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Analysis of the inhomogeneous Willmore equation

Yann Bernard, Glen Wheeler, and Valentina-Mira Wheeler

Abstract. We study a class of fourth-order geometric problems modeling Willmore surfaces, conformally constrained Willmore surfaces, isoperimetrically constrained Willmore surfaces, and biharmonic surfaces in the sense of Chen, among others. We prove several local energy estimates and derive a global gap lemma.

1. Introduction and main results

Let Σ be a smooth, two-dimensional, closed, oriented manifold, and let g_0 be a smooth reference metric on Σ. For any $s \geq 1$, the Sobolev space $W^{k,p}(\Sigma, \mathbb{R}^s)$ is the space of measurable maps $f: \Sigma \to \mathbb{R}^s$ for which

$$\sum_{j=0}^{k} \int_\Sigma |\nabla^j f|_{g_0}^p \, d\mathrm{vol}_{g_0} < \infty.$$

For a closed surface Σ, this space is independent of the reference metric g_0.

The notion of *weak immersion with L^2-bounded second fundamental form* is well understood and has been extensively studied (the interested reader will find a detailed account in [35] and the references therein). This will be the main object of study in this paper, and we now recall the main definition. Let $\vec{\Phi}: \Sigma \to \mathbb{R}^m$, for $m \geq 3$, be measurable and Lipschitz. The associated pull-back metric $g := \vec{\Phi}^* g_{\mathbb{R}^m}$ is given almost everywhere by

$$g(X, Y) := d\vec{\Phi}(X) \cdot d\vec{\Phi}(Y) \quad \forall X, Y \in T\Sigma,$$

where dot indicates the standard scalar product in $\mathbb{R}^m$. We will demand that g be nondegenerate, that is, that there exists a constant $c > 0$ satisfying

$$c^{-1} g_0(X, X) \leq g(X, X) \leq c g_0(X, X) \quad \forall X \in T\Sigma. \tag{1.1}$$

This makes $(\Sigma, \vec{\Phi}^* g_{\mathbb{R}^m})$ a Riemannian 2-manifold with a rough metric. The Gauss map is a bounded measurable map $\vec{n}$ taking values in the Grassmannian $\mathrm{Gr}_{m-2}(\mathbb{R}^m)$ of oriented

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$(m-2)$-planes in $\mathbb{R}^m$ satisfying

$$\vec{n} := \star \frac{\partial_{x^1}\vec{\Phi} \wedge \partial_{x^2}\vec{\Phi}}{|\partial_{x^1}\vec{\Phi} \wedge \partial_{x^2}\vec{\Phi}|},$$

where $\star$ denotes the standard Hodge star operator, and $\{x^1, x^2\}$ is an arbitrary choice of local coordinates.

Finally, to say that the weak immersion $\vec{\Phi}$ has square integrable second fundamental form amounts to requiring that

$$\int_{\Sigma} |d\vec{n}|_g^2 \, d\mathrm{vol}_g < \infty. \tag{1.2}$$

We let

$$\mathcal{E}_{\Sigma} := \big\{\vec{\Phi}\colon \Sigma \to \mathbb{R}^m \text{ measurable and Lipschitz such that (1.1) and (1.2) hold}\big\}.$$

Rescaling if necessary, condition (1.2) ensures that on some local patch, let us say it is the unit disk $D_1(0)$, there holds

$$\int_{D_1(0)} |\nabla\vec{n}|^2 \, dx^1 \, dx^2 < \frac{8\pi}{3}. \tag{1.3}$$

Here $\{x^1, x^2\}$ are local coordinates on $D_1(0)$ and ∇ stands for the usual flat gradient in these coordinates. A well-known result ([18, 33]) states that if $\vec{\Phi} \in \mathcal{E}_{D_1(0)}$ satisfies (1.3), then there exists a bi-Lipschitz homeomorphism ψ of $D_1(0)$ such that the map $\vec{\Phi} \circ \psi\colon D_1(0) \to \mathbb{R}^m$ is conformal, namely,

$$\partial_{x^i}(\vec{\Phi} \circ \psi) \cdot \partial_{x^j}(\vec{\Phi} \circ \psi) = \mathrm{e}^{2\lambda}\delta_{ij}$$

for some conformal factor λ. Without loss of generality, as we are only concerned with locally analyzing the solutions to problems that are independent of parametrization, we will henceforth suppose that $\vec{\Phi}$ itself is conformal.

The present paper is concerned with studying the local analytical properties of the inhomogeneous Willmore equation. To an immersion $\vec{\Phi} \in \mathcal{E}_{\Sigma}$ of an oriented two-dimensional manifold Σ into $\mathbb{R}^m$, for some $m \geq 3$, we assign the second fundamental form $\vec{h} := \pi_{\vec{n}} D^2\vec{\Phi}$, where $\pi_{\vec{n}}$ denotes the projection of vectors in $\mathbb{R}^m$ onto the $(m-2)$-plane defined by the Gauss map $\vec{n}$. The trace of the 2-tensor $\vec{h}$ with respect to g is twice the normal-valued mean curvature vector:

$$\vec{H} := \frac{1}{2}\operatorname{Tr}_g \vec{h}.$$

Willmore immersions are critical points of the Willmore energy

$$\int_{\Sigma} |\vec{H}|^2 \, d\mathrm{vol}_g.$$

The study of Willmore immersions has been steadily gaining momentum over the last century. It would be impossible to give a detailed account of the various works and results that have appeared in recent years. We content ourselves with mentioning the tour de force by Marques and Neves [28], where they prove the celebrated Willmore conjecture [44]: the Clifford torus minimizes, up to Möbius transformations, the Willmore energy in the class of immersed tori in $\mathbb{R}^3$. Although the Willmore conjecture is now resolved, the study of Willmore immersions continues to grow in intensity.

Any critical point of the Willmore energy satisfies the following fourth-order, quasi-linear, strongly coupled system of equations [39, 44]:

$$\Delta_\perp \vec{H} + \langle \vec{h} \cdot \vec{H}, \vec{h} \rangle_g - 2|\vec{H}|^2 \vec{H} = \vec{0}, \tag{1.4}$$

where $\Delta_\perp$ is the negative covariant Laplacian for the connection in the normal bundle. The dot indicates the standard scalar product of vectors in $\mathbb{R}^m$, while the product $\langle \cdot, \cdot \rangle_g$ is the usual contraction product with respect to the metric g for tensors. Naturally, when constraints are imposed on the problem of varying the Willmore energy, the right-hand side of (1.4) is no longer zero. Various examples are provided in [4] and we will below look closer at a few specific cases of relevance in applications. Thus we are motivated to study a problem of the type

$$\Delta_\perp \vec{H} + \langle \vec{h} \cdot \vec{H}, \vec{h} \rangle_g - 2|\vec{H}|^2 \vec{H} = \vec{\mathcal{W}}, \tag{1.5}$$

where the right-hand side $\vec{\mathcal{W}}$ is assumed to be known. Naturally, $\vec{\mathcal{W}}$ has to be normal vector for (1.5) to make sense. It also has to be independent of parametrization. Before going any further, an important observation is in order. When $\vec{\Phi}$ lies in $\mathcal{E}_\Sigma$, it is clear that $\vec{H}$ is square integrable. Even in the case when $\vec{\mathcal{W}} \equiv \vec{0}$, the term $|\vec{H}|^2 \vec{H}$ is already problematic, for it lies in no space that enables us to give a distributional sense to the equation. Nevertheless, one may study the problem and obtain estimates, as is done for example in [43] and the references therein. Another approach was originally devised by Rivière [34]. It relies mainly on the fact that the left-hand side of (1.5) can be factored into an exact divergence, thereby rendering possible the assignment of a distributional sense to (1.4). In [4], it is shown that the divergence structure seemingly hidden in (1.4) is a direct consequence of Noether's theorem applied to the translation invariance of the Willmore energy. The present paper should be understood as a companion to [4]. While in the latter only identities were derived, the present work brings to fruition the reformulations presented in [4] by obtaining local analytical results for problems of type (1.5). The present paper should also be seen as a companion to [43], where only a specific class of right-hand sides $\vec{\mathcal{W}}$ were considered. The class of possible right-hand sides will be here significantly expanded.

As was shown in [34], any conformal immersion $\vec{\Phi}\colon D_1(0) \to \mathbb{R}^m$ that satisfies the Willmore equation (1.4) also satisfies the equation

$$\operatorname{div}(\nabla \vec{H} - 2\pi_{\vec{n}} \nabla \vec{H} + |\vec{H}|^2 \nabla \vec{\Phi}) = \vec{0} \quad \text{on } D_1(0),$$

where $\pi_{\vec{n}}$ denotes projection on the normal bundle. The operators ∇ and div are understood in local coordinates $\{x^1, x^2\}$ on the unit disk $D_1(0)$. This motivates us to consider inhomogeneous Willmore problems of the type

$$\operatorname{div}(\nabla\vec{H} - 2\pi_{\vec{n}}\nabla\vec{H} + |\vec{H}|^2\nabla\vec{\Phi} + \vec{T}) = \vec{v}, \tag{1.6}$$

for some vector field $\vec{T} \in \Gamma(\mathbb{R}^2 \otimes \mathbb{R}^m)$ and some normal vector field $\vec{v} \in \Gamma(\mathbb{R}^m)$. Many known classes of immersed surfaces satisfy a problem of this type:

(1) Willmore immersions with $\vec{v} \equiv 0$ and $\vec{T} \equiv \vec{0}$.

(2) Constrained Willmore immersions:

(i) Varying the Willmore energy $\int_\Sigma |\vec{H}|^2 \, d\mathrm{vol}_g$ in a fixed conformal class (i.e. with infinitesimal, smooth, compactly supported, conformal variations) gives rise to a more general class of surfaces called *conformally constrained Willmore surfaces*, whose corresponding Euler–Lagrange equation [10, 25, 36] is expressed as follows. Let $\vec{h}_0$ denote the trace-free part of the second fundamental form, namely,

$$\vec{h}_0 := \vec{h} - \vec{H} g.$$

A conformally constrained Willmore immersion $\vec{\Phi}$ satisfies

$$\Delta_\perp \vec{H} + (\vec{H} \cdot \vec{h}^i_j)\vec{h}^j_i - 2|\vec{H}|^2\vec{H} = (\vec{h}_0)_{ij} q^{ij}, \tag{1.7}$$

where q is a transverse[1] traceless symmetric 2-form. This tensor q plays the role of Lagrange multiplier in the constrained variational problem. It is shown in [2, 4] that in a conformal parametrization, with conformal parameter λ, (1.7) can also be recast in another form of (1.6), namely by setting $\vec{v} \equiv \vec{0}$ and

$$\vec{T} = -\mathrm{e}^{-2\lambda} M_q \nabla^\perp \vec{\Phi},$$

where $\nabla^\perp \vec{\Phi} := (-\partial_{x^2}\vec{\Phi}, \partial_{x^1}\vec{\Phi})$, and M_q is the matrix

$$M_q := \begin{pmatrix} -q_{12} & q_{11} \\ q_{11} & q_{12} \end{pmatrix}.$$

(ii) Bilayer models [9, 11, 19]. These models also bear the names Helfrich and Canham–Helfrich, and arise in the modeling of the surface of liposomes and vesicles (see [4] and the references therein). One seeks to minimize the Willmore energy under the requirement that the area $A(\Sigma)$, the volume $V(\Sigma)$, and the total curvature

$$M(\Sigma) := \int_\Sigma H \, d\mathrm{vol}_g$$

[1] That is, q is divergence-free: $\nabla^j q_{ji} = 0$ for all i.

be prescribed. This leads to an equation of type (1.5) with

$$\vec{W} = 2(\beta + \alpha H + \gamma K)\vec{n},$$

where K is the Gauss curvature, and α, β, γ are three given parameters acting as Lagrange multipliers.

As shown in [4], this problem can be brought in the form (1.6) with $\vec{v} \equiv \vec{0}$ and $|\vec{T}| \lesssim 1 + |\nabla\vec{n}|$.

(iii) Another instance in which minimizing the Willmore energy arises is the isoperimetric problem [22, 37], which consists in minimizing the Willmore energy under the constraint that the dimensionless isoperimetric ratio $\sigma := 36\pi V^2/A^3$ be a given constant in $(0, 1]$. As both the Willmore energy and the constraint are invariant under dilation, one might fix the volume as $V = 1/(6\sqrt{\pi})$, thereby forcing the area to satisfy $A = \sigma^{1/3}$. This problem is thus equivalent to the bilayer model with $\gamma = 0$ (no constraint imposed on the total curvature, but the volume and area are prescribed separately).

