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Vanishing Theorems for Weakly 1-Complete Manifolds, II

(Dedicated to Professor Jōyō Kanitani for his 80th birthday)

By

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§1. Introduction

This note is a continuation of the previous paper [6] of the author. Here we shall prove the following theorem:

Theorem 1. *Let X be a complex manifold of dimension n, weakly 1-complete with respect to a plurisubharmonic function Ψ. If B is a positive line bundle on X, then we have*

$$H^q(X, \Omega^p(B))=0 \qquad \text{for} \quad p+q>n.$$

We shall also give a differential geometric proof of the following

Theorem 2. *If X is strongly 1-complete, then for any holomorphic vector bundle E on X, we have*

$$H^q(X, \mathcal{O}(E))=0 \qquad \textit{for} \quad q\geqq 1.$$

Since strongly 1-complete manifolds are nothing but Stein manifolds, this is a special case of the famous theorem A for Stein manifolds. Also it is a special case of a theorem of Kazama [3]. We present our proof because it is purely differential geometric.

§2. Proof of Theorem 1

The idea of the proof is completely similar to that given in [6].

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We shall make use of notations in [6], §2, unless we make explicit changes.

What we have to do is the following:

Given a B-valued (p, q)-form φ on X, find a complete Hermitean metric on X and metrics along fibres of B, such that B is $W^{p,q}$-elliptic and $(\varphi, \varphi)<\infty$.

As before, we start with a system of metrics $\{a_j\}$ along the fibres in the sense of [6] such that, for a local coordinate system (z_j^α) in U_j,

$$(1) \qquad (g_{j.\alpha\bar\beta})=\left(\frac{\partial^2 \log a_j}{\partial z_j^\alpha \partial \bar z_j^\beta}\right)>0\,,$$

and define ds^2 by

$$(2) \qquad ds^2=\sum g_{j\alpha\bar\beta}dz_j^\alpha d\bar z_j^\beta\,.$$

(Here we change $a_j^{(0)}$, $g_{j\alpha\bar\beta}^{(0)}$ and $ds_0{}^2$ in [6] into a_j, $g_{j\alpha\bar\beta}$ and ds^2 respectively. Accordingly we change notations as in the following formulas.) Then we choose a suitable function $\lambda(t)$ and set

$$(3) \qquad \begin{cases} A_j=e^{\lambda(\Psi)}a_j,\ \Gamma_{j,\alpha\bar\beta}=\dfrac{\partial^2 \log A_j}{\partial z_j^\alpha \partial \bar z_j^\beta}\,, \\ d\sigma^2=\sum \Gamma_{j\alpha\bar\beta}dz_j^\alpha d\bar z_j^\beta\,, \end{cases}$$

and we assert that the conditions are satisfied by $\{A_j\}$ and $d\sigma^2$.

As for the completeness of $d\sigma^2$ and $W^{p,q}$-ellipticity for $p+q>n$, it is all right as in [6]. Let us examine the condition $(\varphi, \varphi)<\infty$.

Set

$$(4) \qquad u(x)=\sum g_j^{\bar\beta\alpha}\frac{\partial\Psi}{\partial z_j^\alpha}\frac{\partial\Psi}{\partial \bar z_j^\beta}\,,$$

where $(g_j^{\bar\beta\alpha})$ denote the inverse of the matrix $G_j=(g_{j\alpha\bar\beta})$. $u(x)$ does not depend on the local coordinate system (z_j^α) and is a non-negative C^∞ function of $x\in X$. Next we take a matrix function T on U_j, such that $G_j={}^tT\cdot\bar T$, and consider ${}^tT^{-1}(\partial^2\Psi/\partial z_j^\alpha\partial\bar z_j^\beta)\bar T^{-1}$. The eigen values $v_1,\dots, v_n$ of this matrix do not depend on local coordinates. We denote by $v(x)$

the maximum of these eigen values at $x \in X$:

(5) $$v(x) = \max(v_1, \dots, v_n).$$

$v(x)$ is a non-negative continuous function on X.

