\}^{\frac{1}{q_2}} \\
&\le c_8(\Pi, L, T) \\
&:= c_7(\Pi)T^{\frac{q_1-2}{q_1}}L^{\frac{2}{q_1}} + c_7(\Pi)T^{\frac{q_2-1}{q_2}}L^{\frac{1}{q_2}} \quad \text{for all } t \in (0, T_0),
\end{aligned}$$

and that thus the function W defined by

$$W(x,t) := \ln\frac{c_6(\Pi, L, T)}{V(x,t)}, \quad x \in \bar{\Omega},\ t \in (0, T_0),$$

nonnegative throughout $\bar{\Omega}\times(0,T_0)$ thanks to (2.10), satisfies

$$\begin{aligned}
\|W(\cdot,t)\|_{L^1(\Omega)} &= \int_\Omega \ln\Big\{\frac{\delta(L)}{V(\cdot,t)}\cdot\frac{c_6(\Pi, L, T)}{\delta(L)}\Big\} \\
&\le z(t) + \frac{|\Omega|c_6(\Pi, L, T)}{\delta(L)} \\
&\le \frac{1}{c_5(\Pi, L, T)t} + c_9(\Pi, L, T) \quad \text{for all } t \in (0, \hat{t}_1),
\end{aligned} \tag{2.16}$$

with

$$c_9(\Pi, L, T) := c_8(\Pi, L, T) + \frac{|\Omega|c_6(\Pi, L, T)}{\delta(L)}.$$

To derive (1.7) from this, we let $\tau \in (0, T)$ be given and note that we only need to consider the case when $T_0 > \tau$, in which (2.12) warrants that $\hat{t}_1 \ge t_2 \equiv t_2(\Pi, L, T, \tau) := \min\{t_1, \tau\}$.

As (1.3) together with Young's inequality implies that

$$\begin{aligned} W_t &= \Delta W - |\nabla W|^2 - \frac{1}{V}\nabla\cdot(a(x,t)V) - b(x,t) \\ &= \Delta W - |\nabla W|^2 - \nabla\cdot a(x,t) - a(x,t)\cdot\nabla W - b(x,t) \\ &\le \Delta W - \nabla\cdot a(x,t) + \frac{1}{4}|a(x,t)|^2 - b(x,t) \quad \text{in } \Omega\times(0,T_0), \end{aligned} \tag{2.17}$$

according to the comparison principle we may use (2.1) and (2.3) now together with (2.2) and (2.4) to infer on the basis of a corresponding variation-of-constants representation that thanks to (2.16), the Hölder inequality, and (1.5),

$$\begin{aligned} W(\cdot,t) &\le e^{(t-\frac{t_2}{2})\Delta} W\Big(\cdot,\frac{t_2}{2}\Big) \\ &\quad - \int_{\frac{t_2}{2}}^t e^{(t-s)\Delta}\nabla\cdot a(\cdot,s)\,ds + \frac{1}{4}\int_{\frac{t_2}{2}}^t e^{(t-s)\Delta}|a(\cdot,s)|^2\,ds - \int_{\frac{t_2}{2}}^t e^{(t-s)\Delta} b(\cdot,s)\,ds \\ &\le c_4(T)\cdot\Big(t-\frac{t_2}{2}\Big)^{-\frac{n}{2}}\Big\|W\Big(\cdot,\frac{t_2}{2}\Big)\Big\|_{L^1(\Omega)} \\ &\quad + c_1(\Pi,T)\int_{\frac{t_2}{2}}^t (t-s)^{-\frac12-\frac{n}{2p_1}}\|a(\cdot,s)\|_{L^{p_1}(\Omega)}\,ds \\ &\quad + \frac{c_2(\Pi,T)}{4}\int_{\frac{t_2}{2}}^t (t-s)^{-\frac{n}{p_1}}\||a(\cdot,s)|^2\|_{L^{\frac{p_1}{2}}(\Omega)}\,ds \\ &\quad + c_3(\Pi,T)\int_{\frac{t_2}{2}}^t (t-s)^{-\frac{n}{2p_2}}\|b(\cdot,s)\|_{L^{p_2}(\Omega)}\,ds \\ &\le c_4(T)\cdot\Big(t-\frac{t_2}{2}\Big)^{-\frac{n}{2}}\cdot\Big\{\frac{2}{c_5(\Pi,L,T)t_2} + c_9(\Pi,L,T)\Big\} \\ &\quad + c_{10}(\Pi,L,T) \quad \text{in } \Omega, \quad \text{for all } t\in(\tfrac{t_2}{2},T_0), \end{aligned}$$

where

$$\begin{aligned} c_{10}(\Pi,L,T) &:= c_1(\Pi,L,T)L^{\frac{1}{q_1}}\cdot\Big\{\int_0^T \sigma^{-(\frac12+\frac{n}{2p_1})\cdot\frac{q_1}{q_1-1}}\,d\sigma\Big\}^{\frac{q_1-1}{q_1}} \\ &\quad + \frac{c_2(\Pi,L,T)}{4}L^{\frac{2}{q_1}}\cdot\Big\{\int_0^T \sigma^{-\frac{n}{p_1}\cdot\frac{q_1}{q_1-2}}\,d\sigma\Big\}^{\frac{q_1-2}{q_1}} \\ &\quad + c_3(\Pi,L,T)L^{\frac{1}{q_2}}\cdot\Big\{\int_0^T \sigma^{-\frac{n}{2p_2}\cdot\frac{q_2}{q_2-1}}\,d\sigma\Big\}^{\frac{q_2-1}{q_2}} \end{aligned}$$

is finite because of (2.5), and of the fact that (1.4) moreover warrants that $\frac{n}{p_1}\cdot\frac{q_1}{q_1-2}<1$. Since $t_2\le\tau$, by definition of W this particularly means that for all $x\in\Omega$ and $t\in(\tau,T_0)$

we have

$$
\begin{aligned}
V(x,t) &\ge c_6(p,L,T) \\
&\quad \times \exp\Big\{-c_4(T)\cdot\Big(\frac{t_2(\Pi,L,T,\tau)}{2}\Big)^{-\frac{n}{2}} \\
&\qquad \cdot\Big\{\frac{2}{c_5(\Pi,L,T)t_2(\Pi,L,T,\tau)} + c_9(\Pi,L,T)\Big\} - c_{10}(\Pi,L,T)\Big\},
\end{aligned}
$$

and that hence indeed (1.7) holds with some $C(\Pi,L,T,\tau)>0$ independent of T_0, a, b, V_0, and V. ■

Our construction of a counterexample in the case when instead of (1.4) we have (1.8) is much less involved:

Proof of Proposition 1.2. We fix any $\alpha\in(0,1)$ and then use that $2n+\alpha-(1-\alpha)\xi^2\to -\infty$ as $\xi\to\infty$ to pick a nonnegative function $g\in C_0^\infty([0,\infty))$ such that

$$
(\xi^2+1)g(\xi)\ge 2n+\alpha-(1-\alpha)\xi^2 \quad \text{for all } \xi\ge 0. \tag{2.18}
$$

Without loss of generality assuming that $x_0=0$, we then choose $R>0$ and $R_0>R$ such that $\overline{B}_R(0)\subset\Omega\subset B_{R_0}(0)$, and for fixed $T>0$ taking $(T_k)_{k\in\mathbb{N}}\subset(T,T+1)$ such that $T_k\to T$ as $k\to\infty$, we let

$$
b_k(x,t) := -(T_k-t)^{-1}\cdot g((T_k-t)^{-\frac{1}{2}}|x|), \quad x\in\overline{\Omega},\ t\in[0,T],
$$

for $k\in\mathbb{N}$. Then since $T_k>T$, it follows that b_k indeed belongs to $C^\infty(\overline{\Omega}\times[0,T])$ and, with $\omega_n := n|B_1(0)|$, due to the inclusion $\Omega\subset B_{R_0}(0)$ satisfies

$$
\begin{aligned}
\int_\Omega |b_k(x,t)|^p\,dx &\le \omega_n\int_0^{R_0} r^{n-1}\cdot\big\{(T_k-t)^{-1}\cdot g((T_k-t)^{-\frac{1}{2}}r)\big\}^p\,dr \\
&= \omega_n\cdot(T_k-t)^{-p}\int_0^{R_0} r^{n-1}g^p((T_k-t)^{-\frac{1}{2}}r)\,dr \\
&= \omega_n\cdot(T_k-t)^{\frac{n}{2}-p}\int_0^{(T_k-t)^{-\frac{1}{2}}R_0}\xi^{n-1}g^p(\xi)\,d\xi \\
&\le c_1\cdot(T_k-t)^{\frac{n}{2}-p} \quad \text{for all } t\in(0,T) \text{ and } k\in\mathbb{N},
\end{aligned}
$$

where $c_1 := \omega_n\int_0^\infty \xi^{n-1}g^p(\xi)\,d\xi$ is finite according to the boundedness of $\operatorname{supp} g$. Therefore,

$$
\begin{aligned}
\int_0^T \|b_k(\cdot,t)\|_{L^p(\Omega)}^q\,dt &\le c_1^{\frac{q}{p}}\int_0^T (T_k-t)^{(\frac{n}{2}-p)\cdot\frac{q}{p}}\,dt \\
&\le c_1^{\frac{q}{p}}\int_0^{T_k} s^{(\frac{n}{2}-p)\cdot\frac{q}{p}}\,ds \quad \text{for all } k\in\mathbb{N},
\end{aligned}
$$

so that since our assumption (1.8) ensures that $(\frac{n}{2} - p) \cdot \frac{q}{p} > -1$, we can find $c_2 > 0$ fulfilling

$$\int_0^T \|b_k(\cdot,t)\|_{L^p(\Omega)}^q \, dt \le c_2 \quad \text{for all } k \in \mathbb{N}; \tag{2.19}$$

writing $V_0(x) := \min\{T^\alpha, (T+1)^{\alpha-1}R^2\}$, $x \in \overline{\Omega}$, we can thereupon fix $L > 0$ large enough such that besides (1.6) we also have $L \ge c_2$.

