Ann. Inst. H. Poincaré
Anal. Non Linéaire (Online first)
DOI 10.4171/AIHPC/94

Published by EMS Press

Blowup of two-dimensional attractive Bose–Einstein condensates at the critical rotational speed

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Abstract. We study the ground states of a two-dimensional focusing non-linear Schrödinger equation with rotation and harmonic trapping. When the strength of the interaction approaches a critical value from below, the system collapses to a profile obtained from the optimizer of a Gagliardo–Nirenberg interpolation inequality. This was established before in the case of fixed rotation frequency. We extend the result to rotation frequencies approaching, or even equal to, the critical frequency at which the centrifugal force compensates the trap. We prove that the blow-up scenario is to leading order unaffected by such a strong deconfinement mechanism. In particular, the blow-up profile remains independent of the rotation frequency.

1. Introduction

Bose–Einstein condensates (BECs) [9,24] form a remarkable phase of matter where quantum effects can be spectacularly observed on a mesoscopic scale. Indeed, a single quantum wave function being macroscopically occupied, its quantum coherence becomes accessible e.g. to imaging techniques. The flexibility of modern experiments with dilute atomic gases are also remarkable [1, 4, 8, 10, 34, 35], allowing access to reduced dimensionalities (two dimensions or even one), to tune the interactions (allowing for repulsion or attraction between particles), and to mimic external magnetic fields either by rotation or by coupling internal degrees of freedom to optical fields.

In this note we consider such a combination of effects. Namely, we are interested in two-dimensional attractive BECs, where the contact interactions will destabilize the gas towards collapse if they are too strong. The resulting collapse of ground states [20] turns out to be unaffected by the addition of a moderate rotation of the gas [25] (see also [15] for dipolar gases). A fast rotation may however destabilize the gas towards expansion, because the centrifugal force fights the confining potential. These two effects might compete, but we prove that the instability towards collapse always dominates, leading to a blow-up scenario independent of the rotation frequency. This answers a question raised in [25, Remark 2.2].

2020 Mathematics Subject Classification. Primary 35J60; Secondary 35Q40.
Keywords. Rotational Bose–Einstein condensate, ground state, Gagliardo–Nirenberg inequality, blow-up limit.

We will consider the minimization problem

$$E^{\mathrm{NLS}}_{\Omega,a} := \inf\big\{\mathcal{E}^{\mathrm{NLS}}_{\Omega,a}(\phi) \colon \phi \in X(\mathbb{R}^2) \colon \|\phi\|_{L^2} = 1\big\}, \tag{1.1}$$

where $\mathcal{E}^{\mathrm{NLS}}_{\Omega,a}$ is the non-linear Schrödinger (NLS) energy functional with attractive interactions

$$\begin{aligned}\mathcal{E}^{\mathrm{NLS}}_{\Omega,a}(\phi) &= \int_{\mathbb{R}^2} |\nabla\phi(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} |x|^2|\phi(x)|^2 \,\mathrm{d}x + 2\Omega\langle\phi, L\phi\rangle - \frac{a}{2}\int_{\mathbb{R}^2} |\phi(x)|^4 \,\mathrm{d}x \\ &= \int_{\mathbb{R}^2} |(-i\nabla + \Omega x^\perp)\phi(x)|^2 \,\mathrm{d}x + (1-\Omega^2)\int_{\mathbb{R}^2} |x|^2|\phi(x)|^2 \,\mathrm{d}x \\ &\quad - \frac{a}{2}\int_{\mathbb{R}^2} |\phi(x)|^4 \,\mathrm{d}x.\end{aligned}$$

Here, $a > 0$ describes the strength of interactions, $\Omega \geq 0$ is the rotation frequency, $x^\perp = (-x_2, x_1)$, and

$$L = -ix \wedge \nabla = i(x_2\partial_1 - x_1\partial_2)$$

the angular momentum operator. The space $X(\mathbb{R}^2)$ in (1.1) is a functional space in which the energy functional $\mathcal{E}^{\mathrm{NLS}}_{\Omega,a}$ is well defined; see below.

In the case of high rotational speed $\Omega > 1$, it was proved in [3] that there are no ground states for $E^{\mathrm{NLS}}_{\Omega,a}$ for all $a > 0$. Indeed, when the rotational speed is larger than the trapping frequency, the centrifugal force caused by the rotation is stronger than the centripetal force created by the harmonic trap and the gas flies apart. On the other hand, the condensate remains stable when $\Omega < 1$. In this case, one can prove the norm equivalence

$$\|\nabla\phi\|^2_{L^2} + \|x\phi\|^2_{L^2} + 2\Omega\langle L\phi, \phi\rangle \simeq \|\nabla\phi\|^2_{L^2} + \|x\phi\|^2_{L^2}. \tag{1.2}$$

It is then clear that the energy functional is well defined on the weighted Sobolev space

$$\Sigma(\mathbb{R}^2) := H^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, |x|^2 \,\mathrm{d}x),$$

and hence one can take $X(\mathbb{R}^2) \equiv \Sigma(\mathbb{R}^2)$. Using the compact embedding $\Sigma(\mathbb{R}^2) \subset L^r(\mathbb{R}^2)$ for all $r \in [2, \infty)$, one can easily show the existence of a ground state for $E^{\mathrm{NLS}}_{\Omega,a}$ with $0 < a < a_*$ (see e.g. [20] in the case $\Omega = 0$). Here $a_* = \|Q\|^2_{L^2}$ with Q the unique (up to translations) positive solution of the elliptic equation

$$-\Delta Q + Q - Q^3 = 0 \quad \text{in } \mathbb{R}^2. \tag{1.3}$$

The constant a_* also appears in the sharp Gagliardo–Nirenberg inequality

$$\frac{a_*}{2}\int_{\mathbb{R}^2} |\phi(x)|^4 \,\mathrm{d}x \leq \left(\int_{\mathbb{R}^2} |\nabla\phi(x)|^2 \,\mathrm{d}x\right)\left(\int_{\mathbb{R}^2} |\phi(x)|^2 \,\mathrm{d}x\right) \quad \forall \phi \in H^1(\mathbb{R}^2). \tag{1.4}$$

The case of critical rotational speed $\Omega = 1$ is special. The situation becomes more subtle since the centrifugal force caused by the rotation is exactly compensated by the harmonic trap. In particular, the norm equivalence (1.2) is no longer available. Thus, working on $\Sigma(\mathbb{R}^2)$ does not help to find ground states for $E^{\mathrm{NLS}}_{1,a}$. In this case, we study the minimization (1.1) on a larger functional space of magnetic Sobolev functions, namely

$$H^1_{x^\perp}(\mathbb{R}^2) := \{\phi \in L^2(\mathbb{R}^2): (-i\nabla + x^\perp)\phi \in L^2(\mathbb{R}^2)\},$$

hence we set $X(\mathbb{R}^2) = H^1_{x^\perp}(\mathbb{R}^2)$ when $\Omega = 1$. Note that by the Cauchy–Schwarz inequality, we have $\Sigma(\mathbb{R}^2) \subset H^1_{x^\perp}(\mathbb{R}^2)$, but $\Sigma(\mathbb{R}^2) \subsetneq H^1_{x^\perp}(\mathbb{R}^2)$ (for the latter see e.g. [13, Remark 2.1]). By making use of a concentration–compactness argument adapted to magnetic Sobolev spaces (see e.g. [14]), it was proved in [13, 18] that $E^{\mathrm{NLS}}_{1,a}$ has at least one ground state provided that $0 < a < a_*$. By the standard Gagliardo–Nirenberg inequality (1.4) and the diamagnetic inequality (see e.g. [27, Theorem 7.21])

$$\big|\nabla|\phi|(x)\big| \le |(-i\nabla + x^\perp)\phi(x)|, \quad \text{a.e } x \in \mathbb{R}^2, \quad \forall \phi \in H^1_{x^\perp}(\mathbb{R}^2), \tag{1.5}$$

we also have the following magnetic Gagliardo–Nirenberg inequality:

$$\begin{aligned}\frac{a_*}{2}\int_{\mathbb{R}^2} |\phi(x)|^4\,\mathrm{d}x \le{}& \left(\int_{\mathbb{R}^2} |(-i\nabla + x^\perp)\phi(x)|^2\,\mathrm{d}x\right)\\ &\times\left(\int_{\mathbb{R}^2} |\phi(x)|^2\,\mathrm{d}x\right) \quad \forall \phi \in H^1_{x^\perp}(\mathbb{R}^2).\end{aligned} \tag{1.6}$$

The main difference between (1.4) and (1.6) is that there is no optimizer for (1.6), while Q in (1.3) is the unique (up to translations and dilations) optimizer for (1.4). Thanks to (1.6), the energy $E^{\mathrm{NLS}}_{\Omega,a}$ is non-negative for all $0 < a \le a_*$.

1.1. Collapse in NLS theory

In the sequel we are interested in the blow-up behavior of ground states for $E^{\mathrm{NLS}}_{\Omega,a}$ when a approaches a_*. Our first result concerns the blow-up limit with the critical rotation speed $\Omega = 1$.

