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Boundary blow-up solutions of cooperative systems

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Abstract

We study the existence, uniqueness and boundary profile of nonnegative boundary blow-up solution to the cooperative system

$$\begin{cases} \Delta u = g(u-v) & \text{in } \Omega, \\ \Delta v = f(v-\beta u) & \text{in } \Omega, \\ u = v = \infty & \text{on } \partial\Omega \end{cases}$$

in a smooth bounded domain of $\mathbb{R}^N$, where f, g are nondecreasing, nonnegative C^1 functions vanishing in $(-\infty, 0]$ and $\beta > 0$ is a parameter.

Keywords: Cooperative system; Boundary blow-up; Keller–Osserman condition

1. Introduction

For a single equation of the form $\Delta u = f(u)$, $f \geqslant 0$, the existence of solutions which blow up at the boundary of a smoothly bounded domain Ω is equivalent to a growth condition on f known as the Keller–Osserman condition (see [15,17] as well as [9]):

$$\int^{\infty} \frac{dt}{\sqrt{F(t)}} < \infty, \tag{1}$$

where $F(t) = \int_0^t f(s)\,ds$.

If in addition f is nondecreasing, some boundary blow-up solution (BBUS) is maximal: it dominates all other solutions of the equation. In particular, interior uniform estimates can be derived for *any* solution of the original equation, independently of its boundary values.

The current paper is an attempt at generalizing the above theory to autonomous systems of semilinear elliptic equations. Since blow-up solutions are strongly related to the Maximum Principle, it is natural to consider the case of cooperative systems first. Failing short of a theory for general cooperative systems (see Remark 2.4 for further discussion), we study systems of the form

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$$\begin{cases} \Delta u = g(u-v) & \text{in } \Omega, \\ \Delta v = f(v-\beta u) & \text{in } \Omega, \\ u = v = \infty & \text{on } \partial\Omega \end{cases} \tag{2}$$

where Ω is a smoothly bounded domain of Euclidean space, f, g are nondecreasing, nonnegative C^1 functions such that $f = g = 0$ on $\mathbb{R}^-$ and $\beta > 0$ is a parameter. Solutions are sought in the class $C^2(\Omega)$ and the boundary condition is to be understood as

$$\lim_{x\to x_0} u(x) = \lim_{x\to x_0} v(x) = +\infty \quad \text{for all } x_0 \in \partial\Omega.$$

Boundary blow-up solutions of cooperative systems have been considered in [12] (with different nonlinearities than the ones treated here) and some examples of competitive systems have already been studied in [10,11]. Boundary blow-up solutions in different cooperative, competitive or predator–prey systems arise in problems with "refuge", that is, in nonhomogeneous systems where one of the coefficients vanishes on a subset of the domain, see [5,7,8,16]. For yet another type of systems with large solutions, see [6].

We study existence, first order asymptotics and uniqueness of solutions of (2) respectively in Sections 2, 3, 4. In Section 5, we study in more detail a list of relevant examples. Here is a summary of our main results.

Theorem 1.1. *Let Ω denote a smoothly bounded domain of Euclidean space, $f, g : \mathbb{R} \to \mathbb{R}$ nondecreasing, nonnegative C^1 functions such that $f = g = 0$ on $\mathbb{R}^-$ and $\beta > 0$. There exists a solution of the system* (2) *if and only if the following three conditions hold*

- *f satisfies the Keller–Osserman condition* (1)*,*
- *g satisfies the Keller–Osserman condition* (1)*,*
- *$\beta < 1$.*

The asymptotics of solutions is obtained at the price of a technical assumption on the nonlinearities commonly found in the literature (see e.g. [1]). More precisely, let

$$\phi(u) = \int_u^\infty \frac{dt}{\sqrt{2F(t)}},$$

where

$$F(t) = \int_0^t f(s)\,ds.$$

We assume in what follows that f satisfies

$$\liminf_{t\to\infty} \frac{\phi(at)}{\phi(t)} > 1 \quad \forall a \in (0,1). \tag{3}$$

Examples are given by $f(u) = e^u$ or $f(u) = u^p$, $p > 1$. A counter-example is given by $f(u) = u\,(\ln(1+u))^{2p}$, $p > 1$.

For $0 < \beta < 1$ and $\beta < \theta \leqslant 1$, we let $w_\theta > 0$ denote the minimal solution to

$$\begin{cases} \Delta w_\theta = f(\frac{\theta-\beta}{\theta} w_\theta) & \text{in } \Omega, \\ w_\theta = +\infty & \text{on } \partial\Omega \end{cases} \tag{4}$$

(see e.g. [9] for the notion of minimal blow-up solution). Then, we have the following result.

Theorem 1.2. *Make the same assumptions as in Theorem* 1.1*. Assume in addition that f satisfies* (3)*.*

(a) *If f is smaller than g at infinity in the sense that for any $\varepsilon > 0$,*

$$\lim_{t\to+\infty} \frac{f(t)}{g(\epsilon t)} = 0 \tag{5}$$

then for any solution (u, v) *of the system* (2)

$$\lim_{x\to\partial\Omega} \frac{u}{w_1} = 1, \qquad \lim_{x\to\partial\Omega} \frac{v}{w_1} = 1, \tag{6}$$

where w_1 *is the minimal nonnegative solution to* (4) *with* $\theta = 1$.

(b) *If* f *is of the order of* g *at infinity in the sense that for some* $\theta_0 \in (\beta, 1)$,

$$\begin{aligned} &\text{if } \theta \in (\beta, \theta_0), \text{ then} \quad \liminf_{t\to+\infty} \theta \frac{g((1-\theta)t)}{f((\theta-\beta)t)} \geqslant 1, \\ &\text{if } \theta \in (\theta_0, 1), \text{ then} \quad \limsup_{t\to+\infty} \theta \frac{g((1-\theta)t)}{f((\theta-\beta)t)} \leqslant 1 \end{aligned} \tag{7}$$

then for any solution (u, v) *of the system* (2)

$$\lim_{x\to\partial\Omega} \frac{u}{w_{\theta_0}} = \frac{1}{\theta_0}, \qquad \lim_{x\to\partial\Omega} \frac{v}{w_{\theta_0}} = 1, \tag{8}$$

where w_{θ_0} *is the minimal solution to* (4) *with* $\theta = \theta_0$.

Remark 1.3. Condition (7) looks rather unpleasant. Nevertheless, its validity can be easily checked on examples. If e.g. $f(t) = g(t) = t^p$, by the Intermediate Value Theorem there exists $\theta_0 \in (\beta, 1)$ such that $\lim_{t\to+\infty} \theta_0 \frac{g((1-\theta_0)t)}{f((\theta_0-\beta)t)} = \theta_0 \frac{(1-\theta_0)^p}{(\theta_0-\beta)^p} = 1$. Since the quantity $\theta \frac{g((1-\theta)t)}{f((\theta-\beta)t)}$ is nonincreasing in θ, (7) follows. If $f(t) = g(t) = e^t$, then letting $\theta_0 = (1+\beta)/2$, we have $\lim_{t\to+\infty} \theta \frac{g((1-\theta)t)}{f((\theta-\beta)t)} = +\infty$ for $\theta < \theta_0$, while the limit is equal to 0 if $\theta > \theta_0$. Observe that in this case, though (7) holds, there is no value of θ for which the limit is equal to 1.

Remark 1.4. Note that when condition (5) holds, the first order asymptotics of both u and v is independent of the nonlinearity g. The influence of g can be detected in the next terms of the asymptotic expansion of the solution. See Example 1 in Section 5.

On the contrary, when condition (7) holds, f and g already interplay in the value of the constant θ_0 of the leading asymptotics of u and v. See Example 2 in Section 5.

As a consequence of Theorem 1.2, we obtain under an extra convexity assumption:

Corollary 1.5. *Make the same assumptions as in Theorem* 1.1. *Assume in addition that* f *satisfies* (3) *and either condition* (5) *or* (7) *holds.*

(a) *If* f *and* g *are convex, then the system* (2) *has a unique solution.*
(b) *If* Ω *is a ball and* $\frac{f(t)}{t}$, $\frac{g(t)}{t}$ *are nondecreasing in a neighborhood of* $+\infty$, *then the system* (2) *has a unique solution.*

Remark 1.6. Even in the case of the ball, we do not know whether uniqueness remains true under the sole assumption that f and g are nondecreasing.

Notation. For functions $m, n : [0, \infty) \to [0, \infty)$ we say that $m \sim n$ at infinity if

$$\lim_{t\to\infty} \frac{m(t)}{n(t)} = 1$$

and use similar notation when m, n are defined near t_0 and $\lim_{t\to t_0} \frac{m(t)}{n(t)} = 1$.

2. Existence

The proof of the existence of solutions in Theorem 1.1 follows a standard scheme where one first solves the system with a finite boundary condition m and then lets $m \to +\infty$. The former step can be carried out in a more general setting as described next. Consider the system

$$\begin{cases} \Delta u = \mathsf{g}(u, v) & \text{in } \Omega, \\ \Delta v = \mathsf{f}(v, u) & \text{in } \Omega, \\ u = v = \infty & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where f and g are two nonnegative C^1 functions such that $\mathsf{f}(0,0) = \mathsf{g}(0,0) = 0$ and $\partial\mathsf{g}/\partial v \leqslant 0$, $\partial\mathsf{f}/\partial u \leqslant 0$ (the system is then called cooperative).

Proposition 2.1. *Given $m > 0$ the system*

$$\begin{cases} \Delta u = \mathsf{g}(u, v) & \text{in } \Omega, \\ \Delta v = \mathsf{f}(u, v) & \text{in } \Omega, \\ u = v = m & \text{on } \partial\Omega \end{cases} \tag{10}$$

admits a unique minimal nonnegative solution (u, v).

In the previous statement minimality refers to the following property: take any open set $\omega \subseteq \Omega$ and $\bar{u}, \bar{v} \in C^2(\bar{\omega})$ satisfying

$$\begin{cases} \Delta \bar{u} \leqslant \mathsf{g}(\bar{u}, \bar{v}) & \text{in } \omega, \\ \Delta \bar{v} \leqslant \mathsf{f}(\bar{u}, \bar{v}) & \text{in } \omega, \\ \bar{u} \geqslant 0,\ \bar{v} \geqslant 0 & \text{in } \omega, \\ \bar{u} \geqslant u,\ \bar{v} \geqslant v & \text{on } \partial\omega. \end{cases} \tag{11}$$

Then,

$$u \leqslant \bar{u}, \quad v \leqslant \bar{v} \quad \text{in } \omega.$$

To solve the system with finite boundary values we use sub and supersolutions. A convenient reference for monotone methods for equations and systems is [18].

