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Time Periodic Solutions of Some Non-linear Evolution Equations

By

Takao KAKITA*

§1. Introduction

Considered in this paper are non-linear evoiution equations of the form

$$\frac{\partial^2 u}{\partial t^2}+Au+B\left(u, \frac{\partial u}{\partial t}\right)=f(x, t) \qquad \text{in } \Omega\times(-\infty, \infty) \tag{1}$$

together with periodicity conditions

$$u(x, t)=u(x, t+\tau), \quad u_t(x, t)=u_t(x, t+\tau) \tag{2}$$

and Dirichlet boundary conditions

$$D^\alpha u(x, t)=0 \quad \text{on} \quad \partial\Omega \quad \text{for} \quad |\alpha|\leqq m-1. \tag{3}$$

Each A in (1) is a non-linear elliptic operator of order $2m$ in Ω, a fixed bounded domain in $\boldsymbol{R}^N$ (which is similar to the one defined by F.E. Browder ([1])), and

$$\begin{aligned} B(u, u_t) &= \beta_0'(|u|^2)u_t-\Delta u_t, \text{ or more generally} \\ &= \sum_{|\alpha|\leqq m-1}(-1)^{|\alpha|}D^\alpha\beta_\alpha'(|D^\alpha u|^2)D^\alpha u_t \\ &\qquad (\beta_\alpha'(s^2)\geqq\varepsilon_0>0 \text{ for } |\alpha|=1) \end{aligned} \tag{4}$$

where each $\beta_\alpha'(s^2)$ is a non-negative function on $\boldsymbol{R}^1$, of polynomial growth. Each function $f(x, t)$ on $\Omega\times\boldsymbol{R}^1$ is periodic in t (of period $\tau>0$) with values in an appropriate Sobolev space. The purpose of this paper is to prove

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* Department of Mathematics, School of Science and Engineering, Waseda University, Tokyo.

an existence theorem of weak solutions for the equation (1) with conditions (2)-(3), subsequently to [3], where the theorem was proved for A, semi-linear elliptic operators, and $B(u, u_t)=(1+\beta_0'(|u|^2))u_t$. In case A is a quasi-linear elliptic operator of the second order and $B(u, u_t)=\Delta u_t=\sum_{i=1}^{N}\frac{\partial^3 u}{\partial x^2 \partial t}$, initial-boundary value problems for (1) have been solved by M. Tsutsumi ([7]), J.C. Clements ([2]), also boundary value problems with periodic conditions for (1) by Clements ([2]). W.A. Strauss has obtained (in unpublished work) weak solutions periodic in t of the equation

$$\frac{\partial^2 u}{\partial t^2}-\Delta u+|u|^{p-1}u+|u|^{q-1}\frac{\partial u}{\partial t}=f(x, t)$$

where $f(x, t)$ is a function periodic in t, $p\geqq q\geqq 1$ (cf. [6]).[1]

An example of our theorem gives the existence of periodic solutions in t for the equation

$$\frac{\partial^2 u}{\partial t^2}-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)-\Delta\frac{\partial u}{\partial t}+|u|^{q-1}\frac{\partial u}{\partial t}=f(x, t)$$

where $f(x, t)$ is as above and $p/2\geqq q\geqq 2$.

§2. Definitions and Main Theorem

Let $W^{m,p}(\Omega)$ be the Sobolev space

$$\{u(x)\,|\,D^\alpha u(x)\in L^p(\Omega),\ |\alpha|\leqq m\}^{2)}$$

with norm

$$\|u\|_{m,p}=\Big(\sum_{|\alpha|\leqq m}\int_\Omega |D^\alpha u(x)|^p dx\Big)^{1/p}$$

where $D_j=\frac{\partial}{\partial x_j}$, $D^\alpha=D_1^{\alpha_1}\cdots D_N^{\alpha_N}$ for $\alpha=(\alpha_1,\dots\alpha_N)$ and $|\alpha|=\sum_{i=1}^{N}\alpha_i$. $W^{m,p}(\Omega)$ is a separable Banach space for $1<p<\infty$. $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ (the space of all C^∞-functions in Ω with compace support) in $W^{m,p}(\Omega)$. $C^1(\tau)$ is defined as the set of all periodic functions in $C^1(\mathbb{R}^1)$ of

1) This result was informed to the author by Professor Strauss.