(3) Chen surfaces. An isometric immersion $\vec{\Phi}\colon N^n \to \mathbb{R}^{m>n}$ of an n-dimensional Riemannian manifold N^n into Euclidean space is called *biharmonic* if the corresponding mean-curvature vector $\vec{H}$ satisfies

$$\Delta_g \vec{H} = \vec{0}. \tag{1.8}$$

The study of biharmonic submanifolds was initiated by Chen [12] in the mid-1980s as he was seeking a classification of finite-type submanifolds in Euclidean spaces. Independently, Jiang [21] also studied (1.8) in the context of the variational analysis of the biharmonic energy in the sense of Eells and Lemaire. Chen conjectures that a biharmonic immersion is necessarily minimal.[2] Smooth solutions of (1.8) are known to be minimal for $n = 1$ [15], for $(n, m) = (2, 3)$ [14], and for $(n, m) = (3, 4)$ [17]. In [42], it is shown that Chen's conjecture holds up to a growth condition on the Willmore energy. Chen's conjecture has been solved under a variety of hypotheses (see the recent survey paper [13]). The statement remains nevertheless open in general, and in particular for immersed surfaces in $\mathbb{R}^m$. In [3],[3] it is shown that Chen surfaces satisfy an equation of type (1.5) with

$$|\vec{W}| \simeq |\vec{h}|^3.$$

It can more precisely be brought into the form (1.6) with $\vec{v} \equiv \vec{0}$ and $|\vec{T}| \lesssim e^{\lambda}|\nabla\vec{n}|^2$.

[2] The conjecture as originally stated is rather analytically vague: no particular hypotheses on the regularity of the immersion are a priori imposed. Many authors consider only smooth immersions.

[3] This paper is the precursor to the published version [4], which unfortunately, at the referee's request, no longer addresses the question of Chen immersions.

(4) Complete Willmore immersions in asymptotically flat spaces also satisfy a problem of type (1.6). Details may be found in [8].

(5) Equilibria of flow equations. In [23], stability of the sphere is proven for the Willmore flow. Global existence is obtained by contradiction: one assumes that existence time is finite, and then rescales around a point in space-time where the energy concentrates. Local estimates allow one to construct a blowup. The blowup is shown to be an entire Willmore surface with small energy. To this blowup one applies a gap lemma, which implies that any such surface is a standard flat plane. This is in contradiction with the concentration of energy hypothesis, and so no such concentration points can occur, and the flow exists for all time. This argument is by now standard, having been adapted at least to constrained surface diffusion flows [40, 41], locally constrained Willmore flow [31], Willmore flow in Riemannian spaces [26, 32], and a geometric triharmonic heat flow [29].

An appropriate gap lemma combined with local regularity is crucial and so far has been established separately for each of the flows given above. As our work here holds for more general equations than what is currently available, we expect that the results in this paper will apply to a broad class of fourth-order evolution equations. It is an interesting open question to investigate higher-order cases.

Our first main result consists of local energy estimates.

Theorem 1.1. *Let $\vec{\Phi} \in W^{2,2} \cap W^{1,\infty}(D_1(0), \mathbb{R}^m)$ be a conformal immersion with conformal parameter λ satisfying*

$$\|\nabla\lambda\|_{L^{2,\infty}(D_1(0))} < +\infty,$$

where $L^{2,\infty}$ denotes the weak-L^2 Marcinkiewicz space. Suppose that

$$\int_{D_1(0)} |\nabla\vec{n}|^2 \, dx = \varepsilon_0^2. \tag{1.9}$$

Provided that ε_0 is sufficiently small, there is a universal constant $C(\varepsilon_0, \|\nabla\lambda\|_{L^{2,\infty}(D_1(0))})$ for which the following statements hold:

(i) *Let $p \in (1, \infty)$. Suppose that $\vec{\Phi}$ is a solution on $D_1(0)$ of*

$$\operatorname{div}(\nabla\vec{H} - 2\pi_{\vec{n}}\nabla\vec{H} + |\vec{H}|^2\nabla\vec{\Phi} + \vec{T}) = \vec{0}.$$

Then for all $D_\rho(x) \subset D_1(0)$, we have

$$\|\nabla^2\vec{n}\|_{L^p(D_{\rho/2}(x))} \le C(\varepsilon_0, \|\nabla\lambda\|_{L^{2,\infty}(D_1(0))}) \times \big[\|e^\lambda\vec{T}\|_{L^p(D_\rho(x))} + \rho^{\frac{2}{p}-2}\|\nabla\vec{n}\|_{L^2(D_\rho(x))}\big].$$

(ii) *Let $r \in [1, \infty)$. Suppose that $\vec{\Phi}$ is a solution on $D_1(0)$ of*

$$\operatorname{div}(\nabla\vec{H} - 2\pi_{\vec{n}}\nabla\vec{H} + |\vec{H}|^2\nabla\vec{\Phi}) = \vec{v},$$

For all $D_\rho(x) \subset D_1(0)$, we have

(a) *if* $r = 1$ *the second-order estimate*

$$\|\nabla^2\vec{n}\|_{L^p(D_{\rho/2}(x))} \le C(\varepsilon_0, \|\nabla\lambda\|_{L^{2,\infty}(D_1(0))}) \times \big[\|e^{\lambda}\vec{v}\|_{L^r(D_\rho(x))} + \rho^{\frac{2}{p}-2}\|\nabla\vec{n}\|_{L^2(D_\rho(x))}\big]$$

for all $p \in (1,2)$.

(b) *if* $r > 1$ *the third-order estimate*

$$\|\nabla^3\vec{n}\|_{L^r(D_{\rho/2}(x))} \le C(\varepsilon_0, \|\nabla\lambda\|_{L^{2,\infty}(D_1(0))}) \times \big[\|e^{\lambda}\vec{v}\|_{L^r(D_\rho(x))} + \rho^{\frac{2}{r}-3}\|\nabla\vec{n}\|_{L^2(D_\rho(x))}\big].$$

Theorem 1.1 is used to prove the following regularity result:

Corollary 1.1. *Let* $\vec{\Phi} \in W^{2,2} \cap W^{1,\infty}(D_1(0), \mathbb{R}^m)$ *be a conformal immersion satisfying* (1.6) *on the disk* $D_1(0)$. *If* $\vec{T}$ *and* $\vec{v}$ *are smooth, so is* $\vec{\Phi}$.

Finally, we derive an interesting geometric "gap" result, obtained using the same techniques as those leading to Theorem 1.1.

Theorem 1.2. *Let* Σ *be a connected, oriented, complete, immersed surface in* $\mathbb{R}^m$ *whose mean curvature vector satisfies an inhomogeneous Willmore problem of the type*[4]

$$\Delta_\perp\vec{H} + \langle\vec{h}\cdot\vec{H}, \vec{h}\rangle_g - 2|\vec{H}|^2\vec{H} = \mathcal{O}(|\vec{h}|^3).$$

There exists an $\varepsilon_0 > 0$ *such that if*

$$\int_\Sigma |\vec{h}|^2\, d\,\mathrm{vol}_g < \varepsilon_0^2,$$

then Σ *is a flat plane.*

This gap result is to be compared to the one given in [43] (see also [30]).

A word of caution is now in order. Should $\vec{\Phi}$ be a (conformal) Willmore immersion satisfying the small energy condition (1.9), then $\vec{v} \equiv \vec{0}$ and Theorem 1.1 (i) gives the estimate

$$\|\nabla\vec{n}\|_{L^\infty(D_{\rho/2}(x))} \le C\rho^{-1}\|\nabla\vec{n}\|_{L^2(D_\rho(x))}.$$

This estimate, which we will term *parametric ε-regularity*, is the one that was originally derived by Rivière [34]. In conformal parametrization, $|\nabla\vec{n}| \simeq e^{\lambda}\vec{h}$, where $\vec{h}$ is the second fundamental form, so the above reads

$$\|e^{\lambda}\vec{h}\|_{L^\infty(D_{\rho/2}(x))} \le C\rho^{-1}\|e^{\lambda}\vec{h}\|_{L^2(D_\rho(x))}.$$

[4] We use the same notation as in (1.4).

Knowing that our conformal immersion does not "distort" flat disks much, we can further rephrase the latter as

$$\|\vec{h}\|_{L^\infty(D^g_{\rho/2}(x))} \leq C\rho^{-1}\|\vec{h}\|_{L^2_g(D^g_\rho(x))}, \tag{1.10}$$

where $D^g_\rho(x)$ is the metric disk with respect to the induced pull-back metric $g = \vec{\Phi}^* g_{\mathbb{R}^m}$, and L^2_g is the space $(L^2, d\mathrm{vol}_g)$. Estimate (1.10) is to be compared with Kuwert and Schätzle's original estimate in [23], which we will term *ambient ε-regularity*, and which states that if $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$ is a Willmore immersion with

$$\int_{\vec{\Phi}^{-1}(B_\sigma(p))} |\vec{h}|^2 d\mathrm{vol}_g < \varepsilon_0^2$$

for some Euclidean ball $B_\sigma(p) \subset \mathbb{R}^m$, and ε_0 is sufficiently small, then

$$\|\vec{h}\|_{L^\infty(\vec{\Phi}^{-1}(B_{\sigma/2}(p)))} \leq C\sigma^{-1}\|\vec{h}\|_{L^2(\vec{\Phi}^{-1}(B_\sigma(p)))}. \tag{1.11}$$

It is stated in [23, Remark 2.11], and more explicitly in [24, equation (2.18)], that this estimate implies

$$\|\vec{h}\|_{L^\infty(D^g_{\rho/2}(x))} \leq C\rho^{-1}\|\vec{h}\|_{L^2_g(D^g_\rho(x))}, \tag{1.12}$$

which is (1.10). To the authors' knowledge, it is unclear that (1.12) follows from (1.11). The two versions of ε-regularity, parametric and ambient, are in reality distinct, and we do not know how to recover one from the other.

In the same direction, Marque (see [27, Section 2]) has devised a precise example in which parametric ε-regularity holds but ambient ε-regularity fails.

2. Proofs of the results

2.1. Controlling the conformal factor

Using Hélein's method of moving Coulomb frames [18] (in particular Section 5.2), a weak immersion $\vec{\Phi} \in W^{2,2}_{\mathrm{imm}}(D_1(0), \mathbb{R}^m)$ of the unit disk $D_1(0)$ into $\mathbb{R}^m$ can be reparametrized by a diffeomorphism of $D_1(0)$ to become conformal. Our problem being independent of parametrization, we will without loss of generality suppose that $\vec{\Phi}$ is conformal with parameter λ, namely,

$$\partial_{x_i}\vec{\Phi} \cdot \partial_{x_j}\vec{\Phi} = \mathrm{e}^{2\lambda}\delta_{ij}.$$

We will henceforth use the notation ∇, div, and Δ to denote the usual gradient, divergence, and Laplacian operators in flat local coordinates $\{x_1, x_2\}$.

Assume

$$\int_{D_1(0)} |\nabla\vec{n}|^2\, dx =: \varepsilon_0^2 \leq 8\pi/3 \quad \text{and} \quad \|\nabla\lambda\|_{L^{2,\infty}(D_1(0))} < +\infty.$$

We can call upon [18, Lemma 5.1.4] to deduce the existence of an orthogonal frame $\{\vec{e}_1, \vec{e}_2\} \in W^{1,2}(D_1(0))$ satisfying $\star\vec{n} = \vec{e}_1 \wedge \vec{e}_2$ and

$$\|\nabla\vec{e}_1\|_{L^2(D_2(0))} + \|\nabla\vec{e}_2\|_{L^2(D_2(0))} \leq C\|\nabla\vec{n}\|_{L^2(D_2(0))}.$$

As is easily verified, the conformal parameter satisfies

$$\Delta\lambda = \nabla\vec{e}_1 \cdot \nabla^{\perp}\vec{e}_2 \quad \text{in } D_1(0).$$

Let μ satisfy

$$\begin{cases} \Delta\mu = \nabla\vec{e}_1 \cdot \nabla^{\perp}\vec{e}_2 & \text{in } D_1(0), \\ \mu = 0 & \text{on } \partial D_1(0). \end{cases}$$

Standard Wente estimates (cf. [18, Theorem 3.4.1]) give

$$\begin{aligned} \|\mu\|_{L^\infty(D_1(0))} + \|\nabla\mu\|_{L^2(D_1(0))} &\leq \|\nabla\vec{e}_1\|_{L^2(D_1(0))}\|\nabla\vec{e}_2\|_{L^2(D_1(0))} \\ &\leq C\|\nabla\vec{n}\|^2_{L^2(D_1(0))}. \end{aligned} \tag{2.1}$$

The harmonic function $\nu := \lambda - \mu$ satisfies the usual estimate

$$\int_D |\nu - \bar{\nu}|\, dx \leq C\|\nabla\nu\|_{L^1(D_1(0))} \leq C\|\nabla\nu\|_{L^{2,\infty}(D_1(0))},$$

where $\bar{\nu}$ denotes the average of ν on the proper subdisk $D \subset\subset D_1(0)$. Hence

$$\|\nu - \bar{\nu}\|_{L^\infty(D)} \leq C\|\nabla\nu\|_{L^{2,\infty}(D_1(0))}.$$

Combining the latter with (2.1) now yields

$$\begin{aligned} \|\lambda - \bar{\lambda}\|_{L^\infty(D)} &\leq C\|\nabla\lambda\|_{L^{2,\infty}(D_1(0))} + C\|\nabla\vec{n}\|^2_{L^2(D_1(0))} \\ &\leq C(\varepsilon_0, \|\nabla\lambda\|_{L^{2,\infty}(D_1(0))}), \end{aligned}$$

where $\bar{\lambda}$ denotes the average of λ on D. We can summarize this subsection by stating the following lemma.

