Ann. Inst. H. Poincaré
Anal. Non Linéaire (Online first)
DOI 10.4171/AIHPC/87

Published by EMS Press

Stable singularity formation for the inviscid primitive equations

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Abstract. The primitive equations (PEs) model large-scale dynamics of the oceans and the atmosphere. While it is by now well known that the three-dimensional viscous PEs are globally well posed in Sobolev spaces, and that there are solutions to the inviscid PEs (also called the hydrostatic Euler equations) that develop singularities in finite time, the qualitative description of the blowup still remains undiscovered. In this paper, we provide a full description of two blow-up mechanisms, for a reduced PDE that is satisfied by a class of particular solutions to the PEs. In the first one a shock forms, and pressure effects are subleading, but in a critical way: they localize the singularity closer and closer to the boundary near the blow-up time (with a logarithmic-in-time law). This first mechanism involves a smooth blow-up profile and is stable among smooth enough solutions. In the second one the pressure effects are fully negligible; this dynamics involves a two-parameter family of nonsmooth profiles, and is stable only by smoother perturbations.

1. Introduction

We consider the three-dimensional inviscid primitive equations (PEs)

$$u_t + uu_X + vu_Y + wu_Z + p_X - \Omega v = 0, \tag{1a}$$
$$v_t + uv_X + vv_Y + wv_Z + p_Y + \Omega u = 0, \tag{1b}$$
$$p_Z + T = 0, \tag{1c}$$
$$\theta_t + u\theta_X + v\theta_Y + w\theta_Z = 0, \tag{1d}$$
$$u_X + v_Y + w_Z = 0, \tag{1e}$$

set in the domain

$$\mathcal{D} = \mathcal{M} \times [0,1] = \big\{(X,Y,Z)\colon (X,Y) \in \mathcal{M}, 0 \le Z \le 1\big\},$$

where $\mathcal{M} \subseteq \mathbb{R}^2$ is a smooth bounded domain with real analytic boundary. System (1) is supplemented with the initial value (u_0, v_0, θ_0), and satisfies the relevant geophysical

2020 Mathematics Subject Classification. Primary 35B44; Secondary 35Q86, 86A10.
Keywords. Inviscid primitive equations, hydrostatic Euler equations, blowup, stability.

boundary conditions (cf. [25]):

$$
\begin{aligned}
w(t,X,Y,Z) &= 0 \quad \text{on } Z=0 \text{ and } Z=1,\\
\int_0^1 (u,v)(t,X,Y,Z)\,dZ\cdot\vec{n} &= 0 \quad \text{on } (X,Y)\in\partial\mathcal{M},
\end{aligned}
\tag{2}
$$

where $\vec{n}$ is the outward unit normal to $\partial\mathcal{M}$. System (1) is derived as a formal asymptotic limit of the small aspect ratio (the ratio of the depth or the height to the horizontal length scale) from the Boussinesq system (see [1, 29]). With full viscosity, the global existence of strong solutions for the three-dimensional PEs was first established in [9], and later in [20,22,27,28]. The above results were extended to the cases with only horizontal viscosity; see [5–7]. With only vertical viscosity, the ill-posedness in Sobolev spaces is shown in [33]. This ill-posedness can be overcome by considering additional linear (Rayleigh-like friction) damping [8], or Gevrey regularity and some convex conditions on the initial data [16].

In the absence of viscosity, and for adiabatic systems (i.e., constant temperature that can be set by convention to be zero), system (1) is also called the hydrostatic Euler equations. Such a system has a loss of horizontal derivative, making local well-posedness in Sobolev spaces a hard problem for general initial data. Indeed, the linear and nonlinear ill-posedness in any Sobolev space have been established in [33] and in [19], respectively. On the other hand, by assuming either real analyticity or some special structures (local Rayleigh condition) on the initial data, one is able to establish the local well-posedness; see [2, 3, 17, 18, 24, 25, 30]. Moreover, it was proven that smooth solutions to the inviscid PEs, in the absence of rotation, can develop singularities in finite time (cf. [4, 34]). Recently, it has been shown in [21] that the results about ill-posedness and finite-time blowup can be extended to the case with rotation.

The different proofs of the singularity formation rely on the proof of the finite-time blowup for the corresponding two-dimensional model for which the initial data can be lifted to the full three-dimensional equation, where the well-posedness is applicable. To get to the two-dimensional system, the idea in [4, 21] is to observe that if $\Omega = 0$ and initially $\theta_0 = v_0 = 0$, then any smooth enough solution (u, v, w, θ) to system (1), with initial data (u_0, v_0, θ_0) and boundary condition (2), must satisfy $\theta(t, X, Y, Z) \equiv 0$ and $v(t, X, Y, Z) \equiv 0$. Moreover, if initially u_0 is independent of the Y variable, then any smooth enough solution remains independent of the Y variable. Therefore, under the assumption that we have a smooth solution and for the initial data

$$
u_0(X,Y,Z) = u_0(X,Z), \quad v_0(X,Y,Z) = 0, \quad \theta_0(X,Y,Z) = 0,
$$

we obtain the two-dimensional inviscid PE system (also known as the hydrostatic Euler equations)

$$
u_t + u\,u_X + w u_Z + p_X = 0, \tag{3a}
$$

$$
p_Z = 0, \tag{3b}
$$

$$
u_X + w_Z = 0. \tag{3c}
$$

Since system (3) is independent of the Y variable, the horizontal domain $\mathcal{M}$ needs to be Y independent and translation invariant in Y. Hence without loss of generality we may consider system (3) set on

$$\mathcal{D} = \{(X,Z): -L \le X \le L,\ 0 \le Z \le 1\},$$

and the boundary conditions (2) become

$$\begin{gathered} w(t,X,0) = w(t,X,1) = 0, \\ \int_0^1 u(t,-L,Z)\,dZ = \int_0^1 u(t,L,Z)\,dZ = 0. \end{gathered} \tag{4}$$

From (3c) and (4), we know that

$$\begin{gathered} w(t,X,Z) = -\int_0^Z u_X(t,X,\widetilde{Z})\,d\widetilde{Z}, \quad \int_0^1 u_X(t,X,Z)\,dZ = 0, \\ \text{and thus } \int_0^1 u(t,X,Z)\,dZ = 0. \end{gathered} \tag{5}$$

One is able to further simplify system (3). Differentiating (3a) with respect to X, one obtains

$$u_{Xt} + u\,u_{XX} + u_X^2 + w_X u_Z + w u_{XZ} + p_{XX} = 0. \tag{6}$$

Thanks to (5), integrating (6) with respect to Z over the interval $[0,1]$, an integration by parts together with (3b), (3c), and (4) enables us to solve for the pressure:

$$p_{XX} = -\int_0^1 (u^2)_{XX}\,dZ. \tag{7}$$

We consider from now on solutions u that are odd in X, i.e., $u(X) = -u(-X)$, and introduce the trace of their horizontal derivative on the central line:

$$a(t,Z) = -\partial_X u(t,0,Z). \tag{8}$$

Differentiating (6) with respect to X, then injecting (5) and (7), and taking $X = 0$, one obtains the following closed evolution equation for a:

$$a_t - a^2 + \left(\int_0^Z a(t,\widetilde{Z})\,d\widetilde{Z}\right)a_Z + 2\int_0^1 a^2\,dZ = 0, \tag{9a}$$

$$\int_0^1 a(t,Z)\,dZ = 0. \tag{9b}$$

Note that for solutions u having the form

$$u(t,X,Z) = -Xa(t,Z), \quad \text{and thus, } u_X(t,X,Z) = -a(t,Z), \tag{10}$$

system (3) and the boundary condition (4) for u are equivalent to system (9) for a. We emphasize here that the term $2\int_0^1 a^2\,dZ$ comes from the pressure term.

Remark 1. In the presence of rotation, i.e., $\Omega \geq 0$, choosing $-\partial_X v_0(0, Z) = \Omega$, it has been shown in [21] that a defined in (8) still satisfies system (9).

In [4], a family $(\psi_m)_{m>0}$ of initial data has been constructed for which the corresponding solution a to (9) blows up at time $t = 1$ with $a(t, Z) = (1-t)^{-1}\psi_m(Z)$. Lifting this result to a blowup for the original two-dimensional system (3) is however nontrivial given the lack of a well-posedness result in the class of regularity of the profiles ψ_m. In addition, perturbation of these solutions seems challenging given their rigidity. On the other hand, Wong [34] has constructed explicit initial data for (3) that are analytic and for which the corresponding solution will exhibit a singularity in finite time, making $\|u(t,\cdot)\|_{L^\infty} + \|p_X(t,\cdot)\|_{L^\infty} + \|u_X(t,\cdot)\|_{L^\infty}$ infinite at the blow-up time. The purpose of this paper is to provide precise qualitative properties of the singularity formation.

Our main result is the following, showing the existence and stability of a blow-up solution for (9) with a smooth profile.

Theorem 1.1 (Smooth blowup). *Consider the profile $\phi(z) = \phi_0(z) = e^{-z}$. Then there exist $\lambda_0^* > 0$ and $\delta > 0$ such that for all $0 < \lambda_0 \leq \lambda_0^*$ a constant $\kappa > 0$ exists such that, if initially*

$$a_0(Z) = \frac{1}{\lambda_0}\phi\left(\frac{Z}{\nu_0}\right) + \tilde{a}_0(Z), \quad 0 \leq Z \leq 1, \tag{11}$$

with

$$\frac{2}{3\log(\lambda_0^{-1})} \leq \nu_0 \leq \frac{3}{2\log(\lambda_0^{-1})} \quad \textit{and} \quad \|\tilde{a}_0\|_{C^2([0,1])} \leq \kappa, \tag{12}$$

then there exist $T > 0$ and $C > 0$ such that the solution a to (9) with initial data $a(t=0) = a_0$ blows up at time $T > 0$ according to

$$a(t, Z) = \frac{1}{(T-t)}\phi\left(\frac{Z}{\nu(t)}\right) + \tilde{a}(t, Z), \quad \textit{with } \nu(t) = \frac{1}{|\log(T-t)|}, \tag{13}$$

where for all $t \in [0, T)$,

$$\|\tilde{a}(t,\cdot)\|_{L^\infty([0,1])} \leq C(T-t)^{-1}|\log(T-t)|^{-\delta}. \tag{14}$$

Remark 2.
- Note that the general solution u (other than (10)) to (1) might not exist up to time T. If it does, then the divergence $\|(\partial_X u)|_{X=0}(t)\|_{L^\infty([0,1])} \to \infty$ as $t \uparrow T$ implied by (8), (13), and (14) signals the formation of a shock for u along the horizontal X-direction at the point $(X, Z) = (0, 0)$.
- The pressure term is of lower order compared with the other terms as $t \uparrow T$, but not negligible. Indeed, the modulation equations for the scaling parameters (λ, ν) are

$$\begin{cases} \lambda_t = -1 + C_0\nu + \text{h.o.t.}, \\ \nu_t = -C_0\dfrac{\nu^2}{\lambda} + \text{h.o.t.}, \end{cases} \qquad C_0 = 2\int_0^\infty \phi_0^2(z)\,dz = 1. \tag{15}$$

  Here h.o.t. means higher-order terms. The pressure is thus responsible for the behavior (13) of ν corresponding to a self-similarity of the second kind.

- Equation (9a) shares similarities with the viscous Prandtl equation on the axis [13]:

$$\begin{cases} \xi_t - \xi_{yy} - \xi^2 + \left(\int_0^y \xi\right)\xi_Y = 0, \quad Y > 0, \\ \xi(t,0) = 0, \ \xi(0,Y) = \xi_0(Y). \end{cases} \tag{16}$$

  In [10], a blow-up dynamics is found, for which the viscosity is negligible, where the singularity forms on a large spatial scale $y \approx \nu(t) = (T-t)^{-1/2}$; see also [11,12,26]. Note that (9) without pressure is (16) without viscosity. Interestingly, the blow-up dynamics of Theorem 1.1 and that of [10] are genuinely different, due to the absence of a Dirichlet boundary condition for (9), allowing for a blowup at the boundary, and to the absence of pressure and confinement $z \in [0,1]$ in (16), allowing for the transverse spatial scale ν to grow to infinity, so to maintain the divergence-free condition.
- Another instance of a blow-up dynamics with a logarithmic correction for the scale due to subleading but nonnegligible effects happens for the semilinear heat equation [32].

The existence and uniqueness of analytic solutions of system (1) in the domain $\mathcal{D}$ with boundary condition (2) is established in [25]. By virtue of this, Theorem 1.1, and Remark 1, we have the following corollary regarding the existence of an explicit singular solution of system (1) satisfying the boundary condition (2).

Corollary 1.2. *Consider the profile $\phi(z) = e^{-z}$, and suppose that the initial condition (u_0, v_0, θ_0) of system (1) with boundary condition (2) satisfies*

$$u_0(X,Y,Z) = -Xa_0(Z) = -X\left(\frac{1}{\lambda_0}\phi\left(\frac{Z}{\nu_0}\right) + \tilde{a}_0(Z)\right),$$
$$v_0(X,Y,Z) = -\Omega X, \quad \theta_0(X,Y,Z) = 0,$$

where a_0 is defined as in Theorem 1.1. Then the unique analytic solution blows up at time $T > 0$ stated in Theorem 1.1.

In the second result, we study a blow-up regime where the pressure can be neglected around the blow-up time. We show that this can happen through a two-parameter $(\beta, \tilde{\nu})$ family of nonsmooth profiles.

Theorem 1.3 (Nonsmooth blowup). *For any $\beta > 0$, there exists a nonsmooth profile function $\phi_\beta \in \mathcal{C}^{\frac{1}{\beta+1}}([0,\infty))$ with $\phi_\beta > 0$, $\phi_\beta(0) = 1$, and ϕ_β decreasing to 0 on $[0,\infty)$, and a constant $\delta > 0$ such that the following holds true.*

For any $\tilde{\nu}_0^* > 0$, there exists $\lambda_0^* > 0$ and for all $0 < \lambda_0 \le \lambda_0^*$, a constant $\kappa > 0$, such that if initially

$$a_0(Z) = \frac{1}{\lambda_0}\phi_\beta\left(\frac{Z}{\nu_0}\right) + \tilde{a}_0(Z), \quad 0 \le Z \le 1, \tag{17}$$

with

$$0 < \lambda_0 \le \lambda_0^*, \quad 0 < \tilde{\nu}_0 = \frac{\nu_0}{\lambda_0^\beta} \le \tilde{\nu}_0^*, \quad \textit{and} \quad \|\tilde{a}_0\|_{C^1([0,1])} \le \kappa, \tag{18}$$

then there exist $T > 0$, $\tilde{\nu}_\infty > 0$, $C > 0$ *such that the solution* a *to* (9) *with* $a(t=0) = a_0$ *blows up at time* $T > 0$ *with*

$$a(t,Z) = \frac{1}{T-t}\phi_\beta\Big(\frac{Z}{\nu(t)}\Big) + \tilde{a}(t,Z), \quad \textit{with } \nu(t) = \tilde{\nu}_\infty (T-t)^\beta, \tag{19}$$

where for all $t \in [0,T)$,

$$\|\tilde{a}(t)\|_{L^\infty([0,1])} \le C(T-t)^{-1+\delta}. \tag{20}$$

Remark 3.

- The profile ϕ_β is almost explicit; see Proposition 3.1. The parameter $\tilde{\nu}$ comes from the fact that for a fixed $\beta > 0$ the full family of blow-up profile is $(\phi_{\beta,\tilde{\nu}})_{\tilde{\nu}>0}$ where $\phi_{\beta,\tilde{\nu}}(z) = \phi_\beta(z/\tilde{\nu})$.
- The fact that it is not smooth is crucial: the soft[1] singularity of the initial data plays a crucial role in the mechanism underlying the (worst) singularity at time T. This is a feature of the hyperbolic nature of (9). Similar soft singularities playing a role during finite-time blowup are found for the Burgers equation [12], the nonlinear wave equation [23], and the incompressible Euler equations [14, 15]. However, the construction of stable smooth blowup seems in general to be more challenging. For instance, it still remains open for the three-dimensional Euler equations.
- Compared to Theorem 1.1, the modulation equations for the scale parameters (λ,ν) are

$$\begin{cases} \lambda_t = -1 + C_\beta \nu + \text{h.o.t.}, \\ \nu_t = -\beta\dfrac{\nu}{\lambda} - C_\beta(\beta+1)\dfrac{\nu^2}{\lambda} + \text{h.o.t.}, \end{cases} \qquad C_\beta = 2\int_0^\infty \phi_\beta^2(z)\,dz, \tag{21}$$

  for which the pressure effects (the second terms in the right-hand side of (21)) are negligible. Comparing (21) and (15), we see that the $\beta = 0$ case (15) is *critical* for the role of the pressure which drives the blow-up scale ν to 0 and slightly slows down the blow-up speed (through the $C_0\nu$ term in (15)). Interestingly, this critical regime corresponds in fact to a stable blow-up scenario among smooth solutions.