2) Throughout the paper we assume all functions considered are real valued.

period τ. By $<u^*, u>$ we denote the value of $u^* \in X^*$ at $u \in X$ for a Banach space X and its dual X^*. We denote by $L^p(\tau; X)$ the Banach space of functions f which are in L^p over any $I_\tau = [t, t+\tau]$ with values in X and

$$f(t) = f(t+\tau) \qquad \text{in } X \text{ for all } t \in R^1,$$

provided with the norm $(1 \leqq p < \infty)$

$$\|f\|_{L^p(\tau:X)} = \left\{ \int_{I_\tau} \|f(t)\|_X^p \, dt \right\}^{1/p}.$$

As for $L^\infty(\tau; X)$, the usual modification is needed. The $L^p(\Omega)$ norm is denoted by $\|\cdot\|_p$, especially by $\|\cdot\|$ for $p=2$.

Assumption A. Let A be a (non-linear) defferential operator in Ω, given in divergence form

$$Au(x) = \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \dots, D^m u) \tag{5}$$

where $D^k u = \{D^\alpha u\}_{|\alpha|=k}$, and the following conditions are imposed on A_α:

i) each $A_\alpha(x, \xi)$ is a continuous function of (x, ξ) (ξ is a real vector corresponding to $\{D^\alpha u\}_{|\alpha| \leqq m}$);

ii) there exists a continuous function $g_0(s)$ on R^1 such that

$$|A_\alpha(x, u(x), \dots, D^m u(x))| \leqq g_0(\|u\|_{m,p}) \{ \sum_{|\beta| \leqq m} |D^\beta u(x)|^{p-1} + 1 \} \tag{6}$$

for all $u \in W_0^{m,p}(\Omega)$ $(p \geqq 2)$, all α with $|\alpha| \leqq m$ and almost all $x \in \Omega$;

iii) the non-linear Dirichlet form on W,[1]

$$a(u, v) = \sum_{|\alpha| \leqq m} \int_\Omega A_\alpha(x, u(x), \dots, D^m u(x)) D^\alpha v(x) \, dx$$

satisfies, for a continuous function $g_1(s) \geqq 0$ on R^1,

$$|a(u, v)| \leqq g_1(\|u\|_{m,p}) \|v\|_{m,p}, \qquad u, v \in W; \tag{7}$$

iv) for $u \in W$,

$$a(u, u) \geqq c_0(\|u\|_{m,p}) + k_0 \|u\|^2$$

1) We denote $W_0^{m,p}(\Omega)$ by W for simplicity.

where $c_0(s)$ is a continuous function on $\boldsymbol{R}^1$ with $\lim_{s\to\infty} c_0(s)=\infty$ and k_0 is a positive constant;

v) $a(u, u-v)-a(v, u-v)\geqq 0,\ u, v\in W$;

vi) there exists a functional $r(u)$ on W such that

$$a(\psi(t), \psi'(t))\geqq\frac{d}{dt}r(\psi(t)) \tag{8}$$

for any $\psi(t)$, a finite sum of functions $c(t)w$ for $c(t)\in C^1(\tau)$ and $w\in W$, and that for $u\in W$

$$c_1(\|u\|_{m,p})\leqq r(u)\leqq k_1 a(u, u)+k_2 \tag{9}$$

where $c_1(s)$ is a continuous function on $\boldsymbol{R}^1$ with $\lim_{s\to\infty} c_1(s)=\infty$ and k_1, k_2 are some constants.

Assumption B. Let $\beta_0(s)$ be a twice differentiable function on $\boldsymbol{R}^1$ such that for $|\alpha|\leqq m-1$

$$0\leqq\beta'_\alpha(s^2)\leqq C|s|^{q-1},\ \text{in particular,}\ \varepsilon_0\leqq\beta'_\alpha(s^2)\ \ (|\alpha|=1)$$
$$|\beta''_\alpha(s^2)|\leqq C|s|^{q-3}$$

where $2\leqq q\leqq p/2$, C, ε_0 are constants >0.

Now our theorem is stated as follows.

Theorem. *Given $f(t)\in L^2(\tau; W)$ (not zero), there exists a solution $u\in L^\infty(\tau; W)$ of the equation* (1):

$$u_{tt}+Au+B(u, u_t)=f,$$

such that $u_t\in L^2(\tau; L^2(\Omega))$, $u_{tt}\in L^{p'}(\tau; W^*)$ and that

$$B(u, u_t)=\sum_{|\alpha|\leqq m-1}(-1)^{|\alpha|}D^\alpha\beta'_\alpha(|D^\alpha u|^2)D^\alpha u_t$$

where $1/p+1/p'=1$, A and B satisfy Assumption A *and Assumption* B, *respectively.*

§3. Proof of the Theorem

We shall prove the theorem by means of Faedo-Galerkin's method combined with the fixed point theorem and the compactness method. Since W is separable, there exists a countable basis $\{w_n\}$ in W which is orthonormal in $L^2(\Omega)$. Let W_n be the subspace of W spanned by $w_1,\dots, w_n$.