Lemma 2.1. *Let $\vec{\Phi} \in W^{2,2}_{\text{imm}}(D_1(0), \mathbb{R}^m)$ be a conformal weak immersion such that*

$$\int_{D_1(0)} |\nabla\vec{n}|^2\, dx = \varepsilon_0^2 \leq 8\pi/3 \quad \textit{and} \quad \|\nabla\lambda\|_{L^{2,\infty}(D_1(0))} < +\infty,$$

with $\mathrm{e}^\lambda := |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$. Then the following estimate holds for any proper subdisk $D \subset\subset D_1(0)$:

$$\|\mathrm{e}^\lambda\|_{L^\infty(D)}\|\mathrm{e}^{-\lambda}\|_{L^\infty(D)} \leq C(\varepsilon_0, \|\nabla\lambda\|_{L^{2,\infty}(D_1(0))}). \tag{2.2}$$

2.2. Proof of Theorem 1.1 (i)

Per the discussion in the introduction and our aim to study only local properties of solutions to (1.5), we assume without loss of generality that the immersion $\vec{\Phi}$ is conformal, i.e. in local coordinates $\{x^1, x^2\}$ on the unit disk $D_1(0)$ that

$$\partial_{x^i}\vec{\Phi}\cdot\partial_{x^j}\vec{\Phi} = \mathrm{e}^{2\lambda}\delta_{ij},$$

with bounded conformal parameter λ, and such that e^{λ} satisfies the Harnack inequality (2.2). We will first begin by studying an inhomogeneous Willmore equation of the form

$$\operatorname{div}(\nabla\vec{H} - 2\pi_{\vec{n}}\nabla\vec{H} + |\vec{H}|^2\nabla\vec{\Phi} + \vec{T}) = \vec{0}, \quad \text{on } D_1(0), \tag{2.3}$$

where $\vec{T}$ satisfies the following condition for some $p \in (1,\infty)$:

$$\|\mathrm{e}^{\lambda}\vec{T}\|_{L^p(D_1(0))} < \infty. \tag{2.4}$$

Let $D_\rho(x) \subset D_1(0)$. As done in [4], we consider the following two problems:

$$\Delta\vec{X} = \nabla\vec{\Phi}\wedge\vec{T} \quad \text{and} \quad \Delta Y = \nabla\vec{\Phi}\cdot\vec{T} \quad \text{on } D_\rho(x), \tag{2.5}$$

with boundary conditions $\vec{X}|_{\partial D_\rho(x)} = \vec{0}$ and $Y|_{\partial D_\rho(x)} = 0$. Standard Calderon–Zygmund estimates give

$$\|\nabla^2\vec{X}\|_{L^p(D_\rho(x))} + \|\nabla^2 Y\|_{L^p(D_\rho(x))} \lesssim \|\mathrm{e}^{\lambda}\vec{T}\|_{L^p(D_\rho(x))}, \tag{2.6}$$

up to a universal multiplicative constant. Hence,

$$\|\nabla\vec{X}\|_{L^{2,\infty}(D_\rho(x))} + \|\nabla Y\|_{L^{2,\infty}(D_\rho(x))} \lesssim \rho^{2-\frac{2}{p}}\|\mathrm{e}^{\lambda}\vec{T}\|_{L^p(D_\rho(x))}. \tag{2.7}$$

We now follow the procedure outlined in [4]. Integrating (2.3), we infer the existence of a potential $\vec{L}$ satisfying

$$\nabla^\perp\vec{L} = \nabla\vec{H} - 2\pi_{\vec{n}}\nabla\vec{H} + |\vec{H}|^2\nabla\vec{\Phi} + \vec{T} \equiv -\nabla\vec{H} + 2\pi_T\nabla\vec{H} + |\vec{H}|^2\nabla\vec{\Phi} + \vec{T}, \tag{2.8}$$

where π_T is the tangential projection. An elementary computation (cf. [2, equation (II.6)]) reveals that

$$|\pi_T\nabla\vec{H}| \lesssim \mathrm{e}^{\lambda}|\nabla\vec{n}|^2. \tag{2.9}$$

As $\vec{L}$ is defined up to an arbitrary constant, we are certainly free to require that

$$\int_{D_\rho(x)}\vec{L} = \vec{0}.$$

Observe next that

$$\|\nabla\vec{H}\|_{W^{-1,2}(D_\rho(x))} \le \|\vec{H}\|_{L^2(D_\rho(x))} \le \|\mathrm{e}^{-\lambda}\|_{L^\infty(D_\rho(x))}\|\nabla\vec{n}\|_{L^2(D_\rho(x))},$$

and, owing to (2.4) and (2.9),

$$\|\vec{T}+2\pi_T\nabla\vec{H}+|\vec{H}|^2\nabla\vec{\Phi}\|_{L^1(D_\rho(x))}\lesssim\|e^{-\lambda}\|_{L^\infty(D_\rho(x))}\times\big[\|e^{\lambda}\vec{T}\|_{L^1(D_\rho(x))}+\|\nabla\vec{n}\|^2_{L^2(D_\rho(x))}\big],$$

up to a multiplicative constant independent of the parametrization and of the mean curvature, and irrelevant to our purpose. Geared with these last inequalities, we call upon Lemma A.1 and conclude that

$$\|\vec{L}\|_{L^{2,\infty}(D_\rho(x))}\lesssim\|e^{-\lambda}\|_{L^\infty(D_\rho(x))}\big[\|e^{\lambda}\vec{T}\|_{L^1(D_\rho(x))}+\|\nabla\vec{n}\|^2_{L^2(D_\rho(x))}\big],\tag{2.10}$$

where $L^{2,\infty}$ is the weak-L^2 Marcinkiewicz space, seen here as a Lorentz space [38]. Per Lemma 2.1, e^{λ} satisfies a Harnack inequality. The above then yields

$$\|e^{\lambda}\vec{L}\|_{L^{2,\infty}(D_\rho(x))}\lesssim\|e^{\lambda}\vec{T}\|_{L^1(D_\rho(x))}+\|\nabla\vec{n}\|_{L^2(D_\rho(x))}.\tag{2.11}$$

We will use the symbol $\lesssim$ to indicate the presence of a multiplicative constant depending at most only on ε_0 and on $\|\nabla\lambda\|_{L^{2,\infty}(D_1(0))}$.

It is shown in [4] that two important identities hold, namely,

$$\operatorname{div}(\vec{L}\wedge\nabla^\perp\vec{\Phi}+\vec{H}\wedge\nabla\vec{\Phi}+\nabla\vec{X})=\vec{0}\quad\text{and}\quad\operatorname{div}(\vec{L}\cdot\nabla^\perp\vec{\Phi}+\nabla Y)=0.$$

Again, we infer the existence of two potentials $\vec{R}$ and S satisfying

$$\nabla\vec{R}=\vec{L}\wedge\nabla\vec{\Phi}-\vec{H}\wedge\nabla^\perp\vec{\Phi}-\nabla^\perp\vec{X}\quad\text{and}\quad\nabla S=\vec{L}\cdot\nabla\vec{\Phi}-\nabla^\perp Y.\tag{2.12}$$

Owing to (2.11) and (2.7), we find that $\nabla\vec{R}$ and ∇S lie in the weak space $L^{2,\infty}$, namely,

$$\begin{aligned}\|\nabla\vec{R}\|_{L^{2,\infty}(D_\rho(x))}&+\|\nabla S\|_{L^{2,\infty}(D_\rho(x))}\\ &\lesssim\|e^{\lambda}\vec{T}\|_{L^1(D_\rho(x))}+\|\nabla\vec{n}\|^2_{L^2(D_\rho(x))}+\|\nabla\vec{n}\|_{L^2(D_\rho(x))}+\|\nabla\vec{X}\|_{L^{2,\infty}(D_\rho(x))}\\ &\quad+\|\nabla Y\|_{L^{2,\infty}(D_\rho(x))}\\ &\lesssim\rho^{2-\frac{2}{p}}\|e^{\lambda}\vec{T}\|_{L^p(D_\rho(x))}+\|\nabla\vec{n}\|_{L^2(D_\rho(x))},\end{aligned}$$

where $C(\varepsilon_0)$ is a constant depending only on ε_0. In other words,

$$\|\nabla\vec{R}\|_{L^{2,\infty}(D_\rho(x))}+\|\nabla S\|_{L^{2,\infty}(D_\rho(x))}\lesssim M_p,\tag{2.13}$$

where for notational convenience, we have set

$$M_p:=\rho^{2-\frac{2}{p}}\|e^{\lambda}\vec{T}\|_{L^p(D_\rho(x))}+\|\nabla\vec{n}\|_{L^2(D_\rho(x))}.\tag{2.14}$$

It is remarkable that $\vec{R}$ and S are linked together via an interesting system of equations that displays a very particular structural type. It is shown in [4] that[5]

$$\begin{cases}\Delta\vec{R}=\nabla(\star\vec{n})\bullet\nabla^\perp\vec{R}+\nabla(\star\vec{n})\cdot\nabla^\perp S+\operatorname{div}((\star\vec{n})\bullet\nabla\vec{X}+(\star\vec{n})\nabla Y),\\ \Delta S=\nabla(\star\vec{n})\cdot\nabla^\perp\vec{R}+\operatorname{div}((\star\vec{n})\cdot\nabla\vec{X}).\end{cases}\tag{2.15}$$

The apparent notational complication is an artifice of codimension only.

[5] Refer to the appendix for the notation.

System (2.15) is in divergence form. Owing to $\|\vec{n}\|_{L^\infty(D_1(0))} = 1$, to $\|\nabla\vec{n}\|_{L^2(D_1(0))} < \varepsilon_0$, and to (2.6), we can call upon Proposition A.2,[6] which states that for[7] all $s < 2/(2-p)_+$ it holds that

$$\begin{aligned}
&\|\nabla\vec{R}\|_{L^s(D_{5\rho/8}(x))} + \|\nabla S\|_{L^s(D_{5\rho/8}(x))} \\
&\quad\lesssim \rho^{\frac{2}{s}-1}\big[\|\nabla\vec{R}\|_{L^{2,\infty}(D_\rho(x))} + \|\nabla S\|_{L^{2,\infty}(D_\rho(x))} + \rho^{2-\frac{2}{p}}\|\mathrm{e}^\lambda\vec{T}\|_{L^p(D_\rho(x))}\big] \\
&\quad\lesssim \rho^{\frac{2}{s}-1}M_p,
\end{aligned} \tag{2.16}$$

where we have used (2.13). Note that (2.16) holds in particular for $s = 2p$.

A useful identity is derived in [4]; it relays information on $\vec{R}$ and S back to the immersion $\vec{\Phi}$, namely,

$$\mathrm{e}^{2\lambda}\vec{H} = (\nabla\vec{R} + \nabla^\perp\vec{X}) \bullet \nabla^\perp\vec{\Phi} + (\nabla S + \nabla^\perp Y)\cdot\nabla^\perp\vec{\Phi}. \tag{2.17}$$

It follows from this identity and from (2.6) and (2.16) that

$$\|\mathrm{e}^{2\lambda}\vec{H}\|_{L^{2p}(D_{5\rho/8}(x))} \lesssim \rho^{\frac{1}{p}-1}\|\mathrm{e}^\lambda\|_{L^\infty(D_\rho(x))}M_p, \tag{2.18}$$

where as always the symbol $\lesssim$ indicates the presence of a multiplicative constant involving at most ε_0 and $\|\nabla\lambda\|_{L^{2,\infty}(D_1)}$. From (2.18) and the Harnack estimate (2.2), we deduce that

$$\|\vec{H}\wedge\nabla^\perp\vec{\Phi}\|_{L^{2p}(D_{5\rho/8}(x))} \lesssim \rho^{\frac{1}{p}-1}M_p. \tag{2.19}$$

It is shown in the appendix that the Gauss map satisfies the equation

$$\Delta(\star\vec{n}) = \nabla^\perp(\star\vec{n}) \bullet \nabla(\star\vec{n}) - 2\,\mathrm{div}(\vec{H}\wedge\nabla^\perp\vec{\Phi}). \tag{2.20}$$

Using [20, Theorem 10.5.1], there exists some $\vec{v} \in W^{1,2p}(D_{5\rho/8}(x))$ such that

$$\Delta\vec{v} = 2\,\mathrm{div}(\vec{H}\wedge\nabla^\perp\vec{\Phi}) \quad \text{on } D_{5\rho/8}(x) \tag{2.21}$$

and

$$\|\nabla\vec{v}\|_{L^{2p}(D_{5\rho/8}(x))} \lesssim \|\vec{H}\wedge\nabla^\perp\vec{\Phi}\|_{L^{2p}(D_{5\rho/8}(x))} \lesssim \rho^{\frac{1}{p}-1}M_p, \tag{2.22}$$

where we have used (2.19).