2. Formal explanations and organization of the paper

2.1. Formal derivation of the blow-up dynamics

In this subsection we detail formal computations that predict the blow-up solutions of Theorems 1.1 and 1.3. We start by making the hypothesis that u is a blow-up solution of (9) that satisfies the following assumptions:

Assumption 1: $\int_0^1 a^2\,dZ$ is of lower order with respect to the other terms in (9),

Assumption 2: The confinement $Z \in [0,1]$ is irrelevant as the solution is localized near $Z = 0$.

[1]Soft in the sense that it still allows for local-in-time existence of a solution.

Equation (9) then becomes to leading order,

$$a_t - a^2 + \Big(\int_0^Z a(t,\tilde Z)\,d\tilde Z\Big) a_Z = 0, \quad 0 \le Z < \infty. \tag{22}$$

This new equation (22) possesses a family $\{\phi_\beta\}_{\beta\ge 0}$ of self-similar profiles; see Proposition 3.1. Namely, for $\beta \ge 0$, for any $\tilde\nu > 0$ (which is a free parameter),

$$a(t,Z) = \frac{1}{T-t}\phi_\beta\Big(\frac{Z}{\tilde\nu(T-t)^\beta}\Big)$$

is an exact solution of (22). For $\beta = 0$ the profile is smooth and explicit:

$$\phi_0(z) = e^{-z} \tag{23}$$

and for $\beta > 0$ the profile is only of Hölder regularity $C^{\frac{1}{\beta+1}}$ and there holds as $z \to \infty$,

$$\phi_\beta(z) \sim d_\beta z^{-\frac{1}{\beta}}, \quad d_\beta > 0. \tag{24}$$

Formal nonsmooth blow-up solutions. If $\beta > 0$, for any fixed $\tilde\nu > 0$ we let $a_{\mathrm f}(t,Z) = \frac{1}{T-t}\phi_\beta(\frac{Z}{\tilde\nu(T-t)^\beta})$ be the exact blow-up solution of the simplified equation (22). Then $a_{\mathrm f}$ is a formal blow-up solution of the original equation (9). To see it, it suffices to check that this formal solution is consistent with Assumptions 1 and 2 that we made. We compute using (24),

$$|\partial_t a_{\mathrm f}|,\ a_{\mathrm f}^2,\ |\partial_Z^{-1} a_{\mathrm f}\partial_Z a_{\mathrm f}| \approx (T-t)^{-2}, \text{ and } \Big|\int_0^1 a_{\mathrm f}^2\,dZ\Big| \lesssim \begin{cases} (T-t)^{\beta-2} & \text{if } 0<\beta<2,\\ |\log T-t| & \text{if } \beta = 2,\\ 1 & \text{if } \beta > 2,\end{cases}$$

and so Assumption 1 is verified. Assumption 2 also holds true due to the decay (24) of ϕ_β as $z \to \infty$. This formal solution $a_{\mathrm f}(t,Z) = \frac{1}{T-t}\phi_\beta(\frac{Z}{\tilde\nu(T-t)^\beta})$ then corresponds to Theorem 1.3.

Formal smooth blow-up solutions. If $\beta = 0$ then, for any fixed $\tilde\nu > 0$, $a_{\mathrm f}(t,Z) = \frac{1}{T-t}\phi_0(\frac{Z}{\tilde\nu})$ is no longer a formal blow-up solution of (9). Indeed, $|\partial_t a_{\mathrm f}|$, $a_{\mathrm f}^2$ and $|\partial_Z^{-1} a_{\mathrm f}\partial_Z a_{\mathrm f}|$ are of size $\approx (T-t)^{-2}$ but so is $|\int_0^1 a_{\mathrm f}^2\,dZ| \approx (T-t)^{-2}$, so that Assumption 1 is no longer valid.

To cope with that issue, we relax the blow-up time T and take a time-dependent scaling parameter ν. We thus consider the refined formal blow-up solution $a_{\mathrm f}^*(t,Z) = \frac{1}{\lambda(t)}\phi_0(\frac{Z}{\nu(t)})$, where λ and ν are to be found. We now compute the leading-order effects of the term $\int_0^1 a^2\,dZ$. We have, using (23),

$$\partial_t a_{\mathrm f}^* = \frac{-\lambda_t}{\lambda^2}\phi_0\Big(\frac{Z}{\nu}\Big) - \frac{\nu_t}{\lambda\nu^2} Z\phi_0'\Big(\frac{Z}{\nu}\Big), \quad \partial_Z^{-1}(a_{\mathrm f}^*)\partial_Z a_{\mathrm f}^* = \frac{1}{\lambda^2}(\partial_z^{-1}(\phi_0)\partial_z\phi_0)\Big(\frac{Z}{\nu}\Big),$$
$$\int_0^1 a_{\mathrm f}^{*2} = \frac{\nu + O(e^{-\frac{1}{\nu}})}{2\lambda^2}.$$

Hence, using in addition the identity $(\partial_z^{-1}\phi_0)\frac{d}{dz}\phi_0 - \phi_0^2 + \phi_0 = 0$, the error E generated by a_{f}^* is

$$\begin{aligned} E &= \partial_t a_{\mathrm{f}}^* - a_{\mathrm{f}}^{*2} + \partial_Z^{-1}(a_{\mathrm{f}}^*)\partial_Z a_{\mathrm{f}}^* + 2\int_0^1 a_{\mathrm{f}}^{*2}\,dZ \\ &= \frac{-\lambda_t}{\lambda^2}\phi_0\Big(\frac{Z}{\nu}\Big) - \frac{\nu_t}{\lambda\nu^2}Z\phi_0'\Big(\frac{Z}{\nu}\Big) - \frac{1}{\lambda^2}\phi_0\Big(\frac{Z}{\nu}\Big) + \frac{\nu + O(e^{-\frac{1}{\nu}})}{\lambda^2}. \end{aligned} \tag{25}$$

We now choose λ and ν so that a_{f}^* is an approximate solution of (9), or equivalently so that E is suitably small. Since the solution is concentrated near $Z = 0$, we require that $E(t,0) = 0$ and that $\partial_Z E(t,0) = 0$. Injecting (23) into (25), the first requirement imposes that $-\lambda_t - 1 + \nu + O(e^{-\frac{1}{\nu}}) = 0$ and the second that $\lambda_t + \lambda\frac{\nu_t}{\nu} + 1 = 0$. Neglecting the $O(e^{-\frac{1}{\nu}})$ terms, this leads to the dynamical system

$$\begin{cases} \lambda_t = -1 + \nu, \\ \nu_t = -\dfrac{\nu^2}{\lambda}. \end{cases}$$

This dynamical system admits by a direct check solutions $(\lambda_{\mathrm{f}}(t), \nu_{\mathrm{f}}(t))$ such that $\lambda_{\mathrm{f}}(t) \sim T - t$ and $\nu_{\mathrm{f}}(t) \sim 1/|\log T - t|$. We thus take as a refined formal blow-up solution $a_{\mathrm{f}}^*(t,Z) = \frac{1}{\lambda_{\mathrm{f}}(t)}\phi_0(\frac{Z}{\nu_{\mathrm{f}}(t)})$. We check that $|\partial_t a_{\mathrm{f}}^*|$, a_{f}^{*2} and $|\partial_Z^{-1}a_{\mathrm{f}}^*\,\partial_Z a_{\mathrm{f}}^*|$ are of size $\approx (T-t)^{-2}$ and $|\int_0^1 a_{\mathrm{f}}^{*2}\,dZ| \approx (T-t)^{-2}/|\log T - t|$ so that Assumption 1 is now satisfied. Assumption 2 also holds true since $\nu_{\mathrm{f}} \to 0$. This formal solution then corresponds to Theorem 1.1.

2.2. Organization

Section 3 is devoted to the construction in Proposition 3.1 of our family of profiles $(\phi_\beta)_{\beta>0}$. The proof is similar to [10, Proposition 3.2]. In Section 4 we construct the stable blowup in the nonsmooth case, where the decay is exponential in time in similarity variables. The proof uses a *bootstrap* argument by defining a basin of attraction of the blowup in Definition 4.2. Our proof goes in three steps. First, we find and solve the modulation equations in Lemma 4.6 by imposing suitable zero boundary conditions on the perturbation. Next, we estimate the remainder $\tilde{a}$ in an interior region in Lemma 4.7. We use a Lyapunov functional with a spatial weight that penalizes the nonlocal terms inspired from [10]. Its decay in time is due to the repulsivity of the transport field and potential terms generated by the ϕ_β profile, which results in a spectral-gap-like coercivity, in analogy with [31]. Last, in Lemma 4.8 we estimate the solution in an exterior region away from the singularity using the maximum principle. Following the same steps as Section 4, in Section 5 we give the proof of Theorem 1.1. The key difference is due to the modulation equations in Lemma 5.5, which need to go up to quadratic order in the ν equation. The decay rate of the remainder becomes algebraic in this case. A slightly more refined analysis is needed.

3. Profiles

In this section we classify self-similar solutions of

$$a_t - a^2 + \left(\int_0^Z a(t, \tilde{Z})\, d\tilde{Z}\right) a_Z = 0, \quad 0 \le Z < \infty,$$

of the form

$$a(t, Z) = \frac{1}{T-t}\phi_\beta\left(\frac{Z}{(T-t)^\beta}\right)$$

for some $\beta \ge 0$. They correspond to solutions of the profile equation

$$(\beta z + \partial_z^{-1}\phi)\frac{d}{dz}\phi - \phi^2 + \phi = 0, \quad z \in [0, \infty), \tag{26}$$

where $\partial_z^{-1}\phi(z) = \int_0^z \phi(\tilde{z})\, d\tilde{z}$. Note that equation (26) has a scaling invariance: if ϕ solves (26) then so does $\phi_{\tilde{\nu}}$ defined as $\phi_{\tilde{\nu}}(z) = \phi(z/\tilde{\nu})$ for any $\tilde{\nu} > 0$. The classification of bounded solutions to (26) is given by the following proposition:

Proposition 3.1. *For all $\beta \ge 0$, the following holds true:*

- *When $\beta > 0$, equation (26) has a solution $\phi_\beta \in C([0,\infty)) \cap C^\infty((0,\infty))$ satisfying, for a constant $d_\beta > 0$,*

$$\phi_\beta(z) = 1 - z^{\frac{1}{\beta+1}} + z^{\frac{2}{\beta+1}} + o(z^{\frac{2}{\beta+1}}) \quad \text{as } z \to 0 \tag{27}$$

and

$$\phi_\beta(z) = d_\beta z^{-\frac{1}{\beta}} + o(z^{-\frac{1}{\beta}}) \quad \text{as } z \to \infty. \tag{28}$$

- *When $\beta = 0$, equation (26) has the explicit solution*

$$\phi_0(z) = e^{-z}.$$

- *For all $\beta \ge 0$, if $\phi \in C([0,\infty)) \cap C^1((0,\infty))$ solves (26), then either ϕ is unbounded on $[0,\infty)$, or ϕ is constant equal to 0 or 1, or there exists $\tilde{\nu} > 0$ such that $\phi(z) = \phi_\beta(z/\tilde{\nu})$ for all $z \ge 0$.*

Remark 4. Note that $\int_0^\infty \phi^2$ is finite if and only if $0 \le \beta < 2$, while $\int_0^\infty (\phi')^2$ is finite if and only if $0 < \beta < 1$. Note that there might exist solutions to (26) on $[0,\infty)$ that are unbounded, but they are not relevant for our purpose.

Proof of Proposition 3.1. We first consider $\beta > 0$, and assume that $\phi = \phi_\beta \in C([0,\infty)) \cap C^1((0,\infty))$ solves (26) and is bounded.

Step 1: Preliminary properties. We claim that either $\phi = 0$ or $\phi = 1$, or $0 < \phi(z) < 1$ for all $z > 0$. To prove it, we let $\psi(z) = \beta z + \int_0^z \phi(\tilde{z})\, d\tilde{z}$, and claim that $\psi(z_0) \neq 0$ for some $z_0 > 0$. Indeed, if not then $\partial_z \psi = 0$, so that $\phi = -\beta$. But this is not a solution to (26).

Then, defining $X=(\phi,\psi)$ with $\frac{d}{dz}X=(\partial_z\phi,\partial_z\psi)$ and $X(z_0)=X_0$, we have that X solves the following ODE whenever $\psi\neq 0$:

$$\begin{cases} \dfrac{d}{dz}X=\Big(\dfrac{\phi^2-\phi}{\psi},\beta+\phi\Big), \\ X_0=(\phi(z_0),\psi(z_0)). \end{cases} \tag{29}$$

Note that (29) has the explicit solutions $(0,C+\beta z)$ and $(1,C+(\beta+1)z)$ for some $C\in\mathbb{R}$. If X is one of these solutions, since $\psi(z)=\beta z+\int_0^z\phi(\tilde z)\,d\tilde z$, then $C=0$ and these are the $\phi=0$ or $\phi=1$ solutions to (26), respectively.

Next, we introduce the sets

$$Z_1^\pm=\{0<\phi<1,\ \pm\psi>0\},\quad Z_2^\pm=\{\phi>1,\ \pm\psi>0\},\quad Z_3^\pm=\{\phi<0,\ \pm\psi>0\}.$$

Assuming that $\phi\neq 0,1$, there exists $i\in\{1,2,3\}$ and $\iota\in\{\pm\}$ such that $X_0\in Z_i^\iota$. We claim that all cases except $X_0\in Z_1^+$ lead to a contradiction, which will prove the claim of Step 1.

If $X_0\in Z_1^-$ or $X\in Z_2^-$, then we notice by a direct check that both sets are invariant by the backward flow of (29), so that $X(z)\in Z_1^-$ or $X(z)\in Z_2^-$ for all $0<z<z_0$. This implies $\partial_z\psi>0$ for $z<z_0$ and hence $\psi(z)\le\psi(z_0)<0$ which contradicts $\psi(z)=\beta z+\int_0^z\phi\to 0$ as $z\to 0$.

If $X_0\in Z_2^+$, then this set is invariant by the forward flow of (29), so that $X(z)\in Z_2^+$ for all $z>z_0$. This implies $\partial_z\phi(z)>0$ and hence $1<\phi(z_0)\le\phi(z)$ for all $z\ge z_0$. Moreover, by the boundedness of ϕ, we know $\phi(z)\le C$ for some $C>1$. We then get using (29) that for $z\ge z_0$,

$$\partial_z\phi(z)\ge\frac{\phi^2(z_0)-\phi(z_0)}{\psi(z_0)+(\beta+C)(z-z_0)}\gtrsim\frac{1}{1+z}.$$

Integrating this inequality gives $\phi(z)\to+\infty$ as $z\to\infty$, a contradiction since ϕ is bounded.

If $X_0\in Z_3^+$, then we first claim that $X(z)\in Z_3^+$ for all $0<z\le z_0$. By contradiction, if not there would exist $0<z_1<z_0$ such that $X(z)\in Z_3^+$ for $z_1<z\le z_0$ and $X(z_1)\notin Z_3^+$. In this case, $\phi'(z)>0$ for all $z\in(z_1,z_0]$ so that $\phi(z)\le\phi(z_0)<0$. As $X(z_1)\notin Z_3^+$ and $\phi(z_1)<0$ we must have $\psi(z_1)=0$. Since $\psi>0$ on Z_3^+, and $\psi(z)=\beta z+\int_0^z\phi$ with ϕ continuous, we obtain that $0<\psi(z)\le C(z-z_1)$ on $(z_1,z_0]$ for some constant $C>0$. Hence for $z\in(z_1,z_0]$ we have using (29) that

$$\partial_z\phi\ge\frac{\phi^2(z_0)-\phi(z_0)}{C(z-z_1)},$$

which integrated by z implies $\phi(z)\to-\infty$ as $z\downarrow z_1$, a contradiction.

If $X_0\in Z_3^-$, we get similarly to the Z_3^+ case that $X\in Z_3^-$ on $[z_0,\infty)$. Then $\partial_z\phi<0$ for $z\ge z_0$, and ϕ diverges to $-\infty$ by an argument similar to that used for Z_2^+, giving again a contradiction.

