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Global regularity of 2D Navier–Stokes free boundary with small viscosity contrast

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Abstract. This paper studies the dynamics of two incompressible immiscible fluids in two dimensions modeled by the inhomogeneous Navier–Stokes equations. We prove that if initially the viscosity contrast is small then there is global-in-time regularity. This result has been proved recently in Paicu and Zhang [Comm. Math. Phys. 376 (2020)] for $H^{5/2}$ Sobolev regularity of the interface. Here we provide a new approach which allows us to obtain preservation of the natural $C^{1+\gamma}$ Hölder regularity of the interface for all $0 < \gamma < 1$. Our proof is direct and allows for low Sobolev regularity of the initial velocity without any extra technicalities. It uses new quantitative harmonic analysis bounds for C^γ norms of even singular integral operators on characteristic functions of $C^{1+\gamma}$ domains [Gancedo and García-Juárez, J. Funct. Anal. 283 (2022)].

1. Introduction

In this paper we consider incompressible flows in the whole space $\mathbb{R}^2$,

$$\nabla \cdot u = 0, \tag{1.1}$$

of inhomogeneous fluids

$$\partial_t \rho + u \cdot \nabla \rho = 0, \tag{1.2}$$

driven by the Navier–Stokes equations

$$\rho D_t u = \nabla \cdot (\mu \mathbb{D} u - \mathbb{I}_2 P). \tag{INS}$$

Above, u, ρ, μ and P are the velocity field, the density, viscosity and pressure of the fluids. The operator D_t is the total derivative

$$D_t u = \partial_t u + u \cdot \nabla u,$$

the tensor $\mathbb{D}u$ denotes the symmetric part of the gradient

$$\mathbb{D}u = \nabla u + \nabla u^*, \quad \mathbb{D}_{ij} u = \partial_i u_j + \partial_j u_i,$$

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and $\mathbb{I}_2$ is the identity matrix in $\mathbb{R}^2$. The viscosity depends smoothly on the density, $\mu = \tilde{\mu}(\rho)$ with $\tilde{\mu}$ smooth, so that in particular it is also preserved along trajectories,

$$\partial_t \mu + u \cdot \nabla \mu = 0. \tag{1.3}$$

We deal with a moving fluid occupying a bounded domain $D(t) \subset \mathbb{R}^2$ and a second fluid occupying the complement of it $D(t)^c = \mathbb{R}^2 \setminus \overline{D(t)}$. They evolve with the velocity field through the particle trajectories

$$\begin{cases} \frac{dX}{dt}(y,t) = u(X(y,t),t), \\ X(y,0) = y. \end{cases} \tag{1.4}$$

The fluids are immiscible, having different characteristics, principally different densities and viscosities, so that

$$(u,\rho,\mu,P)(x,t) = \begin{cases} (u^{\text{in}},\rho^{\text{in}},\mu^{\text{in}},P^{\text{in}})(x,t), & x \in D(t), \\ (u^{\text{out}},\rho^{\text{out}},\mu^{\text{out}},P^{\text{out}})(x,t), & x \in D(t)^c = \mathbb{R}^2 \setminus \overline{D(t)}. \end{cases} \tag{1.5}$$

A main interest is the dynamics and the regularity of the common boundary between the fluids $\partial D(t)$. The system is assumed to have initial finite kinetic energy

$$\int_{\mathbb{R}^2} \rho(x,0)|u(x,0)|^2 dx < \infty,$$

providing the physically relevant scenario. The classical free boundary physical conditions without capillarity [15],

$$[\![u]\!] = 0 \text{ on } \partial D(t), \tag{1.6}$$

$$[\![\mu \mathbb{D}u - \mathbb{I}_2 P]\!] n = 0 \text{ on } \partial D(t), \tag{1.7}$$

are recovered by considering equations (1.1)–(1.3) in a weak sense, together with the regularity obtained for the solution [20].

1.1. Previous results

Free boundary Navier–Stokes problems have a long history in mathematical science. The one-fluid case (vacuum–fluid interaction where $\mu^{\text{out}} = 0 = \rho^{\text{out}}$) was first considered, where global-in-time existence with gravity for near planar initial data was proved. Recently, low-regularity results for inhomogeneous Navier–Stokes equations in the whole space have given new approaches for the two-fluid case (fluid–fluid interaction). They consider $\mu > 0$, giving global regularity for different scenarios. We describe first the classical vacuum–fluid case and later the fluid–fluid interaction.

The first study of the free boundary Navier–Stokes equations goes back to [37], where fluid–vacuum interaction was studied for closed contours with no gravity ($g = 0$) using

Hölder spaces with the appropriate parabolic scale [38,39]. Local well-posedness in Sobolev spaces was given next for the horizontally flat geometry, where the fluid lies essentially on top of a fixed bottom with nonslip boundary condition and below vacuum [2]. See [1] for similar results in L^p Sobolev spaces.

The long time behavior of solutions was studied in [2], giving existence up to time T depending on the size of the initial, near horizontally flat, data. The first global-in-time existence result for small initial data was given for the surface tension case [3]. This result was extended to the case without surface tension [40,41]. After those results, sharp decay rates of the solution were given in the case of surface tension for asymptotically flat [4] and horizontally periodic geometries [32]. More recently, the results were extended with different approaches without the help of surface tension for both geometries [22,23]. See the recent paper [10], where global well-posedness is shown for this free boundary value problem with the initial domain the half-space and the initial velocity small with respect to a scaling-invariant norm. Contrarily, large size initial data produce finite-time singularities. The Navier–Stokes free boundary blows up in finite time for the two-dimensional vacuum–fluid interaction case [7]. The result considers closed contours producing splash singularities (particle collision on the evolving boundary) in finite time. See [9] for the extension of the blow-up to the three-dimensional case.

The techniques in [37] were extended to the case of two fluids to study the global-in-time well-posedness of problems for small initial velocity [16]. See [35], where the low-regularity case is considered. In [42], decay estimates are obtained for the internal waves case with gravity.

A different approach to studying the interface evolution between immiscible fluids is to use inhomogeneous Navier–Stokes for low-regularity solutions. Parabolicity can be exploited to gain enough regularity for the velocity in the two-fluid case even when the functions defining the fluid properties are given as in (1.5). The approaches with no viscosity jump ($\mu = 1$) are explained first. In two dimensions, there is global regularity for the system (1.1)–(1.2)–(INS) for general smooth positive initial density [27]. In the three-dimensional case, global regularity for large initial data is open as it contains Navier–Stokes as a particular case [18]. If $0 \le \rho(x,0) \in L^\infty$ is allowed and $\sqrt{\rho(x,0)}u(x,0) \in L^2(\mathbb{R}^d)$, $d = 2,3$, there exist global-in-time weak solutions satisfying

$$\int \rho(x,t)|u(x,t)|^2\,dx + 2\int_0^t \int |\nabla u(x,s)|^2\,dx\,ds \le \int \rho(x,0)|u(x,0)|^2\,dx,$$

with $\rho \in L^\infty((0,T)\times\mathbb{R}^d)$, $\rho u \in L^\infty(0,T;L^2(\mathbb{R}^d))$ and $u \in L^2(0,T;\dot{H}^1(\mathbb{R}^d))$ [36]. Throughout the paper, we will use the convention that spaces with a dot denote their homogeneous counterpart. Considering fluids of different constant densities, domains evolving by the fluid velocity were proved to preserve their volume [31]. On the other hand, the propagation of regularity for the free boundary $\partial D(t)$ was proposed as a challenging open question in the same book (1996, P. L. Lions' density patch problem).

Recently, global regularity results in two dimensions, and with smallness assumptions in three dimensions, have been obtained for low-regularity positive density and constant viscosity. Global well-posedness was shown for initial discontinuous densities with sufficiently small jumps and small initial velocities [11, 12]. The case of more regular velocity was considered in [25]. Finally, in [34] the smallness conditions of the density jump were removed. After the results above, global-in-time regularity for fluids of different densities (density patch problem) has been studied. Persistence of $C^{2+\gamma}$ regularity of the free boundary results was shown in two dimensions for $0 < \gamma < 1$, using paradifferential calculus and striated regularity techniques. The works consider positive densities with a small jump first [29] and later without the smallness assumption [30]. Using the approach in [11], propagation of $C^{1+\gamma}$ regularity was given for a small density jump and small initial velocity [14]. The size restriction was removed in [20], providing global-in-time regularity for $C^{1+\gamma}$ two-dimensional contours. This approach does not use paradifferential calculus but bootstrapping arguments, getting propagation of regularity from weak solutions to $C^{1+\gamma}$. It uses an elliptic approach inspired by previous results obtained for two-dimensional Boussinesq temperature fronts [19]. See [28] for the three-dimensional extension with high regularity and smallness in velocity and density jump. In the bounded or periodic case, a new approach has been used to allow the case of possibly vanishing density, with no restriction on the jump size, no gravity and constant viscosity [13]. In this density-zero scenario, the interface evolution would be driven by a Stokes/Navier–Stokes interaction, dealing with a linear Stokes flow for one of the fluids.

For the more singular case of variable viscosity, with density merely bounded, under the additional assumptions that $u_0 \in H^1(\mathbb{T}^2)$ and sufficiently small viscosity variation in L^∞, the weak solutions constructed in [31] satisfy that $u \in L^\infty(0,T;H^1)$, $\sqrt{\rho}u_t \in L^2(0,T;L^2)$, $\rho, \mu \in L^\infty(0,T;L^\infty)$ for all $T > 0$ [17]. However, uniqueness and regularity of these solutions was not known, unless the initial density and viscosity satisfy certain smoothness (at least slightly more than continuity, see [6] and the references therein). Recently, [33] global-in-time regularity for positive density and small viscosity jump is obtained in $\mathbb{R}^2$ under the additional assumption of certain striated regularity for the initial viscosity. In particular, they showed global-in-time propagation of the $H^{5/2}$ regularity of the moving interface for the density and viscosity patch problem. The strategy of the proof uses paradifferential calculus together with striated regularity estimates. The approach is in the spirit of the global regularity result for the two-dimensional vortex patch problem shown in [8].

Main result

In this paper we prove global-in-time well-posedness for the two-dimensional density and viscosity patch problem. We study the evolution of two fluids with different densities and viscosities evolving according to inhomogeneous Navier–Stokes (1.1)–(1.3). The initial density and viscosity functions are bounded from below and from above as follows:

$$0 < \rho^m \le \rho_0(x) \le \rho^M, \quad 0 < \mu^m \le \mu_0(x) \le \mu^M.$$

The initial interface between the fluids is assumed to be a closed $C^{1+\gamma}$ regular curve in the plane. Specifically, we prove the following result.

Theorem 1.1. *Let $D_0 \subset \mathbb{R}^2$ be a bounded domain whose boundary ∂D_0 is non-self-intersecting and of class $C^{1+\gamma}$, $0 < \gamma < 1$. Let $\rho_0^{\text{in}} \in C^\gamma(\bar{D}_0)$, $\rho_0^{\text{out}} \in C^\gamma(\mathbb{R}^2 \setminus D_0)$, with $\rho_0^{\text{out}} - \rho^\infty$, where $\rho^\infty \in \mathbb{R}_+$, and $\mu = \tilde{\mu}(\rho)$ with $\tilde{\mu}$ smooth. Let the initial density be given by*

$$\rho_0(x) = \rho_0^{\text{in}}(x) 1_{D_0}(x) + \rho_0^{\text{out}}(x) 1_{D_0^c}(x) > 0,$$

where 1_{D_0} is the characteristic function of D_0, and let $u_0 \in L^r \cap H^{\gamma+\varepsilon}$, $0 < \varepsilon < \min\{\gamma, 1-\gamma\} < 1$, $1 < r < \min\{\frac{2}{2-\gamma+\varepsilon}, \frac{2}{1+\gamma}\}$ be a divergence-free vector field. Then there exists $\delta > 0$ such that if

$$\left\| 1 - \frac{\mu_0}{\bar{\mu}} \right\|_{L^\infty} \leq \delta, \quad \text{with } \bar{\mu} = \frac{\mu^m + \mu^M}{2}, \tag{1.8}$$

there exists a unique global solution (u, ρ, μ) of (1.1)–(1.3) with $u(x,0) = u_0(x)$, $\rho(x,0) = \rho_0(x)$ and $\mu(x,0) = \mu_0(x)$ such that

$$u \in C(\mathbb{R}_+; H^{\gamma+\varepsilon}) \cap L^1(\mathbb{R}_+; W^{1,\infty}) \cap L^1(\mathbb{R}_+; C^{1+\gamma}(\overline{D(t)}) \cup C^{1+\gamma}(\mathbb{R}^2 \setminus D(t))),$$
$$\partial D \in C(\mathbb{R}_+; C^{1+\gamma}),$$

where $D(t) = X(D_0, t)$, with X the particle trajectories (1.4) associated to the velocity field and

$$\rho(x,t) = \rho^{\text{in}}(x,t) 1_{D(t)}(x) + \rho^{\text{out}}(x,t) 1_{D(t)^c}(x), \quad \rho(X(y,t),t) = \rho_0(y).$$

Moreover, for any $t \geq 0$,

$$\|\sqrt{\rho} u\|_{L^2}^2(t) + \int_0^t \|\sqrt{\mu} \mathbb{D} u\|_{L^2}^2 \, d\tau \leq \|\sqrt{\rho_0} u_0\|_{L^2}^2,$$
$$t^{1-\gamma-\varepsilon} \|\nabla u\|_{L^2}^2 + \int_0^t \tau^{1-\gamma-\varepsilon} \|\sqrt{\rho} D_t u\|_{L^2}^2 \leq C(\|\sqrt{\rho_0} u_0\|_{L^2}, \mu^m, a^M, \delta) \|u_0\|_{\dot{H}^{\gamma+\varepsilon}}^2,$$
$$t^{2-\gamma-\varepsilon} \|D_t u\|_{L^2}^2 + \int_0^t \tau^{2-\gamma-\varepsilon} \|\nabla D_t u\|_{L^2}^2 \leq C(\|\sqrt{\rho_0} u_0\|_{L^2}, a^m, a^M, \delta) \|u_0\|_{\dot{H}^{\gamma+\varepsilon}}^2,$$

and

$$\int_0^t \|\nabla u\|_{L^\infty} \, d\tau + \int_0^t \|\nabla u\|_{\dot{C}^\gamma(\overline{D(t)}) \cup \dot{C}^\gamma(\mathbb{R}^2 \setminus D(t))} \, d\tau \leq C,$$

with $C = C(a^m, \|a_0\|_{C^\gamma(\bar{D}_0) \cap C^\gamma(\mathbb{R}^2 \setminus D_0)}, \|a^{\text{out}} - a^\infty\|_{L^2}, \delta, \|u_0\|_{L^r}, \|u_0\|_{H^{\gamma+\varepsilon}})$, and $a \equiv \rho, \mu$.