For a given $\varphi = \{\varphi_j\} \in C^{p,q}(X, B)$, we express φ_j as

(6) $$\varphi_j = \sum \varphi_{j\alpha_1\dots\alpha_p\bar\beta_1\dots\bar\beta_q} dz_j^{\alpha_1} \wedge \cdots \wedge dz_j^{\alpha_p} \wedge d\bar z_j^{\beta_1} \wedge \cdots \wedge d\bar z_j^{\beta_q}.$$

We omit the suffix j indicating coordinate system in some places, and write down the integrand of (φ, φ) for two sets of metrics:

(7) $$\frac{1}{a_j}\varphi_j \wedge *\bar\varphi_j = K\cdot\frac{1}{a_j}\det(g_{\alpha\bar\beta})\{\sum g^{\bar\gamma_1\alpha_1}\dots g^{\bar\gamma_p\alpha_p} g^{\bar\beta_1\delta_1}\dots g^{\bar\beta_q\delta_q} \times \varphi_{j\alpha_1\dots\alpha_p\bar\beta_1\dots\bar\beta_q}\bar\varphi_{j\gamma_1\dots\gamma_p\bar\delta_1\dots\bar\delta_q}\} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar z^1 \wedge \cdots \wedge d\bar z^n,$$

(8) $$\frac{1}{A_j}\varphi_j \wedge \star\bar\varphi_j = K\cdot\frac{e^{-\lambda(\Psi)}}{a_j}\det(\Gamma_{\alpha\bar\beta})\{\sum \Gamma^{\bar\gamma_1\alpha_1}\dots \Gamma^{\bar\gamma_p\alpha_p}\Gamma^{\bar\beta_1\delta_1}\dots \Gamma^{\bar\beta_q\delta_q} \times \varphi_{j\alpha_1\dots\alpha_p\bar\beta_1\dots\bar\beta_q}\bar\varphi_{j\gamma_1\dots\gamma_p\bar\delta_1\dots\bar\delta_q}\} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar z^1 \wedge \cdots \wedge d\bar z^n.$$

Here $*$ and $\star$ indicate the formation of adjoint forms with respect to ds^2 and $d\sigma^2$ respectively and $(\Gamma^{\bar\alpha\beta})$ is the inverse of $(\Gamma_{\alpha\bar\beta})$. K is a constant common in two formulas.*)

We have $\Gamma = (\Gamma_{\alpha\bar\beta}) = G + W$ where $W \geqq 0$. From this we see that $\Gamma^{-1} \leqq G^{-1}$ as in [6], formula (2.13) and what follows, and hence the sum in { } in the formula (8) is not greater than that in the formula (7). Hence we have only to show that, by a choice of λ, we can achieve

(9) $$\int_X e^{-\lambda(\Psi)}\frac{\det(\Gamma_{\alpha\bar\beta})}{\det(g_{\alpha\bar\beta})} a_0[\varphi] dv < \infty,$$

where dv is the volume element in the metric ds^2 and the non-negative function $a_0[\varphi](x)$ for $x \in X$ is defined by

*) In [6], this constant and another to be multiplied to the expression for $\Lambda\eta$ were missing. It does not affect the main line of the proof.

$$\frac{1}{a_j}\varphi_j \wedge *\bar{\varphi}_j = a_0[\varphi]dv\,. \tag{10}$$

We can assume that $\inf_{x\in X}\Psi(x)=0$. Then we set

$$v(t)=\begin{cases}\sup_{x\in X_t}(a_0[\varphi](x)+u(x)+v(x)) & t>0\,,\\ 0\,, & t\leqq 0\,.\end{cases} \tag{11}$$

Then $v(t)$ is a non-decreasing function of $t\in \boldsymbol{R}$. We take a strictly increasing continuous function $v_1(t)$ such that $v_1(t)\geqq v(t)$. We also choose a non-decreasing continuous function $\rho(t)$ ($\geqq 0$) such that

$$\int_X e^{-\rho(\Psi(x))}dv<\infty\,. \tag{12}$$

Now we have the following lemma:

Lemma. *Given a real valued, continuous and strictly increasing function $\mu(t)$ in $0\leqq t<\infty$ with $\mu(0)=0$, $\mu(t)\to\infty$ for $t\to\infty$, we can find a C^∞ function $\lambda(t)$ in $-\infty<t<\infty$ such that*

$$\lambda'(t)\geqq 0,\ \lambda''(t)\geqq 0 \qquad \text{for all } t,$$

$$\lambda'(t)\geqq \mu(t) \qquad \textit{for } t\geqq c, \textit{ and}$$

$$\lambda'(t)\leqq K\cdot\lambda(t)^2 \textit{ and } \lambda''(t)\leqq K\cdot\lambda(t)^3 \qquad \textit{for } t\geqq c'\,,$$

with some constants c, c' and $K>0$.