Consider the ordinary differential system in W_n

$$(10)\qquad (u^n_{tt}(t), w_j)+a(u^n(t), w_j)+b(u^n(t), u^n_t(t); w_j)$$
$$=(f(t), w_j)\qquad (j=1, 2,\dots, n)$$

with periodic conditions

$$(11)\qquad u^n(t)=u^n(t+\tau),\qquad u^n_t(t)=u^n_t(t+\tau)$$

where

$$b(u^n(t), u^n_t(t); w_j)=(B(u^n(t), u^n_t(t)), w_j)$$
$$=(\beta'_0(|u^n(t)|^2)u^n_t(t), w_j)+((u^n_t, w_j)).$$

The solutions will be of the form

$$(12)\qquad u^n(t)=\sum_{k=1}^{n} c_{n,k}(t)w_k,\qquad c_{n,k}(t)\in C^1(\tau)$$

if they exist. Now the substitution of the $u^n(t)$ into (10), (11) gives the second order differential system of $\boldsymbol{C}_n(t)=(c_{n1}(t),\dots, c_{nn}(t))^*$,[1)]

$$(13)\qquad \boldsymbol{C}''_n(t)+\boldsymbol{F}(\boldsymbol{C}_n(t))+\boldsymbol{H}(\boldsymbol{C}_n(t),\quad \boldsymbol{C}'_n(t))=\boldsymbol{H}_0(t)$$

and the periodic conditions

$$(14)\qquad \boldsymbol{C}_n(t)=\boldsymbol{C}_n(t+\tau),\qquad \boldsymbol{C}'_n(t)=\boldsymbol{C}'_n(t+\tau),$$

where

$$\boldsymbol{F}(\boldsymbol{C}_n(t))=(F_1(\boldsymbol{C}_n(t)),\dots F_n(\boldsymbol{C}_n(t)))^*,$$
$$F_j(\boldsymbol{C}_n(t))=a(u^n(t), w_j);$$
$$\boldsymbol{H}(\boldsymbol{C}_n(t), \boldsymbol{C}'_n(t))=(H_1(\boldsymbol{C}_n(t)),\dots H_n(\boldsymbol{C}_n(t)))^*,$$

1) * denotes the transpose operation of n-vector.

$$H_j(\mathbb{C}_n(t), \mathbb{C}_n'(t)) = b(u^n(t), u_t^n(t); w_j);$$

$$\mathbb{H}_0(t) = (f_1(t), \dots, f_n(t))^*, f_j(t) = (f(t), w_j)$$

$$(j = 1, 2, \dots, n)$$

Lemma 1. *There exists a solution $\mathbb{C}_n(t)$ of* (13), (14).

Proof. Consider a system of λ dependence $(0 \leqq \lambda \leqq 1)$,

$$\mathbb{C}_n''(t) + \delta \mathbb{C}_n'(t) + k\mathbb{C}_n(t) \tag{15}$$

$$= \lambda\{-\mathbb{F}(\mathbb{C}_n(t)) + k\mathbb{C}_n(t) - \mathbb{H}(\mathbb{C}_n(t), \mathbb{C}_n'(t)) + \delta \mathbb{C}_n'(t)\}$$

$$+ \mathbb{H}_0(t)$$

together with (14). Here δ and k are any fixed constants such that $0 < \delta < \delta_0$, $0 < k < k_0$ where δ_0 is a constant satisfying $\delta_0 \|u\| \leqq \|u\|_{1,2}$ for any $u \in H^1(\Omega)$, k_0 a constant in Assumption A-(iv). Let $G_n(t, s)$ be a unique Green's function of (15) for $\lambda = 0$, $\mathbb{H}_0 = 0$ with conditions (14), and define the operator $T_n(\lambda)$ from a Banach space X_n into itself:

$$T_n(\lambda)\mathbb{C}_n = \int_{I_\tau} [\lambda\{-\mathbb{F}(\mathbb{C}_n(s)) + k\mathbb{C}_n(s) - \mathbb{H}(\mathbb{C}_n(s), \mathbb{C}_n'(s)) + \delta \mathbb{C}_n'(s)\} \tag{16}$$

$$+ \mathbb{H}_0(s)] G_n(t, s)\, ds$$

where

$$X_n = \mathbb{C}^1(\tau) \times \cdots \times \mathbb{C}^1(\tau) \qquad (n\text{-copies})$$

with norm
$\|\mathbb{C}_n\|_{X_n} = \sup_{I_\tau} \{|\mathbb{C}_n(t)| + |\mathbb{C}_n'(t)|\}$ ($|\cdot|$: the length of n-vector) for $\mathbb{C}_n \in X_n$.

