The Aronson–Bénilan estimate in Lebesgue spaces

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Abstract. In a celebrated three-page-long paper in 1979, Aronson and Bénilan obtained a remarkable estimate on second-order derivatives for the solution of the porous media equation. Since its publication, the theory of porous medium flow has expanded relentlessly with applications including thermodynamics, gas flow, and groundwater flow, as well as ecological population dynamics. The purpose of this paper is to clarify the use of recent extensions of the Aronson and Bénilan estimate in L^p spaces and of some modifications and improvements, as well as to show certain limitations of their strategy.

1. Introduction

The porous medium equation (PME) is, without a doubt, one of mathematics' evergreens as it typically occurs whenever a quantity, $n(t, x)$, evolves according to a continuity equation,

$$\frac{\partial n}{\partial t} + \mathrm{div}(nu) = 0, \tag{1}$$

where the velocity is given by *Darcy's law*, cf. [27], i.e.,

$$u = -\nabla p.$$

Here, p denotes the *pressure*. Since the equation, in its current form, is not closed, a *constitutive pressure law*, also referred to as an *equation of state*, is chosen to close the equation. Such a law relates the pressure directly to the quantity, n, affected by the pressure, i.e., $p = p(n)$. As mentioned before, its practical applications are manifold, reaching from problems related to groundwater flow [27], non-linear heat transfer [66], and population dynamics, [42, 46, 54], to name just a few. For an extensive and elaborate treatise of the porous medium equation, we refer the reader to the homonymous book by Vázquez [65], and references therein.

One of the most fascinating properties of non-linear diffusion phenomena is the finite speed of propagation. While solutions of the linear diffusion equation have an instantaneous regularising effect, i.e., solutions become positive and smooth after an arbitrarily

short time, solutions of the porous medium equation exhibit behaviour quite different from that of the linear case – solutions remain with limited regularity and compactly supported if they were compactly supported initially, a phenomenon often referred to as *finite speed of propagation*. In [55], the authors introduce a notion of weak solutions and give an existence and uniqueness result for weak solutions to the filtration equation

$$\frac{\partial n}{\partial t} = \frac{\partial^2}{\partial x^2}\phi(n),\tag{2}$$

for a certain class of functions ϕ. Moreover, they show that the equation is satisfied in the classical sense in neighbourhoods of points (t, x), where $n(t, x) > 0$, and that solutions, emerging from compactly supported initial data, have compact support for all times.

In a later paper [48], more properties of the porous medium equation were shown. In particular, invaded regions will remain covered with the density n for all times [48, Lemma 2], every point in space will be invaded by the density after a sufficiently long time [48, Lemma 3], and regions of vacuum do not fill up spontaneously [48, Lemma 3].

Intrigued by the fact that the support of any solution emanating from compactly supported initial data is bounded by two free boundaries, or *interfaces* (cf. [48]), Aronson proved a characterisation of the free boundary speed which is directly related to the pressure gradient which acts as the formal velocity in (1); see [5]. In this paper, it is assumed that some initial data supported on an interval $I = (a_1, a_2)$ are given. In order to establish the aforementioned characterisation of the speed of the moving boundary, Aronson remarks that some control of the quantity "p_{xx}" is required for the analysis. However, the author was able to construct a counterexample explaining that this type of regularity cannot, in general, be expected, which was already known in the case of the explicit Barenblatt–Pattle solution, discovered in 1952; cf. [8, 56]. In fact, in [4] he provides smooth initial data (of C^∞-regularity) that exhibit blow-up of p_{xx} in finite time. Assuming a power law for the pressure, i.e.,

$$p = n^\gamma,\tag{3}$$

Aronson investigates the behaviour of the following one-dimensional Cauchy problem:

$$\begin{cases} \dfrac{\partial p}{\partial t} = \gamma p p_{xx} + p_x^2, \\ p(x, 0) = \cos^2(x), \end{cases}\tag{4}$$

and he proves that, in general, it is not possible to estimate the second derivatives of the solution of (4) in terms of the bounds for the derivatives of the initial data. Indeed, he obtains that the second derivative of the pressure behaves like

$$p_{xx} = \frac{2T}{T - t}, \quad \text{where } T = \frac{\gamma - 1}{2\gamma(\gamma - 1)},\tag{5}$$

which implies that it exceeds any bound in finite time; for more details one can refer to [4, Theorem 1]. Moreover, a similar result can be obtained when the initial data has compact support; cf. [4, p. 301, Example 2].

Thus a different type of control is necessary. In [5, Lemma 2], Aronson proves that if

$$\operatorname{ess\,inf}_I \frac{\partial^2 p}{\partial x^2}(0, x) \geq -\alpha, \tag{6}$$

for some $\alpha \geq 0$, then

$$\frac{\partial^2 p}{\partial x^2}(t, x) \geq -\alpha, \tag{7}$$

for all (t, x) such that $n(t, x) > 0$. To the best of our knowledge, this is the first time this type of lower bound on the Laplacian[1] of the pressure has been obtained. At the same time, this observation acts as the foundation for the more refined version with fewer restrictions on the initial data, obtained in [6] in 1979.

The lower bound on the Laplacian of the pressure is achieved by regularising the initial data in the following way:

$$p_0^n(x) = (k_n \star p_0)(x) + 2^{2-n} K, \tag{8}$$

where K is the Lipschitz constant of p_0 and k_n is a sequence of smoothing kernels converging to a Dirac delta. For the smoothened and strictly positive initial data, classical solutions exist (cf. [55]) and an equation for the second derivative of the pressure, p_{xx}, can be found. Aronson observes that the resulting equation for p_{xx} can be cast in a form whose parabolic operator satisfies the maximum principle presented in [45, Theorem 8]. Ultimately, this allows the uniform bound from below on the second derivative of the pressure to be deduced.

Later, Aronson and Bénilan showed that a similar estimate (nowadays known as the Aronson–Bénilan estimate, i.e., AB-estimate) could be obtained in the multidimensional case (cf. [6]). Under no additional assumptions[2] they showed that

$$\Delta p \geq -\frac{c}{t}, \tag{9}$$

for some constant $c > 0$. Let us note that the same result is mentioned in [5] as a note, since the regularising effect was not the main focus in the derivation of the boundary speed characterisation in one dimension. Later, in 2009 [52], Aronson–Bénilan-type estimates were obtained for the PME on complete Riemann manifolds under the assumptions of a lower bound of the Ricci curvature.

In 1982, Crandall and Pierre generalised the AB-estimate for the initial-value problem associated to the filtration equation (cf. (2)) given by

$$\begin{cases} \partial_t n = \Delta \phi(n), \\ n(0, x) = n_0(x), \end{cases} \tag{10}$$

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[1] At this time, the results by Oleĭnik, Kalashnikov, and Aronson only address the one-dimensional case.
[2] Indeed, the only assumption is for the exponents to be "large enough"; cf. their paper.

for $t > 0$ and $x \in \mathbb{R}^N$, where ϕ is a non-negative, non-decreasing, continuous function with $\phi(0) = 0$; cf. [25]. They prove that if ϕ satisfies an inequality (cf. [25, Eq. (3)] which formally controls the growth of ϕ), the solution n of (10) satisfies

$$\partial_t \phi(n) \geq \frac{K}{t}(\phi(n) + a),\tag{11}$$

for some constants $K > 0$ and $a \geq 0$. In the power-law case, $\phi(n) = n^\gamma$, with $a = 0$ and

$$K \geq \frac{\gamma}{\gamma - 1 + 2/N},$$

obtained from the aforementioned inequality (cf. [25, Eq. (3)]), the solution n underlies the same regularising effect as that of [6]. With this approach, they are able to extend the AB-type estimate to cover a larger class of problems, including, for instance, the Stefan problem (see [17]) which holds for $a = 1$, $K = N/2$. In their proof, they use a technique similar to the one adopted in this work: they study the evolution of the Laplacian of the pressure, Δp, which satisfies an appropriate parabolic inequality. Upon introducing a suitable function $h(p)$ characterised *a posteriori*, they deduce the time estimate on the solution n. Here, different from [25], first we extend the AB-estimate in all the interesting Lebesgue spaces and then, in L^∞, we are able to weaken the condition on the function ϕ, i.e., we do not need to impose their equation (3). Precisely, our necessary conditions to get the AB-type estimate are the same as those in [25]. There are two main differences: first they are not addressing the incompressible limit and then, to assume their equation (3), they are forced to select a specific form of the weight which also ensures that a specific power of ϕ is convex. In our paper, first we do not need any additional regularity on the quantities involved and we do not have to specify the shape of the weight: it solves an inequality, different from [25] where they impose the equality, and we just need to prove that the weight is bounded from above and below. Finally, thanks to Theorem 3.2, we are able to pass to the incompressible limit for all fields of pressure.

