Ann. Inst. H. Poincaré
Anal. Non Linéaire (Online first)
DOI 10.4171/AIHPC/83

Published by EMS Press

Stability of the density patches problem with vacuum for incompressible inhomogeneous viscous flows

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Abstract. We consider the inhomogeneous incompressible Navier–Stokes system in a smooth two- or three-dimensional bounded domain, in the case where the initial density is only bounded. Existence and uniqueness for such initial data was shown recently in Danchin and Mucha [Comm. Pure Appl. Math. 72 (2019)], but the stability issue was left open. After having shown that the solutions constructed therein have exponential decay, a result of independent interest, we prove the stability with respect to initial data, first in Lagrangian coordinates, and then in the Eulerian frame. We actually obtain stability in the energy space for the velocity and in a Sobolev space with negative regularity for the density. Let us underline that, as opposed to prior works, our stability estimates are valid even in the case of a vacuum. In particular, our result applies to the classical density patches problem, where the density is a characteristic function.

1. Introduction

We are interested in the following *inhomogeneous incompressible* Navier–Stokes system:

$$\begin{cases} \varrho_t + v \cdot \nabla \varrho = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \varrho v_t + \varrho v \cdot \nabla v - \mu \Delta v + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}_+ \times \Omega. \end{cases} \tag{1.1}$$

This system describes the motion of incompressible fluids with constant positive viscosity μ and variable density, and originates from simplified models in geophysics. The unknowns are the velocity v, the density ϱ and the pressure P, depending on the time variable $t \geq 0$ and on the space variable $x \in \Omega$, where the fluid domain Ω is a smooth bounded subset of $\mathbb{R}^d$ in the physical dimensions $d = 2, 3$.

The system is supplemented with the initial data

$$\varrho|_{t=0} = \varrho_0 \quad \text{and} \quad v|_{t=0} = v_0. \tag{1.2}$$

At the boundary, we prescribe the no-slip condition

$$v|_{\partial\Omega} = 0. \tag{1.3}$$

2020 Mathematics Subject Classification. Primary 35Q30; Secondary 76N10.
Keywords. Stability, density patches, inhomogeneous flows, rough density, vacuum.

The existence of weak solutions to (1.1) is nowadays well understood and the state of the art on this issue is rather similar to that of the classical incompressible Navier–Stokes system (i.e. with constant density). The analysis goes back to the work of Kazhikhov [28], who showed global existence of weak solutions for initial density bounded away from zero. This constraint was removed by Simon [42]. Later, Lions [34] showed that the density is a renormalized solution to the continuity equation, which in particular allowed the case of density-dependent viscosity to be treated in [18]. Still in the framework of weak solutions, Fanelli and Gallagher [20] recently investigated the fast rotation limit of (1.1) supplemented with a Coriolis force.

Producing "strong solutions" (by strong, we mean solutions having the uniqueness property) requires more constraints on the data: enough regularity and no vacuum, typically. Roughly speaking, according to the classical literature, for smooth enough data and provided the density does not vanish, we have global existence of strong solutions *even for large data* in dimension two, and, like for the constant density case, for small enough initial velocity in dimension three. For such results in the bounded domain case, one can refer to the pioneering work by Ladyzhenskaya and Solonnikov [29] (further extended to less regular data by the first author in [9]).

A number of works have been dedicated to solving (1.1) in $\Omega = \mathbb{R}^d$ in so-called "critical regularity frameworks". The underlying idea (which originates from Fujita and Kato's paper [22] for the constant density case) is that "optimal" functional spaces for well-posedness of (1.1) have to share its scaling invariance, namely, for all $\ell > 0$,

$$(\varrho, v, P)(t, x) \rightsquigarrow (\varrho, \ell v, \ell^2 P)(\ell^2 t, \ell x) \quad \text{and} \quad (\varrho_0, v_0)(x) \rightsquigarrow (\varrho_0, \ell v_0)(\ell x). \tag{1.4}$$

Observing that the couple of homogeneous Besov space $\dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d) \times (\dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d))^d$ indeed possesses this invariance, the first author proved in [8] the well-posedness of (1.1) supplemented with initial velocity v_0 in $\dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)$ and initial density ϱ_0 close to some positive constant in $\dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)$. Note that, owing to the embedding $\dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d) \hookrightarrow \mathcal{C}_b(\mathbb{R}^d)$, this forces the density to be continuous. Subsequent improvements have been brought to this approach (see e.g. [1]) but still the density has to be "almost" continuous. In particular, one cannot consider initial densities that have a jump across an interface, even a smooth one.

Toward considering less regular densities, a first breakthrough has been made by the first two authors in [11, 12]: taking advantage of Lagrangian coordinates (which will be presented below), they established well-posedness results for densities that are possibly discontinuous along interfaces, provided the jump is small enough.

Then, in [38], using a totally different approach, Paicu, Zhang and Zhang succeeded in proving the global existence in $\mathbb{R}^2$ for $v_0 \in H^s$, $s > 0$ and in $\mathbb{R}^3$ for $v_0 \in H^1$ with $\|v_0\|_2 \|\nabla v_0\|_2$ sufficiently small, provided the initial density satisfies

$$0 < c_0 \le \varrho_0 \le C_0 < \infty.$$

In dimension three, this work was extended in [4] to initial velocities that are only in H^s for some $s > 1/2$. Still the density has to be bounded away from zero, and the solution in not time continuous with values in H^s. Very recently, Zhang [44] achieved the critical regularity $\dot{B}^{1/2}_{2,1}$ for the initial velocity, but did not address the uniqueness issue in this setting.

For more results where the initial density is allowed to be discontinuous but still strictly positive, the reader may refer among others to [21,23,27] and to a recent result [3], where an inflow boundary condition is considered. Let us also mention that global well-posedness in the half-space $\mathbb{R}^d_+$ with initial density only bounded but close to a positive constant was shown in [15].

All the above results require the strict positivity of the initial density. To our knowledge, the existence of unique solutions in the presence of vacuum was first proved in [5] for rather high regularity of the initial density and velocity, (namely $\varrho_0 \in L^{3/2} \cap H^2$ and $u_0 \in H^2$) and provided the following compatibility condition is satisfied:

$$-\mu\Delta v_0 + \nabla P_0 = \sqrt{\varrho_0}g \quad \text{for some } g \in L_2 \text{ and } P_0 \in H^1. \tag{1.5}$$

Global existence of unique solutions in a three-dimensional bounded domain or in $\mathbb{R}^3$ under the same compatibility condition and smallness of $\|u_0\|_{\dot{H}^{1/2}}$ was shown in [7].

Condition (1.5) was removed in [30], where local well-posedness in a bounded domain is shown, but still for sufficiently smooth initial density. Global existence in the whole space $\mathbb{R}^3$, again under sufficient regularity of initial density, was proved recently in [25].

An important place in the theory of (1.1) is taken by the so-called *density patch problem*: assuming that

$$\varrho_0 = \alpha_1\chi_{A_0} + \alpha_2\chi_{\Omega\setminus A_0} \tag{1.6}$$

for some nonnegative constants α_1, α_2 and a measurable set A_0, can we say that $\varrho(t)$ has the same structure for all time, with persistence of the regularity of the interface? This problem seems to have been first raised by Lions [34] in the specific case where $\varrho_0 = \chi_{A_0}$ with $A_0 \in \mathbb{R}^2$, and $\sqrt{\varrho_0}u_0 \in L_2$. The original question was whether for all time $\varrho(t) = \chi_{A(t)}$ for some domain $A(t)$ with the same regularity as A_0.

A positive answer has been obtained for C^1 regularity as a consequence of the works of the first two authors in [11,12] if $\alpha_1, \alpha_2 > 0$ are close to each other. Much more complete results have been obtained in the two-dimensional case in [32] (case $\alpha_1 - \alpha_2$ small), and then in [33] for any $\alpha_1, \alpha_2 > 0$. There, the authors actually establish the persistence of high "striated" Sobolev regularity for the density. Similar results have been proved in the three-dimensional case in [31]. The propagation of striated regularity has been adapted to the case where the viscosity depends on the density in [37], where global existence and uniqueness is shown provided the viscosity is close to 1. A two-fluid problem for inhomogeneous incompressible fluids separated by a free interface with possibly different densities and viscosities has been investigated in a recent paper [24], where the global existence of unique solutions is shown provided the ratio of viscosities is close to 1.

Using a different approach, the persistence of Hölder continuity of the interface if α_1, α_2 are close to each other was shown in [16].

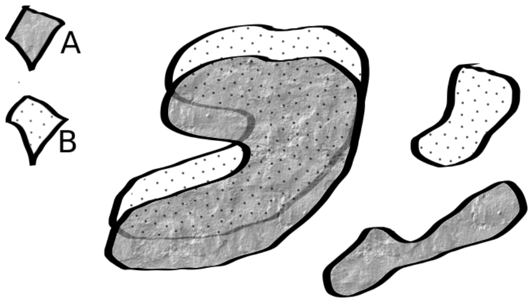

Figure 1. Support of $\varrho_0^{(\mathrm{org})} = \bar{\mathbf{A}}$ and support of $\varrho_0^{(\mathrm{per})} = \bar{\mathbf{B}}$.

Requiring that the initial density is away from zero precludes considering the original Lions problem, namely the case when $\alpha_2 = 0$ in (1.6). Recently in [13], the first and second authors proved the well-posedness of (1.1) for only bounded initial density

$$0 \leq \varrho_0 \leq \varrho^* \tag{1.7}$$

and initial velocity satisfying

$$v_0 \in H_0^1(\Omega), \quad \operatorname{div} v_0 = 0. \tag{1.8}$$

In the two-dimensional case, the solutions are global without any additional condition while, in the three-dimensional case, v_0 has to satisfy some smallness condition (as the results of [13] are of particular importance for our analysis, they will be recalled precisely below). As a by-product, the authors obtained a positive answer to Lions' question in the case $\varrho_0 = \chi_{A_0}$: persistence of Hölder regularity $C^{1,\alpha}$ holds true for any $0 < \alpha < 1$ in two dimensions and $0 < \alpha < \frac{1}{2}$ in three dimensions.

However, the question concerning the stability of the solutions was left open in [13]. In fact, if the density is bounded away from zero then the stability can be proved in the same way as uniqueness, but this is no longer the case if the initial density is allowed to vanish (this has to do with the parabolic character of the momentum equation, which degenerates if the density vanishes). This typically happens if we consider the following model configuration (see Figure 1): the original density is $\varrho_0^{(\mathrm{org})}$ and the perturbation is $\varrho_0^{(\mathrm{per})}$, that is,

$$\varrho_0^{(\mathrm{org})} = \chi_A \quad \text{and} \quad \varrho_0^{(\mathrm{per})} = \chi_B. \tag{1.9}$$

The problem happens whenever

$$A \neq B \quad \text{but the measure of } (A \setminus B) \cup (B \setminus A) \text{ is small.}$$

Then the perturbation is large in $L_\infty(\Omega)$ but small in $L_p(\Omega)$ for all $p < \infty$. There is a need for a special functional framework for stability that captures this situation.

The goal of the present paper is to show the stability of solutions to the inhomogeneous Navier–Stokes system (1.1) with respect to initial conditions of type (1.2), in a regularity setting that includes the density patches problem (1.6) *even if one of the parameters* α_1, α_2 *vanishes*. In particular, we allow the density to be a characteristic function of some set. In the most pathological case the supports of $\varrho_0^{(\mathrm{org})}$ and $\varrho_0^{(\mathrm{per})}$ can even be disjoint.

Let us introduce the notation used in the paper. First, for all $p \in [1,\infty]$ and $k \in \mathbb{N}$, L_p and W_p^k designate Lebesgue and Sobolev spaces, respectively (the dependence with respect to the fluid domain Ω is omitted, and we keep the same notation for vector-valued functions). For the corresponding norms, we use the short notation

$$\|\cdot\|_p := \|\cdot\|_{L_p}, \quad \|\cdot\|_{k,p} := \|\cdot\|_{W_p^k}.$$

Second, for any time interval $I \subset \mathbb{R}$ and Banach space X, we denote by $L_p(I;X)$ the Bochner space of measurable functions ϕ from I to X such that $t \mapsto \|\phi(t)\|_X$ lies in the standard space $L_p(I)$. In some computations, we will agree that $f_p(t)$ denotes a generic function of time which is in $L_p(\mathbb{R}_+)$, and that $f_{p,q}(t)$ stands for a function which is in $L_p(\mathbb{R}_+) \cap L_q(\mathbb{R}_+)$. The precise form of these functions may vary from line to line, but the property of integrability is preserved.

Before stating our main stability result, we have to recall the state of the art concerning global well-posedness for (1.1) supplemented with general data satisfying (1.7) and (1.8).

In dimension $d = 2$, [13, Thm. 2.1] states the following result:

Theorem 1. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$, or a two-dimensional torus. Let $\varrho_0 \in L_\infty(\Omega)$ satisfy (1.7) and let v_0 satisfy (1.8). Then system (1.1) admits a unique solution (ϱ, v, P) such that*

$$\begin{aligned}
&\varrho \in L_\infty(\mathbb{R}_+;L_\infty), \quad v \in L_\infty(\mathbb{R}_+;H^1),\\
&\sqrt{\varrho}v_t, \nabla^2 v, \nabla P \in L_2(\mathbb{R}_+;L_2), \quad \nabla v \in L_{1,\mathrm{loc}}(\mathbb{R}_+;L_\infty),\\
&\sqrt{\varrho}v \in \mathcal{C}(\mathbb{R}_+;L_2) \quad \text{and} \quad \varrho \in \mathcal{C}(\mathbb{R}_+;L_p) \quad \text{for all } 1 \le p < \infty.
\end{aligned} \tag{1.10}$$

For arbitrarily large but finite time $T > 0$, these solutions satisfy in addition, for all $1 \le r < 2$, $1 \le m < \infty$, $s < 1/2$ and $1 \le p < \infty$,

$$\begin{aligned}
&\nabla(\sqrt{t}P), \nabla^2(\sqrt{t}v) \in L_\infty(0,T;L_r) \cap L_2(0,T;L_m), \quad v \in H^s(0,T;L_p),\\
&\sqrt{t\varrho}v_t \in L_\infty(0,T;L_2) \quad \text{and} \quad \nabla v_t \in L_2(0,T;L_2).
\end{aligned} \tag{1.11}$$

In the three-dimensional case, we know the following result from [13, Thm. 2.2]:

Theorem 2. *Let Ω be a smooth bounded domain of $\mathbb{R}^3$, or a three-dimensional torus. Let $\varrho_0 \in L_\infty(\Omega)$ satisfy (1.7) and v_0 satisfy (1.8). There exists $c > 0$ such that if, in addition,*

$$(\varrho^*)^{3/2}\|\sqrt{\varrho_0}v_0\|_2\|\nabla v_0\|_2 \le c\mu^2 \quad \text{with } \varrho^* := \sup_{x\in\Omega}\varrho_0(x), \tag{1.12}$$

then system (1.1) *admits a unique solution* (ϱ, v, P) *satisfying* (1.10) *and, for any finite* $T > 0$, $s < 1/2$ *and* $1 \le p < \infty$,

$$\begin{aligned}&\nabla(\sqrt{t}P), \nabla^2(\sqrt{t}v) \in L_\infty(0,T;L_2) \cap L_2(0,T;L_6), \quad v \in H^s(0,T;L_p),\\&\sqrt{t\varrho}v_t \in L_\infty(0,T;L_2) \quad \textit{and} \quad \nabla v_t \in L_2(0,T;L_2).\end{aligned} \tag{1.13}$$

Although the above solutions are unique, the question of their stability remains open so far. Here we aim to supplement the above statements with a stability result. In order to obtain the most accurate information, it is natural to use Lagrangian coordinates since, in this setting, the density is time independent (it only depends on the position of the particles initially). Therefore, the problem is reduced to the control of the difference of the velocities which, somehow, satisfies a parabolic equation.