It is then clear that thanks to the smoothness features of the constant function V_0 and of $(b_k)_{k\in\mathbb{N}}$, according to standard parabolic theory ([17]) for any $k \in \mathbb{N}$ the problem (1.10) admits a classical solution $V_k \in C^0(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ which, by nonpositivity of b_k and the maximum principle, satisfies

$$V_k \le \min\big\{T^\alpha, (T+1)^{\alpha-1}R^2\big\} \quad \text{in } \overline{\Omega} \times [0,T]. \tag{2.20}$$

To derive (1.11) from this, we let

$$\overline{V}_k(x,t) := (T_k - t)^\alpha f((T_k-t)^{-\frac{1}{2}}|x|), \quad (x,t) \in \overline{\Omega} \times [0,T],\ k \in \mathbb{N},$$

with $f(\xi) := \xi^2 + 1$, $\xi \ge 0$, and use that $f'(\xi) = 2\xi$ and $f''(\xi) = 2$ for all $\xi \ge 0$ in verifying that for each $k \in \mathbb{N}$ and any $(x,t) \in \Omega \times (0,T)$, writing $\xi \equiv \xi(x,t;k) := (T_k - t)^{-\frac{1}{2}}|x|$ we have

$$\begin{aligned}
&\overline{V}_{kt} - \Delta \overline{V}_k - b_k(x,t)\overline{V}_k \\
&\quad = -\alpha(T_k-t)^{\alpha-1} f(\xi) + \frac{1}{2}(T_k-t)^{\alpha-\frac{3}{2}}|x| f'(\xi) \\
&\qquad - (T_k-t)^\alpha \cdot \Big\{(T_k-t)^{-1} f''(\xi) + \frac{n-1}{|x|}(T_k-t)^{-\frac{1}{2}} f'(\xi)\Big\} \\
&\qquad + (T_k-t)^{-1} \cdot g(\xi) \cdot (T_k-t)^\alpha f(\xi) \\
&\quad = (T_k-t)^{\alpha-1} \cdot \Big\{-\alpha f(\xi) + \frac{\xi}{2} f'(\xi) - f''(\xi) - \frac{n-1}{\xi} f'(\xi) + g(\xi) f(\xi)\Big\} \\
&\quad = (T_k-t)^{\alpha-1} \cdot \big\{(1-\alpha)\xi^2 - \alpha - 2n + g(\xi)\cdot(\xi^2+1)\big\} \\
&\quad \ge 0
\end{aligned}$$

due to (2.18). Since for any $x \in \partial B_R(0)$ and all $t \in (0,T)$ we have

$$\overline{V}_k(x,t) = (T_k-t)^\alpha \cdot \big\{(T_k-t)^{-1}R^2 + 1\big\} \ge (T_k-t)^{\alpha-1}R^2 \ge (T+1)^{\alpha-1}R^2 \ge V_k(x,t)$$

according to the inequalities $T_k < T + 1$ and $\alpha < 1$, and thanks to (2.20), and since the latter moreover entails that

$$\overline{V}_k(x,0) \ge T_k^\alpha \cdot \big\{T_k^{-1}|x|^2 + 1\big\} \ge T_k^\alpha \ge T^\alpha \ge V_k(x,0) \quad \text{for all } x \in B_R(0),$$

from the comparison principle we thus infer that $\overline{V}_k \ge V_k$ in $B_R(0) \times (0,T)$ for all $k \in \mathbb{N}$. Therefore, (1.11) results upon observing that since α is positive,

$$\inf_{t\in(0,T)} \overline{V}_k(0,t) = (T_k - T)^\alpha \to 0 \quad \text{as } k \to \infty$$

due to our requirement that $T_k \to T$ as $k \to \infty$. ∎

3. Analysis of (1.1): L^p bounds

Our analysis of (1.1) will now be launched by the observation that according to standard arguments from the theory of Keller–Segel-type cross-diffusion systems, for each $\varepsilon \in (0,1)$ the regularized variant of (1.1) given by

$$\begin{cases} u_{\varepsilon t} = \varepsilon \Delta u_\varepsilon + \Delta(u_\varepsilon \phi(v_\varepsilon)), & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - u_\varepsilon v_\varepsilon, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

admits local-in-time classical solutions enjoying a handy extensibility criterion:

Lemma 3.1. *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and suppose that (1.13) and (1.14) hold. Then for each $\varepsilon \in (0,1)$ there exist $T_{\max,\varepsilon} \in (0,\infty]$ and functions*

$$\begin{cases} u_\varepsilon \in C^0(\bar\Omega \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\bar\Omega \times (0, T_{\max,\varepsilon})) \quad \textit{and} \\ v_\varepsilon \in \bigcap_{q\geq 1} C^0([0, T_{\max,\varepsilon}); W^{1,q}(\Omega)) \cap C^{2,1}(\bar\Omega \times (0, T_{\max,\varepsilon})) \end{cases}$$

such that $u_\varepsilon \geq 0$ and $v_\varepsilon > 0$ in $\bar\Omega \times (0, T_{\max,\varepsilon})$, that $(u_\varepsilon, v_\varepsilon)$ solves (3.1) in the classical sense in $\Omega \times (0, T_{\max,\varepsilon})$, and that

$$\textit{if } T_{\max,\varepsilon} < \infty \quad \textit{then} \quad \limsup_{t \nearrow T_{\max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{3.2}$$

This solution satisfies

$$\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \quad \textit{for all } t \in (0, T_{\max,\varepsilon}) \tag{3.3}$$

and

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)} \quad \textit{for all } t \in (0, T_{\max,\varepsilon}), \tag{3.4}$$

as well as

$$\int_0^{T_{\max,\varepsilon}} \int_\Omega u_\varepsilon v_\varepsilon \leq \int_\Omega v_0. \tag{3.5}$$

Proof. The statements on existence, positivity, and extensibility can be verified by following standard approaches in local existence theories of taxis-type parabolic systems ([2, 19]). The mass conservation property immediately results from an integration in the first subproblem in (3.1), whereas (3.4) is a consequence of the comparison principle. Finally, the inequality (3.5) can be verified upon a time integration of the identity $\frac{d}{dt}\int_\Omega v_\varepsilon = -\int_\Omega u_\varepsilon v_\varepsilon$. ∎

Throughout the sequel, unless otherwise stated we shall tacitly assume that (1.13) holds, and that $n \le 2$ and $\Omega \subset \mathbb{R}^n$ is a smoothly bounded convex domain, noting that the convexity requirement will be needed from Lemma 3.4 on, while the restriction on the spatial dimension will be relied on only in Lemma 3.5 and its sequel. Moreover, once u_0 and v_0 fulfilling (1.14) have been fixed, by $(u_\varepsilon, v_\varepsilon)$ and $T_{\max,\varepsilon}$ we shall exclusively mean the objects provided by Lemma 3.1.

For repeated later reference, let us explicitly state the following elementary implication of our assumptions on ϕ, and especially the requirement that $\phi'(0)$ be positive.

Lemma 3.2. *Let $K > 0$. Then there exist $\lambda(K) > 0$ and $\Lambda(K) > 0$ such that if* (1.14) *holds with $\|v_0\|_{L^\infty(\Omega)} \le K$, we have*

$$\lambda(K)v_\varepsilon \le \phi(v_\varepsilon) \le \Lambda(K)v_\varepsilon \quad \textit{in } \Omega \times (0, T_{\max,\varepsilon}) \tag{3.6}$$

and

$$|\phi'(v_\varepsilon)| \le \Lambda(K) \quad \textit{in } \Omega \times (0, T_{\max,\varepsilon}). \tag{3.7}$$

Proof. Since $\phi(0) = 0$, letting $\Lambda(K) := \|\phi'\|_{L^\infty((0,K))}$ we obtain that besides (3.7), also the right inequality in (3.6) holds due to (3.4). For the same reason, the l'Hôpital rule ensures that $\rho(\xi) := \frac{\phi(\xi)}{\xi}$, $\xi > 0$, extends to a continuous function on $[0, \infty)$ with $\rho(0) = \phi'(0)$, whence combining the positivity of ρ on $(0, K]$ with that of $\phi'(0)$, as both being ensured by (1.13), we obtain that $\lambda(K) := \inf_{\xi \in [0,K]} \rho(\xi)$ is positive and satisfies the left inequality in (3.6). ■

We next derive a space-time L^2 bound for u_ε, weighted by the factor v_ε due to the lower bound from (3.6), by adapting a duality-based strategy which appears well established in the analysis of semilinear parabolic problems, but which has also partially been pursued in some contexts of cross-diffusive systems related to (1.1) ([4, 33]). Unlike in most precedents, however, thanks to (3.5) the information thereby generated will here even include corresponding integrability over the whole existence interval, thus implicitly containing certain decay information.

