Publ. RIMS, Kyoto Univ.
10 (1974), 1-10

One-parameter Family of Radon-Nikodym Theorems for States of a von Neumann Algebra

By

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Abstract

It is shown that any normal state φ of a von Neumann algebra $\mathfrak{M}$ with a cyclic and separating vector Ψ satisfying $\varphi \leqq l\omega_{\Psi}$ for some $l>0$ has a representative vector Φ_{α} in V_{Ψ}^{α} for each $\alpha \in [0,1/2]$ and $\Phi_{\alpha}=Q_{\alpha}\Psi$ for a $Q_{\alpha} \in \mathfrak{M}$ satisfying $\|Q_{\alpha}\| \leqq l^{1/2}$ when $\alpha \in [0, 1/4]$.

§1. Main Theorem

Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ with a unit cyclic and separating vector Ψ. Let Δ_{Ψ} be the modular operator for $\mathfrak{M}$, Ψ. Let V_{Ψ}^{α} denote the closure of $\Delta_{\Psi}^{\alpha}\mathfrak{M}^{+}\Psi$ where $\mathfrak{M}^{+}$ denotes the positive operators in $\mathfrak{M}$ ([2], [6]).

Our main result is the following theorem:

Theorem 1. *For any normal state* φ *of* $\mathfrak{M}$ *such that* $\varphi \leqq l\omega_{\Psi}$ *for some* $l>0$, *there exists a vector* $\Phi_{\alpha} \in V_{\Psi}^{\alpha}$ *for every* $\alpha \in [0, 1/2]$ *such that* $\omega_{\Phi_{\alpha}}=\varphi$.

Combined with Theorem 3(8) of [2], Theorem 1 implies the following:

Theorem 2. *For any normal state* φ *of* $\mathfrak{M}$ *such that* $\varphi \leqq l\omega_{\Psi}$, *there exists a* $Q_{\alpha} \in \mathfrak{M}$ *for* $\alpha \in [0, 1/4]$ *such that* $\omega_{Q_{\alpha}\Psi}=\varphi$, $\|Q_{\alpha}\| \leqq l^{1/2}$.

Operators Q_{α}, such that $Q_{\alpha}\Psi \in V_{\Psi}^{\alpha}$, are characterized in Theorem 3(7) of [2] by the property that $\sigma_{t}^{\psi}(Q_{\alpha})$ has an analytic continuation

Received July 13, 1973.

$\sigma_z^\psi(Q_\alpha)\in\mathfrak{M}$ for $\operatorname{Im} z\in[0, 2\alpha]$ and $\sigma_{i\alpha}^\psi(Q_\alpha)\geqq 0$, where σ_t^ψ denotes the modular automorphisms of $\mathfrak{M}$ relative to Ψ.

The special case $\alpha=0$ gives the non-commutative Radon-Nikodym derivative of Sakai [7]. The case $\alpha=1/4$ gives the Radon-Nikodym derivative satisfying the chain rule [2].

§2. An Application of Carlson's Uniqueness Theorem

Let $\mathrm{f}(z)$ be holomorphic for $\operatorname{Re} z\geqq 0$ and of exponential type: $|\mathrm{f}(z)|\leqq M\mathrm{e}^{\tau|z|}$ for some $\tau>0$ and $M>0$. Let

$$\mathrm{h}(\theta)=\varlimsup_{r\to\infty} r^{-1}\log|\mathrm{f}(re^{i\theta})|, \qquad |\theta|\leqq\pi/2\,. \tag{2.1}$$

Carlson's theorem states that if $\mathrm{h}(\pi/2)+\mathrm{h}(-\pi/2)<2\pi$, then $\mathrm{f}(n)=0$ for $n=0, 1, 2,\ldots$ implies $\mathrm{f}(z)\equiv 0$. [5]