Let us briefly recall how to define Lagrangian coordinates in our setting. First, we introduce the flow $X\colon (t,y) \mapsto X(t,y)$ of v, which is the unique solution of the ODE

$$\begin{cases}\dfrac{\mathrm{d}X}{\mathrm{d}t} = v(t, X(t,y)) & \text{in } \mathbb{R}_+ \times \Omega,\\ X(0,y) = y & \text{in } \Omega.\end{cases} \tag{1.14}$$

Integrating (1.14) yields the following relation between the Eulerian "x" and Lagrangian "y" coordinates:

$$x = X(t,y) = y + \int_0^t v(t', X(t',y))\,\mathrm{d}t'. \tag{1.15}$$

By the standard theory of ODEs, the above change of coordinates is well defined whenever $v \in L_{1,\mathrm{loc}}(\mathbb{R}_+; C^{1,0})$. In the coordinate system (t,y), the unknown functions are named

$$u(t,y) = v(t,X(t,y)), \quad \eta(t,y) = \varrho(t,X(t,y)), \quad Q(t,y) = P(t,X(t,y)).$$

Let us denote

$$\begin{aligned}A_u(t) = \Big(\frac{\mathrm{d}X}{\mathrm{d}y}\Big)^{-1} &= \Big(\mathrm{Id} + \int_0^t \nabla_y u(t',y)\,\mathrm{d}t'\Big)^{-1}\\ &= \Big[\mathrm{cof}\Big(\mathrm{Id} + \int_0^t \nabla_y u(t',y)\,\mathrm{d}t'\Big)\Big]^T,\end{aligned} \tag{1.16}$$

where $\mathrm{cof}(\cdot)$ denotes the cofactor matrix and, in the second equality, we used the fact that $\det A_u = 1$ (see e.g. [13]). For a function $f(t,x)$, denote $\tilde f(t,y) = f(t,X(t,y))$. Then, owing to the chain rule,

$$\begin{aligned}\nabla_x f(t,x) &= A_u^T \nabla_y \tilde f(t,y) =: \nabla_u \tilde f(t,y),\\ \partial_t f(t,x) + v(t,x)\cdot\nabla_x f(t,x) &= \partial_t \tilde f(t,y).\end{aligned} \tag{1.17}$$

In order to transform the divergence operator, observe that Piola's identity (see e.g. [2,6]) ensures that $\mathrm{div}_y\, A_u = 0$ since $\det A_u = 1$. Therefore, for any vector field $z(t,y)$, one may write

$$\mathrm{div}_y(A_u z) = A_u^T : \nabla_y z + z \cdot \mathrm{div}_y(A_u^T) = A_u^T : \nabla_y z. \tag{1.18}$$

Hence, if we denote $\widetilde{w}(t, y) = w(t, X(t, y))$ for any vector field $w(t, x)$, we discover that

$$\operatorname{div}_x w(t, x) = A_u^T : \nabla_y \widetilde{w}(t, y) = \operatorname{div}_y (A_u \widetilde{w}) =: \operatorname{div}_u \widetilde{w}. \tag{1.19}$$

Taking all the above into account we see that in coordinates (t, y), system (1.1) reads

$$\begin{cases} \eta_t = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \eta u_t - \mu \operatorname{div}_u \nabla_u u + \nabla_u Q = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div}_u u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u|_{t=0} = v_0, \ \eta|_{t=0} = \varrho_0 & \text{in } \Omega. \end{cases} \tag{1.20}$$

The main achievement of this paper is the following stability result in the Lagrangian coordinates setting.

Theorem 3. *Let Ω be a smooth bounded domain of $\mathbb{R}^d$ with $d = 2, 3$. Let (ϱ^1, v^1) and (ϱ^2, v^2) be two solutions of (1.1) with initial data (ϱ_0^1, v_0^1) and (ϱ_0^2, v_0^2), respectively, with nonidentically zero bounded ϱ_0^1 and ϱ_0^2, and v_0^1, v_0^2 in $H_0^1(\Omega)$, given either by Theorem 1 or by Theorem 2 (depending on the dimension). Denote by u^1 and u^2 the corresponding velocities in Lagrangian coordinates. Finally, set $\delta u := u^2 - u^1$, $\delta v_0 := v_0^2 - v_0^1$ and $\delta\varrho := \varrho^2 - \varrho^1$.*

Then there exists a positive constant γ depending only on the shape of Ω (i.e. invariant by dilation or isometric transformation of Ω), such that for all $t \geq 0$,

$$e^{\frac{\gamma\mu t}{\varrho^* \mathfrak{d}^2}} \| \min\{\sqrt{\varrho_0^1}, \sqrt{\varrho_0^2}\} \delta u(t)\|_2 + \mu^{1/2} \| e^{\frac{\gamma\mu t}{\varrho^* \mathfrak{d}^2}} \nabla \delta u\|_{L_2(0,t;L_2)} \\ \leq C\big(\|\sqrt{\varrho_0^1}\delta v_0\|_2 + \|\delta\varrho_0\|_2^{1/2}\big), \tag{1.21}$$

where $\mathfrak{d}$ stands for the diameter of Ω, $\varrho^* := \max\{\|\varrho_0^1\|_\infty, \|\varrho_0^2\|_\infty\}$ and $C = C(\Omega, d, \varrho_0^1, \varrho_0^2, u_0^1, u_0^2)$.

Coming back from Lagrangian to Eulerian coordinates, we obtain the following corollary:

Corollary 1. *Under the assumptions of Theorem 3, we have*

$$\sup_{t\in\mathbb{R}_+} \|\delta\varrho(t)\|_{W_p^{-1}} + \sup_{t\in\mathbb{R}_+} e^{\frac{\beta\mu t}{\varrho^* \mathfrak{d}^2}} \|(\sqrt{\varrho^1}\delta v)(t)\|_2 + \mu^{1/2} \| e^{\frac{\beta\mu t}{\varrho^* \mathfrak{d}^2}} \nabla \delta v\|_{L_2(\mathbb{R}_+, L_2)} \\ \leq C_0\big(\|\sqrt{\varrho_0^1}\delta v_0\|_2 + \|\delta\varrho_0\|_2^{1/2}\big)$$

for all $1 < p < \infty$ if $d = 2$ and $1 < p \leq 6$ if $d = 3$, where $\delta v := v^2 - v^1$ and v^1, v^2 are the solutions with data (ϱ_0^1, v_0^1) and (ϱ_0^2, v_0^2) given by Theorems 1 or 2.

A key point in getting rid of any smallness condition is to first establish sufficiently strong time decay of the solutions that have been constructed in Theorems 1 and 2. This will be achieved in Section 3, where by means of rather classical energy arguments, we will even get *exponential* decay, owing to the boundedness of the fluid domain. Before that, in Section 2 we will compare two global solutions satisfying a priori those decay properties, and estimate their difference in Lagrangian coordinates in terms of the difference of the data, getting the result of Theorem 3. Finally, we will rewrite those estimates in Eulerian coordinates to obtain Corollary 1.

In our low regularity setting, the key difficulty for proving stability of solutions to (1.1) comes from the partially hyperbolic nature of the system. In fact, if writing the system satisfied by the difference $(\delta\varrho, \delta v, \delta P)$ of two solutions (ϱ^1, v^1, P^1) and (ϱ^2, v^2, P^2) of (1.1), then the mass equation gives

$$(\partial_t + v^2 \cdot \nabla)\delta\varrho = -\delta v \cdot \nabla\varrho^1.$$

In our framework, where the density is only in $L_\infty(\Omega)$, this forces us to perform estimates for $\delta\varrho$ in a space with regularity index equal to -1. Following the duality approach initiated by Hoff [26] and recently renewed in [14, 36] (for the related compressible Navier–Stokes system), we will actually prove stability estimates for the density in W_p^{-1}, and in L_2 for the velocity.

Proving the exponential decay estimates of Section 3 can be achieved by means of a remarkably simple energy method that is performed directly on the nonlinear problem. This is in sharp contrast with the proof of decay estimates for compressible Navier–Stokes and related models which requires a refined analysis of the linearized system combined with a perturbation argument (see among others [17,19,39,41] and the references therein).

When comparing Theorem 3 with results of [13], a remark is in order concerning the domain. Although Theorems 1 and 2 hold both for a bounded domain with homogeneous Dirichlet conditions or a torus, here we restricted our analysis to the case of a bounded domain to avoid further technical complications. In fact, in the case of Dirichlet boundary conditions, we have the basic Poincaré inequality at hand, which is very helpful to close the estimates globally in time. In the torus case, the corresponding Poincaré inequality has an additional term (namely the total momentum of the solution, see [13, Lem. A.1]) which, although probably harmless, entails serious complications in the proof of decay estimates. Therefore, we leave the torus case for future research.

2. Stability under given decay properties

Throughout this section, we are given two solutions (ϱ^1, v^1, P^1) and (ϱ^2, v^2, P^2) pertaining to data (ϱ_0^1, v_0^1) and (ϱ_0^2, v_0^2), and satisfying the following properties for some $\beta_0 > 0$:

$$\sqrt{\varrho}e^{\beta_0 t} v_t^i \in L_2(\mathbb{R}_+, L_2), \tag{2.1a}$$

$$e^{\beta_0 t}\nabla v^i \in L_1(\mathbb{R}_+; L_\infty) \cap L_4(\mathbb{R}_+; L_3) \cap L_2(\mathbb{R}_+; L_6), \tag{2.1b}$$

$$\sqrt{t}e^{\beta_0 t} v^i \in L_1(\mathbb{R}_+; L_\infty) \cap L_\infty(\mathbb{R}_+; L_\infty), \tag{2.1c}$$

$$\sqrt{t}e^{\beta_0 t}(\nabla^2 v^i, \nabla P^i) \in L_2(\mathbb{R}_+; L_6), \tag{2.1d}$$

$$\sqrt{t}e^{\beta_0 t} v^i_t \in L_{4/3}(\mathbb{R}_+; L_6), \tag{2.1e}$$

$$e^{\beta_0 t} v^i \in L_1(\mathbb{R}_+; L_6) \cap L_\infty(\mathbb{R}_+; L_6), \tag{2.1f}$$

$$\sqrt{t}e^{\beta_0 t}\nabla v^i \in L_2(\mathbb{R}_+; L_\infty), \tag{2.1g}$$

$$e^{\beta_0 t}\nabla^2 v^i \in L_1(\mathbb{R}_+; L_r) \quad \text{for some } r > d. \tag{2.1h}$$

We denote by (η^i, u^i, Q^i) the corresponding solutions in Lagrangian coordinates (hence $\eta^i = \varrho^i_0$).

Lemma 1. *Let* (ϱ, v, P) *solve* (1.1) *and satisfy conditions* (2.1a)–(2.1h). *Then the corresponding Lagrangian solution* (η, u, Q) *also satisfies* (2.1a)–(2.1h).

Proof. The properties involving only the velocity and its first-order space derivatives follow directly from the corresponding ones for v, and from the fact that the matrix A_u^{-1} is bounded. It remains to prove the properties involving second-order space derivatives and the time derivative.

To prove (2.1a), we start from the identity

$$\sqrt{\eta}e^{\beta_0 t}u_t = \sqrt{\varrho \circ X(t,\cdot)}e^{\beta_0 t}(v_t + v\cdot\nabla v)\circ X(t,\cdot).$$

As $X(t,\cdot)$ is measure preserving, the term with v_t may be bounded by means of (2.1a). To bound the other term, it suffices to observe that

$$\begin{aligned}&\|e^{\beta_0 t}\sqrt{\varrho\circ X(t,\cdot)}(v\cdot\nabla v)\circ X(t,\cdot)\|_{L_2(\mathbb{R}_+\times\Omega)}\\ &\quad\le \varrho^*\|e^{\beta_0 t/2}v\|_{L_\infty(\mathbb{R}_+;L_6)}\|e^{\beta_0 t/2}\nabla v\|_{L_2(\mathbb{R}_+;L_3)}.\end{aligned}$$

The first term of the right-hand side may be bounded thanks to (2.1f) and the second one according to (2.1b) and to the boundedness of Ω.

To prove (2.1e), one can again use $u_t = (v_t + v\cdot\nabla v)\circ X(t,\cdot)$ and properties (2.1e)–(2.1g) for v.

In order to prove (2.1h), we differentiate the identity

$$\nabla_y u(t,y) = {}^T(A_u)^{-1}\nabla_x v(t, X(t,y))$$

with respect to y. By the chain rule we obtain

$$\begin{aligned}&\|\nabla_y^2 u\|_{L_1(0,T;L_r)}\\ &\quad\le C\|\nabla_x^2 v\|_{L_1(0,T;L_r)} + C\|\nabla_y(A_u)^{-1}\|_{L_\infty(0,T;L_r)}\|\nabla_x v\|_{L_1(0,T;L_\infty)},\end{aligned} \tag{2.2}$$

which, differentiating (1.15), implies (2.1h) for u due to (2.1h) for v and to (2.1b) for u. In order to prove (2.1d) we proceed similarly, rewriting (2.2) with L_1 norm in time replaced by L_2, and L_r in space replaced by L_6. ∎

In the rest of this section, we aim to estimate

$$\delta u := u^2 - u^1 \quad \text{and} \quad \delta Q := Q^2 - Q^1$$

in terms of the difference of the data. Obviously, denoting $\Delta_u := \mathrm{div}_u \nabla_u$ and $\delta v_0 = v_0^2 - v_0^1$, the couple $(\delta u, \delta Q)$ satisfies

$$\begin{aligned} \varrho_0^1 \delta u_t - \mu \Delta_{u^1} \delta u + \nabla_{u^1} \delta Q &= \mu(\Delta_{u^2} - \Delta_{u^1}) u^2 - (\nabla_{u^2} - \nabla_{u^1}) Q^2 - \delta\varrho_0 u_t^2, \\ \mathrm{div}_{u^1} \delta u &= (\mathrm{div}_{u^1} - \mathrm{div}_{u^2}) u^2, \\ \delta u|_{t=0} &= \delta v_0. \end{aligned} \tag{2.3}$$

Note that

$$\delta\varrho_0 := \varrho_0^2 - \varrho_0^1 = \eta^2 - \eta^1.$$

By (1.20)$_1$, functions η^i are constant in time, so that the perturbation of the density is time independent. As a matter of fact, it is the main motivation for our choosing the Lagrange coordinates approach to deal with the stability issue of system (1.1).