Given that $\mu = \tilde{\mu}(\rho)$, throughout the paper we will use the notation $\mu^{\text{in}} = \tilde{\mu}(\rho^{\text{in}})$, $\mu^{\text{out}} = \tilde{\mu}(\rho^{\text{out}})$, $\mu^\infty = \tilde{\mu}(\rho^\infty)$, and we will have $\mu(X(y,t),t) = \mu_0(y) = \tilde{\mu}(\rho_0(y))$, and therefore

$$\mu(x,t) = \mu^{\text{in}}(x,t) 1_{D(t)}(x) + \mu^{\text{out}}(x,t) 1_{D(t)^c}(x).$$

The first part of the proof consists in getting a priori estimates which are sharp in Sobolev regularity for the initial velocity to propagate $C^{1+\gamma}$ regularity. This is achieved by introducing time weights and interpolation. Then a key step in the proof will be to obtain the L^1-in-time Lipschitz-in-space estimate for the velocity. This is difficult as the gradient of the velocity is given implicitly by a higher-order Riesz transform applied to a discontinuous function on the moving interface. That is, this function depends itself on the gradient of the velocity multiplied by the viscosity jump scalar (2.34). We will overcome this difficulty by propagating further regularity on each domain separately. As part of the argument, we will use the following new quantitative estimate. Consider higher-order Riesz transform operators of even order $2l$, $l \geq 1$, given by

$$R(f)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy, \tag{1.9}$$

where

$$K(x) = \frac{P_{2l}(x)}{|x|^{n+2l}}, \tag{1.10}$$

and $P_{2l}(x)$ is a homogeneous polynomial of degree $2l$ in $\mathbb{R}^2$. Then we have the following result.

Theorem 1.2 ([21]). *Assume $D \subset \mathbb{R}^2$ is a bounded domain of class $C^{1+\gamma}$, $0 < \gamma < 1$. Then the Calderón–Zygmund operator (1.9) with kernel (1.10) applied to the characteristic function of D, 1_D, defines a piecewise C^γ function,*

$$R(1_D) \in C^\gamma(\bar{D}) \cup C^\gamma(\mathbb{R}^2 \setminus D).$$

Moreover, it satisfies the bound

$$\|R(1_D)\|_{\dot{C}^\gamma(\bar{D}) \cup \dot{C}^\gamma(\mathbb{R}^2 \setminus D)} \leq C \mathcal{P}(\|D\|_* + \|D\|_{\text{Lip}}) \|D\|_{\dot{C}^{1+\gamma}}.$$

Above, $\|\cdot\|_*$ measures the arc-chord condition of the boundary of the domain, $\|\cdot\|_{\text{Lip}}$ is the Lipschitz norm, $\|\cdot\|_{\dot{C}^{1+\gamma}}$ is the homogeneous Hölder norm and $\mathcal{P}$ is a polynomial function. If we denote $y(\alpha)$, $\alpha \in [0, 2\pi) = \mathbb{T}$, the parametrization of the boundary ∂D, these quantities are defined as follows

$$\begin{aligned} \|D\|_* &:= \sup_{\alpha \neq \beta} \frac{|\alpha - \beta|}{|y(\alpha) - y(\beta)|}, \\ \|D\|_{\text{Lip}} &:= \sup_{\alpha \neq \beta} \frac{|y(\alpha) - y(\beta)|}{|\alpha - \beta|}, \\ \|D\|_{\dot{C}^{1+\sigma}} &:= \sup_{\alpha \neq \beta} \frac{|y'(\alpha) - y'(\beta)|}{|\alpha - \beta|^\sigma}. \end{aligned}$$

Remark 1.3. By the boundary condition (1.7), one cannot expect to obtain global-in-space further regularity than $\nabla u \in L^\infty(\mathbb{R}^2)$. Indeed, if we denote by τ and n the tangent and normal vectors to the boundary, we have

$$[\![\mu \mathbb{D}_{ij} u - P\delta_{ij}]\!] n_j = 0 \Rightarrow \begin{cases} [\![\mu n \cdot \mathbb{D}u \cdot n]\!] = [\![P]\!], \\ [\![\mu \tau \cdot \mathbb{D}u \cdot n]\!] = 0; \end{cases}$$

thus, if ∇u were continuous, then we would obtain $[\![\mu]\!] = 0$.

Outline of the paper. The rest of the paper is structured as follows. The proof of the main Theorem 1.1 is divided into existence and uniqueness. For the existence, we proceed to obtain the necessary a priori estimates. We separate the process into six steps, bootstrapping the regularity obtained from one to the next. Steps 1–3 consist of energy estimates with time weights, which allow us to obtain high regularity for the velocity despite the low regularity of the density, viscosity and initial velocity. Step 4 bounds the crucial L^1-in-time Lipschitz regularity of the velocity in terms of the higher Hölder regularity on each side, which is studied in Step 5. The previous steps are combined with quantitative estimates of even singular integral operators acting on $C^{1+\gamma}$ domains in Step 6. This concludes the proof of existence. Next, the uniqueness of solutions is shown. The proof is done in Lagrangian variables, due to the discontinuity jumps of the density and viscosity across the fluid interface.

2. Proof of Theorem 1.1

2.1. Existence

The proof of existence follows a standard mollifier and compactness argument (see e.g. [13, 33]). Once the initial data are smoothed out, we show the a priori estimates for the corresponding unique smooth solution. The L^1-in-time Lipschitz-in-space estimate for the velocity implies that the solution exists globally in time. The fact that all these estimates will be uniform in the mollifying parameter gives the necessary compactness to pass to the limit. We proceed to obtain the a priori estimates.

Step 1: $\sqrt{\rho} u \in L^\infty(0, T; L^2)$, $\sqrt{\mu}\mathbb{D}u \in L^2(0, T; L^2)$.

We first obtain the L^2 energy balance

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \rho |u|^2\, dx &= -\int_{\mathbb{R}^2} \mu \partial_j u_i (\partial_j u_i + \partial_i u_j)\, dx \\ &= -2\int_{\mathbb{R}^2} \mu\Big((\partial_1 u_1)^2 + (\partial_2 u_2)^2 + \frac{1}{2}(\partial_1 u_2 + \partial_2 u_1)^2\Big)\, dx \\ &= -\frac{1}{2}\|\sqrt{\mu}\mathbb{D}u\|_{L^2}, \end{aligned}$$

which after integration in time reads

$$\|\sqrt{\rho} u\|_{L^2}^2(t) + \int_0^t \|\sqrt{\mu}\mathbb{D}u\|_{L^2}^2(\tau)\, d\tau \le \|\sqrt{\rho_0} u_0\|_{L^2}^2. \tag{2.1}$$

Step 2: $t^{\frac{1-\gamma-\varepsilon}{2}}\nabla u \in L^\infty(0,T;L^2), t^{\frac{1-\gamma-\varepsilon}{2}} D_t u \in L^2(0,T;L^2)$.

To obtain the result with low-regularity initial data, we use an interpolation argument and time-weighted energy estimates [34]. Consider the linearized problem

$$\begin{aligned} \rho(v_t + u\cdot\nabla v) &= \nabla\cdot(\mu\mathbb{D}v - \mathbb{I}_2 P),\\ \rho_t &= -u\cdot\nabla\rho. \end{aligned}$$

It holds that

$$\|\sqrt{\rho}v\|^2_{L^2}(t) + \int_0^t \|\sqrt{\mu}\mathbb{D}v\|^2_{L^2}(\tau)\,d\tau \le \|\sqrt{\rho_0}v_0\|^2_{L^2}. \tag{2.2}$$

Next we take the inner product of (INS) with $D_t v := v_t + u\cdot\nabla v$ and then integrate by parts to obtain

$$\int_{\mathbb{R}^2}\rho|D_t v|^2\,dx = -\int_{\mathbb{R}^2}\partial_j D_t v_i(\mu\mathbb{D}_{ij}v - P\delta_{ij})\,dx.$$

By the commutator

$$[D_t,\partial_j]f = -\partial_j u\cdot\nabla f$$

and the incompressibility condition, it follows that

$$\begin{aligned} &\int_{\mathbb{R}^2}\rho|D_t v|^2\,dx\\ &\quad= -\int_{\mathbb{R}^2} D_t\partial_j v_i(\mu\mathbb{D}_{ij}v - P\delta_{ij})\,dx - \int_{\mathbb{R}^2}\partial_j u_k\partial_k v_i(\mu\mathbb{D}_{ij}v - P\delta_{ij})\,dx\\ &\quad= -\int_{\mathbb{R}^2} D_t\partial_j v_i\,\mu\mathbb{D}_{ij}v\,dx - \int_{\mathbb{R}^2}\partial_j u_k\partial_k v_i(\mu\mathbb{D}_{ij}v - P\delta_{ij})\,dx. \end{aligned}$$

Noticing that $D_t\mu = 0$, we introduce a time weight t followed by integration in time,

$$\begin{aligned} &\frac{t}{2}\|\sqrt{\mu}\mathbb{D}v\|^2_{L^2}(t) + \int_0^t \tau\|\sqrt{\rho}D_t v\|^2_{L^2}(\tau)\,d\tau\\ &\qquad= \frac12\int_0^t\|\sqrt{\mu}\mathbb{D}v\|^2_{L^2}\,d\tau - \int_0^t\tau\int_{\mathbb{R}^2}\mu\partial_j u_k\partial_k v_i\mathbb{D}_{ij}v\,dx\,d\tau\\ &\qquad\quad+\int_0^t\tau\int_{\mathbb{R}^2}\partial_i u_k\partial_k v_i P\,dx\,d\tau. \end{aligned}\tag{2.3}$$

We take the divergence of (INS) to obtain the following expression for the pressure:

$$P = (-\Delta)^{-1}\nabla\cdot(\rho D_t v) - \nabla\cdot\nabla\cdot(-\Delta)^{-1}(\mu\mathbb{D}v). \tag{2.4}$$

Substituting (2.4) in (2.3) we have

$$\begin{aligned} &\frac{t}{2}\|\sqrt{\mu}\mathbb{D}v\|^2_{L^2}(t) + \int_0^t\tau\|\sqrt{\rho}D_t v\|^2_{L^2}(\tau)\,d\tau\\ &\qquad= \frac12\int_0^t\|\sqrt{\mu}\mathbb{D}v\|^2_{L^2}\,d\tau + I_1 + I_2 + I_3, \end{aligned}\tag{2.5}$$

where

$$I_1 = -\int_0^t \tau \int_{\mathbb{R}^2} \mu \partial_j u_k \partial_k v_i \mathbb{D}_{ij} v \, dx \, d\tau,$$
$$I_2 = \int_0^t \tau \int_{\mathbb{R}^2} \partial_i u_k \partial_k v_i (-\Delta)^{-1} \nabla \cdot (\rho D_t v) \, dx \, d\tau,$$
$$I_3 = -\int_0^t \tau \int_{\mathbb{R}^2} \partial_i u_k \partial_k v_i \nabla \cdot \nabla \cdot (-\Delta)^{-1} (\mu \mathbb{D} v) \, dx \, d\tau.$$

We need estimates for the gradient of v in terms of $D_t v$. Notice that the following identity holds:

$$\nabla \cdot (\mu \mathbb{D} v) = \bar{\mu} \Delta v + \nabla \cdot (\mu \mathbb{D} v - \bar{\mu} \mathbb{D} v),$$

where $\bar{\mu}$ can be taken as $\bar{\mu} = (\mu^M + \mu^m)/2$. Then

$$\nabla v = \frac{1}{\bar{\mu}} \nabla \Delta^{-1} \mathbb{P} \nabla \cdot (\mu \mathbb{D} v) - \nabla \Delta^{-1} \mathbb{P} \nabla \cdot \Big(\Big(\frac{\mu}{\bar{\mu}} - 1 \Big) \mathbb{D} v \Big),$$

where $\mathbb{P}$ denotes the Leray projector,

$$\mathbb{P} f = f - \nabla \Delta^{-1} \nabla \cdot f.$$

Therefore, given condition (1.8), the boundedness of singular integrals in L^q, $1 < q < \infty$, gives

$$\|\nabla v\|_{L^p} \le c(\delta) \|\nabla \Delta^{-1} \mathbb{P} \nabla \cdot (\mu \mathbb{D} v)\|_{L^p}, \quad 2 \le p \le \max\Big\{ \frac{2}{1-\gamma-\varepsilon}, \frac{2}{\gamma-\varepsilon} \Big\}. \tag{2.6}$$

Applying the Leray projector to (INS) we find the relationship between $\mathbb{D} u$ and $D_t u$:

$$\mathbb{P}(\rho D_t v) = \mathbb{P} \nabla \cdot (\mu \mathbb{D} v). \tag{2.7}$$

Recalling the following Gagliardo–Nirenberg inequality in $\mathbb{R}^2$,

$$\|f\|_{L^p} \le c \|f\|_{L^2}^{\frac{2}{p}} \|\nabla f\|_{L^2}^{1-\frac{2}{p}}, \tag{2.8}$$

followed by (2.7), one can find from (2.6) that

$$\|\nabla v\|_{L^p} \le c(\delta) \|\mu \mathbb{D} v\|_{L^2}^{\frac{2}{p}} \|\nabla \nabla \Delta^{-1} \mathbb{P}(\rho D_t v)\|_{L^2}^{1-\frac{2}{p}} \le c(\delta) \|\mu \mathbb{D} v\|_{L^2}^{\frac{2}{p}} \|\rho D_t v\|_{L^2}^{1-\frac{2}{p}}. \tag{2.9}$$

In particular,

$$\|\nabla v\|_{L^4} \le c(\delta) \|\mu \mathbb{D} v\|_{L^2}^{1/2} \|\rho D_t v\|_{L^2}^{1/2}. \tag{2.10}$$