To prove the lemma it suffices to show that the operator $T_n(1)$ has a fixed point in X_n. So we apply Leray-Schauder's theorem to the family of operators $T_n(\lambda)$ $(0 \leqq \lambda \leqq 1)$ on the space X_n. We observe that

$$\|F(\mathbb{C}_n^{(\nu)}) - F(\mathbb{C}_n)\|_\infty \to 0 \tag{17}$$

and

$$\|H(\mathbb{C}_n^{(\nu)}, \mathbb{C}_n^{(\nu)\prime}) - H(\mathbb{C}_n, \mathbb{C}_n')\| \to 0 \tag{18}$$

when $\mathbb{C}_n^{(\nu)} \to \mathbb{C}_n$ in X_n, as $\nu \to \infty$. In fact, (17) is a direct consequence of Assumption A on A_α by measure theoretical arguments. To show (18) we put

$$u^{(\nu)}(x, t) = \sum_{k=1}^{n} c_k^{(\nu)}(t) w_k(x)$$

$$u(x, t) = \sum_{k=1}^{n} c_k(t) w_k(x),$$

dropping the suffix n for brevity in notation. Assumption B implies

$$|\beta_0'(|u^{(\nu)}(x, t)|^2) - \beta_0'(|u(x, t)|^2)|$$
$$\leqq C|u^{(\nu)}(x, t) - u(x, t)|(|u^{(\nu)}(x, t)| + |u(x, t)|)^{q-2}$$

for all t and almost all $x \in \Omega$. Since $\|\mathbb{C}^{(\nu)}\|_\infty$, $\|\mathbb{C}^{(\nu)\prime}\|_\infty$ are bounded on ν, we obtain by Hölder's inequality

$$|(u_t^{(\nu)}(x, t)(\beta_0'(|u^{(\nu)}(x, t)|^2) - \beta_0'(|u(x, t)|^2)), w_j)|$$
$$\leqq C\|u_t^{(\nu)}(t)\| \cdot \|u^{(\nu)}(t) - u(t)\|_{2q}(\|u^{(\nu)}(t)\|_{2q}^{q-2} + \|u(t)\|_{2q}^{q-2})\|w_j\|_{2q}$$
$$\leqq K\|\mathbb{C}^{(\nu)} - \mathbb{C}\|_{X_n}$$

for some constant K. Similarly we have

$$|((u_t(x, t) - u_t^{(\nu)}(x, t))\beta'(|u(x, t)|^2), w_j)| \leqq K\|\mathbb{C}^{(\nu)} - \mathbb{C}\|_{X_n}.$$

Hence $(u_t^{(\nu)}(t)\beta_0'(|u^{(\nu)}(t)|^2), w_j) \to (u_t(t)\beta_0'(|u(t)|^2), w_j)$ uniformly on t as $\nu \to \infty$, which implies (18). Thus the continuity of $T_n(\lambda)$ on X_n follows immediately from (16), (17) and (18).

Next, let $S = \{\mathbb{C} \in X_n \,|\, \|\mathbb{C}_n\| \leqq 1\}$. Then the properties of $G_n(t, s)$ imply that for each λ, $T_n(\lambda)S$ is bounded in X_n and is a set of equi-continuous functions, and that

$$\|(T_n(\lambda_2) - T_n(\lambda_1))S\|_{X_n} \leqq K|\lambda_2 - \lambda_1|$$

for a suitable constant K. Therefore each $T_n(\lambda)$ is a compact operator from X_n into X_n and the family $\{T_n(\lambda) \,|\, 0 \leqq \lambda \leqq 1\}$ is homotopic. We note that the topological degree of $T_n(0)$ is $+1$ since the system (15) for $\lambda = 0$ has a unique solution in X_n. In order to see that the topological degree of $T_n(1)$ is $+1$ (positive) it only remains to show that for each λ

$$\boldsymbol{C}(t)=T_n(\lambda)\boldsymbol{C}(t) \Rightarrow \|\boldsymbol{C}\|_{X_n} \leqq L \tag{19}$$

where L is a constant independent of λ. The proof of (19) is a variant of that of the following lemma, and is omitted. Q.E.D.

Lemma 2. *The solutions u^n of* (10), (11) *have the following estimates*:

$$\sum_{|\alpha|\leqq 1}\int_{I_\tau}\|D^\alpha u_t^n(t)\|^2 dt \leqq K_1, \tag{20}$$

$$\|u_t^n(t)\|, \|u^n(t)\|_{m,p} \leqq K_2 \tag{21}$$

where K_1, K_2 are constants independent of n and t.

