Publ. RIMS, Kyoto Univ.
10 (1975), 721–728

Derivations Determined by Multipliers on Ideals of a C^*-Algebra[1)]

By

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Abstract

Sakai's theorem that every derivation of a simple C^*-algebra is determined by a multiplier is generalized, in the class of separable approximately finite-dimensional C^*-algebras, as follows. It is shown that, in such a C^*-algebra, any derivation can be approximated arbitrarily closely in norm by a derivation which is determined by a multiplier on a nonzero closed two-sided ideal. It is shown, moreover, that the multiplier may be chosen to have norm bounded by fixed multiple of the norm of the derivation.

1.

Examples constructed in [1] and in [6] show that, if a C^*-algebra does not have a minimal closed two-sided ideal, it may have a derivation the restriction of which to no nonzero closed two-sided ideal is determined by a multiplier. Nevertheless, it seems reasonable to expect that the set of such derivations has empty interior. The purpose of this paper is to verify this in a class of C^*-algebras which lends itself particularly to technical analysis.

Theorem. *Let A be the C^*-algebra inductive limit of a sequence of finite-dimensional C^*-algebras, and let D be a derivation of A. Then for each $\varepsilon>0$ there exist a nonzero closed two-sided ideal I_ε of A, a multiplier x_ε of I_ε such that $\|x_\varepsilon\|\leqq 248\|D\|$, and a derivation D_ε of A such that $\|D-D_\varepsilon\|\leqq\varepsilon$ and $D_\varepsilon|I_\varepsilon=\operatorname{ad}x_\varepsilon|I_\varepsilon$.*

Communicated by H. Araki, July 18, 1974. Revised July 31, 1974.

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1) This work was done while the author was a visiting member of the Research Institute for Mathematical Sciences, Kyoto University.

The theorem is a consequence of lemmas 2.2 and 3.6 below.

2.

First we shall show that, roughly, a derivation when sufficiently reduced vanishes asymptotically.

2.1. Lemma. *Let $B_1,\ldots, B_n$ be simple finite-dimensional C^*-algebras, and set $B_1\otimes\cdots\otimes B_n=B$. Let D be a derivation of B such that $DB_1\subset B_1,\ldots, DB_n\subset B_n$. Then*

$$\|D|B_1\|+\cdots+\|D|B_n\|\leqq 2\|D\|.$$

Proof. Write $D=D_1+iD_2$ where D_1 and D_2 are skew-adjoint-preserving derivations of B. Then the proof of 6 of [2] shows that

$$\|D_j|B_1\|+\cdots+\|D_j|B_n\|\leqq\|D_j\|,\qquad j=1, 2.$$

The conclusion follows from the inequalities

$$\|D\|\leqq\|D_1\|+\|D_2\|\leqq 2\|D\|.$$

2.2. Lemma. *Let A be the C^*-algebra inductive limit of a sequence of finite-dimensional C^*-algebras, let D be a derivation of A, and let $\varepsilon>0$. Then there exists a nonzero simple finite-dimensional sub-C^*-algebra B of A such that, if e denotes the unit of B, P_e the map $A\ni a\mapsto eae\in eAe$, and B' the commutant of B in A,*

$$\|P_eD|eB'\|\leqq\varepsilon.$$

Proof. By hypothesis there exists an increasing sequence $A_1\subset A_2\subset\cdots$ of finite-dimensional sub-C^*-algebras of A with union dense in A. It is enough to prove the lemma for D belonging to a dense set of derivations. Therefore, by 2.3 of [3] we may suppose that $D\cup A_k\subset\cup A_k$.

Suppose that the conclusion of the lemma is false. We shall deduce an inequality in contradiction with 2.1.

Choose $n=3, 4,\ldots$ such that $n^{-1}2\|D\|\leqq\varepsilon$, and set $n^{-1}2\|D\|=\delta$. Choose $k_1=1, 2,\ldots$ such that $\|D|A_{k_1}\|>\delta$. Choose $k_2>k_1$ such that $DA_{k_1}\subset A_{k_2}$, and choose a minimal central projection e_2 in A_{k_2} such that

$$\|P_{e_2}D|A_{k_1}\| > \delta .$$

Choose $k_3 > k_2$ such that $DA_{k_2} \subset A_{k_3}$ and choose $k_4 > k_3$ such that $DA_{k_3} \subset A_{k_4}$.