Thus, the terms I_1 and I_3 are readily bounded as

$$I_1 + I_3 \le c \int_0^t \tau \|\nabla u\|_{L^2} \|\nabla v\|_{L^4} \|\mathbb{D} v\|_{L^4} \, d\tau$$
$$\le \frac{1}{4} \int_0^t \tau \|\sqrt{\rho} D_t v\|_{L^2}^2 \, d\tau + c(\delta) \int_0^t \tau \|\nabla u\|_{L^2}^2 \|\sqrt{\mu} \mathbb{D} v\|_{L^2}^2 \, d\tau.$$

Then, denoting by $\mathcal{H}^1$ the Hardy space, we get for I_2 (2.5) the estimate

$$I_2 \leq c \int_0^t \tau \|\partial_i u_k \partial_k v_i\|_{\mathcal{H}^1} \|(-\Delta)^{-1}\nabla \cdot (\rho D_t v)\|_{\text{BMO}}\, d\tau.$$

Since for each i the term $\partial_i u \cdot \nabla v_i$ is the product of a divergence-free function and a curl-free one, we can apply the div-curl lemma to get

$$\|\partial_i u_k \partial_k v_i\|_{\mathcal{H}^1} \leq C \|\nabla u\|_{L^2} \|\nabla v\|_{L^2},$$

which together with the embedding $\dot{H}^1 \hookrightarrow \text{BMO}$ gives

$$\begin{aligned} I_2 &\leq c \int_0^t \tau \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \|\rho D_t v\|_{L^2}\, d\tau \\ &\leq \frac{1}{4} \int_0^t \tau \|\sqrt{\rho} D_t v\|_{L^2}^2\, d\tau + c \int_0^t \tau \|\nabla u\|_{L^2}^2 \|\mathbb{D} v\|_{L^2}^2\, d\tau. \end{aligned}$$

Therefore, we have that (2.5) becomes

$$\begin{aligned} &t \|\sqrt{\mu}\mathbb{D} v\|_{L^2}^2(t) + \int_0^t \tau \|\sqrt{\rho} D_t v\|_{L^2}^2(\tau)\, d\tau \\ &\qquad \leq \|\sqrt{\rho_0} v_0\|_{L^2}^2 + c(\mu^m, \delta) \int_0^t \tau \|\sqrt{\mu}\mathbb{D} v\|_{L^2}^2 \|\sqrt{\mu}\mathbb{D} u\|_{L^2}^2\, d\tau. \end{aligned}$$

Grönwall's lemma followed by (2.1) yields the balance

$$t \|\sqrt{\mu}\mathbb{D} v\|_{L^2}^2 + \int_0^t \tau \|\sqrt{\rho} D_t v\|_{L^2}^2(t)\, d\tau \leq c(\mu^m, \delta, \|\sqrt{\rho_0} u_0\|_{L^2}) \|\sqrt{\rho_0} v_0\|_{L^2}^2. \tag{2.11}$$

We can repeat the steps above without the time weight to obtain

$$\|\sqrt{\mu}\mathbb{D} v\|_{L^2}^2 + \int_0^t \|\sqrt{\rho} D_t v\|_{L^2}^2(\tau)\, d\tau \leq c(\mu^m, \delta, \|\sqrt{\rho_0} u_0\|_{L^2}) \|\sqrt{\mu_0}\mathbb{D} v_0\|_{L^2}^2. \tag{2.12}$$

Thus, the linear operator $T v_0 = \nabla v$ satisfies the bounds $\|T v_0\|_{L^2} \leq c \|\nabla v_0\|_{L^2}$ and $\|T v_0\|_{L^2} \leq c t^{-\frac{1}{2}} \|v_0\|_{L^2}$, and hence we conclude that

$$\|\nabla u\|_{L^2} \leq c(\mu^m, \delta, \|\rho_0 u_0\|_{L^2}) t^{\frac{-1+\gamma+\varepsilon}{2}} \|v_0\|_{\dot{H}^{\gamma+\varepsilon}}.$$

Using Stein's interpolation theorem similarly to [34] for the terms with time integrals, we close the balance in $\dot{H}^{\gamma+\varepsilon}$,

$$\begin{aligned} &t^{1-\gamma-\varepsilon} \|\nabla u\|_{L^2}^2 + \int_0^t \tau^{1-\gamma-\varepsilon} \|\sqrt{\rho} D_t u\|_{L^2}^2(\tau)\, d\tau \\ &\qquad \leq c(\mu^m, \delta, \|\sqrt{\rho_0} u_0\|_{L^2}) \|u_0\|_{\dot{H}^{\gamma+\varepsilon}}^2. \end{aligned} \tag{2.13}$$

Notice that we can combine (2.11) and (2.13) to obtain

$$\max\{t^{1-\gamma-\varepsilon}, t\} \|\nabla u\|_{L^2}^2 + \int_0^t \max\{\tau^{1-\gamma-\varepsilon}, \tau\} \|\sqrt{\rho} D_t u\|_{L^2}^2(\tau)\, d\tau \leq C. \tag{2.14}$$

We will need further time decay. We use the following theorem:

Theorem 2.1 ([24]). *For $1 < r < 2$, and $0 < \alpha < 1$, let $u_0 \in L^r \cap H^\alpha$, $a_0 - a^\infty \in L^2$ and $0 < a^m < a_0 \in L^\infty$ with $a \equiv \rho, \mu$. Then, under the assumption of small viscosity contrast, the inhomogeneous Navier–Stokes with initial data (ρ_0, μ_0, u_0) has a global weak solution and there exists a constant C_α which depends on $\|\rho_0 - \rho^\infty\|_{L^2}$, $\|u_0\|_{L^r}$ and $\|u_0\|_{H^\alpha}$, such that there hold*

$$\|u(t)\|_{L^2}^2 \le C_\alpha (t+e)^{-\frac{2}{r}+1},$$
$$\|\nabla u(t)\|_{L^2}^2 \le C_\alpha (t+e)^{-\frac{2}{r}+\varepsilon},$$
$$\int_0^\infty t^{1-\kappa}(t+e)^{\kappa+\frac{2}{r}-1-\varepsilon}\|u_t\|_{L^2}^2 + \|\mathbb{P}\,\mathrm{div}(\mu\mathbb{D}u)\|_{L^2}^2 + \|(\mathbb{I}_2 - \mathbb{P})\,\mathrm{div}(\mu\mathbb{D}u) - \nabla P\|_{L^2}^2\,dt \le C_\alpha,$$

with any $0 < \varepsilon < 1$ and $0 < \kappa < \alpha$.

Notice that by (2.9) and Young's inequality we have

$$\|D_t u\|_{L^2}^2 \le c(\|u_t\|_{L^2}^2 + \|u\|_{L^2}^2\|\nabla u\|_{L^2}^4).$$

Hence by (2.13), the estimates in Theorem 2.1 above and (2.1) it is possible to get

$$\int_0^t \max\{\tau^{1-\gamma-\varepsilon}, \tau^{\frac{2}{r}-\varepsilon}\}\|D_t u\|_{L^2}^2(\tau)\,d\tau \le C,$$

with $C = C(a^m, a^M, \|a_0 - a^\infty\|_{L^2}, \delta, \|u_0\|_{L^r}, \|u_0\|_{H^{\gamma+\varepsilon}})$. Thus, since $2/r - \varepsilon > 1$, we can improve (2.14) for large times,

$$\max\{t^{1-\gamma-\varepsilon}, t^{\frac{2}{r}-\varepsilon}\}\|\nabla u\|_{L^2}^2 + \int_0^t \max\{\tau^{1-\gamma-\varepsilon}, \tau^{\frac{2}{r}-\varepsilon}\}\|\sqrt{\rho}D_t u\|_{L^2}^2(\tau)\,d\tau \le C. \quad (2.15)$$

Step 3: $t^{1-\frac{\gamma+\varepsilon}{2}}D_t u \in L^\infty(0,T;L^2)$, $t^{1-\frac{\gamma+\varepsilon}{2}}\nabla D_t u \in L^2(0,T;L^2)$, $u \in C(\mathbb{R}_+; H^{\gamma+\varepsilon})$.

We proceed to obtain higher-regularity estimates for $D_t u$. We take D_t in (INS) and then the inner product with $D_t u$ to obtain

$$\int_{\mathbb{R}^2} D_t u \cdot \rho D_t^2 u\,dx = \int_{\mathbb{R}^2} D_t u \cdot D_t \nabla\cdot(\mu\mathbb{D}u)\,dx - \int_{\mathbb{R}^2} D_t u \cdot D_t\nabla P\,dx,$$

which after multiplication by the time weight $t^{2-\gamma-\varepsilon}$ gives

$$\frac{1}{2}\frac{d}{dt}\left(t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}\rho|D_t u|^2\,dx\right) = \frac{2-\gamma-\varepsilon}{2}t^{1-\gamma-\varepsilon}\|\sqrt{\rho}D_t u\|_{L^2}^2 + I_4 + I_5 + I_6, \quad (2.16)$$

with

$$I_4 = t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2} D_t u_i \partial_j D_t(\mu\mathbb{D}_{ij}u)\,dx,$$
$$I_5 = -t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2} D_t u_i \partial_j u_k \partial_k(\mu\mathbb{D}_{ij}u)\,dx,$$
$$I_6 = -t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2} D_t u \cdot D_t\nabla P\,dx.$$

Integration by parts in I_4 provides that

$$\begin{aligned} I_4 &= -t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2} \partial_j D_t u_i \mu D_t \mathbb{D}_{ij} u \, dx \\ &= -t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2} \partial_j D_t u_i \mu (\partial_j D_t u_i + \partial_i D_t u_j) \, dx \\ &\quad + t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2} \partial_j D_t u_i \mu (\partial_j u_k \partial_k u_i + \partial_i u_k \partial_k u_j) \, dx \\ &\leq -\frac{t^{2-\gamma-\varepsilon}}{2}\|\sqrt{\mu}\mathbb{D} D_t u\|_{L^2}^2 + 2t^{2-\gamma-\varepsilon}\|\sqrt{\mu}\mathbb{D} D_t u\|_{L^2}\|\nabla u\|_{L^4}^2, \end{aligned}$$

which by (2.10) (taking v equal to u) gives

$$I_4 \leq -\frac{t^{2-\gamma-\varepsilon}}{4}\|\sqrt{\mu}\mathbb{D} D_t u\|_{L^2}^2 + c(\delta)t^{2-\gamma-\varepsilon}\|\mu\mathbb{D}u\|_{L^2}^2\|\rho D_t u\|_{L^2}^2.$$

Integration by parts in term I_5 yields

$$I_5 \leq 2\mu^M t^{2-\gamma-\varepsilon}\|\nabla D_t u\|_{L^2}\|\nabla u\|_{L^4}^2.$$

The identity

$$\partial_k f_i = \partial_k \Delta^{-1}\partial_j \mathbb{D}_{ij} f - \nabla\cdot\Delta^{-1}\partial_k\partial_i f$$

and the fact that

$$\nabla\cdot D_t u = \nabla u\cdot\nabla u,$$

imply that

$$\|\nabla D_t u\|_{L^2}^2 = \|\mathbb{D} D_t u\|_{L^2}^2 + \|\nabla u\cdot\nabla u\|_{L^2}^2.$$

Therefore, applying Young's inequality we obtain

$$I_5 \leq \frac{t^{2-\gamma-\varepsilon}}{8}\|\sqrt{\mu}\mathbb{D} D_t u\|_{L^2}^2 + c(\mu^m)t^{2-\gamma-\varepsilon}\|\nabla u\|_{L^4}^4.$$

so, using (2.10) again, we have

$$I_4 + I_5 \leq -\frac{t^{2-\gamma-\varepsilon}}{8}\|\sqrt{\mu}\mathbb{D} D_t u\|_{L^2}^2 + c(\mu^m,\delta)t^{2-\gamma-\varepsilon}\|\mu\mathbb{D}u\|_{L^2}^2\|\rho D_t u\|_{L^2}^2. \tag{2.17}$$

For the I_6 term, we first split it as

$$\begin{aligned} I_6 &= -t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2} D_t u\cdot\nabla D_t P\, dx - t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2} D_t u_i \partial_i u_k \partial_k P\, dx \\ &= J_1 + J_2. \end{aligned} \tag{2.18}$$

We proceed with J_2 first. We substitute expression (2.4) for the pressure and integrate by parts to obtain

$$\begin{aligned} J_2 &= t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2} \partial_k D_t u_i \partial_i u_k (-\Delta)^{-1}\nabla\cdot\nabla\cdot(\mu\mathbb{D}u)\, dx \\ &\quad - t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2} D_t u_i \partial_i u_k \partial_k (-\Delta)^{-1}\nabla\cdot(\rho D_t u)\, dx \\ &= K_1 + K_2. \end{aligned}$$

The first term K_1 is bounded as the previous term I_5,

$$K_1 \le \frac{t^{2-\gamma-\varepsilon}}{64}\|\sqrt{\mu}\mathbb{D}D_t u\|_{L^2}^2 + c(\mu^m,\delta)t^{2-\gamma-\varepsilon}\|\mu\mathbb{D}u\|_{L^2}^2\|\rho D_t u\|_{L^2}^2.$$

For the second one, we integrate by parts twice to get

$$\begin{aligned} K_2 = t^{2-\gamma-\varepsilon}&\int_{\mathbb{R}^2}\partial_i u_l\partial_l u_i u_k\partial_k(-\Delta)^{-1}\nabla\cdot(\rho D_t u)\,dx\\ &- t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}\partial_k D_t u_i u_k\partial_i(-\Delta)^{-1}\nabla\cdot(\rho D_t u)\,dx. \end{aligned}$$

Then,

$$K_2 \le ct^{2-\gamma-\varepsilon}\big(\|\nabla u\|_{L^4}^2\|u\|_{L^4}\|\rho D_t u\|_{L^4} + \|\nabla D_t u\|_{L^2}\|u\|_{L^4}\|\rho D_t u\|_{L^4}\big),$$

so, using (2.8) and (2.10) repeatedly, we can bound it by

$$\begin{aligned} K_2 &\le c(\delta)t^{2-\gamma-\varepsilon}\big(\|D_t u\|_{L^2}^{\frac{3}{2}}\|\nabla D_t u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|u\|_{L^2}^{\frac{1}{2}}\\ &\qquad + \|\nabla D_t u\|_{L^2}^{\frac{3}{2}}\|D_t u\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\big)\\ &\le \frac{t^{2-\gamma-\varepsilon}}{64}\|\sqrt{\mu}\mathbb{D}D_t u\|_{L^2}^2 + c(\mu^m,\delta)t^{2-\gamma-\varepsilon}\|D_t u\|_{L^2}^2\|\nabla u\|_{L^2}^2\big(\|u\|_{L^2}^{\frac{2}{3}} + \|u\|_{L^2}^2\big), \end{aligned}$$

thus

$$J_2 \le \frac{t^{2-\gamma-\varepsilon}}{32}\|\sqrt{\mu}\mathbb{D}D_t u\|_{L^2}^2 + c(\mu^m,\delta)t^{2-\gamma-\varepsilon}\|D_t u\|_{L^2}^2\|\nabla u\|_{L^2}^2. \tag{2.19}$$