Theorem 1.1 (Collapse of NLS ground states at the critical rotational speed). *We have, as $a \nearrow a_*$,*

$$E^{\mathrm{NLS}}_{1,a} = (a_* - a)^{1/2}\Big(2\frac{\|xQ_0\|_{L^2}}{a_*^{1/2}} + o(1)\Big), \tag{1.7}$$

where $Q_0 = \|Q\|^{-1}_{L^2} Q$. In addition, for any sequence $\{a_n\}_n$ satisfying $a_n \nearrow a_*$ and any sequence of ground states ϕ_n for E^{NLS}_{1,a_n}, there exist a sequence $\{\theta_n\}_n \subset [0, 2\pi)$ and a sequence $\{x_n\}_n \subset \mathbb{R}^2$ such that the following convergence holds strongly in $H^1 \cap L^\infty(\mathbb{R}^2)$:

$$\begin{aligned}\lim_{n\to\infty} &\frac{(a_* - a_n)^{1/4}}{a_*^{1/4}\|xQ_0\|^{1/2}_{L^2}}\phi_n\Big(\frac{(a_* - a_n)^{1/4}}{a_*^{1/4}\|xQ_0\|^{1/2}_{L^2}}x + x_n\Big)\exp\Big(i\frac{(a_* - a_n)^{1/4}}{a_*^{1/4}\|xQ_0\|^{1/2}_{L^2}}x_n^\perp \cdot x + i\theta_n\Big)\\ &= Q_0(x).\end{aligned} \tag{1.8}$$

As an application of this result, we have the following blow-up behavior of ground states when $\Omega \nearrow 1$ and $a \nearrow a^*$ at the same time.

Corollary 1.2 (Collapse at subcritical rotational speed). *For any sequence $\{\Omega_n\}_n, \{a_n\}_n$ satisfying $\Omega_n \nearrow 1$ and $a_n \nearrow a_*$, and any ground state ϕ_n for $E^{\mathrm{NLS}}_{\Omega_n,a_n}$, there exists a sequence $\{\theta_n\}_n \subset [0,2\pi)$ such that the following convergence holds strongly in $H^1 \cap L^\infty(\mathbb{R}^2)$:*

$$\lim_{n\to\infty} \frac{(a_*-a_n)^{1/4}}{a_*^{1/4}\|xQ_0\|_{L^2}^{1/2}}\phi_n\Big(\frac{(a_*-a_n)^{1/4}}{a_*^{1/4}\|xQ_0\|_{L^2}^{1/2}}x\Big)e^{i\theta_n} = Q_0(x). \tag{1.9}$$

Remark 1.3. We have the following comments:

(1) The convergences of energy and of ground states were proved by Guo and Seiringer [20] when $\Omega = 0$. These convergences were extended to the case $0 < \Omega < 1$ fixed by Lewin, Nam, and the third author [25] (see also further works in [12, 17, 22]). In [19] it is even proved that a fixed rotation rate has no effect at any order. Theorem 1.1 shows that the energy convergence found remains valid in the case of critical rotational speed $\Omega = 1$, at least to leading order. This is noteworthy because the trapping potential, which sets the length-scale of the blow-up behavior, is compensated by the centrifugal force.

(2) The convergence of ground states however has to be stated differently from [20, 25]. The model is translation-invariant for $\Omega = 1$ and thus ground states converge only modulo a magnetic translation (namely, a translation decorated by the suitable phase making it commute with the magnetic Laplacian; see e.g. [33] and references therein).

(3) The only effect of the magnetic/rotation field is to set the blow-up length scale (see the sketch of proof below). This is comparable to the positive particle mass $m > 0$ in the Hartree-type and Thomas–Fermi-type models of stars [21, 29–32].

(4) Our blow-up result, when $\Omega \nearrow 1$ at the same time as $a \nearrow a_*$, answers a question raised in [25, Remark 2.2]. In this situation, although the centrifugal force almost compensates the trapping potential, the small residual trapping favors blowup at the center of the trap. Hence there is no need for a magnetic translation and the ground state convergence is completely similar to the case $0 \le \Omega < 1$ fixed.

Let us briefly describe the strategy of the proof. To prove Theorem 1.1, we first show that the sequence of ground states $\{\phi_n\}_n$ for E^{NLS}_{1,a_n} blows up in the sense that

$$\varepsilon_n := \|\nabla|\phi_n|\|_{L^2}^{-1} \to 0 \quad \text{as } n \to \infty. \tag{1.10}$$

The blow-up length is then set by ε_n (whose precise asymptotic behavior is not known at this point) and we will show that

$$\varphi_n(x) := \varepsilon_n\phi_n(\varepsilon_n x + x_n)e^{i\varepsilon_n x_n^\perp\cdot x + i\theta_n} \to Q_0(x)$$

strongly in $H^1(\mathbb{R}^2)$, i.e. there is convergence modulo a magnetic translation of vector $\{x_n\}_n \subset \mathbb{R}^2$ and the choice of a constant phase $\{\theta_n\}_n \subset [0, 2\pi)$. To prove this, we rely on a property of the Lagrange multiplier associated to ϕ_n, together with the local boundedness of subsolutions obtained by analyzing the corresponding Euler–Lagrange equation. Thanks to the non-degeneracy of Q, we then prove that the imaginary part of φ_n is small in H^1-norm. This implies that the rotation acts on φ_n only as a quadratic external potential. This effectively sets a length scale, and we next prove by matching energy lower and upper bounds that the blow-up length behaves like

$$\frac{(a_* - a_n)^{1/4}}{a_*^{1/4}\|xQ_0\|_{L^2}^{1/2}}.$$

Hence we obtain the energy convergence (1.7). Finally, the L^∞-convergence of ground states follows from H^1-convergence and H^2-boundedness deduced from the variational equation.

To prove Corollary 1.2, we first use an energy argument to show that $E^{\mathrm{NLS}}_{\Omega_n,a_n}$ has the same asymptotic behavior as for $\Omega = 0, 1$. By taking a sequence of ground states for E^{NLS}_{1,a_n} and choosing a suitable trial state for $E^{\mathrm{NLS}}_{\Omega_n,a_n}$, we prove that a ground state for $E^{\mathrm{NLS}}_{\Omega_n,a_n}$ is an approximate ground state for E^{NLS}_{0,a_n}. At this point, the conclusion follows directly from a result proved in [25, Section 3].

1.2. Collapse in the mean-field limit

The focusing NLS functional (1.1) is commonly used to predict the collapse of an attractive system, but it should be seen as an effective, mean-field model [36]. It is of interest to see whether the mean-field and blow-up limits can be exchanged as in [25]. Based on Theorem 1.1 and Corollary 1.2, we give a positive answer to this question, starting from many-body quantum mechanics.

In this framework, a Bose gas with an attractive interaction is described by the N-particle Hamiltonian

$$H_{\Omega,a,N} = \sum_{j=1}^{N}(-\Delta_{x_j} + |x_j|^2 - 2\Omega L_{x_j}) - \frac{a}{N-1}\sum_{1\le i<j\le N} w_N(x_i - x_j),$$

acting on $\mathfrak{H}^N := L^2_{\mathrm{sym}}((\mathbb{R}^2)^N)$. As is customary [36], the two-body interaction w_N is chosen in the form

$$w_N(x) = N^{2\beta} w(N^\beta x)$$

for a fixed parameter $\beta > 0$ and a fixed function w satisfying

$$w(x) = w(-x) \ge 0, \quad (1 + |x|)w,\ \hat{w} \in L^1(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} w(x)\,\mathrm{d}x = 1.$$

We are interested in the large-N behavior of the ground state energy per particle of $H_{\Omega,a,N}$, namely

$$E^{\mathrm{QM}}_{\Omega,a}(N) := N^{-1} \inf_{\Phi_N\in\mathfrak{H}^N, \|\Phi_N\|=1} \langle \Phi_N, H_{\Omega,a,N}\Phi_N\rangle, \tag{1.11}$$

and the associated eigenstates of $H_{\Omega,a,N}$. When $\Omega = 1$, the Hamiltonian $H_{1,a,N}$ is magnetic translation invariant so it probably has no discrete spectrum (see e.g. [2, Proposition 5.4] or a discussion before (1.21) in [28] for a similar model of stars). In the following, we therefore assume that $0 \le \Omega < 1$ and $0 < a < a_*$. We will consider the limit where $a = a_N \nearrow a_*$ at the same time as $\Omega = \Omega_N \nearrow 1$ when $N \to \infty$. In that case, the NLS ground states blow up at the origin to the function Q_0, as shown in Corollary 1.2. We will prove that the many-body ground states condense fully on Q_0. As usual, the convergence of ground states is formulated using k-particles reduced density matrices, defined for any $\Phi_N \in \mathfrak{H}^N$ by a partial trace

$$\gamma_{\Phi_N}^{(k)} := \operatorname*{Tr}_{k+1\to N} |\Phi_N\rangle\langle\Phi_N|.$$

Equivalently, $\gamma_{\Phi_N}^{(k)}$ is the trace class operator on $\mathfrak{H}^k$ with kernel

$$\gamma_{\Phi_N}^{(k)}(x_1,\dots,x_k;y_1,\dots,y_k) = \int_{\mathbb{R}^{2(N-k)}} \Phi_N(x_1,\dots,x_k,Z)\overline{\Phi_N(y_1,\dots,y_k,Z)}\,\mathrm{d}Z.$$

Bose–Einstein condensation is properly expressed by the convergence in trace norm

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_{\Phi_N}^{(k)} - |\phi^{\otimes k}\rangle\langle\phi^{\otimes k}|\right| = 0 \quad \forall k \in \mathbb{N}.$$

We have the following result.

