# Solved and unsolved problems

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The present column is devoted to Number Theory.

## I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

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Consider two positive integers $n \geq 1$ and $a \geq 2$ such that

$$a^{2n} + a^n + 1$$

is a prime. Prove that $n$ is a power of 3.

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The Collatz map is defined as follows:

$$\mathrm{Col}(n) := \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Let

$$t_{m,x} := \min(n > 0 : \mathrm{Col}^m(n) \geq x).$$

That is, $t_{m,x}$ is the smallest integer such that, if we apply the Collatz map $m$ times, the result is larger than $x$.

(a) Find $t_{3,1000}$ and $t_{4,1000}$.

(b) Show that, for $x$ large enough (larger than (say) 1000), we have

$$t_{4,x} \equiv 3 \bmod 4 \quad \text{or} \quad t_{4,x} \equiv 6 \bmod 8.$$

(c) In general, for $m$ odd and $x$ large enough, there exists a constant $X_{m,x}$ such that $t_{m,x}$ is the smallest $n > X_{m,x}$ such that $n \equiv c_m \bmod M_m$. Find $M_m$ and relate $c_m$ to $c_{m-1}$.

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The light-bulb problem: Alice and Bob are in jail for trying to divide by 0. The jailer proposes the following game to decide their freedom: Alice will be shown an $n \times n$ grid of light bulbs. The jailer will point to a light bulb of his choice and Alice will decide whether it should be on or off. Then the jailer will point to another bulb of his choice and Alice will decide on/off. This continues until the very last bulb, when the jailer will decide whether this bulb is on or off. So the jailer controls the order of the selection, and the state of the final bulb. Alice is now removed from the room, and Bob is brought in. Bob's goal is to choose $n$ bulbs such that his selection includes the final bulb (the one determined by the jailer).

Is there a strategy that Alice and Bob can use to guarantee success? What if Bob does not know the orientation in which Alice saw the board (i.e., what if Bob does not know which are the rows and which are the columns)?

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Let $p$ and $q$ be coprime integers greater than or equal to 2. Let $\mathrm{inv}_q(p)$ and $\mathrm{inv}_p(q)$ denote the modular inverse of $p$ mod $q$ and $q$ mod $p$, respectively. That is, $\mathrm{inv}_q(p)p \equiv 1 \bmod q$ and $\mathrm{inv}_p(q)q \equiv 1 \bmod p$.

(a) Show that

$$\mathrm{inv}_p(q) \leq \frac{p}{2} \quad \text{if and only if} \quad \mathrm{inv}_q(p) > \frac{q}{2}.$$

(b) Show by providing an example that, if $1 \leq u < v$ are coprime integers and $\alpha := u/v$, then the statement

$$\mathrm{inv}_p(q) \leq \alpha p \quad \text{if and only if} \quad \mathrm{inv}_q(p) > (1-\alpha)q \qquad (1)$$

is not necessarily true.

(c) What additional assumption should $p$ and/or $q$ satisfy so that the equivalence (1) holds?

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Let $c_n(k)$ denote the Ramanujan sum defined as the sum of $k$th powers of the primitive $n$th roots of unity. Show that, for any integer $m \geq 1$,

$$\sum_{[n,k]=m} c_n(k) = \varphi(m),$$

where the sum is over all ordered pairs $(n,k)$ of positive integers $n,k$ such that their lcm is $m$, and $\varphi$ is Euler's totient function.

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Show that, for every integer $n \geq 1$, we have the polynomial identity

$$\prod_{\substack{k=1\\(k,n)=1}}^{n} (x^{(k-1,n)} - 1) = \prod_{d\mid n} \Phi_d(x)^{\varphi(n)/\varphi(d)},$$

where $\Phi_d(x)$ are the cyclotomic polynomials and $\varphi$ denotes Euler's totient function.

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## II Open problem

### *275*. Chowla's conjecture and its relatives*

*by Terence Tao (UCLA, Department of Mathematics, Los Angeles, USA)*

Let $\lambda\colon \mathbb{N} \to \{-1,+1\}$ denote the Liouville function. In [2], Chowla conjectured that

$$\sum_{n\leq x} \lambda(n+h_1)\cdots\lambda(n+h_k) = o(x) \tag{1}$$

as $x \to \infty$, for any distinct natural numbers $h_1,\ldots,h_k$ (in fact, Chowla made the more general conjecture that

$$\sum_{n\leq x} \lambda(P(n)) = o(x)$$

whenever $P$ is a square-free polynomial mapping from $\mathbb{N}$ to $\mathbb{N}$). Chowla's conjecture was extended to other bounded multiplicative functions by Elliott [3] (see also a technical correction to the conjecture in [8]).

One can view (1) as a less difficult cousin of the notorious *Hardy–Littlewood prime tuples conjecture* [4], which conjectures an asymptotic of the form

$$\sum_{n\leq x} \Lambda(n+h_1)\cdots\Lambda(n+h_k) = \mathfrak{S}x + o(x) \tag{2}$$

where the *singular series* $\mathfrak{S}$ is an explicit product over primes of factors involving the numbers $h_1,\ldots,h_k$. For $k=1$, both conjectures follow readily from the prime number theorem, but they remain open for higher $k$. However, the analogue of (1) (and (2) for $k \leq 2$) were recently established in certain function fields [11], and are also known to hold in the presence of a Siegel zero [1, 5, 6, 15]. The conjecture (1) is also known if one performs enough averaging in the $h_1,\ldots,h_k$ variables [8].