We proceed with J_1 (2.18). After integration by parts, the term J_1 can be written as

$$\begin{aligned} J_1 &= t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}\partial_i u_k\partial_k u_i D_t P\,dx\\ &= \frac{d}{dt}\Big(t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}\partial_i u_k\partial_k u_i P\,dx\Big) - t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}D_t(\partial_i u_k\partial_k u_i)P\,dx\\ &\quad - (2-\gamma-\varepsilon)t^{1-\gamma-\varepsilon}\int_{\mathbb{R}^2}\partial_i u_k\partial_k u_i P\,dx\\ &= K_3 + K_4 + K_5. \end{aligned}\tag{2.20}$$

Commuting the time derivative, the term K_4 is given by

$$\begin{aligned} K_4 &= -2t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}\partial_i D_t u_k\partial_k u_i P\,dx + 2t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}\partial_i u_j\partial_j u_k\partial_k u_i P\,dx\\ &= L_1 + L_2. \end{aligned}\tag{2.21}$$

The term L_1 is bounded as J_2 (2.18),

$$L_1 \le \frac{t^{2-\gamma-\varepsilon}}{64}\|\sqrt{\mu}\mathbb{D}D_t u\|_{L^2}^2 + c(\mu^m,\delta)t^{2-\gamma-\varepsilon}\|D_t u\|_{L^2}^2\|\nabla u\|_{L^2}^2.$$

Next we substitute the pressure (2.4) in L_2 to obtain

$$L_2 \leq ct^{2-\gamma-\varepsilon}\|\nabla u\|_{L^4}^4 + 2t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2} \partial_i u_j \partial_j u_k \partial_k u_i (-\Delta)^{-1}\nabla\cdot(\rho D_t u)\, dx.$$

Then we note that by integrating by parts twice, the second term can be written as

$$\begin{aligned} t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2} &\partial_i u_j \partial_j u_k \partial_k u_i (-\Delta)^{-1}\nabla\cdot(\rho D_t u)\, dx \\ &= \frac{t^{2-\gamma-\varepsilon}}{2}\int_{\mathbb{R}^2} \partial_k u_j \partial_j u_k u_i \partial_i (-\Delta)^{-1}\nabla\cdot(\rho D_t u)\, dx, \end{aligned}$$

and therefore it is bounded as the first term in K_2 above. We conclude that

$$L_2 \leq \frac{t^{2-\gamma-\varepsilon}}{64}\|\sqrt{\mu}\mathbb{D} D_t u\|_{L^2}^2 + c(\mu^m,\delta)t^{2-\gamma-\varepsilon}\|D_t u\|_{L^2}^2\|\nabla u\|_{L^2}^2(1+\|u\|_{L^2}^{\frac{2}{3}}),$$

and thus

$$K_4 \leq \frac{t^{2-\gamma-\varepsilon}}{32}\|\sqrt{\mu}\mathbb{D} D_t u\|_{L^2}^2 + c(\mu^m,\delta)t^{2-\gamma-\varepsilon}\|D_t u\|_{L^2}^2\|\nabla u\|_{L^2}^2. \tag{2.22}$$

Substitution of expression (2.4) for the pressure in K_5 (2.20) and integration by parts gives

$$\begin{aligned} K_5 = (2-\gamma-\varepsilon)t^{1-\gamma-\varepsilon}&\int_{\mathbb{R}^2} \partial_i u_k \partial_k u_i \nabla\cdot\nabla\cdot(-\Delta)^{-1}(\mu\mathbb{D}v)\, dx \\ &+ (2-\gamma-\varepsilon)t^{1-\gamma-\varepsilon}\int_{\mathbb{R}^2} \partial_i u_k u_i \partial_k (-\Delta)^{-1}\nabla\cdot(\rho D_t v)\, dx, \end{aligned}$$

so using (2.10), (2.8),

$$\begin{aligned} K_5 &\leq c(\delta)t^{1-\gamma-\varepsilon}(\|\nabla u\|_{L^2}^2\|\rho D_t u\|_{L^2} + \|\rho D_t u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2}\|u\|_{L^2}^{\frac{1}{2}}) \\ &\leq c(\delta)t^{1-\gamma-\varepsilon}(\|\sqrt{\rho}D_t u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4(1+\|u\|_{L^2}^2)) \\ &\leq c(\delta)t^{1-\gamma-\varepsilon}(\|\sqrt{\rho}D_t u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4). \end{aligned} \tag{2.23}$$

Going back to (2.20), bounds (2.22) and (2.23) provide

$$\begin{aligned} J_1 \leq \frac{d}{dt}\Big(t^{2-\gamma-\varepsilon}&\int_{\mathbb{R}^2} \partial_i u_k \partial_k u_i P\, dx\Big) + \frac{t^{2-\gamma-\varepsilon}}{32}\|\sqrt{\mu}\mathbb{D} D_t u\|_{L^2}^2 \\ &+ c(\mu^m,\delta)t^{2-\gamma-\varepsilon}\|D_t u\|_{L^2}^2\|\nabla u\|_{L^2}^2 + c(\delta)t^{1-\gamma-\varepsilon}(\|D_t u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4). \end{aligned}$$

Recalling the bound for J_2 (2.19), we obtain, for I_6 (2.18),

$$\begin{aligned} I_6 \leq \frac{d}{dt}\Big(t^{2-\gamma-\varepsilon}&\int_{\mathbb{R}^2} \partial_i u_k \partial_k u_i P\, dx\Big) + \frac{t^{2-\gamma-\varepsilon}}{16}\|\sqrt{\mu}\mathbb{D} D_t u\|_{L^2}^2 \\ &+ c(\mu^m,\delta)t^{2-\gamma-\varepsilon}\|D_t u\|_{L^2}^2\|\nabla u\|_{L^2}^2 + c(\delta)t^{1-\gamma-\varepsilon}(\|D_t u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4). \end{aligned} \tag{2.24}$$

Finally, we go back to the balance (2.16) with (2.17) and (2.24):

$$
\begin{aligned}
&\frac{d}{dt}\Big(t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}\rho|D_t u|^2\,dx\Big)+\frac{t^{2-\gamma-\varepsilon}}{8}\|\sqrt{\mu}\mathbb{D}D_t u\|_{L^2}^2\\
&\qquad\le c(\delta)t^{1-\gamma-\varepsilon}(\|D_t u\|_{L^2}^2+\|\nabla u\|_{L^2}^4)+c(\mu^m,\delta)t^{2-\gamma-\varepsilon}\|\nabla u\|_{L^2}^2\|D_t u\|_{L^2}^2\\
&\qquad\quad+\frac{d}{dt}\Big(t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}\partial_i u_k\partial_k u_i P\,dx\Big),
\end{aligned}
$$

and integrate in time to obtain

$$
\begin{aligned}
&t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}\rho|D_t u|^2\,dx+\frac{1}{8}\int_0^t\tau^{2-\gamma-\varepsilon}\|\sqrt{\mu}\mathbb{D}D_t u\|_{L^2}^2\,d\tau\\
&\quad\le c(\rho^m,\mu^m,\delta,\|\sqrt{\rho_0}u_0\|_{L^2})\|u_0\|_{\dot H^{\gamma+\varepsilon}}^2+c(\mu^m,\delta)\int_0^t\tau^{2-\gamma-\varepsilon}\|\nabla u\|_{L^2}^2\|D_t u\|_{L^2}^2\,d\tau\\
&\qquad+t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}\partial_i u_k\partial_k u_i P\,dx,
\end{aligned}
$$

where we have used the previous energy estimate (2.13). Notice that the last term on the right-hand side is like K_5 (2.20) but with an additional factor of t on the time weight. Therefore, from (2.23) and (2.14), we have the bound

$$
\begin{aligned}
&t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}\partial_i u_k\partial_k u_i P\,dx\\
&\quad\le\frac{t^{2-\gamma-\varepsilon}}{2}\|\sqrt{\rho}D_t u\|_{L^2}^2+c(\delta)t^{2-\gamma-\varepsilon}\|\nabla u\|_{L^2}^4\\
&\quad\le\frac{t^{2-\gamma-\varepsilon}}{2}\|\sqrt{\rho}D_t u\|_{L^2}^2+c(\mu^m,\delta,\|\sqrt{\rho_0}u_0\|_{L^2})\|u_0\|_{\dot H^{\gamma+\varepsilon}}^2\frac{t^{2-\gamma-\varepsilon}}{(\max\{t^{1-\gamma-\varepsilon},t\})^2},
\end{aligned}
$$

and thus

$$
\begin{aligned}
&t^{2-\gamma-\varepsilon}\int_{\mathbb{R}^2}\rho|D_t u|^2\,dx+\frac{1}{4}\int_0^t\tau^{2-\gamma-\varepsilon}\|\sqrt{\mu}\mathbb{D}D_t u\|_{L^2}^2\,d\tau\\
&\quad\le c(\mu^m,\delta)\int_0^t\tau^{2-\gamma-\varepsilon}\|\nabla u\|_{L^2}^2\|D_t u\|_{L^2}^2\,d\tau\\
&\qquad+c(\rho^m,\mu^m,\delta,\|\sqrt{\rho_0}u_0\|_{L^2})\|u_0\|_{\dot H^{\gamma+\varepsilon}}^2(1+\min\{t^{\gamma+\varepsilon},t^{-\gamma-\varepsilon}\}).
\end{aligned}
$$

Grönwall's lemma then allows us to conclude that

$$
\begin{aligned}
&t^{2-\gamma-\varepsilon}\|D_t u\|_{L^2}^2+\int_0^t\tau^{2-\gamma-\varepsilon}\|\nabla D_t u\|_{L^2}^2\,d\tau\\
&\quad\le c(\rho^m,\mu^m,\delta,\|\sqrt{\rho_0}u_0\|_{L^2})\|u_0\|_{\dot H^{\gamma+\varepsilon}}^2.
\end{aligned}
\tag{2.25}
$$

Repeating the steps but with the weight $t^{1+\frac{2}{r}-\varepsilon}$ in (2.16) and using (2.15) instead of (2.14), it is analogous to check that the following balance also holds:

$$
t^{1+\frac{2}{r}-\varepsilon}\|D_t u\|_{L^2}^2+\int_0^t\tau^{1+\frac{2}{r}-\varepsilon}\|\nabla D_t u\|_{L^2}^2\,d\tau\le C,
\tag{2.26}
$$

and hence

$$\max\{t^{2-\gamma-\varepsilon}, t^{1+\frac{2}{r}-\varepsilon}\}\|D_t u\|_{L^2}^2 + \int_0^t \max\{\tau^{2-\gamma-\varepsilon}, \tau^{1+\frac{2}{r}-\varepsilon}\}\|\nabla D_t u\|_{L^2}^2\, d\tau \leq C, \quad (2.27)$$

where $C = C(a^m, a^M, \|a_0 - a^\infty\|_{L^2}, \delta, \|u_0\|_{L^r}, \|u_0\|_{H^{\gamma+\varepsilon}})$, $a \equiv \rho, \mu$. Next, to show that $u \in C(\mathbb{R}_+; H^{\gamma+\varepsilon})$, we write (INS) as a forced heat equation,

$$u_t - \Delta u = -\mathbb{P}(\rho u \cdot \nabla u) + \mathbb{P}((1-\rho)u_t) + \mathbb{P}\nabla \cdot ((\mu - 1)\mathbb{D}u),$$

and hence the velocity is given by

$$\begin{aligned} u &= e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta}\big(-\mathbb{P}(\rho u \cdot \nabla u) + \mathbb{P}((1-\rho)u_t) + \mathbb{P}\nabla \cdot ((\mu-1)\mathbb{D}u)\big)(\tau)\, d\tau \\ &= v_1 + v_2 + v_3 + v_4. \end{aligned} \quad (2.28)$$

Ladyzhenskaya's inequality followed by (2.9) gives

$$\|\rho u \cdot \nabla u\|_{L^2} \leq c(\delta)\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|D_t u\|_{L^2}^{\frac{1}{2}},$$

so estimates (2.13) and (2.25) provide

$$\|\mathbb{P}(\rho u \cdot \nabla u)\|_{L^2} \leq c(\rho^m, \mu^m, \delta, \|u_0\|_{H^{\gamma+\varepsilon}})t^{-1+\frac{3}{4}(\gamma+\varepsilon)}.$$

Similarly,

$$\|\mathbb{P}((1-\rho)u_t)\|_{L^2} \leq c(\rho^m, \mu^m, \delta, \|u_0\|_{H^{\gamma+\varepsilon}})t^{-1+\frac{\gamma+\varepsilon}{2}}.$$

Hence, by Young's inequality for convolutions and the decay properties of the heat kernel, we obtain

$$\begin{aligned} &\|v_1\|_{L^\infty_T(\dot{H}^{\gamma+\varepsilon})} \leq c\|u_0\|_{H^{\gamma+\varepsilon}}, \\ &\|v_2 + v_3\|_{L^\infty_T(\dot{H}^{\gamma+\varepsilon})} \\ &\qquad \leq c(\rho^m, \mu^m, \delta, \|u_0\|_{H^{\gamma+\varepsilon}})\left\|\int_0^t (t-\tau)^{-\frac{\gamma+\varepsilon}{2}}(\tau^{-1+\frac{\gamma+\varepsilon}{2}} + \tau^{-1+\frac{4}{3}(\gamma+\varepsilon)})\right\|_{L^\infty_T} \\ &\qquad \leq c(\rho^m, \mu^m, \delta, \|u_0\|_{H^{\gamma+\varepsilon}}). \end{aligned}$$

Estimate (2.13), integration by parts and the arguments above give

$$\begin{aligned} \|v_4\|_{L^\infty_T(\dot{H}^{\gamma+\varepsilon})} &\leq c(\mu^m, \delta, \|u_0\|_{H^{\gamma+\varepsilon}})\left\|\int_0^t (t-\tau)^{-\frac{1+\gamma+\varepsilon}{2}}\tau^{-\frac{1}{2}+\frac{\gamma+\varepsilon}{2}}\right\|_{L^\infty_T} \\ &\leq c(\mu^m, \delta, \|u_0\|_{H^{\gamma+\varepsilon}}). \end{aligned}$$

Therefore, we conclude that

$$\|u\|_{L^\infty(\mathbb{R}_+; H^{\gamma+\varepsilon})} \leq c(\rho^m, \mu^m, \delta, \|u_0\|_{H^{\gamma+\varepsilon}}).$$

The integration in time in (2.28) provides the continuity, following the estimates above.

