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Ergodic Automorphisms of T^∞ Are Bernoulli Transformations

By

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§ 1. Introduction

The purpose of this paper is to prove the fact stated by the title. After Ornstein and Friedman [5], [1], several authors studied the Bernoulli properties of various transformations. Especially Katznelson [3] proved that every ergodic automorphism of a finite-dimensional torus is a Bernoulli transformation extending the results by Sinai-Ornstein-Friedman [9], [1]. The present result is on the way towards the conjecture that every ergodic automorphism of a compact metrizable abelian group is a Bernoulli transformation.

Let X be a compact metrizable group and μ be its normalized Haar measure. Then (X, μ) is a Lebesgue space (cf. [10]). Let σ be an (group) automorphism of X, then σ is an invertible measure-preserving transformation of (X, μ). Our problem is concerned with measure-theoretic properties of σ. We call σ a *Bernoulli transformation* if there exists a measurable partition ξ of X such that $\{\sigma^n\xi\}_{-\infty<n<\infty}$ are independent and $\bigvee_{-\infty}^{\infty} \sigma^n\xi=\varepsilon$ the partition of X into individual points.

We assume $X=\boldsymbol{T}^\infty$ an infinite-dimensional torus i.e. X is a compact metrizable abelian group which is connected, locally connected and infinite-dimensional. Further we assume naturally that the automorphism σ is ergodic i.e. any σ-invariant measurable set has measure 0 or 1.

Our result is the following

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Theorem. *If σ is an ergodic (group) automorphism of an infinite-dimensional torus T^{∞}, then σ is a Bernoulli transformation.*[1)]

§ 2. Preliminary Discussions

Let σ be an ergodic automorphism of $X=T^{\infty}$. To prove the theorem, the character group G of X plays an essential role. G is countable and discrete. Since X is connected, G is torsionfree. Let $\langle g\rangle$ denote the free cyclic group generated by $g\in G$, $g\neq 1$. Then we have $G=\bigotimes_{-\infty}^{\infty}\langle g_n\rangle$ (a direct product of discrete groups).

Conventions. We make use of multiplications for the group operations instead of additions. The units of X and G are denoted by e and 1 respectively.

The automorphism σ of X induces the dual automorphism U of G. We classify elements of G into two classes. The first class is characterized by the condition

(**A**) *for $g\in G$ there exist integers $k\geqq 0, n_0, n_1, \ldots, n_k$ such that $(n_0, \ldots, n_k)\neq(0, \ldots, 0)$ and $g^{n_0}Ug^{n_1}, \ldots U^k g^{n_k}=1$.*

Let G_A be the set of all $g\in G$ satisfying the condition (A), then G_A is U-invariant (i.e. $UG_A=G_A$). It is not hard to see that G_A is a subgroup.

Let $K(g)$ denote the subgroup of G generated by

$$\{U^n\langle g\rangle;\ n\in \boldsymbol{Z}\}^{2)} \quad \text{i.e.} \quad K(g)=\bigcup_{N=1}^{\infty}\prod_{n=-N}^{N}U^n\langle g\rangle.$$

Lemma 1. *If $g\in G\backslash G_A$, then* (i) $K(g)=\bigotimes_{-\infty}^{\infty}U^n\langle g\rangle$ *and* (ii) $G_A\cap K(g)=\{1\}$.

Proof. To prove (i) it is enough to show that $\prod_{-N}^{N}U^n\langle g\rangle=\bigotimes_{-N}^{N}U^n\langle g\rangle$ for each $N\geqq 1$. If $f\in\langle g\rangle\cap U\langle g\rangle$ then $f=g^{n_0}=Ug^{n_1}$ for some n_0 and n_1. Since g does not satisfy (A), $n_0=n_1=0$ and so $f=1$. If $f\in(\langle g\rangle\otimes U\langle g\rangle)\cap U^2\langle g\rangle$ then $f=1$ by the same reason as above, and so on.

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1) After the preparation of the manuscript, the authors were informed that D. Lind obtained the same result which will appear in the Israel Journal.

2) $\boldsymbol{Z}$ denotes the set of all integers.

