# Why should mathematicians ask politicians to avoid the ancient dilemma of pure and applied mathematics?

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*Due to the commercialization of science and technology, there is evidence of politicians paying attention mainly to research with immediate applications and benefits. Furthermore, unrealistic requests are made regarding the direct applicability of results in mathematics, a circumstance that may negatively affect all basic and theoretical studies. This paper not only shows that the distinction between pure and applied mathematics is historically and practically unnecessary and unhelpful, but also emphasizes the romantic idea that pure and applied mathematicians should constitute a community and work together to change politicians' minds.*

The dilemma of "pure versus applied mathematics" dates back to ancient Greece, where the first studies the world of ideas and the second investigates the world of the senses.

Pure mathematics can be regarded as the study of abstract concepts independent of any application to the physical world; see [3]. This description goes back to Plato's metaphysical view of mathematics as the study of ideas or eternal unchanging abstract forms; cf. [4]. Although many of these concepts originate from the real world, the applicability of the results is not the primary concern of pure mathematicians. Mathematicians intend to show the truth of mathematical propositions. In fact, inventing or discovering mathematical structures, generalizing notions, solving important mathematical problems, and briefly seeking "beauty" motivates pure mathematicians to deepen, strengthen, and expand existing mathematics. Hardy [9, pp. 84–85] asserted that "a mathematician, like a painter or a poet, is a maker of patterns" and added: "The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics."

Pythagoras stated that "mathematics is the way to understand the universe." It is mentioned in [10] that "someone who had begun to read geometry with Euclid, when he had learned the first theorem, asked Euclid, 'what shall I get by learning these things?' Euclid called his slave and said, 'Give him a coin since he must profit by what he learns.'" None of the philosophical schools of mathematics – platonism, formalism, logicism, intuitionism – provide a completely convincing and comprehensive explanation of why mathematical structures do describe the real world. Recently, Max Tegmark [19], a mathematician-physicist, offered a different explanation. He believes that the universe itself is an abstract mathematical structure. He introduced the Computable Universe Hypothesis, stating that the mathematical structure that is the external physical reality is defined by computable functions. In addition to these views, the humanism of Reuben Hersh[1] considers mathematics to be part of human culture and history, which originates from the nature of our physiology and physical environment. He believes that mathematical structures adapt to the world around us for the same reason our lungs adapt to the Earth's atmosphere.

Many scientific theories are formulated and expressed using mathematical concepts and symbols. A part of mathematics, named applied mathematics, deals with modeling and simulation of phenomena and related calculations and helps other scientists to better understand nature, and describe and control it. Applied mathematics develops those mathematical methods that are used in various sciences and technologies. The help that applied mathematics provides to other disciplines in order to solve their problems is sometimes so significant that it has given rise to specific branches of mathematics. Examples include data science, mathematical biology, and financial mathematics. Penelope Maddy [14], a contemporary mathematical philosopher, based on historical pieces of evidence in mathematics, says that an applied mathematician invents a model that corresponds to a physical phenomenon that not only is not clearly visible in the phenomenon itself, but is also more complex than it.

Applied mathematics can be regarded as the bridge between pure mathematics and concrete applications of mathematics in the world. It is generally recognized that the practical value of mathematics lies in its applications, but that pure mathematics is indispensable to make such (future) applications possible, and that there are valid intrinsic reasons for doing mathematics, such as curiosity.

[1] https://www.edge.org/conversation/reuben_hersh-what-kind-of-thing-is-a-number

Pure mathematicians provide a solid framework and rigorous scientific basis which enables applied mathematicians to develop efficient methods or invent useful tools to assist physicists, computer scientists, engineers, biologists, medical researchers, economists, etc., in solving real-world problems. Sometimes this may be reversed in, for instance, physics, that is, mathematicians take ideas from physics and incorporate them into their abstract theorems and theories. Archimedes, for example, proved a geometric theorem while inspired by the law of levers in mechanics. Furthermore, some fields such as machine learning help mathematicians better understand the behavior of objects, structures and systems that are too large or too complex, discover patterns, and apply them to formulate conjectures; see [6] for examples in topology and representation theory.