L^p-in-time estimates

We summarize here the $L^p(0,T)$ estimates that will be needed in Steps 4 and 5. It should be noticed that there are different constraints for short and long times. From estimate (2.15), we get that for $2r/(2+r(1-\varepsilon)) < p < 2/(2-\gamma-\varepsilon)$,

$$\int_0^t \|D_t u\|_{L^2}^p \, d\tau \leq \left(\int_0^1 \tau^{1-\gamma-\varepsilon} \|D_t u\|_{L^2}^2 \, d\tau \right)^{\frac{p}{2}} \left(\int_0^1 \tau^{-\frac{p(1-\gamma-\varepsilon)}{2} \frac{2}{2-p}} \, d\tau \right)^{\frac{2-p}{2}}$$
$$+ \left(\int_1^t \tau^{\frac{2}{r}-\varepsilon} \|D_t u\|_{L^2}^2 \, d\tau \right)^{\frac{p}{2}} \left(\int_1^t \tau^{-(\frac{2}{r}-\varepsilon)\frac{p}{2-p}} \, d\tau \right)^{\frac{2-p}{2}},$$

and thus

$$\int_0^t \|D_t u\|_{L^2}^p \, d\tau \leq C(\rho^m, \mu^m, \delta, \|u_0\|_{L^r}, \|u_0\|_{H^{\gamma+\varepsilon}}), \quad \frac{2r}{2+r(1-\varepsilon)} < p < \frac{2}{2-\gamma-\varepsilon}. \tag{2.29}$$

Similarly, we obtain from (2.27) that

$$\int_0^t \|\nabla D_t u\|_{L^2}^p \, d\tau \leq c(\rho^m, \mu^m, \delta, \|\sqrt{\rho_0} u_0\|_{L^2}) \|u_0\|_{\dot{H}^{\gamma+\varepsilon}}^p, \quad \frac{2}{3} < p < \frac{2}{3-\gamma-\varepsilon}, \tag{2.30}$$

and from Theorem 2.1 we have

$$\|\nabla u(t)\|_{L^2} \leq C(t+e)^{-\frac{1}{r}+\frac{\varepsilon}{2}}, \quad \int_0^t \|\nabla u\|_{L^2}^p \, d\tau \leq C, \quad \frac{2r}{2-r\varepsilon} < p \leq 2. \tag{2.31}$$

Additionally, we have the estimate

$$\int_0^t \|\nabla u\|_{L^p} \, d\tau \leq C, \quad \frac{2r}{2-r(1+\varepsilon)} < p < \infty. \tag{2.32}$$

In fact, by interpolation followed by (2.15) and (2.27),

$$\int_0^t \|\nabla u\|_{L^p} \, d\tau \leq c(\delta) \int_0^t \|\nabla u\|_{L^2}^{\frac{2}{p}} \|D_t u\|_{L^2}^{1-\frac{2}{p}} \, d\tau$$
$$\leq C \left(\int_0^1 \frac{d\tau}{\tau^{1-\frac{\gamma+\varepsilon}{2}-\frac{1}{p}}} + \int_1^t \frac{d\tau}{\tau^{\frac{1}{r}+\frac{1}{2}-\frac{\varepsilon}{2}-\frac{1}{p}}} \right) \leq C. \tag{2.33}$$

For clarity in notation, in Steps 4 and 5 we will suppress the dependence on the initial data from the constants.

Step 4: $\nabla u \in L^1(0,T;L^\infty)$.

We first use (2.7) to write the gradient of u as

$$\nabla u = -\nabla\Delta^{-1}\mathbb{P}\nabla\cdot\Big(\Big(\frac{\mu}{\bar\mu}-1\Big)\mathbb{D}u\Big) + \frac{1}{\bar\mu}\nabla\Delta^{-1}\mathbb{P}(\rho D_t u) = I_7 + I_8. \tag{2.34}$$

We proceed first with I_8. Sobolev embedding and (2.7) provide that

$$\begin{aligned}|I_8| &\le c\|\nabla\Delta^{-1}\mathbb{P}(\rho D_t u)\|_{W^{1,\frac{2}{1-\gamma}}}\\ &\le c(\|\nabla\Delta^{-1}\mathbb{P}\nabla\cdot(\mu\mathbb{D}u)\|_{L^{\frac{2}{1-\gamma}}} + \|\nabla\nabla\Delta^{-1}\mathbb{P}(\rho D_t u)\|_{L^{\frac{2}{1-\gamma}}}),\end{aligned}$$

thus the boundedness of singular integrals in L^p followed by (2.9), (2.8), gives

$$|I_8| \le c(\delta)\big(\|\nabla u\|_{L^2}^{1-\gamma}\|D_t u\|_{L^2}^{\gamma} + \|D_t u\|_{L^2}^{1-\gamma}\|\nabla D_t u\|_{L^2}^{\gamma}\big). \tag{2.35}$$

Hölder's inequality with $p = \frac{2}{(2-\gamma)(1-\gamma)}$ on the second term yields

$$|I_8| \le c(\delta)\big(\|\nabla u\|_{L^2}^{1-\gamma}\|D_t u\|_{L^2}^{\gamma} + \|D_t u\|_{L^2}^{\frac{2}{2-\gamma}} + \|\nabla D_t u\|_{L^2}^{\frac{2}{3-\gamma}}\big),$$

so that (2.29)–(2.30) guarantee that the $L^1(0,T)$ norm of the last two the terms is bounded uniformly in T. The $L^1(0,T)$ norm of the first term is controlled by (2.33), and hence for any $t>0$,

$$\int_0^t |I_8|\,d\tau \le C. \tag{2.36}$$

Next, the term I_7 is given in index notation by

$$\begin{aligned}\Big(\nabla\Delta^{-1}\mathbb{P}\nabla\cdot\Big(\frac{\mu}{\bar\mu}-1\Big)\mathbb{D}u\Big)_{i,j} &= \partial_i\Delta^{-1}(\delta_{j,k}-\partial_j\Delta^{-1}\partial_k)\partial_m\Big(\Big(\frac{\mu}{\bar\mu}-1\Big)\mathbb{D}_{k,m}u\Big)\\ &= (R_iR_m\delta_{j,k} - R_iR_jR_kR_m)\Big(\Big(\frac{\mu}{\bar\mu}-1\Big)\mathbb{D}_{k,m}u\Big),\end{aligned}$$

where $\delta_{j,k}$ denotes the Kronecker delta, R_i the Riesz transform and Einstein's summation convention is used. Define the corresponding kernels

$$K_{i,j,k,m}(x) = \mathcal{F}^{-1}\Big(\frac{\xi_i\xi_m}{|\xi|^2}\delta_{j,k} - \frac{\xi_i\xi_j\xi_k\xi_m}{|\xi|^4}\Big)(x),$$

so that

$$\begin{aligned}&\Big(\nabla\Delta^{-1}\mathbb{P}\nabla\cdot\Big(\Big(\frac{\mu}{\bar\mu}-1\Big)\mathbb{D}u\Big)\Big)_{i,j}\\ &\quad= \int_{D(t)} K_{i,j,k,m}(x-y)\Big(\frac{\mu^{\text{in}}(y)}{\bar\mu}-1\Big)\mathbb{D}_{k,m}u(y)\,dy\\ &\qquad+ \int_{\mathbb{R}^2\setminus D(t)} K_{i,j,k,m}(x-y)\Big(\frac{\mu^{\text{out}}(y)}{\bar\mu}-1\Big)\mathbb{D}_{k,m}u(y)\,dy.\end{aligned}$$

In the following we shall use the notation

$$\nabla\Delta^{-1}\mathbb{P}\nabla\cdot(1_{D(t)}\mathbb{D}u) = \int_{D(t)} K(x-y)\cdot\mathbb{D}u(y)\,dy,$$

so we have

$$I_7 = \int_{D(t)} K(x-y)\cdot\Big(\frac{\mu^{\text{in}}(y)}{\bar{\mu}}-1\Big)\mathbb{D}u(y)\,dy + \int_{\mathbb{R}^2\setminus D(t)} K(x-y)\cdot\Big(\frac{\mu^{\text{out}}(y)}{\bar{\mu}}-1\Big)\mathbb{D}u(y)\,dy = I_{7,1}+I_{7,2}. \tag{2.37}$$

We focus on $I_{7,1}$ first. Consider first $x\in\overline{D(t)}$. In the following, whenever $x\in\partial D(t)$ we will define $f(x)$ as the limit from inside $D(t)$. Then,

$$I_{7,1} = \frac{1}{\bar{\mu}}\int_{D(t)} K(x-y)\cdot(\mu^{\text{in}}(y)-\mu^{\text{in}}(x))\mathbb{D}u(y)\,dy + \Big(\frac{\mu^{\text{in}}(x)}{\bar{\mu}}-1\Big)\int_{D(t)} K(x-y)\cdot\mathbb{D}u(y)\,dy = J_3 + \Big(\frac{\mu^{\text{in}}(x)}{\bar{\mu}}-1\Big)J_4. \tag{2.38}$$

Therefore,

$$|J_3| \le c\|\mu^{\text{in}}\|_{\dot{C}^\lambda(\overline{D(t)})}\|\mathbb{D}u\|_{L^{\frac{2}{\lambda-\varepsilon}}},$$

where

$$\varepsilon<\lambda<\gamma.$$

By interpolation,

$$|J_3| \le c\|\mu^{\text{in}}\|^{\frac{\lambda}{\gamma}}_{\dot{C}^\gamma(\overline{D(t)})}\|\mu^{\text{in}}-\bar{\mu}\|^{1-\frac{\lambda}{\gamma}}_{L^\infty}\|\mathbb{D}u\|_{L^{\frac{2}{\lambda-\varepsilon}}},$$

and applying Young's inequality we obtain

$$\begin{aligned}\int_0^t |J_3|\,d\tau &\le c\|\mu^{\text{in}}\|^{\frac{\lambda}{\gamma}}_{L^\infty_t\dot{C}^\gamma(\overline{D(t)})}\delta^{1-\frac{\lambda}{\gamma}}\int_0^t\|\mathbb{D}u\|_{L^{\frac{2}{\lambda-\varepsilon}}}\,d\tau\\ &\le \frac{\gamma-\lambda}{\gamma}\delta\|\mu^{\text{in}}\|^{\frac{\lambda}{\gamma-\lambda}}_{L^\infty_t\dot{C}^\gamma(\overline{D(t)})} + C\frac{\lambda}{\gamma}\Big(\int_0^t\|\mathbb{D}u\|_{L^{\frac{2}{\lambda-\varepsilon}}}\,d\tau\Big)^{\frac{\gamma}{\lambda}}.\end{aligned}$$

The last term is bounded in (2.32), thus

$$\int_0^t |J_3|\,d\tau \le \delta\frac{\gamma-\lambda}{\gamma}\|\mu^{\text{in}}\|^{\frac{\lambda}{\gamma-\lambda}}_{L^\infty_t\dot{C}^\gamma(\overline{D(t)})} + C. \tag{2.39}$$

We proceed with J_4. Without loss of generality, let $\varphi_0(x)$ be a defining function for the domain D_0, $D_0=\{x\in\mathbb{R}^2:\ \varphi_0(x)>0\}$ (see for example [26, p. 119]), and $\varphi(x,t)=\varphi_0(X^{-1}(x,t))$ the corresponding defining function for $D(t)$. Define $\eta(t)$ as the cut-off radius

$$\eta(t) = \min\Big\{\Big(\frac{|\nabla\varphi|_{\inf}}{\|\nabla\varphi\|_{\dot{C}^\gamma}}\Big)^{\frac{1}{\gamma}},1\Big\}, \tag{2.40}$$

where we use the notation

$$|\nabla\varphi|_{\inf} = \inf_{x\in\partial D} |\nabla\varphi(x)|.$$

Then, we split J_4 as

$$\begin{aligned} J_4 &= \int_{D(t)\cap\{|x-y|\le\eta\}} K(x-y)\cdot\mathbb{D}u(y) + \int_{D(t)\cap\{|x-y|\ge\eta\}} K(x-y)\cdot\mathbb{D}u(y) \\ &= J_{4,1} + J_{4,2}. \end{aligned}$$

Since the kernel K is even, the second term on the right below is bounded using [5] (see the geometric lemma) as

$$\begin{aligned} |J_{4,1}| &\le \left|\int_{D(t)\cap\{|x-y|\le\eta\}} K(x-y)\cdot(\mathbb{D}u(y)-\mathbb{D}u(x))\,dy\right| \\ &\quad + \left|\mathbb{D}u(x)\cdot\int_{D(t)\cap\{|x-y|\le\eta\}} K(x-y)\,dy\right| \\ &\le c\|\nabla u\|_{\dot C^\gamma(\overline{D(t)})}\int_0^\eta \frac{dr}{r^{1-\gamma}} + c(\gamma)\|\nabla u\|_{L^\infty} \\ &\le c\|\nabla u\|_{\dot C^\gamma(\overline{D(t)})} + c(\gamma)\|\nabla u\|_{L^\infty}. \end{aligned}$$

The term $J_{4,2}$ is bounded by

$$|J_{4,2}| \le c\left(\int_{\eta(t)}^\infty \frac{dr}{r^{\frac{2+\gamma}{2-\gamma}}}\right)^{\frac{2-\gamma}{2}} \|\nabla u\|_{L^{\frac{2}{\gamma}}} = c\eta^{-\gamma}\|\nabla u\|_{L^{\frac{2}{\gamma}}}.$$

Thus, joining the bounds for $J_{4,1}$ and $J_{4,2}$ we find that

$$|J_4| \le c(\gamma)\|\nabla u\|_{L^\infty} + c\|\nabla u\|_{\dot C^\gamma(\overline{D(t)})} + c\eta^{-\gamma}\|\nabla u\|_{L^{\frac{2}{\gamma}}},$$

and going back to (2.38) with the bound (2.39) we have

$$\begin{aligned} \int_0^t |I_{7,1}|\,d\tau &\le C + \delta\Bigg(\frac{\gamma-\lambda}{\gamma}\|\mu^{\text{in}}\|^{\frac{\lambda}{\gamma-\lambda}}_{L^\infty_t\dot C^\gamma(\overline{D(t)})} + c(\gamma)\int_0^t \|\nabla u\|_{L^\infty}\,d\tau \\ &\quad + c\int_0^t \|\nabla u\|_{\dot C^\gamma(\overline{D(t)})}\,d\tau + c\int_0^t \eta^{-\gamma}\|\nabla u\|_{L^{\frac{2}{\gamma}}}\,d\tau\Bigg). \end{aligned} \tag{2.41}$$

Notice that if $x \notin \overline{D(t)}$, we can define $\tilde{x}$ as a point on the boundary with minimum distance to x, $\tilde{x} = \arg d(x, \partial D(t)) \in \partial D(t)$. Then, by adding and subtracting

$$\mu^{\text{in}}(\tilde{x}) = \lim_{\substack{x\to\tilde{x},\\ x\in D(t)}} \mu^{\text{in}}(x)$$

in (2.38) instead of $\mu^{\text{in}}(x)$,

$$\mathbb{D}u(\tilde{x}) = \lim_{\substack{x\to\tilde{x},\\ x\in D(t)}} \mathbb{D}u(x)$$

for the $J_{4,1}$ term instead of $\mathbb{D}u(x)$, and using the triangle inequality, we obtain the same bounds for J_3 and $J_{4,1}$. Since the bound for $J_{4,2}$ holds equally for $x \notin \overline{D(t)}$, we have that estimate (2.41) holds for any $x \in \mathbb{R}^2$.