1.1. The Aronson–Bénilan estimate and regularity theory of the porous medium equation

In [18], the authors prove that the unique generalised solution to the porous medium equation in two (or more) dimensions, a result due to [64], is continuous. Moreover, they give an explicit expression for the modulus of continuity in space and time (cf. [18, Theorem 1.1]), thus extending the known one-dimensional result on Hölder regularity in space by Aronson (cf. [3, Theorem, p. 465]). It is important to stress that the regularity theory of Caffarelli and Friedman heavily relies on the AB-estimate. On the one hand, it allows them, in some sense, to quantify how the density at the centre of small balls changes in a small time instance (cf. [18, Lemmas 2.2, 2.3]), and to derive an explicit modulus of continuity for the multidimensional porous medium equation. On the other hand, it allows them to study the regularity of the free boundary in any dimension (cf. [20]), generalising the one-dimensional results of [5, 19]. In doing so, they need to quantify how fast

the density begins to intrude into a previously unoccupied domain (cf. [20, Lemma 2.1], which uses the multidimensional AB-estimate). One of the key findings in their paper is the Hölder regularity of the free boundary, $t = f(x)$, which allows them to improve the modulus of continuity solutions which are established to be Hölder continuous too [18, Section 4]. Eight years later, in [21], the authors prove that the free boundary is in fact Lipschitz continuous for all times larger than the first time that the solution contains a ball which includes the initial data. For a smaller time instance, however, Hölder regularity is optimal due to so-called focusing phenomena [2, 7, 21]. The focusing problem is dedicated to understanding how areas of vacuum in the initial data are filled by the evolution of the porous medium equation. While solutions are Hölder continuous, the pressure gradient can blow up in time; cf. [7].

A difficult problem is to perform a linear limit of the PME to recover the heat equation. Indeed, the mentioned Hölder regularity for the solution of the PME cannot be obtained. Only in the recent works [36, 37] have the authors proved that given initial data with low regularity, the solution is L^p in time and belongs to a fractional Sobolev space in $\mathbb{R}^d$ with $d > 1$. Moreover, using the Barenblatt solution, the authors prove that this result is the optimal one.

1.2. Applications to modelling tissue growth

In the last few decades, the study of cancer development has improved due to new analytical tools and due to the introduction of new numerical methods [9–11, 32, 34, 51]. The main difficulty in studying these phenomena is the vast biological complexity related to the presence of different kinds of inter-specifically and intra-specifically interacting cells. Describing the tumour at the macroscopic level, we can distinguish two categories. On the one hand, we can devise partial differential equations (PDEs) to model tumour growth [13, 15, 24, 41, 60, 63], in such a way that cells are represented by densities. On the other hand, tissue growth can be described by a free boundary model [26, 35, 40], where tissue growth is due to the motion of its boundary. Each of these approaches has its advantages: the first approach, also called mechanical models, is widely studied with many numerical and analytical tools. Regarding the second approach, it is closer to the biological vision of the tissue and allows its motion and dynamics to be studied. There is a well-developed technique to establish a link between the two approaches, the so-called incompressible limit, which implies that the pressure becomes stiff [14, 23, 44, 49, 50, 53, 58, 59].

1.2.1. Population-based description of tissue growth.
The simplest way to model tumour growth is by introducing a single equation describing the evolution of the abnormal cell density, $n(x, t)$, where $x \in \mathbb{R}^d$ and $t \in \mathbb{R}^+$, which evolves under pressure forces and cell multiplication according to the equation [16, 58]

$$\frac{\partial n}{\partial t} - \operatorname{div}(n\nabla p) = nG(p), \tag{12}$$

where $p = p(n)$ is the pressure field and $G := G(p)$ models the proliferation of cells and is called the growth function. Suitable assumptions have to be imposed on G, for instance,

$$G'(p) < 0 \quad \text{and} \quad G(p_M) = 0, \tag{13}$$

which imply that the increasing number of the cells is limited by the pressure p, and $p_M > 0$ is called the *homeostatic pressure* [16]. Concerning the pressure, in many papers [14, 44, 49, 50, 58, 59] there is an explicit and assigned relation between p and n. In this work, to be as general as possible, we only assume that

$$p = p(n), \quad p(0) = 0, \quad p'(n) > 0, \tag{14}$$

for $n > 0$. We have two examples in mind: the classical power-law case, where

$$p(n) = n^\gamma \tag{15}$$

(see [14, 28, 57]), and the pressure used in [31] (called DHV throughout), where

$$p(n) = \varepsilon \frac{n}{1 - n}. \tag{16}$$

For a general pressure law, the quantity p satisfies the evolution equation given by

$$\partial_t p = |\nabla p|^2 + qw, \tag{17}$$

where

$$q(p) := np'(n) \quad \text{and} \quad w := \Delta p + G(p). \tag{18}$$

The aforementioned examples give

$$q(p) = \gamma p \ \text{(polytropic law)} \quad \text{and} \quad q(p) = p\left(1 + \frac{p}{\varepsilon}\right) \ \text{(DHV law)}.$$

They differ deeply near $p = 0$ in their behaviours as $\varepsilon \to 0$ and $\gamma \to \infty$ and this is a major issue if one wants to study these limits and establish the Hele–Shaw free boundary problem.

For $G \equiv 0$, Aronson and Bénilan build their estimate on the observation that one can obtain an equation for w [6]. Their argument can be extended to include G and leads to the equation

$$\partial_t w = 2 \sum_{i,j=1}^{N} (\partial_{ij}^2 p)^2 + 2\nabla p \cdot \nabla w - G'(p)|\nabla p|^2 + \Delta(qw) + G'(p)qw.$$

Because

$$2 \sum_{i,j=1}^{N} (\partial_{ij}^2 p)^2 \geq \frac{2}{N}(\Delta p)^2 = \frac{2}{N}(w - G)^2,$$

and since we assume $G' \leq 0$, i.e., (13), we may also write

$$\partial_t w \geq \frac{2}{N}w^2 + 2\nabla p \cdot \nabla w + \Delta(qw) + \left[G'(p)q - \frac{4}{N}G\right]w. \tag{19}$$

This inequality, which is self-contained when $G \equiv 0$, is the very basis of estimates on $|w|_- = \max(0, -w)$ which we analyse in different L^p spaces.

1.2.2. Free-boundary-based description of tissue growth. Besides its huge impact on the regularity theory of solutions to the porous medium equation and the free boundaries thereof, the AB-estimate proves to be a crucial tool for building a bridge between a density-based description and a geometric description of tissue growth. The link between the two models is established through a rigorous study of the *incompressible limit* of the porous medium pressure equation, cf. (17), as the pressure law becomes stiffer and stiffer, i.e., $\gamma \to \infty$ or $\varepsilon \to 0$, in the respective pressure law ((15), (16)). As a result an incompressible model is obtained, satisfying two relations. The first, $p(n - 1) = 0$, implies the absence of any pressure in zones that are not saturated ($\{n < 1\}$), while the second one, also referred to as the *complementarity relation*, yields an equation satisfied by the pressure on $\{p > 0\}$, which is of the form

$$p(\Delta p + G(p)) = 0.$$

It is immediately apparent that strong regularity is needed to obtain such an expression, which is provided by (adaptations) of the AB-estimate – bounds on the Laplacian of the pressure are enough to infer strong compactness of the pressure gradient. This was first observed in [58] to be equivalent to being able to pass to the limit in the porous medium pressure equation and obtain the incompressible limit.