We next define $\vec{v}_0$ and $\vec{v}_1$ such that $\vec{v}_0 + \vec{v}_1 = \star\vec{n} + \vec{v}$, which in accordance with (2.20) and (2.21) satisfy

$$\begin{cases} \Delta\vec{v}_0 = \vec{0}, & \Delta\vec{v}_1 = \nabla^\perp(\star\vec{n}) \bullet \nabla(\star\vec{n}) \quad \text{in } D_{5\rho/8}(x), \\ \vec{v}_0 = \star\vec{n} + \vec{v}, & \vec{v}_1 = \vec{0} \qquad\qquad\qquad\qquad\quad \text{on } \partial D_{5\rho/8}(x). \end{cases}$$

[6] Proposition A.2 is proved for one equation, but it is easily adapted for systems. Details are left to the reader.

[7] That is, $s < 2/(2-p)$ if $p \in (1,2)$, and $s < \infty$ for $p \geq 2$.

To handle $\vec{v}_1$, we apply Wente's inequality in the form of [7, Lemma IV.2] to obtain

$$\begin{aligned}\|\nabla\vec{v}_1\|_{L^2(D_{5\rho/8}(x))} &\lesssim \|\nabla(\star\vec{n})\|_{L^2(D_{5\rho/8}(x))}\|\nabla(\star\vec{n})\|_{L^2(D_{5\rho/8}(x))}\\ &\leq \varepsilon_0\|\nabla\vec{n}\|_{L^2(D_{5\rho/8}(x))}.\end{aligned}\tag{2.23}$$

On the other hand, standard estimates about harmonic function growth give, for $k \in (0, 5/8)$,

$$\begin{aligned}\|\nabla\vec{v}_0\|_{L^2(D_{k\rho}(x))} &\lesssim k\|\nabla\vec{v}_0\|_{L^2(D_{5\rho/8}(x))}\\ &\lesssim k\big(\|\nabla\vec{v}_1\|_{L^2(D_{5\rho/8}(x))} + \|\nabla(\star\vec{n})\|_{L^2(D_{5\rho/8}(x))} + \|\nabla\vec{v}\|_{L^2(D_{5\rho/8}(x))}\big)\\ &\lesssim (k+\varepsilon_0)\|\nabla\vec{n}\|_{L^2(D_{5\rho/8}(x))} + kM_p,\end{aligned}\tag{2.24}$$

where (2.22) and (2.23) were used.

Altogether, (2.22)–(2.24) along with (2.14) easily yield

$$\|\nabla\vec{n}\|_{L^2(D_{k\rho}(x))} \lesssim (k+\varepsilon_0)\|\nabla\vec{n}\|_{L^2(D_\rho(x))} + \rho^{2-\frac{2}{p}}\|\mathrm{e}^\lambda\vec{T}\|_{L^p(D_\rho(x))}.$$

Choosing k and ε_0 small enough, and provided for the time being that $p \in (1, 2)$, then a standard controlled growth argument (see e.g. [16, Lemma III.2.1]) reveals that

$$\begin{aligned}\|\nabla\vec{n}\|_{L^2(D_\tau(x))} &\lesssim \big(\|\mathrm{e}^\lambda\vec{T}\|_{L^p(D_\rho(x))} + \rho^{\frac{2}{p}-2}\|\nabla\vec{n}\|_{L^2(D_\rho(x))}\big)\tau^{2-\frac{2}{p}}\\ &\lesssim \left(\frac{\tau}{\rho}\right)^{2-\frac{2}{p}} M_p \quad \forall\tau < 5\rho/8.\end{aligned}$$

Hence, we see that

$$\|\Delta\vec{v}_1\|_{L^1(D_\tau(x))} \lesssim \|\nabla\vec{n}\|^2_{L^2(D_\tau(x))} \lesssim \left(\frac{\tau}{\rho}\right)^{2-\frac{2}{p}} M_p.$$

Calling upon Proposition A.1 gives that

$$\|\nabla\vec{v}_1\|_{L^s(D_{a\tau}(x))} \lesssim \tau^{\frac{2}{s}-1}\left[\|\nabla\vec{v}_1\|_{L^2(D_\tau(x))} + \left(\frac{\tau}{\rho}\right)^{2-\frac{2}{p}} M_p\right],$$

for any $a \in (0, 1)$ and

$$2 < s < \frac{2}{2-p}.$$

Focusing on $s = 2p$ shows that

$$\|\nabla\vec{v}_1\|_{L^{2p}(D_{a\tau}(x))} \lesssim \tau^{\frac{1}{p}-1}M_p.$$

Using the basic growth property of harmonic functions for $\vec{v}_0$ and (2.22), we obtain

$$\begin{aligned}\|\nabla\vec{n}\|_{L^{2p}(D_{a\tau}(x))} &\lesssim \|\nabla\vec{v}\|_{L^{2p}(D_{a\tau}(x))} + \|\nabla\vec{v}_0\|_{L^{2p}(D_{a\tau}(x))} + \|\nabla\vec{v}_1\|_{L^{2p}(D_{a\tau}(x))}\\ &\lesssim \rho^{\frac{1}{p}-1}M_p + \tau^{\frac{1}{p}-1}\|\nabla\vec{v}_0\|_{L^2(D_{a\tau}(x))} + \tau^{\frac{1}{p}-1}M_p\\ &\lesssim \rho^{\frac{1}{p}-1}M\\ &\quad + \tau^{\frac{1}{p}-1}(\|\nabla\vec{n}\|_{L^2(D_{a\tau}(x))} + \|\nabla\vec{v}\|_{L^2(D_{a\tau}(x))} + \|\nabla\vec{v}_1\|_{L^2(D_{a\tau}(x))})\\ &\lesssim \rho^{\frac{1}{p}-1}M_p.\end{aligned}$$

In particular, we find

$$\|\nabla\vec{n}\|_{L^{2p}(D_{9\rho/16}(x))} \lesssim \rho^{\frac{1}{p}-1} M_p. \tag{2.25}$$

Owing to (2.17), we verify easily that

$$2\vec{H} \wedge \nabla^{\perp}\vec{\Phi} = (\nabla^{\perp}\vec{R} - \nabla\vec{X}) \bullet (\star\vec{n}) + (\nabla^{\perp}S - \nabla Y)(\star\vec{n}), \tag{2.26}$$

so that, using (2.5) and (2.15),

$$\begin{aligned}|\mathrm{div}(\vec{H} \wedge \nabla^{\perp}\vec{\Phi})| &\lesssim |\Delta\vec{X}| + |\Delta Y| + |\nabla\vec{n}|(|\nabla\vec{R}| + |\nabla S| + |\nabla\vec{X}| + |\nabla Y|) \\ &\lesssim |\mathrm{e}^{\lambda}\vec{T}| + |\nabla\vec{n}|(|\nabla\vec{R}| + |\nabla S| + |\nabla\vec{X}| + |\nabla Y|).\end{aligned}$$

Hence, from (2.25), (2.6), and (2.16),

$$\begin{aligned}&\|\mathrm{div}(\vec{H} \wedge \nabla^{\perp}\vec{\Phi})\|_{L^p(D_{9\rho/16}(x))} \\ &\quad\lesssim \|\mathrm{e}^{\lambda}\vec{T}\|_{L^p(D_{9\rho/16}(x))} \\ &\qquad + \|\nabla\vec{n}\|_{L^{2p}(D_{9\rho/16}(x))}(\|\nabla\vec{R}\|_{L^{2p}(D_{9\rho/16}(x))} + \|\nabla S\|_{L^{2p}(D_{9\rho/16}(x))} \\ &\qquad\qquad + \|\nabla\vec{X}\|_{L^{2p}(D_{9\rho/16}(x))} + \|\nabla Y\|_{L^{2p}(D_{9\rho/16}(x))}) \\ &\quad\lesssim \rho^{\frac{2}{p}-2} M_p.\end{aligned} \tag{2.27}$$

As shown in the appendix, the Gauss map $\vec{n}$ satisfies a perturbed harmonic map equation, namely,

$$|\Delta\vec{n}| \leq 2|\mathrm{div}(\vec{H} \wedge \nabla^{\perp}\vec{\Phi})| + \mathcal{O}(|\nabla\vec{n}|^2).$$

Accordingly, from (2.25) and (2.27), we find

$$\begin{aligned}\|\nabla^2\vec{n}\|_{L^p(D_{17\rho/32}(x))} &\lesssim \|\mathrm{div}(\vec{H} \wedge \nabla^{\perp}\vec{\Phi})\|_{L^p(D_{9\rho/16}(x))} \\ &\quad + \|\nabla\vec{n}\|^2_{L^{2p}(D_{9\rho/16}(x))} + \rho^{\frac{2}{p}-2}\|\nabla\vec{n}\|_{L^2(D_{9\rho/16}(x))} \\ &\lesssim \|\mathrm{e}^{\lambda}\vec{T}\|_{L^p(D_\rho(x))} + \rho^{\frac{2}{p}-2}M_p^2 + \rho^{\frac{2}{p}-2}\|\nabla\vec{n}\|_{L^2(D_\rho(x))} \\ &\lesssim \rho^{\frac{2}{p}-2} M_p.\end{aligned} \tag{2.28}$$

To complete the proof of Theorem 1.1 (i), we show that (2.28) remains true when $p \geq 2$. First, when $p = 2$, we have per the above procedure that for all $\delta > 0$,

$$\|\nabla^2\vec{n}\|_{L^{2-\delta}(D_{17\rho/32}(x))} \lesssim \rho^{\frac{2}{2-\delta}-2} M_{2-\delta} \lesssim \rho^{\frac{2}{2-\delta}-2} M_2.$$

Upon letting $\delta \searrow 0$, we find the desired

$$\|\nabla^2\vec{n}\|_{L^2(D_{17\rho/32}(x))} \lesssim \rho^{-1} M_2.$$

Now, if $p > 2$, we first note that by the previous case, it holds that

$$\|\nabla^2\vec{n}\|_{L^2(D_{17\rho/32}(x))} \lesssim \rho^{-1} M_2 \lesssim \rho^{-1} M_p,$$

hence by the Sobolev embedding theorem,

$$\|\nabla\vec{n}\|_{L^{2p}(D_{35\rho/64}(x))} \lesssim \rho^{\frac{1}{p}-1} M_p \quad \forall q < \infty.$$

This is (2.25) and the argument proceeds as in the case $p \in (1,2)$. This completes the proof of Theorem 1.1 (i).