Note that $\|P_{e_2}D\,|\,e_2A'_{k_4}\| > \delta$; otherwise, with f a minimal central projection in $e_2A_{k_4}e_2$ is simple, we would have $f(fA_{k_4}f)' \subset e_2A'_{k_4}$ and so $\|P_fD|f(fA_{k_4}f)'\| \leqq \delta \leqq \varepsilon$, which was assumed not to hold.

The algebra $\cup e_2A_ke_2 \cap e_2A'_{k_4}$ is dense in $e_2A'_{k_4}$; therefore there exists $k_5 > k_4$ such that $\|P_{e_2}D|e_2A_{k_5}e_2 \cap e_2A'_{k_4}\| > \delta$. Choose $k_6 > k_5$ such that $DA_{k_5} \subset A_{k_6}$. Since $DA_{k_3} \subset A_{k_4}$, by 7 of [2] $D(A'_{k_4}) \subset A'_{k_3}$. Since $e_2 \in A_{k_4}$ and $e_2D(e_2)e_2 = 0$ (see proof of 3.4 below), $P_{e_2}D(e_2A'_{k_4}) = e_2D(e_2)A'_{k_4}e_2 + e_2D(A'_{k_4})e_2 = e_2D(A'_{k_4})e_2 \subset e_2A'_{k_3}$. This shows that

$$P_{e_2}D(e_2A_{k_5}e_2 \cap e_2A'_{k_4}) \subset e_2A_{k_6}e_2 \cap e_2A'_{k_3} .$$

Choose a minimal central projection e_6 in $e_2A_{k_6}e_2 \cap e_2A'_{k_3}$ such that

$$\|P_{e_6}D|e_2A_{k_5}e_2 \cap e_2A'_{k_4}\| > \delta .$$

Choose $k_7 > k_6$ and $k_8 > k_7$ such that $DA_{k_6} \subset A_{k_7}$ and $DA_{k_7} \subset A_{k_8}$. As above, $\|P_{e_6}D|e_6A'_{k_8}\| > \delta$, and there exists $k_9 > k_8$ such that $\|P_{e_6}D|e_6A_{k_9}e_6 \cap e_6A'_{k_8}\| > \delta$. Choose $k_{10} > k_9$ such that $DA_{k_9} \subset A_{k_{10}}$. Then, as above,

$$P_{e_6}D(e_6A_{k_9}e_6 \cap e_6A'_{k_8}) \subset e_6A_{k_{10}}e_6 \cap e_6A'_{k_7} ,$$

and so we may choose a minimal central projection e_{10} in $e_6A_{k_{10}}e_6 \cap e_6A'_{k_7}$ such that

$$\|P_{e_{10}}D|e_6A_{k_9}e_6 \cap e_6A'_{k_8}\| > \delta .$$

Continue this process until n projections $e_2 \geqq \cdots \geqq e_{4n-2}$ have been chosen, and denote e_{4n-2} by e. The algebras

$$e_2A_{k_2},\ e_6(e_2A_{k_6}e_2 \cap e_2A'_{k_3}),\ e_{10}(e_6A_{k_{10}}e_6 \cap e_6A'_{k_7}), \dots$$

are pairwise commuting simple finite-dimensional sub-C^*-algebras of A, and so also are the algebras

$$eA_{k_2},\ e(e_2A_{k_6}e_2 \cap e_2A'_{k_3}),\ e(e_6A_{k_{10}}e_6 \cap e_6A'_{k_7}), \dots .$$

The latter algebras have a common unit e; denote them by $B_1,\dots, B_n$ and denote the algebra they generate by B.