1.2.3. Extension to two species. Let us highlight that the mathematical theory of the limit for equations like (12) is well studied both with $G \neq 0$ [31, 49, 58] and without G [12, 38, 39], as well as in the case where nutrients and viscosity are included [28–30, 59]. The limit model turns out to be a free boundary model of Hele–Shaw type.

The model of a single evolution equation, which describes the tumour cell distribution, can be complemented by another species consisting of healthy tissue, and it is given by

$$\frac{\partial n^{(i)}}{\partial t} - \operatorname{div}(n^{(i)} \nabla p) = n^{(1)} F^{(i)}(p) + n^{(2)} G^{(i)}(p), \tag{20}$$

where $i = 1, 2$, $n^{(1)}, n^{(2)}$ denote the population densities, and $G^{(i)}$, $F^{(i)}$ model the reaction or growth phenomena, which are assumed to depend exclusively on the pressure according to experimental observations [16, 62]. The system structure of (20) causes serious analytical difficulties, cf. [14, 22, 33, 43, 61], due to its hyperbolic flavour. The careful study of the pressure equation helps in proving the existence of solutions and obtaining uniform estimates with respect to the stiffness parameter, i.e., γ in the classical power-law case $p = n^{\gamma}$ [43, 53, 57]. Due to the insufficient regularity of the pressure, the incompressible limit can be achieved just in one dimension using Sobolev embeddings. For the DHV pressure law, i.e., assuming that the pressure blows up at a finite threshold, cf. [23, 31, 44], similar mathematical difficulties arise: in order to pass to the limit, strong restrictions have to be imposed.

While the incompressible limit for multiple species remains an interesting open problem for the Darcy law, including viscosity of cells in the model, i.e., altering the velocity

u in (1), changes the analytical properties of the model drastically. In the case of the so-called *Brinkman law* [1], the model reads

$$\begin{cases} \dfrac{\partial n^{(i)}}{\partial t} - \operatorname{div}(n^{(i)} \nabla W) = n^{(1)} F^{(i)}(p) + n^{(2)} G^{(i)}(p), \\ -\nu \Delta W + W = p, \end{cases} \tag{21}$$

where $i = 1, 2$ and ν is the viscosity parameter. On the one hand, the idea to couple the two equations for the individual species through Brinkman's law changes the behaviour dramatically as mentioned before, i.e., classical techniques used in the one-species model fail [59], but on the other hand the pressure field gains regularity and some mathematical difficulties can be overcome. Recently, two results have been obtained in this direction: in [30], the authors are able to establish the incompressible limit in the one-dimensional case by establishing uniform BV-bounds for the two species; then in [29], since the BV-strategy fails in higher dimensions, by employing a non-local compactness criterion [47], the passage to the incompressible limit can be accomplished.

1.3. Plan of the paper

The paper is organised as follows. In Section 2 we derive the L^1-estimate without and with a reaction term G and we show the advantages and limitations of adding a weight h. Then, in Section 3, we conduct similar computations and calculations in the L^∞ space proving that the additional weight helps in generalising the AB-type estimate for all pressure laws. In Section 4 we perform an L^2-estimate showing that it can be closed just for a particular class of pressure laws. Finally, in Section 5 we add few concluding remarks.

2. L^1-type estimate

The form of (19) is well adapted to perform L^1-estimates of the second-order quantity w because it generates, thanks to the Kato inequality, the inequality

$$\partial_t |w|_- \leq -\frac{2}{N} |w|_-^2 + 2 \nabla p \cdot \nabla |w|_- + \Delta(q |w|_-) + \left[G'(p) q - \frac{4}{N} G \right] |w|_-. \tag{22}$$

It turns out that further manipulations lead to restrictions which are more demanding than expected. To explain that, we first treat the case $G \equiv 0$. The extension to the case $G \geq 0$ is stated in the second corresponding subsection.

2.1. L^1-estimates when $G \equiv 0$

When $G \equiv 0$, $w = \Delta p$ and a simple integration of (19) yield

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N} |w|_- \, \mathrm{d}x \leq 2 \frac{N-1}{N} \int_{\mathbb{R}^N} |w|_-^2 \, \mathrm{d}x.$$

Because of the quadratic growth of the right-hand side, this inequality provides us with an L^1 control only in dimension $N = 1$. We thus adopt a different strategy. By adding a positive weight function $h = h(p)$, we aim to study whether or not it helps improve the above result. We shall establish the following theorem:

Theorem 2.1 (Case $G \equiv 0$, L^1-theory). *Assume the pressure law is such that for $p > 0$,*

$$\alpha_1(p) := \int_0^p \left[q(\cdot)h'(\cdot) + \frac{1}{N} h(\cdot) \right] d\rho > 0, \quad \text{with } h(p) := \int_0^p e^{\int_{\cdot}^\rho \frac{1}{q}} d\rho. \tag{23}$$

Then the following a priori estimates hold true:

$$\int_{\mathbb{R}^N} h(p) |\Delta p(t)|_- \, dx + 2 \int_0^t \int_{\mathbb{R}^N} \alpha_1(p(s)) |\Delta p(s)|_-^2 \, dx \, ds$$

$$\leq \int_{\mathbb{R}^N} h(p) |\Delta p(0)|_- \, dx, \tag{24}$$

and, for all $t \leq T$,

$$\int_{\mathbb{R}^N} h(p) |\Delta p(t)|_- \, dx \leq \frac{A(T)}{t} \quad \text{with } A = \frac{1}{2} \sup_{0 \leq t \leq T} \int_{\mathbb{R}^N} \frac{h(p)^2}{\alpha_1(p)} \, dx. \tag{25}$$

Estimate (24) *also holds when $N = 1$ with $h \equiv 1$ and $\alpha_1 \equiv 0$.*

Notice that, because $q'(p)p'(n) = p'(n) + np''(n)$, the condition $\alpha_1 > 0$ is satisfied when, for instance, $p(n)$ is a convex function.

However, it turns out that the result of this theorem is rather weak compared to the L^∞-type estimate. Indeed, the calculation forces us to choose $h(0) = 0$, and therefore the estimate is weak near the free boundary $\{p = 0\}$. However, for the pressure laws we have introduced before, when γ is large or ε is small, the decay rate scales correctly with γ or ε, and this allows for a uniform control of $\partial_t p$. To see this we may give the corresponding expressions for α_1, h, A, explicitly:

- For the power law, we have $q(p) = \gamma p$ and

$$h(p) = \frac{\gamma}{\gamma + 1} p^{\frac{\gamma+1}{\gamma}}, \quad \alpha_1(p) = \gamma \left[1 - \frac{N-1}{N} \frac{1}{\gamma+1} \right] p^{\frac{\gamma+1}{\gamma}} > 0,$$

  and $A = \mathcal{O}(\gamma^{-1})$.

- For the DHV law, the situation is similar and we find

$$h(p) = \varepsilon p - \varepsilon^2 \ln\left(1 + \frac{p}{\varepsilon}\right), \quad \alpha_1(p) = p^2 + \varepsilon \frac{N-1}{N} \left[\varepsilon \ln\left(1 + \frac{p}{\varepsilon}\right) - p \right] > 0,$$

  and $A = \mathcal{O}(\varepsilon^2)$.

