Publ. RIMS, Kyoto Univ.
35 (1999), 315-319

Characterization of the Pull-Back of $\mathscr{D}$-Modules

By

Takayuki TAKAHASHI*

§1. Introduction

Let f: $X \to Y$ be a smooth morphism of smooth algebraic varieties X and Y over $\mathbb{C}$. If a coherent $\mathscr{D}_X$-module $\mathscr{M}$ is the pull-back of a coherent $\mathscr{D}_Y$-module, $\mathscr{M}$ satisfies $\mathrm{Ch}(\mathscr{M}) \subset X \times_Y T^*Y$. Conversely let $\mathscr{M}$ be an algebraic $\mathscr{D}_X$-module such that $\mathrm{Ch}(\mathscr{M}) \subset X \times_Y T^*Y$. It is a natural question to ask when such an $\mathscr{M}$ is the pull-back of a $\mathscr{D}_Y$-module. In this paper we will prove that the condition $\mathrm{Ch}(\mathscr{M}) \subset X \times_Y T^*Y$ implies that $\mathscr{M}$ is always the pull-back of a coherent $\mathscr{D}_Y$-module globally on X if f is a proper smooth morphism with simply connected fiber (Theorem 3.1).

A local result below is known in the analytic category. We denote by X_{an} the associated complex manifold of X and by f_{an}: $X_{an} \to Y_{an}$ the associated holomorphic map. We define $\mathscr{M}_{an}$ by $\mathscr{D}_{X_{an}} \otimes_{\mathscr{D}_X} \mathscr{M}$. Let $\mathscr{I}$ be the ideal of the functions on T^*X that vanish on $X \times_Y T^*Y$. The necessary and sufficient condition for $\mathscr{M}_{an}$ to be isomorphic to $(\mathbb{D}f^*\mathscr{N})_{an}$ locally on X_{an} for a $\mathscr{D}_Y$-module $\mathscr{N}$ is that $\mathscr{M}$ has a good filtration F such that, $\mathscr{I}\mathrm{Gr}^F\mathscr{M}=0$ (in such a case we say that $\mathscr{M}$ is regular singular along $X \times_Y T^*Y$). Note that the condition $\mathrm{Ch}(\mathscr{M}) \subset X \times_Y T^*Y$ does not imply the regular singular condition. In the case of $\mathscr{D}^{\infty}_{X_{an}}$-module, such an $\mathscr{N}$ always exists under the condition $\mathrm{Ch}(\mathscr{M}) \subset X \times_Y T^*Y$ (see Theorem 3.2).

§2. Notations

Let X be a smooth algebraic variety over $\mathbb{C}$. We denote by $\mathscr{D}_X$ the sheaf of rings of algebraic differential operators on X. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, we denote by $\mathrm{Ch}\mathscr{M}$ its characteristics variety. Let Y be another smooth algebraic variety over $\mathbb{C}$, and f a morphism $f: X \to Y$.

We define $\mathscr{D}_{X\to Y}$, $\mathscr{D}_{Y\leftarrow X}$ as follows:

Communicated by M. Kashiwara, January 5, 1999.
1991 Mathematics Subject Classification(s): 32C38, 16S32.
*Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502 Japan.
e-mail: takayuki@kurims.kyoto-u.ac.jp

$$\mathscr{D}_{X\to Y} \overset{\text{def}}{=} \mathscr{H}^{d_Y}_{\Delta_f}(\mathscr{O}^{(0,d_Y)}_{X\times Y}) \text{ and } \mathscr{D}_{Y\leftarrow X} \overset{\text{def}}{=} \mathscr{H}^{d_Y}_{\Delta_f}(\mathscr{O}^{(d_X,0)}_{X\times Y})$$

(where Δ_f is the image of $X \hookrightarrow X \times Y$).