Proof. Since both sides of (10) are linear on w_j, we have, replacing w_j by u_t^n,

$$(u_{tt}^n(t), u_t^n(t))+a(u^n(t), u_t^n(t))+b(u^n(t), u_t^n(t))$$
$$=(f(t), u_t^n(t)).$$

Integrating both sides over I_τ with respect to t and using Assumptions A-(vi), B we obtain

$$\sum_{|\alpha|\leqq 1}\int_{I_\tau}\|D^\alpha u_t^n(t)\|^2 dt \leqq \left(\int_{I_\tau}\|f(t)\|^2 dt\right)^{1/2}\left(\int_{I_\tau}\|u_t^n(t)\|^2 dt\right)^{1/2},$$

from which the estimates (20) follows immediately.

Replacing w_j by u^n in (10) gives

$$(u_{tt}^n(t), u^n(t))+a(u^n(t), u^n(t)) \tag{22}$$
$$+b(u^n(t), u_t^n(t); u^n(t))=(f(t), u^n(t)).$$

We remark that

$$\int_{I_\tau} b(u^n(t), u_t^n(t); u^n(t))\,dt=0$$

because of the periodicity of $u^n(t)$. Then, integrating (22) over I_τ with respect to t and using Assumption A-(iv) we have

(23) $$k_0\int_{I_\tau}\|u^n(t)\|^2dt+\int_{I_\tau}c_0(\|u^n(t)\|_{m,p})\,dt$$

$$\leqq\int_{I_\tau}\|u_t^n(t)\|^2dt+\left(\int_{I_\tau}\|f(t)\|^2dt\right)^{1/2}\left(\int_{I_\tau}\|u^n(t)\|^2dt\right)^{1/2}.$$

This yields that

(24) $$\int_{I_\tau}\|u^n(t)\|^2dt,\qquad \int_{I_\tau}a(u^n(t),\,u^n(t))\,dt$$

have a bound independent of n, because $\int_{I_\tau}c_0(\|u^n(t)\|_{m,p})\,dt$ is bounded from below on n.

Finally substituting u_t^n for w_j in (10) and integrating both sides from s to t $(s<t)$, we obtain by Assumption A-(vi),

(25) $$\frac{1}{2}\|u_t^n(t)\|^2+c_1(\|u^n(t)\|_{m,p})$$

$$\leqq K+\frac{1}{2}\|u_t^n(s)\|^2+k_1a(u^n(s),\,u^n(s))+k_2$$

where K is a constant dependent of f and K_1 in (20). Further, integrating both sides of (25) with respect to s from $t-\tau$ to t and noting that the right hand side is bounded by virtue of (24), we know that

$$\|u_t^n(t)\|,\ c_1(\|u^n(t)\|_{m,p})\leqq K \qquad \text{(independent of } n \text{ and } t).$$

Since $\lim_{s\to\infty}c_1(s)=\infty$, this proves the lemma. Q.E.D.

Now we may infer that

$\{u^n\}$ is bounded in $L^\infty(\tau;\,W)$,

$\{u_t^n\}$ is bounded in $L^2(\tau;\,H^1(\Omega))\cap L^\infty(\tau;\,L^2(\Omega))$.

Then we may extract a subsequence $\{u^\nu\}$ such that

$u^\nu\to u$ (an element of $L^\infty(\tau;\,W)$) weakly star in $L^\infty(\tau;\,W)$,

$u_t^\nu\to u_t$ strongly in $L^2(\tau;\,L^2(\Omega))$;

in addition,

$u^{\nu} \to u$ strongly in $L^{p}(\Omega \times I_{\tau})$

where we have used that the injection mappings

$$i: W^{k,p}(\Omega) \to W^{k-1,p}(\Omega)$$

$$j: H^{1}(\Omega) \quad \to L^{2}(\Omega)$$

are compact.

Making use of these results, we shall prove:

Lemma 3. *For any* $v \in L^{2}(\tau; W)$

$$\int_{I_\tau} b(u^{\nu}, u_t^{\nu}; v)\,dt \to \int_{I_\tau} b(u, u_t; v)\,dt.$$

Proof. By definition, we have

(26) $$b(u^{\nu}, u_t^{\nu}; v) - b(u, u_t; v)$$

$$= (u_t^{\nu} - u_t, \beta_0'(|u|^2)v) + (u_t^{\nu}, (\beta_0'(|u^{\nu}|^2) - \beta_0'(|u|^2))v)$$

