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Ergodic Properties of the Equilibrium Process Associated with Infinitely Many Markovian Particles

By

Tokuzo SHIGA* and Yoichiro TAKAHASHI**

§0. Introduction

Consider a system of independent identically distributed Markov processes which have an invariant measure λ. It is known that if each process starts from each point of a λ-Poisson point process at time zero, these particles are λ-Poisson distributed at every later time $t>0$ [1].

In the present paper we are concerned with *the ergodic properties of the stationary processes obtained from such a system of particles, which is called the equilibrium process.* Sinai's ideal gas model is a special example of the equilibrium processes [4]. In §1 we will give some preliminaries and the definition of the equilibrium process, and §2 is devoted to the study of *the ergodic properties* (*metrical transitivity, mixing properties and pure nondeterminism*) *of the equilibrium processes.* In §3 we will discuss *the Bernoulli property in the strong sense of the shift flow* $\{\Theta_t\}_{-\infty<t<\infty}$ defined in §1. The shift flow induced by the eqilibrium process is a factor flow of $\{\Theta_t\}$. In §4 we prove a central limit theorem. Finally the authors would like to express their hearty gratitude to Professor H. Tanaka for his valuable advice.

§1. Preliminaries

Let $(X, \mathscr{B}_X, \lambda)$ be a σ-finite measure space, and denote by $\mathscr{K}(X)$ a

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* Department of Mathematics, Nara Women's University, Nara.
** College of General Education, University of Tokyo, Tokyo.

family of all the counting measures on X, i.e. each element of $\mathscr{K}(X)$ is an integer-valued measure with a countable set as its support. $\mathscr{K}(X)$ is equipped with a σ-field $\mathscr{G}$ which is generated by $\{\rho\in\mathscr{K}(X)\colon \rho(A)=n\}$, $n\geqq 0$, $A\in\mathscr{B}_X$. An element ρ of $\mathscr{K}(X)$ is represent by $\rho=\sum_i\delta_{xi}$ where $\delta_x(A)=1$ if $x\in A$ and $\delta_x(A)=0$ if $x\notin A$.

Definition 1.1. *Let Π_λ be a probability measure on $(\mathscr{K}(X),\mathscr{G})$. Π_λ is λ-Poisson point process if it satisfies the following conditions;*

(1.1) *for any disjoint system $A_1,\dots,A_n$* of *$\mathscr{B}_X$ such that $\lambda(A_i)<+\infty_{i=1,\dots,n}$ $\rho(A_1),\dots,\rho(A_n)$ are independent random variables on $(\mathscr{K}(X),\mathscr{G},\Pi_\lambda)$ and* $\Pi_\lambda\{\rho;\rho(A_i)=n\}=\frac{[\lambda(A_i)]^n}{n!}\exp[-\lambda(A_i)]$, $i=1,\dots,n$.

Here we summarize some elementary facts on λ-Poisson point process.

Lemma 1.2.

(a) *For any σ-finite measure space $(X,\mathscr{B}_X,\lambda)$ there exists a λ-Poisson point process.*

(b) *A probability measure Π_λ on $(\mathscr{K}(X),\mathscr{G})$ is a λ-Poisson point process if and only if*

(1.2) $\int e^{-<\varphi,\rho>}\Pi_\lambda(d\rho)=e^{-<1-e^{-\varphi},\lambda>}$ [1] *for every non-negative measurable function φ on $(X,\mathscr{B}_X)$,*

and moreover (1.2) *is equivalent to the following condition;*

(1.3) $\int e^{i<\varphi,\rho>}\Pi_\lambda(d\rho)=e^{-<1-e^{i\varphi},\lambda>}$ *for every λ-integrable function φ.*

For each A of $\mathscr{B}_X$, denote by $\mathscr{G}(A)$ the σ-field generated by $\{\rho\in\mathscr{K}(X);\ \rho(B)=n\}$, $n\geqq 0$, $B\in\mathscr{B}_X$, $B\subset A$.

Lemma 1.3.

(a) *If $A_1,\dots,A_n$ are mutually disjoint, $\mathscr{G}(A_1),\dots,\mathscr{G}(A_n)$ are mutually independent σ-fields w.r.t. Π_λ.*

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1) For a function φ and a measure $\lambda<\varphi,\ \lambda>=\int\varphi(x)\lambda(dx)$.