We define the direct image $\mathbb{D}f_*\mathscr{M}$ of a $\mathscr{D}_X$-module $\mathscr{M}$ by f, and the pull-back $\mathbb{D}f^*\mathscr{N}$ of a $\mathscr{D}_Y$-module $\mathscr{N}$ by f as follows:

$$\mathbb{D}f_*\mathscr{M} \overset{\text{def}}{=} \mathbf{R}f_*(\mathscr{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}), \qquad \mathbb{D}f^*\mathscr{N} \overset{\text{def}}{=} \mathscr{D}_{X\to Y} \overset{\mathbf{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{N}.$$

We denote their k-th cohomology groups by:

$$D^k f_*\mathscr{M} \overset{\text{def}}{=} H^k\mathbb{D}f_*\mathscr{M}, \qquad D^k f^*\mathscr{N} \overset{\text{def}}{=} H^k\mathbb{D}f^*\mathscr{N}.$$

We denote by $\mathscr{D}_{X_{an}}$ the sheaf of rings of analytic differential operators on the associated complex manifold X_{an}, and by $\mathscr{D}^\infty_{X_{an}}$ the sheaf of rings of infinite-order analytic differential operators on X_{an}.

Let $f: X \to Y$ be a morphism of algebraic varieties. We define $\mathscr{D}^\infty_{X_{an}\to Y_{an}}$, $\mathscr{D}^\infty_{Y_{an}\leftarrow X_{an}}$ as follows:

$$\mathscr{D}^\infty_{X_{an}\to Y_{an}} \overset{\text{def}}{=} \mathscr{H}^{d_Y}_{\Delta_f^{an}}(\mathscr{O}^{(0,d_Y)}_{X_{an}\times Y_{an}}) \text{ and } \mathscr{D}^\infty_{Y_{an}\leftarrow X_{an}} \overset{\text{def}}{=} \mathscr{H}^{d_Y}_{\Delta_f^{an}}(\mathscr{O}^{(d_X,0)}_{X_{an}\times Y_{an}})$$

(where Δ_f^{an} is the image of $X_{an} \hookrightarrow X_{an} \times Y_{an}$).

For a $\mathscr{D}_X$-module $\mathscr{M}$, We define

$$\mathscr{M}^\infty \overset{\text{def}}{=} \mathscr{D}^\infty_{X_{an}} \otimes_{\mathscr{D}_X} \mathscr{M}.$$

We have the following properties ([SKK]).

1. $\mathscr{D}^\infty_{X_{an}}$ is faithfully flat over $\mathscr{D}_X$.
2. If a coherent $\mathscr{D}_Y$-module $\mathscr{N}$ is non-characteristic for f, then we have

$$(\mathscr{D}_{X\to Y} \otimes_{\mathscr{D}_Y} \mathscr{N})^\infty \xrightarrow{\sim} \mathscr{D}^\infty_{X_{an}\to Y_{an}} \otimes_{\mathscr{D}^\infty_{Y_{an}}} \mathscr{N}^\infty.$$

   In particular, if f is smooth then this holds for any coherent $\mathscr{D}_Y$-module $\mathscr{N}$.
3. If $f: X \to Y$ is proper and if $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module, then we have

$$\mathscr{D}^\infty_{Y_{an}} \otimes_{\mathscr{D}_Y} \mathbf{R}f_*(\mathscr{D}_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathscr{D}_X} \mathscr{M}) \cong \mathbf{R}f_{an*}(\mathscr{D}_{X_{an}} \otimes^{\mathbf{L}}_{\mathscr{D}^\infty_{X_{an}}} (\mathscr{D}^\infty_{X_{an}} \otimes_{\mathscr{D}_X} \mathscr{M})).$$

In particular, for coherent $\mathscr{D}_X$-modules $\mathscr{M}$ and $\mathscr{M}'$, we have

$$\mathscr{M}^\infty \cong \mathscr{M}'^\infty \Rightarrow (\mathbb{D}f_*\mathscr{M})^\infty \cong (\mathbb{D}f_*\mathscr{M}')^\infty.$$

§3. Main Theorem

Theorem 3.1. *Let X and Y be smooth algebraic varieties over $\mathbb{C}$, and let f: $X \to Y$ be a proper smooth morphism of fiber dimension d. Assume that any fiber of*

f_{an}: $X_{an} \to Y_{an}$ is simpley connected. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module such that $\mathrm{Ch}(\mathscr{M}) \subset X \times_Y T^*Y$. *Then $\mathscr{M} \cong \mathbb{D}f^*\mathscr{N}$ for some coherent $\mathscr{D}_Y$-module $\mathscr{N}$.*

To prove this theorem we use the following results.