Step 2: Exact formula. We now assume thanks to Step 1 that $0 < \phi(z) < 1$ for all $z > 0$ and hence $\phi' < 0$ and $\psi > 0$ on $(0, \infty)$. With similar arguments to Step 1 above, we obtain that $\phi(z) \to 0$ as $z \to \infty$, and that $\phi(z) \to 1$ as $z \to 0$. In particular, there exists $z_1 > 0$ such that

$$\phi(z_1) = \frac{1}{2}.$$

We now perform a change of variables on $[0, +\infty)$ by defining ξ such that

$$\frac{d\xi}{dz} = \frac{\xi}{\beta z + \int_0^z \phi(\tilde z)\, d\tilde z}, \quad \xi(z_1) = 1, \quad \text{and} \quad H(\xi) := \phi(z), \tag{30}$$

so that the equation (26) becomes $H - H^2 + \xi \partial_\xi H = 0$, whose solution, since $H(1) = 1/2$, is

$$H = (1 + \xi)^{-1}. \tag{31}$$

Now, differentiating (30) gives

$$\frac{d^2 z}{d\xi^2} = \frac{d}{d\xi}\Big[\frac{\beta z + \int_0^z \phi(\tilde z)\, d\tilde z}{\xi}\Big] = -\frac{1}{\xi}\frac{dz}{d\xi} + \frac{dz}{d\xi}\frac{\beta + \phi(z)}{\xi} = \frac{dz}{d\xi}\Big[\frac{\beta}{\xi} - \frac{1}{1+\xi}\Big].$$

After integration this yields, for C an integration constant,

$$\frac{dz}{d\xi} = \frac{C\xi^\beta}{\xi + 1}. \tag{32}$$

Using (30) and $0 < \beta z + \int_0^z \phi(\tilde z)\, d\tilde z < (1+\beta) z$ we obtain that

$$\lim_{z \downarrow 0} \log \xi(z) = -\int_0^{z_1} \left[\beta z + \int_0^z \phi(\tilde z)\, d\tilde z\right]^{-1} dz = -\infty,$$

so that $\lim_{z\downarrow 0} \xi(z) = 0$. Moreover, we recall that $z(\xi = 1) = z_1$. These two considerations and (32) give

$$z(\xi) = C \int_0^\xi \frac{\tilde\xi^\beta}{\tilde\xi + 1}\, d\tilde\xi, \quad C = z_1 C_\beta, \quad C_\beta = \left(\int_0^1 \frac{\tilde\xi^\beta}{\tilde\xi + 1}\, d\tilde\xi\right)^{-1}. \tag{33}$$

For any $z_1 > 0$, the identities (30), (31), and (33) provide solutions to (26), which are equal up to the scaling transformation $\phi(\cdot) \mapsto \phi(\cdot/\tilde\nu)$. We also proved these are the only possible bounded solutions. To get the asymptotics of ϕ at $z \to 0$, and $z \to \infty$, we integrate (33):

$$z(\xi) = C \int_0^\xi \frac{u^\beta}{1+u}\, du = C \sum_{j \ge 0} \frac{(-1)^j}{\beta + j + 1} \xi^{\beta + j + 1} \quad \text{for } 0 \le \xi < 1.$$

This implies that

$$\xi = \left(\frac{\beta+1}{C}\right)^{\frac{1}{\beta+1}} z^{\frac{1}{\beta+1}} + O\big(z^{1 + \frac{1}{\beta+1}}\big),$$

and consequently, using (30) and (31),

$$\phi(z) = \frac{1}{1 + z^{\frac{1}{\beta+1}}\left(\left(\frac{\beta+1}{C}\right)^{\frac{1}{\beta+1}} + O(z)\right)},$$

which implies (27) upon choosing $z_1 = \frac{\beta+1}{C_\beta}$. Similarly, one can derive the asymptotic as ξ (equivalently z) $\to \infty$.

Step 3: The case $\beta = 0$. This can be solved explicitly by separating variables in the differential equation (26). Indeed, letting $\psi = \partial_z^{-1}\phi$, we get

$$\psi\psi'' = \psi'(\psi' - 1),$$

giving the one-parameter family of solutions $\psi = \tilde{\nu}(e^{z/\tilde{\nu}} - 1)$ for $\tilde{\nu} > 0$, i.e., $\phi = e^{-z/\tilde{\nu}}$. This finishes the proof of the proposition. ∎

For the sake of simplicity, we shall first give the proof of Theorem 1.3 for the case of nonsmooth blowup.

4. Nonsmooth blowup

In this section we focus on the case $\beta > 0$.

4.1. Derivation of the rescaled model in self-similar coordinates

Consider the following rescaling for ν and λ two positive C^1 functions of time:

$$z = \frac{Z}{\nu(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda(t)}, \quad s(0) = s_0. \tag{34}$$

The following computations are done for all $\beta \geq 0$, with $\phi = \phi_\beta$ (we drop the β subscript to ease notation) given by Proposition 3.1. We write the solution $a(t, Z)$ of system (9) as

$$a(t,Z) = \frac{1}{\lambda(s(t))}\left(\phi\left(\frac{Z}{\nu(s(t))}\right) + \varepsilon\left(s(t), \frac{Z}{\nu(s(t))}\right)\right) = \frac{1}{\lambda(s)}(\phi(z) + \varepsilon(s,z)). \tag{35}$$

From the explicit computations,

$$\begin{aligned}
a_t &= \frac{1}{\lambda^2}\left(-\frac{\lambda_s}{\lambda}\phi - \phi'\frac{\nu_s}{\nu}z - \frac{\lambda_s}{\lambda}\varepsilon + \varepsilon_s - \varepsilon_z\frac{\nu_s}{\nu}z\right),\\
a^2 &= \frac{1}{\lambda^2}(\phi^2 + \varepsilon^2 + 2\phi\varepsilon),\\
\left(\int_0^Z a(t,\tilde{Z})\,d\tilde{Z}\right)a_Z &= \frac{1}{\lambda^2}(\partial_z^{-1}\phi\phi' + \partial_z^{-1}\phi\varepsilon_z + \partial_z^{-1}\varepsilon\phi' + \partial_z^{-1}\varepsilon\varepsilon_z),\\
2\int_0^1 a^2(Z)\,dZ &= \frac{2\nu}{\lambda^2}\int_0^{\frac{1}{\nu}}(\phi+\varepsilon)^2(z)\,dz,
\end{aligned}$$

thanks to (26), system (9) gives

$$\varepsilon_s - \frac{\lambda_s}{\lambda}\varepsilon - \frac{\nu_s}{\nu} z\varepsilon_z - 2\phi\varepsilon + \partial_z^{-1}\phi\varepsilon_z + \partial_z^{-1}\varepsilon\phi' - \varepsilon^2 + \partial_z^{-1}\varepsilon\varepsilon_z$$
$$= \Big(\frac{\lambda_s}{\lambda} + 1\Big)\phi + \Big(\beta + \frac{\nu_s}{\nu}\Big) z\phi' - 2\nu \int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2(z)\, dz, \tag{36a}$$
$$\int_0^{\frac{1}{\nu}} (\phi + \varepsilon)(z)\, dz = 0. \tag{36b}$$

The modulation parameters λ and ν are determined by imposing the following vanishing for the expansion of ε, an orthogonality-like condition:

$$\begin{cases} \varepsilon(s, z = 0) = 0, \\ \partial_z \varepsilon(s, z) = O(z^{\frac{2}{\beta+1}-1}) \quad \text{as } z \downarrow 0. \end{cases} \tag{37}$$

In order to have the boundary condition $\varepsilon(z = 0) = 0$, (36a) gives the first modulation equation

$$\frac{\lambda_s}{\lambda} + 1 = 2\nu \int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2(z)\, dz. \tag{38}$$

By taking the derivative of (36a) with respect to z, one obtains

$$\varepsilon_{zs} - \Big(\frac{\lambda_s}{\lambda} + \frac{\nu_s}{\nu} + \phi\Big)\varepsilon_z + \Big(\partial_z^{-1}\phi - \frac{\nu_s}{\nu} z\Big)\varepsilon_{zz} - \phi'\varepsilon + \partial_z^{-1}\varepsilon\phi'' - \varepsilon\varepsilon_z + \partial_z^{-1}\varepsilon\varepsilon_{zz}$$
$$= \Big(\frac{\lambda_s}{\lambda} + 1 + \beta + \frac{\nu_s}{\nu}\Big)\phi' + \Big(\beta + \frac{\nu_s}{\nu}\Big) z\phi''. \tag{39}$$

Since near zero $\phi(z) = 1 - z^{\frac{1}{\beta+1}} + z^{\frac{2}{\beta+1}} + o(z^{\frac{2}{\beta+1}})$, then near zero one has

$$\phi' = \frac{-1}{\beta+1} z^{\frac{-\beta}{\beta+1}} + \frac{2}{\beta+1} z^{\frac{1-\beta}{\beta+1}} + o(z^{\frac{1-\beta}{\beta+1}}),$$
$$z\phi'' = \frac{\beta}{(\beta+1)^2} z^{\frac{-\beta}{\beta+1}} + \frac{2(1-\beta)}{(\beta+1)^2} z^{\frac{1-\beta}{\beta+1}} + o(z^{\frac{1-\beta}{\beta+1}}).$$

The second boundary condition in (37) then gives the second modulation equation

$$\frac{\nu_s}{\nu} = -\beta - (\beta+1)\Big(1 + \frac{\lambda_s}{\lambda}\Big) = -\beta - (\beta+1)2\nu \int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2(z)\, dz. \tag{40}$$

Thus, we can rewrite (39) as

$$\varepsilon_{zs} - \Big(\frac{\lambda_s}{\lambda} + \frac{\nu_s}{\nu} + \phi\Big)\varepsilon_z + \Big(\partial_z^{-1}\phi - \frac{\nu_s}{\nu} z\Big)\varepsilon_{zz} - \phi'\varepsilon + \partial_z^{-1}\varepsilon\phi'' - \varepsilon\varepsilon_z + \partial_z^{-1}\varepsilon\varepsilon_{zz}$$
$$= -(\beta\phi' + (\beta+1)z\phi'')2\nu \int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2(z)\, dz. \tag{41}$$

4.2. Bootstrap argument

We now fix once and for all $\beta > 0$ and its corresponding profile $\phi = \phi_\beta$ given by Proposition 3.1, and pick constants

$$0 < \eta < \min(\beta, 1), \quad \delta = 2\min(\beta, 1) - \eta. \tag{42}$$

Our proof of the main result goes using a bootstrap argument. We start by setting up its framework. The notation $C(a_1, a_2, \dots, a_n)$ stands for a generic constant that depends on its arguments $a_1, \dots, a_n$ and that may change from line to line. Consider a solution a of (9) written in the self-similar coordinates (s, z) given by (34), and assume it is decomposed in the form (35). Therefore, ε and ε_z satisfy (36) and (41), respectively. We control ε with the quantities

$$\mathcal{E}_1^2(s) = \int_0^{z^*} w(z)(\partial_z \varepsilon(s,z))^2\, dz,$$
$$\mathcal{E}_2(s) = \sup_{z^* \le z \le \frac{1}{\nu(s)}} |\varepsilon|.$$

Here $z^* \ge 1$ will be fixed later on, and the weight function is chosen to be

$$w(z) = z^\alpha e^{-Kz}, \tag{43}$$

with K large enough satisfying (76) and (83) below, and α being defined by

$$\alpha = \frac{|1-\beta| - 2 + \frac{\eta}{2}}{\beta + 1}$$
$$= \begin{cases} \dfrac{-\beta - 1 + \frac{\eta}{2}}{\beta+1} = -1 + \dfrac{\eta}{2(\beta+1)} & \text{when } 0 < \beta \le 1, \\ \dfrac{\beta - 3 + \frac{\eta}{2}}{\beta+1} = -1 + \dfrac{4(\beta-1)+\eta}{2(\beta+1)} = 1 - \dfrac{4}{\beta+1} + \dfrac{\eta}{2(\beta+1)} & \text{when } \beta > 1. \end{cases}$$

In particular, notice that $-1 < \alpha < 1$ is true for any $\beta > 0$ and η satisfying (42).

Definition 4.1 (Initial closeness). Let $\lambda_0^*, \tilde{\nu}_0^* > 0$. We say that a_0 is initially close to the blow-up profile if there exist $0 < \lambda_0 \le \lambda_0^*$ and $\nu_0 > 0$ such that

(i) the initial values of the modulation parameters satisfy (note that the first equation fixes the value of s_0)

$$\lambda_0 = e^{-s_0}, \quad \tilde{\nu}_0 = \nu_0 e^{\beta s_0} \le \tilde{\nu}_0^*; \tag{44}$$

(ii) the initial perturbation $\varepsilon(s_0, z) = \varepsilon_0 \in C([0, \frac{1}{\nu_0}]) \cap C^1((0, \frac{1}{\nu_0}])$, given by the decomposition (35), satisfies the integral condition (36b), the boundary condition (37), as well as the two conditions (38) and (40);

(iii) for some small number $\gamma > 0$ satisfying conditions (92) and (95) below, ε_0 satisfies

$$\begin{aligned} \mathcal{E}_1^2(s_0) &= \int_0^{z^*} w(z)(\partial_z \varepsilon_0(z))^2 \, dz < \gamma e^{-\delta s_0}, \\ \mathcal{E}_2^2(s_0) &= \sup_{z^* \le z \le \frac{1}{\nu_0}} |\varepsilon_0|^2 < \frac{1}{16} e^{-\delta s_0}. \end{aligned} \tag{45}$$

Our task is to show that solutions that are initially close to the blow-up profile in the sense of Definition 4.1 will stay close to this blow-up profile up to modulation. The proximity at later times is defined as follows.

Definition 4.2 (Trapped solutions). We say that a solution $a(s, z)$ is trapped on $[s_0, s_1]$ with $s_0 < s_1 < \infty$ if it satisfies the properties of Definition 4.1 at time s_0, and if for some $\widetilde{K} > 1$ and for all $s \in [s_0, s_1]$, $a(s, z)$ can be decomposed as in (35) with

(i) values of the modulation parameters:

$$\frac{1}{\widetilde{K}} e^{-s} < \lambda < \widetilde{K} e^{-s}, \quad \frac{\tilde{\nu}_0}{\widetilde{K}} e^{-\beta s} < \nu < \tilde{\nu}_0 \widetilde{K} e^{-\beta s}; \tag{46}$$

(ii) decay in time of the remainder in the self-similar variables:

$$\begin{aligned} \mathcal{E}_1^2(s) &= \int_0^{z^*} w \varepsilon_z(s)^2 \, dz < \widetilde{K}^2 e^{-\delta s}, \\ \mathcal{E}_2^2(s) &= \sup_{z^* \le z \le \frac{1}{\nu(s)}} |\varepsilon(s)|^2 < \widetilde{K}^2 e^{-\delta s}. \end{aligned} \tag{47}$$

The proof of the Theorem 1.3 relies on the following bootstrap proposition.

Proposition 4.3. *For any $\beta > 0$, $0 < \eta < \min(\beta, 1)$, and $\delta = 2\min(\beta, 1) - \eta$, there exist universal constants $\widetilde{K}, K, z^* \ge 1$ and $\gamma > 0$ such that the following holds true. For any $\tilde{\nu}_0^* > 0$, there exists s_0^* large enough such that for all $s_0 \ge s_0^*$, any solution of (9) which is initially close to the blow-up profile in the sense of Definition 4.1 is trapped on $[s_0, +\infty)$ in the sense of Definition 4.2.*

A standard continuity argument implies that for s_0 large enough, any solution which is initially close to the blow-up profile in the sense of Definition 4.1 is trapped in the sense of Definition 4.2 on some interval $[s_0, s_1]$ with $s_1 > s_0$. Letting $s^* > s_0$ be the supremum of times $s_1 > s_0$ such that the solution is trapped on $[s_0, s_1]$, the purpose now is to show that $s^* = +\infty$. The strategy is to study the trapped regime via several lemmas and show that the solutions cannot escape from the open set defined by Definition 4.2.

Note that the constant s_0^* (defined in Proposition 4.3) will be adjusted during the proof: we will always be able to conclude the proof of the various lemmas by choosing s_0^* large enough. By time-shift invariance, we can always assume the original initial time to be $t = 0$. First, let us derive a priori L^∞ and L^2 bounds of the remainder of trapped solutions.

Lemma 4.4. *Given a solution a of* (9) *that is trapped on the interval* $[s_0, s_1]$ *in the sense of Definition* 4.2*, for all* $s_0 \le s \le s_1$*, we have for* $C^*(\beta, \eta, K, z^*) = C\frac{e^{Kz^*}(z^*)^{-\alpha}}{(1-\alpha)K}$,

$$\int_0^{\frac{1}{\nu}} \varepsilon^2\, dz < C^* \widetilde{K}^2 e^{-\delta s} \frac{1}{\nu} \tag{48}$$

and

$$\sup_{0\le z\le \frac{1}{\nu(s)}} |\varepsilon|^2 < C^* \widetilde{K}^2 e^{-\delta s}. \tag{49}$$

Proof. We divide the proof into two steps.