(b) *If* $\{A_n\}\subset\mathscr{B}_X$ *is non-increasing and* $\bigcap_n A_n=\phi$, $\{\mathscr{G}(A_n)\}$ *is also non-increasing and* $\bigcap_n \mathscr{G}(A_n)=\{\phi,\mathscr{K}(X)\}$ (mod. Π_λ).

Next, we define the equilibrium processes associated with Markovian particles.

Let X be a locally compact separable Hausdorff space and $\mathscr{B}_X$ be the topological Borel field of X. Denote by $\mathscr{W}$ the path space of X, that is, each element of $\mathscr{W}$ is a X-valued right continuous function with left limit defined on $(-\infty,\infty)$, and define the shift operators $\{\theta_t\}_{-\infty<t<\infty}$ of $\mathscr{W}$ as usual; $(\theta_t f)_s=f_{t+s}$ for each f of $\mathscr{W}$.

Put $S=\mathscr{K}(X)$ and $\Omega=\mathscr{K}(\mathscr{W})$. Denote by $\{\Theta_t\}_{-\infty<t<\infty}$ the shift operators on Ω induced by the shift operators $\{\theta_t\}_{-\infty<t<\infty}$ on $\mathscr{W}$, i.e.

$$\Theta_t\omega=\sum_i\delta_{\theta_t f^i}\quad\text{if}\quad\omega=\sum\delta_{f^i}. \tag{1.4}$$

Define S-valued process $\{\xi_t(\omega)\}_{-\infty<t<\infty}$ on Ω as follows;

$$\xi_t(\omega)=\sum_i\delta_{f_t^i}\quad\text{if}\quad\omega=\sum_i\delta_{f^i}. \tag{1.5}$$

Then $\xi_t(\omega)$ is right continuous in t in a natural topology.

In our situation a motion of one particle is given as a Markov process on X and denote by $\{P_t(x,dy)\}$ its transition probabilities.

Assumption.

$\{P_t(x,dy)\}$ *is a conservative Feller Markov process and have a Radon invariant measure* λ, *that is,* $\{P_t(x,dy)\}$ *induces a semi-group of contraction operators* $\{T_t\}$ on $C_\infty(X)$, *and* $\int T_tf(x)\lambda(dx)=\int f(x)\lambda(dx)$ *for every* f *of* $C_0(X)$.[2)]

Under this assumption $\{T_t\}$ is, also, a semi-group of contraction operators on $L^2(X,\mathscr{B}_X,\lambda)$.

Lemma 1.4. *There is only one* σ*-finite measure* Q *on* $(\mathscr{W},\mathscr{B}_{\mathscr{W}})$[3)] *such that*

2) $C_\infty(X)$ is the family of all the continuous functions vanishing at infinity, and $C_0(X)$ is the family of all the continuous functions with compact supports.

3) $\mathscr{B}_{\mathscr{W}}$ is the σ-algebra generated by all the cylindrical subsets of $\mathscr{W}$.

(1.6) *for* $-\infty<t_1<t_2<\cdots<t_n<+\infty$ *and* $\{A_i\}_{i=1,2\ldots,n}$

$$Q[f; f_{t_1}\in A_1, f_{t_2}\in A_2,\ldots, f_{t_n}\in A_n]$$

$$=\int_{A_1}\lambda(dx_1)\int_{A_2}P_{t_2-t_1}(x_1, dx_2)\cdots\int_{A_n}P_{t_n-t_{n-1}}(x_{n-1}, dx_n).$$

In particular Q *is* $\{\theta_t\}$*-invariant.*

Denote by $\mathbb{B}$ the σ-field generated by $\{\omega\in\Omega;\ \omega(A)=n\}$, $n\geqq 0$, $A\in\mathscr{B}_X$ and put $\mathbb{P}=\Pi_Q$ (Q-Poisson point process). We consider $(\Omega,\mathbb{B},\mathbb{P})$ as our basic probability space.