The logarithmically averaged version

$$\sum_{n\leq x} \frac{\lambda(n+h_1)\cdots\lambda(n+h_k)}{n} = o(\log x) \tag{3}$$

turns out to be more tractable, as it can be analyzed by the "entropy decrement method" [12], which has been successfully used to establish the conjecture for $k=2$ (see [12], building upon the breakthrough work [7]) and for odd $k$ (see [14, 16]). The conjecture (3) for arbitrary $k$ is also known to be equivalent [13] to the (logarithmically averaged) Sarnak conjecture [10], which asserts that

$$\sum_{n\leq x} \frac{\lambda(n)F(T^n x_0)}{n} = o(\log x)$$

whenever $T\colon X \to X$ is a compact dynamical system of zero entropy, $x_0$ is a point in $X$, and $F\colon X \to \mathbb{C}$ is continuous. Many special cases of this conjecture are known, unfortunately too many to list here.

The conjecture (3) would also follow from a *higher-order local Fourier uniformity conjecture* [12], which is somewhat complicated to state in full generality here; however, the first unsolved special case of this conjecture asserts that

$$\int_X^{2X} \sup_{\alpha\in\mathbb{R}/\mathbb{Z}} \Big|\sum_{x\leq n\leq x+H} \lambda(n)e(\alpha n)\Big|\,dx = o(XH) \quad \text{as } X\to\infty$$

whenever $1 \leq H = H(X) \leq X$ is such that $H(X) \to \infty$ as $X \to \infty$, where $e(\theta) \coloneqq e^{2\pi i\theta}$. This is currently only established in the regime $H \geq X^{\varepsilon}$ for a fixed $\varepsilon > 0$ (see [9]).

*References*


[1] J. Chinis, Siegel zeros and Sarnak's conjecture, preprint, arXiv: 2105.14653 (2021)

[2] S. Chowla, *The Riemann hypothesis and Hilbert's tenth problem*. Mathematics and its Applications 4, Gordon and Breach Science Publishers, New York (1965)

[3] P. D. T. A. Elliott, On the correlation of multiplicative functions. *Notas Soc. Mat. Chile* **11**, 1–11 (1992)

[4] G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes. *Acta Math.* **44**, 1–70 (1923)

[5] D. R. Heath-Brown, Prime twins and Siegel zeros. *Proc. London Math. Soc. (3)* **47**, 193–224 (1983)

[6] K. Matomäki, J. Merikoski, Siegel zeros, twin primes, Goldbach's conjecture, and primes in short intervals, preprint, arXiv: 2112.11412 (2021)

[7] K. Matomäki and M. Radziwiłł, Multiplicative functions in short intervals. *Ann. of Math. (2)* **183**, 1015–1056 (2016)

[8] K. Matomäki, M. Radziwiłł and T. Tao, An averaged form of Chowla's conjecture. *Algebra Number Theory* **9**, 2167–2196 (2015)

[9] K. Matomäki, M. Radziwiłł and T. Tao, Fourier uniformity of bounded multiplicative functions in short intervals on average. *Invent. Math.* **220**, 1–58 (2020)

[10] P. Sarnak, Mobius randomness and dynamics. *Not. S. Afr. Math. Soc.* **43**, 89–97 (2012)

[11] W. Sawin and M. Shusterman, On the Chowla and twin primes conjectures over $\mathbb{F}_q[T]$. *Ann. of Math. (2)* **196**, 457–506 (2022)

[12] T. Tao, The logarithmically averaged Chowla and Elliott conjectures for two-point correlations. *Forum Math. Pi* **4**, Paper No. e8 (2016)

[13] T. Tao, Equivalence of the logarithmically averaged Chowla and Sarnak conjectures. In *Number Theory—Diophantine Problems, Uniform Distribution and Applications*, Springer, Cham, 391–421 (2017)

[14] T. Tao and J. Teräväinen, Odd order cases of the logarithmically averaged Chowla conjecture. *J. Théor. Nombres Bordeaux* **30**, 997–1015 (2018)

[15] T. Tao, J. Teräväinen, The Hardy–Littlewood–Chowla conjecture in the presence of a Siegel zero, preprint, arXiv:2109.06291 (2021)

[16] T. Tao and J. Teräväinen, The structure of logarithmically averaged correlations of multiplicative functions, with applications to the Chowla and Elliott conjectures. *Duke Math. J.* **168**, 1977–2027 (2019)

## III Solutions

### *260*

Let $f\colon [0,\infty) \to \mathbb{R}$ be a $C^1$-differentiable and convex function with $f(0)=0$.

(i) Prove that, for every $x \in [0,\infty)$, the following inequality holds:

$$\int_0^x f(t)\,dt \le \frac{x^2}{2} f'(x).$$

(ii) Determine all functions $f$ for which we have equality.