To prove (ii) take any $f\in G_A\cap K(g)$. Then f is of the form $f=U^{i_1}g^{m_1}\ldots U^{i_s}g^{m_s}$ and f satisfies the condition (A): $f^{n_0}Uf^{n_1}\ldots U^kf^{n_k}=1$. Putting the above expression of f into the last equation, we see that $g\in G_A$ unless $f=1$.

We will prove the theorem by the following steps:

Case 1. $G=G_A$.

Case 2. $G=K(g)$, $g\in G\backslash G_A$.

Case 3. $G=G_AK(g_1)\ldots K(g_N)$, $g_i\in G\backslash G_A$, $1\leqq i\leqq N$.

Case 4. *General case.*

§ 3. Cases 1 and 2

Case 1. We assume $G=G_A$. Let $G=\{f_1,f_2,\ldots\}$ and G_n the subgroup generated by $\{U^kf_i;\ k\in\mathbf{Z},\ 1\leqq i\leqq n\}$ for $n\geqq 1$.

Lemma 2. Rank $(G_n)<\infty$ *for all* $n\geqq 1$.

Proof. By the condition (A) we have

$$f_i^{n_0(i)}=Uf_i^{n_1(i)}\ldots U^{k_i}f_i^{n_{k_i}(i)},\qquad 1\leqq i\leqq n.$$

Let H be the subgroup generated by $\{U^kf_i;\ 1\leqq k\leqq k_i,\ 1\leqq i\leqq n\}$. Then it is easy to see that for any $k\in\mathbf{Z}$ and $1\leqq i\leqq n$ there is $m\neq 0$ such that $U^kf_i^m\in H$. Therefore for any $g\in G_n$ there is $m\neq 0$ such that $g^m\in H$. This implies that $\mathrm{rank}(G_n)\leqq k_1+\ldots+k_n<\infty$.

Let $X_n=\mathrm{ann}(G_n)$ (the annihilator of G_n), then G_n is the character group of X/X_n. It is known that $\dim(X/X_n)=\mathrm{rank}(G_n)$ (cf. [8]). Thus the factor group X/X_n is a finite-dimensional torus, indeed it is compact, metrizable, abelian, connected, locally connected and finite-dimensional. Since $UG_n=G_n$, we have $\sigma X_n=X_n$ and so σ induces a factor automorphism σ_n of X/X_n, which is obviously ergodic. By the theorem of Katznelson [3], σ_n is a Bernoulli transformation. Since $G_n\subset G_{n+1}$, $\bigcup_{n=1}^{\infty}G_n=G$, we have $X_n\supset X_{n+1}$, $\bigcap_{n=1}^{\infty}X_n=\{e\}$, and hence the following Lemma 3 implies that σ is a Bernoulli transformation.

Lemma 3. (Ornstein [6]). *Let σ be an ergodic invertible measure-preserving transformation on a Lebesgue space $(X, \mathfrak{F}, \mu)$. Assume that there is a sequence of sub-σ-fields $\{\mathfrak{F}_n\}$ such that $\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$, $\sigma(\mathfrak{F}_n) = \mathfrak{F}_n$, $\bigvee_{n=1}^{\infty} \mathfrak{F}_n = \mathfrak{F}$ and each factor $\sigma_n = \sigma|_{\mathfrak{F}_n}$ is a Bernoulli transformation. Then σ is itself a Bernoulli transformation.*

Remark. Ornstein [6] proved the above lemma under the additional assumption that the entropy $h(\sigma_n) < \infty$ for all n. But it is easy to see that we can remove this assumption using

Lemma 4. (Ornstein [7]). *Every non-trivial factor of a Bernoulli transformation is Bernoullian.*

Case 2. We assume $G = K(g)$. By Lemma 1, $K(g) = \bigotimes_{-\infty}^{\infty} U^n \langle g \rangle$. Therefore Pontrjagin duality theorem ([8]) implies $X = \bigotimes_{-\infty}^{\infty} \sigma^n X_0$ (a direct product of compact groups) where $X_0 = \mathrm{ann}\,(\bigotimes_{n \neq 0} U^n \langle g \rangle)$. Then σ is Bernoullian, indeed σ acts on X as $(x_n)_{-\infty}^{\infty} \to (\sigma x_{n-1})_{-\infty}^{\infty}$ and so $\varepsilon(X_0) \otimes \bigotimes_{n \neq 0} \sigma^n(\nu(X_0))$ is a Bernoulli generator of σ, where ν denotes the trivial partition.