Suppose this lemma has been proved. We apply it to $\mu(t)=v_1(t)+2\rho(t)$ and take $\lambda(t)$ as in the lemma. Let us estimate $\det\Gamma/\det G$ in the expression (9). We have

$$\Gamma_{\alpha\bar\beta}=g_{\alpha\bar\beta}+\lambda'(\Psi)\frac{\partial^2\Psi}{\partial z^\alpha\partial\bar z^\beta}+\lambda''(\Psi)\frac{\partial\Psi}{\partial z^\alpha}\,\frac{\partial\Psi}{\partial\bar z^\beta}\,.$$

Hence, if we choose T such that $G={}^tT\cdot\bar T$ and ${}^tT^{-1}(\partial^2\Psi/\partial z^\alpha\partial\bar z^\beta)\,\bar T^{-1}$ is diagonal, then we have

$$(\Gamma_{\alpha\bar\beta})={}^tT\left\{I_n+\lambda'(\Psi)\begin{pmatrix}v_1 & & 0\\ & \ddots & \\ 0 & & v_n\end{pmatrix}+y_\alpha y_{\bar\beta}\right\}\bar T\,,$$

where the column vector (y_α) denotes $\lambda''(\Psi)^{\frac{1}{2}}\,{}^tT^{-1}(\partial\Psi/\partial z^\alpha)$. Hence we have

$$\frac{\det\Gamma}{\det G}=\begin{vmatrix} 1+\lambda'(\Psi)v_1+y_1\bar{y}_1 & y\bar{y}_{12}\cdots\cdots\cdots & y_1\bar{y}_n \\ y_2\bar{y}_1 & 1+\lambda'(\Psi)v_2+y_2\bar{y}_2 & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ y_n\bar{y}_1 & \cdots\cdots & 1+\lambda'(\Psi)v_n+y_n\bar{y}_n \end{vmatrix}$$

$$=\prod_{\alpha=1}^{n}(1+\lambda'(\Psi)v_\alpha)+\sum_{\alpha=1}^{n}\prod_{\beta\neq\alpha}(1+\lambda'(\Psi)v_\beta)\cdot|y_\beta|^2$$

$$\leqq(1+\lambda'(\Psi))v^n+(1+\lambda'(\Psi)v)^{n-1}\lambda''(\Psi)u\,.$$

We see that $\det\Gamma/\det G$ grows, as $\Psi\to\infty$, at most with the order of a polynomial in $\lambda(\Psi)$. $a_0[\varphi]$ does not grow more quickly than $\lambda(\Psi)$, and the growth of the volume of X_t (as $t\to\infty$) has been taken care of by $\rho(t)$. Hence we see that the condition (9) is achieved by our choice of λ.

§3. Proof of the Lemma

Step I. Suppose $x=\mu_1(t)$ is a real valued continuous function of $t\in[0,\infty)$, such that $\mu_1(0)=0$, μ_1 is strictly increasing and $\to\infty$ for $t\to\infty$. Then we assert there is a continuous, strictly increasing function $\mu_2(t)$ in $[0,\infty)$ such that, for $t>0$ μ_2 is of class C^1 and $\mu_1(t)<\mu_2(t)$, and $\mu_2'(t)\leqq K\cdot\mu_2(t)^2$ for $t>c_2$, where c_2 and K are suitable constants.