Proposition 1.5. $\{\Omega,\mathbb{B},\mathbb{P};\ \{\xi_t\}_{-\infty<t<\infty}\}$ *is a right-continuous Markov stationary process with* Π_λ *as its absolute law.*

Proof. It is sufficient to prove the following formula;

(1.7) for $-\infty<t_1<t_2<\cdots<t_n<\infty$ and $\{\varphi_i\}\geqq 0$ measurable functions on X

$$\mathbb{E}[e^{-<\varphi_1,\xi_{t_1}>}\cdots e^{-<\varphi_n,\xi_t>}]=\mathbb{E}[e^{-<\varphi_1,\xi_{t_1}>}\cdots e^{-<\varphi_{n-1},\xi_{t_{n-1}}>}e^{<\log T_{t_n-t_{n-1}}e^{-\varphi_n},\xi_{t_{n-1}}>}].$$ [4)]

Put $\Phi(f)=\sum_{i=1}^{n}\varphi_i(f_{t_i})$.
The left-hand side of (1.7) $=\mathbb{E}[e^{-<\Phi,\omega>}]=\exp-<1-e^{-\Phi}, Q>$

$$=\exp-\int\lambda(dx_1)\int P_{t_2-t_1}(x_1, dx_2)\cdots\int P_{t_n-t_{n-1}}(x_{n-1}, dx_n)[1-e^{-\sum_1^n\varphi_i(x_i)}]$$

$$=\exp-\int\lambda(dx_1)\int P_{t_2-t_1}(x_1, dx_2)\cdots$$

$$\cdots\int P_{t_{n-1}-t_{n-2}}(x_{n-2}, dx_{n-1})[1-e^{-\sum_1^{n-1}\varphi_i(x_i)}T_{t_n-t_{n-1}}e^{-\varphi_n}(x_{n-1})]$$

$$=\mathbb{E}[e^{-<\varphi_1,\xi_{t_1}>}\cdots e^{-<\varphi_{n-1}-\log T_{t_n-t_{n-1}}e^{-\varphi_n},\xi_{t_{n-1}}>}]$$

$=$the right-hand side of (1.7).

In particular $\mathbb{E}[e^{-<\varphi,\xi_t>}]=\exp-<1-e^{-\varphi}, \lambda>$.

Definition 1.6. *The Markov stationary process* $(\Omega,\mathbb{B},\mathbb{P};\ \{\xi_t\}_{-\infty<t<\infty})$

4) $\mathbb{E}$ denotes the expectation by $\mathbb{P}$.

is called the equilibrium process associated with $[\{T_t\}, \lambda]$.

The following calculations are immediate from (1.7).

Proposition 1.7.

(i) $\mathbb{E}[e^{-<\varphi, \xi_t>} | \xi_s] = e^{<\log T_{t-s} e^{-\varphi}, \xi_s>}$ *for* $\forall \varphi \geqq 0$ *and* $s < t$.

(ii) $\mathbb{E}[<\varphi, \xi_t> | \xi_s] = <T_{t-s}\varphi, \xi_s>$ *for* $\forall \varphi \in L^2(X, \mathscr{B}_X, \lambda)$.

(iii) $\mathbb{E}[<\varphi, \xi_t><\psi, \xi_t> | \xi_s] = <T_{t-s}\varphi, \xi_s><T_{t-s}\psi, \xi_s>$
$+ <T_{t-s}(\varphi\psi), \xi_s> - <T_{t-s}\varphi \cdot T_{t-s}\psi, \xi_s>$
for $\forall \varphi, \psi \in L^2(X, \mathscr{B}_X, \lambda) \cap L^1(X, \mathscr{B}_X, \lambda)$.

§2. Ergodic Properties

In this section we discuss the ergodic properties of the equilibrium processes.

Proposition 2.1. *The following* (i)~(iii) *are equivalent.*

(i) $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty<t<\infty})$ *is metrically transitive.*

(ii) $\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_K P_s(x, K)\lambda(dx)ds = 0$ *for every compact subset* K *of* X.