Step 1*: A preliminary estimate.* We claim that there exists $C > 0$ such that for all $\alpha \in (-1, 1)$ and $K \ge 1$, for w given by (43),

$$\int_0^z \frac{1}{w(\tilde z)}\, d\tilde z \le \frac{C}{(1-\alpha)Kw(z)} \quad \text{for all } z > 0. \tag{50}$$

Indeed, recalling $w(z) = z^\alpha e^{-Kz}$, by a rescaling argument and then a change of variables,

$$\sup_{z>0} w(z) \int_0^z \frac{1}{w(\tilde z)}\, d\tilde z = \sup_{z>0} w\left(\frac{z}{K}\right) \int_0^{\frac{z}{K}} \frac{1}{w(\tilde z)}\, d\tilde z = \frac{1}{K} \sup_{z>0} z^\alpha e^{-z} \int_0^z \tilde z^{-\alpha} e^{\tilde z}\, d\tilde z. \tag{51}$$

If $\alpha \le 0$ then for all $z > 0$,

$$z^\alpha e^{-z} \int_0^z \tilde z^{-\alpha} e^{\tilde z}\, d\tilde z \le e^{-z} \int_0^z e^{\tilde z}\, d\tilde z \le 1. \tag{52}$$

If $\alpha > 0$ then an integration by parts gives

$$z^\alpha e^{-z} \int_0^z \tilde z^{-\alpha} e^{\tilde z}\, d\tilde z = \frac{z}{1-\alpha} - \frac{z^\alpha e^{-z}}{1-\alpha} \int_0^z \tilde z^{1-\alpha} e^{\tilde z}\, d\tilde z.$$

Now, since $z \ge \tilde z$, we obtain

$$z^\alpha e^{-z} \int_0^z \tilde z^{-\alpha} e^{\tilde z}\, d\tilde z \le \frac{z}{1-\alpha} - \frac{e^{-z}}{1-\alpha} \int_0^z \tilde z e^{\tilde z}\, d\tilde z = \frac{e^{-z}}{1-\alpha}(e^z - 1) \le \frac{1}{1-\alpha}. \tag{53}$$

Injecting (52) and (53) into (51) shows (50).

Step 2*: Proof of the lemma.* For all $z \in [0, z^*]$, using the boundary condition $\varepsilon(s, z = 0) = 0$, the Cauchy–Schwarz inequality, and (50), from (47) one obtains that

$$\begin{aligned} |\varepsilon(s,z)|^2 &\le \left(\int_0^z w\varepsilon_z^2\, d\tilde z\right)\left(\int_0^z \frac{1}{w}\, d\tilde z\right) \le \mathcal{E}_1^2(s)\left(\int_0^{z^*} \frac{1}{w}\, d\tilde z\right) \\ &\le \frac{C}{(1-\alpha)Kw(z^*)} \mathcal{E}_1^2 \le C^* \mathcal{E}_1^2 \le C^* \widetilde{K}^2 e^{-\delta s}. \end{aligned} \tag{54}$$

This, together with (47), implies (49). Estimate (48) is an immediate consequence of (49) as

$$\int_0^{\frac{1}{\nu}} \varepsilon^2\, dz \le \frac{1}{\nu} \sup_{0\le z\le z^*} |\varepsilon|^2 \le C^* \widetilde{K}^2 e^{-\delta s} \frac{1}{\nu}. \qquad \blacksquare$$

We will need the following technical estimate in this section.

Lemma 4.5. *Suppose that $\beta > 0$, $K \geq 2$, and $z^* \geq 1$. Denote by*

$$\begin{aligned} A &:= \int_0^{z^*} \phi'(z)^2 w(z) \Big(\int_0^z \frac{1}{w}(\tilde{z})\, d\tilde{z} \Big)\, dz \\ &\quad + \int_0^{z^*} \phi''(z)^2 w(z) \Big(\int_0^z \Big(\int_0^{\tilde{z}} \frac{1}{w}(\xi)\, d\xi \Big)^{\frac{1}{2}} d\tilde{z} \Big)^2 dz \\ &:= A_1 + A_2. \end{aligned}$$

For any $\alpha \in (-1, 1)$, if w is given by (43) then

$$A \leq \frac{C(\beta)}{1-\alpha} \begin{cases} K^{-1} & \text{when } 0 < \beta < 1, \\ K^{-1} \ln K & \text{when } \beta = 1, \\ K^{-\frac{2}{\beta+1}} & \text{when } \beta > 1. \end{cases} \tag{55}$$

Proof. We recall that for $i = 1, 2$,

$$|\partial_z^i \phi(z)| \lesssim z^{-\frac{\beta}{\beta+1}+1-i} \text{ for } z \leq 1, \quad |\partial_z^i \phi(z)| \lesssim z^{-\frac{\beta+1}{\beta}+1-i} \text{ for } z \geq 1. \tag{56}$$

We first consider A_1 and decompose:

$$\begin{aligned} A_1 &= A_1^1 + A_1^2 + A_1^3, \\ A_1 &= \int_0^{\frac{1}{K}} \phi'(z)^2 w(z) \Big(\int_0^z \frac{1}{w}(\tilde{z})\, d\tilde{z} \Big)\, dz, \\ A_1^2 &= \int_{\frac{1}{K}}^1 \ldots, \\ A_1^3 &= \int_1^{z^*} \ldots \end{aligned}$$

For $0 < z \leq K^{-1}$ we have $w(z) \approx z^\alpha$ so that using (56),

$$A_1^1 \lesssim \int_0^{K^{-1}} z^{-\frac{2\beta}{\beta+1}} \frac{z}{1-\alpha}\, dz \lesssim C(\beta) \frac{K^{\frac{-2}{\beta+1}}}{1-\alpha}.$$

For $K^{-1} \leq z \leq 1$ we use (50) and (56) so that

$$A_1^2 \lesssim \int_{K^{-1}}^1 z^{-\frac{2\beta}{\beta+1}} \frac{1}{K(1-\alpha)}\, dz \lesssim \frac{C(\beta)}{1-\alpha} \begin{cases} K^{-1} & \text{if } \beta < 1, \\ K^{-1} \log K & \text{if } \beta = 1, \\ K^{\frac{-2}{\beta+1}} & \text{if } \beta > 1. \end{cases}$$

For $z \geq 1$ we use (50) and (56) so that

$$A_1^3 \lesssim \int_1^{z^*} z^{-\frac{2(\beta+1)}{\beta}} \frac{1}{K}\, dz \lesssim \frac{C(\beta)}{1-\alpha} K^{-1}.$$

Summing the three inequalities above shows that

$$A_1 \lesssim \frac{C(\beta)}{1-\alpha} \begin{cases} K^{-1} & \text{if } \beta < 1, \\ K^{-1}\log K & \text{if } \beta = 1, \\ K^{\frac{-2}{\beta+1}} & \text{if } \beta > 1. \end{cases} \tag{57}$$

We turn to A_2. For $z > 0$ we use the Cauchy inequality and that $\tilde{z} \mapsto \int_0^{\tilde{z}} \frac{1}{w}(\xi)\, d\xi$ is increasing to get

$$\begin{aligned} w(z)\left(\int_0^z \left(\int_0^{\tilde{z}} \frac{1}{w}(\xi)\, d\xi\right)^{\frac{1}{2}} d\tilde{z}\right)^2 &\le w(z)\left(\int_0^z 1\, dz\right)\left(\int_0^z \int_0^{\tilde{z}} \frac{1}{w}(\xi)\, d\xi\, d\tilde{z}\right) \\ &\le w(z) z\left(\int_0^z \int_0^z \frac{1}{w}(\xi)\, d\xi\, d\tilde{z}\right) \\ &\le z^2 w(z) \int_0^z \frac{1}{w(\tilde{z})}\, d\tilde{z}. \end{aligned}$$

Notice using (56) that $z\phi''$ has the same asymptotic behavior as ϕ' near $z = 0$ and $z = \infty$. Therefore, by repeating a similar calculation, one obtains the same estimate (57) for the term A_2 as for the A_1 term. These two estimates show (55). ■

In the sequel, we reintegrate over time the modulation equations and the various energy and pointwise estimates, to show that the various upper bounds describing the bootstrap cannot be saturated. Proposition 4.3 follows immediately from the following three lemmas.

Lemma 4.6 (Modulation equations). *For any choice of constants $\widetilde{K}, K, z^* \ge 1$, $\gamma, \tilde{\nu}_0^* > 0$, and $0 < \eta < \min(\beta, 1)$, there exists a large self-similar time s_0^* such that for any $s_0 \ge s_0^*$, for any solution which is trapped on $[s_0, s_1]$, we have for $s \in [s_0, s_1]$,*

$$\left|\frac{\lambda_s}{\lambda} + 1\right| \le Ce^{-\frac{\delta}{2}s}, \qquad \left|\frac{\nu_s}{\nu} + \beta\right| \le Ce^{-\frac{\delta}{2}s}, \tag{58}$$

for $C > 0$ independent of the bootstrap constants, and

$$\frac{1}{2}e^{-s} \le \lambda \le \frac{3}{2}e^{-s}, \qquad \frac{\tilde{\nu}_0}{2}e^{-\beta s} \le \nu \le \frac{3}{2}\tilde{\nu}_0 e^{-\beta s}. \tag{59}$$

Moreover, if $s_1 = \infty$ then there exists some constants $\tilde{\lambda}_\infty, \tilde{\nu}_\infty > 0$ such that

$$\begin{aligned} \lambda &= \tilde{\lambda}_\infty(1 + O(e^{-\frac{\delta}{2}s}))e^{-s}, \\ \nu &= \tilde{\nu}_\infty(1 + O(e^{-\frac{\delta}{2}s}))e^{-\beta s}. \end{aligned} \tag{60}$$

Proof. We divide the proof into three steps.

Step 1*: A preliminary estimate.* We claim that for s_0 large enough, for all $s \in [s_0, s_1]$,

$$\nu \int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2 \, dz \le e^{-(\frac{\delta}{2}+\frac{\eta}{4})s}. \tag{61}$$

To prove it, first we use the decay (28) of $\phi(z)$ as $z \to \infty$, and then inject (46) to obtain

$$\begin{aligned} \nu \int_0^{\frac{1}{\nu}} \phi^2(z) \, dz &\lesssim C(\beta) \begin{cases} \nu & \text{if } \beta < 2, \\ \nu|\log \nu| & \text{if } \beta = 2, \\ \nu^{\frac{2}{\beta}} & \text{if } \beta > 2, \end{cases} \\ &\le C(\beta, \tilde{\nu}_0, \widetilde{K}) \begin{cases} e^{-\beta s} & \text{if } \beta < 2, \\ s e^{-\beta s} & \text{if } \beta = 2, \\ e^{-2s} & \text{if } \beta > 2, \end{cases} \\ &\le C(\beta, \tilde{\nu}_0, \widetilde{K}) s e^{-(\frac{\delta}{2}+\frac{\eta}{2})s} \le \frac{1}{4} e^{-(\frac{\delta}{2}+\frac{\eta}{4})s}, \end{aligned} \tag{62}$$

where we used $\frac{\delta}{2} + \frac{\eta}{2} = \min(\beta, 1)$ for the third inequality, and then took s_0 large enough for the last inequality. Second, using (48) we obtain

$$\nu \int_0^{\frac{1}{\nu}} \varepsilon^2 \, dz < C^* \widetilde{K}^2 e^{-\delta s} \le \frac{1}{4} e^{-(\frac{\delta}{2}+\frac{\eta}{4})s}, \tag{63}$$

where we used $\eta < \delta$ and took s_0 large enough for the last inequality. Combining (62), (63), and the inequality $(x + y)^2 \le 2(x^2 + y^2)$ shows the estimate (61) we claimed.

Step 2*: Computing* λ*.* We rewrite (38) as

$$\lambda_s + \lambda = 2\lambda\nu \int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2(z) \, dz. \tag{64}$$

Injecting (61) into (64) shows the first inequality in (58). Using (46) and (61) gives $\lambda\nu \int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2 \le \widetilde{K} e^{-(1+\frac{\delta}{2}+\frac{\eta}{4})s} \le e^{-(1+\frac{\delta}{2})s}$ for s_0 large enough, implying

$$\frac{d}{ds}(e^s \lambda) = O(e^{-\frac{\delta}{2}s}), \quad \text{hence } \lambda(s) = \left(e^{s_0}\lambda_0 + \int_{s_0}^{s} O(e^{-\frac{\delta}{2}\tilde{s}}) \, d\tilde{s} \right) e^{-s}. \tag{65}$$

Choosing s_0 large enough that $-\frac{1}{2} \le \int_{s_0}^{s} O(e^{-\frac{\delta}{2}\tilde{s}}) \, d\tilde{s} \le \frac{1}{2}$ and injecting (44) into (65) shows

$$\frac{1}{2} e^{-s} \le \lambda(s) \le \frac{3}{2} e^{-s}.$$

In the case $s_1 = \infty$, injecting (44) into (65) and rewriting this identity gives

$$\begin{aligned} \lambda(s) &= \left(1 + \int_{s_0}^{\infty} O(e^{-\frac{\delta}{2}\tilde{s}}) \, d\tilde{s} - \int_{s}^{\infty} O(e^{-\frac{\delta}{2}\tilde{s}}) \, d\tilde{s} \right) e^{-s} \\ &:= \left(\tilde{\lambda}_\infty - \int_{s}^{\infty} O(e^{-\frac{\delta}{2}\tilde{s}}) \, d\tilde{s} \right) e^{-s} = \tilde{\lambda}_\infty (1 + O(e^{-\frac{\delta}{2}s})) e^{-s}. \end{aligned}$$

The two inequalities above show (59) and (60) for λ.

Step 3*: Computing* ν. We rewrite (40) as

$$\nu_s + \beta\nu = -(\beta+1)2\nu^2 \int_0^{\frac{1}{\nu}} (\phi+\varepsilon)^2(z)\,dz.$$

Reasoning exactly as in Step 2, thanks to (61) and (46) one obtains the second inequality in (58) and

$$\frac{d}{ds}(e^{\beta s}\nu) = O(e^{-\frac{\delta}{2}s}), \quad \text{hence } \nu(s) = \left(e^{\beta s_0}\nu_0 + \int_{s_0}^{s} O(e^{-\frac{\delta}{2}\tilde{s}})\,d\tilde{s}\right)e^{-\beta s}.$$

Choosing s_0 large enough so that $-\frac{1}{2} \le \tilde{\nu}_0^{-1}\int_{s_0}^{s} O(e^{-\frac{\delta}{2}\tilde{s}})\,d\tilde{s} \le \frac{1}{2}$ and using (44) we get

$$\frac{\tilde{\nu}_0}{2}e^{-\beta s} \le \nu(s) \le \frac{3}{2}\tilde{\nu}_0 e^{-\beta s}.$$

If $s_1 = \infty$, thanks to (44), one can rewrite

$$\begin{aligned}\nu(s) &= \left(\tilde{\nu}_0 + \int_{s_0}^{\infty} O(e^{-\frac{\delta}{2}\tilde{s}})\,d\tilde{s} - \int_{s}^{\infty} O(e^{-\frac{\delta}{2}\tilde{s}})\,d\tilde{s}\right)e^{-\beta s}\\ &:= \tilde{\nu}_\infty(1 + O(e^{-\frac{\delta}{2}s}))e^{-\beta s}.\end{aligned}$$

The two inequalities above show (59) and (60) for ν. ∎

Lemma 4.7 (Interior estimate). *For any $0 < \eta < \min(\beta, 1)$ and $z^* \ge 1$, there exists $K^* \ge 1$ such that for all $K \ge K^*$ the following holds true. For any constants $\widetilde{K} \ge 1$ and $\gamma, \tilde{\nu}_0^* > 0$, there exists a large self-similar time s_0^* such that for any $s_0 \ge s_0^*$, for any solution which is trapped on $[s_0, s_1]$, we have for $s \in [s_0, s_1]$,*

$$\mathcal{E}_1^2(s) \le 2\gamma e^{-\delta s}. \tag{66}$$

Proof. We recall $w(z) = z^\alpha e^{-Kz}$ and $\alpha = \frac{|1-\beta|-2+\frac{\eta}{2}}{\beta+1}$. Multiplying (41) by $w\varepsilon_z$ and integrating over $[0, z^*]$, one obtains

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}\int_0^{z^*} w\varepsilon_z^2\,dz - \int_0^{z^*}\left(\frac{\lambda_s}{\lambda} + \frac{\nu_s}{\nu} + \phi + \varepsilon\right)w\varepsilon_z^2\,dz\\ &\quad + \int_0^{z^*}\left(\partial_z^{-1}\phi - \frac{\nu_s}{\nu}z + \partial_z^{-1}\varepsilon\right)\varepsilon_{zz}w\varepsilon_z\,dz\\ &\quad - \int_0^{z^*}\phi'\varepsilon w\varepsilon_z\,dz + \int_0^{z^*}\partial_z^{-1}\varepsilon w\varepsilon_z\phi''\,dz\\ &= -2\nu\left(\int_0^{\frac{1}{\nu}}(\phi+\varepsilon)^2(z)\,dz\right)\int_0^{z^*}(\beta\phi' + (\beta+1)z\phi'')w\varepsilon_z\,dz.\end{aligned} \tag{67}$$

Now we estimate all terms in (67).