In [7], the world of mathematics is described as a pyramid at the apex of which mathematical applications to other sciences, commerce, and industry are found. In the middle part of the pyramid, applied mathematics together with data science, mathematical biology, financial mathematics, computer science, scientific computation, information theory, etc., shine. The base of the pyramid is pure mathematics, consisting of logic, number theory, algebra, analysis, and geometry. There is, however, no clear boundary between these sections, and in various places, one can observe an entanglement between pure mathematics, applied mathematics, and mathematical applications. It is notable that if the base of the pyramid is not large enough, the pyramid may not be as stable as required. Some of the achievements of pure mathematicians have found practical applications, while others, so far, have not, but also the latter are still needed to maintain and consolidate the pyramid. We cannot predict where, when, and how a specific piece of pure mathematics will become useful for applications. The bottom of the pyramid is made up of topics related to the fundamentals of mathematics, such as mathematical logic and set theory, which are not considered to be applied in the conventional sense, but without which the pyramid of mathematics cannot be properly erected and stabilized.

In addition, mathematics generally improves decision-making methods in students' minds when we teach them problem-solving techniques, modeling the real world, and cognitive skills. Therefore, it plays an important role in shaping what is named "logical thinking" in humans, apart from what we usually call applications.

Mathematics goes freely beyond the bounds of thought, although it always has an eye for the nature of problems in other branches of science. Ideas in pure mathematics are based on a mental interest in problem-solving toward discovering, establishing, and exploring new structures. "There is nothing more practical than a good theory," said Kurt Lewin, a German-American psychologist [13, p. 169].

History of mathematics has shown that what were once considered mental, abstract, and useless results, were often used at other times by other sciences such as physics, chemistry, computer science, and engineering. To prove this claim, we take a brief look at the history of mathematics and mimic some ideas of [1, 2, 11, 16].

- Number theory is one of the purest fields of mathematics and was considered by many to be a mental game. Fermat's Little Theorem (1600 AD) and Euler's Theorem (1800 AD) are together considered the backbone of the RSA algorithm (named after Rivest, Shamir, and Adleman), which is a public-key cryptosystem. RSA is widely used to secure data transmission, internet communications, e-commerce, and blockchains.
- Conic sections (circle, ellipse, parabola, and hyperbola) were introduced by Apollonius of Perga (250 BC). Some people thought about these as a playground for the mind, until Johannes Kepler (17th century) realized their practical application in describing the orbits of planets.
- In the 19th century, Klein, Beltrami, and Poincaré showed that the geometric structures developed by Nikolai Ivanovich Lobachevsky (hyperbolic geometry) and Bernhard Riemann (elliptic geometry) are as logically consistent as Euclidean geometry. It was a revolutionary discovery and a completely pure one, until in the 20th century Albert Einstein utilized Riemann geometry in his theory of general relativity, where the space-time is curved. Non-Euclidean geometries have since then found applications in cosmology to study the structure and evolution of the universe.
- The complex numbers appeared in the 16th century and were so abstract that they were called imaginary numbers. These numbers gradually found applications in mathematics themselves for polynomial factorization, and in signal processing and electrical circuit calculations. Also, the theory of functions of several complex variables developed by Weierstrass and others in the 19th century has found applications in quantum field theory in the second half of the 20th century.
- Probability theory was established by Girolamo Cardano in the 16th century for solving gambling problems and was well developed by Kolmogorov in 1935 and then applied to statistical mechanics. Probability theory and statistics play a key role in current life, in particular in risk assessment, modeling, and reliability.
- The origins of graphs go back to 1732, when Leonhard Euler posed the seven-bridges problem of Königsberg. Recall that this problem asks whether the seven bridges of the city of Königsberg over its river Preger can all be crossed precisely once during a walk through the city that returns to its starting point. The field of graph theory is considered part of pure mathematics but found its first applications more than one hundred years later when in 1847 Kirchhoff studied electrical networks. Other applications in chemistry, computer science, and social sciences appeared subsequently.
- In the early 20th century, Gottlob Frege analyzed the concepts of arithmetic to show why mathematical reasoning is applicable

as a deductive procedure as applied to statements about the world; see [17, Chapter 1].

