

On Solutions of Initial-Boundary Problem

$$\text{for } u_t = u_{xx} + \frac{1}{1-u}$$

By

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§1. Introduction and Theorem

Various works [1], [2], [3] have been published on the blowing-up of solutions of the Cauchy problem and the initial-boundary value problem of nonlinear partial differential equations. Blowing-up means that the solutions of these problems become infinite in a finite time.

The objective of the present paper is to introduce the concept of quenching which has more general sense than blowing-up and to find some sufficient conditions for quenching of the solutions of the following initial-boundary value problem for $u = u(t, x)$, $t > 0$, $x \in (0, l)$,

$$(1.1a) \quad u_t = u_{xx} + \frac{1}{1-u}, \quad t > 0, \quad x \in (0, l),$$

$$(1.1b) \quad u(t, 0) = u(t, l) = 0, \quad t > 0,$$

$$(1.1c) \quad u(0, x) = 0, \quad x \in (0, l),$$

where l is a positive constant. The above initial-boundary value problem (1.1a~c) is denoted by IVP. Our study may be said to be more illustrative than general, since we restrict ourselves to mixed problems of semilinear heat equations with space dimension one. Nevertheless, we hope that our results will give an insight into a more general situation. The nonlinear perturbation $\frac{1}{1-u}$ ($u \neq 1$) in (1.1a) is locally Lipschitz continuous. Thus IVP has a unique solution which may be local in t .

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The present problems came to our attention in connection with the diffusion equation generated by a polarization phenomena in ionic conductors [4].

We shall define quenching for the solutions of the initial value problems.

Definition 1. Let $u=u(t, x)$ be the solution of the initial value problems which are defined in $t>0, x \in \Omega$. Ω means R^m which stands for the m -dimensional Euclidean space or the bounded domain in R^m .

We shall say that u quenches if $\|u_t\|_c$ becomes infinite in a finite time where $\|\cdot\|_c$ denotes the maximum norm over Ω .

In order to clarify the nature of quenching, let us take some examples.

Example 1. α being constant, the solution of the initial value problem for $u=u(t), t>0$,

$$\begin{cases} \frac{du}{dt} = \frac{1}{1-u}, & t>0 \\ u(0) = \alpha, \end{cases}$$

is $u = 1 + \sqrt{(1-\alpha)^2 - 2t}$, if $\alpha > 1$ and $u = 1 - \sqrt{(1-\alpha)^2 - 2t}$, if $\alpha < 1$. In both cases, we see quenching at $t = \frac{(1-\alpha)^2}{2}$.

Example 2. Let α be as above. The solution of the initial-boundary value problem for $u=u(t, x), t>0, x \in (0, l)$,

$$\begin{cases} u_t = u_{xx} + \frac{1}{1-u}, & t>0, \quad x \in (0, l) \\ u_x(t, 0) = u_x(t, l) = 0, & t>0 \\ u(0, x) = \alpha, & x \in (0, l) \end{cases}$$

is the same as above.

Example 3. Blowing-up in the initial value problems means quenching.

As our main result, we have

Theorem. *In the IVP, suppose $l > 2\sqrt{2}$. Then the solution of the IVP quenches.*

The present paper has two sections apart from this section. In §2, we shall give a Lemma. §3 is devoted to the proof of our Theorem.

§2. Lemma

As a preparation for the proof of Theorem we state the following lemma. Henceforth, let $u = u(t, x)$ be the solution of IVP.

Lemma. *In the IVP, suppose $l > 2\sqrt{2}$. Then u reaches 1 in a finite time at $x = \frac{l}{2}$.*

Proof:

1st Step. We show that $u(t, x)$ is increasing in t for every x in $(0, l)$ as long as u exists. In fact, putting $v = u_t$, we have

$$(2.1) \quad v_t = v_{xx} + \frac{1}{(1-u)^2} \cdot v, \quad x \in (0, l),$$

$$v(t, 0) = v(t, l) = 0,$$

and

$$v(0, x) = 1, \quad x \in (0, l) \quad \text{as long as } u \text{ exists.}$$

We notice that v is a solution of the linear parabolic equation (2.1) and is non-negative on the “parabolic boundary”. Thus v is non-negative everywhere, which implies the required monotonicity of u .

2nd Step. The solution $u_1 = u_1(t, x)$ of the initial-boundary value problem for $u = u(t, x)$,

$$\begin{cases} u_t = u_{xx} + 1, & t > 0, \quad x \in (0, l), \\ u(t, 0) = u(t, l) = 0, & t > 0, \\ u(0, x) = 0, & x \in (0, l) \end{cases}$$

converges its stationary solution $\psi(x) = \frac{1}{2}l(l-x)$ ($0 < x < l$) as $t \rightarrow +\infty$. Thus u_1 crosses 1 in a finite time if $l > 2\sqrt{2}$.