Proof. Choose $a > 0$, $b > 0$ sufficiently large such that the functions

$$u \mapsto \mathsf{g}(u, v) - au, \quad v \mapsto \mathsf{f}(u, v) - bv \quad \text{are decreasing for } 0 \leqslant u, v \leqslant m. \tag{12}$$

Define

$$u_0 \equiv 0, \qquad v_0 \equiv 0 \tag{13}$$

and for $k \geqslant 1$

$$\begin{cases} \Delta u_k - au_k = \mathsf{g}(u_{k-1}, v_{k-1}) - au_{k-1} & \text{in } \Omega, \\ \Delta v_k - bv_k = \mathsf{f}(u_{k-1}, v_{k-1}) - bv_{k-1} & \text{in } \Omega, \\ u_k = v_k = m & \text{on } \partial\Omega. \end{cases} \tag{14}$$

We claim that

$$0 \leqslant u_{k-1} \leqslant u_k \leqslant m \quad \text{in } \Omega$$

and

$$0 \leqslant v_{k-1} \leqslant v_k \leqslant m \quad \text{in } \Omega.$$

Indeed, the property is straightforward if $k = 1$. Take $k \geqslant 2$ and assume by induction that $u_{k-2} \leqslant u_{k-1}$, $v_{k-2} \leqslant v_{k-1}$ in Ω. Then,

$$\begin{aligned} \Delta(u_k - u_{k-1}) - a(u_k - u_{k-1}) &= \mathsf{g}(u_{k-1}, v_{k-1}) - \mathsf{g}(u_{k-2}, v_{k-2}) - a(u_{k-1} - u_{k-2}) \\ &\leqslant \mathsf{g}(u_{k-1}, v_{k-2}) - \mathsf{g}(u_{k-2}, v_{k-2}) - a(u_{k-1} - u_{k-2}) \\ &\leqslant 0 \quad \text{in } \Omega \end{aligned}$$

and hence $u_k - u_{k-1} \geqslant 0$ in Ω. The remaining inequalities are obtained similarly. In particular, the limits

$$u = \lim_{k\to\infty} u_k, \qquad v = \lim_{k\to\infty} v_k$$

exist, solve (10) and satisfy

$$0 \leqslant u \leqslant m, \quad 0 \leqslant v \leqslant m \quad \text{in } \Omega.$$

Let us show now that the solution constructed in this way does not depend on a, b as long as these parameters satisfy (12). For this we argue as follows: suppose that u, v are constructed using a, b and $\tilde{u}$, $\tilde{v}$ are constructed with $\tilde{a}$, $\tilde{b}$ satisfying (12). Let u_k, v_k denote the sequences defined by (13), (14). Arguing by induction we see that if $\tilde{u} \geqslant u_{k-1}$, $\tilde{v} \geqslant v_{k-1}$ then

$$\begin{aligned}\Delta(\tilde{u} - u_k) - a(\tilde{u} - u_k) &= \mathsf{g}(\tilde{u}, \tilde{v}) - \mathsf{g}(u_{k-1}, v_{k-1}) - a(\tilde{u} - u_{k-1}) \\ &\leqslant \mathsf{g}(\tilde{u}, v_{k-1}) - \mathsf{g}(u_{k-1}, v_{k-1}) - a(\tilde{u} - u_{k-1}) \\ &\leqslant 0\end{aligned}$$

and then $\tilde{u} - u_k \geqslant 0$ in Ω. Note that $u \mapsto \mathsf{g}(u, v) - au$ and $v \mapsto \mathsf{f}(u, v) - bv$ are monotone in the appropriate regions because u, v and $\tilde{u}$, $\tilde{v}$ are between 0 and m. Similarly, $\tilde{v} - v_k \geqslant 0$ in Ω and thus $\tilde{u} \geqslant u$, $\tilde{v} \geqslant v$ in Ω. By symmetry we obtain the converse inequality and we deduce that $\tilde{u} = u$, $\tilde{v} = v$.

Minimality. Let $\omega \subset \Omega$ be open and $\bar{u}$, $\bar{v} \in C(\bar{\omega})$ satisfy (11). Choose a, b large enough so that $\mathsf{g}(u, v) - au$ is decreasing in u and $\mathsf{f}(u, v) - bv$ is decreasing in v for all u, v in the range $0 \leqslant u, v \leqslant M$ with $M \geqslant m$, $M \geqslant \max_{\bar{\omega}} \bar{u}$ and $M \geqslant \max_{\bar{\omega}} \bar{v}$.

Consider u_k, v_k defined by (13), (14). Now we show that $\bar{u} \geqslant u_k$, $\bar{v} \geqslant v_k$ in ω for all k. By induction, if $\bar{u} \geqslant u_{k-1}$, $\bar{v} \geqslant v_{k-1}$ in ω then

$$\begin{aligned}\Delta(\bar{u} - u_k) - a(\bar{u} - u_k) &\leqslant \mathsf{g}(\bar{u}, \bar{v}) - \mathsf{g}(u_{k-1}, v_{k-1}) - a(\bar{u} - u_{k-1}) \\ &\leqslant \mathsf{g}(\bar{u}, v_{k-1}) - \mathsf{g}(u_{k-1}, v_{k-1}) - a(\bar{u} - u_{k-1}) \\ &\leqslant 0 \quad \text{in } \omega\end{aligned}$$

and hence $\bar{u} - u_k \geqslant 0$ in ω. $\square$

Proof of Theorem 1.1. *Necessary conditions.* Suppose that (u, v) is a solution to (2) and for given $\gamma > 0$ set $w = \min(\gamma u, v)$. Let χ_A denote the characteristic function of a set A. By Kato's inequality (see [14]),

$$\begin{aligned}\Delta w &\leqslant \gamma \Delta u \chi_{[\gamma u < v]} + \Delta v \chi_{[\gamma u > v]} \\ &= \gamma g(u - v)\chi_{[\gamma u < v]} + f(v - \beta u)\chi_{[\gamma u > v]} \\ &\leqslant \gamma g\big((1-\gamma)u\big)\chi_{[\gamma u < v]} + f\left(\left(1 - \frac{\beta}{\gamma}\right)v\right)\chi_{[\gamma u > v]} \\ &= \gamma g\left(\frac{1-\gamma}{\gamma} w\right)\chi_{[\gamma u < v]} + f\left(\left(1 - \frac{\beta}{\gamma}\right)w\right)\chi_{[\gamma u > v]} \\ &\leqslant \max\left(\gamma g\left(\frac{1-\gamma}{\gamma} w\right), f\left(\left(1 - \frac{\beta}{\gamma}\right)w\right)\right) =: h_1(w).\end{aligned}$$

Hence w is a supersolution to the single equation $\Delta u = h_1(u)$ in Ω with $u = +\infty$ on $\partial\Omega$. Therefore this problem admits a solution and hence h_1 must satisfy the Keller–Osserman condition (1) (see e.g. [9]). Choosing $\gamma = 1$ implies that f satisfies (1) and $\beta < 1$. Then, choosing $\gamma = \beta < 1$ implies that g satisfies (1).

Sufficient conditions. Consider the minimal solution (u_m, v_m) to the truncated problem

$$\begin{cases}\Delta u = g(u - v) & \text{in } \Omega, \\ \Delta v = f(u - \beta v) & \text{in } \Omega, \\ u = v = m & \text{on } \partial\Omega\end{cases} \tag{15}$$

where $m > 0$. Such a solution can easily be constructed by the method of sub and supersolutions, see Proposition 2.1. Let $\gamma \in (\beta, 1)$ and set

$$w_m = \max(\gamma u_m, v_m).$$

Then,

$$\begin{aligned}\Delta w_m &\geqslant \gamma\,\Delta u_m\,\chi_{[\gamma u_m>v_m]} + \Delta v_m\,\chi_{[\gamma u_m<v_m]}\\ &= \gamma g(u_m - v_m)\chi_{[\gamma u_m>v_m]} + f(v_m - \beta u_m)\chi_{[\gamma u_m<v_m]}\\ &\geqslant \gamma g\left(\left(\frac{1}{\gamma}-1\right)w_m\right)\chi_{[\gamma u_m>v_m]} + f\left(\left(1-\frac{\beta}{\gamma}\right)w_m\right)\chi_{[\gamma u_m<v_m]}\\ &\geqslant h_2(w_m),\end{aligned}$$

where $h_2(w) = \min(\gamma g(\frac{1-\gamma}{\gamma}w), f((1-\frac{\beta}{\gamma})w))$. Since h_2 satisfies (1) and is nondecreasing, the boundary blow up equation $\Delta w = h_2(w)$ in Ω, $w = +\infty$ on $\partial\Omega$ has a maximal solution w (obtained e.g. as the limit of (w_n), where w_n denotes the minimal blow-up solution on a subdomain $\Omega_n \Subset \Omega$ with $\bigcup_n \Omega_n = \Omega$). By comparison, $w_m \leqslant w$ in Ω for all $m > 0$. Hence (u_m), (v_m) remain bounded on compact sets of Ω as $m \to \infty$, and by standard elliptic estimates they converge- up to a subsequence – in $C^2_{\mathrm{loc}}(\Omega)$ to a solution of (2). □

Remark 2.2. The proof of Theorem 1.1 implies that whenever solutions exist, one of them is minimal in the class of nonnegative solutions. Moreover this solution (u, v) satisfies

$$\beta u \leqslant v \leqslant u \quad \text{in } \Omega. \tag{16}$$

Indeed let us show that the minimal nonnegative solution (u_m, v_m) to (15) satisfies $v_m \leqslant u_m$ in Ω. For this let us recall that $u_m = \lim_{k\to\infty} u_{m,k}$, $v_m = \lim_{k\to\infty} v_{m,k}$ where $u_{m,k}$, $v_{m,k}$ are defined recursively by (14) starting with the trivial solutions, with $\mathsf{g}(u, v) = g(u - v)$ and $\mathsf{f}(u, v) = f(v - \beta u)$. We chose $a = b$ large so that (12) is satisfied. We claim that $v_{m,k} \leqslant u_{m,k}$. Proceeding inductively, assume $v_{m,k-1} \leqslant u_{m,k-1}$. Then

$$\Delta u_{m,k} - a u_{m,k} = g(u_{m,k-1} - v_{m,k-1}) - a(u_{m,k-1} - v_{m,k-1}) - a v_{m,k-1} \leqslant -a v_{m,k-1}$$

while

$$\Delta v_{m,k} - a v_{m,k} = f(v_{m,k-1} - \beta u_{m,k-1}) - a v_{m,k-1} \geqslant -a v_{m,k-1}.$$

By the maximum principle $u_{m,k} \geqslant v_{m,k}$ in Ω. For the other inequality in (16) we may proceed similarly, but this time it is convenient to work with $\tilde{u}_{m,k}$, $\tilde{v}_{m,k}$ defined by (14) but with the boundary conditions $\tilde{u}_{m,k} = m$ and $\tilde{v}_{m,k} = \beta m$ on $\partial\Omega$. The limit of $\tilde{u}_{m,k}$, $\tilde{v}_{m,k}$ as $k \to +\infty$ and then as $m \to +\infty$ is the minimal nonnegative solution to the system, as can be seen by comparison.

Remark 2.3. For a general system (9) the same proof as that of Theorem 1.1 yields the following necessary condition for existence:

$$\forall \gamma > 0 \quad \max\left(\gamma \mathsf{g}\left(\frac{w}{\gamma}, w\right), \mathsf{f}\left(\frac{w}{\gamma}, w\right)\right) \quad \text{satisfies (1).} \tag{17}$$

Similarly the next condition is sufficient for existence

$$\exists \gamma > 0 \text{ such that} \quad \min\left(\gamma \mathsf{g}\left(\frac{w}{\gamma}, w\right), \mathsf{f}\left(\frac{w}{\gamma}, w\right)\right) \quad \text{satisfies (1).} \tag{18}$$

However, these conditions are not equivalent in general, see Example 3.

Remark 2.4. For general cooperative systems the problem with finite boundary values (10) is always solvable and generates an increasing sequence of approximate solutions $(u_m, v_m)_{m\in\mathbb{N}}$. Going back to (9), obtaining a sharp existence criterion similar to the Keller–Osserman condition (1) is equivalent to characterizing the nonlinearities for which $(u_m, v_m)_{m\in\mathbb{N}}$ remains bounded on compact subsets of Ω. Such a result seems out of reach for general cooperative systems. Still, it would be interesting to answer the following question: assume (9) admits a solution for any domain of the form $\Omega = (-R, R)$, $R > 0$. Is it true that (2) is solvable in any smoothly bounded domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$?

3. Asymptotics

Under the hypotheses on f stated in Theorem 1.1 and the Keller–Osserman condition (1) the problem

$$\Delta u = f(u) \quad \text{in } \Omega, \qquad u = +\infty \quad \text{on } \partial\Omega \tag{19}$$

admits a minimal solution u and a maximal solution U. The maximal solution can be constructed as the limit $U = \lim_{\delta\to 0} u_\delta$ where u_δ is the minimal solution of (19) in the domain $\Omega_\delta = \{\, x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \delta\}$.

The next lemma is well known. It asserts that under hypothesis (3) all solutions to (19) have the same first order boundary behavior, see for instance [1,2] or [3].

Lemma 3.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$. Assume f satisfies* (3). *Then for any solution u of* (19) *we have*

$$\lim_{x\to\partial\Omega} \frac{u(x)}{\psi(d(x))} = 1,$$

where $\psi = \phi^{-1}$ and ϕ is the function appearing in (3).