(iii) $\lim_{t\to\infty} \frac{1}{t}\int_0^t (T_s f, g)_{L^2(\lambda)} ds = 0$ *for all* f *and* g *of* $L^2(X, \lambda)$

Proof. It is easy to show the equivalence of (ii) and (iii). Moreover (i) is equivalent to

$$(2.1) \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}[e^{-\sum_1^n <\varphi_i, \xi_{s_i}>} e^{-\sum_1^m <\psi_j, \xi_{r_j+u}>}] du$$
$$= \mathbb{E}[e^{-\sum_1^n <\varphi_i, \xi_{s_i}>}] \mathbb{E}[e^{-\sum_1^m <\psi_j, \xi_{r_j}>}],$$

for any $-\infty < s_1 < s_2 < \cdots < s_n < \infty$, $-\infty < r_1 < r_2 < \cdots < r_m$ and any $\{\varphi_i\}_{i=1,\dots,n}$, $\{\psi_j\}_{j=1,\dots,m}$ of $C_0^+(X)$.[5]

However it suffices to prove (2.1) for $m=1$ because of the Markov property of $\{\xi_t\}$. For $s_n < r+u$,

5) $C_0^+(X)$ is the family of non-negative elements of $C_0(X)$.

$$\boldsymbol{E}[e^{-\sum_1^n\langle\xi_i,\xi_{s_i}\rangle}e^{-\langle\phi,\xi_{r+u}\rangle}]$$

$$=\exp-\int\lambda(dx)(1-e^{-\varphi_1}T_{s_2-s_1}e^{-\varphi_2}\cdots e^{-\varphi_n}T_{r+u-s_n}e^{-\phi})$$

$$=\exp-\int\lambda(dx)((1-e^{-\varphi_1}T_{s_2-s_1}e^{-\varphi_2}\cdots T_{s_n-s_{n-1}}e^{-\varphi_n})$$

$$+e^{-\varphi_1}T_{s_2-s_1}e^{-\varphi_2}\cdots T_{s_n-s_{n-1}}e^{-\varphi_n}T_{r+u-s_n}(1-e^{-\phi}))$$

$$=\boldsymbol{E}[e^{-\sum_1^n\langle\varphi_i,\xi_{s_i}\rangle}]\boldsymbol{E}[e^{-\langle\phi,\xi_r\rangle}]\times$$

$$\exp\int\lambda(dx)[I-e^{-\varphi_1}T_{s_2-s_1}\cdots T_{s_n-s_{n-1}}e^{-\varphi_n}]T_{r+u-s_n}(1-e^{-\phi}).$$

Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\boldsymbol{E}[e^{-\sum_{i=1}^n\langle\varphi_i,\xi_{s_i}\rangle}e^{-\langle\phi,\xi_{r+u}\rangle}]du=\boldsymbol{E}[e^{-\sum_1^n\langle\varphi_i,\xi_{s_i}\rangle}]\boldsymbol{E}[e^{-\langle\phi,\xi_r\rangle}]\tag{2.2}$$

is equivalent to

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\exp\left[\int\lambda(dx)[I-e^{-\varphi_1}T_{s_2-s_1}e^{-\varphi_2}\cdots T_{s_n-s_{n-1}}e^{-\varphi_n}]\times T_u(1-e^{-\phi})\right]du=1\tag{2.3}$$

or

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\left[\int\lambda(dx)[I-e^{-\varphi_1}T_{s_2-s_1}e^{-\varphi_2}\cdots T_{s_n-s_{n-1}}e^{-\varphi_n}]T_u(1-e^{-\phi})\right]du=0.\tag{2.4}$$

Note $1-e^{-\varphi_1}T_{s_2-s_1}e^{-\varphi_2}\cdots T_{s_n-s_{n-1}}e^{-\varphi_n}\in L^2(X,\lambda)$ and $(1-e^{-\phi})\in L^2(X,\lambda)$. Hence (iii) implies (i). On the other hand it is obvious (2.4) implies (ii) by putting $n=1$, $\varphi_1=\phi\in C_0^+(X)$.

Corollary 2.2. *If $\{\xi_t\}_{-\infty<t<\infty}$ is metrically transitive, then $\lambda(X)=\infty$.*

Proposition 2.3. *The following three statements are equivalent.*

(i) *$(\Omega,\boldsymbol{B},\boldsymbol{P};\{\xi_t\}_{-\infty<t<\infty})$ has the mixing property.*

(ii) $\lim_{t\to\infty}\int_K \lambda(dx)P_t(x, K)=0$ *for every compact subset K of X.*

(iii) $\lim_{t\to\infty}(T_t f, g)_{L^2(X,\lambda)}=0$ *for all f and g of $L^2(X, \lambda)$.*

Since the proof of Proposition 2.3 can be carried out by the similar method as Proposition 2.1, it is omitted.