Potential and transport terms. Integrating by parts yields

$$
\begin{aligned}
&-\int_0^{z^*}\Big(\frac{\lambda_s}{\lambda}+\frac{\nu_s}{\nu}+\phi+\varepsilon\Big)w\varepsilon_z^2\,dz+\int_0^{z^*}\Big(\partial_z^{-1}\phi-\frac{\nu_s}{\nu}z+\partial_z^{-1}\varepsilon\Big)\varepsilon_{zz}w\varepsilon_z\,dz\\
&\quad=\Big(\int_0^{z^*}(\phi+\varepsilon)(\tilde z)\,d\tilde z-\frac{\nu_s}{\nu}z^*\Big)\frac{1}{2}w(z^*)\varepsilon_z^2(z^*)\\
&\qquad+\int_0^{z^*}\Big(-\frac{3}{2}\phi-\frac{3}{2}\varepsilon-\frac{\lambda_s}{\lambda}-\frac{1}{2}\frac{\nu_s}{\nu}-\frac{1}{2}\Big(\partial_z^{-1}\phi-\frac{\nu_s}{\nu}z+\partial_z^{-1}\varepsilon\Big)\frac{w_z}{w}\Big)w\varepsilon_z^2\,dz. \qquad (68)
\end{aligned}
$$

To deal with the boundary term, we compute that the transport field is outgoing at z^*, i.e.,

$$
\int_0^{z^*}(\phi+\varepsilon)(\tilde z)\,d\tilde z-\frac{\nu_s}{\nu}z^*\geq\Big(\beta-\sup_{0\leq z\leq\frac{1}{\nu(s)}}|\varepsilon|\Big)z^*\geq(\beta-\sqrt{C^*}\widetilde{K}e^{-\frac{\delta}{2}s_0})z^*\geq 0, \qquad (69)
$$

where we used (40), (58), and (49), and took s_0 sufficiently large. This implies

$$
\Big(\int_0^{z^*}(\phi+\varepsilon)(\tilde z)\,d\tilde z-\frac{\nu_s}{\nu}z^*\Big)\frac{1}{2}w(z^*)\varepsilon_z^2(z^*)\geq 0. \qquad (70)
$$

Now, since $w=z^\alpha e^{-Kz}$, we know that $\frac{w_z}{w}=\alpha z^{-1}-K=\frac{|1-\beta|-2+\frac{\eta}{2}}{(\beta+1)z}-K$. Using this, (58), and (49) we get the first identity (where the constant involved in the $O()$ depends on $\widetilde{K}$, α, and z^*):

$$
\begin{aligned}
&-2\Big(\frac{\lambda_s}{\lambda}+\frac{\nu_s}{2\nu}+\frac{3}{2}\phi+\frac{3}{2}\varepsilon+\frac{1}{2}(\partial_z^{-1}\phi-\frac{\nu_s}{\nu}z+\partial_z^{-1}\varepsilon)\frac{w_z}{w}\Big)\\
&\quad=2+\beta-3\phi+O(e^{-\frac{\delta}{2}s})-(\partial_z^{-1}\phi+\beta z+O(e^{-\frac{\delta}{2}s}z))(\alpha z^{-1}-K). \qquad (71)
\end{aligned}
$$

We now distinguish between the cases $-1<\alpha\leq 0$ and $0<\alpha<1$. When $-1<\alpha\leq 0$ using the fact that $\partial_z^{-1}\phi\geq\phi z$ due to $\phi'<0$, and then $\phi\leq 1$, we get

$$
\begin{aligned}
(71)&\geq 2+\beta-(3\phi+\alpha\phi)-\alpha\beta+O(e^{-\frac{\delta}{2}s})\geq 2+\beta-3-\alpha-\alpha\beta+O(e^{-\frac{\delta}{2}s})\\
&=\beta-1-(\beta+1)\Big(\frac{|1-\beta|-2+\frac{\eta}{2}}{\beta+1}\Big)+O(e^{-\frac{\delta}{2}s})\geq\delta+\frac{\eta}{2}+O(e^{-\frac{\delta}{2}s}),
\end{aligned}
$$

where we used (42) for the last inequality. When $0<\alpha<1$ notice that then $\beta>1$, and by using the fact that $\phi\leq 1$ and that $\partial_z^{-1}\phi\leq z$, one obtains

$$
\begin{aligned}
(71)&\geq 2+\beta-3\phi-\alpha(\beta+1)+O(e^{-\frac{\delta}{2}s})\\
&\geq -1+\beta-(\beta+1)\Big(\frac{|1-\beta|-2+\frac{\eta}{2}}{\beta+1}\Big)+O(e^{-\frac{\delta}{2}s})\\
&\geq 2-\frac{\eta}{2}+O(e^{-\frac{\delta}{2}s})=\delta+\frac{\eta}{2}+O(e^{-\frac{\delta}{2}s}).
\end{aligned}
$$

In summary, for any $\beta>0$, we get the repulsivity estimate

$$
-2\Big(\frac{\lambda_s}{\lambda}+\frac{\nu_s}{2\nu}+\frac{3}{2}\phi+\frac{3}{2}\varepsilon+\frac{1}{2}\Big(\partial_z^{-1}\phi-\frac{\nu_s}{\nu}z+\partial_z^{-1}\varepsilon\Big)\frac{w_z}{w}\Big)\geq\delta+\frac{\eta}{2}+O(e^{-\frac{\delta}{2}s}),
$$

and hence deduce the spectral-gap-like coercivity estimate

$$
\begin{aligned}
2\int_0^{z^*}\Big(-\frac{3}{2}\phi-\frac{3}{2}\varepsilon-\frac{\lambda_s}{\lambda}-\frac{1}{2}\frac{\nu_s}{\nu}-\frac{1}{2}\Big(\partial_z^{-1}\phi-\frac{\nu_s}{\nu}z+\partial_z^{-1}\varepsilon\Big)\frac{w_z}{w}\Big)w\varepsilon_z^2\,dz&\\
\geq\Big(\delta+\frac{\eta}{2}+O(e^{-\frac{\delta}{2}s})\Big)\mathcal{E}_1(s).&\quad(72)
\end{aligned}
$$

Injecting (70) and (72) into (68) one finally finds that for the potential and transport terms,

$$
\begin{aligned}
-2\int_0^{z^*}\Big(\frac{\lambda_s}{\lambda}+\frac{\nu_s}{\nu}+\phi+\varepsilon\Big)w\varepsilon_z^2\,dz+2\int_0^{z^*}\Big(\partial_z^{-1}\phi-\frac{\nu_s}{\nu}z+\partial_z^{-1}\varepsilon\Big)\varepsilon_{zz}w\varepsilon_z\,dz&\\
\geq\Big(\delta+\frac{\eta}{2}+O(e^{-\frac{\delta}{2}s})\Big)\mathcal{E}_1(s).&\quad(73)
\end{aligned}
$$

Nonlocal terms. Notice that since $\varepsilon(z=0)=0$, by the Cauchy–Schwarz inequality, one obtains that for all $z\in[0,z^*]$,

$$
\varepsilon(z)\leq\mathcal{E}_1\Big(\int_0^z\frac{1}{w}(\tilde z)\,d\tilde z\Big)^{\frac12},\quad(74)
$$

$$
\partial_z^{-1}\varepsilon(z)=\int_0^z\varepsilon(\tilde z)\,d\tilde z\leq\mathcal{E}_1\int_0^z\Big(\int_0^{\tilde z}\frac{1}{w}(\xi)\,d\xi\Big)^{\frac12}d\tilde z.\quad(75)
$$

For $K>1$ large enough so that

$$
\frac{\ln K}{K}\leq K^{-\frac23},\quad(76)
$$

we introduce the parameter $\rho=\min(\frac13,\frac{1}{\beta+1})$, and, thanks to (74), (75), and Lemma 4.5, by the Cauchy–Schwarz inequality, one obtains

$$
2\int_0^{z^*}|\phi'\varepsilon w\varepsilon_z|\,dz\leq2\Big(\int_0^{z^*}|\phi'|^2w\Big(\int_0^z\frac{1}{w}(\tilde z)\,d\tilde z\Big)dz\Big)^{\frac12}\mathcal{E}_1^2\leq\frac{C(\beta,\eta)}{K^\rho}\mathcal{E}_1^2,\quad(77)
$$

where the constant $C(\beta,\eta)$ depends on η through α, and

$$
\begin{aligned}
2\int_0^{z^*}|\partial_z^{-1}\varepsilon w\varepsilon_z\phi''|\,dz\leq2\Big(\int_0^{z^*}|\phi''|^2w\Big(\int_0^z\Big(\int_0^{\tilde z}\frac{1}{w}(\xi)\,d\xi\Big)^{\frac12}d\tilde z\Big)^2dz\Big)^{\frac12}\mathcal{E}_1^2&\\
\leq\frac{C(\beta,\eta)}{K^\rho}\mathcal{E}_1^2.&\quad(78)
\end{aligned}
$$

Source terms. Finally, we consider

$$
4\nu\Big(\int_0^{\frac1\nu}(\phi+\varepsilon)^2(z)\,dz\Big)\int_0^{z^*}(\beta\phi'+(\beta+1)z\phi'')w\varepsilon_z\,dz.
$$

Using (61) one obtains, for C independent of the bootstrap constants,

$$
4\nu\Big(\int_0^{\frac1\nu}(\phi+\varepsilon)^2(z)\,dz\Big)\leq Ce^{-(\frac{\delta}{2}+\frac{\eta}{4})s}.\quad(79)
$$

Thanks to Proposition 3.1, one has the crucial cancellation at the origin,

$$|\beta\phi' + (\beta+1)z\phi''| \lesssim z^{\frac{2}{\beta+1}-1} \quad \text{as } z \to 0.$$

Since $w = z^\alpha e^{-Kz}$ with $\alpha = \frac{|1-\beta|-2+\frac{\eta}{2}}{\beta+1}$, one then has, using Cauchy–Schwarz,

$$\begin{aligned}\int_0^{z^*} (\beta\phi' + (\beta+1)z\phi'')w\varepsilon_z\,dz &\le \left(\int_0^{z^*} (\beta\phi' + (\beta+1)z\phi'')^2 w(z)\,dz\right)^{\frac{1}{2}} \mathcal{E}_1 \\ &\le C(\beta,\eta,z^*,K)\mathcal{E}_1. \end{aligned} \tag{80}$$

Therefore, using (79) and (80), by Young's inequality $xy \le K^\rho x^2/2 + y^2/2K^\rho$, one has

$$\begin{aligned} &4\nu\left(\int_0^{\frac{1}{\nu}} (\phi+\varepsilon)^2(z)\,dz\right)\int_0^{z^*} (\beta\phi' + (\beta+1)z\phi'')w\varepsilon_z\,dz \\ &\quad\le C(\beta,\eta,z^*,K)e^{-(\delta+\frac{\eta}{2})s} + \frac{\mathcal{E}_1^2}{K^\rho}. \end{aligned} \tag{81}$$

Conclusion. Injecting (73), (77), (78), and (81) into (67) shows

$$\begin{aligned} &\frac{d}{ds}\mathcal{E}_1^2 + \left(\delta + \frac{\eta}{2} - \frac{C(\beta,\eta)}{K^\rho} - C(\beta,\eta,K,z^*,\widetilde{K})e^{-\frac{\delta}{2}s}\right)\mathcal{E}_1^2 \\ &\quad\le C(\beta,\eta,z^*,K)e^{-(\delta+\frac{\eta}{2})s}. \end{aligned} \tag{82}$$

Choosing K large enough so that

$$\frac{C(\beta,\eta)}{K^\rho} \le \frac{\eta}{8}, \tag{83}$$

and s_0 large enough so that

$$C(\beta,\eta,K,z^*,\widetilde{K})e^{-\frac{\delta}{2}s_0} \le \frac{\eta}{8},$$

we have that (82) becomes

$$\frac{d}{ds}\left(e^{(\delta+\frac{\eta}{4})s}\mathcal{E}_1^2\right) \le C(\beta,\eta,z^*,K)e^{-\frac{\eta}{4}s}. \tag{84}$$

Integrating (84) between s_0 and s, using (45) gives

$$\begin{aligned} \mathcal{E}_1^2 &\le e^{(\delta+\frac{\eta}{4})(s_0-s)}\mathcal{E}_1^2(s_0) + C(\beta,\eta,z^*,K)e^{-(\delta+\frac{\eta}{4})s} \\ &< \gamma e^{-\delta s} + C(\beta,\eta,z^*,K)e^{-(\delta+\frac{\eta}{4})s} < 2\gamma e^{-\delta s}, \end{aligned}$$

where we have chosen s_0 large enough so that

$$C(\beta,\eta,z^*,K)e^{-\frac{\eta}{4}s_0} < \gamma.$$

This is (66). ∎

Lemma 4.8 (Exterior estimate). *For any* $0 < \eta < \min(\beta, 1)$ *there exists* $\bar{z}^* \geq 1$*, such that the following holds true for all* $z^* \geq \bar{z}^*$*. For any* $K \geq 1$ *there exists* $\gamma^* > 0$ *such that for all* $0 < \gamma \leq \gamma^*$*, for all* $\widetilde{K} \geq 1$ *and* $\tilde{\nu}_0^* > 0$*, for* s_0^* *large enough, for any* $s_0 \geq s_0^*$*, for any solution which is trapped on* $[s_0, s_1]$*, we have for* $s \in [s_0, s_1]$*,*

$$\mathcal{E}_2^2(s) \leq e^{-\delta s}. \tag{85}$$

Proof. We use a standard comparison principle for transport operators, together with a bootstrap argument to control nonlocal effects. To implement this bootstrap argument, we assume in addition that on $[s_0, s_1]$ there holds

$$\mathcal{E}_2(s) \leq \widetilde{K}' e^{-\frac{\delta}{2}s} \tag{86}$$

for some constant $0 < \widetilde{K}' \leq \widetilde{K}$.

Remark 5. The reason we apply this bootstrap argument here is to make z^* independent of $\widetilde{K}$ (see (92)). This is natural since z^*, the boundary between the interior region and the exterior region, should be independent of the size of the remainder, which is $\widetilde{K}$.

Clearly, (86) is satisfied if one chooses $\widetilde{K}' = \widetilde{K}$ from (47). By applying the modulations (38) and (40), we can rewrite (36a) as

$$\varepsilon_s + \mathcal{L}\varepsilon = F, \tag{87}$$

where $\mathcal{L}$ is the transport operator (note that it has a nonlinear part)

$$\mathcal{L}v = -\frac{\lambda_s}{\lambda}v - \frac{\nu_s}{\nu}zv_z - 2\phi v + \partial_z^{-1}\phi v_z - \varepsilon v + \partial_z^{-1}\varepsilon v_z$$

and the source term is

$$F = -\partial_z^{-1}\varepsilon\phi' + 2\nu\left(\int_0^{\frac{1}{\nu}}(\phi+\varepsilon)^2(z)\,dz\right)(-1+\phi-(\beta+1)z\phi').$$

Step 1*: A supersolution for* $\partial_s + \mathcal{L}$ *on* $[z^*, \nu^{-1}]$. We introduce

$$f(s,z) = e^{-\frac{\delta}{2}s}$$

and claim that there exists z^* large enough such that for s_0 large enough, for all $s_0 \leq s \leq s_1$ and $z \geq z^*$,

$$(\partial_s + \mathcal{L})f \geq \frac{\eta}{4}e^{-\frac{\delta}{2}s}. \tag{88}$$

Now we prove (88). We compute using $\partial_z f = 0$, (58), $\phi \lesssim z^{-1/\beta}$ as $z \to \infty$, and (49),

$$\begin{aligned}
(\partial_s + \mathcal{L})f &= \left(-\frac{\delta}{2} - \frac{\lambda_s}{\lambda} - 2\phi - \varepsilon\right)f \\
&= \left(-\frac{\delta}{2} + 1 + O(e^{-\frac{\delta}{2}s}) + O(z^{-\frac{1}{\beta}}) + O(\sqrt{C^*}\widetilde{K}e^{-\frac{\delta}{2}}s)\right)e^{-\frac{\delta}{2}s} \\
&\geq \left(\frac{\eta}{2} + O(e^{-\frac{\delta}{2}s}) + O(z^{-\frac{1}{\beta}}) + O(C(\widetilde{K}, z^*, \beta, \eta)e^{-\frac{\delta}{2}}s)\right)e^{-\frac{\delta}{2}s},
\end{aligned}$$

where we used $\delta = 2\min(\beta, 1) - \eta$. This implies (88) upon choosing z^* large enough and then s_0 large enough.