§ 4. Case 3

We assume that $G = G_A G_0$ where $G_0 = K_1 \dots K_N$ and $K_j = K(g_j)$, $g_j \in G \backslash G_A$, $1 \leqq j \leqq N$. Let k be the largest integer such that (renumbering if necessary)

$$(K_1 \otimes \dots \otimes K_k) \cap K_j \neq \{1\}, \quad k+1 \leqq j \leqq N.$$

Lemma 5. $G_A \cap (K_1 \otimes \dots \otimes K_k) = \{1\}$.

Proof. Take any $f \in G_A \cap (K_1 \otimes \dots \otimes K_k)$, then we have $f = f_1 \dots f_k$, $f_j \in K_j$, $1 \leqq j \leqq k$, and $f^{n_0} U f^{n_1} \dots U^s f^{n_s} = 1$ for $(n_0, \dots, n_s) \neq (0, \dots, 0)$. Hence we get $f_1^{n_0} \dots U^s f_1^{n_s} = (f_2 \dots f_k)^{-n_0} \dots U^s (f_2 \dots f_k)^{-n_s}$ which implies $f_1 = 1$ and inductively all $f_j = 1$, and so $f = 1$.

Thus we have $G = (G_A \otimes K_1 \otimes \dots \otimes K_k) K_{k+1} \dots K_N$. If $k = N$, there is nothing to prove. Indeed we then have $X=X_A\otimes X_1\otimes\ldots\otimes X_N$ where the character group of X_A and X_j are G_A and K_j for $j=1,\ldots,N$, and so Cases 1 and 2 imply that σ is a Bernoulli transformation as a direct product of Bernoulli transformations.

Let us assume $k<N$ and denote

$$G_1=K_1\otimes\ldots\otimes K_k.$$

We will make use of a divisible extension $\bar{G}$ of G, namely $\bar{G}$ is a minimal divisible (=complete) group containing G (cf. [4]). In order to clear the structure of $\bar{G}$ we have the following Lemma 6, of which proof is given in Appendix.

Lemma 6. *Let* $K=\bigotimes_{-\infty}^{\infty} U^n\langle g\rangle$ *where* g *is free. Then there exists a divisible extention* $\bar{K}=\bigotimes_{-\infty}^{\infty}\bar{U}^nQ_0$ *of* K*, where* Q_0 *is an abelian group isomorphic to the (additive) group* $\boldsymbol{Q}$ *of all rational numbers and* $\bar{U}$ *is an automorphism of* $\bar{K}$ *which is an extension of* U.

Using Lemma 6, it is not hard to see that there exists a divisible extension $\bar{G}=(\bar{G}_A\otimes\bar{K}_1\otimes\ldots\otimes\bar{K}_k)\bar{K}_{k+1}\ldots\bar{K}_N$ of G, where $\bar{K}_j=\bigotimes_{-\infty}^{\infty}\bar{U}^nQ_j$, $Q_j\cong\boldsymbol{Q}$ and $\bar{U}$ is an automorphism of $\bar{G}$ which is an extension of U. We remark that $\bar{G}$ is also torsionfree (cf. [4]). Let us put

$$\bar{G}_1=\bar{K}_1\otimes\ldots\otimes\bar{K}_k$$

which is a divisible extension of G_1.

Lemma 7. $G_0\subset\bar{G}_1$.

Proof. In order to prove the lemma it is enough to show that for any fixed $j=k+1,\ldots,N$, there is an integer $n\neq0$ such that $g_j^n\in G_1$.