Theorem 1.4 (Collapse and condensation of the many-body ground states). *Let $0 < \beta < 1/2$ be fixed and $a = a_N = a_* - N^{-\alpha}$ with*

$$0 < \alpha < \min\left\{\frac{4}{5}\beta, 2(1-2\beta)\right\}.$$

Then for every $0 \le \Omega < 1$ we have, as $N \to \infty$,

$$E_{\Omega,a_N}^{\mathrm{QM}}(N) = E_{\Omega,a_N}^{\mathrm{NLS}} + o(E_{\Omega,a_N}^{\mathrm{NLS}}) = (a_* - a_N)^{1/2}\left(2\frac{\|xQ_0\|_{L^2}}{a_*^{1/2}} + o(1)\right). \tag{1.12}$$

Assume in addition that $\Omega = \Omega_N = 1 - N^{-\nu}$ with

$$0 < \nu < \min\left\{1 - 2\beta - \frac{\alpha}{2}, \beta - \frac{5\alpha}{4}\right\}.$$

Let Φ_N be a ground state for $E_{\Omega_N,a_N}^{\mathrm{QM}}(N)$. Then we have

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_{\Phi_N}^{(k)} - |Q_N^{\otimes k}\rangle\langle Q_N^{\otimes k}|\right| = 0$$

for all $k \in \mathbb{N}$, where

$$Q_N(x) = \frac{a_*^{1/4}\|xQ_0\|_{L^2}^{1/2}}{(a_* - a_N)^{1/4}} Q_0\left(\frac{a_*^{1/4}\|xQ_0\|_{L^2}^{1/2}}{(a_* - a_N)^{1/4}}x\right).$$

Remark 1.5. This shows that a result found in [25] remains valid when $\Omega \nearrow 1$ slower than $a \nearrow a_*$ ([25] only deals with $0 \leq \Omega < 1$ fixed). The method is the same as in [25]. The energy estimates do not depend on the rotation parameter. In fact, we also obtain (1.12) for $\Omega = 1$. Furthermore, the convergence of the many-body ground states follows from that of the approximate NLS ground states. In the case $\Omega_N \nearrow 1$, under the additional assumption on the convergence speed of Ω_N in Theorem 1.4, we check that the approximate NLS ground state for $E^{\mathrm{NLS}}_{\Omega_N,a_N}$ is still the one for E^{NLS}_{0,a_N}.

2. Collapse of the NLS ground states

In this section we study the limiting behavior of ground states for (1.1) when a approaches a_* from below. We first deal with the critical speed $\Omega = 1$. The case $\Omega \nearrow 1$ will be given at the end of this section.

2.1. Collapse at the critical speed

Let us consider the case $\Omega = 1$. For simplicity, we denote $\nabla_{x^\perp} := -i\nabla + x^\perp$. Let us start by recalling some useful facts.

Lemma 2.1 (L^2-bound). *We have*

$$2\|\phi\|^2_{L^2} \leq \|\nabla_{x^\perp}\phi\|^2_{L^2} \quad \forall \phi \in H^1_{x^\perp}(\mathbb{R}^2)$$

with equality achieved e.g. by $\phi(x) = \sqrt{\frac{1}{\pi}} e^{-\frac{|x|^2}{2}}$.

This is a consequence of Landau's well-known diagonalization of $(\nabla_{x^\perp})^2$ (see e.g. [37]).

Lemma 2.2 (Compactness modulo translations). *Let $\{\phi_n\}_n$ be a sequence of functions satisfying*

$$\inf_{n\geq 1} \|\phi_n\|_{L^4} \geq C,$$

for some positive constant $C > 0$. We have the following weak convergences:

- *If $\sup_{n\geq 1} \|\phi_n\|_{H^1} < \infty$, then there exist $\phi \in H^1(\mathbb{R}^2)\backslash\{0\}$ and a sequence $\{x_n\}_n \subset \mathbb{R}^2$ such that, up to a subsequence,*

$$\phi_n(x + x_n) \rightharpoonup \phi(x) \quad \text{weakly in } H^1(\mathbb{R}^2) \text{ and almost everywhere in } \mathbb{R}^2.$$

- *If $\sup_{n\geq 1} \|\phi_n\|_{H^1_{x^\perp}} < \infty$, then there exist $\tilde{\phi} \in H^1_{x^\perp}(\mathbb{R}^2)\backslash\{0\}$ and a sequence $\{y_n\}_n \subset \mathbb{R}^2$ such that, up to a subsequence,*

$$e^{iy_n^\perp \cdot x}\phi_n(x + y_n) \rightharpoonup \tilde{\phi}(x) \quad \text{weakly in } H^1_{x^\perp}(\mathbb{R}^2) \text{ and almost everywhere in } \mathbb{R}^2.$$

Here $\phi_n \rightarrow \phi$ weakly in $H^1_{x^\perp}(\mathbb{R}^2)$ means that

$$\int (\nabla_{x^\perp}\phi_n - \nabla_{x^\perp}\phi) \cdot \overline{\nabla_{x^\perp}\varphi}\, \mathrm{d}x + \int (\phi_n - \phi)\bar{\varphi}\, \mathrm{d}x \rightarrow 0 \quad \forall \varphi \in H^1_{x^\perp}(\mathbb{R}^2).$$

Proof. The proof of this lemma can be found in [26, Lemma 6] for the H^1-weak convergence and [13, Lemma 2.6] for the $H^1_{x^\perp}$-weak convergence. ∎

Lemma 2.3 (Energy upper bound). *Let $\{a_n\}_n$ be a positive sequence satisfying $a_n \nearrow a_*$ as $n \to \infty$. Then, for every $0 \le \Omega \le 1$, we have*

$$\lim_{n\to\infty} E^{\mathrm{NLS}}_{\Omega,a_n} = E^{\mathrm{NLS}}_{\Omega,a_*} = 0.$$

More precisely,

$$\limsup_{n\to\infty} \frac{E^{\mathrm{NLS}}_{\Omega,a_n}}{(a_*-a_n)^{1/2}} \le 2\frac{\|xQ_0\|_{L^2}}{a_*^{1/2}}. \tag{2.1}$$

Proof. It is obvious that $E^{\mathrm{NLS}}_{\Omega,a_n} \ge 0$, by the magnetic Gagliardo–Nirenberg inequality (1.6). On the other hand, let Q be the unique positive radial solution of (1.3). By Pohozaev's identity, we have

$$\|\nabla Q\|^2_{L^2} = \frac{1}{2}\|Q\|^4_{L^4} = \|Q\|^2_{L^2} = a_*.$$

Denote $Q_0 = \|Q\|^{-1}_{L^2} Q$. Then

$$\|\nabla Q_0\|^2_{L^2} = \frac{a_*}{2}\|Q_0\|^4_{L^4} = \|Q_0\|^2_{L^2} = 1$$

By the variational principle, we have

$$E^{\mathrm{NLS}}_{\Omega,a_n} \le \mathcal{E}^{\mathrm{NLS}}_{\Omega,a_n}(\lambda Q_0(\lambda\cdot)) = \lambda^2\Big(1-\frac{a_n}{a_*}\Big) + \lambda^{-2}\|xQ_0\|^2_{L^2}$$

for all $\lambda > 0$. Here we have used the fact that $\langle L(\lambda Q_0(\lambda\cdot)), \lambda Q_0(\lambda\cdot)\rangle = 0$ since Q_0 is real valued, where we recall that $L = i(x_2\partial_1 - x_1\partial_2)$. Optimizing over λ, we get

$$E^{\mathrm{NLS}}_{\Omega,a_n} \le 2\frac{\|xQ_0\|_{L^2}}{a_*^{1/2}}(a_*-a_n)^{1/2} \tag{2.2}$$

which implies (2.1) and also $\limsup_{n\to\infty} E^{\mathrm{NLS}}_{\Omega,a_n} \le 0$. ∎

Lemma 2.4 (Blowup). *Let $\{a_n\}_n$ be a positive sequence such that $a_n \nearrow a_*$ as $n \to \infty$ and ϕ_n be a ground state for E^{NLS}_{1,a_n}. Then $\{\phi_n\}_n$ blows up both in $H^1_{x^\perp}(\mathbb{R}^2)$ and in $H^1(\mathbb{R}^2)$ in the sense that*

$$\lim_{n\to\infty}\|\nabla_{x^\perp}\phi_n\|_{L^2} = \lim_{n\to\infty}\|\nabla\phi_n\|_{L^2} = \lim_{n\to\infty}\|\nabla|\phi_n|\|_{L^2} = +\infty.$$

Proof. We first show that $\{\phi_n\}_n$ blows up in $H^1_{x^\perp}(\mathbb{R}^2)$. Assume for contradiction that

$$\sup_{n\ge 1}\|\nabla_{x^\perp}\phi_n\|^2_{L^2} < \infty. \tag{2.3}$$

In particular, $\{\phi_n\}_n$ is then a bounded sequence in $H^1_{x^\perp}(\mathbb{R}^2)$. Observe that there exists $C > 0$ such that

$$\liminf_{n\to\infty} \|\phi_n\|_{L^4} \geq C$$

since otherwise, we have

$$\lim_{n\to\infty} E^{\mathrm{NLS}}_{1,a_n} = \lim_{n\to\infty} \|\nabla_{x^\perp}\phi_n\|^2_{L^2} \geq 2,$$

where the last inequality is due to Lemma 2.1. This, however, is not possible (see Lemma 2.3). Thus, by Lemma 2.2, there exist $\phi \in H^1_{x^\perp}(\mathbb{R}^2)\backslash\{0\}$ and a sequence $\{x_n\}_n \subset \mathbb{R}^2$ such that up to a subsequence,

$$\tilde{\phi}_n(x) := e^{ix_n^\perp\cdot x}\phi_n(x+x_n) \to \phi \quad \text{weakly in } H^1_{x^\perp}(\mathbb{R}^2) \text{ and almost everywhere in } \mathbb{R}^2.$$

We claim that $\|\phi\|^2_{L^2} = 1$. Indeed, we have

$$0 < \|\phi\|^2_{L^2} \leq \liminf_{n\to\infty} \|\tilde{\phi}_n\|^2_{L^2} = \liminf_{n\to\infty} \|\phi_n\|^2_{L^2} = 1.$$

If $\|\phi\|^2_{L^2} < 1$, then by the magnetic translation invariance, we have

$$E^{\mathrm{NLS}}_{1,a_n} = \mathcal{E}^{\mathrm{NLS}}_{1,a_n}(\phi_n) = \mathcal{E}^{\mathrm{NLS}}_{1,a_n}(\tilde{\phi}_n) \geq \mathcal{E}^{\mathrm{NLS}}_{1,a_*}(\tilde{\phi}_n) = \mathcal{E}^{\mathrm{NLS}}_{1,a_*}(\phi) + \mathcal{E}^{\mathrm{NLS}}_{1,a_*}(\tilde{\phi}_n - \phi) + o(1). \quad (2.4)$$