Step 2: Estimate for the source term. We claim that for all $z \geq z^*$ and $s_0 \leq s \leq s_1$, we have

$$|F(s,z)| \leq \frac{\eta}{8}\Big(\frac{1}{4}+\frac{\widetilde{K}'}{4}\Big)e^{-\frac{\delta}{2}s}. \tag{89}$$

To prove this inequality, observe that for $0 \leq z \leq z^*$ we have using (54) and (66) that

$$|\varepsilon(z)| \lesssim \sqrt{2C^*\gamma}e^{-\frac{\delta}{2}s}, \tag{90}$$

while for $z \geq z^*$ one has $|\varepsilon(z)| \leq \widetilde{K}'e^{-\frac{\delta}{2}s}$ thanks to (86). Therefore, thanks to the behavior of ϕ' near ∞, one has

$$\begin{aligned}|\partial_z^{-1}\varepsilon\phi'| \leq \|\varepsilon\|_{L^\infty} z|\phi'(z)| &\leq C\big(\sqrt{2C^*\gamma}e^{-\frac{\delta}{2}s}+\widetilde{K}'e^{-\frac{\delta}{2}s}\big)z^{-\frac{1}{\beta}}\\ &\leq \frac{\eta}{16}\Big(\frac{1}{4}+\frac{\widetilde{K}'}{4}\Big)e^{-\frac{\delta}{2}s},\end{aligned} \tag{91}$$

where we choose z^* large enough and then γ small enough such that

$$C\sqrt{2C^*\gamma}z^{*-\frac{1}{\beta}} \leq \frac{\eta}{64} \quad \text{and} \quad Cz^{*-\frac{1}{\beta}} \leq \frac{\eta}{64}. \tag{92}$$

From Proposition 3.1, we know that $|-1+\phi-(\beta+1)z\phi'| \leq C$ is uniformly bounded. Therefore using (61) we get

$$\left|2\nu\Big(\int_0^{\frac{1}{\nu}}(\phi+\varepsilon)^2(z)\,dz\Big)(-1+\phi-(\beta+1)z\phi')\right| \lesssim e^{-(\frac{\delta}{2}+\frac{\eta}{4})s} \leq \frac{\eta}{64}e^{-\frac{\delta}{2}s} \tag{93}$$

for s_0 large enough. Summing (91) and (93) implies (89).

Step 3: Applying the comparison principle. Let

$$f^{\pm} = \pm\Big(\Big(\frac{1}{4}+\frac{\widetilde{K}'}{4}\Big)f-\varepsilon\Big). \tag{94}$$

Then using (87), (88), and (89) one computes that, for $s \leq s \leq s_1$ and $z \in [z^*, \nu^{-1}]$,

$$(\partial_s+\mathcal{L})f^+ = \Big(\frac{1}{4}+\frac{\widetilde{K}'}{4}\Big)(\partial_s+\mathcal{L})f+F \geq \Big(\frac{1}{4}+\frac{\widetilde{K}'}{4}\Big)\frac{\eta}{4}e^{-\frac{\delta}{2}s}-\frac{\eta}{8}\Big(\frac{1}{4}+\frac{\widetilde{K}'}{4}\Big)e^{-\frac{\delta}{2}s} \geq 0.$$

Similarly, $(\partial_s+\mathcal{L})f^- \leq 0$. Recall that from (69) we know that the particles are always moving from region $0 \leq z \leq z^*$ to region $z^* \leq z \leq \frac{1}{\nu}$. At the boundary $z = z^*$ one has using (90) that

$$f^+(s,z^*) = \Big(\frac{1}{4}+\frac{\widetilde{K}'}{4}\Big)e^{-\frac{\delta}{2}s}-\varepsilon(s,z^*) \geq \Big(\frac{1}{4}-\sqrt{2C^*\gamma}\Big)e^{-\frac{\delta}{2}s} \geq 0$$

provided that

$$C\sqrt{2C^*\gamma} \leq \frac{1}{4}. \tag{95}$$

Similarly, $f^-(s,z^*) \leq 0$. At the point $z(s) = \frac{1}{\nu(s)}$, the characteristics of the full transport field stay on the boundary since

$$\frac{d}{ds}\frac{1}{\nu(s)} = -\frac{\nu_s}{\nu^2} = -\frac{\nu_s}{\nu^2} + \int_0^{\frac{1}{\nu}} (\phi + \varepsilon)\, dz, \tag{96}$$

where we have used (36b). Thanks to (45), we know initially

$$f^+\Big(s_0, \frac{1}{\nu_0}\Big) = \Big(\frac{1}{4} + \frac{\widetilde{K}'}{4}\Big)e^{-\frac{\delta}{2}s_0} - \varepsilon\Big(s_0, \frac{1}{\nu_0}\Big) \geq \Big(\frac{1}{4} + \frac{\widetilde{K}'}{4}\Big)e^{-\frac{\delta}{2}s_0} - \frac{1}{4}e^{-\frac{\delta}{2}s_0} \geq 0,$$

thus $f^+(s, \frac{1}{\nu(s)}) \geq 0$ for all $s \geq s_0$. Similarly $f^-(s, \frac{1}{\nu(s)}) \leq 0$. At initial time $s = s_0$, we have $f^+(s_0, z) \geq 0$ because of (45), and similarly $f^-(s_0, z) \leq 0$. Therefore, one can apply the maximum principle and obtain that $f^+(s,z) \geq 0$ and $f^-(s,z) \leq 0$ for all $s_0 \leq s \leq s_1$ and $z^* \leq z \leq \nu^{-1}$. By their definition (94) this implies

$$|\varepsilon(s,z)| \leq \Big(\frac{1}{4} + \frac{\widetilde{K}'}{4}\Big)e^{-\frac{\delta}{2}s} \quad \text{for all } s_0 \leq s \leq s_1 \text{ and } z^* \leq z \leq \nu^{-1}. \tag{97}$$

Step 4: End of the proof. We first set $\widetilde{K}'_0 = \widetilde{K}$ so that (86) is satisfied with constant $\widetilde{K}' = \widetilde{K}'_0$ because of (86). Then we obtain (97) with constant $\widetilde{K}' = \widetilde{K}'_0$. This implies that (86) is satisfied with constant $\widetilde{K}' = \widetilde{K}'_1$ given by $\widetilde{K}'_1 = \varphi(\widetilde{K}'_0)$ with $\varphi(\widetilde{K}') = (1 + \widetilde{K}')/4$. We iterate this procedure, and obtain constants $\widetilde{K}_2$, then $\widetilde{K}_3, \dots, \widetilde{K}_n$ such that (86) is satisfied with constants $\widetilde{K}_2$, then $\widetilde{K}_3, \dots, \widetilde{K}_n$ with $\widetilde{K}_{k+1} = \varphi(\widetilde{K}_k)$. By iterating a finite (depending on $\widetilde{K}$) number of times k, we obtain $\widetilde{K}'_k \leq 1$ and (86) then implies (85). ∎

We can now end the proof of Proposition 4.3.

Proof of Proposition 4.3. We set $\widetilde{K} = 3$ (or any $\widetilde{K} > 1$). For any $\beta > 0$ and $0 < \eta < \min(1, \beta)$, we pick $z^* \geq \bar{z}^*$ where $\bar{z}^*(\beta, \eta)$ is given by Lemma 4.8, then we pick $K \geq K^*$ where $K^*(\beta, \eta, z^*)$ is given by Lemma 4.7, and then we pick $0 < \gamma \leq \min(\gamma^*, 1)$ where $\gamma^*(\beta, \eta, z^*, K^*)$ is given by Lemma 4.8.

Then for any $\tilde{\nu}_0^* > 0$, there exists s_0^* such that the conclusions of Lemmas 4.6, 4.7, and 4.8 are simultaneously valid for all $s_0 \geq s_0^*$.

For such choices of constants, consider an initial data a_0 trapped in the sense of Definition 4.1. We define

$$s^* = \sup\big\{s_1 \geq s_0,\ a \text{ is trapped in the sense of Definition 4.2 on } [s_0, s_1]\big\}.$$

If $s^* = \infty$, then Proposition 4.3 is proved. We assume by contradiction $s^* < \infty$. Then, applying Lemmas 4.6, 4.7, and 4.8 we obtain at time s^*,

$$\frac{1}{2}e^{-s^*} \leq \lambda(s^*) \leq \frac{3}{2}e^{-s^*}, \quad \frac{\tilde{\nu}_0}{2}e^{-\beta s^*} \leq \nu(s^*) \leq \frac{3}{2}\tilde{\nu}_0 e^{-\beta s^*},$$
$$\mathcal{E}_1^2(s^*) \leq 2\gamma e^{-\delta s^*}, \qquad \mathcal{E}_2^2(s^*) \leq e^{-\delta s^*}.$$

Since $\widetilde{K} = 3$, the bounds of Definitions 4.2 are thus strictly satisfied at time s^*, and by a continuity argument there exists $s' > s^*$ such that these bounds are satisfied on $[s^*, s']$. But this contradicts the definition of s^*. Hence $s^* = \infty$ and Proposition 4.3 is proved. ∎

Remark 6. Notice that when η gets closer to 0, one can get better decay. However, the decay rates of $\mathcal{E}_1$ and $\mathcal{E}_2$ cannot reach or be faster than $e^{-\beta s}$ when $0 < \beta \leq 1$ and e^{-s} when $\beta > 1$. The reasons are that

- for $0 < \beta \leq 1$, the decay of ν is only $e^{-\beta s}$;
- for $\beta > 1$, in order to make (80) integrable near 0, one needs $\frac{4}{\beta+1} - 2 + \alpha > -1 \Leftrightarrow \alpha > \frac{\beta-3}{\beta+1}$. Such a restriction on α makes the spectral gap in (71) strictly less than 2.

We can now end the proof of Theorem 1.3.

Proof of Theorem 1.3. Pick $\beta > 0$. We write $\phi = \phi_\beta$ in the proof for simplicity. Choose then any $0 < \eta < \min(\beta, 1)$ and let the constants $\widetilde{K} > 1$, $K, z^* \geq 1$, and $\gamma > 0$ be given by Proposition 4.3. For a fixed $\tilde{\nu}_0^* > 0$, let s_0^* be given by Proposition 4.3 and define $\lambda_0^* = e^{-2s_0^*}$.

Then, with a parameter κ to be fixed later on in the proof, let

$$\lambda_0 \leq \lambda_0^*/2 \quad \text{and} \quad \tilde{\nu}_0 \leq \tilde{\nu}_0^*/2, \tag{98}$$

and an initial datum a_0 of the form (17) satisfy (18).

Step 1*: Proof assuming a claim.* We claim that for κ small enough, a_0 is initially trapped in the sense of Definition 4.1, with framework parameters η, $\widetilde{K}$, K, z^*, γ, λ_0^*, $\tilde{\nu}_0^*$ defined right above, and decomposition parameters $\bar{\lambda}_0$, $\bar{\nu}_0$ to be determined in Step 2. Assuming this claim, we have using Proposition 4.3 that the solution is trapped in the sense of Definition 4.2 for all $s \in [\bar{s}_0, \infty)$ where $\bar{s}_0 = \log \bar{\lambda}_0^{-1}$.

We unwind the self-similar transformation (34) using (60) and define $T = \int_{\bar{s}_0}^\infty \lambda(s)\, ds < \infty$ so that

$$\begin{aligned} t(s) = \int_{\bar{s}_0}^s \lambda(\tilde{s})\, d\tilde{s} &= T - \int_s^\infty \tilde{\lambda}_\infty e^{-\tilde{s}}(1 + O(e^{-\frac{\delta}{2}\tilde{s}}))\, d\tilde{s} \\ &= T - \tilde{\lambda}_\infty e^{-s} + O(e^{-(1+\frac{\delta}{2})s}) \end{aligned}$$

and hence $\tilde{\lambda}_\infty e^{-s} = (T - t) + O((T - t)^{1+\delta/2})$. We then get using (49) and (60) that

$$\begin{aligned} \|\varepsilon(s)\|_{L^\infty(0,\nu^{-1})} \leq C(T-t)^{\delta/2}, \quad \lambda(s) = T - t + O((T-t)^{1+\delta/2}), \\ \nu = \frac{\tilde{\nu}_\infty}{\tilde{\lambda}_\infty^\beta}(T-t)^\beta + O((T-t)^{\beta+\delta/2}). \end{aligned}$$

Back in original variables (35), this implies the desired results (19) and (20) in the theorem by renaming δ as 2δ and $\tilde{\nu}_\infty$ as $\tilde{\nu}_\infty \tilde{\lambda}_\infty^\beta$.

Step 2: Proof of the claim. For $\bar\lambda_0, \bar\nu_0 > 0$ we define $\bar a_0$ and $\bar\varepsilon_0$ as

$$a_0(Z) = \frac{1}{\lambda_0}\phi\Big(\frac{Z}{\lambda_0^\beta \tilde\nu_0}\Big) + \tilde a_0(Z) = \frac{1}{\bar\lambda_0}\phi\Big(\frac{Z}{\bar\lambda_0^\beta \bar\nu_0}\Big) + \bar a_0(Z),$$
$$\bar\varepsilon_0(z) = \bar\lambda_0 \bar a_0(\bar\lambda_0^\beta \bar\nu_0 z), \quad Z = z\bar\lambda_0^\beta \bar\nu_0.$$

Then, introducing $\mu = \bar\lambda_0^\beta \bar\nu_0 \lambda_0^{-\beta}\tilde\nu_0^{-1}$, we have the two decompositions for $0 < z \le 1$ (ε_1 to ε_4) and $1 \le z \le \bar\lambda_0^{-\beta}\bar\nu_0^{-1}$ ($\bar\varepsilon_1$ and $\bar\varepsilon_2$),

$$\bar\varepsilon_0(z) = \bar\varepsilon_1(z) + \bar\varepsilon_2(z) = \varepsilon_1(z) + \varepsilon_2(z) + \varepsilon_3(z) + \varepsilon_4(z), \tag{99}$$

$$\bar\varepsilon_1(z) = \frac{\bar\lambda_0}{\lambda_0}\phi(\mu z) - \phi(z), \quad \bar\varepsilon_2(z) = \bar\lambda_0 \tilde a_0(\bar\lambda_0^\beta \bar\nu_0 z), \tag{100}$$

$$\begin{aligned} \varepsilon_1(z) &= \frac{\bar\lambda_0}{\lambda_0} - 1 + \bar\lambda_0 \tilde a_0(0) + z^{\frac{1}{\beta+1}}\Big(1 - \frac{\bar\lambda_0}{\lambda_0}\mu^{\frac{1}{\beta+1}}\Big), \\ \varepsilon_2(z) &= \Big(\frac{\bar\lambda_0}{\lambda_0} - 1\Big)(\phi(\mu z) - 1 + (\mu z)^{\frac{1}{\beta+1}}), \end{aligned} \tag{101}$$

$$\varepsilon_3(z) = \phi(\mu z) - \phi(z) + (\mu^{\frac{1}{\beta+1}} - 1)z^{\frac{1}{\beta+1}}, \quad \varepsilon_4 = \bar\lambda_0(\tilde a_0(z\bar\lambda_0^\beta \bar\nu_0) - \tilde a_0(0)).$$

In order for the boundary condition (37) to be satisfied, using the behavior (27) of $\phi(z)$ as $z \to 0$, we require that $\varepsilon_1 = 0$. Using (101) this fixes $\bar\lambda_0, \bar\nu_0$ in an unique manner via the identities

$$\begin{aligned} \bar\lambda_0 &= (\lambda_0^{-1} + \tilde a_0(0))^{-1} = \lambda_0(1 + O(\kappa\bar\lambda_0)), \\ \bar\nu_0 &= \lambda_0^{1+2\beta}\bar\lambda_0^{-1-2\beta}\tilde\nu_0 = \tilde\nu_0(1 + O(\kappa\bar\lambda_0)), \end{aligned} \tag{102}$$

where we used (18). Injecting (102) into (100), using (28) one then obtains that for all $z \ge 1$,

$$|\bar\varepsilon_1(z)| \lesssim \Big|\frac{\bar\lambda_0}{\lambda_0} - 1\Big|\phi(\mu z) + |\mu z - z| \sup_{\tilde z \in [z, \mu z]} |\phi'(\tilde z)| \lesssim \kappa\bar\lambda_0 z^{-\frac{1}{\beta}},$$

and similarly

$$|\partial_z \bar\varepsilon_1(z)| \lesssim \kappa\bar\lambda_0 z^{-\frac{1}{\beta}-1}.$$

Using (18) we have for $z \ge 1$ that $|\bar\varepsilon_2(z)| \le \bar\lambda_0\kappa$ and $|\partial_z \varepsilon_2(z)| \le \kappa\bar\lambda_0^{1+\beta}\bar\nu_0$. Injecting these two inequalities and the two above into the first decomposition in (99) shows

$$|\bar\varepsilon_0(z)| \lesssim \kappa\bar\lambda_0 \quad \text{and} \quad |\partial_z\bar\varepsilon_0(z)| \lesssim \kappa\bar\lambda_0(z^{-1-\frac{1}{\beta}} + \bar\lambda_0^\beta\bar\nu_0) \quad \text{for all } z \in [1, \bar\lambda_0^{-\beta}\bar\nu_0^{-1}]. \tag{103}$$

Next, using the behavior (27) of $\phi(z)$ as $z \to 0$ and then (102), we deduce that for $0 < z \le 1$,

$$|\partial_z \varepsilon_2(z)| \lesssim \Big|\frac{\bar\lambda_0}{\lambda_0} - 1\Big| z^{\frac{2}{\beta+1}-1} \lesssim \kappa\bar\lambda_0 z^{\frac{2}{\beta+1}-1}.$$