Let $G_2=\{g\in G_1K_j;\ g^n\in G_1$ for some $n\neq0\}$, then G_2 is a subgroup of G_1K_j such that $G_1\subset G_2\subset\bar{G}_1$ and $UG_2=G_2$. Take $1\neq f\in G_1\cap K_j$ then f has the form $f=U^{m_1}g_j^{n_1}\ldots U^{m_s}g_j^{n_s}$. Hence multiplying G_2 to the both sides of the last equation and operating some U^m, we have an equation of the form

$$g_j^{n_0}Ug_j^{n_1}\ldots U^pg_j^{n_p}G_2=G_2,\ n_0\neq0,\ n_p\neq0. \qquad (1)$$

Assume that p is the smallest non-negative integer which assures the equation (1). If $p=0$, (1) implies $g_j^{n_0}\in G_2$ and so $g_j^{n_0 n}\in G_1$ for some $n\neq 0$. Assuming $p>0$ we will show a contradiction.

Let us denote $\bar{K}_{j,p}=\bigotimes_{i=1}^{p}\bar{U}^i Q_j$ which is a divisible extension of $K_{j,p}=\bigotimes_{i=1}^{p}U^i\langle g_j\rangle$, and put $\hat{H}_i=G_2\bar{U}^i\bar{K}_{j,p}$ and $H_i=G_2U^iK_{j,p}$. Take any $\bar{g}\in G_2\cap\bar{U}^i\bar{K}_{j,p}$ then $\bar{g}G_2=G_2$ and there is $r\neq 0$ such that $U^{-i}\bar{g}^r\in K_{j,p}$. Hence we have $U^{-i}\bar{g}^r=U_j^{r_1}\dots U^p g_j^{r_p}$ and so $G_2=U^{-i}\bar{g}^rG_2=Ug_j^{r_1}\dots U^p g_j^{r_p}G_2$ for some $r_1,\dots,r_p\in\boldsymbol{Z}$ which implies $\bar{g}=1$ by the minimality of p. Thus we have $\hat{H}_i=G_2\otimes\bar{U}^i\bar{K}_{j,p}$ and $H_i=G_2\otimes U^iK_{j,p}$ for all i. The equation (1) implies that $g_j^{n_0}\in H_0$ and inductively that $U^ig_j^{n_i}\in H_0$ for some $n_i\neq 0$ for all i. Therefore for any $g\in K_j$ there is $n\neq 0$ such that $g^n\in H_0$.

We will prove

$$G_2\bar{K}_j=G_2\otimes\bar{K}_{j,p}. \tag{2}$$

First notice that G_2K_j/G_2 is torsionfree. Indeed if $g^mG_2=G_2$ for $g\in K_j$ and $m\neq 0$ then $g^m\in G_2$ and so $g^{mn}\in G_1$ for some $n\neq 0$, which implies $g\in G_2$. Let $\overline{G_2K_j/G_2}$ be a divisible extension of G_2K_j/G_2, then it is also torsionfree ([4]). We have

$$\hat{H}_i/G_2\subset\overline{G_2K_j/G_2}\subset G_2\bar{K}_j/G_2,\quad i\in\boldsymbol{Z},$$

since $G_2\bar{K}_j/G_2$ $(\supset G_2K_j/G_2)$ is divisible and $\hat{H}_i/G_2$ is a divisible extension of H_i/G_2 $(\subset G_2K_j/G_2)$. Take any $F\in\overline{G_2K_j/G_2}$ then there is $m\neq 0$ such that $F^m\in G_2K_j/G_2$, and so there is $g\in K_j$ such that $F^m=gG_2$. For this g and fixed $i\in\boldsymbol{Z}$, there is $n\neq 0$ such that $g^n\in H_i$. Hence g^n is decomposed into $g^n=fk$, $f\in G_2$, $k\in U^iK_{j,p}$. For this k there is $\bar{k}\in\bar{U}^i\bar{K}_{j,p}$ such that $\bar{k}^{mn}=k$. Therefore we have $F^{mn}=\bar{k}^{mn}G_2$ which implies $F=\bar{k}G_2\in\hat{H}_i/G_2$, because $\overline{G_2K_j/G_2}$ is torsionfree. Thus we have $\hat{H}_i/G_2=\overline{G_2K_j/G_2}=\hat{H}_0/G_2$ for all i, which implies $\hat{H}_i=\hat{H}_0$ for all i and so $\bar{K}_j\subset\hat{H}_0$. This also implies the equation (2).