- Matrix theory was introduced by Cayley in the 19th century with no connection with applications. In 1925, Heisenberg applied matrices to describe his understanding of the atomic structure, and they are now a key tool in all sciences including coding, economics, and wireless communications.
- Linear algebra, developed in the early 20th century, is devoted to studying linear equations and linear mappings in the framework of vector spaces. It is a crucial ingredient in Google's PageRank algorithm and is used in developing artificial intelligence algorithms. In return, some topics of mathematics such as inverse problems have been influenced by artificial intelligence; see [12].
- Partial differential equations were developed in the 18th and early 19th centuries as applied mathematics, but were also pursued by mathematicians of pure temperament. Maxwell, a physicist, introduced a set of coupled partial differential equations, called now the Maxwell equations, and discovered that light is an electromagnetic wave and there must be other such waves with different wavelengths. Later, the radio waves that the theory predicted were found by Hertz.
- The Radon transform is an integral transform that was introduced in 1917 by Johann Radon. About 50 years later, it was used for tomography, a visualization process in which an image is constructed from the projection data connected with cross-sectional scans of a part of the body.
- The Fourier transform is a map of a function space decomposing a function of time or space into functions depending on spatial frequency. Its history goes back to the 19th century when Fourier expanded a function into an infinite series of sines and cosines. The wavelet transform was created by Alfred Haar and Norman Ricker in the first half of the 20th century. The wavelet transform represents a signal in both the time and frequency domains, whilst the Fourier transform represents it only in the frequency domain. The wavelet and Fourier transforms are now used in the design of computer graphics and in medical devices such as MRI machines, or heart, brain and diabetes monitors.
- Group theory was introduced by Lagrange and Galois in their study of symmetry and symmetry transformations. It greatly influenced the development of cryptography, crystallography, and musical set theory.
- The Entscheidungsproblem was a challenge posed in 1928 by David Hilbert, asking whether there is a way to determine the correctness or incorrectness of mathematical statements in a finite number of steps. In the 1930s, Alonzo Church and Alan Turing answered his question in the negative. Turing formulated an abstract machine, called now the Turing machine, which is considered the basis of modern computers.
- Lawrence Klein believes that economics is a mathematical discipline. John Keynes methodologically inspired his revolutionary economic theory from non-Euclidean geometry; cf. [5].
- Classifying and specifying digital images of millions of fingerprints can take up a very large storage space. Wavelet theory makes it possible to compress information quickly, relatively, and simply in such a way that we can compare new individuals in an investigation. In addition, the theory provides faster information retrieval.
- Linear algebra, mathematical analysis, probability, and statistics are used to analyze big data, that is, large or complex data sets. Big data is applied by information specialists, in particular in the modeling of financial markets.
- Biomathematics is the field using mathematical models for understanding phenomena in biology. It uses mainly linear algebra, differential equations, dynamical systems, probability and statistics.

Our daily lives are tied to many advanced technologies such as computers, the internet, and smartphones, while mathematics constitutes their scientific basis. For instance, mathematics is applicable in securing information in financial transactions, removing landmines and tumors, understanding climate change, developing advanced medical devices, redefining architecture, influencing human behavior, improving weather prediction, satellite communications, and searching for a second Earth.[2]

Due to the commercialization of science and technology, most politicians pay attention only to research with immediate applications and direct profit. Such unrealistic requests have negatively affected all basic and theoretical studies and have caused controversy among some mathematicians concerning the importance of both pure and applied mathematics. Free thought is the nature of research at universities, and the demand for immediate application limits it and darkens the horizon for the advancement of science and ultimately technology. Obviously, usefulness is important for research in industrial fields, but in theoretical fields this concept is ambiguous. However, politicians are not usually impressed by the news about settling a conjecture or the eternal beauty of a certain theorem proved by a pure mathematician. And here, applied mathematicians can greatly help the community by convincing them about the importance of math.