2.1. The case of a nice control of vacuum

Compared to the proof of uniqueness that has been performed in [13], the troublemaker is the term $\delta\varrho_0 u_t^2$ in (2.3) since Theorems 1 or 2 only provide us with an information on $\sqrt{\varrho^2} v_t^2$ (hence on $\sqrt{\varrho_0^2} u_t^2$) while we do not necessarily have

$$\mathrm{supp}\, \delta\varrho_0 \subset \mathrm{supp}\, \varrho_0^2. \tag{2.4}$$

In this part, we assume that the initial densities satisfy

$$\|\delta\varrho_0\|_X := \|\delta\varrho_0 / \sqrt{\varrho_0^2}\|_4 < \infty \tag{2.5}$$

and derive a differential inequality which is crucial for proving Theorem 3. The general case, when (2.4) is not valid, is postponed to the next subsection.

The idea is to decompose δu into

$$\delta u = w + z, \tag{2.6}$$

where w stands for a suitable solution to the divergence equation

$$\mathrm{div}_{u^1} w = (\mathrm{div}_{u^1} - \mathrm{div}_{u^2}) u^2 = {}^T\delta A : \nabla u^2 = \mathrm{div}(\delta A\, u^2), \quad w|_{\partial\Omega} = 0, \tag{2.7}$$

with

$$\delta A := A^1 - A^2 \quad \text{and} \quad A^i := A_{u^i}. \tag{2.8}$$

Since by (1.16) we have $\delta A = 0$ initially, we put $w = 0$ at $t = 0$. Although matrices A^1 and A^2 need not be close to Id, they are invertible and uniformly bounded in time.

Lemma 2. *Let u^1, u^2 satisfy* (2.1a)–(2.1h) *and let δA be defined in* (2.8). *Then there exists a solution w to* (2.7) *such that*

$$\begin{aligned}\|e^{\beta_0 t} w(t)\|_2 &\leq f_{1,\infty}(t)\biggl(\int_0^t \|\nabla \delta u(\tau)\|_2^2 \,\mathrm{d}\tau\biggr)^{1/2},\\ \|e^{\beta_0 t} \nabla w(t)\|_2 &\leq f_2(t)\biggl(\int_0^t \|\nabla \delta u(\tau)\|_2^2 \,\mathrm{d}\tau\biggr)^{1/2},\\ \|e^{\beta_0 t} w_t(t)\|_{3/2} &\leq f_{4/3}(t)\biggl(\int_0^t \|\nabla \delta u(\tau)\|_2^2 \,\mathrm{d}\tau\biggr)^{1/2} + f_4(t)\|\nabla \delta u(t)\|_2,\end{aligned} \tag{2.9}$$

where the notation $f_p(t)$ and $f_{p,q}(t)$ was explained at the end of Section 1.

Proof. The vector w will be sought in the form $w = (A^1)^{-1}A^1 w = (A^1)^{-1}\overline{w}$, where $\overline{w}$ is given as a solution to

$$\operatorname{div}\overline{w} = \operatorname{div}(A^1 w) = \operatorname{div}_{u^1} w = (\operatorname{div}_{u^1} - \operatorname{div}_{u^2})u^2 = {}^T\delta A : \nabla u^2 = \operatorname{div}(\delta A\, u^2). \tag{2.10}$$

As a first step in the proof of our claim, let us establish the following bounds:

$$\begin{aligned}\|e^{\beta_0 t}\overline{w}\|_{L_4(0,T;L_2)} &\leq C\|e^{\beta_0 t}\delta A\, u^2\|_{L_4(0,T;L_2)},\\ \|e^{\beta_0 t}\nabla\overline{w}\|_{L_2(0,T;L_2)} &\leq C\|e^{\beta_0 t}\, {}^T\delta A : \nabla u^2\|_{L_2(0,T;L_2)}\\ \text{and } \|e^{\beta_0 t}\overline{w}_t\|_{L_{4/3}(0,T;L_{3/2})} &\leq C\|e^{\beta_0 t}(\delta A\, u^2)_t\|_{L_{4/3}(0,T;L_{3/2})}.\end{aligned} \tag{2.11}$$

The existence of a vector field $\overline{w}$ satisfying (2.10)–(2.11) is ensured by the following lemma:

Lemma 3. *Let A be a matrix-valued function with* $\det A \equiv 1$. *Consider the following divergence equation in a bounded domain with smooth boundary:*

$$\operatorname{div} b = f \text{ in } \mathbb{R}_+ \times \Omega, \quad b = 0 \text{ on } \mathbb{R}_+ \times \partial\Omega,$$

where, for some matrix-valued function d, $f = A^T : \nabla d = \operatorname{div}(Ad)$ and the average of f equals 0.

Then there exists a constant C such that for any $\beta \geq 0$, there exists a solution b to the above equation, such that for all $t \geq 0$, we have

$$\begin{aligned}\|e^{\beta t} b(t)\|_2 &\leq C\|e^{\beta t} A(t)d(t)\|_2,\\ \|e^{\beta t}\nabla b(t)\|_2 &\leq C\|e^{\beta t} A(t)^T \nabla d(t)\|_2,\\ \|e^{\beta t} b_t(t)\|_{3/2} &\leq C\|e^{\beta t}(Ad)_t(t)\|_{3/2}.\end{aligned}$$

Lemma 3 has been proved without exponential weight by the first two authors in [10] (see also [13, Lem. A.3]). In their proof, the function b is given by an explicit formula where the time variable does not come into play. Consequently, a time weight can be treated as a multiplicative parameter (the norms in Lemma 3 only involve the space variable).

Now let us bound the right-hand sides of (2.11). In order to emphasize that we do not need any smallness of $\int_0^t \nabla_y u\,\mathrm{d}\tau$, let us derive an explicit formula for δA.

In the two-dimensional case, starting from (1.16), we immediately obtain

$$\delta A(t) = \begin{bmatrix} \int_0^t \delta u_{2,y_2}\,\mathrm{d}t' & -\int_0^t \delta u_{1,y_2}\,\mathrm{d}t' \\ -\int_0^t \delta u_{2,y_1}\,\mathrm{d}t' & \int_0^t \delta u_{1,y_1}\,\mathrm{d}t' \end{bmatrix}. \tag{2.12}$$

For $d = 3$, one can also use (1.16) to determine δA. As an example, let us compute its first entry. We have

$$(A_u^i)_{11}(t) = \left(1 + \int_0^t u^i_{2,y_2}\,\mathrm{d}t'\right)\left(1 + \int_0^t u^i_{3,y_3}\,\mathrm{d}t'\right) - \int_0^t u^i_{3,y_2}\,\mathrm{d}t' \int_0^t u^i_{2,y_3}\,\mathrm{d}t' \quad \text{for } i = 1, 2.$$

Therefore,

$$\begin{aligned}(\delta A(t))_{11} = & \int_0^t \delta u_{2,y_2}\,\mathrm{d}t' + \int_0^t \delta u_{3,y_3}\,\mathrm{d}t' + \int_0^t \delta u_{2,y_2}\,\mathrm{d}t' \int_0^t u^2_{3,y_3}\,\mathrm{d}t' \\ & + \int_0^t \delta u_{3,y_3}\,\mathrm{d}t' \int_0^t u^1_{2,y_2}\,\mathrm{d}t' - \int_0^t \delta u_{3,y_2}\,\mathrm{d}t' \int_0^t u^2_{2,y_3}\,\mathrm{d}t' \\ & - \int_0^t \delta u_{2,y_3}\,\mathrm{d}t' \int_0^t u^1_{3,y_2}\,\mathrm{d}t'.\end{aligned}$$

The other entries have a similar structure, namely

$$\begin{aligned}(\delta A(t))_{ij} = & \sum_{1\le k,l\le 3} a^{ij}_{kl} \int_0^t \delta u_{k,y_l}\,\mathrm{d}t' \\ & + \sum_{\substack{1\le k,l,m,n\le 3 \\ s\in\{1,2\}}} b^{ij}_{k,l,m,n,s} \int_0^t \delta u_{k,y_l}\,\mathrm{d}t' \int_0^t u^s_{m,y_n}\,\mathrm{d}t',\end{aligned} \tag{2.13}$$

where $a^{ij}_{kl}, b^{ij}_{k,l,m,n,s} \in \{0, 1\}$.

Now, if u^1, u^2 satisfy (2.1b) then, by the Hölder inequality, we obtain

$$\|t^{-1/2}\delta A(t)\|_2 \le C \left\| t^{-1/2} \int_0^t \nabla \delta u\,\mathrm{d}\tau \right\|_2 \le C \left(\int_0^t \|\nabla \delta u\|_2^2\,\mathrm{d}\tau\right)^{1/2}. \tag{2.14}$$

Therefore, by (2.1g), we have

$$\begin{aligned}\|e^{\beta_0 t}\, {}^T\delta A : \nabla u^2(t)\|_2 & \le \|t^{-1/2}\delta A(t)\|_2 \|e^{\beta_0 t} t^{1/2} \nabla u^2\|_\infty \\ & \le C f_2(t) \left(\int_0^t \|\nabla \delta u\|_2^2\,\mathrm{d}\tau\right)^{1/2},\end{aligned} \tag{2.15}$$

which implies (2.9) for $\|\nabla \overline{w}\|_2$.

Similarly, by (2.1c) and (2.14),

$$\|e^{\beta_0 t}\delta A\, u^2\|_2 \le \|t^{-1/2}\delta A\|_2 \|e^{\beta_0 t} t^{1/2} u^2\|_\infty \le f_{1,\infty}(t)\Big(\int_0^t \|\nabla\delta u(\tau)\|_2^2\,\mathrm{d}\tau\Big)^{1/2},$$

whence, applying (2.11) gives

$$\|e^{\beta_0 t}\,\overline{w}(t)\|_2 \le f_1(t)\Big(\int_0^t \|\nabla\delta u(\tau)\|_2^2\,\mathrm{d}\tau\Big)^{1/2}. \tag{2.16}$$

In order to bound $\overline{w}_t$, it suffices to derive an appropriate estimate in $L_{4/3}(0,T;L_{3/2})$ for

$$(\delta A\, u^2)_t = \delta A\, u^2_t + (\delta A)_t u^2.$$

For the first term, thanks to (2.1e) and (2.14) we have

$$\|e^{\beta_0 t}\delta A\, u^2_t\|_{3/2} \le \|t^{-1/2}\delta A\|_2 \|e^{\beta_0 t} t^{1/2} u^2_t\|_6 \le f_{4/3}(t)\Big(\int_0^t \|\nabla\delta u(\tau)\|_2^2\,\mathrm{d}\tau\Big)^{1/2}.$$

The other term can be bounded as

$$\|e^{\beta_0 t}(\delta A)_t u^2\|_{3/2} \le \|(\delta A)_t\|_2 \|e^{\beta_0 t} u^2\|_6.$$

Differentiating (2.13) with respect to t and using (2.1g) for u^1 and u^2, we see that

$$\|\delta A_t(t)\|_2 \le C\bigg(\|\nabla\delta u(t)\|_2 + \Big\|t^{-1/2}\int_0^t \nabla\delta u(\tau)\,\mathrm{d}\tau\Big\|_2 \big(\|t^{1/2}\nabla u^1(t)\|_\infty + \|t^{1/2}\nabla u^2(t)\|_\infty\big)\bigg).$$

In the two-dimensional case, owing to (2.12), one can skip the second term on the right-hand side of the above inequality. Thus, thanks to (2.1f), for both $d = 2, 3$ we conclude that

$$\|\delta A_t u^2(t)\|_{3/2} \le f_{4/3}(t)\Big(\int_0^t \|\nabla\delta u(\tau)\|_2^2\,\mathrm{d}\tau\Big)^{1/2} + f_4(t)\|\nabla\delta u(t)\|_2,$$

which, by (2.11), implies (2.9) for $\|\overline{w}_t\|_{3/2}$. Altogether, this gives the thesis of Lemma 2 for $\overline{w}$, but not yet for w.

In order to get (2.9) for w from the estimates of $\overline{w}$, we first observe that

$$\sup_{t\le T}\|(A^1)^{-1}(t)\|_\infty \le C(1 + \|\nabla u^1\|_{L_1(0,T;L_\infty)}). \tag{2.17}$$

Therefore, as $w = (A^1)^{-1}\overline{w}$, we get

$$\|e^{\beta_0 t} w\|_{L_4(0,T;L_2)} \le C\|e^{\beta_0 t}\overline{w}\|_{L_4(0,T;L_2)}.$$

In order to estimate $\|\nabla w\|_{L_2(0,T;L_2)}$ we proceed as follows:

$$\begin{aligned}
&\|e^{\beta_0 t}\bar{w}\nabla((A^1)^{-1})\|_{L_2(0,T;L_2)}\\
&\quad\le \|\nabla((A^1)^{-1})\|_{L_\infty(0,T;L_r)}\|e^{\beta_0 t}\bar{w}\|_{L_2(0,T;L_{r*})}\\
&\quad\le C\|\nabla^2 u^1\|_{L_1(0,T;L_r)}\|e^{\beta_0 t}\bar{w}\|_{L_2(0,T;L_{r*})} \quad \text{with } \frac{1}{2}=\frac{1}{r}+\frac{1}{r^*},
\end{aligned}$$

where in the last passage we used (1.16). Therefore, by (2.1b) and (2.1h), we have

$$\begin{aligned}
\|e^{\beta_0 t}\nabla w\|_{L_2(0,T;L_2)} &\le \|e^{\beta_0 t}(A^1)^{-1}\nabla\bar{w}\|_{L_2(0,T;L_2)} + \|e^{\beta_0 t}\bar{w}\nabla((A^1)^{-1})\|_{L_2(0,T;L_2)}\\
&\le C[\|e^{\beta_0 t}\nabla\bar{w}\|_{L_2(0,T;L_2)} + \|e^{\beta_0 t}\bar{w}\|_{L_2(0,T;L_{r^*})}].
\end{aligned}$$

Since, in (2.1h), one can take $r > d$, one can always ensure that $r^* < 6$.