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*Solution by the proposers*

(i) We have

$$\int_0^x f(t)\,dt = \int_0^x \big(f(t)-f(0)\big)\,dt.$$

By the Mean Value Theorem,

$$f(t)-f(0) = tf'(c_t)$$

for some $c_t \in (0,t) \subset [0,x]$. Since $f$ is convex, it follows that $f'$ is increasing; hence $f'(c_t) \le f'(x)$. Therefore, $f(t) = tf'(c_t) \le tf'(x)$, and we obtain

$$\int_0^x f(t)\,dt \le \int_0^x tf'(x)\,dt = f'(x)\int_0^x t\,dt = \frac{x^2}{2}f'(x).$$

(ii) We have to find all the solutions to the equation

$$\int_0^x f(t)\,dt = \frac{x^2}{2}f'(x).$$

Denoting

$$g(x) = \int_0^x f(t)\,dt, \quad x \in [0,\infty),$$

the above equation is equivalent to the second-order differential equation

$$g(x) = \frac{x^2}{2}g''(x), \quad x \in [0,\infty).$$

Note that if $g$ is a solution, then

$$g(x) + xg'(x) = \frac{x^2}{2}g''(x) + xg'(x),$$

whence

$$(xg(x))' = \Big(\frac{x^2}{2}g'(x)\Big)'.$$

It follows that

$$\Big(xg(x) - \frac{x^2}{2}g'(x)\Big)' = 0,$$

and so

$$\frac{x^2}{2}g'(x) - xg(x) = C_1, \quad x \in [0,\infty),$$

where $C_1$ is an arbitrary constant. This last equation is equivalent to

$$\frac{x^2g'(x) - 2xg(x)}{x^4} = \frac{2C_1}{x^4}, \quad x \in (0,\infty),$$

or

$$\Big(\frac{g(x)}{x^2}\Big)' = \frac{2C_1}{x^4}, \quad x \in (0,\infty).$$

Consequently,

$$\frac{g(x)}{x^2} = \frac{-2C_1}{3x^3} + C_2, \quad x \in (0,\infty),$$

and we get

$$g(x) = \frac{-2C_1}{3x} + C_2x^2, \quad x \in (0,\infty).$$

On the other hand, $g$ is continuous and $g(0)=0$. This implies that $C_1 = 0$ and $g(x) = C_2x^2$. We conclude that the sought-for functions are necessarily of the form $f(x) = g'(x) = 2C_2x$, i.e., $f(x) = Ax$, where $A$ is an arbitrary real constant.

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Let $y(x)$ be the unknown function of the following fractional-order derivative Cauchy problem:

$$\begin{cases} D^\alpha y = f(x,y), & 0 < \alpha < 1, \\ y(0) = y^*. \end{cases} \tag{1}$$

Find the solution of this problem by solving an equivalent first-order ordinary Cauchy problem, with a solution independent of the kernel of the fractional operator.

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*Solution by the proposer*

Before we give the solution of (1), let us make some preliminary remarks about the most popular definitions of fractional derivatives.

The Riemann–Liouville integral of fractional order $\nu \geq 0$ of a function $f(x)$ is defined as

$$(J^\nu f)(t) = \begin{cases} \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t (t-\tau)^{\nu-1} f(\tau)\, d\tau, & \nu > 0, \\ f(t), & \nu = 0. \end{cases}$$

The corresponding Riemann–Liouville fractional derivative of order $\alpha > 0$ is defined as

$$D^\alpha_{\mathrm{RL}} f(t) = \frac{d^n}{dt^n} J^{n-\alpha} f(t), \quad n \in \mathbb{N},\ n-1 < \alpha \leq n. \tag{2}$$

The main problem with this derivative is that it assigns a nonzero value to a constant function. To avoid this issue, people often use the so-called order-$\alpha$ Caputo fractional derivative, defined as

$$D^\alpha_{\mathrm{C}} f(x) = \begin{cases} \dfrac{d^n f(x)}{dx^n}, & 0 < \alpha \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}}\, d\tau, & t > 0, \\ & 0 \leq n-1 < \alpha < n, \end{cases}$$

where $n$ is an integer, $x > 0$, and $f \in C^n$.

Riemann–Liouville (RL) and Caputo (C) derivatives are the most popular and have been used in many applications; nevertheless, they both have some drawbacks. For this reason, many authors have introduced some more flexible fractional operators. The most general fractional derivative with a given kernel $K(x,\alpha)$ is defined as

$$D^\alpha f(x) = \begin{cases} \dfrac{d^n f(x)}{dx^n}, & 0 < \alpha \in \mathbb{N}, \\ \displaystyle\int_0^x f^{(n)}(\tau) K(x-\tau,\alpha)\, d\tau, & t > 0, \\ & 0 \leq n-1 < \alpha < n. \end{cases} \tag{3}$$

The kernel should be chosen in a such a way that at least the following two conditions are satisfied:

$$\lim_{\alpha \to 0} K(x-\tau,\alpha) = 1 \quad \text{and} \quad \lim_{\alpha \to 1} K(x-\tau,\alpha) = \delta(x-\tau).$$

Moreover, to ensure that one is dealing with a non-singular kernel, one requires that

$$\lim_{x \to \tau} K(x-\tau,\alpha) \neq 0 \quad \forall \alpha.$$

Although several definitions of fractional derivatives are available, they all depend on the proposed kernel, thus implying a subjective and a priori unjustified choice of the fractional operator each time one studies a fractional differential problem. This issue can be avoided by using the following simple definition, which is based on an intuitive interpolation.