Proof of Theorem 2.1. Integrating over $\mathbb{R}^N$, we get

$$\frac{d}{dt} \int_{\mathbb{R}^N} h(p) |w|_- \, dx = \int_{\mathbb{R}^N} \left(h'(p) |w|_- \left(\frac{\partial p}{\partial t}\right) + h(p) \left(\frac{\partial |w|_-}{\partial t}\right) \right) dx$$

$$\leq \int_{\mathbb{R}^N} h'(p)|\nabla p|^2 |w|_- \, \mathrm{d}x - \int_{\mathbb{R}^N} h'(p)q(p)|w|_-^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} \frac{2h}{N}|w|_-^2 \, \mathrm{d}x$$

$$+ \underbrace{\int_{\mathbb{R}^N} h(p)\Delta(q(p)|w|_-) \, \mathrm{d}x}_{I_1} + \underbrace{2 \int_{\mathbb{R}^N} h(p)\nabla p \cdot \nabla|w|_- \, \mathrm{d}x}_{I_2} . \tag{26}$$

We next estimate the integral terms I_1 and I_2 separately. Beginning with I_1, integrating by parts twice, we obtain

$$I_1 = \int_{\mathbb{R}^N} q(p)|w|_- \Delta h(p) \, \mathrm{d}x$$

$$= \int_{\mathbb{R}^N} q(p)h''(p)|\nabla p|^2 |w|_- \, \mathrm{d}x - \int_{\mathbb{R}^N} h'(p)q(p)|w|_-^2 \, \mathrm{d}x.$$

As for the second term I_2, we integrate by parts once, which yields

$$I_2 = -2 \int_{\mathbb{R}^N} h'(p)|\nabla p|^2 |w|_- \, \mathrm{d}x - 2 \int_{\mathbb{R}^N} h(p)|w|_-^2 \, \mathrm{d}x.$$

Substituting the expressions for the two integral terms I_1 and I_2 into (26), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N} h(p)|w|_- \, \mathrm{d}x \leq -2 \int_{\mathbb{R}^N} \alpha_1 |w|_-^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} \beta_1 |w|_- |\nabla p|^2 \, \mathrm{d}x, \tag{27}$$

where

$$\alpha_1 := h'q - h\left(1 - \frac{1}{N}\right) \quad \text{and} \quad \beta_1 := qh'' - h'. \tag{28}$$

In dimension $N = 1$ we can choose $h = 1$, $\alpha_1 = \beta_1 = 0$ which provides us with an L^1-estimate of $|\Delta p|_-$ and proves the last statement of Theorem 2.1.

In higher dimensions, we are unable to do that and we solve $\beta_1(p) = 0$ instead, which gives the expressions for h and α_1 in (23). Then, integrating (27) in t gives the announced estimate (24).

To obtain estimate (25) we use the Cauchy–Schwarz inequality to write

$$\left(\int_{\mathbb{R}^N} h|w|_- \right)^2 \mathrm{d}x \leq A \int_{\mathbb{R}^N} \alpha_1 |w|_-^2 \, \mathrm{d}x, \quad A := \frac{1}{2} \int_{\mathbb{R}^N} \frac{h^2}{\alpha_1} \, \mathrm{d}x.$$

Substituting this information into (27), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N} h(p)|w|_- \, \mathrm{d}x \leq -A^{-1} \left(\int_{\mathbb{R}^N} h|w|_- \, \mathrm{d}x \right)^2 .$$

Since $U(t) = \frac{A}{t}$ is a solution, we conclude that

$$\int_{\mathbb{R}^N} h(p)|w|_- \, \mathrm{d}x \leq A/t,$$

which proves statement (25) and concludes the proof of Theorem 2.1. ∎

2.2. L^1-estimates with $G \neq 0$

With the notation of Theorem 2.1 we define

$$\bar{\delta}_1 = \max_{0 \leq p \leq p_M} \left\{ G\left[2\frac{N-2}{N} - \frac{h'q}{h} \right] + G'q \right\}.$$

Theorem 2.2 (Case $G' \leq 0$, L^1-theory). *With the notation and assumptions of Theorem 2.1, $G' \leq 0$ and $\bar{\delta}_1$ as above, the following a priori estimates hold true:*

$$\int_{\mathbb{R}^N} h(p)|w(t)|_- \, dx + \int_0^t e^{\bar{\delta}_1(t-s)} \int_{\mathbb{R}^N} \alpha_1(p)|w|_-^2 \, dx \, ds$$

$$\leq e^{\bar{\delta}_1 t} \int_{\mathbb{R}^N} h(p)|w(0)|_- \, dx \tag{29}$$

and

$$\int_{\mathbb{R}^N} h(p)|w(t)|_- \, dx \leq A \frac{\bar{\delta}_1 e^{\bar{\delta}_1 t}}{e^{\bar{\delta}_1 t} - 1}, \tag{30}$$

with A as in (25). *Estimate* (29) *also holds when $N = 1$ with $h \equiv 1$ and $\alpha_1 \equiv 0$.*

Notice that the sign of $\bar{\delta}_1$ does not play a role here.

Proof. Still building on inequality (19) and using a positive weight, $h = h(p)$, the evolution in time of the quantity hw is given by

$$\partial_t(h|w|_-) = h'|w|_- \partial_t p + h \partial_t |w|_-$$
$$\leq h'|w|_-(|\nabla p|^2 + qw)$$
$$+ h\left\{ -\frac{2}{N}|w|_-^2 + 2\nabla p \cdot \nabla|w|_- + \Delta(q|w|_-) + \left[G'(p)q - \frac{4}{N}G \right]|w|_- \right\}.$$

Integrating, we get

$$\frac{d}{dt} \int_{\mathbb{R}^N} h|w|_- \, dx \leq \int_{\mathbb{R}^N} h'|\nabla p|^2 |w|_- \, dx - \int_{\mathbb{R}^N} h'q|w|_-^2 \, dx - \int_{\mathbb{R}^N} \frac{2h}{N}|w|_-^2 \, dx$$

$$+ \int_{\mathbb{R}^N} h\Delta(q|w|_-) \, dx + 2 \int_{\mathbb{R}^N} h\nabla p \cdot \nabla|w|_- \, dx$$

$$- \int_{\mathbb{R}^N} \left[\frac{4}{N}G - G'q \right] h|w|_- \, dx$$

$$\leq \int_{\mathbb{R}^N} h'|\nabla p|^2 |w|_- \, dx - \int_{\mathbb{R}^N} \left[\frac{2h}{N} + h'q \right]|w|_-^2 \, dx$$

$$- \int_{\mathbb{R}^N} \left[\frac{4}{N}G - G'q \right] h|w|_- \, dx$$

$$+ \underbrace{\int_{\mathbb{R}^N} q|w|_- \Delta h \, dx}_{I_1} - 2\underbrace{\int_{\mathbb{R}^N} [\nabla h \cdot \nabla p + h\Delta p]|w|_- \, dx}_{I_2}.$$

Integrating by parts twice, the term I_1 can be rewritten as

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^N} [h''|\nabla p|^2 + h'\Delta p]q|w|_- \, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} h''q|\nabla p|^2 |w|_- \, \mathrm{d}x - \int_{\mathbb{R}^N} h'q|w|^2_- \, \mathrm{d}x - \int_{\mathbb{R}^N} h'qG|w|_- \, \mathrm{d}x.
\end{aligned}
$$

Next we simplify the term I_2. Integrating by parts and using the chain rule we obtain

$$
I_2 = \int_{\mathbb{R}^N} h'|\nabla p|^2 |w|_- \, \mathrm{d}x - \int_{\mathbb{R}^N} h|w|^2_- \, \mathrm{d}x - \int_{\mathbb{R}^N} hG|w|_- \, \mathrm{d}x.
$$

Substituting I_1 and I_2 back into our main inequality we get

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N} h|w|_- \, \mathrm{d}x \le -2 \int_{\mathbb{R}^N} \alpha_1^h |w|^2_- \, \mathrm{d}x + \int_{\mathbb{R}^N} \beta_1^h |w|_- |\nabla p|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} \delta_1^h h|w|_- \, \mathrm{d}x,
$$

where

$$
\alpha_1^h = h'q - h\frac{N-1}{N}, \quad \beta_1^h = h''q - h', \quad \text{and} \quad \delta_1^h = G\left[2\frac{N-2}{N} - \frac{h'q}{h}\right] + G'q \le \bar{\delta}_1.
$$

To control the terms on the right-hand side, we argue as in the case $G \equiv 0$. In dimension $N = 1$, we can choose $h = 1$, $\alpha_1^h = \beta_1^h = 0$. Otherwise, it is sufficient to impose that

$$
\alpha_1^h(p) > 0 \quad \text{and} \quad \beta_1^h(p) = 0,
$$

as chosen in Theorem 2.1.