Lemma 3.3. *If* (1.14) *holds, then there exists $C > 0$ such that*

$$\int_0^{T_{\max,\varepsilon}} \int_\Omega u_\varepsilon^2 v_\varepsilon \le C \quad \textit{for all } \varepsilon \in (0, 1). \tag{3.8}$$

Proof. For $\varphi \in L^1(\Omega)$ we abbreviate $\bar{\varphi} := \frac{1}{|\Omega|} \int_\Omega \varphi$, and we let A denote the realization of $-\Delta$ in $L^2_\perp(\Omega) := \{\varphi \in L^2(\Omega) \,|\, \bar{\varphi} = 0\}$, with its domain given by $D(A) := \{\varphi \in W^{2,2}(\Omega) \cap L^2_\perp(\Omega) \,|\, \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega\}$. Then A is self-adjoint and positive, and an application of A^{-1} to the identity

$$\begin{aligned}(u_\varepsilon - \bar{u}_0)_t &= \Delta\big\{\varepsilon(u_\varepsilon - \bar{u}_0) + (u_\varepsilon\phi(v_\varepsilon) - \overline{u_\varepsilon\phi(v_\varepsilon)})\big\} \\ &= -A\big\{\varepsilon(u_\varepsilon - \bar{u}_0) + (u_\varepsilon\phi(v_\varepsilon) - \overline{u_\varepsilon\phi(v_\varepsilon)})\big\}, \quad x \in \Omega,\ t \in (0, T_{\max,\varepsilon}),\end{aligned}$$

as implied by (3.1) and (3.3), upon testing by $u_\varepsilon - \bar{u}_0$ shows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{-\frac{1}{2}}(u_\varepsilon - \bar{u}_0)|^2 \\
&\quad = -\int_\Omega \big\{\varepsilon(u_\varepsilon - \bar{u}_0) + (u_\varepsilon\phi(v_\varepsilon) - \overline{u_\varepsilon\phi(v_\varepsilon)})\big\} \cdot (u_\varepsilon - \bar{u}_0) \\
&\quad = -\varepsilon\int_\Omega (u_\varepsilon - \bar{u}_0)^2 - \int_\Omega u_\varepsilon^2\phi(v_\varepsilon) + \bar{u}_0\int_\Omega u_\varepsilon\phi(v_\varepsilon) \quad \text{for all } t \in (0, T_{\max,\varepsilon}),
\end{aligned}$$

because

$$\int_\Omega \overline{u_\varepsilon\phi(v_\varepsilon)} \cdot (u_\varepsilon - \bar{u}_0) = 0 \quad \text{for all } t \in (0, T_{\max,\varepsilon}),$$

again due to (3.3). In view of Lemma 3.2, after integrating in time and dropping two favorably signed summands we thus obtain that with $K := \|v_0\|_{L^\infty(\Omega)}$ we have

$$\begin{aligned}
\lambda(K)\int_0^t\int_\Omega u_\varepsilon^2 v_\varepsilon &\le \frac{1}{2}\int_\Omega |A^{-\frac{1}{2}}(u_0 - \bar{u}_0)|^2 + \bar{u}_0\int_0^t\int_\Omega u_\varepsilon\phi(v_\varepsilon) \\
&\le \frac{1}{2}\int_\Omega |A^{-\frac{1}{2}}(u_0 - \bar{u}_0)|^2 \\
&\quad + \Lambda(K)\bar{u}_0\int_0^t\int_\Omega u_\varepsilon v_\varepsilon \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

According to (3.5), this entails (3.8) with an obvious choice of C. ■

This enables us to suitably control the interaction-driven contributions that appear in a standard first-order testing procedure applied to the second equation of (3.1). As we are assuming Ω to be convex, corresponding boundary integrals are conveniently signed and hence the overall estimates thereby gained again including the entire time range $(0, T_{\max,\varepsilon})$. An interesting question left open here is how far a large-time relaxation feature similar to that implicitly expressed in (3.9) can be derived also in more general domains; while our existence theory in the context of Theorem 1.3 could readily be extended to such settings by adaptations based on fairly well-established arguments, convexity seems more essential in the parts in which Lemma 3.4 will be applied in the large-time analysis addressing boundedness and stabilization properties of solutions (cf. Lemmas 3.6 and 5.3, for instance).

Lemma 3.4. *Assume* (1.14). *Then there exists* $C > 0$ *such that*

$$\int_0^{T_{\max,\varepsilon}}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \le C \quad \textit{for all } \varepsilon \in (0,1). \tag{3.9}$$

Proof. By straightforward computation using (3.1) and integration by parts (cf. also [36, Lemma 3.2]), we obtain the identity

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \int_\Omega v_\varepsilon|D^2\ln v_\varepsilon|^2 + \frac{1}{2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \\
&\quad = -\int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \frac{1}{2}\int_\Omega \frac{1}{v_\varepsilon}\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} \quad \text{for all } t \in (0, T_{\max,\varepsilon}),
\end{aligned} \tag{3.10}$$

where the rightmost summand is nonpositive, because $\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} \le 0$ on $\partial\Omega \times (0, T_{\max,\varepsilon})$ by convexity of Ω ([23]). As it is well known ([36, Lemma 3.3], [40, Lemma 3.4]) that there exist positive constants c_1 and c_2 such that for all $\varphi \in C^2(\overline{\Omega})$ such that $\varphi > 0$ in $\overline{\Omega}$ and $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$ we have

$$c_1 \int_\Omega \frac{|\nabla\varphi|^4}{\varphi^3} \le \int_\Omega \varphi|D^2 \ln\varphi|^2 \quad \text{and} \quad c_2 \int_\Omega \frac{|D^2\varphi|^2}{\varphi} \le \int_\Omega \varphi|D^2 \ln\varphi|^2,$$

from (3.10) we thus infer that

$$\begin{aligned} &4\frac{d}{dt}\int_\Omega |\nabla\sqrt{v_\varepsilon}|^2 + c_1\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_2\int_\Omega \frac{|D^2 v_\varepsilon|^2}{v_\varepsilon} + \int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \\ &\quad \le -2\int_\Omega \nabla u_\varepsilon\cdot\nabla v_\varepsilon \quad \text{for all } t \in (0, T_{\max,\varepsilon}), \end{aligned} \tag{3.11}$$

where now, after a further integration by parts, we may use Young's inequality to estimate

$$\begin{aligned} -2\int_\Omega \nabla u_\varepsilon\cdot\nabla v_\varepsilon &= 2\int_\Omega u_\varepsilon \Delta v_\varepsilon \\ &\le \frac{c_2}{n}\int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + \frac{n}{c_2}\int_\Omega u_\varepsilon^2 v_\varepsilon \\ &\le c_2\int_\Omega \frac{|D^2 v_\varepsilon|^2}{v_\varepsilon} + \frac{n}{c_2}\int_\Omega u_\varepsilon^2 v_\varepsilon \quad \text{for all } t \in (0, T_{\max,\varepsilon}), \end{aligned}$$

because $|\Delta v_\varepsilon|^2 \le n|D^2 v_\varepsilon|^2$. Therefore,

$$\begin{aligned} &c_1\int_0^t\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \int_0^t\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \\ &\quad \le 4\int_\Omega |\nabla\sqrt{v_0}|^2 + \frac{n}{c_2}\int_0^t\int_\Omega u_\varepsilon^2 v_\varepsilon \quad \text{for all } t \in (0, T_{\max,\varepsilon}), \end{aligned}$$

so that (3.9) results from Lemma 3.3. ∎

To provide a prerequisite for a subsequent L^p regularity argument concerning u_ε, as the second of our key tools we now address the functional inequality announced in (1.12). We underline that its derivation actually does not require any convexity hypothesis, but through the use of a Sobolev embedding property it relies on the assumption that the spatial setting be one- or two-dimensional.