Here we have used the weak convergence in $H^1_{x^\perp}(\mathbb{R}^2)$, the almost everywhere convergence in $\mathbb{R}^2$, and the Brézis–Lieb lemma (see [5]) along with the fact that $\|\tilde{\phi}_n\|_{L^4}$ is bounded uniformly, by the magnetic Gagliardo–Nirenberg inequality (1.6) and (2.3). Again, (1.6) implies that

$$\liminf_{n\to\infty} \mathcal{E}^{\mathrm{NLS}}_{1,a_*}(\tilde{\phi}_n - \phi) \geq 0.$$

Furthermore,

$$\mathcal{E}^{\mathrm{NLS}}_{1,a_*}(\phi) = \|\phi\|^2_{L^2}\mathcal{E}^{\mathrm{NLS}}_{1,a_*}\Big(\frac{\phi}{\|\phi\|_{L^2}}\Big) + \frac{a_*}{2}\Big(\frac{1}{\|\phi\|^2_{L^2}} - 1\Big)\|\phi\|^4_{L^4} > 0$$

since $0 < \|\phi\|_{L^2} < 1$. This contradicts the fact that $E^{\mathrm{NLS}}_{1,a_n} \to 0$ as $n \to \infty$, by Lemma 2.3. Therefore, we must have $\|\phi\|_{L^2} = 1$, hence $\tilde{\phi}_n \to \phi$ strongly in $L^2(\mathbb{R}^2)$. In fact, $\tilde{\phi}_n \to \phi$ strongly in $L^r(\mathbb{R}^2)$ for $r \in [2,\infty)$, because of the $H^1_{x^\perp}(\mathbb{R}^2)$-boundedness. Since $a_n \nearrow a_*$, we have from (2.4) that

$$E^{\mathrm{NLS}}_{1,a_*} \leq \mathcal{E}^{\mathrm{NLS}}_{1,a_*}(\phi) \leq \liminf_{n\to\infty} \mathcal{E}^{\mathrm{NLS}}_{1,a_n}(\phi_n) = \liminf_{n\to\infty} E^{\mathrm{NLS}}_{1,a_n} = E^{\mathrm{NLS}}_{1,a_*}.$$

In particular, ϕ is a ground state for E^{NLS}_{1,a_*}. However, there are no such ground states, as proven in e.g. [13, 18], and we deduce that (2.3) cannot hold.

We now conclude the proof by showing that $\{\phi_n\}_n$ blows up in $H^1(\mathbb{R}^2)$. We have

$$0 = E^{\mathrm{NLS}}_{1,a_*} = \lim_{n\to\infty} E^{\mathrm{NLS}}_{1,a_n} = \lim_{n\to\infty} \mathcal{E}^{\mathrm{NLS}}_{1,a_n}(\phi_n) = \lim_{n\to\infty} \|\nabla_{x^\perp}\phi_n\|^2_{L^2} - \frac{a_n}{2}\|\phi_n\|^4_{L^4}.$$

Since $\|\nabla_{x^\perp}\phi_n\|_{L^2} \to \infty$ as $n \to \infty$, we must have $\|\phi_n\|^4_{L^4} \to \infty$. But then the standard Gagliardo–Nirenberg inequality (1.4) implies that $\|\nabla\phi_n\|_{L^2} \to \infty$ and $\|\nabla|\phi_n|\|_{L^2} \to \infty$ as well. ∎

We are now in the position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof is divided into several steps.

Step 1. *Convergence of the modulus.* We first show that there exists a sequence $\{x_n\}_n \subset \mathbb{R}^2$ such that

$$\varepsilon_n |\phi_n|(\varepsilon_n \cdot + x_n) \to Q_0 \quad \text{strongly in } H^1(\mathbb{R}^2) \text{ as } n \to \infty, \tag{2.5}$$

where ε_n is given by (1.10). Denote

$$v_n(x) := \varepsilon_n |\phi_n|(\varepsilon_n x).$$

We then have

$$\|v_n\|_{L^2} = \|\phi_n\|_{L^2} = 1 \quad \text{and} \quad \|\nabla v_n\|_{L^2} = \varepsilon_n \|\nabla|\phi_n|\|_{L^2} = 1.$$

Hence $\{v_n\}_n$ is a bounded sequence in $H^1(\mathbb{R}^2)$. On the other hand, using the diamagnetic inequality (1.5) we have

$$\mathcal{E}^{\mathrm{NLS}}_{1,a}(\phi) \geq \|\nabla|\phi|\|^2_{L^2} - \frac{a}{2}\|\phi\|^4_{L^4} =: \mathcal{E}^0_a(|\phi|).$$

But the Gagliardo–Nirenberg inequality (1.4) implies

$$\mathcal{E}^0_a(|\phi|) \geq \left(1 - \frac{a}{a_*}\right)\|\nabla|\phi|\|^2_{L^2}.$$

From this and Lemma 2.3, we obtain

$$0 = \lim_{n\to\infty} E^{\mathrm{NLS}}_{1,a_n} = \lim_{n\to\infty} \mathcal{E}^{\mathrm{NLS}}_{1,a_n}(\phi_n) \geq \liminf_{n\to\infty} \mathcal{E}^0_{a_n}(|\phi_n|) \geq 0.$$

In particular, we have $\mathcal{E}^0_{a_n}(v_n) = \varepsilon_n^2 \mathcal{E}^0_{a_n}(|\phi_n|) \to 0$ as $n \to \infty$. Since by definition

$$\|\nabla v_n\|_{L^2} = 1$$

for all $n \geq 1$, we infer that, up to a subsequence,

$$\inf_{n\geq 1} \|v_n\|_{L^4} \geq C$$

for some constant $C > 0$. By Lemma 2.2, there exists $\phi \in H^1(\mathbb{R}^2)\backslash\{0\}$ and $\{y_n\}_n \subset \mathbb{R}^2$ such that up to a subsequence,

$$\tilde{v}_n(x) := v_n(\cdot + y_n) \to \phi \quad \text{weakly in } H^1(\mathbb{R}^2) \text{ and almost everywhere in } \mathbb{R}^2.$$

We next show that $\|\phi\|_{L^2} = 1$. In fact, we first have

$$0 < \|\phi\|^2_{L^2} \leq \liminf_{n\to\infty} \|\tilde{v}_n\|^2_{L^2} = \liminf_{n\to\infty} \|v_n\|^2_{L^2} = 1,$$

where the first inequality comes from the strong convergence in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ (see again [26]). Assume for contradiction that $\|\phi\|_{L^2} < 1$. As in (2.4), we have

$$0 = \lim_{n\to\infty} \mathcal{E}^0_{a_n}(v_n) = \lim_{n\to\infty} \mathcal{E}^0_{a_n}(\tilde{v}_n) \geq \mathcal{E}^0_{a_*}(\phi) + \liminf_{n\to\infty} \mathcal{E}^0_{a_*}(\tilde{v}_n - \phi). \tag{2.6}$$

Again, by the Gagliardo–Nirenberg inequality (1.4), we have

$$\liminf_{n\to\infty} \mathcal{E}^0_{a_*}(\tilde{v}_n - \phi) \geq 0$$

and

$$\mathcal{E}^0_{a_*}(\phi) = \|\phi\|^2_{L^2} \mathcal{E}^0_{a_*}\Big(\frac{\phi}{\|\phi\|_{L^2}}\Big) + \frac{a_*}{2}\Big(\frac{1}{\|\phi\|^2_{L^2}} - 1\Big)\|\phi\|^4_{L^4} > 0$$

since $0 < \|\phi\|_{L^2} < 1$. This is a contradiction with (2.6) and we thus must have $\|\phi\|_{L^2} = 1$. Then $\tilde{v}_n \to \phi$ strongly in $L^2(\mathbb{R}^2)$, up to a subsequence. In fact, $\tilde{v}_n \to \phi$ strongly in $L^r(\mathbb{R}^2)$ for $r \in [2, \infty)$, because of the $H^1(\mathbb{R}^2)$-boundedness. Therefore,

$$0 \leq \mathcal{E}^0_{a_*}(\phi) \leq \liminf_{n\to\infty} \mathcal{E}^0_{a_*}(\tilde{v}_n) \leq \liminf_{n\to\infty} \mathcal{E}^0_{a_n}(v_n) = 0.$$

This shows that

$$\lim_{n\to\infty} \|\nabla \tilde{v}_n\|^2_{L^2} = \lim_{n\to\infty} \frac{a_n}{2}\|\tilde{v}_n\|^4_{L^4} = \lim_{n\to\infty} \frac{a_*}{2}\|\phi\|^4_{L^4} = \|\nabla\phi\|^2_{L^2}.$$

Hence $\tilde{v}_n \to \phi$ strongly in $H^1(\mathbb{R}^2)$, up to a subsequence. Moreover, ϕ is an optimizer of the standard Gagliardo–Nirenberg inequality (1.4). By the uniqueness (up to translations and dilations) of optimizers for (1.4) and the fact that $\tilde{v}_n$ is non-negative, there exist $\lambda > 0$ and $x_0 \in \mathbb{R}^2$ such that $\phi(x) = \lambda Q_0(\lambda(x + x_0))$. Since $\|\nabla\phi\|_{L^2} = 1$, we must have $\lambda = 1$. Again, by uniqueness of Q_0, we conclude that passing to a subsequence is unnecessary. This leads to (2.5) after setting $x_n = \varepsilon_n(y_n - x_0)$.