Assumption B implies

$$|\beta_0'(|u(t)|^2)v(t)| \leqq C|u(t)|^{q-1}|v(t)|.$$

Therefore we obtain

$$\int_{\Omega} |u(t)|^{2(q-1)}|v(t)|^2\,dx \leqq \|u(t)\|_{2q}^{2(p-1)}\|v(t)\|_{2q}^{2},$$

which means

$$\beta_0'(|u(t)|^2)v(t) \in L^{2}(\tau; L^{2}(\Omega)).$$

As $u_t^{\nu} \to u_t$ strongly in $L^{2}(\tau; L^{2}(\Omega))$, we know that

$$\int_{I_\tau} (u_t^{\nu} - u_t, \beta_0'(|u|^2)v)\,dt \to 0.$$

For the second term of the right hand side of (26), since

$$|\beta_0'(|u^{\nu}|^2) - \beta_0'(|u|^2)| \leqq |u^{\nu} - u|(|u^{\nu}|^{q-2} + |u|^{q-2}),$$

we obtain

$$|(u_t^\nu, (\beta_0'(|u^\nu|^2)-\beta_0'(|u|^2))v)|$$

$$\leqq C\|u_t^\nu(t)\|\cdot\|u^\nu(t)-u(t)\|_{2q}(\|u^\nu(t)\|_{2q}^{q-2}+\|u(t)\|_{2q}^{q-2})\|v(t)\|_{2q},$$

taking boundedness of $\|u^\nu(t)\|_{2q}^{q-2}$, $\|u_t^\nu(t)\|$ into consideration,

$$\leqq K\|u^\nu(t)-u(t)\|_{2q}\|v(t)\|_{2q}$$

where K is a constant independent of n and t. Therefore we have

$$\left|\int_{I_\tau}(u_t^\nu, (\beta_0'(|u^\nu|^2)-\beta_0'(|u|^2))v)\,dt\right|$$

$$\leqq K\left(\int_{I_\tau}\|u^\nu(t)-u(t)\|_{2q}^2\,dt\right)^{1/2}\left(\int_{I_\tau}\|v(t)\|_{2q}^2\,dt\right)^{1/2}$$

$$\leqq K_1\|u^\nu-u\|_{L^{2q}(\Omega\times I_\tau)}\cdot\|v\|_{L^2(\tau;L^{2q}(\Omega))},$$

K_1 being another constant independent of n. Since $u^\nu\rightarrow u$ strongly in $L^p(\Omega\times I_\tau)$, the last member in the above inequalities tends to zero. Thus we proved the lemma. Q.E.D.

Finally, to establish the remaining part of our theorem, we need the following assertion.

Lemma 4. *There exists a subsequence $\{\mu\}$ of $\{\nu\}$ such that*

$$\int_{I_\tau}a(u^\mu(t), v(t))\,dt\rightarrow\int_{I_\tau}a(u(t), v(t))\,dt \qquad (\mu\rightarrow\infty)$$

for any $v\in L^2(\tau; W)$.

Proof. Consider a linear form on $L^2(\tau; W)$:

$$v\rightarrow\int_{I_\tau}a(u(t), v(t))\,dt.$$

Since $u\in L^\infty(\tau; W)$, Assumption A-(iii) implies that

$$\left|\int_{I_\tau}a(u(t), v(t))\,dt\right|\leqq\int_{I_\tau}g_1(\|u(t)\|_{m,p})\|v(t)\|_{m,p}\,dt$$

$$\leqq K\|v\|_{L^2(\tau;W)}.$$

Hence the linear form is continuous on $L^2(\tau; W)$, so that there is an element $\mathscr{A}u \in L^2(\tau; W^*)$ such that

$$\int_{I_\tau} a(u(t), v(t))\,dt = \langle\!\langle \mathscr{A}u, v \rangle\!\rangle.$$

Here $\langle\!\langle\ ,\ \rangle\!\rangle$ denotes the pairing between $L^2(\tau; W)$ and $L^2(\tau; W^*)$. The operator $\mathscr{A}$ sending $L^2(\tau; W)$ into $L^2(\tau; W^*)$ satisfies

$$\|\mathscr{A}u\|_{L^2(\tau;W)} = \sup_v \left| \int_{I_\tau} a(u^\nu(t), v(t))\,dt \right| \leqq K$$

where v in the second member runs through the set $\{\|v\|_{L^2(\tau;W)} \leqq 1\}$. Thus there exist a subsequence $\{\mu\} \subset \{\nu\}$ and an element $\xi \in L^2(\tau; W^*)$ such that for $v \in L^2(\tau; W)$

$$\int_{I_\tau} a(u^\mu(t), v(t))\,dt \ \rightarrow \ \langle\!\langle \xi, v \rangle\!\rangle$$

where ξ is an element of $L^2(\tau; W^*)$. We assert $\mathscr{A}u = \xi$. Take any φ, a finite sum of $c_k(t)w_k(x)$ where $c_k \in C^1(\tau)$, $w_k \in W$. Then for large n hold the equalities:

$$-\int_{I_\tau} (u_t^\mu(t), \varphi_t(t))\,dt + \int_{I_\tau} a(u^\mu(t), \varphi(t))\,dt$$

$$+\int_{I_\tau} b(u^\mu(t), u_t^\mu(t); \varphi(t))\,dt = \int_{I_\tau} (f(t), \varphi(t))\,dt.$$