Limiting ourselves to the case $n = 1$, the general structure of the Caputo-type fractional derivative

$$D^\alpha_{\mathrm{C}} f(x) = \int_0^x f'(\tau) K(x-\tau,\alpha)\, d\tau, \quad 0 < \alpha < 1,$$

is based on the kernel $K(x-\tau,\alpha)$, which is a positive function that decays at infinity (to ensure convergence), while according to (2), the general structure of the Riemann–Liouville first-order derivative is

$$D^\alpha_{\mathrm{RL}} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x f(\tau)(x-\tau)^{-\alpha}\, d\tau, \quad 0 < \alpha \leq 1.$$

Usually, to find the solution of (1), we should first choose the kernel of the fractional operator and then solve the fractional problem by using a suitable numerical method, which roughly consists in constructing and solving an equivalent algebraic/differential (of integer order) problem. In any case, the solution will depend not only on the independent variable $x$ and the initial condition $y^*$, but also on the kernel and on the fractional-order parameter:

$$y(x, y^*, K(x-\tau), \alpha).$$

The dependence on the fractional parameter $\alpha$ is essential in solving fractional-order problems. However, the dependence on the kernel leads to an inessential "struggle" about the best choice of the kernel and about its physical/mathematical meaning – an obviously subjective and non-unique choice. Because of this lack of uniqueness, fractional calculus is missing a strong mathematical motivation. On the other hand, there exist many useful mathematical tools, important for the solution of differential problems, that require making choices, such as wavelets, orthogonal polynomials, integral transforms, and many more. Therefore, one can either ignore the uniqueness problem, or try to defend a specific choice by using some reasoning that may still be not sufficient to convince the mathematics community. In the following, we give a solution which is both independent of the choice of the kernel and can be analytically obtained by reduction to an equivalent ordinary differential problem having the same solution as (1).

We search the solution by assuming that the fractional derivative is obtained by linear interpolation between a function and its first-order derivative; consequently, we do not need the integral definition (3) and the accompanying choice of kernel. This is an acceptable assumption, based on the original simple idea in the fundamentals of fractional calculus that the fractional parameter describes a family of interpolation curves. Thus, we set

$$D^{\alpha}y = (1-\alpha)(y-y^*) + \alpha\frac{dy}{dx}$$

so that the initial value problem (1) becomes

$$\alpha\frac{dy}{dx} + (1-\alpha)y = (1-\alpha)y^* + f(x,y).$$

Now, for $0 < \alpha < 1$, we easily obtain the following ordinary differential problem that is equivalent to (1):

$$\begin{cases} \dfrac{dy}{dx} - \Big(1-\dfrac{1}{\alpha}\Big)y = -\Big(1-\dfrac{1}{\alpha}\Big)y^* + \dfrac{1}{\alpha}f(x,y), \\ \qquad\qquad y(0) = y_0. \end{cases} \tag{4}$$

For the moment, we can suppose that the initial conditions of (1) and (4) do not necessarily coincide, i.e., $y^* \neq y_0$. However, some further assumptions can be made when the function $f(x,y)$ is given explicitly. In particular, to achieve a perfect equivalence between (1) and (4), we can set $y^* = (1-\alpha)\tau + \alpha y_0\ \forall\tau\in\mathbb{R}$, and then let $\alpha \to 1$.

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Let $y(x)$ be the unknown function of the following Bernoulli fractional-order Cauchy problem:

$$\begin{cases} D^{\alpha}y = g(x)y^{\beta}, \quad 0<\alpha<1,\ \beta\neq 0,1, \\ y(0) = y^*, \end{cases} \tag{1}$$

where $g(x)$ is a continuous function in the interval $I = [0,\infty)$.

Find the solution of this problem by solving an equivalent first-order ordinary Cauchy problem, with a solution independent of the kernel of the fractional operator.

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*Solution by the proposer*

We search the solution by simply assuming that the fractional derivative is a linear interpolation between a function and its first-order derivative so that we do not need the usual integral definition of the fractional operator which requires us to choose the underlying kernel.

Thus, we set

$$D^{\alpha}y = (1-\alpha)(y-y^*) + \alpha\frac{dy}{dx} \tag{2}$$

so that (1) becomes

$$\alpha\frac{dy}{dx} + (1-\alpha)y = (1-\alpha)y^* + g(x)y^{\beta}. \tag{3}$$

Then taking $0<\alpha<1$ and using equations (2) and (3), we easily get the following ordinary differential problem equivalent to (1):

$$\begin{cases} \dfrac{dy}{dx} - \Big(1-\dfrac{1}{\alpha}\Big)y = -\Big(1-\dfrac{1}{\alpha}\Big)y^* + \dfrac{1}{\alpha}g(x)y^{\beta}, \\ \qquad\qquad\qquad\qquad \beta\neq\{0,1\}, \\ \qquad\qquad y(0) = y_0. \end{cases} \tag{4}$$