If $\tilde{\mathrm{f}}\in\mathscr{D}(R)$ (the set of C^∞-functions with a compact support) and

$$\mathrm{f}(t)=(2\pi)^{-1}\int\tilde{\mathrm{f}}(p)e^{-ipt}\mathrm{d}p\,, \tag{2.2}$$

$$Q(\mathrm{f})=\int\sigma_s^\psi(Q)\,\mathrm{f}(s)\mathrm{d}s\,, \tag{2.3}$$

then $\sigma_t^\psi(Q)$ has an analytic continuation to the $\mathfrak{M}$-valued entire function

$$\sigma_z^\psi(Q(\mathrm{f}))=Q(\mathrm{f}_z), \qquad \mathrm{f}_z(s)=\mathrm{f}(s-z)\,. \tag{2.4}$$

We have also

$$\sigma_{z_1}^\psi(\sigma_{z_2}^\psi[Q(\mathrm{f})])=\sigma_{z_1+z_2}^\psi(Q(\mathrm{f}))\,. \tag{2.5}$$

If $\operatorname{supp}\tilde{\mathrm{f}}\subset[-L, L]$, then

$$|\mathrm{f}(t+iz)|\leqq M_1 e^{L|\operatorname{Re} z|}(1+(t-\operatorname{Im} z)^2)^{-1}$$

for some $M_1>0$ ($M_1=2\max(\|\tilde{\mathrm{f}}\|_1, \|\tilde{\mathrm{f}}''\|_1)$ for example). Hence

$$\|\sigma_{iz}^\psi(Q(\mathrm{f}))\|\leqq M_2 e^{L|\operatorname{Re} z|}\|Q\| \tag{2.6}$$

for $M_2=M_1\pi$.

Let $\tilde{f} \in \mathscr{D}(R)$, $0 \leqq \tilde{f}(p) \leqq 1$ and $\tilde{f}(q)=1$ for $|q| \leqq 1$. Let

$$\tilde{f}_\lambda(p)=\tilde{f}(\lambda p), \qquad f_\lambda(t)=\lambda^{-1} f(t/\lambda).$$

Then by the strong continuity of $\sigma_t^\psi(Q)$ in t, we have

$$\lim_{\lambda \to +0} Q(f_\lambda)=Q. \tag{2.7}$$

Lemma 1. *Let S be an invertible positive self-adjoint operator such that*

$$(Q(f)x, Sy)=(\sigma_{i\alpha}^\psi(Q(f))Sx, y) \tag{2.8}$$

for all $\tilde{f} \in \mathscr{D}(R)$, $Q \in \mathfrak{M}$ and $x, y \in D_S$ where D_S is any core of S (namely $\overline{S|D_S}=S$). Then

$$(Q(f)x, e^{\bar{z}\log S}y)=(\sigma_{i\alpha z}^\psi(Q(f))e^{z\log S}x, y) \tag{2.9}$$

for all complex z, $\tilde{f} \in \mathscr{D}(R)$, $Q \in \mathfrak{M}$, x in the domain of $e^{z\log S}$ and y in the domain of $e^{\bar{z}\log S}$. For real t and $Q \in \mathfrak{M}$,

$$e^{it\log S}Qe^{-it\log S}=\sigma_{-\alpha t}^\psi(Q). \tag{2.10}$$

Proof. By a limiting procedure, (2.8) holds for all x and y in the domain of S. Let D_a be the set of all vectors which have compact supports relative to the spectrum of $\log S$. For any x and y in D_a, $e^{z\log S}x$ and $e^{z\log S}y$ are vector-valued entire functions of z and

$$\|e^{z\log S}x\| \leqq M_x e^{a|\mathrm{Re}\,z|}, \tag{2.11}$$

$$\|e^{\bar{z}\log S}y\| \leqq M_y e^{b|\mathrm{Re}\,z|}, \tag{2.12}$$

for some $a>0$, $b>0$, $M_x>0$ and $M_y>0$. From (2.11), (2.12) and (2.6), it follows that both sides of (2.9) are entire functions of exponential type with $h(\pi/2)=h(-\pi/2)=0$. If $x, y \in D_a$, then $e^{m\log S}x=S^m x \in D_a$ and $e^{m\log S}y=S^m y \in D_a$. Hence, by repeated use of (2.8), we have (2.9) for $z=0, 1, 2, \dots$. By Carlson's theorem, (2.9) holds for all z and $x, y \in D_a$. Since D_a is a core of $e^{\lambda\log S}$ for any real λ, and since $e^{i\lambda\log S}$ is bounded for real λ, (2.9) holds as stated in the Lemma.