Finally, we have

$$\|((A^1)^{-1})_t\bar{w}\|_{3/2} \le \|((A^1)^{-1})_t\|_6\|\bar{w}\|_2 \le \|\nabla u^1\|_6\|\bar{w}\|_2,$$

which together with (2.17) implies

$$\begin{aligned}
\|e^{\beta_0 t}w_t\|_{L_{4/3}(0,T;L_{3/2})} &\le C\|(A^1)^{-1}\|_\infty\|e^{\beta_0 t}\bar{w}_t\|_{L_{4/3}(0,T;L_{3/2})}\\
&\quad + C\|\nabla u^1\|_{L_2(0,T;L_6)}\|e^{\beta_0 t}\bar{w}\|_{L_4(0,T;L_2)}\\
&\le C(\|e^{\beta_0 t}\bar{w}_t\|_{L_{4/3}(0,T;L_{3/2})} + \|e^{\beta_0 t}\bar{w}\|_{L_4(0,T;L_2)}),
\end{aligned}$$

which completes the proof of Lemma 2. ∎

Let us restate equations (2.3) in terms of $(z,\delta Q)$ as follows (assuming that $\mu = 1$ for simplicity):

$$\begin{cases}
\varrho_0^1 z_t - \Delta_{u^1} z + \nabla_{u^1}\delta Q\\
\qquad = (\Delta_{u^2}-\Delta_{u^1})u^2 + (\nabla_{u^1}-\nabla_{u^2})Q^2 - \varrho_0^1 w_t + \Delta_{u^1} w - \delta\varrho_0\, u_t^2,\\
\operatorname{div}_{u^1} z = 0, \quad z|_{t=0} = \delta v_0, \quad z|_{\partial\Omega} = 0.
\end{cases} \tag{2.18}$$

Observe that for a vector field z and functions f, g defined in Lagrangian coordinates we have, according to (1.17) and integration by parts,

$$\begin{aligned}
-\int_\Omega f\operatorname{div}_u z\,dy &= -\int_\Omega f\operatorname{div}(A_u z)\,dy = \int_\Omega A_u z\cdot\nabla_y f\,dy\\
&= \int_\Omega z\cdot A_u^T\nabla_y f\,dy = \int_\Omega z\cdot\nabla_u f\,dy,
\end{aligned} \tag{2.19}$$

which implies

$$-\int_\Omega f\Delta_u g\,dy = -\int_\Omega f\operatorname{div}_u(\nabla_u g)\,dy = \int_\Omega \nabla_u f\cdot\nabla_u g\,dy. \tag{2.20}$$

These identities allow us to test (2.18) by z, while (2.19) implies the following crucial property, thanks to which one does not need to care about the difference of the pressures:

$$\int_\Omega (\nabla_{u^1}\delta Q)\cdot z\,\mathrm{d}y = -\int_\Omega \mathrm{div}_{u^1}\, z\delta Q\,\mathrm{d}y = 0. \tag{2.21}$$

Therefore, also using (2.20), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho_0^1|z|^2\,\mathrm{d}x + \int_\Omega |\nabla_{u^1}z|^2\,\mathrm{d}x = \sum_{k=1}^{5} I_k, \tag{2.22}$$

where

$$I_1 := \int_\Omega ((\Delta_{u^2}-\Delta_{u^1})u^2)\cdot z\,\mathrm{d}y, \qquad I_2 := \int_\Omega ((\nabla_{u^1}-\nabla_{u^2})Q^2)\cdot z\,\mathrm{d}y,$$
$$I_3 := -\int_\Omega \varrho_0^1 w_t\cdot z\,\mathrm{d}y, \qquad I_4 := \int_\Omega (\Delta_{u^1}w)\cdot z\,\mathrm{d}y,$$
$$I_5 := -\int_\Omega \delta\varrho_0 u_t^2\cdot z\,\mathrm{d}y.$$

In order to bound I_1, we combine (2.1g) and (2.14) to write, for all $\varepsilon>0$,

$$\begin{aligned}
|I_1| &= \left|\int_\Omega \mathrm{div}\big(((\delta A)^T A^2 + (A^1)^T\delta A)\nabla u^2\big)\cdot z\,\mathrm{d}y\right| \\
&\le \int_\Omega |(\delta A^T)A^2 + (A^1)^T\delta A|\,|\nabla u^2|\,|\nabla z|\,\mathrm{d}x \\
&\le C\|t^{-1/2}\delta A\|_2\|t^{1/2}\nabla u^2\|_\infty\|\nabla z\|_2 \\
&\le \varepsilon\|\nabla z\|_2^2 + C\varepsilon^{-1}\|t^{1/2}\nabla u^2\|_\infty^2\left(\int_0^t \|\nabla\delta u\|_2^2\,\mathrm{d}\tau\right).
\end{aligned} \tag{2.23}$$

Next, by the Hölder inequality,

$$|I_2(t)| \le \left|\int_\Omega \delta A\,\nabla Q^2\cdot z\,\mathrm{d}x\right| \le C\|t^{-1/2}\delta A\|_2\|t^{1/2}\nabla Q^2\|_4\|z\|_4.$$

Therefore, according to (2.1d), (2.14) and to the Sobolev embedding $H_0^1 \hookrightarrow L_4$, combining the Poincaré and Young inequalities yields, for all $\varepsilon>0$,

$$I_2(t) \le \varepsilon\|\nabla z\|_2^2 + C\varepsilon^{-1}\|t^{1/2}\nabla Q^2\|_4^2\left(\int_0^t \|\nabla\delta u\|_2^2\,\mathrm{d}\tau\right). \tag{2.24}$$

Note that from the Hölder inequality and the Sobolev embedding $H^1(\Omega)\hookrightarrow L_6(\Omega)$, we have

$$\|(\varrho_0^1)^{1/4}z\|_3 \le \|\sqrt{\varrho_0^1}z\|_2^{1/2}\|z\|_6^{1/2} \le C\|\sqrt{\varrho_0^1}z\|_2^{1/2}\|z\|_{H^1}^{1/2}.$$

Therefore, using the Hölder inequality, one can write that

$$\begin{aligned}
I_3(t) &\le \|w_t\|_{3/2}\|\varrho_0^1 z\|_3 \le C\|w_t\|_{3/2}\|(\varrho_0^1)^{1/4}z\|_3 \\
&\le \|w_t\|_{3/2}\|\sqrt{\varrho_0^1}z\|_2^{1/2}\|\nabla z\|_{L_2}^{1/2}.
\end{aligned} \tag{2.25}$$

Next, integrating by parts, we get for all $\varepsilon > 0$,

$$I_4 \leq \int_\Omega |\nabla_{u^1} w|\, |\nabla_{u^1} z| \, \mathrm{d}x \leq \varepsilon \|\nabla_{u^1} z\|_2^2 + C\varepsilon^{-1} \|\nabla_{u^1} w\|_2^2. \tag{2.26}$$

Our "nice control of vacuum" hypothesis (2.5) comes into play only for handling I_5: combining the Hölder and Young inequalities, as well as the embedding $H_0^1 \hookrightarrow L_4$, we write that

$$\begin{aligned} I_5 &\leq \left| \int_\Omega \frac{\delta\varrho_0}{\sqrt{\varrho_0^2}} \sqrt{\varrho_0^2} u_t^2 z \, \mathrm{d}y \right| \\ &\leq \left\| \frac{\delta\varrho_0}{\sqrt{\varrho_0^2}} \right\|_4 \|\sqrt{\varrho_0^2} u_t^2\|_2 \|z\|_4 \\ &\leq C\varepsilon^{-1} \|\sqrt{\varrho_0^2} u_t^2\|_2^2 \|\delta\varrho_0\|_X^2 + \varepsilon \|\nabla z\|_2^2, \end{aligned} \tag{2.27}$$

where $\|\delta\varrho_0\|_X$ is defined in (2.5).

In the end, plugging (2.23), (2.24), (2.25), (2.26) and (2.27) in (2.22), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\sqrt{\varrho_0^1} z(t)\|_2^2 + \|\nabla_{u^1} z\|_2^2 \\ &\quad \leq \varepsilon \|\nabla z\|_2^2 + (\|t^{1/2}\nabla u^2\|_\infty^2 + \|t^{1/2}\nabla Q^2\|_4^2) \int_0^t \|\nabla \delta u\|_2^2 \, \mathrm{d}t \\ &\qquad + \|\sqrt{\varrho_0^2} u_t^2\|_2^2 \|\delta\varrho_0\|_X^2 + C\|\nabla_{u_1} w\|_2^2 + \|w_t\|_{3/2} \|\sqrt{\varrho_0^1} z\|_2^{1/2} \|\nabla z\|_2^{1/2}. \end{aligned}$$

Since $\nabla_{u^1} z = (A^1)^T \nabla z$, we have

$$\|\nabla_{u^1} z\|_2 \geq \|(A^1)^{-1}\|_\infty^{-1} \|\nabla z\|_2.$$

Hence, taking ε small enough, the above inequality implies for some $c_0 > 0$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \|\sqrt{\varrho_0^1} z(t)\|_2^2 + c_0 \|\nabla z\|_2^2 \\ &\quad \leq (\|t^{1/2}\nabla u^2\|_\infty^2 + \|t^{1/2}\nabla Q^2\|_4^2) \int_0^t \|\nabla \delta u\|_2^2 \, \mathrm{d}t + \|\sqrt{\varrho_0^2} u_t^2\|_2^2 \|\delta\varrho_0\|_X^2 \\ &\qquad + C\|\nabla_{u_1} w\|_2^2 + \|w_t\|_{3/2} \|\sqrt{\varrho_0^1} z\|_2^{1/2} \|\nabla z\|_2^{1/2}. \end{aligned} \tag{2.28}$$

Now, using the fact that $\|\nabla \delta u\|_2^2 \leq 2(\|\nabla z\|_2^2 + \|\nabla w\|_2^2)$, multiplying (2.28) with $e^{2\beta t}$ (with $\beta \leq \beta_0$) and adding the second inequality of (2.9), we arrive at

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \|e^{\beta t} \sqrt{\varrho_0^1} z(t)\|_2^2 + (\|e^{\beta t} \nabla z\|_2^2 + \|e^{\beta t} \nabla w\|_2^2) \\ &\quad \leq 2\beta e^{2\beta t} \|\sqrt{\varrho_0^1} z(t)\|_2^2 \\ &\qquad + (\|e^{\beta t} t^{1/2} \nabla u^2\|_\infty^2 + \|e^{\beta t} t^{1/2} \nabla Q^2\|_4^2 + f_1(t)) \int_0^T (\|\nabla z\|_2^2 + \|\nabla w\|_2^2) \, \mathrm{d}t \\ &\qquad + \|e^{\beta t} \sqrt{\varrho_0^2} u_t^2\|_2^2 \|\delta\varrho_0\|_X^2 + \|e^{\beta t} w_t\|_{3/2} \|e^{\beta t} \sqrt{\varrho_0^1} z\|_2^{1/2} \|e^{\beta t} \nabla z\|_2^{1/2}. \end{aligned} \tag{2.29}$$

For small β one can absorb the first term on the right-hand side due to the Poincaré inequality. By (2.1), provided $\beta \le \beta_0$ we have

$$\|e^{\beta t} t^{1/2} \nabla u^2\|_\infty^2 + \|e^{\beta t} t^{1/2} \nabla Q^2\|_4^2 + \|e^{\beta t} \sqrt{\varrho_0^2} u_t^2\|_2^2 \le f_1(t), \tag{2.30}$$

while, by Lemma 2,

$$\begin{aligned} &\|e^{\beta t} w_t\|_{3/2} \|e^{\beta t} \sqrt{\varrho_0^1} z\|_2^{1/2} \|e^{\beta t} \nabla z\|_2^{1/2} \\ &\quad \le \left(f_{4/3}(t) \left(\int_0^t \|\nabla \delta u\|_2^2 \, \mathrm{d}\tau \right)^{1/2} + f_4(t) \|\nabla \delta u\|_2 \right) \|e^{\beta t} \sqrt{\varrho_0^1} z\|_2^{1/2} \|e^{\beta t} \nabla z\|_2^{1/2}. \end{aligned}$$

Taking advantage of Young's inequality we bound the right-hand side of the above inequality as

$$\begin{aligned} &f_{4/3}(t) \left(\int_0^t \|\nabla \delta u\|_2^2 \, \mathrm{d}\tau \right)^{1/2} \|e^{\beta t} \sqrt{\varrho_0^1} z\|_2^{1/2} \|e^{\beta t} \nabla z\|_2^{1/2} \\ &\quad \le f_{4/3}^{4/3}(t) \int_0^t \|\nabla \delta u\|_2^2 \, \mathrm{d}\tau + f_{4/3}^{2/3}(t) \|e^{\beta t} \sqrt{\varrho_0^1} z\|_2 \|e^{\beta t} \nabla z\|_2, \end{aligned}$$

whence

$$\begin{aligned} &f_{4/3}(t) \left(\int_0^t \|\nabla \delta u\|_2^2 \, \mathrm{d}\tau \right)^{1/2} \|e^{\beta t} \sqrt{\varrho_0^1} z\|_2^{1/2} \|e^{\beta t} \nabla z\|_2^{1/2} \\ &\quad \le f_1(t) \int_0^t \|\nabla \delta u\|_2^2 \, \mathrm{d}\tau + \varepsilon \|e^{\beta t} \nabla z\|_2^2 + f_1(t) \|e^{\beta t} \sqrt{\varrho_0^1} z\|_2^2, \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} &f_4(t) \|\nabla \delta u\|_2 \|e^{\beta t} \sqrt{\varrho_0^1} z\|_2^{1/2} \|e^{\beta t} \nabla z\|_2^{1/2} \\ &\quad \le \varepsilon \|\nabla \delta u\|_2^2 + f_4^2(t) \|e^{\beta t} \sqrt{\varrho_0^1} z\|_2 \|e^{\beta t} \nabla z\|_2 \\ &\quad \le f_1(t) \|e^{\beta t} \sqrt{\varrho_0^1} z\|_2^2 + \varepsilon (\|e^{\beta t} \nabla z\|_2^2 + \|\nabla w\|_2^2). \end{aligned} \tag{2.32}$$