Letting $\mu \rightarrow \infty$, we have

$$(27)\qquad -\int_{I_\tau} (u_t(t), \varphi_t(t))\,dt + \langle\!\langle \xi, \varphi \rangle\!\rangle + \int_{I_\tau} b(u(t), u_t(t); \varphi(t))\,dt$$

$$= \int_{I_\tau} (f(t), \varphi(t))\,dt.$$

Let

$$V(\tau; W) = \{v \in L^2(\tau; W) \mid v_t \in L^2(\tau; L^2(\Omega))\}$$

with norm

$$\|v\|_{V(\tau;W)} = \|v\|_{L^2(\tau;W)} + \|v_t\|_{L^2(\tau;L^2(\Omega))}.$$

Since the set of the φ defined above is dense in $V(\tau; W)$, (27) is valid for any $\varphi \in V(\tau; W)$, in particular, for $\varphi = u$. Hence,

$$-\int_{I_\tau} \|u_t(t)\|^2 dt + \langle\!\langle \xi, u \rangle\!\rangle + \int_{I_\tau} b(u(t), u_t(t); u(t))\, dt$$
$$= \int_{I_\tau} (f(t), u(t))\, dt.$$

However, we observe that

$$\int_{I_\tau} (u_t \beta_0'(|u|^2), u)\, dt = \lim \int_{I_\tau} (u_t^\mu \beta_0'(|u^\mu|^2), u^\mu)\, dt$$
$$= 0,$$

from which follows

$$-\int_{I_\tau} \|u_t\|^2 dt + \langle\!\langle \xi, u \rangle\!\rangle = \int_{I_\tau} (f, u)\, dt. \tag{28}$$

Since

$$-\int_{I_\tau} \|u_t^\mu\|^2 dt + \int_{I_\tau} a(u^\mu, u^\mu)\, dt = \int_{I_\tau} (f, u^\mu)\, dt,$$

taking the limit inferior of both sides, and recalling that $u_t^\mu \to u_t$ strongly in $L^2(\tau; L^2(\Omega))$ we obtain that

$$-\int_{I_\tau} \|u_t\|^2 dt + \underline{\lim} \langle\!\langle \mathscr{A} u^\mu, u^\mu \rangle\!\rangle \leqq \int_{I_\tau} (f, u)\, dt. \tag{29}$$

Comparing (29) with (28) yields

$$\langle\!\langle \xi, u \rangle\!\rangle \geqq \underline{\lim} \langle\!\langle \mathscr{A} u^\mu, u^\mu \rangle\!\rangle$$

from which we can conclude in the same way as in [5] that $\xi = \mathscr{A} u$.

Q.E.D.

Now we observe in the proof of Lemma 3 that $B(u, u_t) \in L^2(\tau; W^*)$, so that for any smooth function $\varphi(x, t)$ periodic in t, we have in the distributional sense that

$$\int_{I_\tau}(u_{tt}, \varphi)\,dt+\int_{I_\tau}(Au, \varphi)\,dt+\int_{I_\tau}(B(u, u_t), \varphi)\,dt$$

$$=\int_{I_\tau}(f, \varphi)\,dt,$$

$$u_{tt}=-Au-B(u, u_t)+f\in L^2(\tau; W^*)$$

which completes the proof of our theorem, for $B(u, u_t)=\beta_0'(|u|^2)u_t-\Delta u_t$. When $B(u, u_t)=\sum_{|\alpha|\leqq m-1}(-1)^{|\alpha|}D^\alpha\beta_\alpha'(|D^\alpha u|^2)D^\alpha u_t$, we need some modifications. Consider the system (15) for $\varepsilon_0\delta$ instead of δ and for

$$b(u^n, u_t^n; w_j)=\sum_{|\alpha|\leqq m-1}(\beta_\alpha'(|D^\alpha u^n|^2)D^\alpha u^n, D^\alpha w_j).$$

Then we can obtain the estimates

$$\sum_{|\alpha|\leqq m-1}\int_{I_\tau}\|D^\alpha u_t^n\|^2\,dt\leqq K_1 \tag{30}$$

as in the proof of Lemma 2. Also we know by (20)

$$\sum_{|\alpha|\leqq m}\int_{I_\tau}\|D^\alpha u^n\|_p^p\,dt\leqq K_2\,. \tag{31}$$