By a limiting procedure like (2.7), we obtain (2.10) from (2.9). Q. E. D.

Lemma 2. *If a self-adjoint operator* $S \geqq 0$ *satisfies* (2.8) *for all* $\tilde{\mathrm{f}} \in \mathscr{D}(R)$, $Q \in \mathfrak{M}$ *and* $x, y \in D_S$, *then*

$$(2.13) \qquad (Q(\mathrm{f})x, S^{1/2}y) = (\sigma^{\psi}_{i\alpha/2}(Q(\mathrm{f}))S^{1/2}x, y)$$

for all $\tilde{\mathrm{f}} \in \mathscr{D}(R)$, $Q \in \mathfrak{M}$, x *and* y *in the domain of* $S^{1/2}$.

Proof. Let E be the projection onto the null space of S. By setting $x = (1-E)x'$ in (2.8) for arbitrary x', we obtain

$$(Q(\mathrm{f})(1-E)x', Sy) = 0\,.$$

Hence $EQ(\mathrm{f})(1-E) = 0$. Replacing Q by Q^*, f by f^* and taking the adjoint, we also have $(1-E)Q(\mathrm{f})E = 0$. Hence $EQ(\mathrm{f}) = EQ(\mathrm{f})E = Q(\mathrm{f})E$. Since the set of $Q(\mathrm{f})$ is dense in $\mathfrak{M}$ by (2.7), we have $E \in \mathfrak{M}'$.

Now the proof of Lemma 1 holds for $x, y \in E\mathfrak{H}$, and hence we have (2.9) whenever $\tilde{\mathrm{f}} \in \mathscr{D}(R)$, $Q \in \mathfrak{M}$, $x \in E\mathfrak{H}$, $y \in E\mathfrak{H}$, x is in the domain of $e^{z \log S}E$ and y in the domain of $e^{\bar{z} \log S}E$. Setting $z = 1/2$, and using $[E, Q(\mathrm{f})] = [E, \sigma^{\psi}_{i\alpha z}(Q(\mathrm{f}))] = 0$, we have (2.13). Q. E. D.

§3 Basic Lemmas

For any closable linear operator A with a dense domain, let $|A| = (A^*\bar{A})^{1/2}$ and $\mathrm{u}(A) = (|A|^{-1}A^*)^*$, where the bar denotes the closure. The operator $\mathrm{u}(A)$ is a partial isometry, whose kernel is the kernel of $|A|$, and $\bar{A} = \mathrm{u}(A)|A|$ is the polar decomposition of A.

Lemma 3 *Let* A_1 *and* B_1 *be closed linear operators affiliated with* $\mathfrak{M}$, A_2 *and* B_2 *be closed linear operators affiliated with* $\mathfrak{M}'$, *and* α *be a real number. Assume that either one of the following conditions holds*:

(1) $\alpha \in [0, 1/2]$, Ψ *is in the domains of* A_j, A_j^*, B_j *and* B_j^*, $j = 1, 2$.

(2) Ψ *is in the domains of* A_1, $\Delta_\Psi^\alpha A_1^*$, $\Delta_\Psi^\alpha B_1$, B_1^*, A_2, $\Delta_\Psi^{-\alpha} A_2^*$, $\Delta_\Psi^{-\alpha} B_2$ *and* B_2^*.

Then $A_1\Delta_\Psi^\alpha$, $\Delta_\Psi^\alpha B_1$, $A_2\Delta_\Psi^{-\alpha}$, $\Delta_\Psi^{-\alpha}B_2$ *are closable linear operators with dense domains.*

Proof. Let $\mathfrak{A}_{\Psi 1}$ and $\mathfrak{A}_{\Psi 2}$ be the $*$-algebras of all operators $Q_1 \in \mathfrak{M}$ and $Q_2 \in \mathfrak{M}'$, respectively, such that $\bar{\sigma}_t^\psi(Q_j) \equiv \Delta_\Psi^{it} Q \Delta_\Psi^{-it}$ have analytic continuations to entire functions $\bar{\sigma}_z^\psi(Q_j)$, $j=1, 2$. [2].