Next, we consider the pure non-determinism of the equilibrium processes. In general, let $(\Omega, \mathcal{F}, P; \{z_t\}_{-\infty<t<\infty})$ be a Markov stationary process on X associated with $\{P_t(x, dy), \mu(dx)\}$, where $P_t(x, dy)$ is a transition probability measure and μ is an invariant probability measure. Then the following criterion for the pure non-determinism is applicable.

Lemma 2.4. *$(\Omega, \mathcal{F}, P; \{z_t\}_{-\infty<t<\infty})$ is purely non-deterministic i.e. $\bigcap_t \mathcal{F}_t(z)=\{\phi, \Omega\}$ (mod. P) where $\mathcal{F}_t(z)$ is the σ-field generated by $\{z_s; s\leqq t\}$, if and only if*

$$\lim_{t\to\infty}\int\Big[\int P_t(x, dy)f(y)-\int\mu(dy)f(y)\Big]^2\mu(dx)=0 \quad \text{for } {}^\forall f\in L^2(X, \mu). \tag{2.5}$$

This lemma can be found in [6].

Proposition 2.5. *The following three statements are equivalent.*

(i) $(\Omega, \boldsymbol{B}, \boldsymbol{P}; \{\xi_t\}_{-\infty<t<\infty})$ *is purely non-deterministic.*

(ii) $\lim_{t\to\infty}\int_X \lambda(dx)[P_t(x, K)]^2=0$ *for every compact subset K of X.*

(iii) $\lim_{t\to\infty}\|T_t f\|_{L^2(X,\lambda)}=0$ *for every f of $L^2(X, \lambda)$.*

Proof. By Lemma 2.4 (i) is equivalent to

$$\lim_{t\to\infty}\boldsymbol{E}[(\boldsymbol{E}[e^{-<\varphi,\xi_t>}|\xi_0]-\boldsymbol{E}[e^{-<\xi_0,\varphi>}])^2]=0 \qquad \text{for } {}^\forall\varphi\in C_0^+(X). \tag{2.6}$$

Using Proposition 1.7 and Poisson properties, we have

$$\boldsymbol{E}[(\boldsymbol{E}[e^{-<\varphi,\xi_t>}|\xi_0]-\boldsymbol{E}[e^{-<\xi_0,\varphi>}])^2]=\boldsymbol{E}[(e^{<\log T_t e^{-\varphi},\xi_0>}-e^{-<1-e^{-\varphi},\lambda>})^2]$$

$$=\boldsymbol{E}[e^{<\log(T_t e^{-\varphi})^2,\xi_0>}-e^{-2<1-e^{-\varphi},\lambda>}]=e^{-<1-(T_t e^{-\varphi})^2,\lambda>}-e^{-2<1-e^{-\varphi},\lambda>}$$

$$=(e^{<(T_t(1-e^{-\varphi}))^2,\lambda>}-1)e^{-2<1-e^{-\varphi},\lambda>}.$$

In the last equality we used $<T_t(1-e^{-\varphi}), \lambda>=<1-e^{-\varphi}, \lambda>$. Therefore the equivalence of (2.6) and the statement (ii) is obvious. Moreover the equivalence of (ii) and (iii) is trivial.

Proposition 2.6. *$(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty<t<\infty})$ is purely non-deterministic if and only if*

$\mathbb{E}[\xi_t|\xi_0]$ converges to λ vaguely in probability, i.e. for every φ of $C_0(X)$ $\mathbb{E}[<\varphi, \xi_t>|\xi_0]$ converges to $<\varphi, \lambda>$ in probability.

Proof. By Proposition 1.7 and Poisson properties we have

$$\mathbb{E}[(\mathbb{E}[<\varphi, \xi_t>|\xi_0]-<\varphi, \lambda>)^2]=<(T_t\varphi)^2, \lambda>=\|T_t\varphi\|^2_{L^2(X,\lambda)}.$$

Therefore Proposition 2.6 follows from Proposition 2.5.