The ε terms coming from (2.31) and (2.32) can again be absorbed by the left-hand side of (2.29). Then, plugging (2.30)–(2.32) into (2.29) we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} e^{2\beta t} \|\sqrt{\varrho_0^1} z(t)\|_2^2 + e^{2\beta t} (\|\nabla z\|_2^2 + \|\nabla w\|_2^2) \\ &\quad \le f_1(t) \left(e^{2\beta t} \|\sqrt{\varrho_0^1} z(t)\|_2^2 + \int_0^t e^{2\beta s} (\|\nabla z\|_2^2 + \|\nabla w\|_2^2) \, \mathrm{d}s \right) \\ &\qquad + f_1(t) \|\delta \varrho_0\|_X^2. \end{aligned} \tag{2.33}$$

Denoting

$$G(t) = e^{2\beta t} \|\sqrt{\varrho_0^1} z(t)\|_2^2 + \int_0^t e^{2\beta s} (\|\nabla z\|_2^2 + \|\nabla w\|_2^2) \, \mathrm{d}s,$$

we rewrite (2.33) as

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t) \le f_1(t)G(t) + f_1(t)\|\delta\varrho_0\|_X. \tag{2.34}$$

We have $G(0) = \|\sqrt{\varrho_0^1}\delta v_0\|_2^2$, therefore (2.34) yields

$$G(t) \le \|\sqrt{\varrho_0^1}\delta v_0\|_2^2 e^{\int_0^t f_1(\tau)\,\mathrm{d}\tau} + \|\delta\varrho_0\|_X \int_0^t e^{\int_s^t f_1(\tau)\,\mathrm{d}\tau} f_1(s)\,\mathrm{d}s. \tag{2.35}$$

Notice that the first inequality of (2.9) implies that

$$e^{2\beta t}\|w(t)\|_2^2 \le f_{1,\infty}(t)\int_0^t \|\nabla\delta u(\tau)\|_2^2\,\mathrm{d}\tau \le f_{1,\infty}(t)G(t). \tag{2.36}$$

Combining (2.35) and (2.36), we obtain the following result:

Lemma 4. *Assume that* (2.4) *holds. Then there exist a positive constant* $\beta < \beta_0$ *and* C_0 *depending only on the data such that*

$$\begin{aligned}\sup_{t\in[0,\infty)} e^{2\beta t}\|\sqrt{\varrho_0^1}\delta u\|_2^2 + \mu\int_0^\infty e^{2\beta t}\|\nabla\delta u\|_2^2\,\mathrm{d}t \\ \le C_0(\|\sqrt{\varrho_0^1}\delta v_0\|_2^2 + C_2\|\delta\varrho_0\|_X^2).\end{aligned} \tag{2.37}$$

2.2. The general case

Estimate (2.37) has been obtained under assumption (2.5). It may happen however that the denominator of (2.5) is zero on a subset with positive measure, while the numerator is not. To overcome this obstacle, instead of comparing solutions emanating from (ϱ_0^1, v_0^1) and (ϱ_0^2, v_0^2) directly we compare each of them to an appropriate intermediate solution satisfying (2.5).

To proceed, let us denote by $(\varrho^{3/2}, v^{3/2}, P^{3/2})$ the solution to (1.1) emanating from $(\frac{1}{2}(\varrho_0^1 + \varrho_0^2), v_0^2)$ and by $(\eta^{3/2}, u^{3/2}, Q^{3/2})$ the corresponding solution in Lagrangian coordinates. Then we look at the following differences between solutions (recall that the density component of the solution in Lagrangian coordinates is constant in time):

$$\begin{aligned}(\delta\varrho^{\mathrm{I}}, \delta u^{\mathrm{I}}) &:= \left(\frac{1}{2}(\varrho_0^1 + \varrho_0^2) - \varrho_0^1, u^{3/2} - u^1\right)\\ \text{and}\quad (\delta\varrho^{\mathrm{II}}, \delta u^{\mathrm{II}}) &:= \left(\frac{1}{2}(\varrho_0^1 + \varrho_0^2) - \varrho_0^2, u^{3/2} - u^2\right).\end{aligned}$$

Clearly, we have

$$\delta\varrho^{\mathrm{I}} = -\delta\varrho^{\mathrm{II}} = \frac{1}{2}(\varrho_0^2 - \varrho_0^1) \quad\text{and}\quad \delta u = \delta u^{\mathrm{I}} - \delta u^{\mathrm{II}}, \tag{2.38}$$

and condition (2.4) is fulfilled in both cases, i.e.

$$\operatorname{supp}\delta\varrho_0^{\mathrm{I,II}} \subset \operatorname{supp}\frac{1}{2}(\varrho_0^1 + \varrho_0^2).$$

Let us define

$$\|\delta\varrho_0^{\mathrm{I}}\|_X := \Big\| \frac{\delta\varrho_0^{\mathrm{I}}}{\sqrt{\varrho_0^1+\varrho_0^2}}\Big\|_4 = \Big\| \frac{\delta\varrho_0^{\mathrm{II}}}{\sqrt{\varrho_0^1+\varrho_0^2}}\Big\|_4.$$

Note that, obviously,

$$\Big|\frac{\varrho_0^1-\varrho_0^2}{\sqrt{\varrho_0^1+\varrho_0^2}}\Big|^2 = |\varrho_0^1-\varrho_0^2|\Big|\frac{\varrho_0^1-\varrho_0^2}{\varrho_0^1+\varrho_0^2}\Big| \le |\varrho_0^1-\varrho_0^2|,$$

whence

$$\|\delta\varrho_0^{\mathrm{I}}\|_X \le \|\varrho_0^1-\varrho_0^2\|_2^{1/2}. \tag{2.39}$$

Still assuming that $\mu = 1$ for simplicity and denoting $\delta Q^{\mathrm{I}} := Q^{3/2} - Q^1$, we obtain

$$\begin{cases} \varrho_0^1\delta u_t^{\mathrm{I}} - \Delta_{u^1}\delta u^{\mathrm{I}} + \nabla_{u^1}\delta Q^{\mathrm{I}} \\ \qquad = (\Delta_{u^{3/2}} - \Delta_{u^1})u^{3/2} - (\nabla_{u^{3/2}} - \nabla_{u^1})Q^{3/2} - \delta\varrho^{\mathrm{I}}u_t^{3/2}, \\ \mathrm{div}_{u^1}\,\delta u^{\mathrm{I}} = (\mathrm{div}_{u^1} - \mathrm{div}_{u^{3/2}})u^{3/2}, \\ \delta u^{\mathrm{I}}|_{t=0} = \delta v_0. \end{cases} \tag{2.40}$$

Setting $\delta Q^{\mathrm{II}} = Q^{3/2} - Q^2$, we see that $(\delta u^{\mathrm{II}}, \delta Q^{\mathrm{II}})$ satisfies an analogous system with $(\varrho_0^1, u^1, \delta\varrho^{\mathrm{I}})$ replaced by $(\varrho_0^2, u^2, \delta\varrho^{\mathrm{II}})$ and with the initial condition $\delta u^{\mathrm{II}}|_{t=0} = 0$.

Let us consider the decompositions

$$\delta u^{\mathrm{I}} = z^{\mathrm{I}} + w^{\mathrm{I}} \quad \text{and} \quad \delta u^{\mathrm{II}} = z^{\mathrm{II}} + w^{\mathrm{II}},$$

where the components are defined by (2.7) and (2.18) with obvious replacements of u^1, u^2. Then one can repeat the estimates from the previous subsection. In the end, defining

$$\begin{aligned} G^{\mathrm{I}}(t) &= e^{2\beta t}\|\sqrt{\varrho_0^1}z^{\mathrm{I}}(t)\|_2^2 + \int_0^t e^{2\beta s}(\|\nabla z^{\mathrm{I}}(s)\|_2^2 + \|\nabla w^{\mathrm{I}}(s)\|_2^2)\,\mathrm{d}s, \\ G^{\mathrm{II}}(t) &= e^{2\beta t}\|\sqrt{\varrho_0^2}z^{\mathrm{II}}(t)\|_2^2 + \int_0^t e^{2\beta s}(\|\nabla z^{\mathrm{II}}(s)\|_2^2 + \|\nabla w^{\mathrm{II}}(s)\|_2^2)\,\mathrm{d}s, \end{aligned} \tag{2.41}$$

one obtains the following analogs of (2.35) (recall that $\delta\varrho_0^{\mathrm{I}} = -\delta\varrho_0^{\mathrm{II}}$ and that the velocity component of the initial data for δv^{II} is zero) for some $f^{\mathrm{I}}, f^{\mathrm{II}} \in L_1(\mathbb{R}_+)$:

$$\begin{aligned} G^{\mathrm{I}}(t) &\le \|\sqrt{\varrho_0^1}\delta v_0\|_2^2 e^{\int_0^t f^{\mathrm{I}}(\tau)\,\mathrm{d}\tau} + \|\delta\varrho_0^{\mathrm{I}}\|_X^2 \int_0^t e^{\int_s^t f^{\mathrm{I}}(\tau)\,\mathrm{d}\tau} f^{\mathrm{I}}(s)\,\mathrm{d}s, \\ G^{\mathrm{II}}(t) &\le \|\delta\varrho_0^{\mathrm{I}}\|_X^2 \int_0^t e^{\int_s^t f^{\mathrm{II}}(\tau)\,\mathrm{d}\tau} f^{\mathrm{II}}(s)\,\mathrm{d}s. \end{aligned}$$

Summing the above inequalities and using (2.38) and (2.39) we obtain

$$\begin{aligned} &\sup_{t\in\mathbb{R}_+}\{e^{\beta t}[\|\sqrt{\varrho_0^1}z^{\mathrm{I}}(t)\|_2 + \|\sqrt{\varrho_0^2}z^{\mathrm{II}}(t)\|_2]\} + \|e^{\beta t}\nabla\delta u\|_{L_2(\mathbb{R}_+,L_2)} \\ &\qquad \le C(\|\sqrt{\varrho_0^1}\delta v_0\|_2 + \|\delta\varrho_0\|_2^{1/2}). \end{aligned}$$

Combining this estimate with analogs of (2.36), where $(w, \delta u, G)$ has been replaced by $(w^{\mathrm{I}}, \delta u^{\mathrm{I}}, G^{\mathrm{I}})$ and $(w^{\mathrm{II}}, \delta u^{\mathrm{II}}, G^{\mathrm{II}})$, we obtain (1.21), which completes the proof of Theorem 3 except for the precise form of the exponent β, which is deduced from the results of Section 3.

2.3. Back to the Euler perspective

In this part, we want to translate the stability result obtained in Theorem 3 in terms of Eulerian coordinates, proving Corollary 1. As already pointed out in the introduction, in the case of only bounded initial densities, getting relevant information on $\varrho^1(t,\cdot) - \varrho^2(t,\cdot)$ in some L_p space (even for $p = 1$) is hopeless; it is more suitable to compare functions along two different characteristic fields. Here we want to adopt the language of kinetic theory, considering quantities along characteristics/trajectories, defined in terms of Wasserstein metrics like in e.g. [35, 40].

More precisely, denote by X^1 and X^2 the flows defined by (1.14) for, respectively, v^1 and v^2, and take a smooth function $\phi\colon \Omega \to \mathbb{R}$. Then we consider, for each $t \geq 0$, the quantity

$$I_\phi(t) := \int_\Omega (\varrho^1(t,x) - \varrho^2(t,x))\phi(x)\,\mathrm{d}x.$$

Using the change of variables (of Jacobian 1) $x = X^1(t,y)$ and $x = X^2(t,y)$ for ϱ^1 and ϱ^2, respectively, yields

$$\begin{aligned} I_\phi(t) &= \int_\Omega \big[\varrho_0^1(y)\phi(X^1(t,y)) - \varrho_0^2(y)\phi(X^2(t,y))\big]\,\mathrm{d}y \\ &= \underbrace{\int_\Omega (\varrho_0^1(y) - \varrho_0^2(y))\phi(X^1(t,y))\,\mathrm{d}y}_{A_1(t)} \\ &\quad + \underbrace{\int_\Omega \varrho_0^2(y)(\phi(X^1(t,y)) - \phi(X^2(t,y)))\,\mathrm{d}y}_{A_2(t)}. \end{aligned} \tag{2.42}$$

In order to find the right level of regularity for ϕ, let us first examine the term $A_2(t)$. For suitable functions $\underline{\eta}$, we set

$$\eta^1(t,x) := \underline{\eta}(t, Y^1(t,x)) \quad \text{and} \quad \eta^2(t,x) := \underline{\eta}(t, Y^2(t,x)) \quad \text{with } Y^i(t,\cdot) := (X^i(t,\cdot))^{-1}.$$

Then, again using the fact that X^1 and X^2 are measure preserving, we have

$$\begin{aligned} &\|\eta^1(t) - \eta^2(t)\|_{W_p^{-1}} \\ &\quad = \sup\Big\{\textstyle\int_\Omega (\eta^1(t,x) - \eta^2(t,x))\phi(t,x)\,\mathrm{d}x : \phi \in W^1_{p'}(\Omega),\ \|\phi\|_{W^1_{p'}} \leq 1\Big\} \\ &\quad = \sup\Big\{\textstyle\int_\Omega \underline{\eta}(y)[\phi(X^1(t,y)) - \phi(X^2(t,y))]\,\mathrm{d}y : \phi \in W^1_{p'}(\Omega),\ \|\phi\|_{W^1_{p'}} \leq 1\Big\}. \end{aligned} \tag{2.43}$$

Following [43], in order to handle the term $\phi(X^1(t,y)) - \phi(X^2(t,y))$, we consider the family of intermediate measure-preserving flows $(X^s)_{1\le s\le 2}$ between X^1 and X^2 defined as

$$\frac{\mathrm{d}X^s}{\mathrm{d}t}(t,y) = (2-s)v^1(t,X^s(t,y)) + (s-1)v^2(t,X^s(t,y)), \quad X^s(0,y)=y. \tag{2.44}$$

By the chain rule and the definition of X^s, we have

$$\begin{aligned}\phi(X^2(t,y)) - \phi(X^1(t,y)) &= \int_1^2 \frac{\mathrm{d}}{\mathrm{d}s}\phi(X^s(t,y))\,\mathrm{d}s\\ &= \int_1^2 \Big(\frac{\mathrm{d}}{\mathrm{d}s}X^s(t,y)\Big)\cdot\nabla\phi(X^s(t,y))\,\mathrm{d}s.\end{aligned}$$