Step 2. *A property of Lagrange multipliers.* The minimizer ϕ_n of E^{NLS}_{1,a_n} satisfies the Euler–Lagrange equation

$$(\nabla_{x^\perp})^2\phi_n - a_n|\phi_n|^2\phi_n = \mu_n\phi_n \quad \text{in } \mathbb{R}^2 \tag{2.7}$$

in the distributional sense, namely

$$\int_{\mathbb{R}^2} \nabla_{x^\perp}\phi_n \cdot \nabla_{x^\perp}\chi - a_n|\phi_n|^2\phi_n\chi - \mu_n\phi_n\chi \,\mathrm{d}x = 0 \quad \forall \chi \in C_0^\infty(\mathbb{R}^2),$$

where $\mu_n \in \mathbb{R}$ is the Lagrange multiplier. In this step, we show that $\varepsilon_n^2\mu_n \to -1$ as $n \to \infty$. Indeed, as ϕ_n is a ground state for E^{NLS}_{1,a_n}, using (2.7), we have

$$\mu_n = \|\nabla_{x^\perp}\phi_n\|^2_{L^2} - a_n\|\phi_n\|^4_{L^4} = \mathcal{E}^{\mathrm{NLS}}_{1,a_n}(\phi_n) - \frac{a_n}{2}\|\phi_n\|^4_{L^4} = E^{\mathrm{NLS}}_{1,a_n} - \frac{a_n}{2}\|\phi_n\|^4_{L^4}.$$

Denote

$$\varphi_n(x) = e^{i\theta_n}\psi_n(x) \tag{2.8}$$

with

$$\psi_n(x) := \varepsilon_n \phi_n(\varepsilon_n x + x_n) e^{i\varepsilon_n x_n^\perp \cdot x}$$

and $\theta_n \in [0, 2\pi)$ satisfying

$$\|\varphi_n - Q_0\|_{L^2} = \min_{\theta \in [0,2\pi)} \|e^{i\theta}\psi_n - Q_0\|_{L^2}. \tag{2.9}$$

By (2.5), we have $|\varphi_n| := \varepsilon_n |\phi_n|(\varepsilon_n \cdot + x_n) \to Q_0$ strongly in $H^1(\mathbb{R}^2)$. Therefore,

$$\lim_{n\to\infty} \varepsilon_n^2 \|\phi_n\|_{L^4}^4 = \lim_{n\to\infty} \|\varphi_n\|_{L^4}^4 = \|Q_0\|_{L^4}^4 = \frac{2}{a_*}.$$

Since $0 \le E_{1,a_n}^{\mathrm{NLS}} \to 0$ (see Lemma 2.3) and $a_n \nearrow a_*$, we get

$$\lim_{n\to\infty} \varepsilon_n^2 \mu_n = \lim_{n\to\infty} \varepsilon_n^2 E_{1,a_n}^{\mathrm{NLS}} - \lim_{n\to\infty} \frac{a_n}{2} \varepsilon_n^2 \|\phi_n\|_{L^4}^4 = -1.$$

Step 3. *A subequation for* $|\varphi_n|^2$. We next use (2.7) to derive an equation and a subequation satisfied by φ_n and $|\varphi_n|^2$. To do so, we write

$$\psi_n(x) = \varepsilon_n \tilde{\phi}_n(\varepsilon_n x)$$

with $\tilde{\phi}_n(x) := \phi_n(x + x_n) e^{i x_n^\perp \cdot x}$. A direct computation gives

$$(\nabla_{x^\perp})^2 \tilde{\phi}_n(x) = ((\nabla_{x^\perp})^2 \phi_n)(x + x_n) e^{i x_n^\perp \cdot x},$$

which, by (2.7), implies

$$(\nabla_{x^\perp})^2 \tilde{\phi}_n - a_n |\tilde{\phi}_n|^2 \tilde{\phi}_n = \mu_n \tilde{\phi}_n.$$

Using the identity

$$(\nabla_{x^\perp})^2 \phi = -\Delta\phi + 2L\phi + |x|^2 \phi$$

with $L = i(x_2\partial_1 - x_1\partial_2) = -i x^\perp \cdot \nabla$, we see that $\tilde{\phi}_n$ solves the elliptic equation

$$-\Delta\tilde{\phi}_n + |x|^2\tilde{\phi}_n + 2L\tilde{\phi}_n - a_n|\tilde{\phi}_n|^2\tilde{\phi}_n - \mu_n\tilde{\phi}_n = 0.$$

By the definition of φ_n in (2.8), we get

$$-\Delta\varphi_n + \varepsilon_n^4 |x|^2 \varphi_n + 2\varepsilon_n^2 L\varphi_n - a_n |\varphi_n|^2 \varphi_n - \varepsilon_n^2 \mu_n \varphi_n = 0. \tag{2.10}$$

Observe that (2.10) can be written as

$$(-i\nabla + \varepsilon_n^2 x^\perp)^2 \varphi_n - a_n |\varphi_n|^2 \varphi_n - \varepsilon_n^2 \mu_n \varphi_n = 0$$

which, by [7, Proposition 2.2], implies that $\varphi_n \in L^\infty(\mathbb{R}^2)$ and $\lim_{|x|\to\infty} |\varphi_n(x)| = 0$.

Denote $W_n := |\varphi_n|^2$. Since $|\varphi_n| \in H^1(\mathbb{R}^2)$ (using the diamagnetic inequality (1.5)) and $\varphi_n \in L^\infty(\mathbb{R}^2)$, we have $W_n \in H^1(\mathbb{R}^2)$. Multiplying both sides of (2.10) with $\bar{\varphi}_n$, taking the real part, and using the following identities (in the distributional sense),

$$-\operatorname{Re}(\Delta\varphi_n\bar{\varphi}_n) = -\frac{1}{2}\Delta W_n + |\nabla\varphi_n|^2,$$
$$2\operatorname{Re}(L\varphi_n\bar{\varphi}_n) = L\varphi_n\bar{\varphi}_n + \overline{L\varphi_n}\varphi_n = x^\perp \cdot J(\varphi_n),$$

with $J(\varphi) = i(\varphi\nabla\bar{\varphi} - \bar{\varphi}\nabla\varphi)$ the superfluid current, we obtain

$$-\frac{1}{2}\Delta W_n + |\nabla\varphi_n|^2 + \varepsilon_n^4|x|^2 W_n + \varepsilon_n^2 x^\perp \cdot J(\varphi_n) - a_n W_n^2 - \varepsilon_n^2\mu_n W_n = 0.$$

Using the identity

$$|(-i\nabla + \varepsilon_n^2 x^\perp)\varphi_n|^2 = |\nabla\varphi_n|^2 + \varepsilon_n^2 x^\perp \cdot J(\varphi_n) + \varepsilon_n^4|x|^2 W_n,$$

we deduce that

$$-\frac{1}{2}\Delta W_n - \varepsilon_n^2\mu_n W_n - a_n W_n^2 \le 0 \tag{2.11}$$

in the weak sense, namely

$$\int_{\mathbb{R}^2} \frac{1}{2}\nabla W_n \cdot \nabla\chi - \varepsilon_n^2\mu_n W_n\chi - a_n W_n^2\chi \,\mathrm{d}x \le 0 \quad \forall 0 \le \chi \in C_0^\infty(\mathbb{R}^2).$$

Step 4. *Uniform boundedness of W_n.* To prove the uniform boundedness of the subsolution $W_n = |\varphi_n|^2$ to (2.11), we need its local boundedness. The following formulation is taken from [23, Theorem 4.14] (see [23, Theorem 4.1] and [16, Theorem 8.17] for the proof).

Theorem 2.5 (Local boundedness). *Let D be a connected open set with smooth boundary in $\mathbb{R}^d$. Assume that $a_{jk} \in L^\infty(D)$ satisfies*

$$\lambda|\xi|^2 \le \sum_{j,k} a_{jk}(x)\xi_j\xi_k \le \Lambda|\xi|^2 \quad \forall x \in D,\ \forall \xi \in \mathbb{R}^d,$$

for some positive constants λ and Λ. Let $u \in H^1(D)$ be a non-negative subsolution in D in the following sense:

$$\int_D a_{jk}\partial_j u \partial_k \chi \,\mathrm{d}x \le \int_D f\chi \,\mathrm{d}x \quad \forall \chi \in H_0^1(D),\ \chi \ge 0 \text{ in } D.$$

Suppose that $f \in L^q(D)$ for some $q > \frac{d}{2}$. Then there holds for any $B_R(x_0) \subset D$ and any $p > 0$,

$$\sup_{B_{R/2}(x_0)} u(x) \le C\big(R^{-\frac{d}{p}}\|u\|_{L^p(B_R(x_0))} + R^{2-\frac{d}{q}}\|f\|_{L^q(B_R(x_0))}\big),$$

where $C = C(d, \lambda, \Lambda, p, q)$ is a positive constant.