By a similar estimate, $|\partial_z \varepsilon_3(z)| \lesssim \kappa\bar\lambda_0 z^{\frac{2}{\beta+1}-1}$ for $0 < z \le 1$. Using (18) we obtain $|\partial_z\varepsilon_4| \lesssim \kappa\bar\lambda_0^{1+\beta}\bar\nu_0$. Injecting these inequalities and the one above into the second decomposition and $\varepsilon_1 = 0$ in (99) shows

$$|\partial_z \bar\varepsilon_0(z)| \lesssim \kappa \bar\lambda_0 z^{\frac{2}{\beta+1}-1} + \kappa \bar\lambda_0^{1+\beta} \bar\nu_0 \quad \text{for all } z \in (0,1]. \tag{104}$$

Combining (103) and (104), using $w = z^\alpha e^{-Kz}$ with $\alpha = \frac{|1-\beta|-2+\frac{\eta}{2}}{\beta+1} > -1$ we obtain

$$\int_0^{z^*} w \bar\varepsilon_0^2 \, dz + \sup_{z^* \le z \le \bar\lambda_0^{-\beta} \bar\nu_0^{-1}} \bar\varepsilon_0^2(z) \lesssim \kappa^2 \bar\lambda_0^2 + \kappa^2 \bar\lambda_0^{2+2\beta} \bar\nu_0^2. \tag{105}$$

We now check that a_0 is initially trapped in the sense of Definition 4.1 with decomposition parameters $\bar\lambda_0$ and $\bar\nu_0$, and framework parameters η, $\widetilde K$, K, z^*, γ, λ_0^*, $\tilde\nu_0^*$ defined right before Step 1. We set $\bar s_0 = \log \bar\lambda_0^{-1}$. Estimates (98) and (102) imply (44) for $\bar\lambda_0$, $\bar s_0$, $\bar\nu_0$, $\tilde\nu_0^*$, so that Definition 4.1 (i) is indeed satisfied. The fact that $\varepsilon_1 = 0$ in the second decomposition in (99) and the inequality (104) show that Definition 4.1 (ii) is satisfied. Finally, (105) shows that Definition 4.1 (iii) is also satisfied provided κ has been chosen small enough depending only on λ_0^* and $\tilde\nu_0^*$. Hence a_0 is initially trapped, finishing the proof of the claim. ∎

5. The smooth blow-up case

In this section, we prove Theorem 1.1. We study the limiting critical case when $\beta = 0$, for which

$$\phi_{\beta=0}(z) = \phi(z) = e^{-z}$$

(we drop the β subscript in this section to ease notation). When $\beta = 0$, the vanishing condition (37) becomes

$$\begin{cases} \varepsilon(s, z=0) = 0, \\ \partial_z \varepsilon(s, z=0) = 0, \end{cases} \tag{106}$$

and the modulation equations (38) and (40) become

$$\frac{\lambda_s}{\lambda} + 1 = -\frac{\nu_s}{\nu} = 2\nu \int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2(z) \, dz. \tag{107}$$

Therefore, one can rewrite (36a) and (39) as

$$\begin{aligned} &\varepsilon_s - \frac{\lambda_s}{\lambda}\varepsilon - \frac{\nu_s}{\nu} z \varepsilon_z - 2\phi\varepsilon + \partial_z^{-1}\phi \varepsilon_z + \partial_z^{-1}\varepsilon \phi' - \varepsilon^2 + \partial_z^{-1}\varepsilon\varepsilon_z \\ &\quad = 2\nu \left(\int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2(z) \, dz \right) (-1 + (z+1)\phi) \end{aligned} \tag{108}$$

and

$$\begin{aligned} &\varepsilon_{zs} - (\phi - 1)\varepsilon_z + \left(\partial_z^{-1}\phi - \frac{\nu_s}{\nu} z\right)\varepsilon_{zz} - \phi'\varepsilon + \partial_z^{-1}\varepsilon\phi'' - \varepsilon\varepsilon_z + \partial_z^{-1}\varepsilon\varepsilon_{zz} \\ &\quad = -z\phi 2\nu \int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2(z) \, dz. \end{aligned} \tag{109}$$

The proof of the theorem follows the same strategy as that of Theorem 1.3. It also relies on a bootstrap argument. However, (107) gives $\nu_s = -\nu^2$ to leading order, hence we will have to deal with the slower algebraic decay $\nu \approx s^{-1}$ in comparison with the exponential decays involved in the proof of Theorem 1.3. We first need to adjust Definitions 4.1 and 4.2.

We consider the weight

$$w(z) = z^{-2}. \tag{110}$$

Here, since $\phi(z) = e^{-z}$, explicit computations to control nonlocal terms will avoid the use of a e^{-Kz} factor in the weight.

Definition 5.1 (Initial closeness). Let $\lambda_0^* > 0$ and $\gamma > 0$. We say that a_0 is initially close to the blow-up profile if there exists $\lambda_0 > 0$ and $\nu_0 > 0$ such that the decomposition (35) satisfies

(i) *initial values of the modulation parameters* (note that this fixes the value of s_0):

$$\lambda_0 = s_0 e^{-s_0}, \quad \frac{1}{2s_0} \leq \nu_0 \leq \frac{2}{s_0}; \tag{111}$$

(ii) *compatibility condition for the initial perturbation*: $\varepsilon_0 \in C^2([0, \frac{1}{\nu_0}))$ satisfies the boundary conditions (106) and the integral condition (36b);

(iii) *initial smallness of the remainder in the self-similar variables*: for some small number $\gamma > 0$, with w given by (110),

$$\mathcal{E}_1^2(s_0) = \int_0^{z^*} w\varepsilon_0^2 \, dz < \gamma^2 s_0^{-\frac{4}{3}}, \quad \mathcal{E}_2^2(s_0) = \sup_{z^* \leq z \leq \frac{1}{\nu_0}} |\varepsilon_0|^2 < \frac{1}{4} s_0^{-\frac{4}{3}}. \tag{112}$$

Definition 5.2 (Trapped solutions). Let $s_0^* \geq 0$, $z^* \geq 1$ and $\gamma > 0$. We say that a solution $a(s,z)$ is trapped on $[s_0, s_1]$ with $s_0^* \leq s_0 < s_1 \leq \infty$ if it satisfies the properties of Definition 5.1 at time s_0 and if for all $s \in [s_0, s_1]$, $a(s,z)$ can be decomposed as in (35) with

(i) *values of the modulation parameters*:

$$\frac{1}{4} s e^{-s} < \lambda < 4 s e^{-s}, \quad \frac{1}{4s} < \nu < \frac{4}{s}; \tag{113}$$

(ii) *decay in time of the remainder in the self-similar variables*:

$$\mathcal{E}_1^2(s) = \int_0^{z^*} w\varepsilon_z^2 \, dz < s^{-\frac{4}{3}}, \quad \mathcal{E}_2^2(s) = \sup_{z^* \leq z \leq \frac{1}{\nu(s)}} |\varepsilon|^2 < s^{-\frac{4}{3}}. \tag{114}$$

Remark 7. One could show that the decay rate for $\mathcal{E}_1$ and $\mathcal{E}_2$ is $s^{-1+\eta}$ for any $0 < \eta < 1$. Here for simplicity we take $s^{-\frac{2}{3}}$ as an example.

The heart of our analysis, as in the case $\beta > 0$, is to show that a solution that is initially trapped will remain globally trapped in self-similar time s.

Proposition 5.3. *There exist universal constants $z^* \geq 1$, $\gamma > 0$, and $s_0^* \geq 0$ such that the following holds true. For all $s_0 \geq s_0^*$, any solution of (9) which is initially close to the blow-up profile in the sense of Definition 5.1 is trapped on $[s_0, +\infty)$ in the sense of Definition 5.2.*

The proof of Proposition 5.3 necessitates several lemmas that improve strictly all a priori estimates of Definition 5.2.

Lemma 5.4. *For any $z^* \geq 1$, for s_0^* large enough, if a is trapped on $[s_0, s_1]$ then for all $s_0 \leq s \leq s_1$,*

$$\|\varepsilon(s)\|_{L^\infty([0,\nu^{-1}])} \leq C(z^*)s^{-\frac{2}{3}} \tag{115}$$

and

$$\nu \int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2(z)\, dz \leq 4s^{-1}. \tag{116}$$

Proof. From the vanishing boundary condition, Cauchy–Schwarz, (110), and (114), for $0 < z \leq z^*$,

$$|\varepsilon(z)| = \left| \int_0^z \partial_z \varepsilon \, d\tilde{z} \right| \leq \mathcal{E}_1 \sqrt{\int_0^z z^2 \, d\tilde{z}} \lesssim s^{-\frac{2}{3}} z^{\frac{3}{2}}. \tag{117}$$

This, combined with the second inequality in (114), shows (115). Then, since $\phi = e^{-z}$ we estimate

$$\begin{aligned} \int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2(z)\, dz &\leq \int_0^\infty \phi^2 \, dz + 2 \int_0^{\nu^{-1}} \phi\varepsilon + \int_0^{\nu^{-1}} \varepsilon^2 \, dz \\ &\leq \frac{1}{2} + 2\|\varepsilon\|_{L^\infty([0,\nu^{-1})} + \nu^{-1}\|\varepsilon\|^2_{L^\infty([0,\nu^{-1})} \leq \frac{1}{2} + O(s^{-1/3}) \leq 1 \end{aligned}$$

for s_0^* large enough, where we used (115). The above inequality and (113) show (116). ∎

Lemma 5.5 (Modulation equations). *For any $z^* \geq 1$ and $\gamma > 0$, there exists a large self-similar time s_0^* such that for any $s_0 \geq s_0^*$, for any solution which is trapped on $[s_0, s_1]$, we have for $s \in [s_0, s_1]$,*

$$\left| \frac{\lambda_s}{\lambda} + 1 \right| \leq C s^{-1}, \quad \left| \frac{\nu_s}{\nu} \right| \leq C s^{-1}, \tag{118}$$

for $C > 0$ independent of the bootstrap constants, and

$$\frac{1}{2} s e^{-s} \leq \lambda \leq \frac{3}{2} s e^{-s}, \quad \frac{1}{3s} \leq \nu \leq \frac{3}{s}. \tag{119}$$

Moreover, if $s_1 = \infty$ then there exists a constant $\tilde{\lambda}_\infty > 0$ such that

$$\lambda = \tilde{\lambda}_\infty s e^{-s}(1 + O(s^{-\frac{1}{3}})), \quad \nu = \frac{1}{s} + O(s^{-\frac{4}{3}}). \tag{120}$$

Proof. We divide the proof into three steps.

Step 1*: A preliminary estimate.* We claim that

$$2\int_0^{\frac{1}{\nu}}(\phi+\varepsilon)^2(z)\,dz = 1+O(C(z^*)s^{-\frac{1}{3}}). \tag{121}$$

Indeed, as $\phi(z)=e^{-z}$, we have $2\int_0^\infty \phi^2=1$. Hence, using (113) and (115),

$$\begin{aligned}2\int_0^{\frac{1}{\nu}}(\phi+\varepsilon)^2(z)\,dz &= 1+2\int_{\nu^{-1}}^\infty \phi^2(z)\,dz+4\int_0^{\frac{1}{\nu}}\phi\varepsilon(z)\,dz+2\int_0^{\nu^{-1}}\varepsilon^2\,dz\\ &= 1+O(e^{-2\nu^{-1}})+O(\|\varepsilon\|_{L^\infty([0,\nu^{-1})]})+O(\nu^{-1}\|\varepsilon\|^2_{L^\infty([0,\nu^{-1})]})\\ &= 1+O(C(z^*)s^{-\frac{1}{3}}).\end{aligned}$$

Step 2*: Equation for* ν*.* Injecting (121) into (107) gives

$$-\frac{\nu_s}{\nu^2} = 1+O(C(z^*)s^{-\frac{1}{3}}). \tag{122}$$

Multiplying (122) by ν and using (113) shows the second inequality in (118). Integrating (122) with time, we find

$$\frac{1}{\nu}=\frac{1}{\nu_0}+s-s_0+O(C(z^*)(s^{\frac{2}{3}}-s_0^{\frac{2}{3}}))=\frac{1}{\nu_0}+s-s_0+O(C(z^*)s^{-\frac{1}{3}}(s-s_0)). \tag{123}$$

Therefore, since $\nu_0^{-1}\le 2s_0$, from (111) we infer for s_0^* large enough depending on z^*,

$$\begin{aligned}\frac{1}{\nu}&\le 2s_0+s-s_0+O(C(z^*)s^{-\frac{1}{3}}(s-s_0))\\ &= s+s_0+O(C(z^*)s^{-\frac{1}{3}}(s-s_0))\le 3s.\end{aligned} \tag{124}$$

One finds similarly using $s_0/2\le \nu_0^{-1}$ from (111) that $\frac{1}{\nu}\ge s/3$. This and (124) imply the second inequality in (119). Finally, if $s_1=\infty$ then (123) implies $\nu^{-1}=s+O(s^{2/3})$ and the second inequality in (120) follows.

Step 3*: Equation for* λ*.* Injecting (121) into (107) one finds

$$\frac{\lambda_s}{\lambda}+1=\frac{1}{s}(1+O(s^{-\frac{1}{3}}))(1+O(s^{-\frac{1}{3}}))=\frac{1}{s}+O(s^{-\frac{4}{3}}).$$

This implies the first inequality in (118). Since $\lambda=O(se^{-s})$ from (113), one has

$$\frac{d}{ds}\Big(\frac{e^s\lambda}{s}\Big)=O(s^{-\frac{4}{3}}).$$

We integrate the above equation with time using $\lambda_0=s_0e^{-s_0}$ and find

$$\lambda(s)=se^{-s}\Big(1+\int_{s_0}^s O(s^{-\frac{4}{3}})\,ds\Big)=se^{-s}(1+O(s_0^{-\frac{1}{3}})).$$

This implies the first inequality in (119) for s_0 large enough. If $s_1 = \infty$ then we set $\tilde{\lambda}_\infty = 1 + \int_{s_0}^{\infty} O(s^{-\frac{4}{3}})\, ds$ and rewrite the above equality as

$$\begin{aligned}\lambda(s) &= se^{-s}\Big(1 + \int_{s_0}^{\infty} O(s^{-\frac{4}{3}})\, ds - \int_{s}^{\infty} O(s^{-\frac{4}{3}})\, ds\Big)\\ &= se^{-s}(\lambda_\infty + O(s^{-\frac{1}{3}})).\end{aligned}$$

This is the first inequality in (120). ■

Lemma 5.6 (Interior estimate). *For any $z^* \geq 1$ and $\gamma > 0$, there exists a large self-similar time s_0^* such that for any $s_0 \geq s_0^*$, for any solution which is trapped on $[s_0, s_1]$, we have for $s \in [s_0, s_1]$,*

$$\mathcal{E}_1^2(s) \leq 2\gamma^2 s^{-\frac{4}{3}}. \tag{125}$$

Proof. Recall (110). Multiplying (109) by $w\varepsilon_z$ and integrating over $[0, z^*]$, one obtains that

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}\int_0^{z^*} w\varepsilon_z^2\, dz - \int_0^{z^*} \Big(\frac{\lambda_s}{\lambda} + \frac{\nu_s}{\nu} + \phi + \varepsilon\Big) w\varepsilon_z^2\, dz\\ &\quad + \int_0^{z^*} \Big(\partial_z^{-1}\phi - \frac{\nu_s}{\nu} z + \partial_z^{-1}\varepsilon\Big)\varepsilon_{zz} w\varepsilon_z\, dz\\ &\quad - \int_0^{z^*} \phi'\varepsilon w\varepsilon_z\, dz + \int_0^{z^*} \partial_z^{-1}\varepsilon w\varepsilon_z\phi''\, dz\\ &\quad = -2\nu\Big(\int_0^{\frac{1}{\nu}} (\phi+\varepsilon)^2(z)\, dz\Big)\int_0^{z^*} z\phi w\varepsilon_z\, dz.\end{aligned} \tag{126}$$

We now compute all terms in (67).