If $Q_2 \in \mathfrak{A}_{\Psi 2}$, then $Q_2\Psi$ is in the domains of $A_1\Delta_\Psi^\alpha$, $(A_1\Delta_\Psi^\alpha)^*$, $\Delta_\Psi^\alpha B_1$ and $(\Delta_\Psi^\alpha B_1)^*$:

$$(A_1\Delta_\Psi^\alpha)Q_2\Psi = \bar{\sigma}_{-i\alpha}^\psi(Q_2)A_1\Psi\,, \tag{3.1}$$

$$(A_1\Delta_\Psi^\alpha)^*Q_2\Psi = \bar{\sigma}_{-i\alpha}^\psi(Q_2)\Delta_\Psi^\alpha A_1^*\Psi\,, \tag{3.2}$$

$$(\Delta_\Psi^\alpha B_1)Q_2\Psi = \bar{\sigma}_{-i\alpha}^\psi(Q_2)\Delta_\Psi^\alpha B_1\Psi\,, \tag{3.3}$$

$$(\Delta_\Psi^\alpha B_1)^*Q_2\Psi = \bar{\sigma}_{-i\alpha}^\psi(Q_2)B_1^*\Psi\,, \tag{3.4}$$

where $A_1^*\Psi$ and $B_1\Psi$ are in the domain of Δ_Ψ^α for $\alpha \in [0, 1/2]$ due to

$$\Delta_\Psi^{1/2}A\Psi = J_\Psi A^*\Psi \tag{3.5}$$

for any A affiliated with $\mathfrak{M}$ and for Ψ in the domains of A and A^*, as can be easily proved by a polar decomposition of A and spectral resolution of $|A|$. Since $\mathfrak{A}_{\Psi 2}\Psi$ is dense, $A_1\Delta_\Psi^\alpha$ and $\Delta_\Psi^\alpha B_1$ are closable linear operators with dense domains.

Similarly, $\mathfrak{A}_{\Psi 1}\Psi$ is in the domains of $A_2\Delta_\Psi^{-\alpha}$, $(A_2\Delta_\Psi^{-\alpha})^*$, $\Delta_\Psi^{-\alpha}B_2$, $(\Delta_\Psi^{-\alpha} B_2)^*$ and hence $A_2\Delta_\Psi^{-\alpha}$ and $\Delta_\Psi^{-\alpha}B_2$ are closable linear operators with dense domains. Q.E.D.

Lemma 4. *Let* A_1 *and* B_1 *be closed linear operators affiliated with* $\mathfrak{M}$ *and* A_2 *and* B_2 *be closed linear operators affiliated with* $\mathfrak{M}'$, *such that* $A_1\Delta_\Psi^\alpha$, $\Delta_\Psi^\alpha B_1$, $A_2\Delta_\Psi^{-\alpha}$, $\Delta_\Psi^{-\alpha}B_2$ *are closable linear operators with dense domains. Then*

$$\mathrm{u}(A_1\Delta_\Psi^\alpha) \in \mathfrak{M}, \qquad \mathrm{u}(\Delta_\Psi^\alpha B_1) \in \mathfrak{M}\,, \tag{3.6}$$

$$\mathrm{u}(A_2\Delta_\Psi^{-\alpha}) \in \mathfrak{M}', \qquad \mathrm{u}(\Delta_\Psi^{-\alpha}B_2) \in \mathfrak{M}'\,. \tag{3.7}$$