Therefore we may choose a further subsequence $\{\sigma\}$ of $\{\mu\}$ such that when $\sigma\to\infty$,

$$D^\alpha u_t^\sigma\to D^\alpha u_t \qquad \text{weakly in } L^2(\Omega\times I_\tau) \tag{32}$$

and

$$D^\alpha u^\sigma\to D^\alpha u \qquad \text{strongly in } L^p(\Omega\times I_\tau), \tag{33}$$

both for $|\alpha|\leqq m-1$. Since, for $v\in L^p(\tau; W)$,

$$\begin{aligned}&b(u^\sigma, u_t^\sigma; v)-b(u, u_t; v)\\&=\sum_{|\alpha|\leqq m-1}(D^\alpha u_t^\sigma-D^\alpha u_t, \beta_\alpha'(|D^\alpha u|^2)D^\alpha v)\\&\quad+\sum_{|\alpha|\leqq m-1}(D^\alpha u_t^\sigma(\beta_\alpha'(|D^\alpha u^\sigma|^2)-\beta_\alpha'(|D^\alpha u|^2)), D^\alpha v),\end{aligned}$$

to prove Lemma 3 for $v\in L^p(\tau; W)$ it is enough to show that for each α

$$\int_{I_\tau} (D^\alpha u_t^\sigma - D^\alpha u_t,\ \beta'_\alpha(|D^\alpha u|^2) D^\alpha v)\,dt \to 0$$

(34)

$$\int_{I_\tau} (D^\alpha u_t^\sigma(\beta'_\alpha(|D^\alpha u^\sigma|^2) - \beta'_\alpha(|D^\alpha u|^2)),\ D^\alpha v)\,dt \to 0$$

as $\sigma \to \infty$.

The first assertion is obvious because of (32) and $\beta'_\alpha(|D^\alpha u|^2) D^\alpha v \in L^2(\Omega \times I_\tau)$. For the second one, we can show as in the proof of Lemma 3 that

$$\left|\int_{I_\tau} (D^\alpha u_t^\sigma(\beta'_\alpha(|D^\alpha u|^2) - \beta'_\alpha(|D^\alpha u|^2),\ D^\alpha v)\,dt\right|$$

$$\leqq C \|D^\alpha u_t^\sigma\|_{L^2(\Omega \times I)} \|D^\alpha u^\sigma - D^\alpha u\|_{L^{2q}(\Omega \times I_\tau)}$$

$$\times \{\|D^\alpha u^\sigma\|^{q-2}_{L^{2q}(\Omega \times I_\tau)} + \|D^\alpha u\|^{q-2}_{L^{2q}(\Omega \times I_\tau)}\} \|D^\alpha v\|_{L^{2q}(\Omega \times I_\tau)}.$$

from which (34) follows by virtue of (31), (33). Since Lemma 4 holds for $v \in L^p(\tau;\ W)$ we have completed the proof of the theorem.

Example 1. Define A_α, $a(u, v)$ by

$$A_\alpha(x,\ u, \ldots,\ D^m u) = |D^\alpha u|^{p-2} D^\alpha u$$

and

$$a(u,\ v) = \int_\Omega \sum_{|\alpha| \leqq m} |D^\alpha u|^{p-2} D^\alpha u D^\alpha v\, dx,$$

respectively. It can be easily seen that A_α and $a(u, v)$, then, satisfy Assumption A. Hence an evolution equation

$$\frac{\partial^2 u}{\partial t^2} + \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^\alpha(|D^\alpha u|^{p-2} D^\alpha u) - \Delta \frac{\partial u}{\partial t} + |u|^{q-1} \frac{\partial u}{\partial t} = f(x,\ t)$$

has a solution $u(x, t)$ in $L^\infty(\tau;\ W_0^{m,p}(\Omega))$ provided $2 \leqq q \leqq p/2$ and $f(x,\ t) \in L^2(\tau;\ L^2(\Omega))$.

Example 2. Let A be the operator defined in Example 1. Then an evolution equation

$$\frac{\partial^2 u}{\partial t^2} + Au + \sum_{|\alpha| \leqq m-1} (-1)^{|\alpha|} |D^\alpha u|^{q-1} D^\alpha \frac{\partial u}{\partial t} - \Delta \frac{\partial u}{\partial t} = f(x,\ t)$$

has a solution $u(x, t)$ in $L^{\infty}(\tau; W_0^{m,p}(\Omega))$ provided $2 \leqq q \leqq p/2$ and $f(x, t) \in L^2(\tau; L^2(\Omega))$.

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