From the definition of X^s, we discover that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}X^s(t,y) &= \big(v^2(t,X^s(t,y)) - v^1(t,X^s(t,y))\big)\\ &\quad + \big((2-s)Dv^1(t,X^s(t,y)) + (s-1)Dv^2(t,X^s(t,y))\big)\frac{\mathrm{d}}{\mathrm{d}s}X_s(t,y),\end{aligned}$$

whence, performing a time integration, taking the $L_p(\Omega)$ norm and using that $X^s(t,\cdot)$ is measure preserving, we get

$$\begin{aligned}\Big\|\frac{\mathrm{d}}{\mathrm{d}s}X^s(t,\cdot)\Big\|_p &\le \int_0^t \|\delta v(t',\cdot)\|_p\,\mathrm{d}t'\\ &\quad + \int_0^t \max(\|Dv^1(t',\cdot)\|_\infty, \|Dv^2(t',\cdot)\|_\infty)\Big\|\frac{\mathrm{d}}{\mathrm{d}s}X^s(t',\cdot)\Big\|_p\,\mathrm{d}t'.\end{aligned}$$

In the end, using Grönwall lemma and taking advantage of (2.1b) implies that

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}s}X^s(t,\cdot)\Big\|_p \le C\int_0^t \|\delta v\|_p\,\mathrm{d}t' \quad \text{for all } 1\le p<\infty. \tag{2.45}$$

Furthermore, as said before, $X^s(t,\cdot)$ is measure preserving, and thus $\|\nabla\phi(X^s(t,\cdot))\|_p = \|\nabla\phi\|_p$. Finally, using the embedding $H^1_0 \hookrightarrow L_p$ for all $1\le p<\infty$ if $d=2$, and all $1\le p\le 6$ if $d=3$, we obtain for these values of p and all $t\ge 0$,

$$A_2(t) \le C\|\varrho_0^2\|_\infty\Big(\int_0^\infty e^{2\beta t}\|\nabla\delta v\|_2^2\,\mathrm{d}t\Big)^{1/2}\|\phi\|_{1,p'}. \tag{2.46}$$

The above inequality reveals that one can take the functions ϕ in the space

$$\phi\in W^1_{1^+}(\Omega) \text{ if } d=2 \quad \text{and} \quad \phi\in W^1_{6/5}(\Omega) \text{ if } d=3.$$

Bounding the term $A_1(t)$ is simpler. Under the above assumptions on p, we have $W^1_{p'} \hookrightarrow L_2$. Hence, again using that X^1 is measure preserving, we may write by the Cauchy–Schwarz inequality,

$$A_1(t) \le \|\delta\varrho_0\|_2\|\phi\|_2 \le C\|\delta\varrho_0\|_2\|\phi\|_{1,p'}. \tag{2.47}$$

Altogether, by (2.42), (2.43), (2.46), (2.47) and (1.21), we get for any finite $p>1$ if $d=2$ and for $p\leq 6$ if $d=3$,

$$\sup_{t\in\mathbb{R}_+}\|\delta\varrho(t,\cdot)\|_{W_p^{-1}}\leq C\biggl(\|\varrho_0^2\|_\infty\biggl(\int_0^\infty e^{2\beta t}\|\nabla\delta v\|_2^2\,\mathrm{d}t\biggr)^{1/2}+\|\delta\varrho_0\|_2\biggr). \tag{2.48}$$

Next, let us turn to the velocity estimates. We start from the relation

$$\delta u(t,y)=v^2(t,X^2(t,y))-v^1(t,X^1(t,y)), \tag{2.49}$$

which implies that

$$\nabla_y\delta u(t,y)=\nabla_y X^2(t,y)\cdot\nabla_x v^2(t,X^2(t,y))-\nabla_y X^1(t,y)\cdot\nabla_x v^1(t,X^1(t,y)).$$

Hence, denoting $\delta X:=X^2-X^1$, we may write

$$\begin{aligned}\nabla_y\delta u(t,y)=&\ \nabla_y\delta X(t,y)\cdot\nabla_x v^2(t,X^2(t,y))+\nabla_y X^1(t,y)\cdot\nabla_x\delta v(t,X^2(t,y))\\&+\nabla_y X^1(t,y)\cdot\bigl[\nabla_x v^1(t,X^2(t,y))-\nabla_x v^1(t,X^1(t,y))\bigr]\\=:&\ K_1+K_2+K_3.\end{aligned}$$

First, we observe that

$$\nabla\delta v(t,X^2(t,y))=(A_u^1)^T K_2.$$

Hence, taking the $L_2(\Omega)$ norm, and using that $X^2(t,\cdot)$ is measure preserving, we get

$$\|\nabla\delta v\|_2\leq\|A_u^1\|_\infty\|K_2\|_2\leq\|A_u^1\|_\infty(\|\nabla\delta u\|_2+\|K_1\|_2+\|K_3\|_2). \tag{2.50}$$

Next, from the definition of X^1 and X^2 in terms of the Lagrangian velocity, we have

$$\|K_1\|_2\leq Ct^{1/2}\biggl(\int_0^t\|\nabla\delta u\|_2^2\,\mathrm{d}t'\biggr)^{1/2}\|\nabla v^2\|_\infty.$$

To bound K_3, we first notice that by the mean value theorem and the definition of the intermediate flow X^s, we have

$$K_3=\nabla_y X^1(t,y)\cdot\biggl(\int_1^2\nabla^2 v^1(t,X^s(t,y))\,\mathrm{d}s\biggr)\cdot\Bigl(\frac{\mathrm{d}}{\mathrm{d}s}X^s(t,y)\Bigr).$$

Hence, since X^s is measure preserving, using the Hölder inequality and (2.45) allows us to get

$$\begin{aligned}\|K_3\|_2&\leq\|\nabla_y X^1(t,\cdot)\|_\infty\int_1^2\|\nabla^2 v^1(t,X^s(t))\|_3\Bigl\|\frac{\mathrm{d}}{\mathrm{d}s}X^s(t,\cdot)\Bigr\|_6\,\mathrm{d}s\\&\leq C\|\nabla_y X^1(t,\cdot)\|_\infty\|\nabla^2 v^1(t)\|_3\int_0^t\|\delta v(t')\|_6\,\mathrm{d}t'\\&\leq C\|\nabla_y X^1(t,\cdot)\|_\infty\|t^{1/2}\nabla^2 v^1(t)\|_3\biggl(\int_0^t\|\nabla\delta v\|_2^2\,\mathrm{d}t'\biggr)^{1/2}.\end{aligned}$$

Altogether, remembering (2.50), we obtain for all $t \geq 0$,

$$\|\nabla\delta v(t)\|_2^2 \leq C\Big[\|\nabla\delta u(t)\|_2^2 + \|t^{1/2}\nabla v^2\|_\infty^2 \int_0^t \|\nabla\delta u\|_2^2\,dt' + \|t^{1/2}\nabla^2 v^1\|_3^2 \int_0^t \|\nabla\delta v\|_2^2\,dt'\Big].$$

First multiplying both sides by $e^{2\beta t}$, next integrating in time and using Theorem 3 combined with the properties (2.1), and finally applying the Grönwall lemma, we obtain for all $t \geq 0$,

$$\int_0^t e^{2\beta t'}\|\nabla\delta v\|_2^2\,dt' \leq C_0(\|\sqrt{\varrho_0^1}\delta v_0\|_2^2 + \|\delta\varrho_0\|_2)\exp\Big(C\int_0^t e^{2\beta t'}\|t'^{1/2}\nabla^2 v^1\|_3^2\,dt'\Big).$$

The term in the exponential may be bounded thanks to (2.1d), which leads to the desired inequality for $\nabla\delta v$.

In order to estimate $\varrho_0^1\delta u(t,\cdot)$ in L^2, we may again use the intermediate flow X^s defined in (2.44) to write that

$$\begin{aligned}\sqrt{\varrho_0^1(y)}\delta u(t,y) &= \sqrt{\varrho^1(t,X^1(t,y))}\delta v(t,X^1(t,y)) \\ &\quad + \sqrt{\varrho^1(t,X^1(t,y))}\big(v^2(t,X^2(t,y)) - v^2(t,X^1(t,y))\big) \\ &= \sqrt{\varrho^1(t,X^1(t,y))}\delta v(t,X^1(t,y)) \\ &\quad + \sqrt{\varrho^1(t,X^1(t,y))}\int_1^2 \frac{d}{ds}\big(v^2(t,X^s(t,y))\big)\,ds \\ &= \sqrt{\varrho^1(t,X^1(t,y))}\delta v(t,X^1(t,y)) \\ &\quad + \sqrt{\varrho^1(t,X^1(t,y))}\int_1^2 Dv^2(t,X^s(t,y))\frac{d}{ds}X^s(t,y)\,ds.\end{aligned}$$

Taking the L^2 norm and using the Hölder inequality as well as (2.45) with $p = 3$ and, finally, remembering that $X^1(t,\cdot)$ is measure preserving, this implies that for all $t \geq 0$, we have

$$\|(\sqrt{\varrho^1}\delta v)(t)\|_2 \leq \|\sqrt{\varrho_0^1}\delta u(t)\|_2 + C\sqrt{\varrho^*}\|Dv^2(t)\|_6 \int_0^t \|\delta v(t')\|_3\,dt'.$$

Hence, multiplying both sides by $e^{\beta t}$ and using Sobolev embedding, we discover that

$$e^{\beta t}\|(\sqrt{\varrho^1}\delta v)(t)\|_2 \leq e^{\beta t}\|\sqrt{\varrho_0^1}\delta u(t)\|_2 + C\sqrt{\varrho^* t}e^{\beta t}\|Dv^2(t)\|_6\|\nabla\delta v\|_{L_2(\mathbb{R}_+;L_2)}.$$

The term with $\nabla\delta v$ may be bounded according to our previous estimate. We claim that one may find $\beta > 0$ so that $\sup_{t\in\mathbb{R}_+}\sqrt{t}e^{\beta t}\|Dv^2(t)\|_6$ is bounded. Indeed, again using Sobolev embedding, we may write that

$$\|Dv^2\|_6 \leq C\|\nabla v^2\|_{H^1},$$

while the maximal regularity properties of the Stokes system ensure that we have

$$\sup_{t\in\mathbb{R}_+} \sqrt{t}e^{\beta t}\|\nabla v^2(t)\|_{H^1} \leq C\sqrt{\varrho^*}\sup_{t\in\mathbb{R}_+} \sqrt{t}e^{\beta t}\|(\sqrt{\varrho}\dot{v}^2)(t)\|_{L^2},$$

where $\dot{v}^2 := v_t^2 + v^2\cdot\nabla v^2$. Obviously, we have

$$\sqrt{t}e^{\beta t}\|(\sqrt{\varrho}\dot{v}^2)(t)\|_{L^2} \leq \sqrt{t}e^{\beta t}\|(\sqrt{\varrho}v_t^2)(t)\|_{L^2} + \sqrt{\varrho^*}\|\sqrt{t}e^{\beta t}v^2(t)\|_\infty\|\nabla v^2(t)\|_2.$$

Lemma 7 allows us to bound the supremum on $\mathbb{R}_+$ of the first term on the right-hand side, while the last one is bounded thanks to (2.1c) (if $\beta\leq\beta_0$) and (3.4). This completes the proof of Corollary 1.

3. Decay estimates

The goal of this section is to show that the solutions provided by Theorems 1 and 2 indeed satisfy properties (2.1a)–(2.1h). This will be an immediate consequence of Lemmas 6, 7 and 8 below.

Before tackling the proof, we need to recall elementary properties of the solutions to (1.1). The first one is the conservation of any L_p norm of the density, a consequence of the divergence-free property of the velocity field: we have $\|\varrho(t)\|_p = \|\varrho_0\|_p$ for all $p\in[1,\infty]$.

Of course, as v in $H_0^1(\Omega)$, we have the Poincaré inequality:

$$\|v\|_2 \leq C_P\|\nabla v\|_2. \tag{3.1}$$

3.1. Decay of space derivatives

The starting point is the following decay estimate, which is a direct consequence of diffusion and the Poincaré inequality:

Lemma 5. *Let (ϱ, v) be a solution to (1.1) given either by Theorem 1 or by Theorem 2. Then*

$$\text{for all } t\geq 0,\quad \int_\Omega \varrho(t)|v(t)|^2\,\mathrm{d}x \leq e^{-2\beta_1 t}\int_\Omega \varrho_0|v_0|^2\,\mathrm{d}x,\quad \text{where } \beta_1 = \frac{\mu}{\varrho^* C_P^2}. \tag{3.2}$$

Proof. The proof is independent of the space dimension. We start with the classical energy estimate that is obtained testing the momentum equation by v, namely

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho|v|^2\,\mathrm{d}x + \mu\int_\Omega |\nabla v|^2\,\mathrm{d}x = 0. \tag{3.3}$$

Hence, remembering (3.1), we get

$$\frac{C_P^2}{2\mu}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho|v|^2\,\mathrm{d}x + \int_\Omega |v|^2\,\mathrm{d}x \leq 0.$$

Multiplying both sides by ϱ^* and using the obvious fact that $\varrho^*|v|^2 \geq \varrho|v|^2$, we obtain

$$\frac{\varrho^* C_P^2}{2\mu}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho|v|^2\,\mathrm{d}x + \int_\Omega \varrho|v|^2\,\mathrm{d}x \leq 0,$$

from which we conclude (3.2). ■

Our next aim is to establish a decay estimate for the gradient of the velocity.

Lemma 6. *Let (ϱ, v) be a solution to (1.1) given either by Theorem 1 or by Theorem 2. Then there exist positive constants C_0 and $C_{0,p}$ depending only on the data (and on p for the second one), and*[1] *$\beta_2 < \beta_1$ such that for all $t \geq 0$, we have*

$$\|\nabla v(t)\|_2 \leq C_0 e^{-\beta_2 t}, \tag{3.4}$$

$$\|v(t)\|_p \leq C_{0,p} e^{-\beta_2 t}, \tag{3.5}$$

$$\int_0^\infty e^{2\beta_2 t}\big[\|\sqrt{\varrho(t)}v_t(t)\|_2^2 + \|\nabla^2 v(t)\|_2^2 + \|\nabla P(t)\|_2^2\big]\,\mathrm{d}t \leq C_0, \tag{3.6}$$

where, in (3.5), one can take any $p \in [1,\infty)$ if $d = 2$, and any $p \in [1,6]$ if $d = 3$.