For the moment, we search a general solution of (1) by assuming that $y^*\neq y_0$. Solving separately the two equations

$$\begin{aligned} &\frac{dy}{dx} - \Big(1-\frac{1}{\alpha}\Big)y = -\Big(1-\frac{1}{\alpha}\Big)y^*, \\ &\frac{dy}{dx} - \Big(1-\frac{1}{\alpha}\Big)y = \frac{1}{\alpha}g(x)y^{\beta}, \quad \beta\neq\{0,1\}, \end{aligned}$$

we obtain the respective solutions

$$\begin{aligned} y(x) &= y^* + k_1 e^{(1-\frac{1}{\alpha})x}, \\ y(x)^{1-\beta} &= e^{(1-\beta)(1-\frac{1}{\alpha})x}\Big[k_2 + \frac{1-\beta}{\alpha}\int_0^x g(\xi)e^{\frac{1-\beta}{\alpha}\xi}d\xi\Big], \\ &\qquad \beta\neq\{0,1\}. \end{aligned}$$

Consequently, the solution of problem (4), which is also the solution of problem (1), takes the forms listed below.

1. Let $y_0\neq y^*$ and $\beta\neq\{0,1\}$. Then

$$\begin{aligned} y(x) &= y^* + \frac{1}{2}(y_0-y^*)e^{(1-\frac{1}{\alpha})x} \\ &\quad + e^{(1-\beta)(1-\frac{1}{\alpha})x}\Big[\frac{1}{2}(y_0-y^*) + \frac{1-\beta}{\alpha}\int_0^x g(\xi)e^{\frac{1-\beta}{\alpha}\xi}d\xi\Big]. \end{aligned}$$

2. Let $y_0 = y^*$ and $\beta\neq\{0,1\}$. Then

$$y(x) = y^* + e^{(1-\beta)(1-\frac{1}{\alpha})x}\Big[\frac{1-\beta}{\alpha}\int_0^x g(\xi)e^{\frac{1-\beta}{\alpha}\xi}d\xi\Big].$$

3. Let $y_0 = 0$, $y^*\neq 0$ and $\beta\neq\{0,1\}$. In this case, note that, in order to solve the given fractional-order Cauchy problem (1) via an equivalent ordinary differential problem, we can simply set $y_0 = 0$ in (4) and thus obtain the solution

$$\begin{aligned} y(x) &= y^* - \frac{1}{2}y^*e^{(1-\frac{1}{\alpha})x} \\ &\quad + e^{(1-\beta)(1-\frac{1}{\alpha})x}\Big[-\frac{1}{2}y^* + \frac{1-\beta}{\alpha}\int_0^x g(\xi)e^{\frac{1-\beta}{\alpha}\xi}d\xi\Big]. \end{aligned}$$

4. Let $y_0 = y^* = 0$ and $\beta\neq\{0,1\}$. Then

$$y(x) = \frac{1-\beta}{\alpha}e^{(1-\beta)(1-\frac{1}{\alpha})x}\int_0^x g(\xi)e^{\frac{1-\beta}{\alpha}\xi}d\xi.$$

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Let $g$ be a real-valued $C^2$-function defined on $(0,\infty)$, strictly increasing, such that $g(x) > 1$ for all $x\in(0,\infty)$ and $g(2) < 4$. Consider the boundary value problem

$$y'' = -g(x)y, \quad y(0) = 1, \quad y'(0) = 0.$$

Prove that the solution $y$ has exactly one zero in $(0, \pi/2)$, i.e., there exists a unique point $x_0 \in (0, \pi/2)$ such that $y(x_0) = 0$, and give a positive lower bound for $x_0$.

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*Solution by the proposer*

First suppose that $y(x) > 0$ for $x \in (0, \pi/2)$. The function $z(x) = \cos x$ is the solution to the auxiliary initial value problem

$$z'' = -z, \quad z(0) = 1, \quad z'(0) = 0.$$

Therefore,

$$y''z - yz'' = (1 - g(x))zy.$$

Integrating this equality over the interval $(0, \pi/2)$, we obtain

$$\begin{aligned}\int_0^{\pi/2} (1 - g(x))zy\,dx &= \int_0^{\pi/2} (y''z - yz'')\,dx \\ &= (y'z - yz')|_0^{\pi/2} = y(\pi/2),\end{aligned}$$

and $y(\pi/2) \geq 0$ by the continuity of $y$. But

$$(1 - g(x))zy < 0 \quad \text{in } (0, \pi/2),$$

so we reached a contradiction. Thus, $y$ has at least one zero in $(0, \pi/2)$.

Next, observe that $\pi/2 < 2$ and by assumption $g(2) < 4$. Consider the function $w(x) = \cos(2x)$, which is a solution to the second auxiliary initial value problem

$$w'' = -4w, \quad w(0) = 1, \quad w'(0) = 0$$

and satisfies $w(x) > 0$ for $x \in (0, \pi/4)$. Suppose that $y$ has (at least) one zero in $(0, \pi/4)$. Denote the smallest such zero by $x_1$. Then $y(x)$ is positive for $x \in (0, x_1)$ (recall that $y(0) = 1$) and $y(x_1) = 0$; hence $y'(x_1) \leq 0$. An argument analogous to the one above shows that

$$\begin{aligned}\int_0^{x_1} (g(x) - 4)wy\,dx &= \int_0^{\pi/2} (w''y - wy'')\,dx \\ &= (w'y - wy')|_0^{x_1} = -w(x_1)y'(x_1).\end{aligned}$$

Note that $-w(x_1)y'(x_1) \geq 0$, but the integrand $(g(x) - 4)wy < 0$, so again we reached a contradiction. Thus, $y$ has no zero in $(0, \pi/4)$, so $\pi/4$ is a positive lower bound for the zeros of $y$.