The term $I_{7,2}$ (2.37) is decomposed further: $I_{7,2} = J_5 + J_6$, with

$$J_5 = \int_{(\mathbb{R}^2\setminus D(t))\cap\{|x-y|>1\}} K(x-y)\cdot\Big(\frac{\mu^{\text{out}}(y)}{\bar{\mu}} - 1\Big)\mathbb{D}u(y)\,dy$$

and

$$J_6 = \int_{(\mathbb{R}^2\setminus D(t))\cap\{|x-y|<1\}} K(x-y)\cdot\Big(\frac{\mu^{\text{out}}(y)}{\bar{\mu}} - 1\Big)\mathbb{D}u(y)\,dy.$$

At this point, it is direct to bound J_5:

$$\begin{aligned}\int_0^t |J_5|\,d\tau &\le c\Big\|\frac{\mu^{\text{out}}}{\bar{\mu}} - 1\Big\|_{L^\infty}\int_0^t \|\nabla u\|_{L^{\frac{2}{\gamma}}}\,d\tau\\ &\le c\delta\int_0^t \eta^{-\gamma}\|\nabla u\|_{L^{\frac{2}{\gamma}}}\,d\tau.\end{aligned}$$

The term J_6 is handled as $I_{7,1}$ (2.37) to get the estimate

$$\begin{aligned}\int_0^t |I_{7,2}|\,d\tau \le C + \delta\Big(&\frac{\gamma-\lambda}{\gamma}\|\mu^{\text{out}}\|^{\frac{\lambda}{\gamma-\lambda}}_{L^\infty_t\dot{C}^\gamma(\mathbb{R}^2\setminus D(t))} + c(\gamma)\int_0^t \|\nabla u\|_{L^\infty}\,d\tau\\ &+ c\int_0^t \|\nabla u\|_{\dot{C}^\gamma(\mathbb{R}^2\setminus D(t))}\,d\tau + c\int_0^t \eta^{-\gamma}\|\nabla u\|_{L^{\frac{2}{\gamma}}}\,d\tau\Big).\end{aligned}$$

Going back to (2.37), the above estimate and (2.41) allow us to get

$$\begin{aligned}\int_0^t |I_7|\,d\tau \le C + \delta\Big(&\frac{\gamma-\lambda}{\gamma}\|\mu\|^{\frac{\lambda}{\gamma-\lambda}}_{L^\infty_t\dot{C}^\gamma(\overline{D(t)})\cap\dot{C}^\gamma(\mathbb{R}^2\setminus D(t))} + c(\gamma)\int_0^t \|\nabla u\|_{L^\infty}\,d\tau\\ &+ c\int_0^t \|\nabla u\|_{\dot{C}^\gamma(\overline{D(t)})\cap\dot{C}^\gamma(\mathbb{R}^2\setminus D(t))}\,d\tau + c\int_0^t \eta^{-\gamma}\|\nabla u\|_{L^{\frac{2}{\gamma}}}\,d\tau\Big).\end{aligned}$$

Splitting (2.34), the above estimate together with (2.36) provides

$$\begin{aligned}\int_0^T \|\nabla u\|_{L^\infty}\,dt \le C + \frac{c\delta}{1-c(\gamma)\delta}\Big(&\|\mu\|^{\frac{\lambda}{\gamma-\lambda}}_{L^\infty_T\dot{C}^\gamma(\overline{D(t)})\cap\dot{C}^\gamma(\mathbb{R}^2\setminus D(t))}\\ &+\int_0^T \eta(t)^{-\gamma}\|\nabla u\|_{L^{\frac{2}{\gamma}}}\,dt\\ &+\int_0^T \|\nabla u\|_{\dot{C}^\gamma(\overline{D(t)})\cap\dot{C}^\gamma(\mathbb{R}^2\setminus D(t))}\,dt\Big).\end{aligned} \tag{2.42}$$

Step 5: $\nabla u \in L^1(0,T;\dot{C}^\gamma(\overline{D(t)})\cap\dot{C}^\gamma(\mathbb{R}^2\setminus D(t)))$.

Recalling expression (2.34) for ∇u, we write

$$\nabla u(x+h) - \nabla u(x) = I_{9,1} + I_{9,2} + I_{10}, \tag{2.43}$$

where

$$I_{9,1}=\int_{D(t)}(K(x+h-y)-K(x-y))\Big(\frac{\mu^{\rm in}(y)}{\bar\mu}-1\Big)\mathbb{D}u(y)\,dy,$$
$$I_{9,2}=\int_{\mathbb{R}^2\setminus D(t)}(K(x+h-y)-K(x-y))\Big(\frac{\mu^{\rm out}(y)}{\bar\mu}-1\Big)\mathbb{D}u(y)\,dy$$

and

$$I_{10}=\frac{1}{\bar\mu}\big(\nabla\Delta^{-1}\mathbb{P}(\rho D_t u)(x+h)-\nabla\Delta^{-1}\mathbb{P}(\rho D_t u)(x)\big).$$

We deal first with the term I_{10}. Classical Sobolev embedding together with the Gagliardo–Nirenberg inequality (2.8) gives

$$\begin{aligned}|I_{10}|&\le c\|\nabla\Delta^{-1}\mathbb{P}(\rho D_t u)\|_{\dot W^{1,\frac{2}{1-\gamma}}}|h|^\gamma\\&\le c\|D_t u\|_{L^{\frac{2}{1-\gamma}}}|h|^\gamma\\&\le c\|D_t u\|_{L^2}^{1-\gamma}\|\nabla D_t u\|_{L^2}^{\gamma}|h|^\gamma,\end{aligned}\tag{2.44}$$

so repeating the steps for (2.35) we conclude that I_{10} is uniformly bounded in $L^1(0,T)$.

Next we deal with $I_{9,1}$ and $I_{9,2}$. Assume that x and $x+h$ belong to $\overline{D(t)}$. We proceed first with the term $I_{9,1}$. We decompose $I_{9,1}$ as

$$I_{9,1}=L_1+L_2+L_3+L_4+L_5,\tag{2.45}$$

with

$$\begin{aligned}L_1=&\int_{D(t)\cap\{|x-y|<2|h|\}}K(x+h-y)\\&\qquad\times\Big(\Big(\Big(\frac{\mu^{\rm in}}{\bar\mu}-1\Big)\mathbb{D}u\Big)(y)-\Big(\Big(\frac{\mu^{\rm in}}{\bar\mu}-1\Big)\mathbb{D}u\Big)(x+h)\Big)\,dy,\\L_2=&\int_{D(t)\cap\{|x-y|<2|h|\}}K(x-y)\Big(\Big(\Big(\frac{\mu^{\rm in}}{\bar\mu}-1\Big)\mathbb{D}u\Big)(x)-\Big(\Big(\frac{\mu^{\rm in}}{\bar\mu}-1\Big)\mathbb{D}u\Big)(y)\Big)\,dy,\\L_3=&\int_{D(t)\cap\{|x-y|<2|h|\}}K(x-y)\Big(\Big(\Big(\frac{\mu^{\rm in}}{\bar\mu}-1\Big)\mathbb{D}u\Big)(x+h)-\Big(\Big(\frac{\mu^{\rm in}}{\bar\mu}-1\Big)\mathbb{D}u\Big)(x)\Big)\,dy,\\L_4=&\int_{D(t)\cap\{|x-y|\ge2|h|\}}(K(x+h-y)-K(x-y))\\&\qquad\times\Big(\Big(\Big(\frac{\mu^{\rm in}}{\bar\mu}-1\Big)\mathbb{D}u\Big)(y)-\Big(\Big(\frac{\mu^{\rm in}}{\bar\mu}-1\Big)\mathbb{D}u\Big)(x+h)\Big)\,dy,\\L_5=&\Big(\frac{\mu^{\rm in}(x+h)}{\bar\mu}-1\Big)\mathbb{D}u(x+h)\int_{D(t)}(K(x+h-y)-K(x-y))\,dy.\end{aligned}$$

The terms L_1 and L_2 are directly bounded by

$$|L_1|+|L_2|\le c\Big\|\Big(\frac{\mu^{\rm in}}{\bar\mu}-1\Big)\mathbb{D}u\Big\|_{\dot C^\gamma(\overline{D(t)})}|h|^\gamma.$$

Choosing $2|h| \leq \min_{t\in[0,T]} \eta(t)$, we have (see [5])

$$|L_3| \leq c \Big\| \Big(\frac{\mu^{\text{in}}}{\bar{\mu}} - 1\Big)\mathbb{D}u \Big\|_{\dot{C}^\gamma(\overline{D(t)})} |h|^\gamma.$$

By the mean value theorem we also have

$$|L_4| \leq c \Big\| \Big(\frac{\mu^{\text{in}}}{\bar{\mu}} - 1\Big)\mathbb{D}u \Big\|_{\dot{C}^\gamma(\overline{D(t)})} |h|^\gamma.$$

The term L_5 is more singular and we have to use contour dynamics to control it. According to Theorem 1.2,

$$|L_5| \leq c \Big\| \Big(\frac{\mu^{\text{in}}}{\bar{\mu}} - 1\Big)\mathbb{D}u \Big\|_{L^\infty(\overline{D(t)})} \mathscr{P}(\|D(t)\|_{\text{Lip}} + \|D(t)\|_*)\|D(t)\|_{\dot{C}^{1,\gamma}} |h|^\gamma,$$

where $\mathscr{P}$ is a polynomial. Let $z_0(\alpha)$ be a $C^{1,\gamma}$ parametrization of the initial domain D_0, so that its evolution via the particle trajectories (1.4),

$$z(\alpha, t) = X(z_0(\alpha), t),$$

gives the parametrization of $D(t)$. Then, since

$$\|\partial_\alpha z\|_{\dot{C}^\gamma} \leq (\|\partial_\alpha z_0\|_{C^\gamma} + 1)^{1+\gamma} \|\nabla X\|_{C^\gamma(\overline{D(t)})},$$

we obtain

$$|L_5| \leq c \Big\| \Big(\frac{\mu^{\text{in}}}{\bar{\mu}} - 1\Big)\mathbb{D}u \Big\|_{L^\infty(\overline{D(t)})} \mathscr{P}(\|\nabla\varphi\|_{L^\infty} + |\nabla\varphi|_{\inf}^{-1}) \|\nabla X\|_{C^\gamma(\overline{D(t)})} |h|^\gamma,$$

where the constant c only depends on the initial domain. Then we notice that

$$\Big\| \Big(\frac{\mu^{\text{in}}}{\bar{\mu}} - 1\Big)\mathbb{D}u \Big\|_{\dot{C}^\gamma(\overline{D(t)})} \leq c\|\mu^{\text{in}}\|_{\dot{C}^\gamma(\overline{D(t)})} \|\nabla u\|_{L^\infty} + c\delta\|\nabla u\|_{\dot{C}^\gamma(\overline{D(t)})}.$$

The same approach is taken to control $I_{9,2}$ but using the decomposition $\widetilde{L}_1 - \widetilde{L}_5$, where the domain $D(t)$ is replaced by $\mathbb{R}^2 \setminus D(t)$ and using $((\frac{\mu^{\text{out}}}{\bar{\mu}} - 1)\mathbb{D}u)(\tilde{x})$ and $((\frac{\mu^{\text{out}}}{\bar{\mu}} - 1)\mathbb{D}u)(\tilde{x}_h)$ instead, where $\tilde{x}_h = \arg d(x + h, \partial D(t))$. In fact, since for $y \in \mathbb{R}^2 \setminus D(t)$ one has that $|y - \tilde{x}| \leq 2|y - x|$, $|y - \tilde{x}_h| \leq 2|x + h - y|$, we obtain

$$|\widetilde{L}_1| + |\widetilde{L}_2| \leq c \Big\| \Big(\frac{\mu^{\text{out}}}{\bar{\mu}} - 1\Big)\mathbb{D}u \Big\|_{\dot{C}^\gamma(\mathbb{R}^2\setminus D(t))} |h|^\gamma.$$