Proof. Performing a suitable time, space, density and velocity rescaling reduces the proof to the case $\varrho^* = \mu = 1$, in a domain of diameter 1. Hence, we will only prove the lemma in this case for notational simplicity, since reverting to the original variables will lead to the form of β_2 announced in the footnote. Now, testing the momentum equation by v_t yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^2\,\mathrm{d}x + \int_\Omega \varrho|v_t|^2\,\mathrm{d}x &= -\int_\Omega \varrho v_t\cdot(v\cdot\nabla v)\,\mathrm{d}x \\ &\leq \frac{1}{2}\int_\Omega \varrho|v_t|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \varrho|v\cdot\nabla v|^2\,\mathrm{d}x.\end{aligned}$$

The classical theory for the Stokes system yields, for some C_Ω depending only on the shape of Ω,

$$\|\nabla^2 v\|_2^2 + \|\nabla P\|_2^2 \leq C_\Omega(\|\varrho v\cdot\nabla v\|_2^2 + \|\varrho v_t\|_2^2).$$

Hence we have (remember that $\varrho^* = 1$)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^2\,\mathrm{d}x &+ \frac{1}{2}\int_\Omega\Big(\varrho|v_t|^2 + \frac{1}{C_\Omega}(|\nabla^2 v|^2 + |\nabla P|^2)\Big)\,\mathrm{d}x \\ &\leq \frac{3}{2}\int_\Omega \varrho|v\cdot\nabla v|^2\,\mathrm{d}x.\end{aligned} \tag{3.7}$$

[1] One can take β_2 in the form $\beta_2 = c_\Omega\mu/(\varrho^*\mathfrak{d}^2)$, where $\mathfrak{d}$ stands for the diameter of Ω and c depends only on the shape of Ω.

Adding this inequality to (3.3) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big[|\nabla v|^2+\frac12\varrho|v|^2\Big]\,\mathrm{d}x+\int_\Omega\Big[|\nabla v|^2+\frac12\varrho|v|^2\Big]\,\mathrm{d}x\\ +\frac12\int_\Omega\Big(\varrho|v_t|^2+\frac{1}{C_\Omega}(|\nabla^2 v|^2+|\nabla P|^2)\Big)\,\mathrm{d}x\\ \leq\frac32\int_\Omega\varrho|v\cdot\nabla v|^2\,\mathrm{d}x+\frac12\int_\Omega\varrho|v|^2\,\mathrm{d}x.\end{aligned}\tag{3.8}$$

In order to estimate the first term on the right-hand side in the two-dimensional case, we first write

$$\int_\Omega\varrho|v\cdot\nabla v|^2\,\mathrm{d}x\leq\|\sqrt{\varrho}v\|_2\|v\|_\infty\|\nabla v\|_4^2.$$

Hence, using the two interpolation inequalities

$$\begin{aligned}\|z\|_4&\leq C\|z\|_2^{1/2}\|\nabla z\|_2^{1/2},\\ \|z\|_\infty&\leq C\|z\|_2^{1/2}\|\nabla^2 z\|_2^{1/2}\end{aligned}\tag{3.9}$$

and remembering that $\varrho^*=1$, we get

$$\begin{aligned}\frac32\int_\Omega\varrho|v\cdot\nabla v|^2\,\mathrm{d}x&\leq C\|\sqrt{\varrho}v\|_2\|v\|_2^{1/2}\|\nabla v\|_2\|\nabla^2 v\|_2^{3/2}\\ &\leq\frac{1}{4C_\Omega}\|\nabla^2 v\|_2^2+C\|\sqrt{\varrho}v\|_2^4\|v\|_2^2\|\nabla v\|_2^4.\end{aligned}$$

The first term can be absorbed by the left-hand side of (3.8) and one can bound $\|v\|_2$ from (3.1). Hence, taking advantage of inequality (3.2), we get for some $C_0>0$ depending only on $\|\sqrt{\varrho_0}v_0\|_2$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big[|\nabla v|^2+\frac12\varrho|v|^2\Big]\,\mathrm{d}x+\int_\Omega\Big[|\nabla v|^2+\frac12\varrho|v|^2\Big]\,\mathrm{d}x\\ +\frac12\int_\Omega\Big(\varrho|v_t|^2+\frac{1}{2C_\Omega}(|\nabla^2 v|^2+|\nabla P|^2)\Big)\,\mathrm{d}x\\ \leq C_0e^{-2\beta_1 t}[1+\|\nabla v\|_2^6].\end{aligned}\tag{3.10}$$

The crucial observation is that $\|\nabla v(t)\|_2$ may bounded uniformly in time in terms of the data (see [13, Props. 3.1 & 3.3]). Therefore, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big[|\nabla v|^2+\frac12\varrho|v|^2\Big]\,\mathrm{d}x\\ +\int_\Omega\Big(|\nabla v|^2+\frac12\varrho|v|^2+\frac12\varrho|v_t|^2+\frac{1}{4C_\Omega}(|\nabla^2 v|^2+|\nabla P|^2)\Big)\,\mathrm{d}x\\ \leq C_0e^{-2\beta_1 t}.\end{aligned}\tag{3.11}$$

This leads for any $\gamma_1 \in \mathbb{R}$ to

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big[e^{2\gamma_1 t}\int_\Omega \Big(|\nabla v|^2+\frac12\varrho|v|^2\Big)\,\mathrm{d}x\Big]\\
&\quad+(1-2\gamma_1)\int_\Omega e^{2\gamma_1 t}\Big(|\nabla v|^2+\frac12\varrho|v|^2+\frac12\varrho|v_t|^2+\frac{1}{4C_\Omega}(|\nabla^2 v|^2+|\nabla P|^2)\Big)\,\mathrm{d}x\\
&\quad\le C_0 e^{-2(\beta_1-\gamma_1)t}.
\end{aligned}
\tag{3.12}
$$

In the three-dimensional case, the only difference is the slightly more complicated treatment of the right-hand side of (3.8). Nevertheless, by using the Hölder inequality and the Gagliardo–Nirenberg inequality, we arrive (still using that $\varrho^*=1$) at

$$
\begin{aligned}
\int_\Omega \varrho|v\cdot\nabla v|^2 &\le \|\varrho^{1/4}v\|_4^2\|\nabla v\|_4^2 \le C\|\sqrt{\varrho}v\|_2^{1/2}\|v\|_6^{3/2}\|\nabla v\|_2^{1/2}\|\nabla^2 v\|_2^{3/2}\\
&\le C\|\sqrt{\varrho}v\|_2^{1/2}\|\nabla v\|_2^2\|\nabla^2 v\|_2^{3/2}\\
&\le \frac{1}{2C_\Omega}\|\nabla^2 v\|_2^2+C\|\sqrt{\varrho}v\|_2^2\|\nabla v\|_2^8\\
&\le \frac{1}{2C_\Omega}\|\nabla^2 v\|_2^2+C_0e^{-2\beta_1 t},
\end{aligned}
$$

where in the last passage we have used (3.2) and the uniform boundedness of $\|\nabla v(t)\|_2$. Again, the first term can be absorbed by the left-hand side of (3.8) and we obtain (3.12).

Now, choosing e.g. $\gamma_1=\min(1/4,\beta_1/2)$ and integrating (3.12) on $[0,t]$ yields

$$
\begin{aligned}
&e^{2\gamma_1 t}\Big(\|\nabla v(t)\|_2^2+\frac12\|\sqrt{\varrho(t)}v(t)\|_2^2\Big)\\
&\quad+\frac12\int_0^t e^{2\gamma_1 s}\Big(\|\nabla v\|_2^2+\frac12\|\sqrt{\varrho}v\|_2^2+\frac12\|\sqrt{\varrho}v_t\|_2^2+\frac{1}{4C_\Omega}(\|\nabla^2 v\|_2^2+\|\nabla P\|_2^2)\Big)\,\mathrm{d}s\\
&\le \frac{C_0}{\beta_1-\gamma_1}+\|\nabla v_0\|_2^2+\frac12\|\sqrt{\varrho_0}v_0\|_2^2,
\end{aligned}
$$

which readily gives (3.4) and (3.6). As for (3.5), it just results from the Poincaré inequality and Sobolev embedding. ∎

3.2. Decay of time derivatives

Here we estimate time and time-space derivatives.

Lemma 7. *Let (ϱ, v) be a solution to (1.1) given either by Theorem 1 or by Theorem 2. Then there exists a positive constant $\beta_3<\beta_2$ (still of the form $\beta_3=c_\Omega\mu/(\varrho^*\mathfrak{d}^2)$) such that*

$$
\sup_{t\in\mathbb{R}_+}\int_\Omega te^{2\beta_3 t}\varrho|v_t|^2\,\mathrm{d}x+\int_0^\infty e^{2\beta_3 t}\big[\|\sqrt{t}\nabla v_t\|_2^2+\|\sqrt{t}v_t\|_2^2\big]\,\mathrm{d}t\le C_0 \tag{3.13}
$$

for some positive constant C_0 depending only on the data.

Proof. We keep the assumption $\mathfrak{d} = \mu = \varrho^* = 1$. Now differentiating the momentum equation in time and multiplying by $\sqrt{t}e^{\beta t}$, we obtain

$$\varrho(\sqrt{t}e^{\beta t}v_t)_t + \sqrt{t}e^{\beta t}\varrho_t v_t - \frac{1}{2\sqrt{t}}e^{\beta t}\varrho v_t - \beta\sqrt{t}e^{\beta t}\varrho v_t + \sqrt{t}e^{\beta t}\varrho_t v\cdot\nabla v$$
$$+ \sqrt{t}e^{\beta t}\varrho v_t\cdot\nabla v + \sqrt{t}e^{\beta t}\varrho v\cdot\nabla v_t - \Delta(\sqrt{t}e^{\beta t}v_t) + \nabla(\sqrt{t}e^{\beta t}P_t) = 0. \quad (3.14)$$

Testing (3.14) with $\sqrt{t}e^{\beta t}v_t$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega te^{2\beta t}\varrho|v_t|^2\,\mathrm{d}x + \int_\Omega te^{2\beta t}|\nabla v_t|^2\,\mathrm{d}x = \sum_{i=1}^5 I_i, \quad (3.15)$$

where, using $\varrho_t = -\operatorname{div}(\varrho v)$ in the third term on the left-hand side of (3.14), we have

$$I_1 = \frac{1}{2}\int_\Omega e^{2\beta t}\varrho|v_t|^2\,\mathrm{d}x,$$
$$I_2 = -\int_\Omega(\sqrt{t}e^{\beta t}\varrho_t v\cdot\nabla v)\cdot(\sqrt{t}e^{\beta t}v_t)\,\mathrm{d}x,$$
$$I_3 = -\int_\Omega(\sqrt{t}e^{\beta t}\varrho v_t\cdot\nabla v)\cdot(\sqrt{t}e^{\beta t}v_t)\,\mathrm{d}x,$$
$$I_4 = \int_\Omega \varrho v\cdot\nabla|\sqrt{t}e^{\beta t}v_t|^2\,\mathrm{d}x,$$
$$I_5 = \beta\int_\Omega te^{2\beta t}\varrho|v_t|^2\,\mathrm{d}x.$$

Now we estimate the right-hand side of (3.15). From the continuity equation, we have

$$I_2 = \int_\Omega te^{2\beta t}\operatorname{div}(\varrho v)(v\cdot\nabla v)\cdot v_t\,\mathrm{d}x = -\int_\Omega te^{2\beta t}\varrho v\cdot\nabla[(v\cdot\nabla v)\cdot v_t]\,\mathrm{d}x.$$

Therefore,

$$|I_2| \le \int_\Omega te^{2\beta t}\varrho|v|\big(|\nabla v|^2|v_t| + |v|\,|\nabla^2 v|\,|v_t| + |v|\,|\nabla v|\,|\nabla v_t|\big)\,\mathrm{d}x$$
$$=: I_{21} + I_{22} + I_{23}.$$

For I_{21} in the two-dimensional case, we have

$$|I_{21}| \le \int_\Omega(\sqrt{t\varrho}|v|\,|\nabla v|^2\sqrt{t\varrho}|v_t|e^{2\beta t})\,\mathrm{d}x \le \|v\|_\infty^2\|\sqrt{t\varrho}e^{\beta t}v_t\|_2^2 + te^{2\beta t}\|\nabla v\|_4^4,$$

so using (3.9) and (3.4) we get

$$|I_{21}| \le \|v\|_\infty^2\|\sqrt{t\varrho}e^{\beta t}v_t\|_2^2 + t\|\nabla v\|_2^2e^{2\beta t}\|\nabla^2 v\|_2^2$$
$$\le \|v\|_\infty^2\|\sqrt{t\varrho}e^{\beta t}v_t\|_2^2 + Ce^{-2\beta_2 t}\|e^{\beta t}\nabla^2 v\|_2^2. \quad (3.16)$$

In the three-dimensional case we do not have (3.9), but using the Hölder inequality and the embedding $H_0^1 \hookrightarrow L_6$ we can write

$$
\begin{aligned}
|I_{21}| &\leq \sqrt{t} e^{2\beta t} \int_\Omega \sqrt{t\varrho}|v_t|\,|v|\,|\nabla v|^2\,\mathrm{d}x \\
&\leq \sqrt{t} e^{2\beta t} \|\sqrt{t\varrho} v_t\|_4 \|v\|_6 \|\nabla v\|_{24/7}^2 \\
&\leq \sqrt{t} e^{\beta t} \|\sqrt{t\varrho} v_t\|_2^{1/4} \|\sqrt{t} e^{\beta t} v_t\|_6^{3/4} \|v\|_6 \|\nabla v\|_{24/7}^2 \\
&\leq \frac{1}{10} \|\sqrt{t} e^{\beta t} \nabla v_t\|_2^2 + C t^{4/5} e^{8\beta t/5} \|\sqrt{t\varrho} v_t\|_2^{2/5} \|\nabla v\|_{24/7}^{16/5} \|\nabla v\|_2^{8/5}.
\end{aligned}
$$

Then, using the Gagliardo–Nirenberg inequality $\|\nabla v\|_{24/7}^{16/5} \leq C\|\nabla v\|_2^{6/5} \|\nabla^2 v\|_2^2$ and (3.4), we discover that

$$
\begin{aligned}
|I_{21}| &\leq \frac{1}{10} \|\sqrt{t} e^{\beta t} \nabla v_t\|_2^2 + t^{4/5} e^{-4\beta t/5} \|\sqrt{t\varrho} e^{\beta t} v_t\|_2^{2/5} \|\nabla v\|_2^{14/5} \|e^{\beta t} \nabla^2 v\|_2^2 \\
&\leq \frac{1}{15} \|\sqrt{t} e^{\beta t} \nabla v_t\|_2^2 + C_0 e^{-(4\beta + 14\beta_2)t/5} \|e^{\beta t} \nabla^2 v\|_2^2 \|\sqrt{t\varrho} e^{\beta t} v_t\|_2^{2/5}.
\end{aligned}
$$

Therefore, there exists $c > 0$ such that

$$
|I_{21}| \leq \frac{1}{10} \|\sqrt{t} e^{\beta t} \nabla v_t\|_2^2 + C_0 e^{-ct} \|e^{\beta t} \nabla^2 v\|_2^2 (1 + \|\sqrt{t\varrho} e^{\beta t} v_t\|_2^2). \tag{3.17}
$$