Potential and transport terms. Integrating by parts yields

$$\begin{aligned}&-\int_0^{z^*} \Big(\frac{\lambda_s}{\lambda} + \frac{\nu_s}{\nu} + \phi + \varepsilon\Big) w\varepsilon_z^2\, dz + \int_0^{z^*} \Big(\partial_z^{-1}\phi - \frac{\nu_s}{\nu} z + \partial_z^{-1}\varepsilon\Big)\varepsilon_{zz} w\varepsilon_z\, dz\\ &= \Big(\int_0^{z^*} (\phi+\varepsilon)(\tilde{z})\, d\tilde{z} - \frac{\nu_s}{\nu} z^*\Big)\frac{1}{2} w(z^*)\varepsilon_z^2(z^*)\\ &\quad + \int_0^{z^*} \Big(-\frac{3}{2}\phi - \frac{3}{2}\varepsilon - \frac{\lambda_s}{\lambda} - \frac{1}{2}\frac{\nu_s}{\nu} - \frac{1}{2}\Big(\partial_z^{-1}\phi - \frac{\nu_s}{\nu} z + \partial_z^{-1}\varepsilon\Big)\frac{w_z}{w}\Big) w\varepsilon_z^2\, dz.\end{aligned} \tag{127}$$

For the boundary term, we know that $\|\varepsilon\|_{L^\infty} \leq C(z^*) s^{-2/3}$ from (115) and thus using (118),

$$\begin{aligned}\int_0^{z^*} (\phi+\varepsilon)(\tilde{z})\, d\tilde{z} - \frac{\nu_s}{\nu} z^* &\geq 1 - e^{-z^*} - \|\varepsilon\|_{L^\infty} z^* + O(C(z^*)s^{-1})\\ &\geq 1 - e^{-z^*} - C(z^*)s^{-\frac{2}{3}} \geq 0\end{aligned} \tag{128}$$

when s_0 is large enough. Since the weight function is $w = z^{-2}$, one has using $\phi = e^{-z}$, (118), and (115),

$$\begin{aligned}
&\Big(-\frac{3}{2}\phi - \frac{3}{2}\varepsilon - \frac{\lambda_s}{\lambda} - \frac{1}{2}\frac{\nu_s}{\nu} - \frac{1}{2}\Big(\partial_z^{-1}\phi - \frac{\nu_s}{\nu}z + \partial_z^{-1}\varepsilon\Big)\frac{w_z}{w}\Big)\\
&\quad = \Big(-\frac{3}{2}e^{-z} + O(s^{-\frac{2}{3}}) + 1 + O(s^{-1}) + O(s^{-1})\\
&\qquad - \frac{1}{2}\Big(1 - e^{-z} + O(s^{-1}z) + O(s^{-\frac{2}{3}}z)\Big)\Big(-\frac{2}{z}\Big)\Big)\\
&\quad = 1 - \frac{3}{2}e^{-z} + \frac{1-e^{-z}}{z} + O(s^{-\frac{2}{3}}) \geq \frac{1}{2} + O(s^{-\frac{2}{3}})
\end{aligned} \tag{129}$$

(where we have used the fact that $1 - e^{-z} \geq ze^{-z}$. Injecting (128) and (129) into (127) shows

$$\begin{aligned}
&-\int_0^{z^*}\Big(\frac{\lambda_s}{\lambda} + \frac{\nu_s}{\nu} + \phi + \varepsilon\Big)w\varepsilon_z^2\,dz + \int_0^{z^*}\Big(\partial_z^{-1}\phi - \frac{\nu_s}{\nu}z + \partial_z^{-1}\varepsilon\Big)\varepsilon_{zz}w\varepsilon_z\,dz\\
&\quad \geq \Big(\frac{1}{2} + O(s^{-\frac{2}{3}})\Big)\mathcal{E}_1(s).
\end{aligned} \tag{130}$$

The nonlocal terms. By direct computations, using $|\varepsilon(z)| \leq \mathcal{E}_1\sqrt{\int_0^z w^{-1}}$, Cauchy–Schwarz, and $\phi = e^{-z}$, one gets

$$\begin{aligned}
\int_0^{z^*}|\phi'\varepsilon w\varepsilon_z|\,dz &\leq \Big(\int_0^{z^*}|\phi'|^2 w\Big(\int_0^z \frac{1}{w}(\tilde z)\,d\tilde z\Big)\,dz\Big)^{\frac{1}{2}}\mathcal{E}_1^2\\
&\leq \Big(\int_0^{\infty}\frac{1}{3}ze^{-2z}\,dz\Big)^{\frac{1}{2}}\mathcal{E}_1^2 = \frac{1}{2}\sqrt{\frac{1}{3}}\mathcal{E}_1^2
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{z^*}|\partial_z^{-1}\varepsilon w\varepsilon_z\phi''|\,dz &\leq \Big(\int_0^{z^*}|\phi''|^2 w\Big(\int_0^z\Big(\int_0^{\tilde z}\frac{1}{w}(\xi)\,d\xi\Big)^{\frac{1}{2}}d\tilde z\Big)^2 dz\Big)^{\frac{1}{2}}\mathcal{E}_1^2\\
&\leq \Big(\int_0^{\infty}\frac{4}{75}z^3e^{-2z}\,dz\Big)^{\frac{1}{2}}\mathcal{E}_1^2 = \frac{1}{5\sqrt{2}}\mathcal{E}_1^2.
\end{aligned}$$

The source term. Using (116) and Cauchy–Schwarz,

$$\begin{aligned}
\Big|\nu\Big(\int_0^{\frac{1}{\nu}}(\phi+\varepsilon)^2(z)\,dz\Big)\int_0^{z^*}z\phi w\varepsilon_z\,dz\Big| &\leq Cs^{-1}\mathcal{E}_1\sqrt{\int_0^{z^*}z^2\phi^2 w\,dz}\\
&\leq Cs^{-1}\mathcal{E}_1 \leq \frac{\mathcal{E}_1^2}{200} + Cs^{-2}.
\end{aligned}$$

Conclusion. Injecting (130) and the three inequalities above into (126) yields

$$\frac{d}{ds}\mathcal{E}_1^2 + \Big(1 - \sqrt{\frac{1}{3}} - \frac{\sqrt{2}}{5} - \frac{1}{200} - C(z^*)s_0^{-\frac{2}{3}}\Big)\mathcal{E}_1^2 \leq \frac{C}{s^2}.$$

Choose s_0 large enough that $C(z^*)s^{-\frac{3}{2}} \le \frac{1}{200}$. Then $1 - \sqrt{\frac{1}{3}} - \frac{\sqrt{2}}{5} - \frac{1}{200} - C(z^*)s_0^{-\frac{2}{3}} \ge \frac{1}{8}$, and consequently,

$$\frac{d}{ds}\mathcal{E}_1^2 + \frac{1}{8}\mathcal{E}_1^2 \le \frac{C}{s^2}.$$

When s_0 is large enough, we have

$$\frac{d}{ds}(s^2\mathcal{E}_1^2 e^{\frac{s}{16}}) = e^{\frac{s}{16}}s^2\frac{d}{ds}\mathcal{E}_1^2 + 2se^{\frac{s}{16}}\mathcal{E}_1^2 + \frac{s^2}{16}\mathcal{E}_1^2 e^{\frac{s}{16}} \le e^{\frac{s}{16}}s^2\frac{d}{ds}\mathcal{E}_1^2 + e^{\frac{s}{16}}\frac{s^2}{8}\mathcal{E}_1^2 \le Ce^{\frac{s}{16}}.$$

Integrating both sides from s_0 to s, since $\mathcal{E}_1^2(s_0) \le \gamma^2 s_0^{-\frac{4}{3}}$, when s_0 is large enough, one obtains

$$\mathcal{E}_1^2(s) \le \left(s_0^2\mathcal{E}_1^2(s_0)e^{\frac{s_0}{16}} + Ce^{\frac{s}{16}}\right)e^{-\frac{s}{16}}\frac{1}{s^2} \le 2\gamma^2 s^{-\frac{4}{3}}.$$ ■

The following lemma is similar to Lemma 4.8.

Lemma 5.7 (Exterior estimate). *There exists* $\bar{z}^* \ge 1$, *and for any* $z^* \ge \bar{z}^*$, *a* $\gamma^* > 0$ *such that for* $0 < \gamma \le \gamma^*$ *the following holds true. There exists* s_0^* *large enough such that if a solution is trapped on* $[s_0, s_1]$ *with* $s_0 \ge s_0^*$, *for any time* $s_0 \le s \le s_1$ *we have*

$$\mathcal{E}_2^2(s) \le \frac{1}{4}s^{-\frac{4}{3}}. \tag{131}$$

Proof. The proof relies on the maximum principle. We rewrite (108) as

$$\varepsilon_s + \mathcal{L}\varepsilon = F, \tag{132}$$

where the transport operator $\mathcal{L}$ (note that it has a nonlinear part) and the source term are

$$\mathcal{L}v = -\frac{\lambda_s}{\lambda}v - \frac{\nu_s}{\nu}zv_z - 2\phi v + \partial_z^{-1}\phi v_z - \varepsilon v + \partial_z^{-1}\varepsilon v_z,$$

$$F = -\partial_z^{-1}\varepsilon\phi' + 2\nu\left(\int_0^{\frac{1}{\nu}}(\phi+\varepsilon)^2(z)\,dz\right)(-1 + (z+1)\phi).$$

Step 1*: A supersolution for* $\partial_s + \mathcal{L}$ *on* $[z^*, \nu^{-1}]$. We introduce

$$f(s,z) = \frac{1}{2}s^{-\frac{2}{3}}$$

and claim that there exists z^* large enough such that for s_0 large enough, for all $s_0 \le s \le s_1$ and $z \ge z^*$,

$$(\partial_s + \mathcal{L})f \ge \frac{s^{-\frac{2}{3}}}{4}. \tag{133}$$

To prove (133), we compute using (118) and (115),

$$\begin{aligned}(\partial_s + \mathcal{L})f &= \left(-\frac{2}{3s} - \frac{\lambda_s}{\lambda} - 2e^{-z} - \varepsilon\right)\frac{s^{-\frac{2}{3}}}{2}\\ &= (O(s^{-1}) + 1 + O(s^{-1}) + O(e^{-z^*}) + O(s^{-\frac{2}{3}}))\frac{s^{-\frac{2}{3}}}{2},\end{aligned}$$

which implies (133) upon taking z^* large enough and then s_0^* large enough.

Step 2: Estimate for the source term. We claim that for z^* large enough and then for γ small enough, for all $s_0 \le s \le s_1$ and $z \in [z^*, \nu^{-1})$,

$$|F(s,z)| \le \frac{s^{-\frac{2}{3}}}{8}. \tag{134}$$

We now prove this inequality. We inject the improved bootstrap bound (125) into the computation (117) and get

$$|\varepsilon(z)| \le C\gamma s^{-\frac{2}{3}} z^{\frac{3}{2}} \quad \text{for } z \in [0, z^*]. \tag{135}$$

Also, $|\varepsilon(z)| \le s^{-\frac{2}{3}}$ for $z \in [z^*, \nu^{-1})$ using (114). Therefore, using $\phi(z) = e^{-z}$,

$$|\partial_z^{-1} \varepsilon \phi'| \le \|\varepsilon\|_{L^\infty} z |\phi'(z)| \le C(C(z^*)\gamma s^{-\frac{2}{3}} + s^{-\frac{2}{3}}) z^* e^{-z^*} \le \frac{s^{-\frac{2}{3}}}{100}, \tag{136}$$

where we choose z^* large enough and then γ small enough. Next, using (116), for all $z \ge z^*$,

$$\left| 2\nu \left(\int_0^{\frac{1}{\nu}} (\phi + \varepsilon)^2(z)\, dz \right) (-1 + (z+1)\phi) \right| \le 8 s^{-1} (1 + (z+1)e^{-z}) \le \frac{C}{s}. \tag{137}$$

Combining (136) and (137) and taking s_0^* large enough shows (134).

Step 3: End of the proof. We introduce

$$f^{\pm} = \pm (f - \varepsilon). \tag{138}$$

Then using (132), (133), and (134) one obtains that for $s_0 \le s \le s_1$ and $z \in [z^*, \nu^{-1}]$,

$$\begin{gathered} (\partial_s + \mathcal{L}) f^+ = (\partial_s + \mathcal{L}) f + F \ge \frac{s^{-\frac{2}{3}}}{4} - \frac{s^{-\frac{2}{3}}}{8} \ge 0 \\ \text{and similarly } (\partial_s + \mathcal{L}) f^- \le 0. \end{gathered} \tag{139}$$

Similar to (69), thanks to (120), one has

$$\begin{aligned} -\frac{\nu_s}{\nu} z^* + \int_0^{z^*} (\phi + \varepsilon)(\tilde z)\, d\tilde z &\ge \Big(\nu - \sup_{0 \le z \le \frac{1}{\nu(s)}} |\varepsilon| \Big) z^* \\ &\ge \Big(\frac{1}{s} + O(s^{-\frac{4}{3}}) - \sqrt{C^*} \tilde K e^{-\frac{\delta}{2} s} \Big) z^* \ge 0, \end{aligned}$$

provided that s_0 is large enough. From this we know that the particles are always moving from region $0 \le z \le z^*$ to $z^* \le z \le \frac{1}{\nu}$. At the boundary $z = z^*$ one has using (135) that

$$\begin{gathered} f^+(s, z^*) = \frac{s^{-\frac{2}{3}}}{2} - \varepsilon(s, z^*) \ge \Big(\frac{1}{2} - C z^{*\frac{3}{2}} \gamma s^{-\frac{2}{3}} \Big) s^{-\frac{2}{3}} \ge 0 \\ \text{and similarly } f^-(s, z^*) \le 0, \end{gathered} \tag{140}$$

provided γ is small enough depending on z^*. At initial time $s = s_0$, we have using (112) that for all $z \in [z^*, \nu_0^{-1}]$,

$$f^+(s_0, z) \geq \frac{s_0^{-\frac{2}{3}}}{2} - \|\varepsilon_0\|_{L^\infty[z^*,\nu_0^{-1}]} \geq \frac{s_0^{-\frac{2}{3}}}{2} - \frac{s_0^{-\frac{2}{3}}}{4} \geq 0, \quad \text{and similarly } f^-(s_0, z) \leq 0. \tag{141}$$

From (96), we know that the particle on the boundary point $z = \frac{1}{\nu}$ does not move. This together with (141) implies that $f^+(s, \frac{1}{\nu}) \geq 0$ and $f^-(s, \frac{1}{\nu}) \leq 0$. Therefore, in view of (139), (140), and (141) one can apply the maximum principle and obtain that $f^+(s,z) \geq 0$ and $f^-(s,z) \leq 0$ for all $s_0 \leq s \leq s_1$ and $z^* \leq z \leq \nu^{-1}$. By the definition (138) of $f^\pm$ this implies the desired estimate (131) and completes the proof of the lemma. ∎

We can now end the proof of Proposition 5.3.

Proof of Proposition 5.3. Proposition 5.3 is implied by Lemmas 5.5, 5.6, and 5.7. The reasoning is similar, and actually simpler since fewer parameters are involved, to the proof of Proposition 4.3, which was done for the case $\beta > 0$. Thus, we omit it. ∎

We can now end the proof of Theorem 1.1.

Proof of Theorem 1.1. We take $\beta = 0$ and $\phi(z) = \phi_0(z) = e^{-z}$. Let the constants $z^*, s_0^* \geq 1$ and $\gamma > 0$ be given by Proposition 5.3. Then, with κ fixed shortly after, let

$$\lambda_0 \leq \lambda_0^*/2 \quad \text{and} \quad \frac{2}{3\log(\lambda_0^{-1})} \leq \nu_0 \leq \frac{3}{2\log(\lambda_0^{-1})}$$

and an initial datum a_0 of the form (11) satisfy (12). We claim that for $\kappa > 0$ small enough, there exist parameters $\bar\lambda_0 = \lambda_0(1 + O(\lambda_0\kappa))$ and $\bar\nu_0 = \nu_0(1 + O(\lambda_0\kappa))$ such that a_0 is trapped in the sense of Definition 5.1 with framework parameters z^*, s_0^*, γ defined just above and decomposition parameters $\bar\lambda_0$, $\bar\nu_0$. The proof of this claim is so similar (and simpler since the profile is smooth) to the proof of the analogous claim in the proof of Theorem 1.3 for $\beta > 0$, that we omit the details and refer the reader to that proof.

Thus, applying Proposition 5.3, one obtains that the solution a is trapped for all self-similar times $s \in [s_0, \infty)$. We invert the self-similar transformation (34) using (120) and define $T = \int_{\bar s_0}^\infty \lambda(s)\, ds < \infty$ so that

$$t(s) = \int_{\bar s_0}^s \lambda(\tilde s)\, d\tilde s = T - \int_s^\infty \tilde\lambda_\infty \tilde s e^{-\tilde s}(1 + O(\tilde s^{-\frac{1}{3}}))\, d\tilde s = T - \tilde\lambda_\infty s e^{-s} + O(s^{\frac{2}{3}} e^{-s}),$$

and hence $\tilde\lambda_\infty s e^{-s} = (T-t) + O((T-t)|\log(T-t)|^{-1/3})$. We then get using (115) and (120) that $\|\varepsilon\|_{L^\infty(0,\nu^{-1})} \leq C|\log(T-t)|^{-2/3}$, $\lambda = (T-t) + O((T-t)|\log(T-t)|^{-1/3})$, and $\nu = |\log(T-t)|^{-1} + O(|\log(T-t)|^{-4/3})$. Injecting these estimates into the original variables (35) shows the desired estimates (13) and (14) with $\delta = \frac{1}{3}$. ∎

Acknowledgments. S.I. would like to thank the programs "Mathematical problems in fluid dynamics" at MSRI, spring 2021, and "Hamiltonian methods in dispersive and wave evolution equations" at ICERM, fall 2021, for providing him with an inspirational environment in which to work on this project.

Funding. The work of C. Collot was funded by the CY Initiative of Excellence (Grant "Investissements d'Avenir" ANR-16-IDEX-0008). The work of S.I. was supported by NSERC grant (371637-2019).

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Received 25 January 2022; revised 6 December 2022; accepted 2 February 2023.

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