Lemma 3.2. *Suppose $f \sim g$ at infinity and that f satisfies* (3). *Let u be any solution to* (19) *and v any solution to* (19) *with nonlinearity replaced by g. Then*

$$\lim_{x\to\partial\Omega} \frac{u}{v} = 1.$$

Proof. Let $G(t) = \int_0^t g(s)\,ds$, $\phi_g(u) = \int_u^\infty \frac{dt}{\sqrt{2G(t)}}$, $\psi_g = \phi_g^{-1}$. Let ϕ_f and ψ_f denote the corresponding functions associated to f. By Lemma 3.1 it suffices to prove that

$$\lim_{\delta\to 0} \frac{\psi_f(\delta)}{\psi_g(\delta)} = 1.$$

Since $f \sim g$ at infinity we also have $F \sim G$ at infinity and therefore $\phi_f \sim \phi_g$ at infinity. It follows from this and the fact that ϕ_f satisfies the condition (3) that ϕ_g satisfies this condition too.

Let $m > 1$. Condition (3) on ϕ_g implies that there exists $\eta > 1$ and $\delta_0 > 0$ such that

$$\psi_g(\delta) \leqslant m\psi_g(\eta\delta) \quad \forall 0 < \delta < \delta_0. \tag{20}$$

Since $\phi_f \sim \phi_g$ at infinity we have

$$\lim_{\delta\to 0} \frac{\delta}{\phi_g(\psi_f(\delta))} = 1.$$

Hence taking $\delta_1 > 0$ small,

$$\delta \leqslant \eta\phi_g\big(\psi_f(\delta)\big) < \delta_0 \quad \forall 0 < \delta < \delta_1.$$

Since ψ_g is nonincreasing we deduce

$$\psi_g(\delta) \geqslant \psi_g\big(\eta\phi_g\big(\psi_f(\delta)\big)\big) \quad \forall 0 < \delta < \delta_1$$

and by (20)

$$\psi_g(\delta) \geqslant \frac{1}{m}\psi_f(\delta) \quad \forall 0 < \delta < \delta_1.$$

It follows that

$$\limsup_{\delta\to 0} \frac{\psi_f(\delta)}{\psi_g(\delta)} \leqslant m$$

and since $m > 1$ was arbitrary, that

$$\limsup_{\delta\to 0} \frac{\psi_f(\delta)}{\psi_g(\delta)} \leqslant 1.$$

The corresponding inequality for the liminf is proved by reversing the roles of ψ_f and ψ_g. $\square$

The next lemma asserts that under the condition (3) the boundary behavior of solutions to (19) depends continuously on multiplicative perturbations of the nonlinearity.

Lemma 3.3. *Assume f satisfies* (3). *Given $\gamma > 0$ let u_γ denote any solution of*

$$\Delta u_\gamma = f(\gamma u_\gamma) \quad \text{in } \Omega, \qquad u_\gamma = +\infty \quad \text{on } \partial\Omega.$$

Then

$$\limsup_{\gamma\to 1}\limsup_{x\to\partial\Omega}\frac{u_\gamma}{u_1} \leqslant 1 \leqslant \liminf_{\gamma\to 1}\liminf_{x\to\partial\Omega}\frac{u_\gamma}{u_1}.$$

Proof. Given $\gamma > 0$ let $f_\gamma(u) = f(\gamma u)$, $F_\gamma(t) = \int_0^t f_\gamma(s)\,ds = \frac{1}{\gamma}F(\gamma t)$, $\phi_\gamma(u) = \int_u^\infty \frac{dt}{\sqrt{2F_\gamma(t)}} = \frac{1}{\sqrt{\gamma}}\phi(\gamma u)$ and $\psi_\gamma = (\phi_\gamma)^{-1}$. Note that $\psi_\gamma(\delta) = \frac{1}{\gamma}\psi(\sqrt{\gamma}\delta)$.

By Lemma 3.1 it is enough to establish

$$\limsup_{\gamma\to 1}\limsup_{\delta\to 0}\frac{\psi_\gamma(\delta)}{\psi(\delta)} \leqslant 1 \leqslant \liminf_{\gamma\to 1}\liminf_{\delta\to 0}\frac{\psi_\gamma(\delta)}{\psi(\delta)}.$$

Let $m > 1$ and $\delta_0 > 0$, $\eta > 1$ be such that (20) holds. Then if $\sqrt{\gamma} \leqslant \eta$ it follows that

$$\frac{1}{\gamma}\psi(\delta) \leqslant \frac{m}{\gamma}\psi(\eta\delta) \leqslant \frac{m}{\gamma}\psi(\sqrt{\gamma}\delta) = m\psi_\gamma(\delta) \quad \forall 0 < \delta < \delta_0$$

and therefore

$$\liminf_{\delta\to 0}\frac{\psi_\gamma(\delta)}{\psi(\delta)} \geqslant \frac{1}{\gamma m} \quad \forall 0 < \gamma < \eta^2.$$

As $m > 1$ is arbitrary we deduce

$$\liminf_{\gamma\to 1}\liminf_{\delta\to 0}\frac{\psi_\gamma(\delta)}{\psi(\delta)} \geqslant 1.$$

Similarly, let $m > 1$ and $\delta_0 > 0$, $\eta > 1$ such that (20) holds. If $\sqrt{\gamma} \geqslant \frac{1}{\eta}$ we have

$$\psi_\gamma(\delta) = \frac{1}{\gamma}\psi(\sqrt{\gamma}\delta) \leqslant \frac{1}{\gamma}\psi(\delta/\eta) \leqslant \frac{m}{\gamma}\psi(\delta)$$

for $\delta > 0$ small and therefore

$$\limsup_{\delta\to 0}\frac{\psi_\gamma(\delta)}{\psi(\delta)} \leqslant \frac{m}{\gamma} \quad \forall \gamma \geqslant \frac{1}{\eta^2}.$$

Hence

$$\limsup_{\gamma\to 1}\limsup_{\delta\to 0}\frac{\psi_\gamma(\delta)}{\psi(\delta)} \leqslant 1. \qquad \square$$

Proof of Theorem 1.2, part (a). Let (u, v) be any solution to (2) and w_1 be the minimal nonnegative solution to (4) with $\theta = 1$. For simplicity we write $w = w_1$. First we note that we have

$$w \leqslant v \leqslant u. \tag{21}$$

Indeed for the minimal solution (u, v), we always have $u \geqslant v$ by (16). Consequently,

$$\Delta v = f(v - \beta u) \leqslant f\big((1-\beta)v\big)$$

so v is a supersolution of (4) and since w is the minimal nonnegative solution it follows that $w \leqslant v$. Let

$$z_\theta = \max(\theta u, v),$$

where

$$\beta < \theta < 1.$$

By Kato's inequality we have

$$\Delta z_\theta \geqslant h_\theta(z)$$

with h_θ given by

$$h_\theta(w) = \min\left(\theta g\left(\frac{1-\theta}{\theta}\, w\right), f\left(\frac{\theta-\beta}{\theta} w\right)\right). \tag{22}$$

Let w_θ be the minimal solution to (4) and $\tilde{w}_\theta$ be the maximal solution to

$$\Delta \tilde{w}_\theta = h_\theta(\tilde{w}_\theta) \quad \text{in } \Omega, \qquad \tilde{w}_\theta = +\infty \quad \text{on } \partial\Omega.$$

Then $z_\theta \leqslant \tilde{w}_\theta$ in Ω. Note that under condition (5), we have $h_\theta(w) = f(\frac{\theta-\beta}{\theta} w)$ for large w. It follows from Lemma 3.2 that

$$\lim_{x\to\partial\Omega} \frac{\tilde{w}_\theta}{w_\theta} = 1$$

and therefore

$$\limsup_{x\to\partial\Omega} \frac{z_\theta}{w_\theta} \leqslant 1$$

for any $\theta \in (\beta, 1)$. It follows from the previous inequality that

$$\limsup_{x\to\partial\Omega} \frac{z_\theta}{w} \leqslant \limsup_{x\to\partial\Omega} \frac{z_\theta}{w_\theta} \limsup_{x\to\partial\Omega} \frac{w_\theta}{w} \leqslant \limsup_{x\to\partial\Omega} \frac{w_\theta}{w}.$$

Letting now $\theta \to 1$ and using Lemma 3.3 we deduce that

$$\limsup_{\theta\to 1} \limsup_{x\to\partial\Omega} \frac{z_\theta}{w} \leqslant 1.$$

This together with (21) yields the conclusion. □

Proof of Theorem 1.2, part (b). We use Kato's inequality with

$$z_\theta = \max(\theta u, v),$$

where

$$\beta < \theta < \theta_0.$$

We have

$$\Delta z_\theta \geqslant h_\theta(z)$$

with h_θ given by (22). By assumption (7), given $\varepsilon > 0$, $h_\theta(t) \geqslant (1-\varepsilon) f(\frac{\theta-\beta}{\theta} t)$ for t large. In particular, there exists a neighborhood V of $\partial\Omega$, $V \subset \Omega$ such that

$$\Delta z_\theta \geqslant (1-\varepsilon) f\left(\frac{\theta-\beta}{\theta} z_\theta\right).$$

Let $w_{\varepsilon,\theta}$ denote the maximal solution of

$$\begin{cases} \Delta w_{\varepsilon,\theta} = (1-\varepsilon) f(\frac{\theta-\beta}{\theta} w_{\varepsilon,\theta}) & \text{in } V, \\ w_{\varepsilon,\theta} = +\infty & \text{on } \partial V. \end{cases}$$

Then, $z_\theta \leqslant w_{\varepsilon,\theta}$ in V. By Lemma 3.3

$$\limsup_{\varepsilon\to 0, \theta\to\theta_0} \limsup_{x\to\partial\Omega} \frac{w_{\varepsilon,\theta}}{w} \leqslant 1, \tag{23}$$

where w is the minimal solution of (4) with $\theta=\theta_0$. Thus,

$$\limsup_{\theta\to\theta_0}\limsup_{x\to\partial\Omega}\frac{z_\theta}{w}\leqslant 1. \tag{24}$$

Let $\theta\in(\theta_0,1)$ and $\tilde z_\theta=\min(\theta u,v)$. Then, as before, $\tilde z_\theta\geqslant\tilde w_{\varepsilon,\theta}$ where now $\tilde w_{\varepsilon,\theta}$ is the minimal solution of

$$\begin{cases}\Delta\tilde w_{\varepsilon,\theta}=(1-\varepsilon)f(\frac{\theta-\beta}{\theta}\tilde w_{\varepsilon,\theta}) & \text{in } V,\\ \tilde w_{\varepsilon,\theta}=+\infty & \text{on } \partial\Omega,\\ \tilde w_{\varepsilon,\theta}=\tau & \text{on } \partial V\setminus\partial\Omega\end{cases}$$

and $\tau>0$ is a fixed small constant. Using Lemma 3.3 one proves that

$$\liminf_{\varepsilon\to0,\theta\to\theta_0}\liminf_{x\to\partial\Omega}\frac{\tilde w_{\varepsilon,\theta}}{w}\geqslant 1, \tag{25}$$

whence

$$\liminf_{\theta\to\theta_0}\liminf_{x\to\partial\Omega}\frac{\tilde z_\theta}{w}\geqslant 1. \tag{26}$$

Collecting (24) and (26), the theorem is proved. □

4. Uniqueness

In this section, we prove Corollary 1.5, which states the uniqueness of the solutions of (2) provided that f,g are nondecreasing, nonnegative C^1 functions such that $f=g=0$ on $\mathbb{R}^-$ that satisfy (3), that either condition (5) or (7) holds, and that

(a) either f,g are convex functions;
(b) or Ω is a ball and $\frac{f(t)}{t}$, $\frac{g(t)}{t}$ nondecreasing in a neighborhood of $+\infty$, and f,g nondecreasing everywhere.

We begin with the proof of the uniqueness result assuming that f,g are convex functions.

Proof of Corollary 1.5, part (a). Let $\varepsilon>0$. Consider (u,v) the minimal BBUS solution and (u_1,v_1) another solution to (2). Actually, by (6) or (8) we have that $(1+\varepsilon)u>u_1\geqslant u$, $(1+\varepsilon)v>v_1\geqslant v$ in a neighborhood of $\partial\Omega$.