Suppose that u does not reach 1 in a finite time if $l > 2\sqrt{2}$. Then IVP has a global solution, i.e., u satisfies $0 \leq u \leq 1$ in $(0, l) \times [0, +\infty)$ by virtue of the monotonicity of u . Comparing u with u_1 , we get $u \geq u_1$ in $(0, l) \times [0, +\infty)$ since $\frac{1}{1-\lambda} \geq 1$ in $0 \leq \lambda \leq 1$. This contradicts the assumption. We shall denote the time when u reaches 1 by $t = T_0$.

3rd Step. u satisfies (i) $u_x(t, 0) > 0$ by virtue of positivity of u ; (ii) $u_x(t, \frac{l}{2}) = 0$ since u is symmetric with respect to $x = \frac{l}{2}$. Putting $\pi = u_x$, we have

$$\pi_t = \pi_{xx} + \frac{1}{(1-u)^2} \cdot \pi, \quad t \in [0, T_0), \quad x \in \left(0, \frac{l}{2}\right),$$

$$\pi(t, 0) > 0, \quad \pi\left(t, \frac{l}{2}\right) = 0, \quad t \in [0, T_0),$$

and

$$\pi(0, x) = 0, \quad x \in \left(0, \frac{l}{2}\right).$$

Repeating the same argument as in 1st Step, we see that

$$(2.2) \quad \pi = u_x(t, x) > 0, \quad t \in [0, T_0), \quad x \in \left(0, \frac{l}{2}\right).$$

Combining (2.2) and (ii), we get that u takes its maximum at $x = \frac{l}{2}$ for any $t \in [0, T_0)$. This completes the proof.

§3. Proof of Theorem

1st Step

1. a) Put $\mu = \mu(t) = u\left(t, \frac{l}{2}\right)$ in $[0, T_0)$. μ satisfies

$$(3.1) \quad \frac{d\mu}{dt} \leq \frac{1}{1-\mu} \quad \text{in } [T_0 - \varepsilon, T_0)$$

for sufficiently small $\varepsilon (> 0)$ since $u_{xx}\left(t, \frac{l}{2}\right) \leq 0$ in $[0, T_0]$. Put $T_1 = T_0 - \varepsilon$ and $\Omega_\varepsilon = (0, l) \times [T_1, T_0]$. Comparing $\mu(t)$ with $v = v(t) = 1 - \sqrt{2} \sqrt{T_0 - t}$ in $[T_1, T_0]$, we get

$$(3.2) \quad \mu \geq v, \quad \text{in } [T_1, T_0]$$

since v satisfies (see Example 1)

$$\frac{dv}{dt} = \frac{1}{1-v}, \quad t \in [T_1, T_0]$$

and

$$\lim_{t \rightarrow T_0} v(t) = 1.$$

(3.2) implies that there exists the domain D_ε in which u satisfies

$$u(t, x) \geq v(t).$$

Denote the complement of D_ε by E_ε and put $E_\varepsilon^{(1)} = E_\varepsilon \cap \left\{ \left(0, \frac{l}{2}\right) \times [T_1, T_0] \right\}$ and $E_\varepsilon^{(2)} = E_\varepsilon \cap \left\{ \left(\frac{l}{2}, l\right) \times [T_1, T_0] \right\}$. For D_ε , there may be two cases:

Case (a) D_ε has no interior points, i.e., there holds

$$u_{xx}\left(t, \frac{l}{2}\right) = 0 \quad \text{in } [T_1, T_0].$$

Case (b) D_ε has interior points.

For the case (a), u quenches obviously. Henceforth we consider only the case (b).

1. b) Denote the boundary between D_ε and $E_\varepsilon^{(i)}$ by $x = s^{(i)}(t)$ ($t \in [T_1, T_0]$) for $i = 1, 2$. Then $x = s^{(i)}(t)$ satisfies

$$(i) \quad \lim_{t \rightarrow T_0} s^{(i)}(t) = \frac{l}{2};$$

$$(ii) \quad u_x(t, s^{(i)}(t)) \cdot \dot{s}^{(i)}(t) = -u_{xx}(t, s^{(i)}(t)), \quad t \in [T_1, T_0] \quad \text{where } \dot{s}^{(i)}(t) \text{ means}$$

$\frac{ds^{(i)}(t)}{dt}$ for $i = 1, 2$. In fact, there holds

$$(3.3) \quad u = v \quad \text{on} \quad x = s^{(i)}(t), \quad t \in [T_1, T_0].$$

Differentiating both sides of (3.3) and using (3.3), we get

$$(3.4) \quad u_t(t, s^{(i)}(t)) + u_x(t, s^{(i)}(t)) \cdot \dot{s}^{(i)}(t) = \frac{1}{1 - u(t, s^{(i)}(t))}.$$

By virtue of (1.1 a) on $x = s^{(i)}(t)$ and (3.3) we have (ii).