Here α *is real.*

Proof. For $T=A_2\Delta_{\Psi}^{-\alpha}$ or $\Delta_{\Psi}^{-\alpha}B_2$, we have

$$Q(\mathrm{f})^*\bar{T}y=\bar{T}\sigma_{-i\alpha}^{\psi}(Q(\mathrm{f})^*)y$$

for y in the domain of T and hence for y in the domain of $\bar{T}$. Since $\sigma_{-i\alpha}^{\psi}(Q(\mathrm{f})^*)^*=\sigma_{i\alpha}^{\psi}(Q(\mathrm{f}))$, we have

$$(Q(\mathrm{f})x,\ \bar{T}y)=(\sigma_{i\alpha}^{\psi}(Q(\mathrm{f}))T^*x,\ y) \tag{3.8}$$

for all x in the domain of T^* and y in the domain of $\bar{T}$. We also have

$$(Q(\mathrm{f})y,\ T^*x)=(Q(\mathrm{f})^*T^*x,\ y)^*=(\sigma_{-i\alpha}^{\psi}(Q(\mathrm{f})^*)x,\ \bar{T}y)^*$$
$$=(\sigma_{i\alpha}^{\psi}(Q(\mathrm{f}))\bar{T}y,\ x)$$

for x in the domain of T^* and y in the domain of $\bar{T}$. Hence the positive self-adjoint operator $S=T^*\bar{T}$ satisfies

$$(Q(\mathrm{f})x,\ Sy)=(\sigma_{2i\alpha}^{\psi}(Q(\mathrm{f}))Sx,\ y) \tag{3.9}$$

for all x and y in the domain of S. (See (2.5).) By Lemma 2, $|T|=S^{1/2}$ satisfies

$$(Q(\mathrm{f})x,\ |T|y)=(\sigma_{i\alpha}^{\psi}(Q(\mathrm{f}))|T|x,\ y) \tag{3.10}$$

for all x and y in the domain of $|T|$. From (3.8) and (3.10), we have

$$(Q(\mathrm{f})x,\ \mathrm{u}(T)|T|y)=(\sigma_{i\alpha}^{\psi}(Q(\mathrm{f}))|T|\,|T|^{-1}T^*x,\ y)$$
$$=(Q(\mathrm{f})\mathrm{u}(T)^*x,\ |T|y)$$

for x in the domain of T^* and y in the domain of $|T|$.

Since $\mathrm{u}(T)^*\mathrm{u}(T)$ is the projection onto the closure of the range of $|T|$, we have

$$Q(\mathrm{f})^*\mathrm{u}(T)=\mathrm{u}(T)Q(\mathrm{f})^*\mathrm{u}(T)^*\mathrm{u}(T). \tag{3.11}$$

$1-\mathrm{u}(T)^*\mathrm{u}(T)$ is the projection onto the kernel of T and hence onto the kernel of $S=T^*T$. By the proof of Lemma 2, (3.9) implies that $[Q(\mathrm{f}),\ \mathrm{u}(T)^*\mathrm{u}(T)]=0$. Hence (3.11) implies

$$Q(\mathrm{f})^*\mathrm{u}(T)=\mathrm{u}(T)\mathrm{u}(T)^*\mathrm{u}(T)Q(\mathrm{f})=\mathrm{u}(T)Q(\mathrm{f}).$$

Hence $u(T)\in\mathfrak{M}'$.

A similar proof holds for $A_1\Delta_\Psi^\alpha$ and $\Delta_\Psi^\alpha B_1$, where $\mathfrak{M}$ is replaced by $\mathfrak{M}'$. Q.E.D.

Lemma 5. *The vectors*

$$Q(\mathrm{f})\sigma_{-2i\alpha}^{\psi}(Q(\mathrm{f})^*)\Psi=Q(\mathrm{f})\Delta_\Psi^{2\alpha}Q(\mathrm{f})^*\Psi \tag{3.12}$$

for $Q\in\mathfrak{M}$ *and* $\tilde{\mathrm{f}}\in\mathscr{D}(R)$ *are in* V_Ψ^α *and dense in* V_Ψ^α *for* $\alpha\in[0, 1/2]$. *The vectors*

$$Q'(\mathrm{f})\bar{\sigma}_{i-2i\alpha}^{\psi}(Q'(\mathrm{f})^*)=Q'(\mathrm{f})\Delta_\Psi^{2\alpha-1}Q'(\mathrm{f})^*\Psi \tag{3.13}$$

for $Q'\in\mathfrak{M}'$ *and* $\tilde{\mathrm{f}}\in\mathscr{D}(R)$ *are in* V_Ψ^α *and dense in* V_Ψ^α *for* $\alpha\in[0, 1/2]$.