Remark 2.7. *If $\{T_t\}$ has no finite invariant measure, the corresponding equilibrium process is metrically transitive.*

Remark 2.8. *If the equilibrium process associated with $[\{T_t\}, \lambda]$ is metrically transitive and $\{T_t\}$ are symmetric on $L^2(X, \lambda)$, then it is purely non-deterministic.*

Remark 2.9. *The equilibrium process associated with uniform motions on R^n are mixing, but not purely non-deterministic. However the equilibrium processes associated with all the additive processes on R^n except uniform motions are purely non-deterministic.*

Remark 2.10. *Let $(\Omega, \mathscr{F}, P_x, \{x_t\}_{t\geq 0})$ be a Hunt Markov process corresponding to $\{T_t\}$. If the equilibrium process associated with $[\{T_t\}, \lambda]$ is metrically transitive, for almost all x (w.r.t. λ) and any compact subset K, $P_x[\omega; \tau_{K^c}<+\infty]=1$, where τ_{K^c} denote the first hitting time for K^c.*

§3. The Bernoulli Property of the Shifs Flow

It is easy to see that $\{\Theta_t\}_{-\infty<t<\infty}$, which is defined by (1.4) in §1, is a flow on the probability space $(\Omega=\mathscr{K}(\mathscr{W}), \mathbb{B}, \mathbb{P}=\Pi_Q)$. So, we discuss the Bernoulli property in the strong sense of the flow $\{\Theta_t\}_{-\infty<t<\infty}$.

Definition 3.1. *$(\Omega, \mathbb{B}, \mathbb{P}; \{\Theta_t\}_{-\infty<t<\infty})$ is called Bernoulli flow if it* *satisfies the following conditions*;

(3.1) *There exists a system of σ-subfields $\{\zeta_r^s\}$, $r<s$, of $\mathbb{B}$ which satisfies*

(i) $\Theta_t \cdot \zeta_r^s = \zeta_{r+t}^{s+t}$ *for every $r<s$ and t,*

(ii) $\zeta_r^t = \zeta_r^s \vee \zeta_s^t$ *for $r<s<t$,*

(iii) ζ_r^s *and* ζ_s^t *are mutually independent for $r<s<t$,*

(iv) $\bigvee_{r<s} \zeta_r^s = \mathbb{B}$ (mod. $\mathbb{P}$)

The following lemma is essentially due to H. Tanaka, and is a generalization of the Sinai-Volkoviskii's result on the K-property of the ideal gas model [4].

Lemma 3.2. *Suppose that there exists a real measurable function $\tau(f)$ on the σ-finite measure space $(\mathcal{W}, \mathcal{B}_{\mathcal{W}}, Q)$ such that for almost all $f(Q)$*

(a) $-\infty < \tau(f) < +\infty$

(b) $\tau(f) = t + \tau(\theta_t f)$ *for all t of R^1.*[6)]

Then, $(\Omega, \mathbb{B}, \mathbb{P}; \{\Theta_t\}_{-\infty<t<\infty})$ is a Bernoulli flow.

Proof. May assume every f of $\mathcal{W}$ satisfies the conditions (a), (b). Put $\mathcal{W}_r^s = \{f;\ -r \geqq \tau(f) > -s\}$. Then $\theta_t \mathcal{W}_r^s = \{f;\ -r-t \geqq \tau(f) > -s-t\}$ by the condition (b). Obviously we have

$$\theta_t \mathcal{W}_r^s = \mathcal{W}_{r+t}^{s+t}, \qquad \bigcup_{r<s} \mathcal{W}_r^s = \mathcal{W} \tag{3.2}$$

So, we denote by ζ_r^s the σ-subfield $\mathcal{G}(\mathcal{W}_r^s)$ which is generated by $\{\omega;\ \omega(A) = n\}$, $n \geqq 0$, $A \in \mathcal{B}_{\mathcal{W}}$, $A \subset \mathcal{W}_r^s$. Noting $\Theta_t\{\omega;\ \omega(A) = n\} = \{\omega;\ \omega(\theta_t A) = n\}$, we can see $\zeta_{r+t}^{s+t} = \Theta_t \cdot \zeta_s^r = \mathcal{G}(\theta_t \mathcal{W}_r^s)$. Therefore ζ_r^s satisfies the conditions (i)$\sim$(iv) in Definiton 3.1 by Lemma 1.3.