Therefore, since by convexity $\frac{f(t)}{t}$, $\frac{g(t)}{t}$ are increasing functions $((1+\varepsilon)u,(1+\varepsilon)v)$ is a supersolution of (2):

$$\begin{cases}\Delta(1+\varepsilon)u\leqslant g((1+\varepsilon)u-(1+\varepsilon)v) & \text{in } \Omega,\\ \Delta(1+\varepsilon)v\leqslant f((1+\varepsilon)v-\beta(1+\varepsilon)u) & \text{in } \Omega,\\ (1+\varepsilon)u=(1+\varepsilon)v=\infty & \text{on } \partial\Omega.\end{cases}$$

Therefore, $w:=u_1-(1+\varepsilon)u$, $z:=v_1-(1+\varepsilon)v$ satisfy $w\leqslant0$, $z\leqslant0$ in a neighborhood of $\partial\Omega$, and

$$\Delta w-g'(\xi_1)w+g'(\xi_1)z\geqslant0\quad\text{in } \Omega,$$
$$\Delta z-f'(\xi_2)z+\beta f'(\xi_2)w\geqslant0\quad\text{in } \Omega,$$

for some $\xi_1\geqslant0$, $\xi_2\geqslant0$. Since $g'(\xi_1)\geqslant0$, $f'(\xi_2)\geqslant0$ and $(-1+\beta)f'(\xi_2)\leqslant0$ we can apply the maximum principle for cooperative systems (see for example Appendix A of this paper or [19], Theorem 3.15 and its following remark) to conclude that $w\leqslant0$ and $z\leqslant0$ in Ω. Letting $\varepsilon\to0$, we obtain the desired inequality. □

We now prove the uniqueness result relaxing the hypotheses on f,g if Ω is a ball. This result compares with the uniqueness result in [9]. We begin with a lemma concerning the boundary behavior of the minimal solution (u,v) of our problem, and that is interesting in itself, since it is true for general domains and gives some insight of the boundary behavior of solutions.

Lemma 4.1. *Let u, v denote the minimal boundary blow-up solution of* (2). *Then,*

$$\lim_{x\to\partial\Omega}\big(u(x)-v(x)\big)=+\infty, \tag{27}$$

$$\lim_{x\to\partial\Omega}\big(v(x)-\beta u(x)\big)=+\infty. \tag{28}$$

Proof. We establish (27), the other limit being similar. Fix $A>0$. Consider the problem

$$\Delta u_{A,m}=g(u_{A,m}-v_{A,m}), \tag{29}$$

$$\Delta v_{A,m}=f(v_{A,m}-\beta u_{A,m}), \tag{30}$$

$$u_{A,m}=m+A,\quad v_{A,m}=m\quad \text{on } \partial\Omega. \tag{31}$$

For a given $A>0$, if m is large enough then $(m+A,m)$ is a supersolution to this problem and by the classical iterative method described in Proposition 2.1, we can construct a solution to this approximated problem. As we did in the proof of Theorem 1.1, $(u_{A,m},v_{A,m})$ converges to the minimal BBUS solution when $m\to+\infty$.

In addition, we have that

$$\Delta(u_{A,m}-v_{A,m})\leqslant g(u_{A,m}-v_{A,m}), \tag{32}$$

$$u_{A,m}-v_{A,m}=A\quad \text{on } \partial\Omega. \tag{33}$$

Therefore $u_{A,m}-v_{A,m}$ is a supersolution to a single equation problem. Set w_A for the solution to

$$\Delta w_A=g(w_A)\quad \text{in } \Omega, \tag{34}$$

$$w_A=A\quad \text{on } \partial\Omega. \tag{35}$$

Therefore, everywhere in Ω

$$w_A(x)\leqslant u_{A,m}(x)-v_{A,m}(x). \tag{36}$$

We now let $m\to+\infty$, then $A\to+\infty$ that leads to

$$w(x)\leqslant u(x)-v(x), \tag{37}$$

where w is the minimal BBUS solution to (34). □

We now complete the proof of the uniqueness result.

Proof of Corollary 1.5, part (b). To fix ideas, assume that Ω is the unit ball. Consider (u,v) the minimal solution to (2) and (u_1,v_1) the maximal solution (obtained as the limit $(u_1,v_1)=\lim_{\delta\to0}(u_\delta,v_\delta)$ where (u_δ,v_δ) is the minimal solution in the ball with radius $1-\delta$). Then it suffices to show that $u\equiv u_1$ and $v\equiv v_1$. It is worth to observe that u, v, u_1, v_1 are radial functions.

Consider $r\in(0,1)$. Let Ω_r be the ball of radius r. Then for each $x\in\Omega_r$ there exist $\xi,\xi'\in\mathbb{R}$ such that

$$\Delta(u_1-u)=g(u_1-v_1)-g(u-v)=g'(\xi)(u_1-u)-g'(\xi)(v_1-v), \tag{38}$$

$$\Delta(v_1-v)=f(v_1-\beta u_1)-f(v-\beta u)=f'(\xi')(v_1-v)-\beta f'(\xi')(u_1-u). \tag{39}$$

By the maximum principle for cooperative systems, we have

$$\sup_{\Omega_r}(u_1-u)\leqslant \max\big(u_1(r)-u(r),v_1(r)-v(r)\big),$$

$$\sup_{\Omega_r}(v_1-v)\leqslant \max\big(u_1(r)-u(r),v_1(r)-v(r)\big).$$

This ensures that the function

$$r\mapsto M(r):=\max\big(u_1(r)-u(r),v_1(r)-v(r)\big)$$

is nondecreasing in $(0,1)$.

Assume that

$$u(0) < u_1(0) \quad \text{or} \quad v(0) < v_1(0) \tag{40}$$

for if $u(0) = u_1(0)$ and $v_1(0) = v(0)$ by uniqueness for the system of ODEs we would have $u \equiv u_1$ and $v \equiv v_1$ in $(0, 1)$ and the proof is over.

By Lemma 4.1 there is R_0 such that $\min(u - v, v - \beta u)(t) \geqslant a$ for all $t \geqslant R_0$, where a is such that both $\frac{f(t)}{t}$, $\frac{g(t)}{t}$ nondecreasing for $t \geqslant a$.

We argue in a slightly different way in the following cases:

$$\text{there exists } r \in (R_0, 1) \text{ such that } u_1(r) - u(r) > v_1(r) - v(r), \tag{41}$$

$$\text{there exists } r \in (R_0, 1) \text{ such that } u_1(r) - u(r) < v_1(r) - v(r), \tag{42}$$

$$u_1(r) - u(r) = v_1(r) - v(r) \quad \text{for all } r \in (R_0, 1). \tag{43}$$

To begin with we assume that (41) holds. In this case choose $R_1 \in (R_0, 1)$ such that

$$u_1(R_1) - u(R_1) > v_1(R_1) - v(R_1). \tag{44}$$

Define

$$w := u_1 - (1 + \varepsilon)u, \qquad z := v_1 - (1 + \varepsilon)v$$

and take $\varepsilon > 0$ small enough such that

$$w(R_1) > z(R_1)$$

and by (40)

$$w(R_1) > 0 \quad \text{or} \quad z(R_1) > 0.$$

Thus in particular $w(R_1) > 0$. We choose $r_\varepsilon > R_1$ close to 1, such that $w(r_\varepsilon) < 0$ and $z(r_\varepsilon) < 0$. This is possible by (6) or (8).

In the annulus $\{r\colon\ R_1 < r < r_\varepsilon\}$ we then have

$$\begin{cases} \Delta((1 + \varepsilon)u - u_1) \leqslant g((1 + \varepsilon)u - (1 + \varepsilon)v) - g(u_1 - v_1), \\ \Delta((1 + \varepsilon)v - v_1) \leqslant f((1 + \varepsilon)v - \beta(1 + \varepsilon)u) - f(v_1 - \beta u_1). \end{cases}$$

Therefore, (w, z) satisfy in the annulus

$$\Delta w - g'(\xi_1)w + g'(\xi_1)z \geqslant 0,$$
$$\Delta z - f'(\xi_2)z + \beta f'(\xi_2)w \geqslant 0.$$

By the maximum principle in the annulus $R_1 \leqslant r \leqslant r_\varepsilon$ we have

$$\max(w, z) \leqslant \max\bigl(w(R_1), z(R_1)\bigr) = w(R_1)$$

and hence

$$u_1(r) - (1 + \varepsilon)u(r) \leqslant u_1(R_1) - (1 + \varepsilon)u(R_1).$$

Note that as $\varepsilon \to 0$, r_ε can be taken to approach 1. We then let $\varepsilon \to 0$ to obtain $u_1(r) - u(r) \leqslant u_1(R_1) - u(R_1)$ for $R_1 \leqslant r < 1$. By continuity $M(r) = u_1(r) - u(r)$ in some interval around R_1. Since $r \mapsto M(r)$ is nondecreasing we deduce that $M(r) = \lambda = \text{const}$ in some interval of the form $[R_1, R_1 + \sigma]$ with $\sigma > 0$. From Eq. (38) we also have $g(u_1 - v_1) = g(u - v)$ in that interval. But $g(t)$ is strictly increasing for $t \geqslant a$ and $u(r) - v(r) \geqslant a$, $u_1(r) - v_1(r) \geqslant a$ for $r \geqslant R_0$ if we choose R_0 close enough to 1. Hence $v_1 - v = \lambda = \text{const}$ in $[R_1, R_1 + \sigma]$. This contradicts (44).

A similar argument rules out the case (42) and therefore we are in the situation (43). The previous argument with R_1 replaced by R_0 then yields

$$\max(w, z) \leqslant \max\bigl(w(R_0), z(R_0)\bigr), \quad R_0 \leqslant r \leqslant r_\varepsilon,$$

which implies

$$u_1(r) - (1 + \varepsilon)u(r) \leqslant \max\bigl(u_1(R_0) - (1 + \varepsilon)u(R_0), v_1(R_0) - (1 + \varepsilon)v(R_0)\bigr)$$

for $R_0 \leqslant r \leqslant r_\varepsilon$. Letting $\varepsilon \to 0$ we have

$$M(r) = u_1(r) - u(r) \leqslant M(R_0), \quad R_0 \leqslant r < 1,$$

and since M is nondecreasing we conclude that M is constant in $[R_0, 1)$. Thus $u_1 - u = \lambda$ and $v_1 - v = \mu$ are constant in $[R_0, 1)$. Going back to the system we then have in the annulus $R_0 < r < 1$

$$0 = \Delta(u_1 - u) = g(u_1 - v_1) - g(u - v),$$
$$0 = \Delta(v_1 - v) = f(v_1 - \beta u_1) - f(v - \beta u).$$

Since f, g are strictly increasing functions in the appropriate range we then have $u_1 - v_1 = u - v$ and $v_1 - \beta u_1 = v - \beta u$ in $R_0 < r < 1$. Hence $\lambda = \mu = 0$ and therefore $u_1 - u = 0$ and $v_1 - v = 0$ in $[0, 1)$. This completes the proof. □

5. Examples

Example 1. The first example falls in case (a) of Theorem 1.2.

$$\begin{cases} \Delta u = e^{u-v} - 1 & \text{in } \Omega, \\ \Delta v = (v - \beta u)^p & \text{in } \Omega, \\ u = v = \infty & \text{on } \partial\Omega, \end{cases} \tag{45}$$

where $p > 1$, $0 < \beta < 1$.

By Theorems 1.1, 1.2 and Corollary 1.5, problem (45) has a unique solution (u, v), which we know the leading order asymptotics of. We investigate here how the first equation affects the next terms in the asymptotic expansions of u, v. We do this for simplicity in the case $p > 3$.

Proposition 5.1. *Assume $0 < \beta < 1$ and $p > 3$. Let d denote the function distance to the boundary. Then the unique solution (u, v) of (45) has the behavior*

$$u = cd^{-\alpha} + e_1 \log d + f_1 + O(d^\varepsilon),$$
$$v = cd^{-\alpha} + e_2 \log d + f_2 + O(d^\varepsilon)$$

where $\varepsilon > 0$ is suitably small and the constants are uniquely determined by the equations

$$\alpha = \frac{2}{p-1}, \tag{46}$$
$$(1-\beta)^p c^{p-1} = \alpha(\alpha + 1), \tag{47}$$
$$e_1 - e_2 = -\alpha - 2, \qquad e_2 - \beta e_1 = 0, \tag{48}$$
$$e^{f_1 - f_2} = c\alpha(\alpha + 1), \qquad c\alpha(\alpha + 1)p\frac{f_2 - \beta f_1}{(1-\beta)c} = -e_2. \tag{49}$$

Proof. The argument relies on constructing a sub- and a supersolution having a suitable boundary behavior. The reader is invited to check that the sub- and the supersolution that we construct depend continuously on Ω: this means that if a comparison principle is available (for systems with standard boundary conditions), then the solution (u, v) of (45) can be compared e.g. on an increasing sequence of domains $\Omega_n \Subset \Omega$ with the supersolution $\bar{u}_n, \bar{v}_n$ blowing-up on $\partial\Omega_n$. Letting $n \to \infty$ and checking that $\bar{u}_n(x), \bar{v}_n(x)$ converge pointwise to the supersolution $\bar{u}, \bar{v}$ blowing-up on $\partial\Omega$, we obtain the desired inequality: $u \leqslant \bar{u}$, $v \leqslant \bar{v}$. A similar approximation by outer domains enables us to compare (u, v) with a given subsolution.