Lemma 3.5. *Let $n \le 2$ and $G \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $p \ge 2$. Then there exists $C(p, G) > 0$ such that for any $\varphi \in C^1(\overline{G})$ and $\psi \in C^1(\overline{G})$ fulfilling $\varphi \ge 0$ and $\psi > 0$ in $\overline{G}$,*

$$\begin{aligned} \int_G \frac{\varphi^p}{\psi}|\nabla\psi|^2 &\le \eta\int_G \varphi^{p-2}\psi|\nabla\varphi|^2 + \eta\int_G \varphi\psi \\ &\quad + C(p,G)\cdot\Big(1+\frac{1}{\eta}\Big)\cdot\bigg\{\int_G \varphi^p + \bigg\{\int_G \varphi\bigg\}^{2p-1}\bigg\}\cdot\int_G \frac{|\nabla\psi|^4}{\psi^3} \\ &\quad \textit{for all } \eta > 0. \end{aligned} \tag{3.12}$$

Proof. Since we are assuming that $n \le 2$, and that thus $W^{1,1}(G)$ is continuously embedded into $L^2(G)$, a corresponding Sobolev inequality yields $c_1(G) > 0$ fulfilling

$$\|\rho\|_{L^2(G)} \le c_1(G)\|\nabla\rho\|_{L^1(G)} + c_1(G)\|\rho\|_{L^1(G)} \quad \text{for all } \rho \in W^{1,1}(G),$$

so that since Hölder's and Young's inequalities imply that

$$\begin{aligned} c_1(G)\|\rho\|_{L^1(G)} &\le c_1(G)\|\rho\|_{L^2(G)}^{\frac{2p-2}{2p-1}}\|\rho\|_{L^{\frac{1}{p}}(G)}^{\frac{1}{2p-1}} \\ &= \left\{\frac{1}{2}\|\rho\|_{L^2(G)}\right\}^{\frac{2p-2}{2p-1}} \cdot 2^{\frac{2p-2}{2p-1}} c_1(G)\|\rho\|_{L^{\frac{1}{p}}(G)}^{\frac{1}{2p-1}} \\ &\le \frac{1}{2}\|\rho\|_{L^2(G)} + 2^{2p-2}c_1^{2p-1}(G)\|\rho\|_{L^{\frac{1}{p}}(G)} \quad \text{for all } \rho \in L^2(G), \end{aligned}$$

it follows that

$$\|\rho\|_{L^2(G)} \le c_2(p,G)\|\nabla\rho\|_{L^1(G)} + c_2(p,G)\|\rho\|_{L^{\frac{1}{p}}(G)} \quad \text{for all } \rho \in W^{1,1}(G)$$

with $c_2(p,G) := \max\{2c_1(G), (2c_1(G))^{2p-1}\}$. On the right-hand side of the estimate

$$\int_G \frac{\varphi^p}{\psi}|\nabla\psi|^2 \le \left\{\int_G \frac{|\nabla\psi|^4}{\psi^3}\right\}^{\frac{1}{2}} \cdot \left\{\int_G \varphi^{2p}\psi\right\}^{\frac{1}{2}}, \tag{3.13}$$

valid whenever $0 \le \varphi \in C^1(\bar{G})$ and $0 < \psi \in C^1(\bar{G})$ by the Cauchy–Schwarz inequality, we can therefore control the second factor according to

$$\begin{aligned} \left\{\int_G \varphi^{2p}\psi\right\}^{\frac{1}{2}} &= \|\varphi^p\sqrt{\psi}\|_{L^2(\Omega)} \\ &\le c_2(p,G)\int_G \left|p\varphi^{p-1}\sqrt{\psi}\nabla\varphi + \frac{\varphi^p}{2\sqrt{\psi}}\nabla\psi\right| + c_2(p,G)\cdot\left\{\int_G \varphi\psi^{\frac{1}{2p}}\right\}^p \\ &\le pc_2(p,G)\int_G \varphi^{p-1}\sqrt{\psi}|\nabla\varphi| + \frac{c_2(p,G)}{2}\int_G \frac{\varphi^p}{\sqrt{\psi}}|\nabla\psi| \\ &\quad + c_2(p,G)\cdot\left\{\int_G \varphi\psi^{\frac{1}{2p}}\right\}^p. \end{aligned} \tag{3.14}$$

Here, three applications of the Hölder inequality show that

$$pc_2(p,G)\int_G \varphi^{p-1}\sqrt{\psi}|\nabla\varphi| \le pc_2(p,G)\cdot\left\{\int_G \varphi^p\right\}^{\frac{1}{2}} \cdot \left\{\int_G \varphi^{p-2}\psi|\nabla\varphi|^2\right\}^{\frac{1}{2}}$$

and

$$\frac{c_2(p,G)}{2}\int_G \frac{\varphi^p}{\sqrt{\psi}}|\nabla\psi| \le \frac{c_2(p,G)}{2}\cdot\left\{\int_G \varphi^p\right\}^{\frac{1}{2}} \cdot \left\{\int_G \frac{\varphi^p}{\psi}|\nabla\psi|^2\right\}^{\frac{1}{2}},$$

as well as

$$c_2(p,G)\cdot\left\{\int_G \varphi\psi^{\frac{1}{2p}}\right\}^p = c_2(p,G)\cdot\left\{\int_G (\varphi\psi)^{\frac{1}{2p}}\cdot\varphi^{\frac{2p-1}{2p}}\right\}^p$$
$$\le c_2(p,G)\cdot\left\{\int_G \varphi\right\}^{\frac{2p-1}{2}}\cdot\left\{\int_G\varphi\psi\right\}^{\frac12}.$$

Inserting (3.14) into (3.13) and using Young's inequality, we hence infer that for each $\eta>0$,

$$\begin{aligned}\int_G \frac{\varphi^p}{\psi}|\nabla\psi|^2 &\le pc_2(p,G)\cdot\left\{\int_G\frac{|\nabla\psi|^4}{\psi^3}\right\}^{\frac12}\cdot\left\{\int_G\varphi^p\right\}^{\frac12}\cdot\left\{\int_G\varphi^{p-2}\psi|\nabla\varphi|^2\right\}^{\frac12}\\ &\quad+\frac{c_2(p,G)}{2}\cdot\left\{\int_G\frac{|\nabla\psi|^4}{\psi^3}\right\}^{\frac12}\cdot\left\{\int_G\varphi^p\right\}^{\frac12}\cdot\left\{\int_G\frac{\varphi^p}{\psi}|\nabla\psi|^2\right\}^{\frac12}\\ &\quad+c_2(p,G)\cdot\left\{\int_G\frac{|\nabla\psi|^4}{\psi^3}\right\}^{\frac12}\cdot\left\{\int_G\varphi\right\}^{\frac{2p-1}{2}}\cdot\left\{\int_G\varphi\psi\right\}^{\frac12}\\ &\le\frac{\eta}{2}\int_G\varphi^{p-2}\psi|\nabla\varphi|^2+\frac{p^2c_2^2(p,G)}{2\eta}\cdot\left\{\int_G\varphi^p\right\}\cdot\int_G\frac{|\nabla\psi|^4}{\psi^3}\\ &\quad+\frac12\int_G\frac{\varphi^p}{\psi}|\nabla\psi|^2+\frac{c_2^2(p,G)}{8}\cdot\left\{\int_G\varphi^p\right\}\cdot\int_G\frac{|\nabla\psi|^4}{\psi^3}\\ &\quad+\frac{\eta}{2}\int_G\varphi\psi+\frac{c_2^2(p,G)}{2\eta}\cdot\left\{\int_G\varphi\right\}^{2p-1}\cdot\int_G\frac{|\nabla\psi|^4}{\psi^3},\end{aligned}$$

which readily implies (3.12) with $C(p,G):=p^2c_2^2(p,G)$. ∎

We are now prepared to make sure that despite the diffusion degeneracy in the first equation of (3.1), the respective first solution components remain bounded with respect to the norm in any L^p space with $p\ge 2$. Our derivation of this will rely on two things, namely Lemma 3.5 and the corresponding decay features expressed in (3.9) and, again, in (3.5).

Lemma 3.6. *Given any $p\ge2$, one can pick $C(p)>0$ such that if (1.14) holds, then*

$$\int_\Omega u_\varepsilon^p(\cdot,t)\le C(p)\quad\textit{for all } t\in(0,T_{\max,\varepsilon})\textit{ and }\varepsilon\in(0,1). \tag{3.15}$$

Proof. We first employ Lemma 3.4 to fix $c_1>0$ such that

$$\int_0^{T_{\max,\varepsilon}}\int_\Omega\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}\le c_1\quad\text{for all }\varepsilon\in(0,1), \tag{3.16}$$

and to make adequate use of this together with the outcome of Lemma 3.5, we utilize Young's inequality when testing the first equation in (3.1) by u_ε^{p-1} to find that

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p = p\int_\Omega u_\varepsilon^{p-1}\Delta\{\varepsilon u_\varepsilon+u_\varepsilon\phi(v_\varepsilon)\}$$

$$= -p(p-1)\varepsilon \int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 - p(p-1)\int_\Omega u_\varepsilon^{p-2}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2$$
$$- p(p-1)\int_\Omega u_\varepsilon^{p-1}\phi'(v_\varepsilon)\nabla u_\varepsilon\cdot\nabla v_\varepsilon$$
$$\le -\frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-2}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2$$
$$+ \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^p \frac{\phi'^2(v_\varepsilon)}{\phi(v_\varepsilon)}|\nabla v_\varepsilon|^2 \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \tag{3.17}$$