Theorem 3.2 ([SKK]). *Let s: $Y \to X$ be a section of f: $X \to Y$, and $\mathscr{M}$ a coherent $\mathscr{D}_X$-module. If* $\mathrm{Ch}(\mathscr{M}) \subset X \times_Y T^*Y$, *then*

$$\mathscr{M}^\infty \cong (\mathbf{D}f^*\mathbf{D}s^*)^\infty.$$

in a neighborhood of $s(Y)$.

Theorem 3.3 ([SKK]). *Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be coherent $\mathscr{D}_X$-modules such that* $\mathrm{Ch}\mathscr{M}_i \subset X \times_Y T^*Y$ $(i=1,2)$, *then $\mathscr{H}om_{\mathscr{D}^\infty_{X_{an}}}(\mathscr{M}_1^\infty, \mathscr{M}_2^\infty)$ is locally constant along the fiber of f.*

Lemma 3.4. *Under the same assumption as in Theorem* 3.1, *there exists locally on Y a coherent $\mathscr{D}_Y$-module $\mathscr{N}$ such that $\mathscr{M}^\infty \cong (\mathbb{D}f^*\mathscr{N})^\infty$.*

Proof. There exists a section $s: Y \to X$ locally on Y. Since the fiber of f is simply connected, the sheaves $\mathscr{H}om_{\mathscr{D}^\infty_{X_{an}}}(\mathscr{M}^\infty, (\mathbb{D}f^*\mathbb{D}s^*\mathscr{M})^\infty)$ and $\mathscr{H}om_{\mathscr{D}^\infty_{X_{an}}}((\mathbb{D}f^*\mathbb{D}s^*\mathscr{M})^\infty, \mathscr{M}^\infty)$ are constant along the fiber of f. Therefore the isomorphism $\mathscr{M}^\infty \xrightarrow{\sim} (\mathbb{D}f^*\mathbb{D}s^*\mathscr{M})^\infty$ on $s(Y)$ and its inverse extend to the whole X.

§4. Proof of Main Theorem

Because the fiber of $f: X \to Y$ is simply connected, we have

$$D^k f_*(\mathscr{O}_X) \cong \mathscr{O}_Y \otimes R^{k+d} f_*(\mathbb{C}_X) \cong \begin{cases} \mathscr{O}_Y & \text{for } k=-d, \\ 0 & \text{for } k=1-d. \end{cases}$$

For any coherent $\mathscr{D}_Y$-module $\mathscr{N}$, $\mathbb{D}f_*\mathbb{D}f^*\mathscr{N}$ is isomorphic to $\mathscr{N} \otimes^{\mathbf{L}} \mathbb{D}f_*(\mathscr{O}_X)$. Hence we have

$$D^k f_* \mathbb{D}f^*\mathscr{N} \cong \begin{cases} \mathscr{N} & \text{for } k=-d, \\ 0 & \text{for } k=1-d. \end{cases} \tag{4.1}$$

We shall prove Main Theorem (Theorem 3.1). First we shall prove Main Theorem in the following special case.

Lemma 4.1. *Under the same assumption as in Theorem* 3.1, *assume further $D^{-d}f_*\mathscr{M}=0$. Then $\mathscr{M}=0$.*

Proof. From Lemma 3.4, there exists a coherent $\mathscr{D}_Y$-module $\mathscr{N}$ such that $\mathscr{M}^\infty \cong (\mathbb{D}f^*\mathscr{N})^\infty$. Hence we have

$$(\mathbb{D}f_*\mathscr{M})^\infty \cong \mathbb{D}f_{an*}(\mathscr{M}^\infty) \cong \mathbb{D}f_{an*}((\mathbb{D}f^*\mathscr{N})^\infty) \cong (\mathbb{D}f_*\mathbb{D}f^*\mathscr{N})^\infty.$$