We finally note that the standard comparison principle for systems can be used, since for $g(u, v) = e^{u-v}$, $f(u, v) = (v - \beta u)^p$ we have $\frac{\partial g}{\partial u} + \frac{\partial g}{\partial v} \geqslant 0$, $\frac{\partial g}{\partial v} \leqslant 0$ and $\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \geqslant 0$, $\frac{\partial f}{\partial u} \leqslant 0$. So it remains to construct the sub- and supersolution of (45). Let $\delta > 0$ be small and define $U_\delta = \{x \in \Omega\colon d(x) < \delta\}$.

We use as a supersolution

$$\bar{u} = cd^{-\alpha} + e_1 \log d + f_1 + g_1 d^\varepsilon \quad \text{and} \quad \bar{v} = cd^{-\alpha} + e_2 \log d + f_2 + g_2 d^\varepsilon$$

where $0 < \varepsilon < \alpha$ and $g_1, g_2 > 0$ are to be fixed later on.

In U_δ we have

$$\Delta \bar{u} = c\alpha(\alpha+1)d^{-\alpha-2} - e_1 d^{-2} - g_1\varepsilon(1-\varepsilon)d^{\varepsilon-2} - c\alpha d^{-\alpha-1}\Delta d + e_1 d^{-1}\Delta d + g_1\varepsilon d^{\varepsilon-1}\Delta d,$$
$$\Delta \bar{v} = c\alpha(\alpha+1)d^{-\alpha-2} - e_2 d^{-2} - g_2\varepsilon(1-\varepsilon)d^{\varepsilon-2} - c\alpha d^{-\alpha-1}\Delta d + e_2 d^{-1}\Delta d + g_2\varepsilon d^{\varepsilon-1}\Delta d$$

and

$$\begin{aligned} e^{\bar{u}-\bar{v}} &= c\alpha(\alpha+1)d^{-\alpha-2}e^{(g_1-g_2)d^\varepsilon} \\ &= c\alpha(\alpha+1)d^{-\alpha-2} + c\alpha(\alpha+1)d^{-\alpha-2}\left(e^{(g_1-g_2)d^\varepsilon}-1\right). \end{aligned}$$

We take g_1, g_2 of the form

$$g_1 = ta_1, \qquad g_2 = ta_2$$

where $t \geqslant 1$ and $a_1, a_2 > 0$ are fixed such that

$$\beta a_1 < a_2 < a_1.$$

Using convexity

$$e^{\bar{u}-\bar{v}} \geqslant c\alpha(\alpha+1)d^{-\alpha-2} + c\alpha(\alpha+1)(a_1-a_2)td^{\varepsilon-\alpha-2}$$

and hence

$$\Delta\bar{u} - \left(e^{\bar{u}-\bar{v}}-1\right) \leqslant -c\alpha(\alpha+1)(a_1-a_2)td^{\varepsilon-\alpha-2} + Cd^{-\alpha-1} + Ctd^{\varepsilon-2} \quad \text{in } U_\delta$$

where C depends only on p, β, Ω, a_1 and a_2.

Again, using convexity

$$(\bar{v}-\beta\bar{u})^p \geqslant c\alpha(\alpha+1)d^{-\alpha-2} + c\alpha(\alpha+1)p\frac{f_2-\beta f_1}{(1-\beta)c}d^{-2} + c\alpha(\alpha+1)p\frac{a_2-\beta a_1}{(1-\beta)c}td^{\varepsilon-2}.$$

Since $p > 3$ we have $\alpha \in (0,1)$. Hence, using (49) we find

$$\Delta\bar{v} - (\bar{v}-\beta\bar{u})^p \leqslant -c\alpha(\alpha+1)p\frac{a_2-\beta a_1}{(1-\beta)c}td^{\varepsilon-2} + Ctd^{\varepsilon-1} + Cd^{-\alpha-1} \quad \text{in } U_\delta.$$

Then there is $\delta > 0$ such that $\bar{u}, \bar{v}$ is a supersolution of the system in the set $U_\delta = \{x \in \Omega\colon d(x) < \delta\}$ for any $t \geqslant 1$. Having fixed δ we now select t large such that

$$\bar{u} \geqslant u \quad \text{and} \quad \bar{v} \geqslant v \quad \text{on } d(x) = \delta.$$

It follows by comparison that $u \leqslant \bar{u}$ and $v \leqslant \bar{v}$ in U_δ.

The construction of a subsolution $\underline{u}$, $\underline{v}$ is similar. We take

$$\underline{u} = cd^{-\alpha} + e_1\log d + f_1 - a_1td^\varepsilon \quad \text{and} \quad \underline{v} = cd^{-\alpha} + e_2\log d + f_2 - a_2td^\varepsilon$$

where $0 < \varepsilon < \alpha$, $a_1 > 0$, $a_2 > 0$ are chosen such that

$$\beta a_1 < a_2 < a_1$$

and $t > 0$ is to be fixed later on. Let us introduce

$$\sigma = a_2 - \beta a_1 > 0.$$

Later on we will need σ to be small.

Let $\delta > 0$ be small and $U_\delta = \{x \in \Omega\colon d(x) < \delta\}$. Recall that the unique solution u, v to (45) satisfies $u \geqslant 0$, $v \geqslant 0$. We take C large so that if

$$t = C\delta^{-\alpha-\varepsilon} \tag{50}$$

then

$$\underline{u} \leqslant 0 \quad \text{and} \quad \underline{v} \leqslant 0 \quad \text{at } d(x) = \delta.$$

We have in U_δ

$$\Delta \underline{v} = c\alpha(\alpha+1)d^{-\alpha-2} - e_2 d^{-2} + a_2 t\varepsilon(1-\varepsilon)d^{\varepsilon-2} - c\alpha d^{-\alpha-1}\Delta d + e_2 d^{-1}\Delta d - a_2 t\varepsilon d^{\varepsilon-1}\Delta d.$$

Given $\ell > 0$ there is $A > 0$ such that

$$(1+h)^p \leqslant 1 + ph + Ah^2 \quad \forall -1 \leqslant h \leqslant \ell.$$

Using this inequality with

$$h = \frac{f_2 - \beta f_1}{(1-\beta)c} d^\alpha + \frac{\sigma t}{(1-\beta)c} d^{\alpha+\varepsilon}$$

we find

$$(\underline{v} - \beta\underline{u})^p \leqslant c\alpha(\alpha+1)d^{-\alpha-2} + c\alpha(\alpha+1)p\frac{f_2-\beta f_1}{(1-\beta)c}d^{-2} + c\alpha(\alpha+1)p\frac{\sigma t}{(1-\beta)c}d^{\varepsilon-2}$$
$$+ A\left(c\alpha(\alpha+1)p\frac{f_2-\beta f_1}{(1-\beta)c}d^{-2} + c\alpha(\alpha+1)p\frac{\sigma t}{(1-\beta)c}d^{\varepsilon-2}\right)^2,$$

provided that $h \in [-1, \ell]$. This condition is indeed satisfied if we take $\ell > 0$ large but fixed and then $\delta > 0$ small, since t is given by (50). It follows that in U_δ

$$\Delta\underline{v} - (\underline{v} - \beta\underline{u})^p \geqslant -Cd^{-\alpha-1} - C\varepsilon t d^{\varepsilon-1} + Ct\sigma d^{\varepsilon-2} - ACd^{\alpha-2} - AC\sigma^2 t^2 d^{2\varepsilon+\alpha-2}.$$

By taking $\sigma > 0$ sufficiently small and then $\delta > 0$ small we finally obtain

$$\Delta\underline{v} - (\underline{v} - \beta\underline{u})^p \geqslant 0 \quad \text{in } U_\delta.$$

Now let us verify that $\Delta\underline{u} - e^{\underline{u}-\underline{v}} \geqslant 0$ in U_δ. It will then follow that $(\underline{u}, \underline{v})$ is a subsolution of (45). First we have

$$\Delta\bar{u} = c\alpha(\alpha+1)d^{-\alpha-2} - e_1 d^{-2} + a_1 t\varepsilon(1-\varepsilon)d^{\varepsilon-2} - c\alpha d^{-\alpha-1}\Delta d + e_1 d^{-1}\Delta d - a_1 t\varepsilon d^{\varepsilon-1}\Delta d$$

In addition

$$e^{\underline{u}-\underline{v}} = c\alpha(\alpha+1)d^{-\alpha-2} + c\alpha(\alpha+1)d^{-\alpha-2}\left(e^{-(a_1-a_2)td^\varepsilon} - 1\right).$$

If $\gamma > 0$ is sufficiently small then

$$e^x \leqslant 1 + \gamma x \quad \text{for } -\frac{1}{2\gamma} \leqslant x \leqslant 0.$$

Therefore

$$e^{\underline{u}-\underline{v}} \leqslant c\alpha(\alpha+1)d^{-\alpha-2} - \gamma c\alpha(\alpha+1)d^{\varepsilon-\alpha-2}(a_1-a_2)t$$

in U_δ provided that

$$(a_1 - a_2)td^\varepsilon \leqslant \frac{1}{2\gamma} \quad \text{in } U_\delta. \tag{51}$$

To this end we choose $\gamma = \kappa\delta^\alpha$ with $\kappa > 0$ small. Recalling that t is given by (50) we see that (51) is satisfied in U_δ. It then follows that

$$\Delta\underline{u} - \left(e^{\underline{u}-\underline{v}} - 1\right) \geqslant -Cd^{-2} - Ctd^{\varepsilon-1} + Ct\gamma d^{-\alpha+\varepsilon-2} \quad \text{in } U_\delta$$

and so

$$\Delta\underline{u} - \left(e^{\underline{u}-\underline{v}} - 1\right) \geqslant 0 \quad \text{in } U_\delta$$

if we fix $\delta > 0$ sufficiently small. By comparison we deduce that $u \geqslant \underline{u}$ and $v \geqslant \underline{v}$ in U_δ. □

Example 2. Our second example falls in case (b) of Theorem 1.2. Let $\alpha > 0$, $\beta > 0$ and consider

$$\begin{cases} \Delta u = e^{u-\alpha v} & \text{in } \Omega, \\ \Delta v = e^{v-\beta u} & \text{in } \Omega, \\ u = v = +\infty & \text{on } \partial\Omega. \end{cases} \tag{52}$$

We shall see that the constants involved in the leading asymptotics of the solution (u, v) depend on both nonlinearities f and g. We also compute the next term in the asymptotics of (u, v) and we observe that it is independent of the geometry of $\partial\Omega$.

Existence of solutions for the system (52) does not follow directly from Theorem 1.1, since the nonlinearities $g = \exp$ and $f = \exp$ do not vanish at 0. To obtain the existence we thus need to construct a suitable subsolution.