Here, abbreviating $K := \|v_0\|_{L^\infty(\Omega)}$ we may draw on Lemma 3.2 in estimating

$$\phi(v_\varepsilon) \ge \lambda(K)v_\varepsilon \quad \text{and} \quad \frac{\phi'^2(v_\varepsilon)}{\phi(v_\varepsilon)} \le \frac{\Lambda^2(K)}{\lambda(K)v_\varepsilon} \quad \text{in } \Omega\times(0,T_{\max,\varepsilon}),$$

so that since an application of Lemma 3.5 to $\eta := \min\{\frac{p(p-1)\Lambda^2(K)}{2\lambda(K)},1\}$ provides $c_2(p)>0$ such that for all $\varphi\in C^1(\overline{\Omega})$ and $\psi\in C^1(\overline{\Omega})$ with $\varphi\ge 0$ and $\psi>0$ in $\overline{\Omega}$ we have

$$\frac{p(p-1)\Lambda^2(K)}{2\lambda(K)}\int_\Omega \frac{\varphi^p}{\psi}|\nabla\psi|^2 \le \frac{p(p-1)\lambda(K)}{2}\int_\Omega \varphi^{p-2}\psi|\nabla\varphi|^2 + \int_\Omega \varphi\psi$$
$$+ c_2(p)\cdot\left\{\int_\Omega \varphi^p + \left\{\int_\Omega \varphi\right\}^{2p-1}\right\}\cdot\int_\Omega \frac{|\nabla\psi|^4}{\psi^3},$$

thanks to (3.3) this entails that for all $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$,

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p \le \int_\Omega u_\varepsilon v_\varepsilon + c_2(p)\cdot\left\{\int_\Omega u_\varepsilon^p + \left\{\int_\Omega u_0\right\}^{2p-1}\right\}\cdot\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}.$$

For each $\varepsilon\in(0,1)$, the functions given by

$$y_\varepsilon(t) := \int_\Omega u_\varepsilon^p(\cdot,t) + \left\{\int_\Omega u_0\right\}^{2p-1}, \quad t\in[0,T_{\max,\varepsilon}),$$

as well as

$$g_\varepsilon(t) := \int_\Omega u_\varepsilon(\cdot,t)v_\varepsilon(\cdot,t) \quad \text{and} \quad h_\varepsilon(t) := c_2(p)\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^4}{v_\varepsilon^3(\cdot,t)}, \quad t\in(0,T_{\max,\varepsilon}),$$

thus satisfy

$$y_\varepsilon'(t) \le g_\varepsilon(t) + h_\varepsilon(t)y_\varepsilon(t) \quad \text{for all } t\in(0,T_{\max,\varepsilon}),$$

which upon an ODE comparison argument implies that

$$y_\varepsilon(t) \le y_\varepsilon(0)e^{\int_0^t h_\varepsilon(s)\,ds} + \int_0^t e^{\int_s^t h_\varepsilon(\sigma)d\sigma}g_\varepsilon(s)\,ds \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \tag{3.18}$$

Since

$$\int_s^t h_\varepsilon(\sigma)d\sigma \le c_1c_2(p) \quad \text{for all } t\in(0,T_{\max,\varepsilon}),\ s\in[0,t), \text{ and } \varepsilon\in(0,1)$$

by (3.16), and since

$$\int_0^t g_\varepsilon(s)\,ds \le \int_\Omega v_0 \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1)$$

due to (3.5), from (3.18) we thus obtain

$$\int_\Omega u_\varepsilon^p(\cdot,t) \le \left\{ \int_\Omega u_0^p + \left\{ \int_\Omega u_0 \right\}^{2p-1} \right\} \cdot e^{c_1 c_2(p)} \\ + e^{c_1 c_2(p)} \int_\Omega v_0 \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1)$$

to conclude as intended. ∎

4. Analysis of (1.1): Positivity properties of v_ε and higher-order estimates

Thanks to Lemma 3.6, we are now in the position to draw the intended conclusion from Proposition 1.1, and to thereby obtain a pointwise lower estimate for the second solution components, which indeed is uniform with respect to the approximation parameter:

Corollary 4.1. *Assume* (1.14)*. Then for all $T > 0$ and $\tau \in (0,T)$ there exists $C(T,\tau) > 0$ such that*

$$v_\varepsilon(x,t) \ge C(T,\tau) \quad \textit{for all } x \in \Omega,\ t \in (\tau, T) \cap (0, T_{\max,\varepsilon}), \textit{ and } \varepsilon \in (0,1). \tag{4.1}$$

Proof. Since $v_0 \not\equiv 0$, this immediately results from Proposition 1.1 upon applying Lemma 3.6 to, e.g., $p := 2$. ∎

Independently from the latter, through standard parabolic regularity arguments the outcome of Lemma 3.6 furthermore entails uniform bounds for the taxis gradients in (3.1):

Lemma 4.2. *If* (1.14) *holds, then there exists $C > 0$ such that*

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all } t \in (0, T_{\max,\varepsilon}) \textit{ and } \varepsilon \in (0,1). \tag{4.2}$$

Proof. According to well-known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω ([35]), fixing any $p > 2$ we can find $c_1 > 0$ such that for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} &\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \\ &\quad = \left\| \nabla e^{t(\Delta-1)} v_0 - \int_0^t \nabla e^{(t-s)(\Delta-1)} \{u_\varepsilon(\cdot,s) v_\varepsilon(\cdot,s) - v_\varepsilon(\cdot,s)\}\,ds \right\|_{L^\infty(\Omega)} \\ &\quad \le c_1 \|v_0\|_{W^{1,\infty}(\Omega)} \\ &\qquad + c_1 \int_0^t \left(1 + (t-s)^{-\frac12 - \frac{n}{2p}}\right) e^{-(t-s)} \|u_\varepsilon(\cdot,s) v_\varepsilon(\cdot,s) - v_\varepsilon(\cdot,s)\|_{L^p(\Omega)}\,ds. \end{aligned} \tag{4.3}$$

Since (3.4) implies that

$$\begin{aligned}
&\|u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s) - v_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \\
&\quad\le \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)}\|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} + |\Omega|^{\frac{1}{p}}\|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} \\
&\quad\le c_2\|v_0\|_{L^\infty(\Omega)} + |\Omega|^{\frac{1}{p}}\|v_0\|_{L^\infty(\Omega)} \quad \text{for all } s \in (0,T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1),
\end{aligned}$$

with $c_2 := \sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T_{\max,\varepsilon})} \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)}$ being finite by Lemma 3.6, from (4.3) we directly obtain (4.2). ∎

Relying on information on actual nondegeneracy of diffusion in (3.1), as implied by Corollary 4.1 throughout any region of the form $\Omega \times ((\tau,T)\cap(0,T_{\max,\varepsilon}))$ with $0<\tau<T$, by means of a straightforward temporal cut-off procedure we can now utilize Lemma 4.2 to establish local-in-time L^∞ bounds for u_ε through the outcome of a Moser-type iterative reasoning.

Lemma 4.3. *Suppose that (1.14) holds. Then for all $T>0$ and $\tau\in(0,T)$ one can find $C(T,\tau)>0$ such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C(T,\tau) \quad \textit{for all } t\in(\tau,T)\cap(0,T_{\max,\varepsilon}) \textit{ and } \varepsilon\in(0,1). \tag{4.4}$$

Proof. We fix $\zeta \in C^\infty([0,\infty))$ such that $\zeta\equiv 0$ on $[0,\frac{\tau}{2}]$ and $\zeta\equiv 1$ on $[\tau,\infty)$, and then from (3.1) we obtain that $w_\varepsilon(x,t) := \zeta(t)\cdot u_\varepsilon(x,t)$, $(x,t)\in\overline{\Omega}\times[0,T_{\max,\varepsilon})$, $\varepsilon\in(0,1)$, satisfies

$$\begin{aligned}
w_{\varepsilon t} &= \nabla\cdot(D_\varepsilon(x,t)\nabla w_\varepsilon) \\
&\quad + \nabla\cdot f_\varepsilon(x,t) + g_\varepsilon(x,t), \quad x\in\Omega,\ t\in(0,T_{\max,\varepsilon}),\ \varepsilon\in(0,1),
\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}
D_\varepsilon(x,t) &:= \varepsilon + \phi(v_\varepsilon(x,t)), \\
f_\varepsilon(x,t) &:= \zeta(t)u_\varepsilon(x,t)\phi'(v_\varepsilon(x,t))\nabla v_\varepsilon(x,t), \\
g_\varepsilon(x,t) &:= \zeta'(t)u_\varepsilon(x,t)
\end{aligned}$$

for $(x,t)\in\Omega\times(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$. Here, since

$$\begin{aligned}
&\lambda(\|v_0\|_{L^\infty(\Omega)})\cdot v_\varepsilon \le D_\varepsilon(x,t) \\
&\quad\le 1+\Lambda(\|v_0\|_{L^\infty(\Omega)})\cdot\|v_0\|_{L^\infty(\Omega)} \quad \text{for all } x\in\Omega,\ t\in(0,T_{\max,\varepsilon}), \text{ and } \varepsilon\in(0,1)
\end{aligned}$$

by Lemma 3.2 and (3.4), using Corollary 4.1 we infer the existence of $c_1(T,\tau)>0$ and $c_2>0$ such that

$$c_1(T,\tau)\le D_\varepsilon(x,t)\le c_2 \quad \text{for all } x\in\Omega,\ t\in(\tfrac{\tau}{2},T)\cap(0,T_{\max,\varepsilon}), \text{ and } \varepsilon\in(0,1).$$