Proposition 3.3. *Suppose that $\{T_t\}$ is transient in following sense; $\int_0^\infty (T_t\varphi, \varphi)_{L^2(X,\lambda)} dt < +\infty$ for every φ of $C_0^+(X)$. Then, $(\Omega, \mathbb{B}, \mathbb{P}; \{\Theta_t\})$ is a Bernoulli flow.*

Proof. First, we will show

6) Such a random time $\tau(w)$ is called L-time which was introduced by M. Nagasawa [7].

(3.3) $\int_{-\infty}^{\infty}\varphi(f_s)\,ds<+\infty$ for almost all f (Q) for every φ of $C_0^+(X)$.

For any φ and ψ of $C_0^+(X)$,

$$\int_{\mathscr{W}}\left[\int_{-\infty}^{\infty}\varphi(f_s)\,ds\cdot\psi(f_0)\right]Q(df)=\int_{\mathscr{W}}\left[\int_0^{\infty}\varphi(f_s)\,ds\ \ \psi(f_0)\right]Q(df)$$

$$+\int_{\mathscr{W}}\left[\int_{-\infty}^{0}\varphi(f_s)\,ds\ \ \psi(f_0)\right]Q(df)$$

$$=\int_0^{\infty}(T_s\varphi,\ \psi)_{L^2(X,\lambda)}\,ds+\int_0^{\infty}(\varphi,\ T_s\psi)_{L^2(X,\lambda)}\,ds<+\infty$$

Therefore (3.3) holds.

Next, choose a countable sequence $\{\varphi_n\}$ of $C_0^+(X)$ such that $\bigcup_n\{x\in X;\ \varphi_n(x)>0\}=X$. Putting $\mathscr{W}_1=\{f;\,0<\int_{\infty-}^{\infty}\varphi_1(f_s)\,ds<+\infty\}$, $\theta_t\mathscr{W}_1=\mathscr{W}_1$. And define $\mathscr{W}_{n+1}$ by $\{f;\,0<\int_{-\infty}^{\infty}\varphi_{n+1}(f_s)\,ds<+\infty\}\backslash\mathscr{W}_n$. Thus we have a sequence of disjoint subsets of $\mathscr{W}$ which are $\{\theta_t\}$-invariant. So define $\tau(f)=\sup\{t;\int_{-\infty}^{t}\varphi_n(f_u)\,du\leqq\frac{1}{2}\int_{-\infty}^{\infty}\varphi_n(f_u)\,du\}$ if $f\in\mathscr{W}_n$. Then we have $\{f;\ -\infty<\tau(f)<+\infty\}=\bigcup_n\mathscr{W}_n=\mathscr{W}$ (mod. Q), and if $f\in\mathscr{W}_n$ $\theta_sf\in\mathscr{W}_n$ and $\tau(\theta_sf)=\sup\{t;\ \int_{-\infty}^{t}\varphi_n(f_{u+s})\,du\leqq\frac{1}{2}\int_{-\infty}^{\infty}\varphi_n(f_u)\,du\}=\tau(f)-s$. Therefore $\tau(f)$ satisfies the conditions of Lemma 3.2.

Remark 3.4. *The equilibrium process* $\{\xi_t\}$ *induces a factor flow of* $\{\Theta_t\}$. *Since a Bernoulli flow* $\{\Theta_t\}$ *in the sense of* (3.1) *is a Bernoulli flow in the weak sense (i.e. the automorphism* Θ_t *is Bernoulli for each* $t\neq0$), *the shift flow induced by* $\{\xi_t\}$ *is also a Bernoulli flow in the weak sense by the theorem of Ornstein.* [2]. *But, perhaps, it may be a Bernoulli flow in the sense of* (3.1).

Remark 3.5. *In the ideal gas model of Sinai-Volkoviskii* [4], *the path space* $\mathscr{W}$ *is identified to* $R^n\times R^n$ and ζ_r^s *in* (3.1) *is the* σ*-algebra generated by the functions* $\omega\rightsquigarrow\omega(E)$, $E\subset V_r^s$, *where* $V_r^s=\{f=(q_1,\dots,q_n,v_1,\dots,v_n)\in R^n\times R^n\mid -r\geqq\sum_1^n q_iv_i\geqq -s\}$. *In this case the function* $\tau(f)$ *in Lemma* 3.2 *is given by the last exit time of the set* V_0^∞ *for each* $f\in R^n\times R^n$.