2.3. Proof of Theorem 1.1 (ii)

In this section we will build upon the results previously derived in order to obtain regularity estimates for an inhomogeneous Willmore equation of the type

$$\operatorname{div}(\nabla\vec{H} - 2\pi_{\vec{n}}\nabla\vec{H} + |\vec{H}|^2\nabla\vec{\Phi}) = \vec{v},$$

where we suppose that

$$\mathrm{e}^{\lambda}\vec{v} \in L^r(D_1(0)) \quad \text{for some } r \geq 1.$$

Let $D_\rho(x) \subset D_1(0)$. In order to recover (2.3), we let $\vec{V}$ satisfy the problem

$$\begin{cases} -\Delta\vec{V} = \vec{v} & \text{in } D_\rho(x), \\ \vec{V} = \vec{0} & \text{on } \partial D_\rho(x). \end{cases} \tag{2.29}$$

Using the Harnack inequality (2.2), we easily deduce

$$\begin{cases} \|\mathrm{e}^{\lambda}\nabla\vec{V}\|_{L^{2,\infty}(D_\rho(x))} \lesssim \|\mathrm{e}^{\lambda}\vec{v}\|_{L^1(D_\rho(x))}, & r = 1 \\ \|\mathrm{e}^{\lambda}\nabla\vec{V}\|_{L^{2r/(2-r)}(D_\rho(x))} \lesssim \|\mathrm{e}^{\lambda}\vec{v}\|_{L^r(D_\rho(x))}, & r \in (1,2). \end{cases}$$

We are now back in the case studied in the previous section with $\vec{T} := \nabla\vec{V}$. In particular, when $r = 1$, we find

$$\|\mathrm{e}^{\lambda}\vec{T}\|_{L^p(D_\rho(x))} \lesssim \rho^{\frac{2}{p}-1}\|\mathrm{e}^{\lambda}\vec{T}\|_{L^{2,\infty}(D_\rho(x))} \lesssim \rho^{\frac{2}{p}-1}\|\mathrm{e}^{\lambda}\vec{v}\|_{L^1(D_\rho(x))} \quad \forall p \in (1,2),$$

from which (2.28) yields

$$\|\nabla^2\vec{n}\|_{L^p(D_{\rho/2}(x))} \lesssim \rho^{\frac{2}{p}-2} M \quad \forall p \in (1,2),$$

where

$$M = \rho\|\mathrm{e}^{\lambda}\vec{v}\|_{L^1(D_\rho(x))} + \|\nabla\vec{n}\|_{L^2(D_\rho(x))}.$$

Consider next the case $r \in (1,2)$. This time, estimate (2.28) gives

$$\|\nabla^2\vec{n}\|_{L^{2r/(2-r)}(D_{\rho/2}(x))} \lesssim \rho^{\frac{2}{r}-3} M_r, \tag{2.30}$$

with

$$M_r = \rho^{3-\frac{2}{r}}\|\mathrm{e}^{\lambda}\vec{v}\|_{L^r(D_\rho(x))} + \|\nabla\vec{n}\|_{L^2(D_\rho(x))}.$$

Next, from (2.29), we find

$$\|\nabla \vec{V}\|_{L^{r^*}(D_\rho(x))} + \|\nabla^2 \vec{V}\|_{L^r(D_\rho(x))} \lesssim \|\vec{v}\|_{L^r(D_\rho(x))},$$

where for notational convenience, we have set $r^* := 2r/(2-r) > 2$. Since $\vec{T} := \nabla \vec{V}$, the latter and (2.2) yield

$$\|e^\lambda \vec{T}\|_{L^{r^*}(D_\rho(x))} + \|e^\lambda \nabla \vec{T}\|_{L^r(D_\rho(x))} \lesssim \|e^\lambda \vec{v}\|_{L^r(D_\rho(x))}. \tag{2.31}$$

Recall that $\vec{X}$ and Y satisfy (2.5):

$$\Delta \vec{X} = \nabla \vec{\Phi} \wedge \vec{T} \quad \text{and} \quad \Delta Y = \nabla \vec{\Phi} \cdot \vec{T} \quad \text{on } D_\rho(x),$$

with boundary conditions $\vec{X}|_{\partial D_\rho(x)} = \vec{0}$ and $Y|_{\partial D_\rho(x)} = 0$. Standard Calderon–Zygmund estimates and again (2.2) give

$$\begin{aligned} &\rho\|\nabla^2 \vec{X}\|_{L^{r^*}(D_\rho(x))} + \rho\|\nabla^2 Y\|_{L^{r^*}(D_\rho(x))} \\ &\qquad + \rho^{\frac{2}{r^*}-2}(\|\nabla \vec{X}\|_{L^2(D_\rho(x))} + \|\nabla Y\|_{L^2(D_\rho(x))}) \lesssim \|e^\lambda \vec{v}\|_{L^r(D_\rho(x))} \end{aligned} \tag{2.32}$$

and, moreover, using (2.31),

$$\begin{aligned} &\|\nabla \Delta \vec{X}\|_{L^r(D_\rho(x))} + \|\nabla \Delta Y\|_{L^r(D_\rho(x))} \\ &\quad \lesssim \|\nabla^2 \vec{\Phi}\|_{L^2(D_\rho(x))}\|\vec{T}\|_{L^{r^*}(D_\rho(x))} + \|\nabla \vec{\Phi}\|_{L^\infty(D_\rho(x))}\|\nabla \vec{T}\|_{L^r(D_\rho(x))} \\ &\quad \lesssim \|e^\lambda \vec{v}\|_{L^r(D_\rho(x))}. \end{aligned} \tag{2.33}$$

Note that we have used the fact that $\nabla^2 \vec{\Phi} = e^\lambda \mathcal{O}(|\nabla \vec{n}|)$.

From (2.17), we easily verify that

$$\begin{aligned} |\nabla \operatorname{div}(\vec{H} \wedge \nabla^\perp \vec{\Phi})| \lesssim |\nabla \Delta \vec{X}| + |\nabla \Delta Y| + |\nabla \vec{n}|(|\nabla^2 \vec{R}| + |\nabla^2 S| + |\nabla^2 \vec{X}| + |\nabla^2 Y|) \\ + |\nabla^2 \vec{n}|(|\nabla \vec{R}| + |\nabla S| + |\nabla \vec{X}| + |\nabla Y|). \end{aligned}$$

Estimates (2.30), (2.32), and (2.33) then show that[8]

$$\|\nabla \operatorname{div}(\vec{H} \wedge \nabla^\perp \vec{\Phi})\|_{L^r(D_{\rho/2}(x))} \lesssim \rho^{\frac{2}{r}-3} M_r. \tag{2.34}$$

We next move on to finding a third-order estimate for $\vec{n}$. Recall that

$$\Delta(\star\vec{n}) = \nabla^\perp(\star\vec{n}) \bullet \nabla(\star\vec{n}) - 2\operatorname{div}(\vec{H} \wedge \nabla^\perp \vec{\Phi}),$$

from which, with the help of (2.30) and (2.34), we easily deduce the estimate

$$\begin{aligned} \|\nabla^3 \vec{n}\|_{L^r(D_{\rho/3}(x))} &\lesssim \|\nabla \vec{n}\|_{L^2(D_\rho(x))}\|\nabla^2 \vec{n}\|_{L^{r^*}(D_\rho(x))} + \|\nabla \operatorname{div}(\vec{H} \wedge \nabla^\perp \vec{\Phi})\|_{L^r(D_{\rho/2}(x))} \\ &\quad + \rho^{\frac{2}{r}-3}\|\nabla \vec{n}\|_{L^2(D_\rho(x))} \\ &\lesssim \rho^{\frac{2}{r}-3} M_r. \end{aligned}$$

This is the statement of Theorem 1.1 (ii) in the case $r \in (1,2)$. The case $r \geq 2$ is handled mutatis mutandis the end of the proof of Theorem 1.1 (i) and follows easily from the case $r \in (1,2)$.

[8] Recall that we are considering the case $r \in (1,2)$.

2.4. On smoothness of the solution: Proof of Corollary 1.1

Let us suppose that $\vec{\Phi} \in W^{2,2}_{\mathrm{imm}}(D_1(0), \mathbb{R}^m)$ satisfies the equation

$$\operatorname{div}(\nabla \vec{H} - 2\pi_{\vec{n}} \nabla \vec{H} + |\vec{H}|^2 \nabla \vec{\Phi} + \vec{T}) = \vec{v} \quad \text{on } D_1(0),$$

where $\vec{T}$ and $\vec{v}$ are smooth. As we are interested in obtaining a local result, we may always rescale so as to guarantee that the small energy assumption

$$\|\nabla \vec{n}\|_{L^2(D_1(0))} < \varepsilon_0$$

holds for some ε_0 sufficiently small. We proved in the last section that $\nabla \vec{n} \in \bigcap_{p<\infty} W^{2,p}$. Owing to the Liouville equation[9]

$$-\Delta \lambda = \mathrm{e}^{2\lambda} K = \mathcal{O}(|\nabla \vec{n}|^2),$$

it follows that $\mathrm{e}^{\pm\lambda}$ lie in $\bigcap_{p<\infty} W^{2,p}$. Hence $\nabla^2 \vec{\Phi} \equiv \mathcal{O}(\mathrm{e}^{\lambda} |\nabla \vec{n}|) \in \bigcap_{p<\infty} W^{2,p}$.

Thus, $\vec{H} \in \bigcap_{p<\infty} W^{2,p}$. From this and (2.8), it follows that $\vec{L} \in \bigcap_{p<\infty} W^{2,p}$, and hence by (2.12) that ∇S and $\nabla \vec{R}$ lie in $\bigcap_{p<\infty} W^{2,p}$.

The function $\vec{V}$ defined in (2.29) is smooth. By definition, so is $\vec{U} := \vec{T} + \nabla \vec{V}$. Using (2.5), we deduce that $\nabla \vec{X}$ and ∇Y belong to $\bigcap_{p<\infty} W^{4,p}$. We see in the paragraph following (2.17) that $\vec{R}$ and S also belong to $\bigcap_{p<\infty} W^{2,p}$. In turn, (2.15) yields the immediate improvement that ∇S and $\nabla \vec{R}$ lie in $\bigcap_{p<\infty} W^{3,p}$. Accordingly, per (2.26), we have

$$\vec{H} \wedge \nabla^{\perp} \vec{\Phi} \in \bigcap_{p<\infty} W^{3,p}.$$

Per (2.20), we now see that $\Delta \vec{n}$ belongs to the space $\bigcap_{p<\infty} W^{2,p}$, and therefore that $\nabla \vec{n} \in \bigcap_{p<\infty} W^{3,p}$. The regularity has thus been improved and the process may be repeated indefinitely until eventually reaching that $\vec{n}$ and thus the immersion $\vec{\Phi}$ itself are smooth.

2.5. Remarks about the critical case

As its name indicates, the critical case is far more delicate to handle, and, as far as the authors know, there is no general method to prove the regularity of solutions to the inhomogeneous Willmore equation

$$\operatorname{div}(\nabla \vec{H} - 2\pi_{\vec{n}} \nabla \vec{H} + |\vec{H}|^2 \nabla \vec{\Phi} + \vec{T}) = \vec{0}, \tag{2.35}$$

with a generic inhomogeneity $\mathrm{e}^{\lambda} \vec{T} \in L^1$, if it is only known that the second fundamental form is square integrable. There are of course special cases, such as the Willmore immersions (with $\vec{T} \equiv \vec{0}$) and more generally the conformally constrained Willmore immersions (which include Willmore and CMC immersions) whose $\mathrm{e}^{\lambda} \vec{T}$ has a very specific form;

[9] K denotes the Gauss curvature.

see [2]. The conformally constrained Willmore immersions have an inhomogeneous term $\vec{T}$ for which the solutions to (2.5) are identically vanishing. In turn, this guarantees that system (2.15) is of Wente type and can thus be made subcritical just as we have done for $e^{\lambda}\vec{T} \in L^{p>1}$.

But even if we assume from the onset that the solution to (2.35) is sufficiently regular,[10] the presence of an inhomogeneity $\vec{T}$, and thus of nonzero solutions of (2.5), will in general prevent us from reaching estimates of the type appearing in Theorem 1.1. This difficulty can only be resolved on a case-by-case basis. We will content ourselves in this short section with mentioning one specific type of inhomogeneity for which Theorem 1.1 can be obtained.

Let us write the inhomogeneity $\vec{T}$ in the form

$$\vec{T} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \partial_{x^1}\vec{\Phi} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \partial_{x^2}\vec{\Phi} + \begin{pmatrix} \vec{U}_1 \\ \vec{U}_2 \end{pmatrix},$$

where $\vec{U}_1$ and $\vec{U}_2$ are two normal vectors. One easily verifies that

$$\nabla\vec{\Phi} \wedge \vec{T} = e^{2\lambda}(A_2 - B_1)(\star\vec{n}) - \vec{U}_1 \wedge \partial_{x^1}\vec{\Phi} - \vec{U}_2 \wedge \partial_{x^2}\vec{\Phi}$$

and

$$\nabla\vec{\Phi} \cdot \vec{T} = e^{2\lambda}(A_1 + B_2).$$

Accordingly, if the functions $(A_1 + B_2)$, $(A_2 - B_1)$, and the normal projection $\pi_{\vec{n}}\vec{T}$ lie in the space $L^{1+\delta}$ for some $\delta > 0$, we can apply to (2.5) the same technique as used in the proof of Theorem 1.1. This holds of course even if the functions A_1, A_2, B_1, and B_2 are only merely integrable.

In general, it is not possible to obtain a subcritical-type energy estimate. However, as we have seen above, there are exceptions when $\vec{T}$ has a specific form. Another important exceptional case occurs when $\vec{T}$ depends on the geometry of the problem, and if the solution is already known to be regular enough, say $\vec{\Phi} \in W^{2,2+\delta}(D_1(0))$, for some positive $\delta \in (0,1)$. We only focus on the specific situation when

$$e^{\lambda}\vec{T} = \mathcal{O}(|\nabla\vec{n}|^2)$$

for an inhomogeneous Willmore problem of the type

$$\operatorname{div}(\nabla\vec{H} - 2\pi_{\vec{n}}\nabla\vec{H} + |\vec{H}|^2\nabla\vec{\Phi} + \vec{T}) = \vec{0},$$

and assuming as usual that

$$\|\nabla\vec{n}\|_{L^2(D_1(0))} < \varepsilon_0$$

for some ε_0 chosen sufficiently small.

[10] Even if ever so slightly, say $\vec{H} \in L^{2+\delta}$ for some $\delta > 0$.