Finally, suppose that $y$ has more than one zero in $(\pi/4, \pi/2)$, namely, there exist at least two points $x_2, x_3 \in (\pi/4, \pi/2)$ such that $x_2 < x_3$ and $y(x_2) = y(x_3) = 0$. Take the function

$$v(x) = a\cos(2x + b),$$

where $a, b$ are chosen in such a way that $v(x_2) = 0$ and $v(x)$ is negative for $x \in (x_2, x_3)$; see Figure 1.

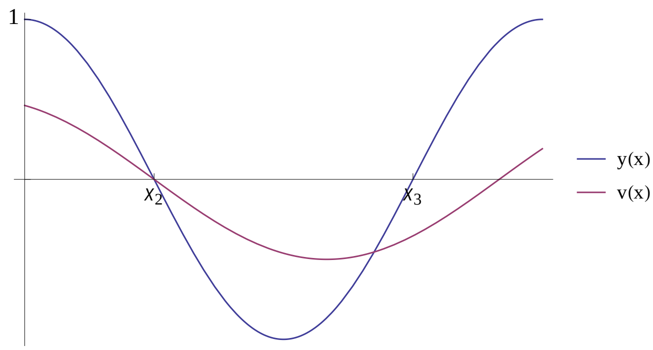


*Figure 1.* Assuming $y$ has more than one zero in $(\pi/4, \pi/2)$

We have $v'' = -4v$, so $y''v - yv'' = (4 - g(x))yv$, which is positive on the interval $(x_2, x_3)$. Therefore,

$$\int_{x_2}^{x_3} (y''v - yv'')\,dx > 0.$$

On the other hand,

$$\int_{x_2}^{x_3} (y''v - yv'')\,dx = (y'v - yv')|_{x_2}^{x_3} = y'(x_3)v(x_3),$$

and the right-hand side is negative since $y(x_3) = 0$ and $y(x) < 0$ for $x \in (x_2, x_3)$.

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We propose an interesting stochastic-source scattering problem in acoustics. The stochastic nature for such problems forces us to deal with stochastic partial differential equations (SPDEs), rather than partial differential equations (PDEs) which hold for the corresponding deterministic counterparts. In particular, the results of our proposed model will be applied to establish existence and uniqueness for the stochastic solution of a finite element approximation of the stochastic-source Helmholtz equation.

Consider the following approximation problem of a stochastic-source Helmholtz equation:

$$\begin{aligned}\Delta u + k^2 u &= f \quad \text{in } D, \\ u &= 0, \quad x \in \partial D,\end{aligned} \tag{1}$$

where $f = \sum_a f_a H_a$ is a generalized stochastic source. For the stochastic problem (1), we use the equations

$$u = \sum_a u_a H_a, \quad f = \sum_a f_a H_a,$$

and we get the collection of deterministic problems

$$\begin{aligned}\Delta u_a + k^2 u_a &= f_a \quad \text{in } D, \\ u_a &= 0, \quad x \in \partial D.\end{aligned} \tag{2}$$

Assume that $u_a \in H_0^1(D)$ solves problem (2). Then prove that, for all $v \in H_0^1(D)$, the solution $u_a \in H_0^1(D)$ satisfies

$$-\int_D \nabla u_a \cdot \nabla v \, dx + \int_D k^2 u_a v \, dx = \int_D f_a v \, dx.$$

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*Solution by the proposers*

We decompose our problem into a hierarchy of deterministic evolution (BVPs), and we give their corresponding variational formulations.

For $|a| = 0$, we get

$$\begin{cases} \Delta u_0 + k^2 u_0 = f_0 & \text{in } D, \\ \qquad\quad u_0 = 0 & \text{on } \partial D. \end{cases} \tag{3}$$

The estimation of a solution of problem (3) is

$$\|u_0\|_{H^1(D)} \leq c_0 \|f_0\|_{L^2(D)}.$$

For $|a| = 1$, we get

$$\begin{aligned} \Delta u_1 + k^2 u_1 = f_1 \quad & \text{in } D, \\ u_1 = 0 \quad & \text{on } \partial D. \end{aligned} \tag{4}$$

We take an arbitrary $v \in H_0^1(D)$ and multiply equation (4) by $v$. Then we get

$$(\Delta u_1) v + k^2 u_1 v = f_1 v \tag{5}$$

and integrate over $D$. Every term is integrable since $u_a \in H_0^1(D)$, and hence we have $\Delta u_1 \in H_0^1(D)$ and $v \in H_0^1(D)$, so

$$(\Delta u_1) v \in H_0^1(D), \quad k^2 \in L^\infty(D), \quad u_1 \in H_0^1(D), \quad v \in H_0^1(D).$$

Therefore, $k^2 u_1 v \in H_0^1(D)$ and $f_1 \in L^2(D)$, so $f_1 v \in H_0^1(D)$. We obtain

$$\int_D (\Delta u_1) v \, dx + \int_D k^2 u_1 v \, dx = \int_D f_1 v \, dx.$$

We use Green's formula according to which

$$\int_D (\Delta u_1) v \, dx = -\int_D \nabla u_1 \cdot \nabla v \, dx + \int_{\partial D} \gamma_1(u_1) \gamma_0(v) \, d\Gamma$$

since $v \in H_0^1(D)$ is equivalent to $\gamma_0(v) = 0$.