Since $B\subset eAe$, P_eD is a derivation from B into eAe. We shall now show that P_eDB_i commutes with B_j for distinct $i, j=1,\dots, n$. By 7 of [2] it is enough to consider the case $i<j$, which is clear from the relations

$$B_2\subset e'\cap(e_2A_{k_3}e_2)'\subset(eA_{k_3}e)',\ B_3\subset e'\cap(e_6A_{k_1}e_6)'\subset(eA_{k_1})',\dots,$$

$$P_eDB_1=P_eD(eA_{k_2})=eD(A_{k_2})e\subset eA_{k_3}e,$$

$$P_eDB_2=P_eD(e(e_2A_{k_6}e_2\cap e_2A'_{k_3}))=eD(e_2A_{k_6}e_2\cap e_2A'_{k_3})e\subset eA_{k_7}e,\dots.$$

From $\|P_{e_2}D|A_{k_1}\|>\delta$, simplicity of $e_2A_{k_2}$, and $e\in e_2A'_{k_2}$ follows $\|P_eD|A_{k_1}\|>\delta$, whence $\|P_eD|eA_{k_1}\|>\delta$. From $\|P_{e_6}D|e_2A_{k_5}e_2\cap e_2A'_{k_4}\|>\delta$, simplicity of $e_6(e_2A_{k_6}e_2\cap e_2A'_{k_3})$ and $e\in e_6(e_2A_{k_6}e_2\cap e_2A'_{k_3})'$ follows $\|P_eD|e_2A_{k_5}e_2\cap e_2A'_{k_4}\|>\delta$, whence $\|P_eD|e(e_2A_{k_5}e_2\cap e_2A'_{k_4})\|>\delta$. It is possible to continue in this way.

Let P be a projection of norm one from eAe onto B; then (see e.g. 2 of [2]) $\bar{D}=PP_eD$ is a derivation of B such that $\bar{D}B_1\subset B_1,\dots, \bar{D}B_n\subset B_n$. Since $P_eD(eA_{k_1})\subset B_1$, $P_eD(e(e_2A_{k_5}e_2\cap e_2A'_{k_4}))\subset B_2,\dots,$ the preceding paragraph shows that $\|\bar{D}|B_1\|>\delta,\dots, \|\bar{D}|B_n\|>\delta$. Hence by 2.1,

$$2\|D\|=n\delta<\|\bar{D}|B_1\|+\cdots+\|\bar{D}|B_n\|\leqq 2\|\bar{D}\|\leqq 2\|D\|.$$

This contradiction completes the proof of the lemma.

3.

We shall now use relations between derivations of an algebra and of a reduced subalgebra, established in [3], to complement the preceding result.

3.1. Lemma. *Let A be the C^*-algebra inductive limit of a sequence of finite-dimensional C^*-algebras, let e be a projection in A, and let D be a derivation of eAe. Then there exists a derivation D_0 of A such that $\|D_0\|\leqq 3\|D\|$ and $D_0|eAe=D$.*

Proof. This is the statement of 4.5 of [3], except for the estimate of the norm of the extension, which can be obtained by examining the proof of 4.5 of [3].

3.2. Lemma. *Let A be the C^*-algebra inductive limit of a sequence of finite-dimensional C^*-algebras, let D be a derivation of A, and let e be a projection in A. Suppose that $D|eAe=0$. Denote by I the closed two-sided ideal of A generated by e. Then there exists a multiplier z of I such that $\|z\|\leqq 16\|D\|$ and $D|I=\mathrm{ad}\, z|I$.*

Proof. This is the statement of 4.4 of [3], except for the estimate of the norm of the multiplier, which can be obtained by examining the construction described in 4.4 of [3].