Proposition 5.2. *The system* (52) *has a solution if and only if* $\alpha\beta < 1$. *Moreover if* $\alpha\beta < 1$ *then* (52) *has a unique solution* (u, v) *and it satisfies*

$$u = -c_1 \log d + e_1 + o(1), \qquad v = -c_2 \log d + e_2 + o(1) \quad \text{as } x \to \partial\Omega \tag{53}$$

where

$$d(x) = \operatorname{dist}(x, \partial\Omega)$$

and

$$c_1 = 2\frac{1+\alpha}{1-\alpha\beta}, \qquad c_2 = 2\frac{1+\beta}{1-\alpha\beta}, \tag{54}$$

$$e_1 = \frac{\log c_1 - \alpha \log c_2}{1-\alpha\beta}, \qquad e_2 = \frac{\log c_2 - \beta \log c_1}{1-\alpha\beta}. \tag{55}$$

Proof. Regarding the existence part we let the reader check that Theorem 1.1 still holds when f, g do not vanish at $(0, 0)$, provided there exists a bounded subsolution of the problem. We construct such a subsolution for (52) as follows. Take $K > 0$ large so that

$$|x|^2 - K \leqslant 0 \quad \forall x \in \Omega$$

and choose $\gamma > 0$ such that

$$\frac{1}{\alpha} > \gamma > \beta.$$

Let

$$\underline{u} = A\big(|x|^2 - K\big), \qquad \underline{v} = A\gamma\big(|x|^2 - K\big)$$

with $A > 1$ such that $A\gamma > 1$. Then

$$\Delta\underline{u} = 2NA \geqslant 1, \qquad \Delta\underline{v} = 2NA\gamma \geqslant 1,$$

and

$$\exp(\underline{u} - \alpha\underline{v}) = \exp\big(A\big(|x|^2 - K\big)(1 - \alpha\gamma)\big) \leqslant 1$$

since $|x|^2 - K \leqslant 0$ and $1 - \alpha\gamma > 0$. Similarly

$$\exp(\underline{v} - \beta\underline{u}) = \exp\big(A\big(|x|^2 - K\big)(\gamma - \beta)\big) \leqslant 1.$$

The fact that $\alpha\beta < 1$ is a necessary and sufficient condition for existence follows from the above discussion, Theorem 1.1 and the change of unknown $\tilde{u} = u$, $\tilde{v} = \alpha v$.

For the rest of the proof we assume that $\alpha\beta < 1$. Regarding the asymptotic behavior of solutions, first we establish that any solution (u, v) to (52) satisfies (53). For this purpose we first construct appropriate sub- and supersolutions. For $\delta > 0$ define

$$U_\delta = \big\{x \in \Omega\colon\ d(x) < \delta\big\}.$$

Let $\delta_0 > 0$ be a small fixed number. Then there is $\tilde{d}$ smooth in $\overline{\Omega}$ such that $\tilde{d} > 0$ in Ω and

$$\tilde{d} \equiv d \quad \text{in } U_{\delta_0}.$$

Let $0 < \sigma < 1$ and w be the solution to

$$\begin{cases} -\Delta w = \tilde{d}^{\sigma-2} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{56}$$

Using comparison with appropriate powers of d one can obtain

$$a_1 \tilde{d}^\sigma \leqslant w \leqslant a_2 \tilde{d}^\sigma \quad \text{in } \Omega \tag{57}$$

for some constants $a_1, a_2 > 0$. See also Gilbarg, Trudinger [13] Theorem 4.9 and Exercise 4.6.

Step 1. For appropriate choices of $\gamma \in (\alpha, 1/\beta)$ and $K > 0$

$$\underline{u} = -c_1 \log \tilde{d} + e_1 - \gamma K w, \qquad \underline{v} = -c_2 \log \tilde{d} + e_2 - K w, \tag{58}$$

form a subsolution of (67), where the constants c_1, c_2, e_1, e_2 are given by (54), (55). Indeed,

$$\Delta \underline{u} = c_1 \tilde{d}^{-2} |\nabla \tilde{d}|^2 - c_1 \tilde{d}^{-1} \Delta \tilde{d} + \gamma K \tilde{d}^{\sigma-2},$$

while

$$e^{\underline{u} - \alpha \underline{v}} = c_1 \tilde{d}^{-2} e^{-(\gamma-\alpha)K w}.$$

Therefore

$$\Delta \underline{u} - e^{\underline{u} - \alpha \underline{v}} = c_1 \tilde{d}^{-2} \big(|\nabla \tilde{d}|^2 - e^{-(\gamma-\alpha)K w} - \tilde{d} \Delta \tilde{d} \big) + \gamma K \tilde{d}^{\sigma-2}. \tag{59}$$

Using the inequality

$$e^{-t} \leqslant 1 - \frac{t}{3} \quad \forall 0 \leqslant t \leqslant 1 \tag{60}$$

and (57) we have

$$-e^{-(\gamma-\alpha)K w} \geqslant -1 + \frac{1}{3}(\gamma - \alpha) a_1 K \tilde{d}^\sigma$$

whenever

$$\tilde{d}^\sigma \leqslant \frac{1}{(\gamma-\alpha) a_2 K}.$$

This holds in U_δ if $\delta > 0$ is small and

$$\delta^\sigma K \leqslant \frac{1}{(\gamma - \alpha) a_2}.$$

We note that from the start γ can be chosen close to α so that

$$\frac{c_1 + \|\tilde{d} \Delta \tilde{d}\|_{L^\infty}}{\min(1, \gamma)} \leqslant \frac{1}{(\gamma - \alpha) a_2}. \tag{61}$$

We now decrease $\delta > 0$ further to achieve $|\nabla \tilde{d}| = 1$ in U_δ and

$$\delta \frac{3 \|\Delta \tilde{d}\|_{L^\infty}}{(\gamma - \alpha) a_1} \leqslant c_1 + \|\tilde{d} \Delta \tilde{d}\|_{L^\infty}. \tag{62}$$

With $\delta > 0$ now being fixed we choose K such that

$$\frac{c_1 + \|\tilde{d} \Delta \tilde{d}\|_{L^\infty}}{\min(1, \gamma)} \leqslant K \delta^\sigma \leqslant \frac{1}{(\gamma - \alpha) a_2}. \tag{63}$$

Then in U_δ we have by (59), (60)

$$\Delta \underline{u} - e^{\underline{u} - \alpha \underline{v}} \geqslant c_1 \tilde{d}^{\sigma-2} \left(\frac{1}{3} (\gamma - \alpha) a_1 K - \tilde{d}^{1-\sigma} \Delta \tilde{d} \right) + \gamma K \tilde{d}^{\sigma-2} \geqslant 0$$

by (62) and (63). In $\Omega \setminus U_\delta$

$$\Delta \underline{u} - e^{\underline{u}-\alpha\underline{v}} \geqslant -c_1\tilde{d}^{-2} - c_1\tilde{d}^{-1}\Delta\tilde{d} + \gamma K\tilde{d}^{\sigma-2} \geqslant 0$$

thanks to (63). Similar calculations imply that $\Delta\underline{v} - e^{\underline{v}-\beta\underline{u}} \geqslant 0$ in Ω.

Step 2. Let $(\underline{u}, \underline{v})$ denote the subsolution (58). Then for any solution (u, v) of (52) we have

$$u \geqslant \underline{u}, \qquad v \geqslant \underline{v} \quad \text{in } \Omega. \tag{64}$$

To prove this statement, for $\varepsilon > 0$ small consider the domain $\Omega_\varepsilon = \Omega \cup \{x \in \mathbb{R}^N\colon \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$. Using Step 1, we can construct a subsolution $(\underline{u}_\varepsilon, \underline{v}_\varepsilon)$ to (52) in the domain Ω_ε. Note that $\underline{u}_\varepsilon$, $\underline{v}_\varepsilon$ depend continuously on ε for $\varepsilon > 0$ small. Substituting u by λu (for a given $\lambda > 0$) in the system (52), we obtain the equivalent form

$$\begin{cases} \Delta u = \mathsf{g}_\lambda(u, v) & \text{in } \Omega,\\ \Delta v = \mathsf{f}_\lambda(u, v) & \text{in } \Omega,\\ u = v = \infty & \text{on } \partial\Omega \end{cases} \tag{65}$$

where $\mathsf{g}_\lambda(u, v) = \frac{1}{\lambda}e^{\lambda u-\alpha v}$, $\mathsf{f}_\lambda(u, v) = e^{v-\lambda\beta u}$. Note that $\frac{\partial\mathsf{g}_\lambda}{\partial u} = 1 \geqslant 0$, $\frac{\partial\mathsf{g}_\lambda}{\partial v} = -\frac{\alpha}{\lambda} \leqslant 0$ and $\frac{\partial\mathsf{g}_\lambda}{\partial u} + \frac{\partial\mathsf{g}_\lambda}{\partial v} = 1 - \frac{\alpha}{\lambda}$. Similarly $\frac{\partial\mathsf{f}_\lambda}{\partial v} = 1 \geqslant 0$, $\frac{\partial\mathsf{f}_\lambda}{\partial u} = -\beta\lambda \leqslant 0$ and $\frac{\partial\mathsf{f}_\lambda}{\partial u} + \frac{\partial\mathsf{f}_\lambda}{\partial v} = 1 - \beta\lambda$. Since $\alpha\beta < 1$ it is possible to choose $\lambda > 0$ such that $\frac{\partial\mathsf{g}_\lambda}{\partial u} + \frac{\partial\mathsf{g}_\lambda}{\partial v} \geqslant 0$ and $\frac{\partial\mathsf{f}_\lambda}{\partial u} + \frac{\partial\mathsf{f}_\lambda}{\partial v} \geqslant 0$. With these conditions the maximum principle holds for the system (65) and since $\underline{u}_\varepsilon - u \to -\infty$, $\underline{v}_\varepsilon - v \to -\infty$ as $x \to \partial\Omega$ we deduce

$$u \geqslant \underline{u}_\varepsilon \quad \text{and} \quad v \geqslant \underline{v}_\varepsilon \quad \text{in } \Omega.$$

Letting $\varepsilon \to 0$, we obtain (64).

Step 3. Following the same argument as in the two previous steps one can show that for appropriate choices of $\gamma > 0$ and $K > 0$

$$\bar{u} = -c_1 \log\tilde{d} + e_1 + \gamma K w, \qquad \bar{v} = -c_2 \log\tilde{d} + e_2 + K w,$$

where w is the solution of (56), is a supersolution of (67), and for any solution u, v of (67) we have

$$u \leqslant \bar{u} \quad \text{and} \quad v \leqslant \bar{v} \quad \text{in } \Omega. \tag{66}$$

Step 4. Let u_i, v_i, $i = 1, 2$, be two solutions to (52). Then $u_1 = u_2$ and $v_1 = v_2$. Indeed u_i/λ, v_i are also solutions to (65). Moreover by (64) and (66) and the fact that $w(x) \to 0$ as $x \to \partial\Omega$ we have that $(u_1 - u_2)/\lambda \to 0$ and $v_1 - v_2 \to 0$ as $x \to \partial\Omega$. By the maximum principle for (65) we conclude that $u_1 = u_2$ and $v_1 = v_2$. □

Example 3. Our next system is not of the form (2). It demonstrates how our technique can still be used in more general settings. It also provides an example where conditions (17) and (18) in Remark 2.3 are not equivalent. Consider $p, q, r, s > 0$ and the system

$$\begin{cases} \Delta u = \frac{u^p}{v^q} & \text{in } \Omega,\\ \Delta v = \frac{v^r}{u^s} & \text{in } \Omega,\\ u > 0, \quad v > 0 & \text{in } \Omega,\\ u = v = +\infty & \text{on } \partial\Omega. \end{cases} \tag{67}$$

Proposition 5.3. *Problem* (67) *has a solution if and only if*

$$p > 1, \quad r > 1 \quad \textit{and} \quad (p-1)(r-1) > qs. \tag{68}$$

Moreover, under condition (68) *the system has a unique solution and it satisfies*

$$u = c_1 d^{-\gamma}\big(1 + \mathrm{o}(1)\big) \quad \textit{and} \quad v = c_2 d^{-\lambda}\big(1 + \mathrm{o}(1)\big) \quad \textit{as } x \to \partial\Omega \tag{69}$$

where

$$\gamma = 2\frac{r - 1 + q}{(p-1)(r-1) - sq}, \qquad \lambda = 2\frac{p - 1 + s}{(p-1)(r-1) - sq} \tag{70}$$

and c_1, c_2 are given by

$$c_1^{(p-1)(r-1)-qs} = \big(\gamma(\gamma+1)\big)^{r-1}\big(\lambda(\lambda+1)\big)^{q}, \tag{71}$$
$$c_2^{(p-1)(r-1)-qs} = \big(\gamma(\gamma+1)\big)^{s}\big(\lambda(\lambda+1)\big)^{p-1}. \tag{72}$$

Remark 5.4. Let $m = \frac{p+s-1}{q+r-1}$. Observe the following distinction between the *balanced system* ($m = 1$) and the case $m > 1$

- balanced system: introduce $n = r - s = p - q$, then problem (67) reads also
$$\begin{cases} \Delta u = (\frac{u}{v})^q u^n & \text{in } \Omega, \\ \Delta v = (\frac{v}{u})^s v^n & \text{in } \Omega, \\ u > 0, \quad v > 0 & \text{in } \Omega, \\ u = v = +\infty & \text{on } \partial\Omega. \end{cases} \tag{73}$$
Consider z the BBUS for the single equation $\Delta z = z^n$. Then (z, z) is the BBUS for the balanced system.
- unbalanced system: perform the change of variable $w = u^m$. Then
$$\Delta w \geqslant \left(\frac{w}{v}\right)^q w^{1-q+\frac{p-1}{m}} = \left(\frac{w}{v}\right)^q w^n,$$
$$\Delta v = \left(\frac{v}{w}\right)^s v^n,$$
with $n = r - \frac{s}{m} = 1 - q + \frac{p-1}{m}$. Observe that $n = 1 + \frac{(p-1)(r-1)-sq}{p+s-1} > 1$. Let again z denote the BBUS for the single equation $\Delta z = z^n$, then using an ordering lemma (see Lemma 5.5 below), we have
$$u^m \leqslant v \leqslant z.$$

Before proving Proposition 5.3 we need to establish some preliminary results. The first one is the following comparison lemma.