We rewrite the inequality, after using the Cauchy–Schwarz inequality, as

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N} h|w|_- \, \mathrm{d}x \le -A^{-1}\left(\int_{\mathbb{R}^N} h|w|_-\right)^2 \, \mathrm{d}x + \bar{\delta}_1 \int_{\mathbb{R}^N} h|w|_- \, \mathrm{d}x.
$$

It remains to observe that

$$
A\frac{\bar{\delta}_1 e^{\bar{\delta}_1 t}}{e^{\bar{\delta}_1 t} - 1}
$$

is a solution and we obtain the statements of Theorem 2.2.   ∎

3. L^∞-type estimate

The other extreme L^p-space is that used in the original paper, and establishes a bound in L^∞ of $|w|_-$. It uses the strong form of the equation satisfied by w. Namely, starting from (19), we can write

$$
\partial_t w \ge \frac{2}{N}w^2 + 2\nabla(p+q)\cdot\nabla w + q\Delta w + w\Delta q + \left[G'(p)q - \frac{4}{N}G\right]w.
$$

Using that

$$\Delta q = q''(p)|\nabla p|^2 + q'(p)\Delta p = q''(p)|\nabla p|^2 + q'(p)(w - G),$$

we find

$$\partial_t w \geq \Big[q' + \frac{2}{N}\Big]w^2 + 2\nabla(p+q)\cdot\nabla w + q''|\nabla p|^2 w + q\Delta w$$
$$+ \Big[G'q - \Big(\frac{4}{N} + q'\Big)G\Big]w. \tag{31}$$

A first result that can be deduced directly from this calculation is the following:

Theorem 3.1 (Lower bound on Δp, special case). *Assume that $q'' \geq 0$ and assume there are constants $\bar{\delta} \in \mathbb{R}$, $\alpha_0 > 0$ such that $G'q - (\frac{4}{N} + q')G \leq \bar{\delta}$ and $q' + \frac{2}{N} \geq \alpha_0$; then we have*

$$\Delta p + G \geq -\frac{1}{\alpha_0}\frac{\bar{\delta}e^{\bar{\delta}t}}{e^{\bar{\delta}t} - 1}.$$

This result applies to homogeneous pressure laws $q(p) = \gamma p$ and $\alpha_0 = O(\gamma)$ and matches that of [58] (see [58, Eqs. (2.14)]). It also applies to the DHV law, $q = p + \varepsilon^{-1}p^2$, $q' = 1 + 2\varepsilon^{-1}p$, and $q'' = 2\varepsilon^{-1}$, but then $\alpha_0 = O(1)$ (when $p \approx 0$) does not give uniform decay as $\frac{1}{\varepsilon}$ as needed to study the Hele–Shaw limit.

To treat more general pressure laws we can refine the argument and, as in the L^1-case, we begin with the porous medium equation and then we include the growth term.

3.1. L^∞-estimates when $G \equiv 0$

We begin by estimating the Laplacian of the pressure, $w = \Delta p$, when $G \equiv 0$. We are going to prove the following theorem.

Theorem 3.2 (Lower bound on Δp, $G \equiv 0$). (1) *Assume that*

$$\Big(\frac{q}{p}\Big)' \geq 0 \quad and \quad \tilde{\alpha}_0 := \min_{0 \leq p \leq p_M}\frac{pq'(p)}{q(p)} > 0;$$

then

$$\frac{q(p)}{p}\Delta p \geq -\frac{1}{\tilde{\alpha}_0 t}.$$

(2) *Assume that $q'(p) > -1$ for $p \in [0, p_M]$, and*

$$\tilde{\alpha}_0 = \frac{\min_{0 \leq p \leq p_M} q'(p)}{1 + \max_{0 \leq p \leq p_M} q'(p)} + \frac{2}{N}\frac{1 + q'(0)}{(1 + \max_{0 \leq p \leq p_M} q'(p))^2} > 0.$$

Then we have

$$\Big(1 + \min_{0 \leq p \leq p_M} q'(r)\Big)\Delta p \geq -\frac{1}{\tilde{\alpha}_0 t}.$$

With the first set of assumptions, the estimate is compatible with the Hele–Shaw asymptotics in the two examples of the power law (then $\tilde{\alpha}_0 = 1$ and $\frac{q}{p} = \gamma$) and DHV law (then $\tilde{\alpha}_0 = 1$ and $\frac{q}{p} = \frac{p+\varepsilon}{\varepsilon}$). The second set of assumptions is an explicit example motivated by [25].

To understand whether there is some slack in the estimate, we compute the evolution of the quantity hw, where $h = h(p)$ is assumed to be a positive weight function. We obtain

$$
\begin{aligned}
\frac{\partial(hw)}{\partial t} &= wh'\frac{\partial p}{\partial t} + h\frac{\partial w}{\partial t} \\
&\geq wh'(|\nabla p|^2 + qw) \\
&\quad + h\left(\left[q' + \frac{2}{N}\right]w^2 + q''|\nabla p|^2 w + 2\nabla(p+q)\cdot\nabla w + q\Delta w\right),
\end{aligned}
$$

where we used the equation satisfied by the pressure, (17), and (31) for w. Upon rearranging the terms we get

$$
\begin{aligned}
\frac{\partial(hw)}{\partial t} &\geq w^2\left[hq' + qh' + \frac{2}{N}\right] + w|\nabla p|^2(h' + q''h) \\
&\quad + q\underbrace{h\Delta w}_{I_1} + \underbrace{2h(1+q')\nabla p\cdot\nabla w}_{I_2}.
\end{aligned}
\tag{32}
$$

The terms involving the linear operators have to be rewritten in terms of the new quantity hw rather than w. Therefore, from the first term we get

$$
\begin{aligned}
I_1 &= \Delta(hw) - w\Delta h - 2\nabla h\cdot\nabla w \\
&= \Delta(hw) - w(h''|\nabla p|^2 + h'w) - 2h'\nabla p\cdot\nabla w \\
&= \Delta(hw) - w(h''|\nabla p|^2 + h'w) - 2\frac{h'}{h}\nabla p\cdot(\nabla(hw) - w\nabla h) \\
&= \Delta(hw) - 2\frac{h'}{h}\nabla p\cdot\nabla(hw) - w(h''|\nabla p|^2 + h'w) + 2(hw)\frac{(h')^2}{h^2}|\nabla p|^2 \\
&= \Delta(hw) - 2\frac{h'}{h}\nabla p\cdot\nabla(hw) + w|\nabla p|^2\left(2\frac{(h')^2}{h} - h''\right) - h'w^2.
\end{aligned}
$$

For the second one we have

$$
\begin{aligned}
I_2 &= 2\nabla(p+q)\cdot h\nabla w \\
&= 2\nabla(p+q)\cdot(\nabla(hw) - h'w\nabla p) \\
&= 2\nabla(p+q)\cdot\nabla(hw) - 2(1+q')h'w|\nabla p|^2.
\end{aligned}
$$

Substituting the simplified expressions for I_1 and I_2 into (32), we have

$$
\frac{\partial(hw)}{\partial t} \geq \alpha_\infty^h(hw)^2 + \beta_\infty^h(hw)|\nabla p|^2 + \mathcal{L}_\infty^h(hw),
\tag{33}
$$

where

$$
\alpha_\infty^h = \frac{q'}{h} + \frac{2}{Nh^2} \quad\text{and}\quad \beta_\infty^h = q'' - \frac{qh''}{h} + 2\frac{q(h')^2}{h^2} - 2\frac{q'h'}{h} - \frac{h'}{h},
$$

as well as

$$\mathcal{L}_\infty^h(hw) = q\Delta(hw) + 2\Big(1 + q' - \frac{qh'}{h}\Big)\nabla p \cdot \nabla(hw).$$

In order to find a subsolution and to close the estimate, it is enough to ensure that

$$\alpha_\infty^h = \frac{q'}{h} + \frac{2}{Nh^2} \geq \alpha_0 > 0, \quad \frac{\beta_\infty^h}{h} = \Big(\frac{q}{h}\Big)'' + \Big(\frac{1}{h}\Big)' \leq 0, \tag{34}$$

where α_0 is a constant. We propose two strategies to fulfill these requirements.

(1) Assume $(\frac{q}{p})' \geq 0$. With this assumption we can simply choose

$$h = \frac{q}{q'(0)p}, \quad \Big(\frac{q}{h}\Big)' = q'(0),$$

because $\frac{1}{h}$ is non-increasing. Then we compute

$$\alpha_0 := q'(0) \simeq \min_{0 \leq p \leq p_M}\Big[\frac{pq'(p)}{q(p)} + \frac{2}{N}\frac{q'(0)p^2}{q(p)^2}\Big],$$

and this gives our first statement in Theorem 3.2 after simplifying a coefficient $q'(0)$.