Take any $\bar{g}\in\overline{G_2\cap K_j}$, where $\overline{G_2\cap K_j}$ is a divisible extension of $G_2\cap K_j$, then there is $n\neq 0$ such that $\bar{g}^n\in G_2\cap K_j$ and so $\bar{g}^n(G_2\cap\bar{K}_j)=G_2\cap\bar{K}_j$. Since $\bar{K}_j/G_2\cap\bar{K}_j\cong G_2\bar{K}_j/G_2=(G_2\otimes\bar{K}_{j,p})/G_2\cong\bar{K}_{j,p}$ is torsionfree we have $\bar{g}\in G_2\cap\bar{K}_j$. Therefore we have $\overline{G_2\cap K_j}\subset G_2\cap\bar{K}_j\subset G$, which implies $G_2 \cap K_j = \{1\}$ and so $G_1 \cap K_j = \{1\}$ because the character group X of G is locally connected (cf. [8]). Thus we arrive at a contradiction, which proves Lemma 7.

Now let us prove the Bernoulli property of (X, σ). Let $\hat{X} = X_A \otimes \bar{X}_1$ be the character group of $G_A \otimes \bar{G}_1$ and $\bar{\sigma}$ the dual automorphism of $\hat{X}$ induced by $\bar{U}$. Then the factors $(X_A, \bar{\sigma})$ and $(\bar{X}_1, \bar{\sigma})$ are Bernoullian by Case 1 and the same reason as Case 2 respectively. Therefore $(\hat{X}, \bar{\sigma})$ is also Bernoullian. Since (X, σ) is a factor of $(\hat{X}, \bar{\sigma})$ (i.e. $X = \hat{X}/\mathrm{ann}\,(G)$), (X, σ) is a Bernoulli transformation by Lemma 4.

§ 5. General Case

There is a sequence $\{g_n\} \subset G \backslash G_A$ such that putting

$$G_n = G_A K(g_1) K(g_2) \dots K(g_n), \quad n \geqq 1,$$

we have $UG_n = G_n$, $G_n \subset G_{n+1}$ and $\bigcup_{n=1}^{\infty} G_n = G$. Let $X_n = \mathrm{ann}(G_n)$, then $\sigma X_n = X_n$ and G_n is the character group of X/X_n. Hence σ on X/X_n is Bernoullian by Case 3. Since $X_n \supset X_{n+1}$ and $\bigcap_{n=1}^{\infty} X_n = \{e\}$, Lemma 3 implies that σ on X is itself a Bernoulli transformation. Thus the proof of our theorem is completed.

§ 6. Examples

Lemma 1 and the argument of Case 2 imply that if $G \neq G_A$ then the entropy $h(\sigma) = \infty$. This applies also to an automorphism of a finite-dimensional torus, and we have $G = G_A$ for it because it has a finite entropy. The first example is like a finite-dimensional one.

Example 1. Let $\{n_i;\ i \geqq 1\}$ be an infinite non-decreasing sequence of integers such that $n_1 \geqq 2$. Let σ_i be an ergodic automorphism of the torus T^{n_i} and φ_i be a continuous homomorphism from T^{n_i} into $T^{n_{i+1}}$ for all $i \geqq 1$. Define an infinite-dimensional torus $T^{\infty} = \bigotimes_{i=1}^{\infty} T^{n_i}$. Denoting $x = (x_1, x_2, \dots) \in T^{\infty}$ where $x_i \in T^{n_i}$, $i \geqq 1$, we define a mapping

$$\sigma(x)=(\sigma_1(x_1),\ \varphi_1(x_1)\sigma_2(x_2),\ \varphi_2(x_2)\sigma_3(x_3),\ \ldots).$$

It is easy to see that σ is an automorphism of the topological group T^∞. Since the subgroup $\bigotimes_{i=k+1}^{\infty} T^{n_i}$ is σ-invariant, σ induces the factor automorphism $\sigma^{(k)}$ of the factor group $T^{(k)}=T^\infty/\bigotimes_{i=k+1}^{\infty} T^{n_i}=T^{n_1}\otimes\ldots\otimes T^{n_k}$:

$$\sigma^{(k)}(x_1,\ \ldots,\ x_k)=(\sigma_1(x_1),\ \varphi_1(x_1)\sigma_2(x_2),\ \ldots,\ \varphi_{k-1}(x_{k-1})\sigma_k(x_k)).$$

It can be proved inductively using the following lemma that each $\sigma^{(k)}$ is ergodic. Hence σ is itself ergodic.