Lemma 5.5. *Let $u_1, v_1 \in C^2(\Omega)$, $u_1, v_1 > 0$ in Ω, be a subsolution to* (67).
Similarly, let (u_2, v_2) be a supersolution to (67) *and assume*
$$\limsup_{x\to\partial\Omega} \frac{u_1}{u_2} \leqslant 1 \quad \text{and} \quad \limsup_{x\to\partial\Omega} \frac{v_1}{v_2} \leqslant 1. \tag{74}$$
If $p, q, r, s > 0$ satisfy (68) *then*
$$u_1 \leqslant u_2 \quad \text{and} \quad v_1 \leqslant v_2 \quad \text{in } \Omega. \tag{75}$$

Proof. Consider $\tilde u_i = \log u_i$, $\tilde v_i = \log v_i$. Then $\tilde u_1$, $\tilde v_1$ satisfy
$$\begin{cases} \Delta\tilde u_1 + |\nabla\tilde u_1|^2 \geqslant e^{(p-1)\tilde u_1 - q\tilde v_1} & \text{in } \Omega, \\ \Delta\tilde v_1 + |\nabla\tilde v_1|^2 \geqslant e^{(r-1)\tilde v_1 - s\tilde u_1} & \text{in } \Omega \end{cases}$$
and $\tilde u_2$, $\tilde v_2$ satisfy the corresponding reversed inequalities. It is convenient to introduce one more change of variables: $\lambda U_i = \tilde u_i$ and $V_i = \tilde v_i$ where λ is such that
$$\frac{r-1}{s} > \lambda > \frac{q}{p-1}. \tag{76}$$
Then
$$\begin{cases} \Delta U_1 + \lambda|\nabla U_1|^2 \geqslant \mathsf{g}_\lambda(U_1, V_1) & \text{in } \Omega, \\ \Delta V_1 + |\nabla V_1|^2 \geqslant \mathsf{f}_\lambda(U_1, V_1) & \text{in } \Omega \end{cases} \tag{77}$$
where $\mathsf{g}_\lambda(u, v) = \frac{1}{\lambda} e^{(p-1)\lambda u - qv}$, $\mathsf{f}_\lambda(u, v) = e^{(r-1)v - s\lambda u}$.

The inequalities (75) are equivalent to $U_1 \leqslant U_2$ and $V_1 \leqslant V_2$ in Ω. Suppose that one of these inequalities fail. We deal first with the case $\sup_\Omega(U_1 - U_2) \geqslant \sup_\Omega(V_1 - V_2)$. Then $\sup_\Omega(U_1 - U_2) > 0$ and, since $\limsup_{x\to\partial\Omega}(U_1 -$

$U_2) \leqslant 0$ by (74), there is $x_0 \in \Omega$ where $U_1 - U_2$ attains its maximum. Then $\nabla U_1(x_0) = \nabla U_2(x_0)$ and $\Delta U_1(x_0) - \Delta U_2(x_0) \leqslant 0$. Using the first inequality in (77) – and its analogue for U_2 – we obtain for some $\eta_1, \eta_2 \geqslant 0$

$$\begin{aligned} 0 &\geqslant \Delta(U_1 - U_2)(x_0) + \lambda\left(\left|\nabla U_1(x_0)\right|^2 - \left|\nabla U_2(x_0)\right|^2\right) \\ &\geqslant \mathsf{g}_\lambda\left(U_1(x_0), V_1(x_0)\right) - g_\lambda\left(U_2(x_0), V_2(x_0)\right) \\ &= \frac{\partial \mathsf{g}_\lambda(\eta_1, \eta_2)}{\partial u}\left(U_1(x_0) - U_2(x_0)\right) + \frac{\partial \mathsf{g}_\lambda(\eta_1, \eta_2)}{\partial v}\left(V_1(x_0) - V_2(x_0)\right). \end{aligned}$$

Since $\frac{\partial \mathsf{g}_\lambda(\eta_1,\eta_2)}{\partial v} \leqslant 0$ and $V_1(x_0) - V_2(x_0) \leqslant \sup_\Omega(V_1 - V_2) \leqslant U_1(x_0) - U_2(x_0)$ we deduce

$$0 \geqslant \left(\frac{\partial \mathsf{g}_\lambda(\eta_1, \eta_2)}{\partial u} + \frac{\partial \mathsf{g}_\lambda(\eta_1, \eta_2)}{\partial v}\right)\left(U_1(x_0) - U_2(x_0)\right). \tag{78}$$

But

$$\frac{\partial \mathsf{g}_\lambda(\eta_1, \eta_2)}{\partial u} + \frac{\partial \mathsf{g}_\lambda(\eta_1, \eta_2)}{\partial v} = \left(\lambda(p-1) - q\right)e^{(p-1)\eta_1 - q\eta_2} > 0$$

by (76). This gives a contradiction with (78).

The remaining case, that is when $\sup_\Omega(U_1 - U_2) \leqslant \sup_\Omega(V_1 - V_2)$, is analogous so we skip it. □

Proposition 5.3 will be obtained through a blow-up argument, using an idea from [4]. We start out by studying the associated limiting problem. We write $x \in \mathbb{R}^N$ as $x = (x_1, x')$ with $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$. Let $\mathbb{R}^N_+ = \{(x_1, x'):\ x_1 > 0\}$.

Proposition 5.6. *Assume condition* (68) *and let* $\gamma, \lambda, c_1, c_2$ *be defined by* (70)–(72). *Suppose* $u, v \in C^2(\mathbb{R}^N_+)$, $u, v > 0$ *solve* (67) *in* $\mathbb{R}^N_+$ *and satisfy*

$$\frac{1}{C}x_1^{-\gamma} \leqslant u \leqslant Cx_1^{-\gamma} \quad \text{and} \quad \frac{1}{C}x_1^{-\lambda} \leqslant v \leqslant Cx_1^{-\lambda} \tag{79}$$

for some $C > 0$. *Then* $u \equiv c_1 x_1^{-\gamma}$ *and* $v \equiv c_2 x_1^{-\lambda}$.

Proof. We start proving that $u \leqslant c_1 x_1^{-\gamma}$ in $\mathbb{R}^N_+$. Let $\sigma > 0$ satisfy

$$\frac{q}{r-1} < \sigma < \frac{p-1}{s}. \tag{80}$$

For $t > 0$ set

$$u_t = tc_1x_1^{-\gamma}, \qquad v_t = t^\sigma c_2 x_1^{-\lambda}$$

and note that for $t > 1$ the pair (u_t, v_t) is a supersolution of (67) in $\mathbb{R}^N_+$. Let

$$t_0 = \inf\left\{t > 1:\ u \leqslant u_t \text{ and } v \leqslant v_t \text{ in } \mathbb{R}^N_+\right\}.$$

Note that by (79) t_0 is well defined and

$$u \leqslant u_{t_0} \quad \text{and} \quad v \leqslant v_{t_0} \quad \text{in } \mathbb{R}^N_+. \tag{81}$$

We wish to show that $t_0 \leqslant 1$. Assume by contradiction that $t_0 > 1$. Let t_n be a sequence such that $t_n \to t_0$ and for each n, either $u \leqslant u_{t_n}$ fails or $v \leqslant v_{t_n}$ does. At least one of these inequalities has to fail for infinitely many n's, and we work out the details in the former case. Passing to a subsequence if necessary there are x_n such that

$$u(x_n) > u_{t_n}(x_n). \tag{82}$$

We write $x_n = (x_{1,n}, x_n')$ and define $r_n = x_{1,n}$ and the functions

$$u_n(y) = r_n^\gamma u\left(r_n y + (0, x_n')\right), \qquad v_n(y) = r_n^\lambda v\left(r_n y + (0, x_n')\right).$$

Then u_n, v_n also satisfy the bounds (79). By standard elliptic estimates, up to a new subsequence $u_n \to u^*$ and $v_n \to v^*$ uniformly on compact sets of $\mathbb{R}^N_+$ and (u^*, v^*) is a solution to (67) in $\mathbb{R}^N_+$ and satisfies (79). From (81) we find

$$u^* \leqslant u_{t_0} \quad \text{and} \quad v^* \leqslant v_{t_0} \quad \text{in } \mathbb{R}^N_+ \tag{83}$$

and (82) implies that

$$u^*(e_1) = u_{t_0}(e_1).$$

This yields $\Delta u^*(e_1) \leqslant \Delta u_{t_0}(e_1)$. On the other hand, since $t_0 > 1$ we have $\Delta u_{t_0} < u_{t_0}^p / v_{t_0}^q$ in all of $\mathbb{R}^N_+$. Hence

$$\frac{u^*(e_1)^p}{v^*(e_1)^q} = \Delta u^*(e_1) \leqslant \Delta u_{t_0}(e_1) < \frac{u_{t_0}(e_1)^p}{v_{t_0}(e_1)^q}$$

which leads to $v^*(e_1) > v_{t_0}(e_1)$, contradicting (83). □

Proof of Proposition 5.3. *Step 1.* Here we show that (68) is necessary for existence. Consider $\alpha, \beta \in (0,1)$. Suppose that (u, v) is a solution of (67) and define

$$w = \min\left(u^\alpha, v^\beta\right).$$

Then,

$$\Delta w \leqslant \alpha w^{\frac{\alpha-1+p}{\alpha}} w^{-\frac{q}{\beta}} \chi_{[u^\alpha < v^\beta]} + \beta w^{\frac{\beta-1+r}{\beta}} w^{-\frac{s}{\alpha}} \chi_{[u^\alpha > v^\beta]}.$$

Hence the function

$$w \mapsto \max\left(w^{\frac{\alpha-1+p}{\alpha} - \frac{q}{\beta}}, w^{\frac{\beta-1+r}{\beta} - \frac{s}{\alpha}}\right)$$

must satisfy the Keller–Osserman condition (1). Therefore for any $\alpha, \beta \in (0,1)$

$$\frac{p-1}{q} > \frac{\alpha}{\beta} \quad \text{or} \quad \frac{\alpha}{\beta} > \frac{s}{r-1}.$$

This implies (68).

Step 2. Condition (68) is sufficient for the existence of a solution.

First we show that there exists a subsolution $\underline{u}, \underline{v} > 0$ of (67). Consider

$$\underline{u} = a\left(|x|^2 + 1\right), \qquad \underline{v} = b\left(|x|^2 + 1\right).$$

Then $\Delta \underline{u} \geqslant \underline{u}^p / \underline{v}^q$ if

$$2N \geqslant a^{p-1} b^{-q} A \quad \text{where } A = \sup_\Omega \left(|x|^2 + 1\right)^{p-q} \tag{84}$$

and similarly, in order that $\Delta \underline{v} \geqslant \underline{v}^r / \underline{u}^s$ it is sufficient that

$$2N \geqslant a^{-s} b^{r-1} B, \quad B = \sup_\Omega \left(|x|^2 + 1\right)^{r-s}.$$

Setting $a^s = b^{r-1} B/(2N)$ and inserting in (84) shows that it is enough that

$$(2N)^{p-1+s} \geqslant b^{(p-1)(r-1)-qs} B^{p-1} A^s,$$

which can be achieved for small $b > 0$ thanks to the condition $(p-1)(r-1) - qs > 0$.