§4. A Central Limit Theorem

Finally we will prove a central limit theorem related to the equilibrium process. Denote by $G\varphi(x)=\int_0^\infty T_t\varphi(x)dt$ if the integral is well-defined.

Proposition 4.1. *Consider any function* $\varphi\in L^2(X, \lambda)$ *which satisfies* $(G|\varphi|, |\varphi|)_{(L^2X,\lambda)}<+\infty$, *and* $(G(|\varphi|G|\varphi|), |\varphi|)_{L^2(X,\lambda)}<+\infty$.
Then

$$\lim_{t\to\infty} P\left[\omega ; \alpha<\frac{\int_0^t <\varphi, \xi_s> ds-t<\varphi, \lambda>}{\sqrt{2(\varphi, G\varphi)_{L^2(\lambda)}\cdot t}}<\beta\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-\frac{x^2}{2}}dx \quad for \ {}^\forall\alpha<{}^\forall\beta.$$

Proof. It suffices to show

$$\lim_{t\to\infty} E\left[\exp iz\frac{1}{\sqrt{t}}\left(\int_0^t <\varphi, \xi_s> ds-t<\varphi, \lambda>\right)\right] \tag{4.1}$$

$$=\exp(-z^2(\varphi, G\varphi)_{L^2(X,\lambda)}).$$

$\Phi(f)=\int_0^t \varphi(f_s)ds$ is a function on $\mathscr{W}$ and satisfies $<\Phi, \omega>=\int_0^t <\varphi, \xi_s(\omega)> ds$. $E[\exp iz\int_0^t <\varphi, \xi_s> ds]=E[e^{iz<\Phi,\omega>}]=\exp\int_{\mathscr{W}}(e^{iz\Phi(f)}-1)Q(df)$ by Lemma 1.2. Thus we have

$$E\left[\exp iz\frac{1}{\sqrt{t}}\left(\int_0^t <\varphi, \xi_s> ds-t<\varphi, \lambda>\right)\right] \tag{4.2}$$

$$=\exp\int_{\mathscr{W}}(e^{\frac{iz}{\sqrt{t}}\Phi(f)}-1-iz\sqrt{t}<\varphi, \lambda>)Q(df).$$

Noting $e^{ix}-1-ix-\frac{(ix)^2}{2}=O(|x|^3)$,

$$\int_{\mathscr{W}}(e^{\frac{iz}{\sqrt{t}}\Phi(f)}-1-iz\sqrt{t}<\varphi, \lambda>)Q(df) \tag{4.3}$$

$$=\left(\frac{iz}{\sqrt{t}}\int_{\mathscr{W}}\Phi(f)Q(df)-iz\sqrt{t}<\varphi,\lambda>\right)+\frac{1}{2}\int_{\mathscr{W}}\left(\frac{iz}{\sqrt{t}}\Phi(f)\right)^2Q(df)$$

$$+O\left(\left|\int_{\mathscr{W}}\left(\frac{z}{\sqrt{t}}\Phi(f)\right)^3Q(df)\right|\right).$$

The first term vanishes because of $\int_{\mathscr{W}}\Phi(f)Q(df)=\int_0^t\left[\int_{\mathscr{W}}\varphi(f_s)Q(df)\right]ds$ $=t<\varphi,\lambda>$.

$$\int_{\mathscr{W}}\Phi(f)^2Q(df)=\int_0^t du\int_0^t ds\int_{\mathscr{W}}\varphi(f_s)\varphi(f_u)Q(df)$$

$$=2\int_0^t ds\int_0^s du\int_{\mathscr{W}}\varphi(f_s)\varphi(f_u)Q(df)$$

$$=2\int_0^t ds\int_0^s dv(\varphi, T_v\varphi).$$

Therefore the second term of the right-hand side of (4.3) converges to $-z^2(G\varphi,\varphi)_{L^2(X,\lambda)}$. By the similar calculation,

$$\left|\int\Phi(f)^3Q(df)\right|=6\left|\int_0^t\int_0^s\int_0^{s-u}(T_v\varphi T_u\varphi,\varphi)_{L^2(\lambda)}du\,dv\,ds\right|$$

$$\leqq 6t(G(|\varphi|G|\varphi|),|\varphi|)_{L^2(\lambda)}.$$

Therefore the third term converges to zero. Thus, we can complete the proof of Proposition 4.1.

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