Let $M > 0$ and denote $D_M = \{x \in \mathbb{R}^2 : |x| > M\}$. Applying Theorem 2.5 to (2.11) with $D = D_M$, $a_{jk} = \frac{1}{2}\delta_{jk}$, $f = \varepsilon_n^2 \mu_n W_n + a_n W_n^2$, $p = q = 2$, $R = 2$, and $B_2(x_0) \subset D_M$, we get

$$\sup_{B_1(x_0)} W_n(x) \leq C(\|W_n\|_{L^2(B_2(x_0))} + \|W_n^2\|_{L^2(B_2(x_0))}) \tag{2.12}$$

for some universal constant $C > 0$. Since $B_2(x_0) \subset D_M$, we deduce

$$\begin{aligned} \|W_n\|_{L^2(B_2(x_0))} + \|W_n^2\|_{L^2(B_2(x_0))} &\leq \|W_n\|_{L^2(|x|>M)} + \|W_n^2\|_{L^2(|x|>M)} \\ &\to \|Q_0^2\|_{L^2(|x|>M)} + \|Q_0^4\|_{L^2(|x|>M)}. \end{aligned}$$

Here we have used $\varepsilon_n^2 \mu_n \to -1$ and the fact that $W_n \to Q_0^2$ in $L^2(\mathbb{R}^2)$ and $W_n^2 \to Q_0^4$ in $L^2(\mathbb{R}^2)$ because

$$\begin{aligned} \|W_n - Q_0^2\|_{L^2} &\leq \big\||\varphi_n| - Q_0\big\|_{L^4} \big\||\varphi_n| + Q_0\big\|_{L^4}, \\ \|W_n^2 - Q_0^4\|_{L^2} &\leq \big\||\varphi_n| - Q_0\big\|_{L^8} \big\||\varphi_n| + Q_0\big\|_{L^8} \big\||\varphi_n|^2 + Q_0^2\big\|_{L^4}, \end{aligned}$$

and $|\varphi_n| \to Q_0$ strongly in $L^r(\mathbb{R}^2)$ for all $r \in [2, \infty)$. The latter follows from the strong convergence $|\varphi_n| \to Q_0$ in $H^1(\mathbb{R}^2)$ and Sobolev embedding. In particular, for $\epsilon > 0$, there exist $n_\epsilon \in \mathbb{N}$ and M_ϵ sufficiently large such that for all $n \geq n_\epsilon$ and all $M \geq M_\epsilon$,

$$\|W_n\|_{L^2(B_2(x_0))} + \|W_n^2\|_{L^2(B_2(x_0))} \leq \frac{\epsilon}{C},$$

which together with (2.12) yields

$$\sup_{B_1(x_0)} W_n(x) \leq \epsilon$$

for all $B_1(x_0) \subset D_{M_\epsilon}$. As $B_1(x_0)$ is arbitrarily in D_{M_ϵ}, we get (by possibly increasing M_ϵ)

$$W_n(x) \leq \epsilon \quad \text{for all } |x| > M_\epsilon \text{ and all } n \text{ sufficiently large.} \tag{2.13}$$

Applying Theorem 2.5 again to (2.11) with $D = \mathbb{R}^2$, $a_{jk} = \frac{1}{2}\delta_{jk}$, $f = \mu_n \varepsilon_n^2 W_n + a_n W_n^2$, $p = q = 2$, and $R = 2M_\epsilon$, we get

$$\sup_{B_{M_\epsilon}(0)} W_n(x) \leq C(M_\epsilon^{-1} \|W_n\|_{L^2(B_{2M_\epsilon}(0))} + M_\epsilon \|W_n^2\|_{L^2(B_{2M_\epsilon}(0))})$$

for some universal constant $C > 0$. This implies

$$\sup_{B_{M_\epsilon}(0)} W_n(x) \leq C(M_\epsilon) \quad \text{for all } n \text{ sufficiently large.} \tag{2.14}$$

Collecting (2.13) and (2.14), we prove

$$0 \leq \sup_{x \in \mathbb{R}^2} W_n(x) \leq C \quad \text{for all } n \text{ sufficiently large,} \tag{2.15}$$

where $C > 0$ is a constant independent of n.

Step 5. Uniform exponential decay of W_n. We now prove the uniform exponential decay of W_n. Since $C_0^\infty(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, we can test (2.11) against non-negative functions in $H^1(\mathbb{R}^2)$. The following calculation is done formally by testing (2.11) with $e^{\alpha|x|}W_n$ for some constant $\alpha > 0$ to be chosen shortly. Strictly speaking, this requires a standard truncation argument. First we replace $e^{\alpha|x|}$ by

$$\chi_\delta(x) := e^{\alpha\frac{|x|}{1+\delta|x|}}, \quad \delta > 0$$

and perform the usual computation. Then we let $\delta \to 0$ to obtain the desired estimate. For more details, see e.g. [6, Theorem 8.1.1]. Note that χ_δ is bounded, Lipschitz continuous, and $|\nabla\chi_\delta| \leq \alpha\chi_\delta$; hence $\chi_\delta W_n \in H^1(\mathbb{R}^2)$.

We have

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}^2}\Delta W_n e^{\alpha|x|}W_n\,\mathrm{d}x \\
&\qquad -\mu_n\varepsilon_n^2\int_{\mathbb{R}^2}W_n e^{\alpha|x|}W_n\,\mathrm{d}x - a_n\int_{\mathbb{R}^2}W_n^2 e^{\alpha|x|}W_n\,\mathrm{d}x \leq 0.
\end{aligned} \tag{2.16}$$

Observe that

$$\begin{aligned}
\int_{\mathbb{R}^2}\Delta W_n e^{\alpha|x|}W_n\,\mathrm{d}x &= \int_{\mathbb{R}^2}e^{\alpha|x|}\Big(\frac{1}{2}\Delta(W_n^2) - |\nabla W_n|^2\Big)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^2}W_n^2\Delta(e^{\alpha|x|})\,\mathrm{d}x - \int_{\mathbb{R}^2}|\nabla W_n|^2 e^{\alpha|x|}\,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^2}W_n^2\Big(\alpha^2 + \frac{\alpha}{|x|}\Big)e^{\alpha|x|}\,\mathrm{d}x - \int_{\mathbb{R}^2}|\nabla W_n|^2 e^{\alpha|x|}\,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^2}|\nabla(W_n e^{\alpha|x|/2})|^2\,\mathrm{d}x = \frac{\alpha^2}{4}\int_{\mathbb{R}^2}W_n^2 e^{\alpha|x|}\,\mathrm{d}x + \int_{\mathbb{R}^2}|\nabla W_n|^2 e^{\alpha|x|}\,\mathrm{d}x \\
-\frac{1}{2}\int_{\mathbb{R}^2}W_n^2\Big(\alpha^2 + \frac{\alpha}{|x|}\Big)e^{\alpha|x|}\,\mathrm{d}x.
\end{aligned}$$

In particular, we have

$$\int_{\mathbb{R}^2}\Delta W_n e^{\alpha|x|}W_n\,\mathrm{d}x = \frac{\alpha^2}{4}\int_{\mathbb{R}^2}W_n^2 e^{\alpha|x|}\,\mathrm{d}x - \int_{\mathbb{R}^2}|\nabla(W_n e^{\alpha|x|/2})|^2\,\mathrm{d}x,$$

hence (see (2.16))

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}|\nabla(W_n e^{\alpha|x|/2})|^2\,\mathrm{d}x - \frac{\alpha^2}{8}\int_{\mathbb{R}^2}W_n^2 e^{\alpha|x|}\,\mathrm{d}x - \mu_n\varepsilon_n^2\int_{\mathbb{R}^2}W_n^2 e^{\alpha|x|}\,\mathrm{d}x \\
&\qquad - a_n\int_{\mathbb{R}^2}W_n^3 e^{\alpha|x|}\,\mathrm{d}x \leq 0
\end{aligned}$$

so

$$\int_{\mathbb{R}^2}\Big(-\mu_n\varepsilon_n^2 - \frac{\alpha^2}{8} - a_n W_n\Big)W_n^2 e^{\alpha|x|}\,\mathrm{d}x \leq 0.$$

We pick $\alpha = 1$ and choose $M > 0$ so large that $W_n(x) \leq \frac{1}{4a_*}$ for all $|x| \geq M$ and all n sufficiently large (see (2.13)). As $\mu_n \varepsilon_n^2 \to -1$ (by Step 1), we get

$$-\mu_n \varepsilon_n^2 - \frac{1}{8} - a_n W_n(x) \geq \frac{1}{2}$$

for all $|x| \geq M$ and all n sufficiently large. Thus we obtain

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^2 \setminus B_M(0)} W_n^2 e^{|x|}\,\mathrm{d}x &\leq \int_{B_M(0)} \left|-\mu_n \varepsilon_n^2 - \frac{1}{8} - a_n W_n\right| W_n^2 e^{|x|}\,\mathrm{d}x \\ &\leq C e^M \|W_n\|_{L^2}^2 \leq C e^M\end{aligned}$$

for all n sufficiently large, where we have used (2.15) to get the second estimate. This proves that

$$\int_{\mathbb{R}^2} W_n^2 e^{|x|}\,\mathrm{d}x \leq C \tag{2.17}$$

for all n sufficiently large, where $C > 0$ is independent of n. From this, we get

$$\begin{aligned}\int_{\mathbb{R}^2} |\varphi_n|^2 e^{|x|/4}\,\mathrm{d}x &= \int_{\mathbb{R}^2} W_n e^{|x|/2} e^{-|x|/4}\,\mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^2} W_n^2 e^{|x|}\,\mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}^2} e^{-|x|/2}\,\mathrm{d}x\right)^{1/2} \leq C\end{aligned} \tag{2.18}$$

for all n sufficiently large. A consequence of this uniform exponential decay and $|\varphi_n| \to Q_0$ in $H^1(\mathbb{R}^2)$ is that $|x|\,|\varphi_n| \to |x| Q_0$ strongly in $L^2(\mathbb{R}^2)$.