Let $H_0^1(D)$ be a stochastic Hilbert space. If we now assume the bilinear form on $H_0^1(D) \times H_0^1(D)$,

$$a(u_1, v) = \int_D (-\nabla u_1 \cdot \nabla v + k^2 u_1 v) \, dx,$$

and the linear functional on $H_0^1(D)$,

$$\ell(v) = \int_D f_1 v \, dx,$$

then the variational formulation of problem (5) is

$$a(u_1, v) = \ell(v) \quad \forall v \in H_0^1(D). \tag{6}$$

The estimation of a solution of problem (6) is

$$\|u_1\|_{H^1(D)} \leq c_1 \|f_1\|_{L^2(D)}.$$

For $|a| = n$, we get

$$\begin{cases} \Delta u_n + k^2 u_n = f_n & \text{in } D, \\ \qquad\quad u_n = 0 & \text{on } \partial D. \end{cases} \tag{7}$$

The estimation of a solution of problem (7) is

$$\|u_n\|_{H^1(D)} \leq c_n \|f_n\|_{L^2(D)}.$$

Via the above variational formulations and taking into account $u = \sum_a u_a H_a$, we can prove that the solution $u$ of the stochastic boundary value problem (1) satisfies the following inequality:

$$\|u\|_{H^1(D)} \leq c_0 \|f_0\|_{L^2(D)} + c_1 \|f_1\|_{L^2(D)} + \cdots + c_n \|f_n\|_{L^2(D)},$$

where $c_i$, $i = 0, 1, \ldots, n$, are considered to be in agreement with appropriate built-in weights. The solution $u$ belongs to the space $\{H^1(D), \Omega, F, \mu\}$ which is a stochastic Hilbert space with $\mu$ the probability measure defined by $H_a(\omega)$.

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For a Newtonian incompressible fluid, the Navier–Stokes momentum equation, in vector form, reads [4]

$$\begin{aligned} \rho\Big(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\Big) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}, \\ \mathbf{u} = \mathbf{u}(\mathbf{x}, t), \quad \mathbf{u} \colon \mathbf{R}^n \times (0, \infty) \to \mathbf{R}^n. \end{aligned} \tag{1}$$

Here, $\rho$ is the fluid density, $\mathbf{u}$ is the velocity vector field, $p$ is the pressure, $\mu$ is the viscosity, and $\mathbf{F}$ is an external force field.

(i) Assuming that both the pressure drop $\nabla p$ and the external field $\mathbf{F}$ are negligible, it is easy to show that equation (1) reduces to

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u}, \tag{2}$$

and finally to equation (3), where $\nu = \frac{\mu}{\rho}$ is the so-called kinematic viscosity [5].

(ii) Regarding the one-dimensional viscous Burgers equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad u = u(x, t), \tag{3}$$

prove that an analytical solution can be obtained by means of the Tanh Method [2, 3, 5] as

$$u(x, t) = \lambda\Big[1 - \tanh\Big(\frac{\lambda}{2\nu}(x - \lambda t)\Big)\Big], \quad \lambda > 0.$$

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*Solution by the proposers*

Notice that, for $\nabla p = \mathbf{F} = \mathbf{0}$, equation (1) becomes

$$\rho\Big(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\Big) = \mu\nabla^2\mathbf{u}. \tag{4}$$

Since, for an incompressible fluid, $\rho$ is a nonzero constant, one can divide both sides of equation (4) by $\rho$ and thus obtain equation (2). Now consider the motion of a one-dimensional viscous fluid with fluid velocity $u$ along the $x$-axis as time passes, $u = u(x,t)$. In this case, equation (2) transforms into equation (3). Introduce the transformation of $u$ given by

$$\begin{cases} u(x,t) = u(\zeta), \\ \qquad \zeta = \mu(x-\lambda t), \quad \mu > 0,\ \lambda \neq 0, \end{cases} \tag{5}$$

with $\mu$ representing the wave number and $\lambda$ the velocity. Then transformation (5) reduces equation (3) to the following ODE for $u(\zeta)$:

$$-\lambda u'(\zeta) + u(\zeta)u'(\zeta) - \nu\mu u''(\zeta) = 0. \tag{6}$$

Integrating equation (6) and taking the integration constant to be zero, we obtain

$$-\lambda u(\zeta) + \frac{1}{2}u^2(\zeta) - \nu\mu u'(\zeta) = 0. \tag{7}$$

The idea behind the Tanh method uses a key property of the functional derivatives all being written in terms of the Tanh function [2, 3]. The following identity is used:

$$\operatorname{sech}^2\zeta = 1 - \tanh^2\zeta, \quad \zeta \in \mathbb{R}. \tag{8}$$