(2) Assume $q'(p) > -1$. Then we impose, from (34),

$$\Big(\frac{q}{h}\Big)' + \frac{1}{h} = 1 + q'(0) \quad \text{and} \quad h(0) = 1. \tag{35}$$

Because of the degeneracy at $p = 0$, the condition $h(0) = 1$ is imposed since we can compute

$$1 + q'(0) = \frac{q'(0)}{h(0)} + q(0)(h^{-1})'(0) + \frac{1}{h(0)} = \frac{q'(0) + 1}{h(0)}.$$

We can analyse the differential equation (35).

Lemma 3.3. *Assume that $q'(p) > -1$ for $p \in [0, p_M]$. Then the solution of (35) satisfies*

$$\frac{1 + \min_{0 \leq r \leq p} q'(r)}{1 + q'(0)} \leq h(p) \leq \frac{1 + \max_{0 \leq r \leq p} q'(r)}{1 + q'(0)} \quad \forall p \in (0, p_M).$$

Remark 3.4 (Power law, DHV pressure, general pressure laws). Notice that if q is smooth as in DHV, $q'(0) = 1$, but for the power law, $q'(0) = \gamma$. As a matter of fact, for the power-law pressure and the singular pressure we are able to provide explicit expressions for h, i.e.,

$$h_\gamma(p) = 1 \quad \text{and} \quad h_\varepsilon(p) = 1 + p - \varepsilon \log(p + \varepsilon) + \varepsilon \log \varepsilon. \tag{36}$$

However, we also emphasise that the bounds established in the preceding theorem allow us to prove an Aronson–Bénilan-type estimate for more general pressure laws $p = p(n)$, under rather weak assumptions on q, thus extending the known cases.

Proof of Lemma 3.3. First, we change variables and set, in (35),

$$u(p) := \frac{1}{h(p)}.$$

In the new variable, it becomes

$$qu' + (q' + 1)u = 1 + q'(0), \tag{37}$$

with $u(0) = 1$. The rest of the argument, i.e., the proof of the upper and lower bounds on h, is by contradiction. To this end, we define

$$U(p) = a\frac{1 + q'(0)}{1 + \max_{0 \le r \le p} q'(r)}.$$

By construction, it is a non-increasing function and it satisfies $u(0) = 1 > U(0) = a$. Assume there exists a point $p^* \in (0, p_M)$, that we can choose to be minimal, such that

$$u(p^*) = U(p^*). \tag{38}$$

Therefore, at this point we have to have $u'(p^*) \le 0$, as well as $(1 + q'(p^*))u(p^*) = (1 + q'(p^*))U(p^*)$. Revisiting (37), we see that

$$\begin{aligned}
1 + q'(0) &= (qu)'(p^*) + u(p^*)\\
&= q(p^*)u'(p^*) + (q(p^*) + 1)u(p^*)\\
&\le (1 + q(p^*))U(p^*)\\
&\le a(1 + q'(0)),
\end{aligned}$$

having used the fact that $qu'(p^*) \le 0$ and the definition of U. It is clear that this is a contradiction, since $0 < a < 1$, and proves that $u > aU$. Finally, taking $a \to 1$, we obtain the upper bound. The lower bound is obtained in the same way. ∎

The derivation of Theorem 3.2 is now as usual because $-\frac{1}{\alpha_0 t}$ is a subsolution of (33) and, using Lemma 3.3, we can choose

$$\begin{aligned}
\alpha_0 = \min_{0 \le p \le p_M} q'(p)&\frac{1 + q'(0)}{1 + \max_{0 \le p \le p_M} q'(p)}\\
&+ \frac{2}{N}\left(\frac{1 + q'(0)}{1 + \max_{0 \le p \le p_M} q'(p)}\right)^2.
\end{aligned}$$

Then, using Lemma 3.3 a second time,

$$\frac{1 + \min_{0 \le r \le p} q'(r)}{1 + q'(0)}w \ge \min_p h(p)w \ge -\frac{1}{\alpha_0 t},$$

which gives the result of Theorem 3.2.

3.2. L^∞-estimates when $G \neq 0$

Next we proceed by incorporating reaction terms, cf. (12), and prove the following theorem:

Theorem 3.5 (Lower bound on Δp, general G). *With the assumptions and notation of Theorem* 3.2, *we have, with the constant* δ_∞^h *defined below,*

$$\left(1 + \min_{0 \leq p \leq p_M} q'(r)\right)(\Delta p + G) \geq -\frac{1}{\tilde{\alpha}_0} \frac{\bar{\delta}_\infty^h e^{\bar{\delta}_\infty^h t}}{e^{\bar{\delta}_\infty^h t} - 1}.$$

Still using (31), we compute the evolution of the quantity hw, i.e.,

$$\frac{\partial(hw)}{\partial t} \geq h'|\nabla p|^2 w + \left(\frac{h'q}{h^2}\right)(hw)^2 + hq''|\nabla p|^2 w + \left(\frac{q'}{h}\right)(hw)^2$$
$$+ \underbrace{2h\nabla(p+q) \cdot \nabla w}_{I_1} + q\underbrace{h\Delta w}_{I_2} + (hw)\left(G'q - \left(\frac{4}{N} + q'\right)G\right). \quad (39)$$

Next, we rewrite the terms I_1 and I_2 using the new variable, hw. The first term, I_1, becomes

$$I_1 = 2\nabla(p+q) \cdot \nabla(hw) - 2w|\nabla p|^2(1+q')h'.$$

Regarding the second one, we obtain

$$\begin{aligned}
I_2 &= \Delta(hw) - w\Delta h - 2\nabla h \cdot \nabla w \\
&= \Delta(hw) - w(h''|\nabla p|^2 + h'\Delta p) - 2h'\nabla p \cdot \nabla w \\
&= \Delta(hw) - w(h''|\nabla p|^2 + h'w - h'G) - 2\frac{h'}{h}\nabla p \cdot (\nabla(hw) - wh'\nabla p) \\
&= \Delta(hw) - 2\frac{h'}{h}\nabla p \cdot \nabla(hw) + w|\nabla p|^2\left(2\frac{h'^2}{h} - h''\right) - (hw)^2\left(\frac{h'}{h^2}\right) + (hw)\left(\frac{h'G}{h}\right).
\end{aligned}$$

Substituting the terms I_1 and I_2 back into (39), we get

$$\frac{\partial(hw)}{\partial t} \geq \alpha_\infty^h(hw)^2 + \beta_\infty^h|\nabla p|^2 w + \mathcal{L}_\infty^h(hw) + \delta_\infty^h(hw), \quad (40)$$

where α_∞^h, β_∞^h, and $\mathcal{L}_\infty^h(hw)$ are as in inequality (33), and

$$\delta_\infty^h(p) := G'q - \left(\frac{4}{N} + q'\right)G + \frac{qh'G}{h} \leq \bar{\delta}_\infty < \infty. \quad (41)$$

In order to close the L^∞-type bound, it suffices to apply a slight variation of the proof of Theorem 3.2 incorporating additional terms related to the growth. Again, we have to require that

$$\alpha_\infty^h \geq \alpha_0 > 0 \quad \text{and} \quad \beta_\infty^h = 0, \quad (42)$$

which are identical to the conditions given by (34), (35).

Therefore Theorem 3.5 follows as in Section 2.2 because

$$-\frac{1}{\tilde{\alpha}_0} \frac{\bar{\delta}^h_\infty e^{\bar{\delta}^h_\infty t}}{e^{\bar{\delta}^h_\infty t} - 1}$$

is a subsolution of the corresponding equation.

4. L^2-type estimate

We now investigate the L^2 space which has been used in situations where the L^∞ estimate cannot be applied because the growth term depends on other quantities and cannot be differentiated with uniform control. As we shall see, the advantage of working in L^2 is to provide additional dissipation terms which do not appear in L^1 or L^∞, while keeping an estimate compatible with the free boundary in opposition to L^1.