The term $\widetilde{L}^4$ is done analogously. For $\widetilde{L}^3$, notice that the only nontrivial case happens when $d(x, \partial D(t)) < 2|h|$ (otherwise $\widetilde{L}_3 = 0$). Also, since the domain is restricted to a ball, the integral of the kernel is bounded in the same way ([5]). We thus have

$$|\widetilde{L}_3| \leq c \Big\| \Big(\frac{\mu^{\text{out}}}{\bar{\mu}} - 1\Big)\mathbb{D}u \Big\|_{\dot{C}^\gamma(\mathbb{R}^2\setminus D(t))} |\tilde{x}_h - \tilde{x}|^\gamma \leq c \Big\| \Big(\frac{\mu^{\text{out}}}{\bar{\mu}} - 1\Big)\mathbb{D}u \Big\|_{\dot{C}^\gamma(\mathbb{R}^2\setminus D(t))} |h|^\gamma,$$

where in the last step we use that $|\tilde{x}_h - \tilde{x}| \le |\tilde{x} - (x+h)| + |(x+h) - \tilde{x}_h| \le 6|h|$. The term $\widetilde{L}_5$ follows directly from Theorem 1.2,

$$|\widetilde{L}_5| \le c \Big\| \Big(\frac{\mu^{\text{out}}}{\bar{\mu}} - 1 \Big) \mathbb{D}u \Big\|_{L^\infty(\mathbb{R}^2 \setminus D(t))} \mathcal{P}(\|D(t)\|_{\text{Lip}} + \|D(t)\|_*) \|D(t)\|_{\dot{C}^{1,\gamma}} |h|^\gamma.$$

The case $x, x+h \in \mathbb{R}^2 \setminus D(t)$ is done analogously. It yields all the desired estimates to control the Hölder norm for ∇u. In order to gather them all, we will denote

$$\dot{C}^\gamma_D = \dot{C}^\gamma(\overline{D(t)}) \cap \dot{C}^\gamma(\mathbb{R}^2 \setminus D(t))$$

for clarity in notation, and analogously for C^γ_D, to be used in the following. Therefore, from splitting (2.43) and the above estimates it is possible to get

$$\begin{aligned} \|\nabla u\|_{\dot{C}^\gamma_D} \le{}& \frac{c\delta}{1-c\delta} \|\nabla u\|_{L^\infty} \mathcal{P}(\|\nabla \varphi\|_{L^\infty} + |\nabla \varphi|^{-1}_{\inf}) \|\nabla X\|_{C^\gamma_D} \\ &+ \frac{c}{1-c\delta} \|\mu\|_{\dot{C}^\gamma_D} \|\nabla u\|_{L^\infty} + \frac{c}{1-c\delta} \big(\|D_t u\|^{\frac{2}{2-\gamma}}_{L^2} + \|\nabla D_t u\|^{\frac{2}{3-\gamma}}_{L^2} \big). \end{aligned} \tag{2.46}$$

Step 6: Closing all estimates.

We now introduce the bound (2.46) back into (2.42) to obtain

$$\begin{aligned} \int_0^t \|\nabla u\|_{L^\infty} d\tau \le{}& c\delta^2 \int_0^t \|\nabla u\|_{L^\infty} \mathcal{P}(\|\nabla \varphi\|_{L^\infty} + |\nabla \varphi|^{-1}_{\inf}) \|\nabla X\|_{C^\gamma_D} d\tau \\ &+ c\delta \int_0^t \big(\|D_t u\|^{\frac{2}{2-\gamma}}_{L^2} + \|\nabla D_t u\|^{\frac{2}{3-\gamma}}_{L^2} \big) d\tau + c\delta e^{c \int_0^t \|\nabla u\|_{L^\infty} d\tau} \\ &+ c\delta \int_0^t \frac{\|\nabla \varphi\|_{\dot{C}^\gamma}}{|\nabla \varphi|_{\inf}} \|\nabla u\|_{L^{\frac{2}{\gamma}}} d\tau + C. \end{aligned} \tag{2.47}$$

Above we have used the definition of η (2.40) and that μ is transported by the flow:

$$\|\mu\|_{\dot{C}^\gamma_D} \le \|\mu_0\|_{\dot{C}^\gamma_D} e^{\gamma \int_0^t \|\nabla u\|_{L^\infty} d\tau}.$$

We denote

$$y(t) = \int_0^t \|\nabla u\|_{L^\infty} d\tau,$$

and notice that

$$\begin{aligned} \|\nabla X\|_{L^\infty} &\le \|\nabla X_0\|_{L^\infty} e^{y(t)}, \\ \|\nabla \varphi\|_{L^\infty} &\le \|\nabla \varphi_0\|_{L^\infty} e^{y(t)}, \\ |\nabla \varphi|_{\inf} &\ge |\nabla \varphi_0|_{\inf} e^{-y(t)}. \end{aligned} \tag{2.48}$$

Furthermore, recalling the definition $\varphi(x,t) = \varphi_0(X^{-1}(x,t))$, we have

$$\nabla \varphi(X(x,t),t) = (\nabla X(x,t))^{-1} \nabla \varphi_0(x),$$

$$
\begin{aligned}
&\frac{\nabla\varphi(X(x,t),t)-\nabla\varphi(X(y,t),t)}{|X(x,t)-X(y,t)|^\gamma}\\
&\quad=\frac{(\nabla X(x,t))^{-1}-(\nabla X(y,t))^{-1}}{|x-y|^\gamma}\nabla\varphi_0(x)\Big(\frac{|x-y|}{|X(x,t)-X(y,t)|}\Big)^\gamma\\
&\qquad+(\nabla X(y,t))^{-1}\frac{\nabla\varphi_0(x)-\nabla\varphi_0(y)}{|x-y|^\gamma}\Big(\frac{|x-y|}{|X(x,t)-X(y,t)|}\Big)^\gamma,
\end{aligned}
$$

and hence,

$$
\|\nabla\varphi\|_{\dot{C}^\gamma}\le(\|(\nabla X)^{-1}\|^2_{L^\infty}\|\nabla X\|_{\dot{C}^\gamma_D}\|\nabla\varphi_0\|_{L^\infty}+\|(\nabla X)^{-1}\|_{L^\infty}\|\nabla\varphi_0\|_{\dot{C}^\gamma})\|\nabla X^{-1}\|^\gamma_{L^\infty}.
$$

Using (2.48), we conclude that

$$
\|\nabla\varphi\|_{\dot{C}^\gamma}\le c(\|\nabla X\|_{\dot{C}^\gamma_D}+1)e^{c(\gamma)y(t)}. \tag{2.49}
$$

Next, we propagate further regularity,

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla X\|_{L^\infty}\le\|\nabla u\|_{L^\infty}\|\nabla X\|_{L^\infty},\\
&\frac{d}{dt}\|\nabla X\|_{\dot{C}^\gamma_D}\le\|\nabla u\|_{L^\infty}\|\nabla X\|_{\dot{C}^\gamma_D}+\|\nabla X\|^{1+\gamma}_{L^\infty}\|\nabla u\|_{\dot{C}^\gamma_D},
\end{aligned}
$$

and substitute estimate (2.46) to obtain

$$
\begin{aligned}
\frac{d}{dt}\|\nabla X\|_{C^\gamma_D}\le{}&\|\nabla u\|_{L^\infty}\|\nabla X\|_{C^\gamma_D}\\
&+\delta\|\nabla u\|_{L^\infty}\mathcal{P}(\|\nabla\varphi\|_{L^\infty}+|\nabla\varphi|^{-1}_{\inf})\|\nabla X\|^{1+\gamma}_{L^\infty}\|\nabla X\|_{C^\gamma_D}\\
&+c\|\mu\|_{\dot{C}^\gamma_D}\|\nabla u\|_{L^\infty}\|\nabla X\|^{1+\gamma}_{L^\infty}\\
&+c\|\nabla X\|^{1+\gamma}_{L^\infty}\big(\|D_t u\|^{\frac{2}{2-\gamma}}_{L^2}+\|\nabla D_t u\|^{\frac{2}{3-\gamma}}_{L^2}\big).
\end{aligned}
$$

We denote

$$
\begin{aligned}
x(t)&=\|\nabla X\|_{C^\gamma_D},\\
a(t)&=\|\nabla u\|_{L^\infty}+\delta\|\nabla u\|_{L^\infty}\mathcal{P}(\|\nabla\varphi\|_{L^\infty}+|\nabla\varphi|^{-1}_{\inf})\|\nabla X\|^{1+\gamma}_{L^\infty},\\
d(t)&=c\|\mu\|_{\dot{C}^\gamma_D}\|\nabla u\|_{L^\infty}\|\nabla X\|^{1+\gamma}_{L^\infty}+c\|\nabla X\|^{1+\gamma}_{L^\infty}\big(\|D_t u\|^{\frac{2}{2-\gamma}}_{L^2}+\|\nabla D_t u\|^{\frac{2}{3-\gamma}}_{L^2}\big),
\end{aligned} \tag{2.50}
$$

so that the above inequality can be rewritten as

$$
\dot{x}(t)\le a(t)x(t)+d(t),
$$

and after integration,

$$
x(t)\le g(t)+\int_0^t a(s)x(s)\,ds,
$$

where

$$g(t) = x(0) + \int_0^t d(s)\, ds. \tag{2.51}$$

Hence, applying Grönwall's lemma, we find that

$$x(t) \le \|g\|_{L^\infty_T} e^{\int_0^t a(s)\, ds}. \tag{2.52}$$

Next, the terms $a(t)$ and $g(t)$ are estimated using (2.48) and (2.29)–(2.30),

$$\begin{aligned} \|g\|_{L^\infty_T} &\le c\|\mu\|_{L^\infty_T(\dot{C}^\gamma_D)} y(T) e^{(1+\gamma)y(T)} + C(1 + e^{(1+\gamma)y(T)}) \\ &\le C(1 + y(T))e^{cy(T)} \\ &\le Ce^{cy(T)}, \\ \int_0^t a(s)\, ds &\le y(t) + c\delta y(t) e^{cy(t)}. \end{aligned}$$

Thus, by (2.52), we have

$$x(t) \le Ce^{cy(T)} e^{c\delta y(t) e^{cy(t)}}. \tag{2.53}$$

We introduce the bound (2.53) in (2.47), together with (2.29)–(2.32) and(2.48), to get

$$y(t) \le c\delta^2 y(T) e^{cy(T)} e^{c\delta y(t) e^{cy(t)}} + \delta C e^{y(T)} + C(1+\delta),$$

that is,

$$y(T) \le \delta C_1 e^{C_2 y(T)} e^{C_3 \delta y(T) e^{C_4 y(T)}} + C_5.$$

Assume that

$$\delta \le \min\Big\{ \frac{e^{-2C_4 C_5}}{2C_3 C_5}, \frac{C_5 e^{-2C_2 C_5 - 1}}{2C_1} \Big\}. \tag{2.54}$$

Then we proceed by contradiction. If there exists a first time T such that $y(T) = 2C_5$, we would then have

$$2C_5 \le C_5 \Big(\frac{C_1}{C_5} \delta e^{2C_2 C_5 + 1} + 1 \Big) \le \frac{3}{2} C_5.$$

Therefore, we conclude that there exists a $\delta > 0$ satisfying (2.54) such that for any $T > 0$ it holds that

$$y(T) = \int_0^T \|\nabla u\|_{L^\infty}\, dt < 2C_5,$$

and hence, from (2.48), (2.53) and (2.46), for all $t > 0$,

$$\begin{aligned} \eta(t)^\gamma &\ge Ce^{-cC_1} > 0, \\ \|\nabla X\|_{C^\gamma_D} &\le C, \\ \int_0^t \|\nabla u\|_{\dot{C}^\gamma_D}\, d\tau &\le C. \end{aligned}$$

■

2.2. Uniqueness

As in the case of constant viscosity [11–13], the uniqueness of solutions is proved in Lagrangian variables. This is due to the low regularity of the density and viscosity, produced by their jumps across the interface.

We denote by (ρ_0, μ_0, v, Q) the solution to (INS) in Lagrangian coordinates,

$$\rho_0(y) = \rho(X(y,t),t), \qquad \mu_0(y) = \mu(X(y,t),t),$$
$$v(y,t) = u(X(y,t),t), \quad Q(y,t) = P(X(y,t),t),$$

where X is the flow defined by (1.4). Note that

$$\nabla X(y,t) = \mathbb{I}_2 + \int_0^t \nabla v(y,\tau)\, d\tau,$$

and denote

$$A(t) = (\nabla X(\cdot,t))^{-1}.$$

Then, in Lagrangian variables the operators ∇, $\nabla\cdot$ are given as follows. If we denote $\tilde{f}(y,t) = f(X(y,t),t)$, then

$$\nabla_u \tilde{f}(y,t) := (\nabla f)(X(y,t),t) = A^* \nabla \tilde{f}(y,t),$$
$$\nabla_u \cdot \tilde{f}(y,t) := (\nabla\cdot f)(X(y,t),t) = \nabla\cdot(A\tilde{f}(y,t)),$$

and furthermore, since $\det A(t) = 1$ due to the incompressibility condition, the following identity holds for vector fields (see e.g. [11]):

$$\nabla\cdot(A\tilde{f}) = A^* : \nabla\tilde{f}. \tag{2.55}$$

Hence, (INS) in $y \in \mathbb{R}^2$, $0 < t < T$, is rewritten as

$$\begin{aligned} &\rho_0 \partial_t v = \nabla_u \cdot (\mu_0 \mathbb{D}_u v - Q\mathbb{I}_2),\\ &\nabla_u \cdot v = 0. \end{aligned}$$

The equivalence of these formulations is guaranteed assuming that

$$\int_0^T \|\nabla v\|_{L^\infty}\, d\tau \le c < 1. \tag{2.56}$$

In that case, one can write

$$A(t) = \sum_{j=0}^{\infty} (-1)^j \left(\int_0^t \nabla v(\cdot,\tau)\, d\tau \right)^j. \tag{2.57}$$

Let $(\rho^1, \mu^1, u^1, P^1, X^1)$, $(\rho^2, \mu^2, u^2, P^2, X^2)$ be two solutions as in Theorem 1.1 for the same initial data. In Lagrangian coordinates, we denote their difference by

$$\delta v = v^2 - v^1, \quad \delta Q = Q^2 - Q^1,$$

so that

$$\begin{aligned}\rho_0 \delta v_t &= \nabla_{u^1} \cdot (\mu_0 \mathbb{D}_{u^1} \delta v - \delta Q \mathbb{I}_2) + \nabla_{u^2} \cdot (\mu_0 \mathbb{D}_{u^2} v^2 - Q^2 \mathbb{I}_2) \\ &\quad - \nabla_{u^1} \cdot (\mu_0 \mathbb{D}_{u^1} v^2 - Q^2 \mathbb{I}_2), \\ \nabla_{u^1} \cdot \delta v &= (\nabla_{u^1} - \nabla_{u^2}) \cdot v^2, \\ \delta v|_{t=0} &= 0.\end{aligned}$$