Since, apart from that, a combination of Lemma 3.6 with (3.4) and Lemma 4.2 shows that

$$\begin{aligned}
&\sup_{\varepsilon\in(0,1)}\ \sup_{t\in(0,T_{\max,\varepsilon})} \left\{\|w_\varepsilon(\cdot,t)\|_{L^p(\Omega)} + \|f_\varepsilon(\cdot,t)\|_{L^p(\Omega)} + \|g_\varepsilon(\cdot,t)\|_{L^p(\Omega)}\right\} \\
&\qquad < \infty \quad \text{for all } p\in[2,\infty),
\end{aligned}$$

and since $w_\varepsilon(\cdot,\frac{\tau}{2})\equiv 0$ for all $\varepsilon\in(0,1)$ according to our choice of ζ, an application of [30, Lemma A.1] yields $c_3(T,\tau)>0$ such that

$$\|w_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\le c_3(T,\tau)\quad \text{for all } t\in(\tfrac{\tau}{2},T)\cap(0,T_{\max,\varepsilon}) \text{ and } \varepsilon\in(0,1).$$

As $w_\varepsilon(\cdot,t)\equiv u_\varepsilon(\cdot,t)$ in Ω for all $t\in(\tau,T)\cap(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$, this implies (4.4). ∎

The latter especially rules out any blow-up in the approximate problems:

Lemma 4.4. *If (1.14) holds, then $T_{\max,\varepsilon}=+\infty$ for all $\varepsilon\in(0,1)$.*

Proof. This immediately follows from Lemma 4.3 when combined with (3.2). ∎

Apart from that, Lemma 4.3 can be combined with Lemma 4.2 in the course of an essentially straightforward bootstrap procedure so as to yield temporally local higher-order regularity properties.

Lemma 4.5. *Assume (1.14). Then for all $T>0$ and any $\tau\in(0,T)$ there exist $\theta=\theta(T,\tau)\in(0,1)$ and $C(T,\tau)>0$ such that*

$$\|u_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[\tau,T])}\le C(T,\tau)\quad \textit{for all } \varepsilon\in(0,1) \tag{4.6}$$

and

$$\|v_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[\tau,T])}\le C(T,\tau)\quad \textit{for all } \varepsilon\in(0,1). \tag{4.7}$$

Proof. We rewrite the first equation of (3.1) according to

$$u_{\varepsilon t}=\nabla\cdot A_\varepsilon(x,t,\nabla u_\varepsilon),\quad x\in\Omega,\ t>0,\ \varepsilon\in(0,1),$$

with

$$\begin{aligned}A_\varepsilon(x,t,\xi):=\varepsilon\xi+\phi(v_\varepsilon(x,t))\xi\\ +\phi'(v_\varepsilon(x,t))u_\varepsilon(x,t)\nabla v_\varepsilon(x,t),\quad (x,t,\xi)\in\Omega\times(0,\infty)\times\mathbb{R},\ \varepsilon\in(0,1),\end{aligned}$$

and employ Corollary 4.1 along with Lemma 3.2, (3.4), and Lemmas 4.3 and 4.2 to find $c_1(T,\tau)>0$, $c_2(T,\tau)>0$, and $c_3(T,\tau)>0$ such that whenever $\varepsilon\in(0,1)$,

$$A_\varepsilon(x,t,\xi)\cdot\xi\ge c_1(T,\tau)|\xi|^2-c_2(T,\tau)\quad \text{for all } (x,t,\xi)\in\Omega\times(\tfrac{\tau}{8},T)\times\mathbb{R}^n \text{ and } \varepsilon\in(0,1)$$

and

$$|A_\varepsilon(,x,t,\xi)|\le c_3(T,\tau)|\xi|+c_3(T,\tau)\quad \text{for all } (x,t,\xi)\in\Omega\times(\tfrac{\tau}{8},T)\times\mathbb{R}^n \text{ and } \varepsilon\in(0,1).$$

Again based on Lemma 4.3, by means of a standard result on Hölder regularity of bounded solutions to scalar parabolic equations ([29]) we thus obtain $\theta_1=\theta_1(T,\tau)\in(0,1)$ and $c_4(T,\tau)>0$ such that

$$\|u_\varepsilon\|_{C^{\theta_1,\frac{\theta_1}{2}}(\bar\Omega\times[\frac{\tau}{4},T])}\le c_4(T,\tau)\quad \text{for all } \varepsilon\in(0,1),$$

whereupon parabolic Schauder theory applies to the second equation of (3.1) to yield $\theta_2 = \theta_2(T,\tau) \in (0,1)$ and $c_5(T,\tau) > 0$ fulfilling

$$\|v_\varepsilon\|_{C^{2+\theta_2, 1+\frac{\theta_2}{2}}(\bar\Omega\times[\frac{\tau}{2},T])} \le c_5(T,\tau) \quad \text{for all } \varepsilon \in (0,1). \tag{4.8}$$

This information in turn enables us to go back to the first equation in (3.1), now written in the form

$$u_{\varepsilon t} = \{\varepsilon + \phi(v_\varepsilon)\}\Delta u_\varepsilon + \{2\phi'(v_\varepsilon)\nabla v_\varepsilon\}\cdot\nabla u_\varepsilon \\ + \{\phi'(v_\varepsilon)\Delta v_\varepsilon + \phi''(v_\varepsilon)|\nabla v_\varepsilon|^2\}u_\varepsilon, \quad x\in\Omega,\ t>0,\ \varepsilon\in(0,1),$$

to conclude again from parabolic Schauder theory and the estimates provided by Corollary 4.1, (3.4), and Lemma 4.2 that (4.6) holds with some $\theta = \theta(T,\tau) \in (0,1)$ and $C(T,\tau) > 0$. In view of (4.8), the proof thereby becomes complete. ∎

As a last preparation for our limit passage, let us once more go back to Lemma 3.6 to obtain the following information on Hölder regularity of v_ε down to the temporal origin.

Lemma 4.6. *If* (1.14) *is satisfied, then for each $T > 0$ there exist $\theta = \theta(T) \in (0,1)$ and $C(T) > 0$ such that*

$$\|v_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[0,T])} \le C(T) \quad \textit{for all } \varepsilon\in(0,1). \tag{4.9}$$

Proof. This immediately follows from standard parabolic regularity theory ([29]) after applying Lemma 3.6 to any fixed $p \ge 2$. ∎

A solution of (1.1) in the flavor of the statement from Theorem 1.3 can now be obtained by a standard extraction process, followed by a suitably arranged argument asserting continuity of the corresponding first component with respect to weak L^p topologies.

Lemma 4.7. *Assume* (1.14)*. Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$, as well as functions u and v on $\bar\Omega\times(0,\infty)$, such that $\varepsilon_j \searrow 0$ as $j\to\infty$, that* (1.15) *holds with $u > 0$ and $v > 0$ in $\bar\Omega\times(0,\infty)$, and that*

$$u_\varepsilon \to u \quad \textit{in } C^{2,1}_{loc}(\bar\Omega\times(0,\infty)), \tag{4.10}$$
$$u_\varepsilon \rightharpoonup u \quad \textit{in } L^p_{loc}(\bar\Omega\times[0,\infty)) \quad \textit{for all } p\ge 1, \tag{4.11}$$
$$v_\varepsilon \to v \quad \textit{in } C^0_{loc}(\bar\Omega\times[0,\infty)) \textit{ and in } C^{2,1}_{loc}(\bar\Omega\times(0,\infty)), \quad \textit{and that} \tag{4.12}$$
$$\nabla v_\varepsilon \overset{\star}{\rightharpoonup} \nabla v \quad \textit{in } L^\infty(\Omega\times(0,\infty)) \tag{4.13}$$

as $\varepsilon = \varepsilon_j \searrow 0$. In the classical sense, these functions satisfy $u_t = \Delta(u\phi(v))$ and $v_t = \Delta v - uv$ in $\Omega\times(0,\infty)$ with $\frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0$ on $\partial\Omega\times(0,\infty)$ and $v(x,0) = v_0(x)$ for all $x\in\Omega$, and moreover (1.16) *holds. Apart from that,*

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0, \quad \textit{as well as } \|v(\cdot,t)\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)}, \quad \textit{for all } t>0, \tag{4.14}$$

and

$$\int_0^\infty \int_\Omega uv \le \int_\Omega v_0. \tag{4.15}$$

Proof. The existence of $(\varepsilon_j)_{j\in\mathbb{N}}$ and nonnegative functions u and v with the properties in (1.15) and (4.10)–(4.13) follows from Lemmas 4.5, 4.6, 3.6, and 4.2 by means of a straightforward extraction procedure, whereupon the claimed classical solution features can then immediately be verified by taking $\varepsilon = \varepsilon_j \searrow 0$ in (3.1) and using (4.10), (4.12) and the continuity of ϕ, ϕ', and ϕ''. Strict positivity of u and v throughout $\overline{\Omega} \times (0,\infty)$ can then a posteriori be deduced by applying the classical strong maximum principle to the identities $v_t = \Delta v - uv$ and $u_t = \Delta(u\phi(v))$, while (4.14) and (4.15) result from (3.3), (3.4), and (3.5) in conjunction with (4.10), (4.12), and Fatou's lemma.