This transforms equation (6) into a polynomial equation for successive powers of the Tanh function. Introducing the new variable

$$y = \tanh\zeta, \tag{9}$$

solution(s) can be sought in the form

$$u(y) = \sum_{n=0}^{N} a_n y^n. \tag{10}$$

Chain differentiation yields

$$\begin{aligned} \frac{d}{d\zeta} &= \frac{d}{dy}\frac{dy}{d\zeta} = \operatorname{sech}^2\zeta\frac{d}{dy} \\ &\underset{(8)}{=} (1-\tanh^2\zeta)\frac{d}{dy} \underset{(9)}{=} (1-y^2)\frac{d}{dy}. \end{aligned} \tag{11}$$

The positive integer value of $N$ is determined after substituting expressions (10) and (11) into equation (7) and balancing the resulting highest-order terms.

Once $N$ is determined, substituting expression (10) in equation (7), one obtains an algebraic system for the coefficients $a_n$, $n = 0, 1, \dots, N$. Depending on the problem under consideration, $\mu$ is either determined or not, while $\lambda$ is always a function of $\mu$.

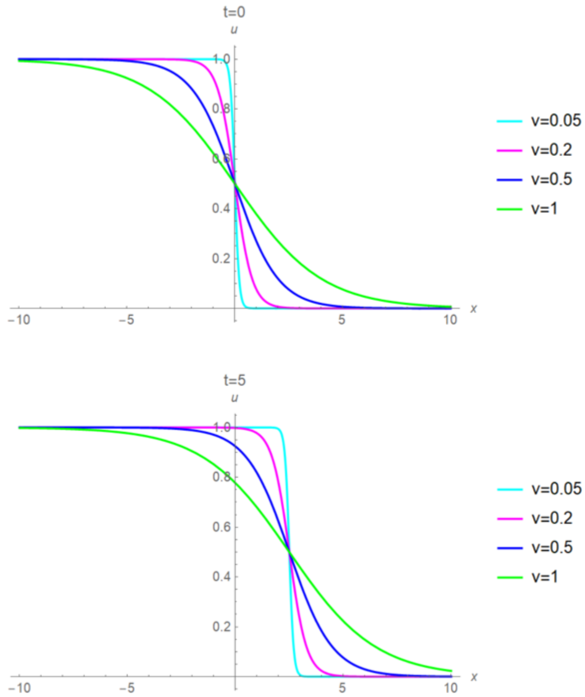


*Figure 2.* Different right-moving, kink-shaped solutions of the viscous Burgers equation, for $\nu \in \{0.05, 0.2, 0.5, 1\}$ and $\lambda = 0.5$, on the interval $x \in [-10, 10]$ and with $t \in \{0, 5\}$. Small viscosity effects lead to a steeper waveform, whereas larger viscosity effects lead to smoother and wider waveforms.

In the present case, $N$ is found to be equal to 1; hence substituting expression (10) in equation (7) and setting the coefficients of the like powers of $y$ equal to zero leads to the algebraic system

$$\begin{cases} \dfrac{a_1^2}{2} + a_1\nu\mu = 0, \\ a_0a_1 - a_1\lambda = 0, \\ \dfrac{a_0^2}{2} - a_0\lambda - a_1\nu\mu = 0, \end{cases}$$

the solution of which is

$$\begin{cases} \mu = \dfrac{\lambda}{2\nu}, \quad \lambda > 0, \\ a_0 = \lambda, \\ a_1 = -\lambda. \end{cases} \tag{12}$$

Combining (12) and (10) and using expression (9), one obtains

$$u(\zeta) = \lambda(1 - \operatorname{Tanh}\zeta),$$

and finally

$$u(x,t) = \lambda\Big[1 - \operatorname{Tanh}\Big(\frac{\lambda}{2\nu}(x-\lambda t)\Big)\Big].$$

Figure 2 displays right-moving analytical solutions of the viscous Burgers equation for different values of the kinematic viscosity $\nu$.

*References*


[1] L. C. Evans, *Partial differential equations*. Graduate Studies in Mathematics 19, American Mathematical Society, Providence (1998)

[2] W. Malfliet and W. Hereman, The tanh method. I. Exact solutions of nonlinear evolution and wave equations. *Phys. Scripta* **54**, 563–568 (1996)

[3] A.-M. Wazwaz, The tanh method for traveling wave solutions of nonlinear equations. *Appl. Math. Comput.* **154**, 713–723 (2004)

[4] M. A. Xenos, An Euler–Lagrange approach for studying blood flow in an aneurysmal geometry. *Proc. A.* **473**, Article ID 20160774 (2017)

[5] M. A. Xenos and A. C. Felias, Nonlinear dynamics of the KdV-B equation and its biomedical applications. In *Nonlinear Analysis, Differential Equations, and Applications*, Springer Optim. Appl. 173, Springer, Cham, 765–793 (2021)


*We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions to Michael Th. Rassias by email to mthrassias@yahoo.com.*

*We also solicit your new problems with their solutions for the next "Solved and unsolved problems" column, which will be devoted to* Probability Theory.