3.3. Lemma. *Let A be the C^*-algebra inductive limit of a sequence of finite-dimensional C^*-algebras, let D be a derivation of A, and let e be a projection in A. Suppose that $D(eAe)\subset eAe$ and that $\|D|eAe\|\leqq \varepsilon$. Denote by I the closed two-sided ideal of A generated by e. Then there exist a derivation D_1 of A and a multiplier z of I such that $\|D-D_1\|\leqq 3\varepsilon$, $\|z\|\leqq 16\|D\|+48\varepsilon$, and $D_1|I=\mathrm{ad}\, z|I$.*

Proof. By 3.1 there exists a derivation D_0 of A such that $\|D_0\|\leqq 3\|D|eAe\|\leqq 3\varepsilon$ and $D_0|eAe=D|eAe$. Set $D-D_0=D_1$. Then $\|D-D_1\|=\|D_0\|\leqq 3\varepsilon$, and $D_1|eAe=0$. Hence by 3.2 there exists a multiplier z of I such that

$$\|z\|\leqq 16\|D_1\|\leqq 16\|D\|+16\|D_0\|\leqq 16\|D\|+48\varepsilon$$

and $D_1|I=\mathrm{ad}\, z|I$.

3.4. Lemma. *Let A be the C^*-algebra inductive limit of a sequence of finite-dimensional C^*algebras, let D be a derivation of A, and let e be a projection in A. Suppose that $\|P_eD|eAe\|\leqq\varepsilon$, where P_e denotes the map $A\ni a\mapsto eae\in eAe$. Denote by I the closed two-sided ideal of A generated by e. Then there exist a derivation D_1 of A and a multiplier w of I such that $\|D-D_1\|\leqq 3\varepsilon$, $\|w\|\leqq 82\|D\|+48\varepsilon$, and $D_1|I=\mathrm{ad}\, w|I$.*

Proof. Since $e^2=e$, we have $D(e)e+eD(e)=D(e)$, $2eD(e)e=eD(e)e$,

$eD(e)e=0$; hence $[[D(e), e], e]=D(e)$. Set $D-\mathrm{ad}[D(e), e]=D_2$. Then $D_2(eAe)\subset eAe$, so $D_2|eAe=P_eD_2|eAe$. Again since $eD(e)e=0$, $P_e\mathrm{ad}[D(e), e]|eAe=0$. Hence $D_2|eAe=P_eD_2|eAe=P_eD|eAe$; $\|D_2|eAe\|\leqq\varepsilon$.

By 3.3 there exist a derivation D_3 of A and a multiplier z of I such that $\|D_2-D_3\|\leqq 3\varepsilon$, $\|z\|\leqq 16\|D_2\|+48\varepsilon\leqq 80\|D\|+48\varepsilon$, and $D_3|I=\mathrm{ad}\, z|I$.

Set $D_3+\mathrm{ad}[D(e), e]=D_1$ and $z+[D(e), e]=w$. Then $\|D-D_1\|=\|D_2-D_3\|\leqq 3\varepsilon$, w is a multiplier of I with $\|w\|\leqq\|z\|+\|[D(e), e]\|\leqq 82\|D\|+48\varepsilon$, and $D_1|I=\mathrm{ad}\, w|I$.

3.5. Lemma. *Let A be a C^*-algebra with unit, let D be a derivation of A, and let B be a simple finite-dimensional sub-C^*-algebra of A with the same unit as A. Suppose that $\|D|B'\|\leqq\varepsilon$, where B' denotes the commutant of B in A. Then there exists $y\in A$ such that $\|y\|\leqq\|D\|$ and $\|D-\mathrm{ad}\, y\|\leqq 3\varepsilon$.*

Proof. Let U be a finite subgroup of the unitary group of B generating B as a linear space. Following [5], set $n^{-1}\sum_{u\in U}D(u)u^*=y$, where n is the number of elements of U. Then $D|B=\mathrm{ad}\, y|B$. (If $v\in U$ then $vyv^*+D(v)v^*=n^{-1}\sum_{u\in U}vD(u)u^*v^*+D(v)v^*=n^{-1}\sum_{u\in U}(D(vu)u^*v^*-D(v)uu^*v^*)+D(v)v^*=y$; $D(v)=[y, v]$.) Moreover, $\|y\|\leqq\|D\|$.