Proof. Since $q=Q(\mathrm{f})\sigma_{-2i\alpha}^{\psi}(Q(\mathrm{f})^*)$ has an analytic continuation

$$\sigma_z^{\psi}(q)=\sigma_z^{\psi}[Q(\mathrm{f})](\sigma_{\bar{z}+2i\alpha}^{\psi}[Q(\mathrm{f})])^*$$

which is obviously positive for $z=i\alpha$, $q\Psi$ is in V_Ψ^α by Theorem 3 (7) of [2].

By definition, $\Delta_\Psi^\alpha Q^2\Psi$, $Q\in\mathfrak{M}^+$ is dense in V_Ψ^α. If $Q(\mathrm{f}_\lambda)^*=Q(\mathrm{f}_\lambda)$ are uniformly bounded and $Q(\mathrm{f}_\lambda)\to Q$ strongly, then $Q(\mathrm{f}_\lambda)^2\to Q^2$ strongly. Since $d(\alpha)=\|\Delta_\Psi^\alpha(Q(\mathrm{f}_\lambda)^2-Q^2)\Psi\|^2$ is convex in α and $d(0)=d(1/2)\to 0$, we have $\Delta_\Psi^\alpha Q(\mathrm{f}_\lambda)^2\Psi\to\Delta_\Psi^\alpha Q^2\Psi$. Hence the vectors

$$\Delta_\Psi^\alpha Q(\mathrm{g})^2\Psi=Q(\mathrm{f})\sigma_{-2i\alpha}^{\psi}(Q(\mathrm{f})^*)\Psi$$

for $Q=Q^*\in\mathfrak{M}$, $\tilde{\mathrm{g}}\in\mathscr{D}(R)$ and $\mathrm{g}^*=\mathrm{g}$ are dense in V_Ψ^α where $\mathrm{f}(t)=\mathrm{g}(t+i\alpha)$, $\sigma_{-i\alpha}^{\psi}(Q(\mathrm{g}))=Q(\mathrm{f})$. This completes the proof of the first half.

The second half is obtained from the first half by

$$J_\Psi Q(\mathrm{f})\sigma_{-2i\beta}^{\psi}(Q(\mathrm{f})^*)\Psi=Q'(\mathrm{f}^*)\bar{\sigma}_{2i\beta}^{\psi}(Q'(\mathrm{f}^*)^*)\Psi$$

for $Q'=J_\Psi QJ_\Psi\in\mathfrak{M}'$ $(Q=J_\Psi Q'J_\Psi)$ and $\beta=(1/2)-\alpha$, due to $J_\Psi V_\Psi^\beta=V_\Psi^\alpha$ (Theorem 3(4) of [2]). Q.E.D.

Lemma 6. *Let* A_1, A_2, B_1, B_2 *be as in Lemma* 3 *and* $\alpha\in[0, 1]$. *Then*

$$|A_1\Delta_\Psi^\alpha|\Psi\in V_\Psi^{\alpha/2},\qquad |\Delta_\Psi^\alpha B_1|\Psi\in V_\Psi^{\alpha/2}, \tag{3.14}$$

(3.15) $$|A_2\Delta_\Psi^{-\alpha}|\Psi\in V_\Psi^{(1-\alpha)/2}, \qquad |\Delta_\Psi^{-\alpha}B_2|\Psi\in V_\Psi^{(1-\alpha)/2}.$$