As $e^{\lambda}\vec{T} \in L^{1+\frac{\delta}{2}}$, it follows from Theorem 1.1 that $\nabla\vec{n}$ lies in $W^{1,1+\frac{\delta}{2}} \subset L^{2\frac{2+\delta}{2-\delta}}$, and thus $e^{\lambda}\vec{T}$ lies in $W^{1,2\frac{2+\delta}{6-\delta}}$, which is a proper subset of $L^{1+\frac{\delta}{2}}$. Calling again on Theorem 1.1, the integrability of $\nabla\vec{n}$ is improved accordingly. This procedure may be repeated until reaching that $\nabla^2\vec{n}$ belongs to all L^p spaces, with p finite, i.e. that $\vec{n}$ belongs to $C^{1,\alpha}$ for all $\alpha < 1$. Standard arguments then imply that $\vec{n}$, and thus the immersion $\vec{\Phi}$, are smooth.

2.6. Gap phenomenon: Proof of Theorem 1.2

Let us suppose that Σ is a complete, connected, noncompact, oriented, immersed surface into $\mathbb{R}^{m\geq 3}$ satisfying an inhomogeneous Willmore equation (1.4) of the form

$$\Delta_{\perp}\vec{H} + \langle\vec{h}\cdot\vec{H},\vec{h}\rangle_g - 2|\vec{H}|^2\vec{H} = \vec{\mathcal{W}}, \tag{2.36}$$

with the same notation as before, and where $\vec{\mathcal{W}}$ is a normal field with the property that

$$\vec{\mathcal{W}} = \mathcal{O}(|\vec{h}|^3), \quad \text{i.e. } |\vec{\mathcal{W}}| \leq c|\vec{h}|^3, \tag{2.37}$$

for some constant c. We suppose further that

$$\int_{\Sigma} |\vec{h}|_g^2 \, d\mathrm{vol}_g < \varepsilon_0^2, \tag{2.38}$$

for some ε_0^2 chosen to be small enough (at least smaller than $8\pi/3$). A well-known result of Müller and Šverák [33] guarantees that Σ is embedded and conformally equivalent to $\mathbb{R}^2$. Accordingly, we parametrize Σ by a conformal immersion $\vec{\Phi}\colon \mathbb{R}^2 \hookrightarrow \mathbb{R}^m$ with conformal parameter λ, and such that $\vec{\Phi} \in W^{2,2}(\mathbb{R}^2)$.

Just as was done in Section 1, in the flat coordinates of $\mathbb{R}^2$, the inhomogeneous Willmore equation (2.36) can be recast in the form

$$\operatorname{div}(\nabla\vec{H} - 2\pi_{\vec{n}}\nabla\vec{H} + |\vec{H}|^2\nabla\vec{\Phi}) = \vec{v} \quad \text{on } \mathbb{R}^2,$$

where

$$\vec{v} := e^{2\lambda}\vec{\mathcal{W}}.$$

Per (2.37), note that

$$|e^{\lambda}\vec{v}| \simeq |e^{\lambda}\vec{h}|^3 \simeq |\nabla\vec{n}|^3, \tag{2.39}$$

where, as always, $\vec{n}$ is the Gauss map associated with $\vec{\Phi}$. The smallness hypothesis (2.38) translates into

$$\|\nabla\vec{n}\|_{L^2(\mathbb{R}^2)} < \varepsilon_0, \tag{2.40}$$

for some $\varepsilon_0 > 0$ sufficiently small.

Owing to the Liouville equation

$$-\Delta\lambda = e^{2\lambda}K = \mathcal{O}(|\nabla\vec{n}|^2) \in L^1(\mathbb{R}^2),$$

it follows that $\nabla\lambda$ lies in the space $L^{2,\infty}(\mathbb{R}^2)$ with norm controlled by $\|\nabla\vec{n}\|_{L^2(\mathbb{R}^2)}$. We can in particular repeat the analysis leading to Lemma 2.1 to deduce that

$$\|e^{\lambda}\|_{L^\infty(\mathbb{R}^2)}\|e^{-\lambda}\|_{L^\infty(\mathbb{R}^2)} \le C(\varepsilon_0). \tag{2.41}$$

This Harnack-type inequality will be used in our argument.

As we did in the proof of Theorem 1.1 (ii), we let

$$-\Delta\vec{V} = \vec{v} \quad \text{on } \mathbb{R}^2,$$

and $\vec{T} := \nabla\vec{V}$. As the equation

$$\operatorname{div}(\nabla\vec{H} - 2\pi_{\vec{n}}\nabla\vec{H} + |\vec{H}|^2\nabla\vec{\Phi} + \vec{T}) = \vec{0}$$

holds on $\mathbb{R}^2$, we can repeat the analysis done in the proof of Theorem 1.1 (i) and deduce the existence of $\vec{R}$ and S satisfying

$$\begin{cases} \Delta\vec{R} = \nabla(\star\vec{n}) \bullet \nabla^\perp\vec{R} + \nabla(\star\vec{n}) \cdot \nabla^\perp S + \operatorname{div}((\star\vec{n}) \bullet \nabla\vec{X} + (\star\vec{n})\nabla Y), \\ \Delta S = \nabla(\star\vec{n}) \cdot \nabla^\perp\vec{R} + \operatorname{div}((\star\vec{n}) \cdot \nabla\vec{X}), \end{cases} \tag{2.42}$$

where, as before, $\vec{X}$ and Y satisfy

$$\Delta\vec{X} = \nabla\vec{\Phi} \wedge \vec{T} \quad \text{and} \quad \Delta Y = \nabla\vec{\Phi} \cdot \vec{T} \quad \text{on } \mathbb{R}^2. \tag{2.43}$$

We have

$$\|\Delta\vec{X}\|_{L^q(\mathbb{R}^2)} + \|\Delta Y\|_{L^q(\mathbb{R}^2)} + \|\nabla\vec{X}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla Y\|_{L^{q^*}(\mathbb{R}^2)} \lesssim \|e^{\lambda}\vec{T}\|_{L^q(\mathbb{R}^2)}, \tag{2.44}$$

for $q \in (1,2)$ and $q^* := 2q/(2-q)$.

Applying Wente's inequality to (2.42) as in [7, Lemma IV.2], we find

$$\begin{aligned} \|\nabla\vec{R}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla S\|_{L^{q^*}(\mathbb{R}^2)} &\le \|\nabla\vec{n}\|_{L^2(\mathbb{R}^2)}(\|\nabla\vec{R}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla S\|_{L^{q^*}(\mathbb{R}^2)}) \\ &\quad + \|\nabla\vec{X}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla Y\|_{L^{q^*}(\mathbb{R}^2)}, \end{aligned}$$

which, owing to (2.40) and (2.44), yields

$$\begin{aligned} \|\nabla\vec{R}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla S\|_{L^{q^*}(\mathbb{R}^2)} &\le C(\varepsilon_0)(\|\nabla\vec{X}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla Y\|_{L^{q^*}(\mathbb{R}^2)}) \\ &\le C(\varepsilon_0)\|e^{\lambda}\vec{T}\|_{L^q(\mathbb{R}^2)}. \end{aligned} \tag{2.45}$$

We saw in the previous section that

$$2\vec{H} \wedge \nabla^\perp\vec{\Phi} = (\nabla^\perp\vec{R} - \nabla\vec{X}) \bullet (\star\vec{n}) + (\nabla^\perp S - \nabla Y)(\star\vec{n}),$$

hence

$$|\operatorname{div}(2\vec{H} \wedge \nabla^\perp\vec{\Phi})| \le |\Delta\vec{X}| + |\Delta Y| + |\nabla\vec{n}|(|\nabla\vec{R}| + |\nabla S| + |\nabla\vec{X}| + |\nabla Y|).$$

This gives, using (2.44) and (2.45),

$$\begin{aligned}
&\|\mathrm{div}(2\vec{H}\wedge\nabla^{\perp}\vec{\Phi})\|_{L^q(\mathbb{R}^2)}\\
&\quad\leq \|\Delta\vec{X}\|_{L^q(\mathbb{R}^2)}+\|\Delta Y\|_{L^q(\mathbb{R}^2)}\\
&\qquad+\|\nabla\vec{n}\|_{L^2(\mathbb{R}^2)}(\|\nabla\vec{R}\|_{L^{q^*}(\mathbb{R}^2)}+\|\nabla S\|_{L^{q^*}(\mathbb{R}^2)}+\|\nabla\vec{X}\|_{L^{q^*}(\mathbb{R}^2)}+\|\nabla Y\|_{L^{q^*}(\mathbb{R}^2)})\\
&\quad\leq 2\|\mathrm{e}^{\lambda}\vec{T}\|_{L^q(\mathbb{R}^2)}+C(\varepsilon_0)(\|\nabla\vec{X}\|_{L^{q^*}(\mathbb{R}^2)}+\|\nabla Y\|_{L^{q^*}(\mathbb{R}^2)})\\
&\quad\leq C(\varepsilon_0)\|\mathrm{e}^{\lambda}\vec{T}\|_{L^q(\mathbb{R}^2)}.
\end{aligned}\tag{2.46}$$

Recall that

$$\Delta\vec{n}=\mathrm{div}(2\vec{H}\wedge\nabla^{\perp}\vec{\Phi})+\mathcal{O}(|\nabla\vec{n}|^2).$$

According to (2.46), to (2.40), and to the Sobolev embedding theorem, we thus have

$$\begin{aligned}
\|\nabla^2\vec{n}\|_{L^q(\mathbb{R}^2)}&\leq\|\mathrm{div}(2\vec{H}\wedge\nabla^{\perp}\vec{\Phi})\|_{L^q(\mathbb{R}^2)}+\|\nabla\vec{n}\|_{L^2(\mathbb{R}^2)}\|\nabla\vec{n}\|_{L^{q^*}(\mathbb{R}^2)}\\
&\leq C(\varepsilon_0)\|\mathrm{e}^{\lambda}\vec{T}\|_{L^q(\mathbb{R}^2)}+\varepsilon_0\|\nabla^2\vec{n}\|_{L^q(\mathbb{R}^2)},
\end{aligned}$$

thereby yielding

$$\|\nabla^2\vec{n}\|_{L^q(\mathbb{R}^2)}\leq C(\varepsilon_0)\|\mathrm{e}^{\lambda}\vec{T}\|_{L^q(\mathbb{R}^2)}.\tag{2.47}$$

We now call upon the Gagliardo–Nirenberg interpolation inequality, (2.40), and (2.47) to find

$$\|\nabla\vec{n}\|_{L^p(\mathbb{R}^2)}\leq\|\nabla^2\vec{n}\|^{\alpha}_{L^q(\mathbb{R}^2)}\|\nabla\vec{n}\|^{1-\alpha}_{L^2(\mathbb{R}^2)}\leq C(\varepsilon_0)\varepsilon_0^{1-\alpha}\|\mathrm{e}^{\lambda}\vec{T}\|^{\alpha}_{L^q(\mathbb{R}^2)},$$

for

$$\frac{1}{p}=\frac{1}{2}+\Big(\frac{1}{q}-1\Big)\alpha\quad\text{and}\quad 0\leq\alpha\leq 1.$$

Equivalently,

$$\||\nabla\vec{n}|^3\|_{L^b(\mathbb{R}^2)}\leq C(\varepsilon_0)\varepsilon_0^{3(1-\alpha)}\|\mathrm{e}^{\lambda}\vec{T}\|^{3\alpha}_{L^q(\mathbb{R}^2)}$$

for

$$\frac{1}{b}=\frac{3}{2}+3\Big(\frac{1}{q}-1\Big)\alpha.$$

As $\mathrm{e}^{\lambda}\Delta\vec{V}=-\mathrm{e}^{\lambda}\vec{v}=\mathcal{O}(|\nabla\vec{n}|^3)$ and $\vec{T}=\nabla\vec{V}$, the latter yields

$$\|\mathrm{e}^{\lambda}\Delta\vec{V}\|_{L^b(\mathbb{R}^2)}\leq C(\varepsilon_0)\varepsilon_0^{3(1-\alpha)}\|\mathrm{e}^{\lambda}\nabla\vec{V}\|^{3\alpha}_{L^q(\mathbb{R}^2)},$$

hence, using (2.41),

$$\|\Delta\vec{V}\|_{L^b(\mathbb{R}^2)}\leq C(\varepsilon_0)\varepsilon_0^{3(1-\alpha)}\|\mathrm{e}^{\lambda}\|^{3\alpha-1}_{L^{\infty}(\mathbb{R}^2)}\|\nabla\vec{V}\|^{3\alpha}_{L^q(\mathbb{R}^2)}.\tag{2.48}$$

Let $\delta\in(0,2/3)$. We specialize to

$$q=2-\delta\quad\text{and}\quad 3\alpha=\frac{1}{1-\delta}.$$

This gives

$$\frac{1}{b} = \frac{3}{2} - \frac{1}{2-\delta}, \qquad \text{so that } b \in (1,2).$$

Using the Sobolev embedding theorem in (2.48) then gives

$$\|\nabla \vec{V}\|^{1-\delta}_{L^{\frac{2-\delta}{1-\delta}}(\mathbb{R}^2)} \le C(\varepsilon_0)\varepsilon_0^{2-3\delta}\|\mathrm{e}^{\lambda}\|^{\delta}_{L^\infty(\mathbb{R}^2)}\|\nabla \vec{V}\|_{L^{2-\delta}(\mathbb{R}^2)}. \tag{2.49}$$

Since

$$\frac{1-\delta}{2-\delta} + \frac{1}{2-\delta} = 1,$$

we interpolate (2.49) to find

$$\|\nabla \vec{V}\|^{2(1-\delta)}_{L^2(\mathbb{R}^2)} \le C(\varepsilon_0)\varepsilon_0^{2-3\delta}\|\mathrm{e}^{\lambda}\|^{\delta}_{L^\infty(\mathbb{R}^2)}\|\nabla \vec{V}\|^{2-\delta}_{L^{2-\delta}(\mathbb{R}^2)}.$$

Letting $\delta \searrow 0$ reveals that

$$\|\nabla \vec{V}\|_{L^2(\mathbb{R}^2)} \le C(\varepsilon_0)\varepsilon_0\|\nabla \vec{V}\|_{L^2(\mathbb{R}^2)}.$$

Since ε_0 can be adjusted at will, the latter implies that $\nabla \vec{V} \equiv \vec{0}$, hence that $\vec{T} \equiv \vec{0}$, and therefore that $\nabla \vec{n} \equiv \vec{0}$ by (2.47). This guarantees that Σ is a flat plane, as announced.

