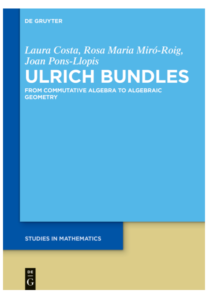
Ulrich Bundles: From Commutative Algebra to Algebraic Geometry by Laura Costa, Rosa María Miró-Roig and Joan Pons-Llopis

Reviewed by Flaminio Flamini



Vector bundles on algebraic varieties constitute an important and ubiquitous topic in mathematics, not only because of their many applications to other disciplines, e.g., commutative algebra, mathematical physics, complex analysis, just to mention a few, but mainly because they offer a fundamental point of view for a thorough understanding of the geometry and topology of algebraic varieties.

It is in this context that the book under review, written by internationally recognized experts on vector bundles and their moduli spaces, stands as a fundamental handbook focusing on particular classes of vector bundles, the so-called *arithmetically Cohen–Macaulay* (aCM for short) and more specifically *Ulrich bundles*, which recently have been the object of an intense and fruitful research activity. In the category of all vector bundles with which a given projective variety X can be endowed, a particularly important class is that of aCM bundles, which are basically those bundles with a large number of vanishing cohomology groups; among them, the subfamily of bundles having further the largest allowed number of linearly independent global sections are Ulrich vector bundles on X, and these constitute the main topic of the text.

The study of these bundles has had strong input from commutative algebra, beginning with the work of B. Ulrich in 1984, in connection with *maximally graded Cohen–Macaulay* modules with respect to the associated coordinate ring of X. The attention of algebraic geometers was later drawn by a celebrated paper of D. Eisenbud, F. O. Schreyer and J. Weyman in 2003 where, among other things, the Chow form of X is computed using Ulrich bundles under the assumption that X supports bundles of this type.

Besides these facts, several basic questions on Ulrich vector bundles enter the game (cf. *Eisenbud–Schreyer's conjecture* in Question 3.1.2 of the book): Is it true that any projective variety X supports an Ulrich vector bundle? If so, what is the minimum possible rank of an Ulrich bundle on X (the so-called *Ulrich complexity of X*)? Further basic questions are: If X supports Ulrich bundles of a given rank and of some other extra discrete data, i.e., Chern classes, do they fit in a *moduli space*? If so, what are the local and global properties of such moduli spaces?

From the list of previous questions, it is then clear why Ulrich's bundle theory has become a very active research area, with a constant offer of new results, but also of many still open questions. In approaching this theory, a variety of techniques have been used from several authors, depending on either the rank or the type of variety X on which some of these issues are studied. All these different approaches are scattered throughout an extensive literature making it difficult for an interested mathematician, and in particular a young mathematician who faces these problems for the first time, to keep track of the current state of the art in this topic.

It is with this in mind that the text under review presents itself as a fundamental tool: it provides a detailed introduction to the subject, attempting to emphasize the main reasons for the interest in these bundles, the wide range of areas that are involved in the development of their study, as well as the central questions that remain open in this area; it also gives an as comprehensive and organic as possible collection of all the major techniques used, scattered throughout the literature.

The detailed plan of the book is the following. To make the text as self-contained as possible, Chapter 1 is used to fix notation and introduce general objects and results in vector bundle theory that are used throughout the book. Chapter 2 deals with aCM bundles. First, fundamental preliminaries about these bundles are introduced and the important concepts of varieties of *finite*, *tame*, or *wild representation type* are given (cf. Definition 2.1.14); then the chapter focuses on classification of aCM bundles on relevant projective varieties: projective spaces and Horrock's theorem, smooth hyperquadrics and Knörrer's theorem, Grassmann varieties and, at last, low-rank aCM bundles on some smooth hypersurfaces. Chapter 3 focuses on the main object of the text: Ulrich bundles on a projective variety X. The chapter starts with a beautiful historical background on the subject, then provides the different definitions of Ulrich bundles scattered in the literature. The authors discuss first examples and basic properties of such bundles. Finally, to exhibit preliminary examples towards the Eisenbud–Schreyer conjecture, they review the classification of Ulrich bundles on smooth curves C of degree d and genus g in projective spaces, highlighting the different behavior of families of high-rank, indecomposable Ulrich bundles on C, according to the genus g being either 0, or 1 or, greater than or equal to 2. The rest of the book gives an outlook on the state of the art of the aforementioned questions, stressing the wide range of techniques involved in answering them. Precisely, Chapter 4 deals with the case of complete intersection varieties, in particular hypersurfaces, connections with representation theory and Cayley–Chow forms. Chapter 5 is devoted to the study of Ulrich bundles on smooth projective surfaces in the setting of the Enriques–Kodaira classification. Chapter 6 treats Ulrich bundles of any rank r at least 2 on a smooth complete intersection of two hyperquadrics in the 5-dimensional projective space, as well as low-rank Ulrich bundles on smooth Fano threefolds of index 2 (or even, Del Pezzo threefolds), or on smooth prime Fano threefolds. Chapter 7 discusses some relevant higher-dimensional cases, namely Segre, Grassmann and flag varieties.

Each chapter is endowed with a short final section of further references, historical remarks, and lists of suggested additional reading. Moreover, to make the text as self-contained as possible, the book concludes with an appendix, where important results concerning derived categories and Fourier–Mukai transforms sometimes used in the text are collected, and with an extensive and detailed bibliography.

To sum-up, the book provides a thorough, self-contained and well-organized introduction to a variety of fundamental concepts, techniques and examples concerning Ulrich vector bundles. The clarity of presentation, together with the detailed description of several key examples, makes the book suitable and versatile not only for established researchers with a particular interest in the topic, but also for young researchers, as a motivation to learn these basic techniques in a practical way. For all these reasons, I consider the book a valuable, highly recommended addition to the literature.

Laura Costa, Rosa María Miró-Roig and Joan Pons-Llopis, *Ulrich Bundles: From Commutative Algebra to Algebraic Geometry.* De Gruyter, 2021, 282 pages, Hardback ISBN 978-3-11-064540-8.

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DOI 10.4171/MAG/122