To see this, we consider the inverse function $t=f(x)$ of μ_1. f is continuous and strictly increasing in $[0,\infty)$, and $f(x)\to\infty$ for $x\to\infty$. We set

$$g(x)=\begin{cases} 0 & \text{for } x=0, \\ (1/x)\displaystyle\int_0^x f(z)dz & \text{for } x>0, \end{cases}$$

then $g(x)$ has the same properties as listed for f. Moreover for $x>0$ we have

$$g(x)<f(x),$$

$$g \text{ is } C^1 \quad \text{and} \quad g'(x)=\frac{xf(x)-\int_0^x f(z)\,dz}{x^2}>0\,.$$

Consider the inverse function $x=\mu_2(t)$ of $t=g(x)$. Then μ_2 is again continuous, strictly increasing in $[0, \infty)$, $\mu_2(t)>\mu_1(t)$ for $t>0$ and μ_2 is C^1 for $t>0$. We have

$$\mu_2'(t)=(g'(x))^{-1}=\frac{\mu_2(t)^2}{xf(x)-\int_0^x f(z)\,dz}\,,$$

where $x=\mu_2(t)$. We set

$$h(x)=xf(x)-\int_0^x f(z)dz\,,$$

then for $x>y>0$

$$h(x)-h(y)=xf(x)-yf(y)-\int_y^x f(z)dz$$

$$=(x-y)f(x)-\int_y^x f(z)dz+y(f(x)-f(y))>0\,.$$

Hence we can find $c_1>0$ and $K>0$ such that

$$h(x)\geqq 1/K \qquad \text{for} \quad x>c_1\,,$$

and we see

$$\mu_2'(t)\leqq K\cdot\mu_2(t)^2 \qquad \text{for} \quad t>c_2 \quad (=g(c_1))\,.$$

We note that μ_2 is of class C^∞ in $(0, \infty)$ if μ_1 is so.

Step II. We set

$$\mu_3(t)=K\cdot\int_0^t \mu_2(\tau)^2 d\tau+L\,,$$

where K is as above and L is to be chosen suitably. Then we have

$$\mu_3'(t)=K\cdot\mu_2(t)^2>0 \qquad \text{for} \quad t>0$$

and

$$\mu_3'(t) > \mu_2'(t) \qquad \text{for} \quad t > c_2 .$$

Hence we can achieve, by a suitable choice of L, that

$$\mu_2(t) \leqq \mu_3(t) \qquad \text{for} \quad t \geqq 0 .$$

Then we see

$$\mu_3'(t) = K \cdot \mu_2(t)^2 \leqq K \cdot \mu_3(t)^2 ,$$

$$\mu_3''(t) = 2K\mu_2(t)\mu(t)_2' \leqq 2K^2\mu_3(t)^3 \qquad \text{for} \quad t > c_2 .$$

Step III. As $\mu_1(t)$ in Step I, we take a function which is C^∞ in $(0, \infty)$ and $\geqq \mu$. In order to obtain the desired function $\lambda(t)$, we take a non-decreasing C^∞ function $\mu_4(t)$ on the whole line $-\infty < t < \infty$ such that

$$\mu_4(t) = \begin{cases} 0 & \text{for} \quad t < 1 , \\ 1 & \text{for} \quad t > 2 , \end{cases}$$

and set

$$\lambda(t) = \mu_3(t) \cdot \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} \mu_4(\sigma) d\sigma \right) d\tau .$$

It is easy to see that this $\lambda(t)$ fills the need of the lemma.

§4. Strongly 1-Complete Manifold

X is strongly 1-complete if there exists a strictly plurisubharmonic function Ψ which exhausts X. It is well known that such X is a Stein manifold.

Proposition 1. *On a strongly* 1*-complete manifold* X, *every holomorphic vector bundle is positive in the strong sense.*

Proof. Ψ will denote the exhaustion function. For a holomorphic vector bundle E, we take Hermitean metrics $\{h_j\}$ along fibres of E.

We form the curvature form of $\{h_j\}$ and form $H=(H_{\bar{\nu}\bar{\beta},\mu\alpha})$ given by the formula (2.15) of [6]. Since $(v_{\alpha\bar{\beta}})=(\partial^2\Psi/\partial z^\alpha\partial\bar{z}^\beta)$ is positive definite at every point of X, we can find a C^∞ increasing convex function $\lambda(t)$ of $t\in\boldsymbol{R}$ such that

$$H+\lambda'(\Psi)(h_{\bar{\nu}\mu}v_{\alpha\bar{\beta}})>0$$

at every point of X.