We proceed again by departing from (19) which we write as

$$\partial_t \frac{|w|^2_-}{2} \leq -\frac{2}{N}|w|^3_- + \nabla p \cdot \nabla |w|^2_- + |w|_- \Delta(q|w|_-) + \left[G'q - \frac{4}{N}G\right]|w|^2_-. \quad (43)$$

We distinguish the two cases $G \equiv 0$ or not.

4.1. L^2-estimates when $G \equiv 0$. No weight

We are going to prove the following theorem:

Theorem 4.1. *Assume* $\inf_{0 \leq p \leq p_M} \left(\frac{2}{N} - 1 + \frac{q'}{2}\right) =: \alpha_0 \geq 0$ *and* $q'' \leq 0$; *then*

$$\int_{\mathbb{R}^N} |w(t)|^2_- \, dx + 2 \int_0^t \int_{\mathbb{R}^N} \left(\frac{2}{N} - 1 + \frac{q'}{2}\right)|w(s)|^3_- \, dx \, ds \leq \int_{\mathbb{R}^N} |w(t=0)|^2_- \, dx.$$

When, for $t \in [0, T]$, *we have* $\alpha_0 > 0$, *for solutions with compact support in x, it holds that*

$$\int_{\mathbb{R}^N} |w(t)|^2_- \, dx \leq \frac{C(T)}{t^2} \quad \forall t \in [0, T].$$

For the power law, $q' = \gamma$, we recover the condition obtained in [43], that is,

$$\gamma \geq 2 - \frac{4}{N}.$$

In the case of the DHV pressure, cf. (16), the assumption $q'' \leq 0$ is not met.

Notice also that the argument can be localised, see [28], and this allows us to remove the compact support assumption.

Proof of Theorem 4.1. Integrating (43), and using two integration by parts, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^N} \frac{|w|^2_-}{2} \, dx \leq -\frac{2}{N} \int_{\mathbb{R}^N} |w|^3_- \, dx + \int_{\mathbb{R}^N} |w|^3_- \, dx \underbrace{- \int_{\mathbb{R}^N} \nabla|w|_- \nabla(q|w|_-) \, dx}_{I}. \quad (44)$$

Integrating by parts, the first term, I, can be rewritten, i.e.,

$$
\begin{aligned}
I &= -\int_{\mathbb{R}^N} |w|_- \nabla q \cdot \nabla |w|_- \, dx - \int_{\mathbb{R}^N} q |\nabla |w|_-|^2 \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \Delta q |w|_-^2 \, dx - \int_{\mathbb{R}^N} q |\nabla |w|_-|^2 \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} q'' |\nabla p|^2 |w|_-^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} q' |w|_-^3 \, dx - \int_{\mathbb{R}^N} q |\nabla |w|_-|^2 \, dx,
\end{aligned}
$$

where we used the chain rule $\Delta q = q'' |\nabla p|^2 + q' w$.

Substituting the simplified expressions for I into (44), we obtain

$$
\frac{d}{dt} \int_{\mathbb{R}^N} \frac{|w|_-^2}{2} \, dx \le -\int_{\mathbb{R}^N} \left(\frac{2}{N} - 1 + \frac{q'}{2} \right) |w|_-^3 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} q'' |\nabla p|^2 |w|_-^2 \, dx \\
- \int_{\mathbb{R}^N} q |\nabla |w|_-|^2 \, dx.
$$

Since $q \ge 0$, this ensures that the estimate can be closed if $q' > 2 - \frac{4}{N}$ for all $p \in (0, p_M)$ and $q'' \le 0$. Indeed, we deduce the inequality

$$
\int_{\mathbb{R}^N} |w(t)|_-^2 \, dx \le -\int_{\mathbb{R}^N} \alpha_0 |w(t)|_-^3. \tag{45}
$$

The conclusions of the theorem follow by time integration (first estimate) or using a subsolution $\frac{C(T)}{t^2}$ (second estimate with regularising effect). ∎

It is interesting to investigate whether adding a weight can help us to include more general pressure laws.

4.2. L^2-estimates when $G \equiv 0$. With weights

In order to be as general as possible, we add a weight $h = h(p)$.

Theorem 4.2. *Assume there exists a positive weight $0 < c \le h(p) \le c^{-1}$ such that the two differential inequalities*

$$
\alpha_2^h := \frac{4}{N} + \frac{2h'q}{h} - 2 + q' \ge 0 \quad and \quad \beta_2^h := \frac{h''q + q''h - h'}{h} \le 0
$$

are met. Then

$$
\int_{\mathbb{R}^N} h |w|_-^2 (t) \, dx + \int_0^t \int_{\mathbb{R}^N} \alpha_2^h h |w|_-^3 \, dx \, dt \le \int_{\mathbb{R}^N} h |w|_-^2 (0) \, dx.
$$

Note that the same regularisation effect as in Theorem 4.1 can be obtained if $\alpha_2^h > \alpha_0$ for some constant $\alpha_0 > 0$.

Proof. Using the equation satisfied by the pressure, (17), and (43) for $|w|_-^2$, we compute the evolution in time of $h|w|_-^2$:

$$\frac{\partial}{\partial t}(h|w|_-^2) = h'|w|_-^2(|\nabla p|^2 + qw) + h\partial_t|w|_-^2$$

$$\leq h'|\nabla p|^2|w|_-^2 - h'q|w|_-^3$$

$$+ 2h\left(-\frac{2}{N}|w|_-^3 + 2|w|_-\nabla p \cdot \nabla|w|_- + |w|_-\Delta q|w|_-\right). \quad (46)$$

Integrating over $\mathbb{R}^N$ and with an integration by parts for the last two terms, we get

$$\frac{d}{dt}\int_{\mathbb{R}^N}(h|w|_-^2)\,dx \leq \int_{\mathbb{R}^N} h'|\nabla p|^2|w|_-^2\,dx - \int_{\mathbb{R}^N}\left(\frac{4}{N} + \frac{h'q}{h}\right)h|w|_-^3\,dx$$

$$+ 4\int_{\mathbb{R}^N} h|w|_-\nabla p \cdot \nabla|w|_-\,dx - 2\int_{\mathbb{R}^N}\nabla(h|w|_-)\nabla(q|w|_-)\,dx$$

$$\leq -\int_{\mathbb{R}^N} h'|\nabla p|^2|w|_-^2\,dx - \int_{\mathbb{R}^N}\left(\frac{4}{N} - 2 + \frac{h'q}{h}\right)h|w|_-^3\,dx$$

$$\underbrace{- 2\int_{\mathbb{R}^N}\nabla(h|w|_-)\nabla(q|w|_-)\,dx}_{I}. \quad (47)$$

Next we need to address the term I. We compute

$$I = \int_{\mathbb{R}^N}[h'\nabla p|w|_- + h\nabla|w|_-] \cdot [q'\nabla p|w|_- + q\nabla|w|_-]\,dx$$

$$= \int_{\mathbb{R}^N}\left[h'q'|\nabla p|^2|w|_-^2 + \frac{h'q + hq'}{2}\nabla p \cdot \nabla|w|_-^2 + hq|\nabla|w|_-|^2\right]dx$$

$$= \int_{\mathbb{R}^N}\left[h'q'|\nabla p|^2|w|_-^2 + \frac{h'q + hq'}{2}|w|_-^3\right.$$

$$\left. - \left[\frac{h'q + hq'}{2}\right]'|\nabla p|^2|w|_-^2 + hq|\nabla|w|_-|^2\right]dx. \quad (48)$$

Reorganising the terms, we get

$$I = \int_{\mathbb{R}^N}\left[\frac{h'q + hq'}{2}|w|_-^3 - \frac{h''q + hq''}{2}|\nabla p|^2|w|_-^2 + hq|\nabla|w|_-|^2\right]dx.$$

Finally, substituting I into (47), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^N}(h|w|_-^2)\,dx \leq -\int_{\mathbb{R}^N}\alpha_2^h h|w|_-^3\,dx + \int_{\mathbb{R}^N}\beta_2^h|\nabla p|^2 h|w|_-^2\,dx$$

$$- 2\int_{\mathbb{R}^N} hq|\nabla|w|_-|^2\,dx,$$

where

$$\alpha_2^h = \frac{4}{N} + \frac{2h'q}{h} - 2 + q' \quad \text{and} \quad \beta_2^h = \frac{h''q + q''h - h'}{h}.$$

By the assumption

$$\alpha_2^h \geq 0 \quad \text{and} \quad \beta_2^h \leq 0, \tag{49}$$

the statement holds true.   ■

Remark 4.3. For the condition $\alpha_2^h \geq 0$, the weight h does not help when it is positive because the term generated by the weight, $\frac{2h'q}{h^3}$, vanishes for $p = 0$. However, choosing $h = p$ should improve the range of parameters but we will lose the estimate near the free boundary. For the power law we reach the conditions

$$\alpha_2^h = \frac{4}{p^2 N} + \frac{3\gamma}{p^2} - \frac{2}{p^2} \geq 0,$$

which is less restrictive than when $h = 1$, while the condition $\beta_2^h \leq 0$ is fulfilled.