Proof. Since $\mathfrak{A}_{\Psi 1}\Psi$ is in the domain of T for $T=A_2\Delta_\Psi^{-\alpha}$ and for $T=\Delta_\Psi^{-\alpha}B_2$, it is also in the domain of $|T|$. For $Q\in\mathfrak{M}$ and $\tilde{\mathrm{f}}\in\mathscr{D}(R)$, we have

$$(Q(\mathrm{f})\sigma_{-i\alpha}^{\psi}(Q(\mathrm{f})^*)\Psi,\ |T|\Psi)=(\sigma_{i\alpha}^{\psi}(Q(\mathrm{f}))|T|\sigma_{-i\alpha}^{\psi}(Q(\mathrm{f})^*)\Psi,\ \Psi)$$

$$=(|T|\sigma_{-i\alpha}^{\psi}(Q(\mathrm{f})^*)\Psi,\ \sigma_{-i\alpha}^{\psi}(Q(\mathrm{f})^*)\Psi)\geqq 0.$$

by (3.10). Since $Q(\mathrm{f})\sigma_{-i\alpha}^{\psi}(Q(\mathrm{f})^*)\Psi$ are dense in $V_\Psi^{\alpha/2}$, we have $|T|\Psi\in V_\Psi^{(1-\alpha)/2}$ due to Theorem 3(5) of [2].

Similarly we have

$$(Q'(\mathrm{f})\bar{\sigma}_{i\alpha}^{\psi}(Q'(\mathrm{f})^*)\Psi,\ |T'|\Psi)$$

$$=(|T'|\bar{\sigma}_{i\alpha}^{\psi}(Q'(\mathrm{f})^*)\Psi,\ \bar{\sigma}_{i\alpha}^{\psi}(Q'(\mathrm{f})^*)\Psi\geqq 0$$

for $T'=A_1\Delta_\Psi^{\alpha}$ and for $T'=\Delta_\Psi^{\alpha}B_1$. Hence $|T'|\Psi\in(V_\Psi^{(1-\alpha)/2})'=V_\Psi^{\alpha/2}$.

Q.E.D.

§4. Proof of Theorem 1

A vector Φ is called a representative of a state φ if the vector state ω_Φ is φ.

Lemma 7. *If normal state φ has a representative vector in $V_\Psi^{1/2}$, then it has a representative vector Φ_α in V_Ψ^{α} for each $\alpha\in[1/4, 1/2]$.*

Proof. Let $\Phi\in V_\Psi^{1/2}$ and $\omega_\Phi=\varphi$. There exists a self-adjoint operator $A_2\geqq 0$ affiliated with $\mathfrak{M}'$ such that Ψ is in the domain of A_2 $(=A_2^*)$ and $\Phi=A_2\Psi$. [8]. By Lemma 3, $A_2\Delta_\Psi^{-\beta}$ is a closable linear operator with a dense domain for $0\leqq\beta\leqq 1/2$. Let

(4.1) $$\Phi_\alpha=|A_2\Delta_\Psi^{-\beta}|\Psi, \qquad \alpha=(1-\beta)/2\in[1/4, 1/2].$$

By Lemma 6, $\Phi_\alpha\in V_\Psi^{\alpha}$. By Lemma 4, $u\equiv\mathrm{u}(A_2\Delta_\Psi^{-\beta})\in\mathfrak{M}'$. Furthermore, $u^*u|A_2\Delta_\Psi^{-\beta}|=|A_2\Delta_\Psi^{-\beta}|$ and

$$u\Phi_\alpha=u|A_2\Delta_\Psi^{-\beta}|\Psi=A_2\Delta_\Psi^{-\beta}\Psi=A_2\Psi=\Phi.$$

Hence, as states of $\mathfrak{M}$, we have the following equalities:

$$\varphi = \omega_{\Phi} = \omega_{\Phi_\alpha} .$$

Q.E.D.

Lemma 8. *If* $Q' \in \mathfrak{M}'$ *and* $Q'\Psi \in V_{\Psi}^{1/4}$, *then* $(Q')^*\Psi$ *is in the domain of* Δ_{Ψ}^{-1}.

Proof. By Theorem 4(2) of [2],

$$Q'\Psi = J_{\Psi}Q'\Psi = \Delta_{\Psi}^{-1/2}(Q')^*\Psi .$$

Since $Q'\Psi$ is in the domain of $\Delta_{\Psi}^{-1/2}$ for any $Q' \in \mathfrak{M}'$, $(Q')^*\Psi$ must be in the domain of Δ_{Ψ}^{-1}. Q.E.D.