In recent years, great resources (grants, projects) are concentrated on applied research; it is not surprising that this happens. In fact, applied mathematics (including statistics and computer sciences), as well as the speed of computer simulations, are now such that problems from other disciplines can be tackled these days that were completely out of reach a few decades ago. However, it is necessary to support basic research. Furthermore, applying scientometric indicators such as the h-index for granting mathematicians causes some unfair decisions, since not only such indicators do not

[2] https://ima.org.uk/case-studies/mathematics-matters

consider the "hardness" of the topics on which mathematicians are working, but also in some countries applied mathematicians usually have scientometric indicators higher than pure mathematicians, and therefore such a comparison is not appropriate. Also, the highly-visible Fields medals are traditionally awarded for research in pure mathematics, which is somehow annoying because it may split the community and drive applied people away from the general community.

In some countries, there is a cultural difference between applied and pure math departments, where each side may view the other with some sort of suspicion, sometimes even disdain.

Indeed, pure mathematicians are criticized by some of their colleagues who work in applied fields, along the following lines:

- The goal of pure mathematicians is to expand the boundaries of knowledge, while science, industry, and society have more serious problems to explore.
- Research in pure mathematics is not directly aimed at solving society's problems.
- Pure research refers to a vague future in which applications may or may not materialize.

In contrast, pure mathematicians mainly argue as follows:

- Looking this deep into the immediate future applications of pure mathematics is not reasonable. Any kind of real innovative application of mathematics is based on basic research in pure mathematics.
- Pure mathematicians establish and develop precise and correct mathematics used in applications.
- Pure mathematicians ensure that mathematics can be reliably and confidently used in science, industry, and society.

The distinction made between pure and applied mathematicians comes back to their different criteria for doing mathematics. For example, an applied mathematician may use quasi-empirical arguments, such as a non-rigorous method to effectively solve an important problem, while a pure mathematician never does that; see [15]. However, computers and automated reasoning may help pure mathematicians formulate and prove their conjectures. For example, the proof of the Four Color Problem by Kenneth Appel and Wolfgang Haken utilized computers to test a large number of cases. Thus pure and applied mathematics can interact with each other.

Practically, any discussion on the distinction between pure and applied mathematics makes a negative impression on the possible collaborations between pure and applied mathematicians aimed at solving deep problems and improving their faculty; see [20]. In the past there was no such distinction – Kepler, Euler, and Gauss were of course both. In mathematics departments of many prestigious universities, research in pure mathematics is considered as important as research in interdisciplinary and emerging disciplines in applied mathematics. Because they look at mathematics as an integrated system: a device in which the existence of each part, whether pure mathematics or applied mathematics, is essential for its proper functioning. Pure and applied mathematicians constitute a community that needs to work together.

We should pay attention to both pure and applied aspects of mathematics and measure their achievements solely based on depth, breadth, and impact. Ian Stewart [18] said that the very distinction between pure and applied mathematics is looking increasingly artificial, dated, and unhelpful. Sometimes at conferences, after a presentation of results in pure mathematics, a person in the audience may ask: "What is the application of these results?" There is no problem in asking this question as long as it is aimed at helping to connect more pure and applied branches of mathematics. But in many cases such a question arises from a misunderstanding of mathematics, its history, and its structure (a continuum), and shows superficial thinking; see [8].

It is better to consider pure and applied mathematics as two sides of the same coin and recognize and cherish all achievements that have rich and important mathematical content. Pure and applied mathematicians should stand together to explain the word of importance and make science effective progress.

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