Lemma 8. *Let σ_i be an ergodic automorphism of a compact abelian metrizable group X_i ($i=1, 2$) and φ be a continuous homomorphism from X_1 into X_2. Define an automorphism σ of $X_1\otimes X_2$ by*

$$\sigma(x_1,\ x_2)=(\sigma_1(x_1),\ \varphi(x_1)\sigma_2(x_2)).$$

Then σ is ergodic.

Proof. Let G_i be the character group of X_i ($i=1, 2$), then $G_1\otimes G_2$ is the character group of $X_1\otimes X_2$. Denote the dual automorphisms of σ, σ_1 and σ_2 by U_σ, U_{σ_1} and U_{σ_2} respectively. Since

$$\sigma^n(x_1,\ x_2)=(\sigma_1^n(x_1),\ \psi_n(x_1)\sigma_2^n(x_2))$$

where

$$\psi_n(x_1)=\varphi(\sigma_1^{n-1}(x_1))\sigma_2(\varphi(\sigma_1^{n-2}(x_1))\ \cdots\ \sigma_2^{n-1}(\varphi(x_1)),$$

we have

$$U_\sigma^n(g_1\otimes g_2)=(U_{\sigma_1}^n g_1)(g_2\circ\psi_n)\otimes U_{\sigma_2}^n g_2$$

for $g_i\in G_i$, $i=1, 2$. Hence for $g_1\otimes g_2\neq 1\otimes 1$

$$\begin{aligned}&(g_1\otimes g_2,\ U_\sigma^n(g_1\otimes g_2))_{L^2(X_1\otimes X_2)}\\&\quad=(g_1,\ (U_{\sigma_1}^n g_1)(g_2\circ\psi_n))_{L^2(X_1)}(g_2,\ U_{\sigma_2}^n g_2)_{L^2(X_2)}=0,\end{aligned}$$

because U_{σ_1} and U_{σ_2} have no finite orbit except 1. Thus U_σ has no finite orbit except $1\otimes 1$, and hence σ is ergodic.

Next we will show that $G=G_A$ where G is the character group of

T^∞. Let G_i be the character group of T^{n_i}, $i\geqq 1$, then $G=\bigotimes_{i=1}^{\infty} G_i$. Let U and $U^{(k)}$ denote the dual automorphisms of G and $G^{(k)}=G_1\otimes\ldots\otimes G_k$ (a subgroup of G) induced by σ and $\sigma^{(k)}$ respectively. Each $g\in G$ has the form $g=\bigotimes_{i=1}^{\infty} g_i$ where $g_i=1$, $i\geqq k+1$, for some k. Hence we can consider $g\in G^{(k)}$ and $Ug=U^{(k)}g$. Since $\sigma^{(k)}$ is an automorphism of a finite-dimensional torus, we have $g\in G_A$ for $U^{(k)}$ and so $g\in G_A$ for U.

Example 2. Let $T^\infty=\bigotimes_{-\infty}^{\infty} T_i$, $T_i\cong T^1$, be an infinite-dimensional torus and $G=\bigotimes_{-\infty}^{\infty} G_i$ the character group of T^∞. Let U_0 be the shift automorphism of G: $(U_0g)_i=g_{i-1}$, $-\infty<i<\infty$. Let U_1' be an automorphism of $G_0\otimes G_1$ which is dual to an ergodic automorphism of $T_0\otimes T_1\cong T^2$. Define an automorphism U_1 of G by

$$(U_1g)_i=\begin{cases} g_i, & \text{if } i\neq 0, 1,\\ (U_1'(g_0, g_1))_i, & \text{if } i=0, 1.\end{cases}$$