Consider now the minimal solution (u_m, v_m) to the truncated problem (10) with $g(u,v) = u^p/v^q$, $f(u,v) = v^r/u^s$ where $m > 0$, which existence is guaranteed by Proposition 2.1. Let $\alpha, \beta > 1$ be such that

$$\frac{p-1}{q} > \frac{\alpha}{\beta} \quad \text{and} \quad \frac{\alpha}{\beta} > \frac{s}{r-1} \tag{85}$$

which is possible thanks to (68). Now let

$$w_m = \max\left(u_m^\alpha, v_m^\beta\right).$$

Then by Kato's inequality and since $\alpha, \beta > 1$

$$\Delta w_m \geqslant \alpha w_m^{\frac{\alpha-1+p}{\alpha}} w_m^{-\frac{q}{\beta}} \chi_{[u^\alpha < v^\beta]} + \beta w_m^{\frac{\beta-1+r}{\beta}} w_m^{-\frac{s}{\alpha}} \chi_{[u^\alpha > v^\beta]} \geqslant h(w_m)$$

where

$$h(w) = \min\bigl(\alpha w^{\frac{\alpha-1+p}{\alpha}-\frac{q}{\beta}}, \beta w^{\frac{\beta-1+r}{\beta}-\frac{s}{\alpha}}\bigr).$$

But thanks to (85) we have $\frac{\alpha-1+p}{\alpha} - \frac{q}{\beta} > 1$ and $\frac{\beta-1+r}{\beta} - \frac{s}{\alpha} > 1$, which implies that h satisfies the Keller–Osserman condition (1). Hence the problem $\Delta w = h(w)$ in Ω has a solution w with $w = \infty$ on $\partial\Omega$. It follows that $w_m \leqslant w$, which shows that u_m and v_m remain bounded on compact sets of Ω, and in a standard way one obtains a solution to (67).

From now on we assume that (68) holds.

Step 3. For any solution (u, v) of (67) we have

$$\frac{1}{C} d^{-\gamma} \leqslant u \leqslant C d^{-\gamma} \quad \text{and} \quad \frac{1}{C} d^{-\lambda} \leqslant v \leqslant C d^{-\lambda} \quad \text{in } \Omega \tag{86}$$

for some constant $C > 0$.

We let $t > 1$ and $\sigma > 0$ such that (80) holds. Let $0 < \varepsilon < \min(\gamma, \lambda)$ and define

$$\bar{u} = t\bigl(c_1 d^{-\gamma} - k_1 d^{\varepsilon-\gamma}\bigr), \qquad \bar{v} = t^\sigma\bigl(c_2 d^{-\lambda} - k_2 d^{\varepsilon-\lambda}\bigr)$$

where $\gamma, \lambda, c_1, c_2$ are defined by (70)–(72) and $k_1, k_2 > 0$ will be specified later on. Let $\delta > 0$ be small such that in $U_\delta = \{x \in \Omega\colon d(x) < \delta\}$, d is smooth and $|\nabla d| = 1$. Then, in U_δ,

$$\Delta\bar{u} = t c_1 \gamma(1+\gamma) d^{-2-\gamma}\left[1 - \frac{1}{1+\gamma} d\Delta d - k_1 \frac{(\gamma-\varepsilon)(\gamma-\varepsilon+1)}{c_1\gamma(1+\gamma)} d^\varepsilon + k_1 \frac{\gamma-\varepsilon}{c_1\gamma(1+\gamma)} d^{1+\varepsilon}\Delta d\right].$$

By choosing k_1 such that

$$k_1 \geqslant 2\|\Delta d\|_{L^\infty} \frac{c_1\gamma}{(\gamma-\varepsilon)(\gamma-\varepsilon+1)}$$

and $\delta < 1$ such that

$$\delta\|\Delta d\|_{L^\infty} \leqslant \frac{1}{2}$$

we have

$$\Delta\bar{u} \leqslant t c_1\gamma(1+\gamma) d^{-2-\gamma} \quad \text{in } U_\delta.$$

Additionally,

$$\frac{\bar{u}^p}{\bar{v}^q} = t^{p-\sigma q} c_1^p c_2^{-q} d^{-2-\gamma} \frac{(1-k_1 d^\varepsilon/c_1)^p}{(1-k_2 d^\varepsilon/c_2)^q}.$$

Therefore

$$\Delta\bar{u} - \frac{\bar{u}^p}{\bar{v}^q} = t c_1\gamma(1+\gamma) d^{-2-\gamma}\left[1 - t^{p-1-\sigma q} \frac{(1-k_1 d^\varepsilon/c_1)^p}{(1-k_2 d^\varepsilon/c_2)^q}\right].$$

Since $p - 1 - \sigma q > 0$ thanks to (80) we may fix $t_0 > 1$ and find a uniform δ small such that for all $t > t_0$:

$$\Delta\bar{u} - \frac{\bar{u}^p}{\bar{v}^q} \leqslant 0 \quad \text{in } U_\delta.$$

Similarly

$$\Delta\bar{v} - \frac{\bar{v}^r}{\bar{u}^s} \leqslant 0 \quad \text{in } U_\delta$$

for all $t > t_0$. By decreasing δ we also can achieve

$$\inf_{d(x)=\delta} \bar{u}(x) > 0 \quad \text{and} \quad \inf_{d(x)=\delta} \bar{v}(x) > 0.$$

Now we take t_0 large enough such that $\bar u \geqslant u$ and $\bar v \geqslant v$ on $\{x \in \Omega\colon d(x) = \delta\}$. By Lemma 5.5, we deduce that $u \leqslant \bar u$ and $v \leqslant \bar v$ in U_δ which implies the upper bounds in (86). The lower bounds are obtained similarly.

Step 4. Any solution (u, v) of (67) satisfies the boundary behavior (69).

Let $x_n \in \Omega$ be such that $x_n \to x_0 \in \partial\Omega$. Without loss of generality we may assume that $\nu(x_0) = -e_1$. Let $r_n = d(x_n)$ and $\hat x_n \in \partial\Omega$ be the point on $\partial\Omega$ closest to x_n.

Define

$$u_n(x) = r_n^\gamma u(r_n x + \hat x_n), \qquad v_n(x) = r_n^\lambda v(r_n x + \hat x_n).$$

Then (u_n, v_n) solves (67) in $\Omega_n = (\Omega - \hat x_n)/r_n$. As $n \to \infty$, Ω_n approaches the half space $\mathbb{R}^+_N$. Moreover, letting $d_n(x) = \operatorname{dist}(x, \partial\Omega_n)$ we have $d_n(x) = d(r_n x)/r_n$. Using this and (86)

$$\frac{1}{C} d_n^{-\gamma} \leqslant u_n \leqslant C d_n^{-\gamma} \quad \text{and} \quad \frac{1}{C} d_n^{-\lambda} \leqslant v_n \leqslant C d_n^{-\lambda} \quad \text{in } \Omega_n.$$

Using standard elliptic estimates and the above inequalities, up to a subsequence, $u_n \to u^*$, $v_n \to v^*$ uniformly on compact sets of $\mathbb{R}^N_+$ where (u^*, v^*) is a solution to (67) in $\mathbb{R}^N_+$ satisfying

$$\frac{1}{C} x_1^{-\gamma} \leqslant u^* \leqslant C x_1^{-\gamma} \quad \text{and} \quad \frac{1}{C} x_1^{-\lambda} \leqslant v^* \leqslant C x_1^{-\lambda} \quad \text{in } \mathbb{R}^N_+.$$

By Proposition 5.6 we have $u^* \equiv c_1 x_1^{-\gamma}$ and $v^* \equiv c_2 x_1^{-\lambda}$. Hence

$$d(x_n)^\gamma u(x_n) = u_n\big((x_n - \hat x_n)/r_n\big) \to u^*(e_1).$$

It follows that $\lim_{n\to\infty} d(x_n)^\gamma u(x_n) = c_1$. Similarly $\lim_{n\to\infty} d(x_n)^\lambda v(x_n) = c_2$.

Step 5. From Step 4 and Lemma 5.5 we deduce that (67) has a unique solution, which satisfies (69). □

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Appendix A. Maximum principle for cooperative systems

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set and consider the linear system

$$\begin{aligned} \Delta u &\geqslant a_{11} u + a_{12} v \quad \text{in } \Omega, \\ \Delta v &\geqslant a_{21} u + a_{22} v \quad \text{in } \Omega. \end{aligned} \tag{87}$$

We assume that $a_{ij} \in L^\infty(\Omega)$ satisfy

$$\begin{aligned} &a_{11} + a_{12} \geqslant 0, \quad a_{12} \leqslant 0 \text{ in } \Omega, \\ &a_{21} + a_{22} \geqslant 0, \quad a_{21} \leqslant 0 \text{ in } \Omega. \end{aligned}$$

Theorem A.1. *Suppose $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy* (87) *and*

$$u \leqslant M, \quad v \leqslant M \quad \text{on } \partial\Omega$$

where $M \geqslant 0$. Then

$$u \leqslant M, \quad v \leqslant M \quad \text{in } \Omega.$$

Proof. First we assume $M = 0$. Let $\varepsilon > 0$ and

$$\tilde u = u + \varepsilon e^{A x_1}, \qquad \tilde v = v + \varepsilon e^{A x_1}$$

where $A > 0$ is a large constant. Then

$$\Delta\tilde{u} \geqslant a_{11}u + a_{12}v + \varepsilon A^2 e^{x_1} = a_{11}\tilde{u} + a_{12}\tilde{v} + \varepsilon e^{x_1}\big(A^2 - a_{11} - a_{12}\big) > a_{11}\tilde{u} + a_{12}\tilde{v}$$

in Ω if A is taken large, and similarly

$$\Delta\tilde{v} > a_{21}\tilde{u} + a_{22}\tilde{v}.$$

Moreover $\tilde{u} \leqslant \varepsilon K$ and $\tilde{v} \leqslant \varepsilon K$ on $\partial\Omega$ where

$$K = \max_{\partial\Omega} e^{Ax_1}.$$

We claim that

$$\tilde{u} \leqslant \varepsilon K \quad \text{and} \quad \tilde{v} \leqslant \varepsilon K \quad \text{in } \Omega.$$

Suppose that the conclusion fails and that $\max_{\overline{\Omega}} u > \varepsilon K$ and $\max_{\overline{\Omega}} u \geqslant \max_{\overline{\Omega}} v$. Let $x_0 \in \Omega$ be a point where $\tilde{u}$ attains its maximum. Then

$$0 \geqslant \Delta\tilde{u}(x_0) > a_{11}\tilde{u}(x_0) + a_{12}\tilde{v}(x_0) \geqslant (a_{11} + a_{12})\tilde{u}(x_0) \geqslant 0$$

which is impossible. Thus we have

$$\max_{\overline{\Omega}}\big(u + \varepsilon e^{Ax_1}\big) \leqslant \varepsilon K, \qquad \max_{\overline{\Omega}}\big(v + \varepsilon e^{Ax_1}\big) \leqslant \varepsilon K.$$

Letting $\varepsilon \to 0$ we obtain

$$\max_{\overline{\Omega}} u \leqslant 0, \qquad \max_{\overline{\Omega}} v \leqslant 0.$$

Now we assume $M \geqslant 0$. Consider $\tilde{u} = u - M$, $\tilde{v} = v - M$. Then

$$\Delta\tilde{u} \geqslant a_{11}u + a_{12}v = a_{11}\tilde{u} + a_{12}\tilde{v} + M(a_{11} + a_{12}) \geqslant a_{11}\tilde{u} + a_{12}\tilde{v}$$

by the assumptions $M \geqslant 0$, $a_{11} + a_{12} \geqslant 0$. Similarly

$$\Delta\tilde{v} \geqslant a_{21}\tilde{u} + a_{22}\tilde{v}.$$

Applying the previous case we deduce $\tilde{u} \leqslant 0$ and $\tilde{v} \leqslant M$ which yields

$$u \leqslant M \quad \text{and} \quad v \leqslant M \quad \text{in } \Omega. \qquad \square$$

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