1.c) Obviously we have the following inequalities

$$(3.5a) \quad \frac{1}{1-u} \geq \frac{1}{\sqrt{2\sqrt{T_0-t}}} \quad \text{in } D_\varepsilon,$$

and

$$(3.5b) \quad \frac{1}{1-u} < \frac{1}{\sqrt{2\sqrt{T_0-t}}} \quad \text{in } E_\varepsilon.$$

2nd Step.

2.a) Let $p = p(t, x)$ be $\frac{1}{2(T_0-t)}$ in D_ε and $\frac{1}{(1-u)^2}$ in E_ε . Then the solution $v_1 = v_1(t, x)$ of the initial-boundary value problem for $v = v(t, x)$ in Ω_ε ,

$$\begin{cases} v_t = v_{xx} + p \cdot v & \text{in } \Omega_\varepsilon \\ v(t, 0) = v(t, l) = 0, & t \in [T_1, T_0], \\ v(T_1, x) = \beta(x) = u_t(T_1, x) > 0, & x \in (0, l), \end{cases}$$

exists and satisfies $0 < v_1 \leq v$ in Ω_ε by virtue of (3.5 a) and the maximum principle (cf. 1st Step in the proof of Lemma).

2.b) Put $W = W(t, x) = \sqrt{T_0-t} \cdot v_1$. Denoting W in D_ε by $W^{(1)}$, we have $W_t^{(1)} = W_{xx}^{(1)}$ in D_ε . Furthermore it should be noted that $W(t, x) > 0$ in Ω_ε .

3rd Step.

3.a) We shall deal with the following initial-boundary value problem for $V = V(t, x)$ in $(-\infty, +\infty) \times [T_1, T_0]$.

$$(3.6a) \quad V_t = V_{xx} \quad \text{in } (-\infty, +\infty) \times [T_1, T_0]$$

(3.6 b) $V = W^{(1)}$ in D_ε

(3.6 c) $V = \sqrt{\varepsilon} \cdot \beta(x)$, $x \in [0, s^{(1)}(T_1)] \cup (s^{(2)}(T_1), l]$

(3.6 d) $V = 0$, $x \in (-\infty, 0) \cup (l, +\infty)$.

In what follows we impose on the solution $V(t, x)$ the following conditions at infinity: $V(t, x)$ and $V_x(t, x)$ are bounded as $x \rightarrow \pm \infty$ uniformly with respect to t in $[T_1, T_0]$. We see the solution $\hat{W} = \hat{W}(t, x)$ of (3.6) uniquely exists. Uniqueness of \hat{W} is shown by Holmgren's theorem. Using the Green's function

$$K(t, x; \tau, \xi) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp \left\{ -\frac{(x-\xi)^2}{4(t-\tau)} \right\},$$

\hat{W} is represented by

$$\begin{aligned} \hat{W}(t, x) = & \int_{T_1}^t [K(t, x; \tau, s^{(1)}(\tau))W_\xi^{(1)}(\tau, s^{(1)}(\tau)) \\ & - W^{(1)}(\tau, s^{(1)}(\tau))K_\xi(t, x; \tau, s^{(1)}(\tau))]d\tau \\ (3.7) \quad & + \int_0^{s^{(1)}(T_1)} K(t, x; T_1, \xi)\sqrt{\varepsilon} \cdot \beta(\xi)d\xi \\ & + \int_{T_1}^t K(t, 0; \tau, s^{(1)}(\tau))W^{(1)}(\tau, s^{(1)}(\tau)) \cdot \dot{s}^{(1)}(\tau)d\tau, \\ & -\infty < x < s^{(1)}(t), t \in [T_1, T_0]. \end{aligned}$$

Also in $s^{(2)}(t) < x < +\infty, t \in [T_1, T_0]$, we have the similar expression as (3.7).

(3. b) Using the positivity of β, W and maximum principle, we have

$$\hat{W}(t, x) \geq 0 \quad \text{in } (-\infty, +\infty) \times [T_1, T_0].$$

Thus from (3.6) and (3.5 b) we see

$$\hat{W}(t, x) \geq W(t, x) > 0 \quad \text{in } \Omega_\varepsilon.$$

4th Step. We claim that

$$\lim_{t \rightarrow T_0} \hat{W}\left(t, \frac{l}{2}\right) > 0.$$

On the contrary, we suppose that

$$\lim_{t \rightarrow T_0} \hat{W}\left(t, \frac{l}{2}\right) = 0,$$

which implies that $0 \equiv \hat{W}(t, x) \geq W(t, x) \geq 0$ in Ω_ε by the strong maximum principle [5]. This is a contradiction. Thus we get that

$$\lim_{t \rightarrow T_0} \frac{d\mu(t)}{dt} = \lim_{t \rightarrow T_0} v\left(t, \frac{l}{2}\right) \geq \lim_{t \rightarrow T_0} v_1\left(t, \frac{l}{2}\right) = \lim_{t \rightarrow T_0} \frac{\hat{W}\left(t, \frac{l}{2}\right)}{\sqrt{t - T_0}} = +\infty$$

This completes the proof.

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