A. Appendix

A.1. Notational conventions

We append an arrow to all the elements belonging to $\mathbb{R}^m$. To simplify the notation, by $\vec{\Phi} \in X(D_1(0))$ is meant $\vec{\Phi} \in X(D_1(0), \mathbb{R}^m)$ whenever X is a function space. Similarly, we write $\nabla\vec{\Phi} \in X(D_1(0))$ for $\nabla\vec{\Phi} \in \mathbb{R}^2 \otimes X(D_1(0), \mathbb{R}^m)$.

We let differential operators act on elements of $\mathbb{R}^m$ componentwise. Thus, for example, $\nabla\vec{\Phi}$ is the element of $\mathbb{R}^2 \otimes \mathbb{R}^m$ with $\mathbb{R}^m$-valued components $(\partial_{x^1}\vec{\Phi}, \partial_{x^2}\vec{\Phi})$. If S is a scalar and $\vec{R}$ an element of $\mathbb{R}^m$, then we let

$$\begin{aligned}
\vec{R} \cdot \nabla\vec{\Phi} &:= (\vec{R} \cdot \partial_{x^1}\vec{\Phi}, \vec{R} \cdot \partial_{x^2}\vec{\Phi}),\\
\nabla^{\perp} S \cdot \nabla\vec{\Phi} &:= \partial_{x^1} S \partial_{x^2}\vec{\Phi} - \partial_{x^2} S \partial_{x^1}\vec{\Phi},\\
\nabla^{\perp}\vec{R} \cdot \nabla\vec{\Phi} &:= \partial_{x^1}\vec{R} \cdot \partial_{x^2}\vec{\Phi} - \partial_{x^2}\vec{R} \cdot \partial_{x^1}\vec{\Phi},\\
\nabla^{\perp}\vec{R} \wedge \nabla\vec{\Phi} &:= \partial_{x^1}\vec{R} \wedge \partial_{x^2}\vec{\Phi} - \partial_{x^2}\vec{R} \wedge \partial_{x^1}\vec{\Phi}.
\end{aligned}$$

Analogous quantities are defined according to the same logic.

Two operations between multivectors are useful. The *interior multiplication* $\llcorner$ maps a pair comprising a q-vector γ and a p-vector β to a $(q-p)$-vector. It is defined via

$$\langle \gamma \llcorner \beta, \alpha \rangle = \langle \gamma, \beta \wedge \alpha \rangle \quad \text{for each } (q-p)\text{-vector } \alpha.$$

Let α be a k-vector. The *first-order contraction* operation $\bullet$ is defined inductively through

$$\alpha \bullet \beta = \alpha \llcorner \beta \quad \text{when } \beta \text{ is a 1-vector,}$$

and

$$\alpha \bullet (\beta \wedge \gamma) = (\alpha \bullet \beta) \wedge \gamma + (-1)^{pq} (\alpha \bullet \gamma) \wedge \beta$$

when β and γ are respectively a p-vector and a q-vector.

A.2. On the Gauss map

It is shown in [6, Appendix B] that the Gauss map $\star\vec{n} := \mathrm{e}^{-2\lambda}\partial_{x^1}\vec{\Phi} \wedge \partial_{x^2}\vec{\Phi}$ satisfies (for a conformal parametrization $\vec{\Phi}$ with conformal coefficient λ) the identity

$$\Delta\vec{n} = 2\mathrm{e}^{2\lambda}K\vec{n} + 2 \star \mathrm{e}^{2\lambda}\vec{h}_{12} \wedge (\vec{h}_{11} - \vec{h}_{22}) + 2 \star (\nabla^{\perp}\vec{\Phi} \wedge \nabla\vec{H}), \tag{A.1}$$

where

$$\vec{h}_{ij} := \pi_{\vec{n}} \nabla_{ij} \vec{\Phi}$$

are the components of the second fundamental form, and K is the Gauss curvature.

One easily verifies that

$$(\star\vec{n}) \bullet \nabla_j(\star\vec{n}) = -\vec{h}_{ij} \wedge \nabla^i\vec{\Phi}.$$

Let ϵ^{jk} be the Kronecker symbol. Differentiating the above gives

$$\begin{aligned} \nabla^{\perp}((\star\vec{n}) \bullet \nabla(\star\vec{n})) &= \epsilon^{jk}\nabla_k[(\star\vec{n}) \bullet \nabla_j(\star\vec{n})] \\ &= -\epsilon^{jk}\vec{h}_{ij} \wedge \vec{h}^i_k - \epsilon^{jk}\nabla_k\vec{h}_{ij} \wedge \nabla^i\vec{\Phi}. \end{aligned} \tag{A.2}$$

Owing to the Codazzi–Mainardi identity in the form

$$\nabla_k\vec{h}_{ij} = \nabla_i\vec{h}_{jk} + (\vec{h}_{kj} \cdot \vec{h}_{il} - \vec{h}_{ij} \cdot \vec{h}_{kl})\nabla^l\vec{\Phi},$$

we recast (A.2) in the form[11]

$$\begin{aligned} \epsilon^{jk}\nabla_k[(\star\vec{n}) \bullet \nabla_j(\star\vec{n})] &= -\epsilon^{jk}\vec{h}_{ij} \wedge \vec{h}^i_k + \epsilon^{jk}(\vec{h}_{ij} \cdot \vec{h}_{kl})\nabla^l\vec{\Phi} \wedge \nabla^i\vec{\Phi} \\ &= 2\mathrm{e}^{2\lambda}\vec{h}_{12} \wedge (\vec{h}_{11} - \vec{h}_{22}) + 2K\mathrm{e}^{2\lambda}(\star\vec{n}). \end{aligned}$$

Injecting this into (A.1) and slightly rearranging yields

$$\Delta(\star\vec{n}) = \nabla^{\perp}(\star\vec{n}) \bullet \nabla(\star\vec{n}) - 2\,\mathrm{div}(\vec{H} \wedge \nabla^{\perp}\vec{\Phi}). \tag{A.3}$$

[11] Recall that the Kronecker symbol is an antisymmetric tensor, while the second fundamental form is symmetric.

A.3. Some useful elliptic results

The following result is established in [5, Appendix].

Lemma A.1. *Let D be a disk and suppose that $G = G_1 + G_2$ satisfies*

$$\operatorname{div} G = 0 \quad \textit{on } D,$$

where

$$G_1 \in W^{-1,2}(D, \mathbb{R}^2), \quad G_2 \in L^1(D, \mathbb{R}^2).$$

Then there exists an element L in the space $L^{2,\infty}(D, \mathbb{R})$ such that

$$G = \nabla^{\perp} L$$

and

$$\|L - L_D\|_{L^{2,\infty}(D)} \leq C(\|G_1\|_{W^{-1,2}(D)} + \|G_2\|_{L^1(D)}),$$

where L_D denotes the average of L on the disk D, and C is a universal constant.

The following propositions are decisive for our estimates.

Proposition A.1. *Let $u \in W^{1,2}(D_\rho(x))$ satisfy $\Delta u = F$ on $D_\rho(x)$. Suppose that*

$$\|F\|_{L^1(D_\tau(x))} \leq C_F \tau^q$$

for some constants $C_F > 0$, $q > 0$, and for all $\tau < a\rho$, for some $a \in (0, 1)$. Then we have

$$\|\nabla u\|_{L^s(D_{b\rho}(x))} \lesssim \rho^{\frac{2}{s}-1}(\|\nabla u\|_{L^2(D_\rho(x))} + C_F \rho^q)$$

for any $b \in (0, a/2)$ and any

$$2 < s < \frac{2-q}{1-q}.$$

Proof. Consider the maximal function

$$Mg(y) := \sup_{\tau>0} \tau^{-q} \int_{D_\tau(y)} |g(z)|\, dz. \tag{A.4}$$

Calling upon the assumption on F, we derive that for $y \in D_{a\rho/2}(x)$, there holds

$$M(\chi_{D_{a\rho/2}(y)} \Delta u(z))(y) \leq \sup_{0<\tau<\frac{a\rho}{2}} \tau^{-q} \|F\|_{L^1(D_\tau(y))} \leq C_F. \tag{A.5}$$

On the other hand, we have

$$\|\Delta u\|_{L^1(D_{a\rho/2}(x))} = \|F\|_{L^1(D_{a\rho/2}(x))} \lesssim C_F \rho^q. \tag{A.6}$$

Let $|z|^{-1}$ denote the standard Riesz transform. It is stated in [1, Proposition 3.2] that

$$\||z|^{-1} * \chi_{D_{a\rho/2}(y)} \Delta u\|_{L^{\alpha,\infty}(D_{a\rho/2}(x))} \lesssim \|M(\chi_{D_{a\rho/2}(y)} \Delta u)\|^{1-\frac{1}{\alpha}}_{L^\infty(D_{a\rho/2}(x))} \|\Delta u\|^{\frac{1}{\alpha}}_{L^1(D_{a\rho/2}(x))},$$

where $\alpha := (2-q)/(1-q) > 2$. Hence, according to (A.5) and (A.6), we have

$$\| |z|^{-1} * \chi_{D_{a\rho/2}(y)} \Delta u \|_{L^{\alpha,\infty}(D_{a\rho/2}(x))} \lesssim C_F \rho^{q/\alpha}. \tag{A.7}$$

We let $y \in D_{a\rho/2}(x)$ and we decompose $u = u_0 + u_1$ with

$$\begin{cases} \Delta u_0 = 0, \quad \Delta u_1 = \chi_{D_{a\rho/2}(y)} \Delta u & \text{in } D_{a\rho/2}(x), \\ u_0 = u, \quad u_1 = 0 & \text{on } \partial D_{a\rho/2}(x). \end{cases}$$

Let $s \in (2, \alpha)$. Using standard estimates for the harmonic function u_0 and the estimate (A.7) gives for any $b \in (0, a/2)$ that

$$\begin{aligned} \|\nabla u\|_{L^s(D_{b\rho}(x))} &\leq \|\nabla u_0\|_{L^s(D_{b\rho}(x))} + \|\nabla u_1\|_{L^s(D_{b\rho}(x))} \\ &\lesssim \rho^{\frac{2}{s}-\frac{4}{3}} \|\nabla u_0\|_{L^{3/2}(D_{a\rho/2}(x))} + \|\nabla u_1\|_{L^s(D_{b\rho}(x))} \\ &\lesssim \rho^{\frac{2}{s}-\frac{4}{3}} \|\nabla u\|_{L^{3/2}(D_{a\rho/2}(x))} + \| |z|^{-1} * \chi_{D_{a\rho/2}(y)} \Delta u \|_{L^s(D_{b\rho}(x))} \\ &\lesssim \rho^{\frac{2}{s}-1} \|\nabla u\|_{L^2(D_{a\rho/2}(x))} + \rho^{\frac{2}{s}-\frac{2}{\alpha}} \| |z|^{-1} * \chi_{D_{a\rho/2}(y)} \Delta u \|_{L^{\alpha,\infty}(D_{a\rho/2}(x))} \\ &\lesssim \rho^{\frac{2}{s}-1} (\|\nabla u\|_{L^2(D_\rho(x))} + C_F \rho^q). \end{aligned} \tag{A.8}$$

■

Proposition A.2. *Let $D_\rho(x) \subset D_1(0)$, and let $u \in W^{1,(2,\infty)}(D_\rho(x))$ satisfy the equation*

$$\Delta u = \nabla b \cdot \nabla^\perp u + \operatorname{div}(b \nabla f) \quad \text{on } D_\rho(x), \tag{A.9}$$

where $f \in W_0^{2,p}(D_\rho(x))$ for some $p > 1$. Suppose moreover that

$$b \in W^{1,2} \cap L^\infty(D_\rho(x)) \quad \text{with } \|\nabla b\|_{L^2(D_\rho(x))} < \varepsilon_0 \text{ and } \|b\|_{L^\infty(D_\rho(x))} \leq 1, \tag{A.10}$$

for some ε_0 chosen to be "small enough". Then

$$\|\nabla u\|_{L^s(D_{5\rho/8}(x))} \leq C(\varepsilon_0) \big[\rho^{\frac{2}{s}-1} \|\nabla u\|_{L^{2,\infty}(D_\rho(x))} + \rho^{\frac{2}{s}-\frac{2}{p}+1} \|\nabla^2 f\|_{L^p(D_\rho(x))} \big],$$

for some constant $C(\varepsilon_0)$ depending only on ε_0, and where $s < 2/(2-p)$ if $p \in (1,2)$, or $s < \infty$ if $p \geq 2$.