When we replace h_j by $e^{-\lambda(\Psi)}h_j$, then H is replaced by $H+\lambda'(\Psi)(h_{\bar{\nu}\mu}v_{\alpha\bar{\beta}})+\lambda''(\Psi)\left(h_{\bar{\nu}\mu}\frac{\partial\Psi}{\partial z^\alpha}\frac{\partial\Psi}{\partial\bar{z}^\beta}\right)$, and we see that E is positive.

Now we quote standard arguments in the discussion of vector bundle valued cohomologies: If E is a holomorphic bundle of vector spaces $\boldsymbol{C}^r$ over a complex manifold M, we denote the associated $\boldsymbol{P}^{r-1}$-bundle by $P(E)$, i.e. $P(E)=\{E-(0\text{-section})\}/\boldsymbol{C}^*$. $L(E)$ denotes the complex line bundle over $P(E)$, associated to the principal bundle $E-(0)\rightarrow P(E)$. It is well known that

(A) $H^q(M, \mathcal{O}(W\otimes S^kE^*))\cong H^q(P(E), \mathcal{O}(\pi^*W\otimes L(E)^{-k}))$,

where S^kE^* denotes the k-ple symmetric product of the dual bundle E^* of E, W denotes any complex line bundle over M, and π means the projection $P(E)\rightarrow M$.

(B) Canonical bundles K_M and $K_{P(E)}$ of M and $P(E)$ are in the relation

$$K_{P(E)}=L(E)^r\pi^*(K_X\det E^*)\,.$$

(C) We understand positivity and semi-positivity of a holomorphic vector bundle in terms of the curvature, say in the sense of Griffiths [2]. Then,

$$L(E^*)^{-1} \text{ is positive if } E\rightarrow M \text{ is positive.}$$

([4] Theorem 2.1 and Proposition 2.2, or [2], (1.9), (2.36) and (2.38). Compactness of M is not necessary in these arguments.)

On the other hand, as was first pointed out to the author by Hironaka,

(D) If $\pi: Y \to X$ is a proper holomorphic map and if X is weakly 1-complete with respect to Ψ, then so is Y with respect to $\pi^*\Psi$.

Combining these facts with our Theorem 1, we obtain the counterpart of [4], Theorem 2.3 for weakly 1-complete manifolds. In particular, Corollary 2.4 becomes

Proposition 2. *Let X be a weakly 1-complete manifold, E a holomorphic vector bundle over X and F a complex line bundle over X. If either $E>0$ and $K_X \cdot \det E \cdot F^{-1} \leqq 0$ or $E \geqq 0$ and $K_X \cdot \det E \cdot F^{-1} < 0$, then we have $H^q(X, \mathcal{O}(S^k E \otimes F)) = 0$ for $q \geqq 1$.*

If X is strongly 1-complete, E and $(K_X \det E)^{-1}$ are positive by Proposition 1 for every E. Hence we conclude:

Theorem 2. *If X is strongly 1-complete, then for any holomorphic vector bundle E on X, we have*

$$H^q(X, \mathcal{O}(E)) = 0 \qquad \text{for} \quad q \geqq 1 .$$

It is true that the argument leading to [5], Theorem 1 (with correction in [1]) and a result of H. Kazama give our Theorem 2, directly from Proposition 1. Kazama makes use of an approximation theorem which is typical in the theory of functions of several complex variables. The present proof intends to avoid the direct use of this method. The author does not know if the definition of positivity due to Kobayashi and Ochiai has a nice function theoretic characterization in case of noncompact weakly 1-complete manifold. This is the reason why he adopted the definition of positivity in terms of curvatures.

A. Fujiki has pointed out that the argument of J. Le Potier (Comptes Rendus Acad. Sc. Paris, 276 (1973) Ser. A pp. 535–537) and ours, combined together, will give the counterpart of Potier's Theorem 1 for weakly 1-complete base manifold. My thanks are due to him for pointing out this and for calling my attention to that our $v(t)$ may not be continuous.

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