We will now prove that for $T > 0$ small enough,

$$\int_0^T \int_{\mathbb{R}^2} |\nabla \delta v|^2 \, dy \, dt = 0. \tag{2.58}$$

First, let $\delta v = w + z$, where w is assumed to be the solution to the equation

$$\nabla_{u^1} \cdot w = (\nabla_{u^1} - \nabla_{u^2}) \cdot v^2 = \nabla \cdot (\delta A v^2). \tag{2.59}$$

Then the equations for z become

$$\begin{aligned}\rho_0 z_t - \nabla_{u^1} \cdot (\mu_0 \mathbb{D}_{u^1} z) &= -\nabla_{u^1} \delta Q + \nabla_{u^2} \cdot (\mu_0 \mathbb{D}_{u^2} v^2 - Q^2 \mathbb{I}_2) \\ &\quad - \nabla_{u^1} \cdot (\mu_0 \mathbb{D}_{u^1} v^2 - Q^2 \mathbb{I}_2) \\ &\quad - \rho_0 w_t + \nabla_{u^1} \cdot (\mu_0 \mathbb{D}_{u^1} w), \\ \nabla_{u^1} \cdot z &= 0,\end{aligned}$$

and thus we have

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho_0} z\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\mu_0} \mathbb{D}_{u^1} z\|_{L^2}^2 = \sum_{j=1}^4 I_j, \tag{2.60}$$

with

$$\begin{aligned}I_1 &= \int_{\mathbb{R}^2} z \cdot \big(\nabla_{u^2} \cdot (\mu_0 \mathbb{D}_{u^2} v^2) - \nabla_{u^1} \cdot (\mu_0 \mathbb{D}_{u^1} v^2)\big) \, dy, \\ I_2 &= \int_{\mathbb{R}^2} z \cdot (\nabla_{u^1} Q^2 - \nabla_{u^2} Q^2) \, dy, \\ I_3 &= -\int_{\mathbb{R}^2} \rho_0 w_t \cdot z \, dy, \\ I_4 &= \int_{\mathbb{R}^2} \nabla_{u^1} \cdot (\mu_0 \mathbb{D}_{u^1} w) \cdot z \, dy,\end{aligned}$$

where we have used that $\nabla_{u^1} \cdot z = 0$. We proceed to estimate each of these terms. Notice first that if v^1, v^2 satisfy (2.56), then from (2.57) we obtain

$$\|\delta A(t)\|_{L^2} \leq C t^{\frac{1}{2}} \|\nabla \delta v\|_{L^2_t L^2}. \tag{2.61}$$

Next, we integrate by parts to obtain

$$|I_1| \leq \int_{\mathbb{R}^2} \mu_0 |\nabla z| \big|(\delta A A_2^* + A_1 \delta A^*) \nabla v^2 + \delta A (\nabla v^2)^* A_2 + A_1 (\nabla v^2)^* \delta A\big| \, dy,$$

and therefore

$$|I_1| \le C\|\nabla z\|_{L^2} t^{\frac{1}{2}} \|\nabla v^2\|_{L^\infty} t^{-\frac{1}{2}} \|\delta A\|_{L^2}.$$

Thus, using (2.61) and integrating in time,

$$\int_0^T |I_1|\, dt \le C\|\nabla \delta v\|^2_{L^2_T L^2} \|t^{\frac{1}{2}} \nabla v^2\|^2_{L^2_T L^\infty} + \frac{\mu^m}{8}\|\nabla z\|^2_{L^2_T L^2}.$$

Similarly, integration by parts and identity (2.55) provide that

$$I_2 = \int_{\mathbb{R}^2} \delta A^* : \nabla z Q^2\, dy.$$

Now we substitute the pressure by its expression in (2.4) to get

$$\begin{aligned} I_2 = &\int_{\mathbb{R}^2} \delta A^* : \nabla z (\nabla \cdot \nabla \cdot \Delta^{-1})(\mu^2 \mathbb{D} u^2)(X^2(y))\, dy \\ &- \int_{\mathbb{R}^2} \delta A^* : \nabla z (\Delta^{-1} \nabla \cdot)(\rho^2 D_t u^2)(X^2(y))\, dy, \end{aligned}$$

and using (2.55) again we integrate by parts back in the second term,

$$\begin{aligned} I_2 = &\int_{\mathbb{R}^2} \delta A^* : \nabla z (\nabla \cdot \nabla \cdot \Delta^{-1})(\mu^2 \mathbb{D} u^2)(X^2(y))\, dy \\ &+ \int_{\mathbb{R}^2} \delta A z \cdot (\nabla \Delta^{-1} \nabla \cdot)(\rho^2 D_t u^2)(X^2(y)) \nabla X^2(y)\, dy. \end{aligned}$$

Therefore,

$$\begin{aligned} |I_2| \le\ & C\|t^{-\frac{1}{2}} \delta A\|_{L^2} \|t^{\frac{1}{2}} (\nabla \cdot \nabla \cdot \Delta^{-1})(\mu^2 \mathbb{D} u^2)\|_{L^\infty} \|\nabla z\|_{L^2} \\ &+ C\|t^{-\frac{1}{2}} \delta A\|_{L^2} \|t^{\frac{1}{2}} \nabla \Delta^{-1} \nabla \cdot (\rho^2 D_t u^2)\|_{L^6} \|z\|^{\frac{2}{3}}_{L^2} \|\nabla z\|^{\frac{1}{3}}_{L^2}, \end{aligned}$$

where we have used (2.8). The Calderon–Zygmund and Young inequalities provide

$$\begin{aligned} \int_0^T |I_2|\, dt \le\ & C\|\nabla \delta v\|^2_{L^2_T L^2} \|t^{\frac{1}{2}} (\nabla \cdot \nabla \cdot \Delta^{-1})(\mu^2 \mathbb{D} u^2)\|^2_{L^2_T L^\infty} + \frac{\mu^m}{16}\|\nabla z\|^2_{L^2_T L^2} \\ &+ C\|\nabla \delta v\|_{L^2_T (L^2)} \|t^{\frac{1}{2}} \rho^2 D_t u^2\|_{L^{\frac{6}{5}}_T L^6} \|z\|^{\frac{2}{3}}_{L^\infty_T L^2} \|\nabla z\|^{\frac{1}{3}}_{L^2_T L^2} \\ \le\ & C\|\nabla \delta v\|^2_{L^2_T L^2} \big(\|t^{\frac{1}{2}} (\nabla \cdot \nabla \cdot \Delta^{-1})(\mu^2 \mathbb{D} u^2)\|^2_{L^2_T L^\infty} + \|t^{\frac{1}{2}} \rho^2 D_t u^2\|^2_{L^{\frac{6}{5}}_T L^6}\big) \\ &+ \frac{1}{4}\|z\|^2_{L^\infty_T L^2} + \frac{\mu^m}{8}\|\nabla z\|^2_{L^2_T L^2}. \end{aligned}$$

We are left to show that $t^{\frac{1}{2}} (\nabla \cdot \nabla \cdot \Delta^{-1})(\mu^2 \mathbb{D} u^2) \in L^2(0, T; L^\infty)$ and $t^{\frac{1}{2}} D_t u^2 \in L^{\frac{6}{5}}(0, T; L^6)$. The latter follows from (2.8) and the regularity estimates in Theorem 1.1. For the first, it is enough to define $y(T) = \int_0^T \tau \|\nabla u\|^2_{L^\infty}\, d\tau$ and repeat Steps 4 and 5, by noticing that instead of (2.35) now we would have

$$t\|\nabla u\|_{L^{\frac{4}{1-\gamma}}} + t\|D_t u\|_{L^{\frac{4}{1-\gamma}}} \le c(T),$$

which follows by the regularity estimates in Theorem 1.1. The constant $c(T)$ above is continuous, increasing and such that $c(0) = 0$. We are done with I_1 and I_2. To deal with I_3 and I_4, we need to study w first.

Lemma 2.2. *Let $A(t)$ be a matrix-valued function on $[0,T]\times\mathbb{R}^2$ satisfying*

$$\det A = 1.$$

There exists a constant c such that if

$$\|\mathbb{I}_2 - A\|_{L^\infty_T L^\infty} + \|A_t\|_{L^{\frac{6}{5}}_T L^6} \le c,$$

then for all functions g in $L^2(0,T;L^2)$ satisfying

$$g = \nabla\cdot R, \quad R \in L^\infty_T L^2, \quad R_t \in L^{\frac{6}{5}}_T L^{\frac{3}{2}},$$

the equation

$$\nabla\cdot(Aw) = g \quad \textit{in } [0,T]\times\mathbb{R}^2$$

has a solution w in the space

$$W_T = \{w \in L^\infty_T L^2,\ \nabla w \in L^2_T L^2,\ w_t \in L^{\frac{6}{5}}_T L^{\frac{3}{2}}\},$$

that satisfies

$$\begin{aligned}
\|w\|_{L^\infty_T L^2} &\le C\|R\|_{L^\infty_T L^2},\\
\|\nabla w\|_{L^2_T L^2} &\le C\|g\|_{L^2_T L^2},\\
\|w_t\|_{L^{\frac{6}{5}}_T L^{\frac{3}{2}}} &\le C\|R\|_{L^\infty_T L^2} + C\|R_t\|_{L^{\frac{6}{5}}_T L^{\frac{3}{2}}}.
\end{aligned}$$

Proof. The proof follows as in [13, Lemma A.2] with minor modifications. ∎

Lemma 2.3. *The solution w to (2.59) given by Lemma 2.2 satisfies*

$$\|w\|_{L^\infty_T L^2} + \|\nabla w\|_{L^2_T L^2} + \|w_t\|_{L^{\frac{6}{5}}_T L^{\frac{3}{2}}} \le c(T)\|\nabla\delta v\|_{L^2_T L^2}, \tag{2.62}$$

where $c(T)$ is a continuous increasing function of T with $c(0) = 0$.

Proof. Using (2.9) and the estimates in Theorem 1.1, we have $\nabla v \in L^{\frac{6}{5}}(0,T;L^6)$, and thus there exists a constant c such that if

$$\|\nabla v^1\|_{L^1_T L^\infty} + \|\nabla v^1\|_{L^{\frac{6}{5}}_T L^6} \le C,$$

then, by Lemma 2.2 and identity (2.55),

$$\begin{aligned}
\|w\|_{L^\infty_T L^2} &\le C\|\delta A v^2\|_{L^\infty_T L^2},\\
\|\nabla w\|_{L^2_T L^2} &\le C\|\delta A^* : \nabla v^2\|_{L^2_T L^2},\\
\|w_t\|_{L^{\frac{6}{5}}_T L^{\frac{3}{2}}} &\le C\|\delta A v^2\|_{L^\infty_T L^2} + C\|(\delta A v^2)_t\|_{L^{\frac{6}{5}}_T L^{\frac{3}{2}}}.
\end{aligned}$$

Using Hölder's inequality and (2.61) repeatedly, we obtain

$$\begin{aligned}
\|\delta A v^2\|_{L^\infty_T L^2} &\leq C\|t^{\frac{1}{2}} v^2\|_{L^\infty_T L^\infty}\|\nabla\delta v\|_{L^2_T L^2} \\
&\leq C\|t^{\frac{1}{2}} v^2\|_{L^\infty_T W^{1,\frac{2}{1-\gamma}}}\|\nabla\delta v\|_{L^2_T L^2}, \\
\|\delta A^* : \nabla v^2\|_{L^2_T L^2} &\leq C\|\nabla\delta v\|_{L^2_T L^2}\|t^{\frac{1}{2}}\nabla v^2\|_{L^2_T L^\infty}, \\
\|\delta A_t v^2\|_{L^{\frac{6}{5}}_T L^{\frac{3}{2}}} &\leq C\|\delta A_t\|_{L^2_T L^2}\|v^2\|_{L^3_T L^6} \\
&\leq C\|\nabla\delta v\|_{L^2_T L^2}\|v^2\|_{L^3_T L^6}, \\
\|\delta A v^2_t\|_{L^{\frac{6}{5}}_T L^{\frac{3}{2}}} &\leq \|\nabla\delta v^2\|_{L^2_T L^2}\|t^{\frac{1}{2}} v^2_t\|_{L^{\frac{6}{5}}_T L^6},
\end{aligned}$$

so, by the regularity provided in Theorem 1.1, the proof is concluded. ∎

Now we go back to estimate the terms I_3, I_4 in (2.60). Hölder's inequality and Lemma 2.3 provide that

$$\begin{aligned}
\int_0^T |I_3|\,dt &\leq C\|w_t\|_{L^{\frac{6}{5}}_T L^{\frac{3}{2}}}\|z\|_{L^6_T L^3} \leq c(T)\|\nabla\delta v\|_{L^2_T L^2}\|z\|^{\frac{2}{3}}_{L^\infty_T L^2}\|\nabla z\|^{\frac{1}{3}}_{L^2_T L^2} \\
&\leq c(T)\|\nabla\delta v\|^2_{L^2_T L^2} + \frac{1}{4}\|z\|^2_{L^\infty_T L^2} + \frac{\mu^m}{8}\|\nabla z\|^2_{L^2_T L^2}
\end{aligned}$$

and

$$\int_0^T |I_4|\,dt \leq \frac{\mu^m}{8}\|\nabla z\|^2_{L^2_T L^2} + c(T)\|\nabla\delta v\|^2_{L^2_T L^2}.$$

Hence, joining the estimates for I_1 to I_4 and going back to (2.60), we obtain that for small $T > 0$,

$$\sup_{t\in[0,T]}\|z\|^2_{L^2} + \int_0^T \|\nabla z\|^2_{L^2}\,dt \leq c(T)\int_0^T \|\nabla\delta v\|^2_{L^2}\,dt. \tag{2.63}$$

Recalling that $\delta v = w + z$ and the estimate for w (2.62), we obtain

$$\int_0^T \|\nabla\delta v\|^2_{L^2}\,dt \leq c(T)\int_0^T \|\nabla\delta v\|^2_{L^2}\,dt,$$

and hence we conclude (2.58), i.e. that for $T > 0$ small enough, $\|\nabla\delta v\|_{L^2_T L^2} = 0$. Plugging this back into (2.63) and (2.62) allows us to conclude that

$$v^1 \equiv v^2 \quad \text{on } [0,T]\times\mathbb{R}^2.$$

One can now go back to Eulerian coordinates, while the passage to arbitrary $T > 0$ follows from standard connectivity arguments. ∎

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