Step 6. *H^1-strong convergence.* By the definition of φ_n (see (2.8)), we have

$$\phi_n(x) = \varepsilon_n^{-1} \varphi_n(\varepsilon_n^{-1}(x - x_n)) e^{-i x_n^{\perp} \cdot x - i\theta_n}.$$

Since ϕ_n is a ground state for E_{1,a_n}^{NLS}, we see that

$$E_{1,a_n}^{\mathrm{NLS}} = \mathcal{E}_{1,a_n}^{\mathrm{NLS}}(\phi_n) = \|\nabla \phi_n\|_{L^2}^2 + \|x \phi_n\|_{L^2}^2 + 2\langle L\phi_n, \phi_n\rangle - \frac{a_n}{2}\|\phi_n\|_{L^4}^4. \tag{2.19}$$

This implies the following identity (see again (2.8)):

$$\varepsilon_n^2 E_{1,a_n}^{\mathrm{NLS}} = \|\nabla \varphi_n\|_{L^2}^2 + 2\varepsilon_n^2 \langle L\varphi_n, \varphi_n\rangle + \varepsilon_n^4 \|x\varphi_n\|_{L^2}^2 - \frac{a_n}{2}\|\varphi_n\|_{L^4}^4. \tag{2.20}$$

By the Cauchy–Schwarz inequality, we have

$$|2\varepsilon_n^2 \langle L\varphi_n, \varphi_n\rangle| \leq 2\varepsilon_n^2 \|\nabla \varphi_n\|_{L^2} \|x\varphi_n\|_{L^2} \leq \frac{1}{2}\|\nabla \varphi_n\|_{L^2}^2 + 2\varepsilon_n^4 \|x\varphi_n\|_{L^2}^2,$$

which implies

$$\|\nabla \varphi_n\|_{L^2}^2 \leq 2\big(\varepsilon_n^2 E_{a_n}^{\mathrm{NLS}} + \varepsilon_n^4 \|x\varphi_n\|_{L^2}^2 + \frac{a_n}{2}\|\varphi_n\|_{L^4}^4\big).$$

Since $E^{\mathrm{NLS}}_{1,a_n} \to 0$, $\varepsilon_n \to 0$, $|x|\,|\varphi_n| \to |x|Q_0$ strongly in $L^2(\mathbb{R}^2)$, and $|\varphi_n| \to Q_0$ strongly in $L^4(\mathbb{R}^2)$, we infer that $\{\varphi_n\}_n$ is bounded uniformly in $H^1(\mathbb{R}^2)$.

From (2.20), we also have

$$\|\nabla\varphi_n\|^2_{L^2} - \frac{a_*}{2}\|\varphi_n\|^4_{L^4} = \varepsilon_n^2 E^{\mathrm{NLS}}_{1,a_n} - 2\varepsilon_n^2\langle L\varphi_n, \varphi_n\rangle - \varepsilon_n^4\|x\varphi_n\|^2_{L^2} - \frac{a_* - a_n}{2}\|\varphi_n\|^4_{L^4}.$$

Using the uniform boundedness of $\{\varphi_n\}_n$ in $H^1(\mathbb{R}^2)$, the strong convergence $|x|\,|\varphi_n| \to |x|Q_0$ in $L^2(\mathbb{R}^2)$, and $a_n \nearrow a_*$, we deduce that

$$\lim_{n\to\infty} \|\nabla\varphi_n\|^2_{L^2} - \frac{a_*}{2}\|\varphi_n\|^4_{L^4} = 0.$$

Since $\|\varphi_n\|_{L^2} = 1$ and $|\varphi_n| \to Q_0$ strongly in $L^r(\mathbb{R}^2)$ for all $r \in [2,\infty)$, there exists $\{z_n\}_n \subset \mathbb{R}^2$ such that

$$\varphi_n(x + z_n) \to e^{i\theta}Q_0(x)$$

strongly in $H^1(\mathbb{R}^2)$, for some $\theta \in [0, 2\pi)$. Using the fact that $\|Q_0(\cdot + z_n) - Q_0\|_{H^1} \to 0$ if and only if $|z_n| \to 0$, we get $|z_n| \to 0$. This in turn implies that $\varphi_n \to e^{i\theta}Q_0$ strongly in $H^1(\mathbb{R}^2)$ since

$$\begin{aligned}\|\varphi_n - e^{i\theta}Q_0\|_{H^1} &= \|\varphi_n(\cdot + z_n) - e^{i\theta}Q_0(\cdot + z_n)\|_{H^1}\\ &\le \|\varphi_n(\cdot + z_n) - e^{i\theta}Q_0\|_{H^1} + \|Q_0 - Q_0(\cdot + z_n)\|_{H^1} \to 0.\end{aligned}$$

Now we write

$$\varphi_n(x) = q_n(x) + i r_n(x)$$

with q_n and r_n the real and imaginary parts of φ_n respectively. By (2.9), we have the following orthogonality condition

$$\int_{\mathbb{R}^2} Q_0 r_n \,\mathrm{d}x = 0. \tag{2.21}$$

Since $\|\varphi_n - e^{i\theta}Q_0\|^2_{L^2} \to 0$, we have

$$\int_{\mathbb{R}^2} (\mathrm{Re}(\varphi - e^{i\theta}Q_0))^2 + (\mathrm{Im}(\varphi_n - e^{i\theta}Q_0))^2 \,\mathrm{d}x \to 0.$$

In particular, we get

$$\int_{\mathbb{R}^2} (r_n - Q_0\sin\theta)^2 \,\mathrm{d}x \to 0.$$

Using (2.21), we have

$$\int_{\mathbb{R}^2} r_n^2 + Q_0^2\sin^2\theta \,\mathrm{d}x \to 0.$$

This shows that

$$\int_{\mathbb{R}^2} r_n^2 \,\mathrm{d}x \to 0 \quad \text{and} \quad \sin^2\theta = 0$$

or $\theta = 0$ or $\theta = \pi$. In the following, we consider only the case $\theta = 0$. The case $\theta = \pi$ can be treated similarly by changing φ_n to $-\varphi_n$. For $\theta = 0$, we have $\varphi_n \to Q_0$ strongly in $H^1(\mathbb{R}^2)$. In particular,

$$\int_{\mathbb{R}^2} (q_n - Q_0)^2 \, \mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} r_n^2 \, \mathrm{d}x \to 0.$$

This, together with the exponential decay of W_n, yields

$$\int_{\mathbb{R}^2} |x|^2 (q_n - Q_0)^2 \, \mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} |x|^2 r_n^2 \, \mathrm{d}x \to 0. \tag{2.22}$$

In fact, by the exponential decay of W_n (see (2.18)), we have

$$\int_{\mathbb{R}^2} |x|^2 r_n^2 \, \mathrm{d}x \le \left(\int_{\mathbb{R}^2} |x|^4 r_n^2 \, \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^2} r_n^2 \, \mathrm{d}x \right)^{1/2} \to 0,$$

and similarly for $q_n - Q_0$.

Step 7. Smallness of the imaginary part. Observe that

$$\begin{aligned} \langle L\varphi_n, \varphi_n \rangle = \mathrm{Re} \langle L\varphi_n, \varphi_n \rangle &= \int_{\mathbb{R}^2} x^\perp \cdot \mathrm{Im}(\bar{\varphi}_n \nabla \varphi_n) \, \mathrm{d}x \\ &= \int_{\mathbb{R}^2} x^\perp \cdot (q_n \nabla r_n - r_n \nabla q_n) \, \mathrm{d}x \\ &= 2 \int_{\mathbb{R}^2} x^\perp q_n \nabla r_n \, \mathrm{d}x \end{aligned} \tag{2.23}$$

which implies

$$|\langle L\varphi_n, \varphi_n \rangle| \le 2 \|x q_n\|_{L^2} \|\nabla r_n\|_{L^2} \le C \|\nabla r_n\|_{L^2}.$$

Here we have used the fact that $|x| q_n$ is bounded uniformly in $L^2(\mathbb{R}^2)$ since $|x|\,|\varphi_n| \to |x| Q_0$ strongly in $L^2(\mathbb{R}^2)$. We deduce from the above and (2.20) that

$$\varepsilon_n^2 E^{\mathrm{NLS}}_{1,a_n} \ge \int_{\mathbb{R}^2} |\nabla q_n|^2 + |\nabla r_n|^2 - \frac{a_*}{2} (q_n^4 + r_n^4 + 2 q_n^2 r_n^2) \, \mathrm{d}x - C \varepsilon_n^2 \|\nabla r_n\|_{L^2}.$$

We have

$$\begin{aligned} \frac{a_*}{2} \int_{\mathbb{R}^2} (r_n^4 + 2 q_n^2 r_n^2) \, \mathrm{d}x &\le a_* \int_{\mathbb{R}^2} |\varphi_n|^2 r_n^2 \, \mathrm{d}x \\ &= \int_{\mathbb{R}^2} Q^2 r_n^2 \, \mathrm{d}x + a_* \int (|\varphi_n|^2 - Q_0^2) r_n^2 \, \mathrm{d}x \\ &= \int_{\mathbb{R}^2} Q^2 r_n^2 \, \mathrm{d}x + o(1) \|r_n\|_{H^1}^2. \end{aligned}$$

Here we have used that

$$\left| \int_{\mathbb{R}^2} (|\varphi_n|^2 - Q_0^2) r_n^2 \, \mathrm{d}x \right| \le \big\| |\varphi_n|^2 - Q_0^2 \big\|_{L^2} \|r_n\|_{L^4}^2 \le C \big\| |\varphi_n|^2 - Q_0^2 \big\|_{L^2} \|r_n\|_{H^1}^2$$

and

$$\left\| |\varphi_n|^2 - Q_0^2 \right\|_{L^2} \le \left\| |\varphi_n| - Q_0 \right\|_{L^4} \left\| |\varphi_n| + Q_0 \right\|_{L^4} \to 0$$

as $|\varphi_n| \to Q_0$ strongly in $H^1(\mathbb{R}^2)$, hence in $L^4(\mathbb{R}^2)$ by Sobolev embeddings. On the other hand, by the standard Gagliardo–Nirenberg inequality (1.4), we have

$$\int_{\mathbb{R}^2} |\nabla q_n|^2 - \frac{a_*}{2} q_n^4 \, dx \ge \|\nabla q_n\|_{L^2}^2 (1 - \|q_n\|_{L^2}^2) = (1 + o(1)) \|r_n\|_{L^2}^2,$$

where we have used that $q_n \to Q_0$ strongly in $H^1(\mathbb{R}^2)$, $\|q_n\|_{L^2}^2 + \|r_n\|_{L^2}^2 = 1$ as $\|\varphi_n\|_{L^2}^2 = 1$, and $\|\nabla Q_0\|_{L^2}^2 = 1$. Thus we get

$$\begin{aligned} \varepsilon_n^2 E_{1,a_n}^{\mathrm{NLS}} &\ge \int_{\mathbb{R}^2} |\nabla r_n|^2 - Q^2 r_n^2 + r_n^2 \, dx + o(1) \|r_n\|_{H^1}^2 - C \varepsilon_n^2 \|\nabla r_n\|_{L^2} \\ &= \langle \mathcal{L} r_n, r_n \rangle + o(1) \|r_n\|_{H^1}^2 - C \varepsilon_n^2 \|\nabla r_n\|_{L^2}, \end{aligned}$$

where $\mathcal{L} := -\Delta - Q^2 + 1$.