By taking the $-d$-th cohomology, we have

$$(D^{-d}f_*\mathcal{M})^\infty \cong (D^{-d}f_*\mathbb{D}f^*\mathcal{N})^\infty,$$

where the left-hand side is 0 by assumption, and the right-hand side is isomorphic to $\mathcal{N}^\infty$ by (4.1). Hence we have

$$\mathcal{N}^\infty = 0,$$

and the faithfully flatness of $\mathcal{D}^\infty_{Y_{an}}$ over $\mathcal{D}_Y$ implies

$$\mathcal{N} = 0.$$

Therefore $\mathcal{M}^\infty \cong (\mathbb{D}f^*\mathcal{N})^\infty = 0$. Finally the faithfully flatness of $\mathcal{D}^\infty_{X_{an}}$ over $\mathcal{D}_X$ implies $\mathcal{M}=0$. □

Let us prove Main Theorem in the general case. For any coherent $\mathcal{D}_Y$-module $\mathcal{N}$, we have

$$\mathbb{R}\mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{N}, \mathbb{D}f_*\mathcal{M}[-d]) \cong \mathbb{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N}, \mathcal{M}).$$

By taking the 0-th cohomology, we have

$$\mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{N}, D^{-d}f_*\mathcal{M}) \cong \mathrm{Hom}_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N}, \mathcal{M}).$$

Hence, by setting $\mathcal{N}=D^{-d}f_*\mathcal{M}$, there exists a morphism

$$\alpha : \mathbb{D}f^*\mathcal{N} \to \mathcal{M}$$

such that the composition

$$\mathcal{N} \xrightarrow{\sim} D^{-d}f_*\mathbb{D}f^*\mathcal{N} \xrightarrow{D^{-d}f_*\alpha} D^{-d}f_*\mathcal{M}$$

coincides with the isomorphism $\mathcal{N} \xrightarrow{\sim} D^{-d}f_*\mathcal{M}$. From the short exact sequence

$$0 \to \mathcal{L} \to \mathbb{D}f^*\mathcal{N} \to \mathrm{Im}(\alpha) \to 0, \tag{4.2}$$

where $\mathcal{L}$ is the kernel of α, we obtain the exact sequence

$$0 \to D^{-d}f_*\mathcal{L} \to D^{-d}f_*\mathbb{D}f^*\mathcal{N} \xrightarrow{\beta} D^{-d}f_*(\mathrm{Im}\ \alpha).$$

Because the composition of

$$D^{-d}f_*\mathbb{D}f^*\mathcal{N} \xrightarrow{\beta} D^{-d}f_*(\mathrm{Im}\alpha) \to D^{-d}f_*\mathcal{M}$$

is an isomorphism, β is injective. Therefore $D^{-d}f_*\mathcal{L}=0$. Because $\mathrm{Ch}(\mathcal{L}) \subset X \times_Y T^*Y$ by the exact sequence (4.2), Lemma 4.1 shows $\mathcal{L}=0$, and hence α is injective. So we have a short exact sequence

$$0 \to \mathbb{D}f^*\mathcal{N} \xrightarrow{\alpha} \mathcal{M} \to \mathcal{M}' \to 0,$$

where $\mathcal{M}'$ is the cokernel of α. The associated long exact sequence gives

$$0 \longrightarrow D^{-d}f_*\mathbb{D}f^*\mathcal{N} \xrightarrow{\sim} D^{-d}f_*\mathcal{M} \longrightarrow D^{-d}f_*\mathcal{M}' \longrightarrow D^{1-d}f_*\mathbb{D}f^*\mathcal{N}.$$

Because $D^{1-d}f_*\mathbb{D}f^*\mathcal{N}=0$ by (4.1), $D^{-d}f_*\mathcal{M}'$ must vanish. Lemma 4.1 shows $\mathcal{M}'=0$. Hence

$$\alpha : \mathbb{D}f^*\mathcal{N} \longrightarrow \mathcal{M}$$

is an isomorphism. This completes the proof of Main Theorem.

References

[SKK] Sato, M., Kawai, T. and Kashiwara, M., Microfunctions and pseudo-differential equations, Hyperfunctions and pseudo-differential equations, *Lecture Notes in Math.*, Springer-Verlag, **287** (1973) 265-529.