It thus remains to derive the initial trace feature expressed in (1.16) for each $p \ge 1$, and to achieve this, assuming without loss of generality that $p > 1$ we let $\psi \in (L^p(\Omega))^\star \cong L^{\frac{p}{p-1}}(\Omega)$ and $\eta > 0$ be given and pick any $\psi_\eta \in C_0^\infty(\Omega)$ such that, in accordance with Lemma 3.6 and (4.10), we have

$$\begin{aligned} \|u(\cdot,t)\|_{L^p(\Omega)} \cdot \|\psi - \psi_\eta\|_{L^{\frac{p}{p-1}}(\Omega)} &\le \frac{\eta}{3} \quad \text{for all } t > 0 \\ \text{and} \quad \|u_0\|_{L^p(\Omega)} \cdot \|\psi - \psi_\eta\|_{L^{\frac{p}{p-1}}(\Omega)} &\le \frac{\eta}{3}. \end{aligned} \tag{4.16}$$

We thereafter choose $t_\eta \in (0,1)$ suitably small such that with Λ taken from Lemma 3.2 we have

$$\left\{\int_\Omega u_0\right\} \cdot \Lambda(\|v_0\|_{L^\infty(\Omega)}) \cdot \|v_0\|_{L^\infty(\Omega)} \cdot \|\Delta\psi_\eta\|_{L^\infty(\Omega)} \cdot t_\eta \le \frac{\eta}{3}, \tag{4.17}$$

and we claim that these selections guarantee that

$$\left|\int_\Omega u(\cdot,t)\psi - \int_\Omega u_0\psi\right| \le \eta \quad \text{for all } t \in (0,t_\eta). \tag{4.18}$$

In fact, since ψ_η belongs to $C_0^\infty(\Omega)$, when testing the first equation in (3.1) against ψ_η we do not encounter nontrivial boundary integrals and hence obtain

$$\begin{aligned} \int_\Omega u_\varepsilon(\cdot,t)\psi_\eta - \int_\Omega u_0\psi_\eta &= \varepsilon\int_0^t\int_\Omega u_\varepsilon\Delta\psi_\eta \\ &\quad + \int_0^t\int_\Omega u_\varepsilon\phi(v_\varepsilon)\Delta\psi_\eta \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \end{aligned} \tag{4.19}$$

Here, fixing any $t \in (0,t_\eta)$ we may invoke (4.10) to see that

$$\int_\Omega u_\varepsilon(\cdot,t)\psi_\eta \to \int_\Omega u(\cdot,t)\psi_\eta \quad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$

while combining (4.11) with (4.12) and the continuity of ϕ readily implies that

$$\varepsilon\int_0^t\int_\Omega u_\varepsilon\Delta\psi_\eta \to 0 \quad \text{and} \quad \int_0^t\int_\Omega u_\varepsilon\phi(v_\varepsilon)\Delta\psi_\eta \to \int_0^t\int_\Omega u\phi(v)\Delta\psi_\eta \quad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Accordingly, (4.19) entails that due to (4.14), Lemma 3.2, and (4.17),

$$\begin{aligned}
&\left|\int_\Omega u(\cdot,t)\psi_\eta - \int_\Omega u_0\psi_\eta\right| \\
&\quad= \left|\int_0^t \int_\Omega u\phi(v)\Delta\psi_\eta\right| \\
&\quad\le \int_0^t \|u(\cdot,s)\|_{L^1(\Omega)}\|\phi(v(\cdot,s))\|_{L^\infty(\Omega)}\|\Delta\psi_\eta\|_{L^\infty(\Omega)}\,ds \\
&\quad\le \left\{\int_\Omega u_0\right\}\cdot\Lambda(\|v_0\|_{L^\infty(\Omega)})\cdot\|v_0\|_{L^\infty(\Omega)}\cdot\|\Delta\psi_\eta\|_{L^\infty(\Omega)}\cdot t \\
&\quad\le \frac{\eta}{3},
\end{aligned}$$

because $t \in (0, t_\eta)$. In view of (4.16), we thus obtain that, indeed,

$$\begin{aligned}
&\left|\int_\Omega u(\cdot,t)\psi - \int_\Omega u_0\psi\right| \\
&\quad= \left|\int_\Omega u(\cdot,t)\cdot(\psi-\psi_\eta) + \left\{\int_\Omega u(\cdot,t)\psi_\eta - \int_\Omega u_0\psi_\eta\right\} + \int_\Omega u_0\cdot(\psi_\eta-\psi)\right| \\
&\quad\le \frac{\eta}{3}+\frac{\eta}{3}+\frac{\eta}{3} = \eta,
\end{aligned}$$

and that hence the verification of (4.18), as thereby achieved, completes the proof. ∎

Our main result concerning global solvability in (1.1) has thereby been accomplished:

Proof of Theorem 1.3. We only need to take (u, v) as provided by Lemma 4.7. ∎

5. Large-time behavior in (1.1). Proofs of Theorems 1.4 and 1.5

Our analysis of the large-time behavior in (1.1) is rooted in the following consequence of (4.15) on the total variation of u when considered as a $W_N^{2,\infty}(\Omega)$-valued function over $[0,\infty)$. Here and below, for definiteness in our corresponding argument, we shall let the Banach space $W_N^{2,\infty}(\Omega)$, as introduced before Theorem 1.4, be equipped with the norm given by $\|\varphi\|_{W^{2,\infty}(\Omega)} := \max_{|\alpha|\le 2}\|D^\alpha\varphi\|_{L^\infty(\Omega)}$, $\varphi \in W_N^{2,\infty}(\Omega)$.

Lemma 5.1. *Let $K > 0$. Then there exists $C(K) > 0$ with the property that if (1.14) holds with $\|v_0\|_{L^\infty(\Omega)} \le K$, for any choice of $(t_k)_{k\in\mathbb{N}} \subset [0,\infty)$ such that $t_{k+1} \ge t_k$ for all $k \in \mathbb{N}$, we have*

$$\sum_{k\in\mathbb{N}} \|u(\cdot,t_{k+1}) - u(\cdot,t_k)\|_{(W_N^{2,\infty}(\Omega))^\star} \le C(K)\int_\Omega v_0. \tag{5.1}$$

Proof. For fixed $\psi \in W^{2,\infty}_N(\Omega)$, an integration by parts in (3.1) shows that

$$\int_\Omega u_\varepsilon(\cdot,t_{k+1})\cdot\psi - \int_\Omega u_\varepsilon(\cdot,t_k)\cdot\psi$$
$$= \varepsilon\int_{t_k}^{t_{k+1}}\int_\Omega u_\varepsilon\Delta\psi + \int_{t_k}^{t_{k+1}}\int_\Omega u_\varepsilon\phi(v_\varepsilon)\Delta\psi \quad \text{for all } k\in\mathbb{N} \text{ and } \varepsilon\in(0,1),$$

and that hence, by (4.10), (4.11), (4.12), and the continuity of ϕ,

$$\int_\Omega u(\cdot,t_{k+1})\cdot\psi - \int_\Omega u(\cdot,t_k)\cdot\psi = \int_{t_k}^{t_{k+1}}\int_\Omega u\phi(v)\Delta\psi \quad \text{for all } k\in\mathbb{N}.$$

Since $\phi(v)\le\Lambda(K)v$ according to Lemma 3.2, (4.12), and our assumption, this implies that

$$\left|\int_\Omega\{u(\cdot,t_{k+1})-u(\cdot,t_k)\}\cdot\psi\right| \le \Lambda(K)\|\Delta\psi\|_{L^\infty(\Omega)}\int_{t_k}^{t_{k+1}}\int_\Omega uv \quad \text{for all } k\in\mathbb{N},$$

so that estimating $\|\Delta\psi\|_{L^\infty(\Omega)}\le n\|\psi\|_{W^{2,\infty}(\Omega)}$ we obtain

$$\|u(\cdot,t_{k+1})-u(\cdot,t_k)\|_{(W^{2,\infty}_N(\Omega))^\star} \le n\Lambda(K)\int_{t_k}^{t_{k+1}}\int_\Omega uv \quad \text{for all } k\in\mathbb{N}$$

and thus

$$\sum_{k\in\mathbb{N}}\|u(\cdot,t_{k+1})-u(\cdot,t_k)\|_{(W^{2,\infty}_N(\Omega))^\star} \le n\Lambda(K)\int_0^\infty\int_\Omega uv,$$

because $(t_k,t_{k+1})\cap(t_l,t_{l+1})=\emptyset$ for all $k\in\mathbb{N}$ and $l\in\mathbb{N}$ with $k\ne l$. The claim therefore results upon recalling (4.15). ∎

Thanks to the quantitative dependence on v_0, this does not only imply large-time stabilization of each individual trajectory in its first component, but it moreover provides some information on the distance between the associated limit and the initial data.