It is unclear to us how to choose the weight h for the DHV law in L^2.

Adding the reaction term G, we do not gain anything. Precisely, without the weight h, we obtain the same conditions on q' and q'' as those in Theorem 4.1. Since G is bounded and decreasing, cf. (13), the term which involves G can always be controlled. We have decided not to report all the calculations because they can be derived easily from those in Section 4.1.

To conclude the paper, we have decided to add the reaction term $G > 0$ and we can derive the L^2-estimate for $h|w|_-$.

4.3. L^2-estimates when $G \neq 0$. With weights

This section is dedicated to proving the following theorem.

Theorem 4.4. *Assume there exists a positive weight $0 < c \leq h(p) \leq c^{-1}$ such that the two differential inequalities $\alpha_2^h \geq 0$, $\beta_2^h \leq 0$ are satisfied, with α_2^h and β_2^h defined as in Theorem 4.2.*

Then there holds

$$\int_{\mathbb{R}^N} h|w|_-^2(t)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^N} \alpha_2^h h|w|_-^3\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \int_{\mathbb{R}^N} h|w|_-^2(0)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^N} \bar{\delta}_2^h h|w|_-^2\,\mathrm{d}x\,\mathrm{d}t,$$

where

$$\bar{\delta}_2^h = \sup_{0 \leq p \leq p_M} \left\{ 2G'q + G\left(2\left(1 - \frac{4}{N}\right) - \frac{h'q + q'h}{h}\right) \right\}.$$

As before, the same regularisation effect as in Theorem 4.1 is obtained if we can guarantee that $\alpha_2^h > \alpha_0$ for a positive constant $\alpha_0 > 0$. Theorem 4.4 proves that adding the reaction, the estimate does not gain anything.

Proof of Theorem 4.4. Starting from (46) and including the additional growth terms G, the evolution of $h|w|_-^2$ becomes

$$\frac{\partial}{\partial t}(h|w|_-^2) \leq h'|\nabla p|^2|w|_-^2 - h'q|w|_-^3$$

$$+ 2h\left(-\frac{2}{N}|w|_-^3 + \nabla p \cdot \nabla |w|_-^2 + |w|_-\Delta(q|w|_-)\right)$$

$$+ 2h\left(G'q - \frac{4}{N}G\right)|w|_-^2. \tag{50}$$

Integrating in space and by an integration by parts, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} h|w|_-^2\,\mathrm{d}x \leq -\int_{\mathbb{R}^N}\left(\frac{4}{N} + \frac{qh'}{h} - 2\right)h|w|_-^3\,\mathrm{d}x - \int_{\mathbb{R}^N} h'|w|_-^2|\nabla p|^2\,\mathrm{d}x$$

$$- 2\underbrace{\int_{\mathbb{R}^N} \nabla(h|w|_-)\cdot\nabla(q|w|_-)\,\mathrm{d}x}_{I}$$

$$+ 2\int_{\mathbb{R}^N} h|w|_-^2\left(G'q + G\left(1 - \frac{4}{N}\right)\right)\mathrm{d}x. \tag{51}$$

Using (48) and the definition of w, cf. (18), the term I simplifies to

$$I = \frac{1}{2}\int_{\mathbb{R}^N}(h'q + hq')|w|_-^3\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}(h''q + hq'')|\nabla p|^2|w|_-^2\,\mathrm{d}x$$

$$+ \frac{1}{2}\int_{\mathbb{R}^N}(h'q + hq')G|w|_-^2\,\mathrm{d}x + \int_{\mathbb{R}^N} hq|\nabla|w|_-|^2\,\mathrm{d}x. \tag{52}$$

Substituting everything into (51), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\frac{h|w|^2}{2}\,\mathrm{d}x \leq -\int_{\mathbb{R}^N}\alpha_2^h h|w|_-^3\,\mathrm{d}x + \int_{\mathbb{R}^N}|\nabla p|^2\beta_2^h h|w|_-^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\delta_2^h h|w|_-^2\,\mathrm{d}x,$$

where

$$\alpha_2^h = \frac{4}{N} + \frac{2qh'}{h} - 2 + q' \quad \text{and} \quad \beta_2^h = \frac{h''q + q''h - h'}{h},$$

as well as

$$\delta_2^h = 2G'q + G\left(2\left(1 - \frac{4}{N}\right) - \frac{h'q + q'h}{h}\right) \leq \bar{\delta}_2 < \infty,$$

where $\bar{\delta}_2$ is as in the statement. By the assumption,

$$\alpha_2^h \geq 0 \quad \text{and} \quad \beta_2^h \leq 0, \tag{53}$$

and the statement holds true. ∎

Remark 4.5. Different techniques have been used to perform an L^2-bound. For instance, in [28] the authors compute exactly the square of the expression $(w - G)$ and then they apply Young's inequality to the term

$$2\left(1 - \frac{2}{N}\right)G|w|_-^2.$$

This choice allows them to get an additional small constant $\mu > 0$ in the expression for α_2^h, which can help in getting a weaker condition to close the estimate. However, there is an additional term which involves the growth term, i.e.,

$$C \int_{\mathbb{R}^N} G^2 |w|_- \, dx,$$

with $C > 0$ a constant. Now the estimate can only be closed by assuming the standard conditions on G, cf. (13), plus an additional condition on the domain. Finally, the expression for δ_2^h changes a bit, i.e.,

$$(\delta_2^h)_Y = \frac{2G'q}{h} - G\left(\frac{h'q + q'h}{h^2}\right) \leq \tilde{\delta}_2 < \infty,$$

which can be bounded by controlling h'.

Remark 4.6. In the power-law case, regarding the conditions on α_2^h and β_2^h, we can refer to Remark 4.3. The last one, substituting $q = p\gamma$ and $h(p) = c_1 + p$, δ_2^h becomes

$$\delta_2^h = \gamma p(2G' - c_1 G) - (p + c_1)(G\gamma + 2G) < \infty. \tag{54}$$

Note that this expression is always non-positive for non-negative and decreasing growth terms; cf. (13).

5. Conclusions

The Aronson–Bénilan estimate has proven to be a fundamental tool for studying regularity and asymptotics in several problems related to the porous media equation. Even if it has been used mainly to control the Laplacian of the pressure from below by a term such as $\Delta p(t) \geq -\frac{C}{t}$, one may use it in other Lebesgue spaces. We have systematically studied the restrictions on the parameters and the conclusions that one can draw in L^1, L^2, and L^∞ (original work of Aronson and Bénilan). In particular, we considered two specific forms of the pressure law: the power law and DHV law.

Our conclusions are that the L^∞ setting provides the widest range of parameters, generating the strongest estimate. For instance, it can be applied to both pressure laws. On the other hand, the L^2 estimate requires restrictions on the parameters (which exclude the DHV law), but is enough to estimate the Laplacian of the pressure for the power law. Because of integration by parts, and because a dissipation term occurs explicitly, it is however useful for some strongly coupled problems where L^∞ bounds are not possible. The L^1 estimate turns out to be the simplest but is only useful in space dimension $N = 1$.

When weights are included in order to treat more general equations of state, we improve the results in [25] and we obtain estimates correctly scaled with respect to the Hele–Shaw limit, which, with our notation, is expressed as $|w|_- \approx 0$ for $\gamma \gg 1$ or $\varepsilon \ll 1$.

If one wishes to estimate the quantity $p\Delta p(t)$, losing regularity near the free boundary, then one can drastically extend the range of possible pressure laws.

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