The remaining two parts of I_2 are simpler: we have

$$
\begin{aligned}
I_{22} &= \int_\Omega \sqrt{t\varrho} e^{\beta t} |\nabla^2 v|\,|v|^2 \sqrt{t\varrho} e^{\beta t} |v_t|\,\mathrm{d}x \\
&\leq \|\sqrt{t\varrho} e^{\beta t} \nabla^2 v\|_2 \|v\|_6^2 \|\sqrt{t\varrho} e^{\beta t} v_t\|_6 \\
&\leq C\|\sqrt{t} e^{\beta t} \nabla^2 v\|_2 \|\nabla v\|_2^2 \|\sqrt{t} e^{\beta t} \nabla v_t\|_2 \\
&\leq \frac{1}{15} \|\sqrt{t} e^{\beta t} \nabla v_t\|_2^2 + C_0 e^{-4\beta_2 t} \|\sqrt{t} e^{\beta t} \nabla^2 v\|_2^2
\end{aligned} \tag{3.18}
$$

and

$$
\begin{aligned}
I_{23} &\leq \|\sqrt{t\varrho} e^{\beta t} \nabla v_t\|_2 \|v\|_6^2 \|\sqrt{t\varrho} e^{\beta t} \nabla v\|_6 \\
&\leq C\|\sqrt{t} e^{\beta t} \nabla v_t\|_2 \|\nabla v\|_2^2 \|\sqrt{t} e^{\beta t} \nabla^2 v\|_2 \\
&\leq \frac{1}{15} \|\sqrt{t} e^{\beta t} \nabla v_t\|_2^2 + C_0 e^{-4\beta_2 t} \|\sqrt{t} e^{\beta t} \nabla^2 v\|_2^2.
\end{aligned} \tag{3.19}
$$

Hence, putting (3.17), (3.18) and (3.19) together, we obtain for some $c > 0$,

$$
I_2 \leq \frac{1}{5} \|\sqrt{t} e^{\beta t} \nabla v_t\|_2^2 + C_0 e^{-ct} \|e^{\beta t} \nabla^2 v\|_2^2 (1 + \|\sqrt{t\varrho} e^{\beta t} v_t\|_2^2). \tag{3.20}
$$

Next we estimate I_3 as follows:

$$\begin{aligned} I_3 &\le \|\nabla v\|_2 \|\sqrt{t\varrho} e^{\beta t} v_t\|_4^2 \\ &\le C\|\nabla v\|_2 \|\sqrt{t\varrho} e^{\beta t} v_t\|_2^{1/2} \|\sqrt{t} e^{\beta t} v_t\|_6^{3/2} \\ &\le C\|\nabla v\|_2 \|\sqrt{t\varrho} e^{\beta t} v_t\|_2^{1/2} \|\sqrt{t} e^{\beta t} \nabla v_t\|_2^{3/2} \\ &\le \frac{1}{10}\|\sqrt{t} e^{\beta t} \nabla v_t\|_2^2 + C\|\sqrt{t\varrho} e^{\beta t} v_t\|_2^2 \|\nabla v\|_2^4 \\ &\le \frac{1}{10}\|\sqrt{t} e^{\beta t} \nabla v_t\|_2^2 + C_0 e^{-4\beta_2 t} \|\sqrt{t\varrho} e^{\beta t} v_t\|_2^2 \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} I_4 &\le 2\int_\Omega \varrho|v|\,|\sqrt{t} e^{\beta t} \nabla v_t|\,|\sqrt{t} e^{\beta t} v_t|\,\mathrm{d}x \\ &\le \frac{1}{10}\|\sqrt{t} e^{\beta t} \nabla v_t\|_2^2 + C\|v\|_\infty^2 \|\sqrt{t\varrho} e^{\beta t} v_t\|_2^2. \end{aligned} \tag{3.22}$$

Finally, as (3.1) also applies to v_t, one may write for sufficiently small β,

$$I_5 \le \frac{1}{10}\|\sqrt{t} e^{\beta t} \nabla v_t\|_2^2. \tag{3.23}$$

Combining (3.15), (3.20), (3.21), (3.22) and (3.23) we arrive at

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega t e^{2\beta t}\varrho|v_t|^2\,\mathrm{d}x &+ \frac{1}{2}\int_\Omega t e^{2\beta t}|\nabla v_t|^2\,\mathrm{d}x \\ &\le C_0(e^{-ct}\|e^{\beta t}\nabla^2 v\|_2^2 + \|v\|_\infty^2)\|\sqrt{t\varrho} e^{\beta t} v_t\|_2^2 \\ &\quad + C_0(\|\sqrt{\varrho} e^{\beta t} v_t\|_2^2 + e^{-ct}\|e^{\beta t}\nabla^2 v\|_2^2). \end{aligned} \tag{3.24}$$

By virtue of Lemma 6, the last line is integrable on $\mathbb{R}_+$ for any $\beta \le \beta_2$, as well as the prefactor of the second line (observe that $H^2 \hookrightarrow L_\infty$). Hence, the Grönwall inequality ensures that

$$\sup_{t\in\mathbb{R}_+}\int_\Omega t e^{2\beta t}\varrho|v_t|^2\,\mathrm{d}x + \int_{\mathbb{R}_+} e^{2\beta t}\|\sqrt{t}\nabla v_t\|_2^2\,\mathrm{d}t < \infty. \tag{3.25}$$

Now the Poincaré inequality implies the bound for $\|e^{\beta t}\sqrt{t} v_t\|_{L_2(\mathbb{R}_+;L_2)}$, which completes the proof. ∎

3.3. Shift of integrability and control of $\|\nabla v\|_\infty$

Using the decay estimates we have proved so far will finally enable us to establish similar properties for higher-order norms:

Lemma 8. *Let* (ϱ, v) *be a solution to* (1.1) *given either by Theorem* 1 *or by Theorem* 2. *Then the following properties hold true:*

$$\sqrt{t}e^{\beta_2 t}(\nabla^2 v, \nabla P) \in L_p(\mathbb{R}_+; L_s(\Omega)) \qquad 2 \le s \le 6 \text{ and } p = \frac{4s}{3s-6} \text{ if } d = 3, \tag{3.26}$$

$$\sqrt{t}e^{\beta_2 t}(\nabla^2 v, \nabla P) \in L_p(\mathbb{R}_+; L_s(\Omega)) \qquad 2 \le s < \infty \text{ and } p = \frac{2s}{s-2} \text{ if } d = 2, \tag{3.27}$$

$$e^{\beta_4 t}\nabla^2 v \in L_1(\mathbb{R}_+; L_r(\Omega)) \qquad \text{for some } r > d, \tag{3.28}$$

$$e^{\beta_4 t}\nabla v \in L_1(\mathbb{R}_+; L_\infty(\Omega)), \tag{3.29}$$

for some $0 < \beta_4 = \frac{c\mu}{\varrho^* \mathfrak{d}^2} < \beta_2$*, where* β_2 *is defined in Lemma* 6.

Proof. We multiply $(1.1)_1$ by $\sqrt{t}e^{\beta t}$, where $\beta = \beta_2$, and rewrite it as the Stokes system

$$-\Delta\sqrt{t}e^{\beta t}v + \nabla\sqrt{t}e^{\beta t}P = -\sqrt{t}e^{\beta t}\varrho v_t - \sqrt{t}e^{\beta t}\varrho v\cdot\nabla v, \quad \operatorname{div}\sqrt{t}e^{\beta t}v = 0. \tag{3.30}$$

We start by proving (3.26). By the interpolation inequality

$$\|f\|_q \le \|f\|_2^{(6-q)/2q}\|f\|_6^{(3q-6)/2q}, \quad 2 \le q \le 6,$$

it is enough to prove

$$\sqrt{t}e^{\beta t}(\nabla^2 v, \nabla P) \in L_\infty(\mathbb{R}_+; L_2(\Omega)) \cap L_2(\mathbb{R}_+; L_6(\Omega)). \tag{3.31}$$

By (3.13) and Sobolev embedding, we have

$$\sqrt{\varrho t}e^{\beta t}v_t \in L_\infty(\mathbb{R}_+; L_2(\Omega)) \cap L_2(\mathbb{R}_+; L_6(\Omega)). \tag{3.32}$$

Therefore, the elliptic regularity of (3.30) implies

$$\begin{aligned}
&\|\sqrt{t}e^{\beta t}(\nabla^2 v, \nabla P)\|_{L_\infty(\mathbb{R}_+;L_2)}\\
&\quad\le C_0 + C\|\varrho v\cdot\nabla\sqrt{t}e^{\beta t}v\|_{L_\infty(\mathbb{R}_+;L_2)}\\
&\quad\le C_0 + C\|e^{\beta t}v\|_{L_\infty(\mathbb{R}_+;L_6)}\|\sqrt{t}\nabla v\|_{L_\infty(\mathbb{R}_+;L_3)}\\
&\quad\le C_0 + C\sqrt{t}e^{-d\beta t/6}\|e^{\beta t}v\|_{L_\infty(\mathbb{R}_+;L_6)}\|\nabla v\|^{1-d/6}_{L_\infty(\mathbb{R}_+;L_2)}\|\sqrt{t}e^{\beta t}\nabla^2 v\|^{d/6}_{L_\infty(\mathbb{R}_+;L_2)}.
\end{aligned}$$

So, by the Young inequality and (3.5), we get

$$\|\sqrt{t}e^{\beta t}(\nabla^2 v, \nabla P)\|_{L_\infty(\mathbb{R}_+;L_2)} \le C_0. \tag{3.33}$$

Similarly, starting from (3.30) and thanks to inequality (3.13) and the embedding $H_0^1 \hookrightarrow L_6$, we have

$$\begin{aligned}
\|\sqrt{t}e^{\beta t}(\nabla^2 v, \nabla P)\|_{L_2(\mathbb{R}_+;L_6)} &\le C_0 + C\|\sqrt{t}e^{\beta t}v\cdot\nabla v\|_{L_2(\mathbb{R}_+;L_6)}\\
&\le C_0 + C\|e^{\beta t/2}\sqrt{t}v\|_{L_\infty(\mathbb{R}_+;L_\infty)}\|e^{\beta t/2}\nabla v\|_{L_2(\mathbb{R}_+;L_6)}.
\end{aligned}$$

In the three-dimensional case we have

$$\|\sqrt{t}e^{\beta t/2}v\|_{L_\infty(\mathbb{R}_+;L_\infty)} \le C\|\sqrt{t}e^{\beta t/2}\nabla v\|^{1/2}_{L_\infty(\mathbb{R}_+;L_2)}\|\sqrt{t}e^{\beta t/2}\nabla v\|^{1/2}_{L_\infty(\mathbb{R}_+;L_6)},$$

which implies

$$\begin{aligned}
&\|\sqrt{t}e^{\beta t}\nabla^2 v\|_{L_2(\mathbb{R}_+;L_6)}\\
&\quad\le C_0 + C\|e^{\beta t/2}\sqrt{t}\nabla v\|^{1/2}_{L_\infty(\mathbb{R}_+;L_2)}\|e^{\beta t/2}\nabla v\|_{L_2(\mathbb{R}_+;L_6)}\|\sqrt{t}e^{\beta t/2}\nabla v\|^{1/2}_{L_\infty(\mathbb{R}_+;L_6)}\\
&\quad\le C_0 + C_0\|e^{\beta t}\nabla v\|^{1/2}_{L_\infty(\mathbb{R}_+;L_2)}\|\sqrt{t}e^{\beta t/2}\nabla v\|^{1/2}_{L_\infty(\mathbb{R}_+;L_6)}\\
&\quad\le C_0 + C_0\|e^{\beta t}\nabla v\|_{L_\infty(\mathbb{R}_+;L_2)} + \|\sqrt{t}e^{\beta t}\nabla^2 v\|_{L_\infty(\mathbb{R}_+;L_2)},
\end{aligned}$$

where in the first passage we have used (3.6) and Sobolev embedding to estimate $\|e^{\beta t/2}\nabla v\|_{L_2(\mathbb{R}_+;L_6)}$.

Therefore, by (3.4) and (3.33), choosing β small enough, we obtain

$$\|\sqrt{t}e^{\beta t}(\nabla^2 v, \nabla P)\|_{L_2(\mathbb{R}_+;L_6)} \le C_0.$$

This completes the proof of (3.31), and thus also of (3.26). The easier two-dimensional case (that is, (3.27)) is left to the reader.

In order to show the next part of the lemma, we note that for $\beta_4, \delta > 0$ such that $\beta_4 + \delta < \beta_2$ we have

$$\begin{aligned}
\int_1^{+\infty} e^{\beta_4 t}\|\nabla^2 v\|_{L_s}\,\mathrm{d}t &\le \Big(\int_1^{+\infty} e^{-\delta p' t}\,\mathrm{d}t\Big)^{1/p'}\Big(\int_1^{+\infty} e^{(\beta_4+\delta)pt}\|\nabla^2 v(t)\|^p_{L_s}\,\mathrm{d}t\Big)^{1/p}\\
&\le C\|e^{\beta_2 t}\sqrt{t}\nabla^2 v\|_{L_p(\mathbb{R}_+;L_s)}.
\end{aligned}$$

For small times we can write

$$\int_0^1 \|\nabla^2 v(t)\|_r\,\mathrm{d}t \le \Big(\int_0^1 t^{-\alpha p'}\,\mathrm{d}t\Big)^{1/p'}\int_0^1 t^{\alpha p}\|\nabla^2 v(t)\|^p_r\,\mathrm{d}t.$$

We can choose for instance $p = \frac{8}{5}$, which corresponds to $s = 4$ in (3.26), then $\alpha = \frac{5}{16}$ so that $\alpha p = \frac{1}{2}$ and $\alpha p' < 1$, so the first integral is again finite. This completes the proof of (3.28).

As for (3.29), it results directly from (3.4) and (3.28) owing to a suitable Gagliardo–Nirenberg inequality that yields

$$\|e^{\beta_4 t}\nabla v\|_{L_1(\mathbb{R}_+;L_\infty)} \le C(\|e^{\beta_4 t}\nabla v\|_{L_1(\mathbb{R}_+;L_2)} + \|e^{\beta_4 t}\nabla^2 v\|_{L_1(\mathbb{R}_+;L_r)}).$$

The right-hand side is finite whenever $\beta_4 < \beta_2$. ■

Funding. The first two authors have been partially supported by the ANR project INFAMIE (ANR-15-CE40-0011). The second (PBM) and third (TP) authors were partially supported by National Science Centre grant No2018/29/B/ST1/00339 (Opus).

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Received 15 June 2022; revised 2 December 2022; accepted 13 December 2022.

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