Lemma 5.2. *Let* $K>0$. *Then there exists* $\Xi(K)>0$ *such that whenever* (1.14) *holds with* $\|v_0\|_{L^\infty(\Omega)}\le K$, *the function* u *obtained in Lemma* 4.7 *has the property that*

$$u(\cdot,t)\to u_\infty \quad \text{in } (W^{2,\infty}_N(\Omega))^\star \quad \text{as } t\to\infty, \tag{5.2}$$

with some $u_\infty\in(W^{2,\infty}_N(\Omega))^\star$ *which satisfies*

$$\|u_\infty-u_0\|_{(W^{2,\infty}_N(\Omega))^\star} \le \Xi(K)\int_\Omega v_0. \tag{5.3}$$

Proof. Given any unbounded $(t_k)_{k\in\mathbb{N}}\subset(0,\infty)$ such that $t_{k+1}>t_k$ for all $k\in\mathbb{N}$, from (5.1) we obtain that $(u(\cdot,t_k))_{k\in\mathbb{N}}$ forms a Cauchy sequence in $(W^{2,\infty}_N(\Omega))^\star$, and that hence (5.2) holds with some $u_\infty\in(W^{2,\infty}_N(\Omega))^\star$. The characterization in (5.3) thereupon results from a second application of (5.1), this time to the particular sequence $(0,t,2t,\dots)$,

$t > 0$, which namely ensures the existence of $c_1(K) > 0$ such that under the hypotheses stated above we have

$$\|u(\cdot,t) - u_0\|_{(W_N^{2,\infty}(\Omega))^\star} \le c_1(K) \int_\Omega v_0 \quad \text{for all } t > 0,$$

and thereby establishes (5.3) due to (5.2). ∎

Also with regard to the large-time behavior in the second solution component, we shall first content ourselves with a topological framework somewhat more moderate than the one appearing in Theorem 1.4:

Lemma 5.3. *If* (1.14) *holds, then for v as in Lemma* 4.7 *we have*

$$v(\cdot,t) \to 0 \quad \text{in } L^1(\Omega) \quad \text{as } t \to \infty. \tag{5.4}$$

Proof. This can be seen by means of an argument similar to that performed to a slightly more complex variant in [38, Section 4]: From Lemma 3.4, (3.4), and Lemma 4.7 we obtain that $\int_0^\infty \int_\Omega |\nabla v|^4$ is finite, and that hence, according to a Poincaré inequality,

$$\int_t^{t+1} \|v(\cdot,s) - \overline{v(\cdot,s)}\|_{L^4(\Omega)}\, ds \to 0 \quad \text{as } t \to \infty,$$

where again $\bar\varphi := \frac{1}{|\Omega|}\int_\Omega \varphi$ for $\varphi \in L^1(\Omega)$. Since furthermore $c_1 := \sup_{t>0} \|u(\cdot,t)\|_{L^{\frac{4}{3}}(\Omega)}$ is finite by (1.15), and since

$$\int_t^{t+1} \int_\Omega uv \to 0 \quad \text{as } t \to \infty$$

according to (4.15), in view of the mass conservation property from (4.14), and thanks to the Hölder inequality, this implies that

$$\begin{aligned}
\overline{u_0} \int_t^{t+1} \|v(\cdot,s)\|_{L^1(\Omega)}\, ds &= \left| \int_t^{t+1} \int_\Omega u(x,s)\overline{v(\cdot,s)}\, dx\, ds \right| \\
&= \left| \int_t^{t+1} \int_\Omega uv - \int_t^{t+1} \int_\Omega u(x,s)(v(x,s) - \overline{v(\cdot,s)})\, dx\, ds \right| \\
&\le \int_t^{t+1} \int_\Omega uv + c_1 \int_t^{t+1} \|v(\cdot,s) - \overline{v(\cdot,s)}\|_{L^4(\Omega)}\, ds \\
&\to 0 \quad \text{as } t \to \infty.
\end{aligned}$$

Since $\overline{u_0}$ is positive according to (1.14), from this we immediately infer (5.4) upon noting that $0 \le t \mapsto \|v(\cdot,t)\|_{L^1(\Omega)}$ is nonincreasing due to the fact that v solves its respective subproblem in (1.1) classically in $\Omega \times (0,\infty)$ by Lemma 4.7. ∎

By means of straightforward interpolation relying on (1.18) and (1.15), from the latter and Lemma 5.2 we readily obtain our main result on stabilization in (1.1):

Proof of Theorem 1.4. Since (1.15) guarantees that $(u(\cdot,t))_{t>0}$ is relatively compact with respect to the weak topology in each of the spaces $L^p(\Omega)$ with $p>1$, taking u_∞ as in Lemma 5.2 we obtain the inclusion $u_\infty \in \bigcap_{p\ge 1} L^p(\Omega)$ and (1.18) as direct consequences of (5.2) when combined with the continuity of the embedding $(W_N^{2,\infty}(\Omega))^\star \hookrightarrow L^p(\Omega)$ for any such p, while the identity thereupon follows from (1.18) and (4.14).

Likewise, (1.19) results from the boundedness of $(v(\cdot,t))_{t>0}$ in $W^{1,\infty}(\Omega)$, as implied by (1.15), in conjunction with the statement on L^1 decay made in Lemma 5.3. ∎

Returning to (5.3), we can finally make sure that under a suitable smallness condition on v_0, the large-time limit thus obtained cannot be constant:

Lemma 5.4. *Let $u_0 \in W^{1,\infty}(\Omega)$ be nonnegative with $u_0 \not\equiv$ const. Then given $K>0$, one can find $\delta(K)>0$ such that if $v_0 \in W^{1,\infty}(\Omega)$ is nonnegative with $\sqrt{v_0} \in W^{1,2}(\Omega)$ and $\|v_0\|_{L^\infty(\Omega)} \le K$, as well as*

$$\int_\Omega v_0 \le \delta(K), \tag{5.5}$$

then the corresponding limit $u_\infty \in (W_N^{2,\infty}(\Omega))^\star$ from Lemma 5.2 has the property that $u_\infty \not\equiv$ const.

Proof. Since u_0 is continuous and not constant, we can fix numbers $c_1>0$, $c_2>c_1$, and $R>0$, as well as points $x_1\in\Omega$ and $x_2\in\Omega$, such that $B_{2R}(x_i)\subset\Omega$ for $i\in\{1,2\}$, and that $u_0\le c_1$ in $B_{2R}(x_1)$ and $u_0\ge c_2$ in $B_{2R}(x_2)$. It is then possible to pick $c_3>0$, as well as nonnegative functions $\psi_i\in C_0^\infty(\Omega)$, $i\in\{1,2\}$, which are such that supp $\psi_i\subset B_{2R}(x_i)$, that $\psi_i\equiv c_3$ in $B_R(x_i)$ and $\|\psi_i\|_{W_N^{2,\infty}(\Omega)}=1$ for $i\in\{1,2\}$. For fixed $K>0$, we then take $\Xi(K)$ as in Lemma 5.2 and claim that then the intended conclusion holds if we let

$$\delta(K) := \frac{c_3\kappa\cdot|B_R(0)|}{2\Xi(K)} \tag{5.6}$$

with $\kappa := \frac{c_2-c_1}{2}$.

Indeed, assuming on the contrary that $0\le v_0\in W^{1,\infty}(\Omega)$ with $\sqrt{v_0}\in W^{1,2}(\Omega)$ and $\|v_0\|_{L^\infty(\Omega)}\le K$ satisfied (5.5), but had the property that for the associated limit u_∞ we had $u_\infty\equiv a$ for some $a\in\mathbb{R}$, by definition of κ we would either have $a\le c_2-\kappa$ or $a\ge c_1+\kappa$. In the latter of these cases, however, we could use the localization features of ψ_1 together with (5.6) to estimate

$$\begin{aligned}
\|u_\infty-u_0\|_{(W_N^{2,\infty}(\Omega))^\star} \ge \int_\Omega (a-u_0)\cdot\psi_1 &= \int_{B_{2R}(x_1)} (a-u_0)\cdot\psi_1 \\
&\ge \int_{B_R(x_1)} (a-u_0)\cdot c_3 \\
&\ge \int_{B_R(x_1)} \{(c_1+\kappa)-c_1\}\cdot c_3 \\
&= c_3\kappa\cdot|B_R(0)| = 2\Xi(K)\delta(K),
\end{aligned}$$

which in view of (5.3) is absurd. As it can be shown in quite a similar manner that also the inequality $a \le c_2 - \kappa$ is impossible, it follows that, in fact, u_∞ cannot coincide with any constant. ■

Our reasoning thereby becomes complete:

Proof of Theorem 1.5. The claimed result has precisely been asserted by Lemma 5.4. ■

Acknowledgments. The author warmly thanks Wenbin Lv as well as the anonymous reviewers for several useful comments.

Funding. The author acknowledges support of the Deutsche Forschungsgemeinschaft (Project No. 462888149).

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Received 13 December 2021; revised 25 May 2022; accepted 23 June 2022.

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