Then we define automorphism $U=U_0U_1$ of G. We take $U_1'(g_0, g_1)=(g_0^2g_1, g_0g_1)$ for simplicity. Notice that U_1' is given by the matrix $\begin{pmatrix} 2 & 1\\ 1 & 1\end{pmatrix}$ in the usual notation of addition. Hence we have

$$\begin{aligned} &U(\ldots, g_{-1}, g_0, \quad g_1, \quad g_2, \quad g_3, \ldots)\\ &=(\ldots, g_{-2}, g_{-1}, g_0^2g_1, g_0g_1, g_2, \ldots).\end{aligned}$$

First we will show that U has no finite orbit except 1 and so the dual automorphism of T^∞ induced by U is ergodic. Indeed if $U^ng=g$ for $g=(g_i)_{-\infty<i<\infty}\in G$ then $g_i=g_{n+i}$ for $i\geqq 2$, $g_i=g_{-n+i}$ for $i\leqq 0$ and $g_0g_1=g_{n+1}$. Since $g_i=1$ except a finite number of i, we have $g_i=1$ for all $-\infty<i<\infty$ i.e. $g=1$.

Next we will show that $G_A=\{1\}$. Assume $g^{n_0}Ug^{n_1}\ldots U^kg^{n_k}=1$ for $g=(g_i)_{-\infty<i<\infty}\in G$. Then we have $g_{k+i}^{n_0}g_{k+i-1}^{n_1}\cdots g_i^{n_k}=1$ for $i\geqq 2$ and $g_i^{n_0}g_{i-1}^{n_1}\cdots g_{i-k}^{n_k}=1$ for $i\leqq 0$, and so $g_i=1$ for $i\neq 1$. We have also $g_i^{n_0}(g_0^2g_1)^{n_1}\cdots(g_{-k+1}^2g_{-k+2}^2\cdots g_0^2g_1)^{n_k}=1$ and so $g_1=1$. Thus we get $g=1$.

Appendix

We suppose that the fact stated in Lemma 6 is well known. But we can not find it in literatures, so we will give here its proof for the completeness.

Since each $U^n\langle g\rangle$ is isomorphic to the (additive) group $\mathbb{Z}$ of all integers, there is a divisible extension $\bar{K}=\bigotimes_{-\infty}^{\infty} Q_n$ of K where each Q_n is isomorphic to Q (cf. [4]). Let us define $\bar{U}$ as follows. For each $f\in\bar{K}$ there is k such that $\bar{f}^k\in K$, and then there is unique $\bar{f}\in\bar{K}$ such that $\bar{f}^k= Uf^k$. It is easy to see that f does not depend on the choice of k. Thus $\bar{U}f=\bar{f}$ defines a transformation $\bar{U}$ on $\bar{K}$.

Let us now prove that $\bar{U}$ is an automorphism of $\bar{K}$. Let $f, h\in\bar{K}$ and take i and k such that f^i, $h^k\in K$. Then $(\bar{U}fh)^{ik}=U(fh)^{ik}=Uf^{ik}Uh^{ik}= (\bar{U}f)^{ik}(\bar{U}h)^{ik}=(\bar{U}f\bar{U}h)^{ik}$ and so $\bar{U}fh=\bar{U}f\bar{U}h$; $\bar{U}$ is a homomorphism. Let $f\in\bar{K}$ and $f^k\in K$. Take $\bar{f}\in\bar{K}$ such that $\bar{f}^k=U^{-1}f^k$. Then $(\bar{U}\bar{f})^k=U\bar{f}^k =f^k$ and so $\bar{U}\bar{f}=f$; $\bar{U}$ is onto. Assume $\bar{U}f=1$ and $f^k\in K$, then $Uf^k= (\bar{U}f)^k=1$ and so $f^k=1$ which implies $f=1$; $\bar{U}$ is one-to-one.

Next let us prove $\bar{U}Q_n=Q_{n+1}$. Take any $f\in Q_n$ and k such that $f^k\in U^n\langle g\rangle$. Then $(\bar{U}f)^k=Uf^k\in U^{n+1}\langle g\rangle$ and hence $\bar{U}f\in Q_{n+1}$. Conversely take any $f\in Q_{n+1}$ and k such that $f^k\in U^{n+1}\langle g\rangle$. Then $(\bar{U}^{-1}f)^k=U^{-1}f^k\in U^n\langle g\rangle$ and so $f\in\bar{U}Q_n$. This completes the proof.

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