Proof of Theorem 1: It is well-known that $\varphi \leqq l\omega_{\Psi}$ implies the existence of $A_2 \in (\mathfrak{M}')^+$ such that $\Phi = A_2\Psi$ $(\in V_{\Psi}^{1/2})$ is a representative of φ. By Lemma 7, φ has a representative $\Phi_{1/4}$ in $V_{\Psi}^{1/4}$. (This is also obtained in Theorem 6 of [2].) By Theorem 3(8) of [2], there exists $Q \in \mathfrak{M}$ such that $\Phi_{1/4} = Q\Psi$. By Theorem 4(2) of [2], $\Phi_{1/4} = J_{\Psi}\Phi_{1/4} = (J_{\Psi}QJ_{\Psi})\Psi$. Set $Q' = J_{\Psi}QJ_{\Psi} \in \mathfrak{M}'$. By Lemma 8 $(Q')^*\Psi$ is in the domain of Δ_{Ψ}^{-1} and hence in the domain of $\Delta_{\Psi}^{-\beta}$ for any $\beta \in [0, 1]$. By Lemma 3, $Q'\Delta_{\Psi}^{-\beta}$ is a closable linear operator with a dense domain. Let

$$\Phi_\alpha = |Q'\Delta_{\Psi}^{-\beta}|\Psi, \qquad \alpha = (1-\beta)/2 \in [0, 1/2] .$$

By the same argument as the proof of Lemma 7, Φ_α is a representative vector of the state φ in V_{Ψ}^{α}. Q.E.D.

Remark. If $\varphi \leqq l\omega_{\Psi}$, then there exists $A_2 \in (\mathfrak{M}')^+$ such that $\|A_2\| \leqq l^{1/2}$ and $\omega_{\Phi} = \varphi$ for $\Phi = A_2\Psi$. For any $\alpha \in [0, 1/2]$, $\omega_{\Phi} = \omega_{\Phi_\alpha}$ implies the existence of a partial isometry $v_\alpha \in \mathfrak{M}'$ such that $\Phi_\alpha = v_\alpha\Phi = Q_\alpha\Psi$, $Q_\alpha = v_\alpha A_2$. Then $Q_\alpha \in \mathfrak{M}'$ and $\|Q_\alpha\| \leqq l^{1/2}$.

§5. Additional Remarks

The following Lemma is a variation of Lemma 6, which will be used in [4].

Lemma 9. *Let* $h\in\mathfrak{M}^{+}$. *Then*

$$|h^{1/n}\Delta_{\Psi}^{1/(2n)}|^{n}\Psi\in V_{\Psi}^{1/4}. \tag{5.1}$$

Proof. By setting $T=h^{1/n}\Delta_{\Psi}^{1/(2n)}$ and replacing $Q\in\mathfrak{M}$ by $Q'\in\mathfrak{M}'$ in the proof of Lemma 4, we obtain

$$(Q'(\mathrm{f})x,\ |T|y)=(\sigma_{-i/(2n)}^{\psi}(Q'(\mathrm{f}))|T|x,\ y)$$

for all x and y in the domain of $|T|$. (cf. (3.10).) By repeated use, we have

$$(Q'(\mathrm{f})x,\ |T|^{n}y)=(\sigma_{-i/2}^{\psi}(Q'(\mathrm{f}))|T|^{n}x,\ y)$$

for all x and y in the domain of $|T|^{n}$.

By replacing $|T|$ by $|T|^{n}$, $Q\in\mathfrak{M}$ by $Q'\in\mathfrak{M}'$, $\sigma_{-i\alpha}^{\psi}$ by $\bar{\sigma}_{i-i\alpha}^{\psi}$ and setting $\alpha=1/2$ in the proof of Lemma 6, we obtain

$$|T|^{n}\Psi\in(V_{\Psi}^{1/4})'=V_{\Psi}^{1/4}.$$

Q.E.D.

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