If $b\in B'$ then for each $u\in U$,

$$[D(u), b]=D(ub)-uD(b)-D(bu)+D(b)u=[D(b), u];$$

since

$$[y, b]=n^{-1}\sum_{u\in U}[D(u)u^*, b]=n^{-1}\sum_{u\in U}[D(u), b]u^*$$

we have

$$\|[y, b]\|=\|n^{-1}\sum_{u\in U}[D(b), u]u^*\|\leqq 2\|D(b)\|\leqq 2\varepsilon\|b\|.$$

This shows that $\|(D-\mathrm{ad}\, y)|B'\|\leqq\varepsilon+2\varepsilon=3\varepsilon$. Since $(D-\mathrm{ad}\, y)|B=0$, by the proof of 4.1 of [3] we have $\|D-\mathrm{ad}\, y\|\leqq 3\varepsilon$.

3.6. Lemma. *Let A be the C^*-algebra inductive limit of a sequence of finite-dimensional C^*-algebras, let D be a derivation of A, and let*

B be a simple finite-dimensional sub-C^*-algebra of A, with unit e. Suppose that

$$\|P_e D|eB'\| \leqq \varepsilon,$$

where P_e denotes the map $A \ni a \mapsto eae \in eAe$ and B' denotes the commutant of B in A. Denote by I the closed two-sided ideal of A generated by e. Then there exist a derivation D_1 of A and a multiplier x of I such that $\|D-D_1\| \leqq 9\varepsilon$, $\|x\| \leqq 247\|D\|+144\varepsilon$, and $D_1|I = \mathrm{ad}\,x|I$.

Proof. By 3.5 there exists $y \in eAe$ such that $\|y\| \leqq \|P_e D\| \leqq \|D\|$ and $\|(P_e D - \mathrm{ad}\,y)|eAe\| \leqq 3\varepsilon$. Then $\|P_e(D-\mathrm{ad}\,y)|eAe\| \leqq 3\varepsilon$, whence by 3.4 there exist a derivation D_2 of A and a multiplier w of I such that $\|D-\mathrm{ad}y-D_2\| \leqq 9\varepsilon$, $\|w\| \leqq 82\|D-\mathrm{ad}\,y\|+144\varepsilon \leqq 264\|D\|+144\varepsilon$, and $D_2|I = \mathrm{ad}\,w|I$. Set $D_2+\mathrm{ad}\,y = D_1$, $y+w=x$. Then $\|D-D_1\| \leqq 9\varepsilon$, $\|x\| \leqq \|y\|+\|w\| \leqq 247\|D\|+144\varepsilon$, and $D_1|I = (D_2+\mathrm{ad}\,y)|I = (\mathrm{ad}\,w+\mathrm{ad}\,y)|I = \mathrm{ad}\,x|I$.

4. Questions and Remarks

4.1. Examination of the proof of 1 shows that I_ε may be chosen so that $I_\varepsilon \not\subset J$, where J is a given proper closed two-sided ideal of A. It is not at all clear though whether I_ε can be chosen to be essential, i.e., with zero annihilator.

A related question is whether D_ε and I_ε may be chosen so that also the image of D_ε in A/I_ε is determined by a multiplier. A weaker requirement is that D_ε may be chosen so that for some composition series (I_α) of A the derivation of each $I_{\alpha+1}/I_\alpha$ induced by D_ε is determined by a multiplier.

4.2. A modification of the techniques of this paper, incorporating the methods of [4], shows that if an automorphism of a separable approximately finite-dimensional C^*-algebra leaves closed two-sided ideals invariant and in each irreducible representation is extendible to the weak closure, then it is approximable arbitrarily closely in norm by an automorphism determined by a multiplier on a nonzero closed two-sided ideal, which may be chosen not to lie in a given proper closed two-sided ideal.

It follows that such an automorphism is extendible in any representation to an inner automorphism of the weak closure. (One constructs a composition series in each quotient of which the automorphism is close in norm to an automorphism determined by a multiplier, and applies the theorems of Kadison and Ringrose that an automorphism close in norm to the identity is the exponential of a derivation, and of Sakai and Kadison that a derivation is extendible in any representation to an inner derivation of the weak closure.) This generalizes the implication (iii)⇒(ii) of Theorem 3.2 of [4], to the class of C^*-algebras considered, i.e., inductive limits of sequences of finite-dimensional C^*-algebras.

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