Proof. Suppose first that $p \in (1,2)$. Then for every $D_\sigma(z) \subset D_\rho(x)$, it holds that

$$\|\nabla f\|_{L^2(D_\sigma(z))} \lesssim \sigma^{2-\frac{2}{p}} \|\nabla f\|_{L^{2p/(2-p)}(D_\rho(x))} \lesssim \sigma^{2-\frac{2}{p}} \|\nabla^2 f\|_{L^p(D_\rho(x))}. \tag{A.11}$$

Let us fix once and for all some point $x_0 \in D_{3\rho/4}(x)$ and some radius $0 < r \leq \rho/4$, so that the disk $D_r(x_0)$ of radius r and centered on the point x_0 is contained in $D_\rho(x)$. With the help of the theorem of Fubini, we may always find some $r_0 \in (r/2, r)$ such that

$$\begin{aligned} \int_{\partial D_{r_0}(x_0)} |\nabla u|^{\frac{3}{2}} \lesssim \frac{1}{r} \int_{D_r(x_0)} |\nabla u|^{\frac{3}{2}} &\lesssim r^{-\frac{1}{2}} \|\nabla u\|^{\frac{3}{2}}_{L^{2,\infty}(D_r(x_0))} \\ &\lesssim r_0^{-\frac{1}{2}} \|\nabla u\|^{\frac{3}{2}}_{L^{2,\infty}(D_\rho(x))}. \end{aligned} \tag{A.12}$$

We next define $u = u_0 + u_1$, where the new variables, in accordance with (A.9), satisfy

$$\begin{cases} \Delta u_0 = \operatorname{div}(b\nabla f), \ \ \Delta u_1 = \nabla b \cdot \nabla^{\perp} u & \text{in } D_{r_0}(x_0), \\ \ \ u_0 = u, \qquad\qquad\quad u_1 = 0 & \text{on } \partial D_{r_0}(x_0). \end{cases}$$

Let

$$\bar{u} := \frac{1}{2\pi r_0} \int_{\partial D_{r_0}(x_0)} u.$$

Standard elliptic theory, our assumptions on b and f, and the Sobolev embedding theorem give

$$\begin{aligned} \|\nabla u_0\|_{L^2(D_{r_0}(x_0))} &\lesssim \|b\nabla f\|_{L^2(D_{r_0}(x_0))} + \|u - \bar{u}\|_{H^{1/2}(\partial D_{r_0}(x_0))} \\ &\lesssim \|\nabla f\|_{L^2(D_{r_0}(x_0))} + r_0^{\frac{1}{3}} \|\nabla u\|_{L^{3/2}(\partial D_{r_0}(x_0))} \\ &\lesssim r_0^{2-\frac{2}{p}} \|\nabla^2 f\|_{L^p(D_\rho(x))} + \|\nabla u\|_{L^{2,\infty}(D_\rho(x))}, \end{aligned} \tag{A.13}$$

where (A.11) and (A.12) were used.

To handle u_1, we apply Wente's inequality in the form of [7, Lemma IV.2] to obtain

$$\|\nabla u_1\|_{L^2(D_{r_0}(x_0))} \lesssim \|\nabla b\|_{L^2(D_{r_0}(x_0))} \|\nabla u\|_{L^{2,\infty}(D_{r_0}(x_0))} \le \varepsilon_0 \|\nabla u\|_{L^{2,\infty}(D_\rho(x))}. \tag{A.14}$$

Altogether, (A.13) and (A.14) yield that ∇u belongs to $L^2(D_{r_0}(x_0))$. In particular,

$$\|\nabla u\|_{L^2(D_{r_0}(x_0))} \lesssim r_0^{2-\frac{2}{p}} \|\nabla^2 f\|_{L^p(D_\rho(x))} + \|\nabla u\|_{L^{2,\infty}(D_\rho(x))}. \tag{A.15}$$

Now let $k \in (0, 1)$. Using (A.11) again, standard elliptic theory and growth estimates give

$$\begin{aligned} \|\nabla u_0\|_{L^2(D_{kr_0}(x_0))} &\lesssim \|b\nabla f\|_{L^2(D_{r_0}(x_0))} + k\|\nabla u_0\|_{L^2(D_{r_0}(x_0))} \\ &\lesssim \|\nabla f\|_{L^2(D_{r_0}(x_0))} + k\|\nabla u_0\|_{L^2(D_{r_0}(x_0))} \\ &\lesssim r_0^{2-\frac{2}{p}} \|\nabla^2 f\|_{L^p(D_\rho(x))} + k\|\nabla u_0\|_{L^2(D_{r_0}(x_0))}. \end{aligned} \tag{A.16}$$

For u_1, we apply Wente's inequality this time as in [18, Theorem 3.4.1] so as to find

$$\begin{aligned} \|\nabla u_1\|_{L^2(D_{r_0}(x_0))} &\lesssim \|\nabla b\|_{L^2(D_{r_0}(x_0))} \|\nabla u\|_{L^2(D_{r_0}(x_0))} \\ &\lesssim \varepsilon_0 \|\nabla u\|_{L^2(D_{r_0}(x_0))}, \end{aligned} \tag{A.17}$$

again up to some multiplicative constant without bearing on the sequel. Hence, combining (A.16) and (A.17) we obtain the estimate

$$\|\nabla u\|_{L^2(D_{kr_0}(x_0))} \lesssim (k + \varepsilon_0 + k\varepsilon_0) \|\nabla u\|_{L^2(D_{r_0}(x_0))} + r_0^{2-\frac{2}{p}} \|\nabla^2 f\|_{L^p(D_\rho(x))}. \tag{A.18}$$

Because ε_0 is a small adjustable parameter, we may always choose k so that $(\varepsilon_0 + k\varepsilon_0)$ is small enough. A standard controlled-growth argument (see [16, Lemma III.2.1]) along with (A.15) enables us to conclude that for some constant $C(\varepsilon_0)$, it holds that

$$\begin{aligned}\|\nabla u\|_{L^2(D_\sigma(x_0))} &\le C(\varepsilon_0)\sigma^{2-\frac{2}{p}}\big[r_0^{\frac{2}{p}-2}\|\nabla u\|_{L^2(D_{r_0}(x_0))} + \|\nabla^2 f\|_{L^p(D_\rho(x))}\big]\\ &\le C(\varepsilon_0)\sigma^{2-\frac{2}{p}}\big[r_0^{\frac{2}{p}-2}\|\nabla u\|_{L^{2,\infty}(D_\rho(x))} + \|\nabla^2 f\|_{L^p(D_\rho(x))}\big],\end{aligned}$$

for

$$x_0 \in D_{3\rho/4}(x) \quad \text{and} \quad \sigma \in (0, r_0).$$

In particular, for $r_0 = \rho/2$, we find

$$\|\nabla u\|_{L^2(D_\sigma(x_0))} \le C(\varepsilon_0)\sigma^{2-\frac{2}{p}}\big[\rho^{\frac{2}{p}-2}\|\nabla u\|_{L^{2,\infty}(D_\rho(x))} + \|\nabla^2 f\|_{L^p(D_\rho(x))}\big], \tag{A.19}$$

for

$$x_0 \in D_{\rho/2}(x) \quad \text{and} \quad \sigma \in (0, \rho/2).$$

Our proof proceeds next in two distinct cases.

Case 1: $p \in (1, 2)$. We recast equation (A.9) in the form

$$-\Delta u = b\Delta f + \nabla b \cdot (\nabla^\perp u + \nabla f).$$

From (A.11) and (A.19), we have

$$\begin{aligned}\|\Delta u\|_{L^1(D_\sigma(x))} &\lesssim \|\Delta f\|_{L^1(D_\sigma(x))} + \|\nabla u\|_{L^2(D_\sigma(x))} + \|\nabla f\|_{L^2(D_\sigma(x))}\\ &\le C(\varepsilon_0)\big[\rho^{\frac{2}{p}-2}\|\nabla u\|_{L^{2,\infty}(D_\rho(x))} + \|\nabla^2 f\|_{L^p(D_\rho(x))}\big]\sigma^{2-\frac{2}{p}}.\end{aligned}$$

Calling upon Proposition A.1 yields

$$\begin{aligned}&\|\nabla u\|_{L^s(D_{b\sigma}(x))}\\ &\quad\lesssim \sigma^{\frac{2}{s}-1}\big[\|\nabla u\|_{L^2(D_\sigma)} + \sigma^{2-\frac{2}{p}}\big[\rho^{\frac{2}{p}-2}\|\nabla u\|_{L^{2,\infty}(D_\rho(x))} + \|\nabla^2 f\|_{L^p(D_\rho(x))}\big]\big]\\ &\quad\lesssim \sigma^{\frac{2}{s}-\frac{2}{p}+1}\big[\rho^{\frac{2}{p}-2}\|\nabla u\|_{L^{2,\infty}(D_\rho(x))} + \|\nabla^2 f\|_{L^p(D_\rho(x))}\big]\\ &\quad\lesssim C(\varepsilon_0)\big[\sigma^{\frac{2}{s}-1}\|\nabla u\|_{L^{2,\infty}(D_\rho(x))} + \sigma^{\frac{2}{s}-\frac{2}{p}+1}\|\nabla^2 f\|_{L^p(D_\rho(x))}\big],\end{aligned}$$

for any $b \in (0, 1)$ and

$$2 < s < \frac{2}{2-p}.$$

Hence for $b < 1/2$, we deduce that

$$\|\nabla u\|_{L^s(D_{b\rho}(x))} \lesssim C(\varepsilon_0)\big[\rho^{\frac{2}{s}-1}\|\nabla u\|_{L^{2,\infty}(D_\rho(x))} + \rho^{\frac{2}{s}-\frac{2}{p}+1}\|\nabla^2 f\|_{L^p(D_\rho(x))}\big]. \tag{A.20}$$

Case 2: $p \geq 2$. Let $s \in (2, \infty)$ be arbitrary. Choose $0 < \delta < 2/s$. Then, setting $q = 2 - \delta$, we have

$$\|\nabla^2 f\|_{L^q(D_\rho(x))} \lesssim \rho^{\frac{2}{q}-\frac{2}{p}} \|\nabla^2 f\|_{L^p(D_\rho(x))}.$$

Since $s < 2/(2-q)$, we have per the above discussion (previous case and notably (A.20)) that

$$\begin{aligned}\|\nabla u\|_{L^s(D_{b\rho}(x))} &\leq C(\varepsilon_0)\big[\rho^{\frac{2}{s}-1}\|\nabla u\|_{L^{2,\infty}(D_\rho(x))} + \rho^{\frac{2}{s}-\frac{2}{q}+1}\|\nabla^2 f\|_{L^q(D_\rho(x))}\big] \\ &\leq C(\varepsilon_0)\big[\rho^{\frac{2}{s}-1}\|\nabla u\|_{L^{2,\infty}(D_\rho(x))} + \rho^{\frac{2}{s}-\frac{2}{p}+1}\|\nabla^2 f\|_{L^p(D_\rho(x))}\big].\end{aligned}$$

In other words, (A.20) holds for all $p \in (1, \infty)$, with any $s < 2/(2-p)$ if $p \in (1, 2)$ and any $s < \infty$ if $p \geq 2$. We combine these facts by writing that (A.8) holds for all $s < 2/(2-p)_+$, which concludes the proof. ∎

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Yann Bernard
School of Mathematics, Monash University, 3800 Victoria, Australia;
yann.bernard@monash.edu.au

Glen Wheeler
Institute for Mathematics and Its Applications, University of Wollongong, 2522 New South Wales, Australia; glenw@uow.edu.au

Valentina-Mira Wheeler
Institute for Mathematics and Its Applications, University of Wollongong, 2522 New South Wales, Australia; vwheeler@uow.edu.au