We now use the non-degeneracy property of Q. It is well known (see [27, Theorem 11.8 and Corollary 11.9]) that Q is the first eigenfunction of $\mathcal{L}$ and the corresponding eigenvalue 0 is non-degenerate. In particular, we have

$$\langle \mathcal{L} u, u \rangle \ge \lambda_2 \|u\|_{L^2}^2$$

for all u orthogonal to Q, where $\lambda_2 > 0$ is the second eigenvalue of $\mathcal{L}$. This, together with the fact that

$$\langle \mathcal{L} u, u \rangle \ge \|u\|_{H^1}^2 - \|Q\|_{L^\infty}^2 \|u\|_{L^2}^2,$$

yields

$$\langle \mathcal{L} u, u \rangle \ge C \|u\|_{H^1}^2$$

for some constant $C > 0$ and all u orthogonal to Q. Thanks to this estimate and the orthogonality condition (2.21), we get

$$\varepsilon_n^2 E_{1,a_n}^{\mathrm{NLS}} \ge C_1 \|r_n\|_{H^1}^2 - C_2 \varepsilon_n^2 \|\nabla r_n\|_{L^2}$$

for some positive constants C_1 and C_2. This implies that

$$\|r_n\|_{H^1}^2 \le C(\varepsilon_n^2 E_{1,a_n}^{\mathrm{NLS}} + \varepsilon_n^4). \tag{2.24}$$

On the other hand, from (2.2), the magnetic Gagliardo–Nirenberg inequality (1.6) and the diamagnetic inequality (1.5), we have

$$\begin{aligned} C(a_* - a_n)^{1/2} \ge E_{1,a_n}^{\mathrm{NLS}} = \mathcal{E}_{1,a_n}^{\mathrm{NLS}}(\phi_n) &\ge \frac{a_* - a_n}{a_*} \|\nabla_{x^\perp} \phi_n\|_{L^2}^2 \\ &\ge \frac{a_* - a_n}{a_*} \|\nabla |\phi_n|\|_{L^2}^2 = \frac{a_* - a_n}{a_*} \varepsilon_n^{-2}, \end{aligned}$$

which implies

$$E_{1,a_n}^{\mathrm{NLS}} \le C(a_* - a_n)^{1/2} \le C \varepsilon_n^2 \tag{2.25}$$

for some constant $C > 0$. This together with (2.24) yields

$$\|r_n\|_{H^1} \le C \varepsilon_n^2. \tag{2.26}$$

Step 8. *Identifying the blow-up limit.* Coming back to (2.23), we have

$$\begin{aligned}\langle L\varphi_n, \varphi_n\rangle &= 2\int_{\mathbb{R}^2} x^{\perp}\cdot\nabla r_n q_n \,\mathrm{d}x\\ &= 2\int_{\mathbb{R}^2} x^{\perp}\cdot\nabla r_n Q_0 \,\mathrm{d}x + 2\int_{\mathbb{R}^2} x^{\perp}\cdot\nabla r_n (q_n - Q_0)\,\mathrm{d}x\\ &= 2\int_{\mathbb{R}^2} x^{\perp}\cdot\nabla r_n (q_n - Q_0)\,\mathrm{d}x\end{aligned}$$

where we have used the fact that $x^{\perp}\cdot\nabla Q_0 = 0$ since Q_0 is radial and (2.22). This shows that

$$|\langle L\varphi_n, \varphi_n\rangle| \leq \|\nabla r_n\|_{L^2}\|x(q_n - Q_0)\|_{L^2} \leq o(1)\|\nabla r_n\|_{L^2} \leq o(\varepsilon_n^2). \tag{2.27}$$

Here we have used (2.22) in the second inequality and (2.26) in the last one.

From (2.19) and the Gagliardo–Nirenberg inequality (1.4), we have

$$E^{\mathrm{NLS}}_{1,a_n} \geq 2\langle L\phi_n, \phi_n\rangle + \|x\phi_n\|^2_{L^2} = 2\langle L\varphi_n, \varphi_n\rangle + \varepsilon_n^2\|x\varphi_n\|^2_{L^2}.$$

Denote

$$\beta_n := \frac{\varepsilon_n}{(a_* - a_n)^{1/4}}.$$

From (2.25), we have

$$\beta_n^2 \geq C > 0.$$

Moreover, using (2.2), we also have

$$C \geq \frac{E^{\mathrm{NLS}}_{1,a_n}}{(a_* - a_n)^{1/2}} \geq \frac{2}{(a_* - a_n)^{1/2}}\langle L\varphi_n, \varphi_n\rangle + \beta_n^2\|x\varphi_n\|^2_{L^2}.$$

Thanks to (2.27) and the fact that $|x|\,|\varphi_n| \to |x|Q_0$ strongly in $L^2(\mathbb{R}^2)$, we deduce

$$C \geq \beta_n^2(\|xQ_0\|^2_{L^2} + o(1)).$$

In particular, we deduce that $\{\beta_n\}_n$ is bounded above and below away from zero. Passing to a subsequence, we have $\beta_n \to \beta > 0$ as $n \to \infty$.

By (2.20), we have

$$\begin{aligned}E^{\mathrm{NLS}}_{1,a_n} &\geq \frac{a_* - a_n}{2}\|\phi_n\|^4_{L^4} + 2\langle L\phi_n, \phi_n\rangle + \|x\varphi_n\|^2_{L^2}\\ &= \frac{(a_* - a_n)^{1/2}}{2\beta_n^2}\|\varphi_n\|^4_{L^4} + 2\langle L\varphi_n, \varphi_n\rangle + (a_* - a_n)^{1/2}\beta_n^2\|x\varphi_n\|^2_{L^2}.\end{aligned}$$

Since $\varphi_n \to Q_0$ strongly in $H^1(\mathbb{R}^2)$, $|x|\,|\varphi_n| \to |x|Q_0$ strongly in $L^2(\mathbb{R}^2)$, and (2.27), we infer that

$$\frac{E^{\mathrm{NLS}}_{1,a_n}}{(a_* - a_n)^{1/2}} \geq \frac{1}{2\beta^2}\|Q_0\|^4_{L^4} + \beta^2\|xQ_0\|^2_{L^2} + o(1).$$

Optimizing over $\beta > 0$ and noticing that $\|Q_0\|_{L^4}^4 = \frac{2}{a_*}$ we get

$$\liminf_{n\to\infty} \frac{E_{1,a_n}^{\mathrm{NLS}}}{(a_* - a_n)^{1/2}} \geq 2\frac{\|xQ_0\|_{L^2}}{a_*^{1/2}} \quad \text{and} \quad \beta = \frac{1}{a_*^{1/4}\|xQ_0\|_{L^2}^{1/2}}.$$

From this and the energy upper bound (2.1), we obtain (1.7) and (1.8).

Step 9. L^∞-convergence. We finally prove the L^∞-convergence. To this end, we first show the uniform exponential decay for $\nabla\varphi_n$, namely

$$\int_{\mathbb{R}^2} |\nabla\varphi_n|^2 e^{|x|/4}\,\mathrm{d}x \leq C \tag{2.28}$$

for all n sufficiently large. We provide below a formal calculation and a regularizing argument is needed to justify it rigorously (see Step 5). We multiply both sides of (2.10) by $e^{\alpha|x|}\bar{\varphi}_n$, integrate over $\mathbb{R}^2$, and take the real part to get

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^2} &-\Delta\varphi_n e^{\alpha|x|}\bar{\varphi}_n + \varepsilon_n^4|x|^2 e^{\alpha|x|}|\varphi_n|^2 + 2\varepsilon_n^2 L\varphi_n e^{\alpha|x|}\bar{\varphi}_n\\ &- a_n|\varphi_n|^4 e^{\alpha|x|} - \varepsilon_n^2\mu_n|\varphi_n|^2 e^{\alpha|x|}\,\mathrm{d}x = 0.\end{aligned}$$

Arguing as in [25, Lemma 3.2], we have

$$\mathrm{Re}\int_{\mathbb{R}^2} -\Delta\varphi_n e^{\alpha|x|}\bar{\varphi}_n\,\mathrm{d}x = \int_{\mathbb{R}^2} |\nabla(e^{\alpha|x|/2}\varphi_n)|^2\,\mathrm{d}x - \frac{\alpha^2}{2}\int_{\mathbb{R}^2} e^{\alpha|x|}|\varphi_n|^2\,\mathrm{d}x.$$

In particular, we get

$$\begin{aligned}0 = &\int_{\mathbb{R}^2} |\nabla(e^{\alpha|x|/2}\varphi_n)|^2\,\mathrm{d}x + \varepsilon_n^4\int_{\mathbb{R}^2} |x|^2 e^{\alpha|x|}|\varphi_n|^2\,\mathrm{d}x\\ &+ \int_{\mathbb{R}^2} e^{\alpha|x|}\Big(-a_n|\varphi_n|^2 - \varepsilon_n^2\mu_n - \frac{\alpha^2}{4}\Big)|\varphi_n|^2\,\mathrm{d}x\\ &+ 2\varepsilon_n^2\int_{\mathbb{R}^2} L\varphi_n e^{\alpha|x|}\bar{\varphi}_n\,\mathrm{d}x.\end{aligned}$$

Since $L(e^{\alpha|x|/2}) = 0$, we have

$$\begin{aligned}\left|2\varepsilon_n^2\int_{\mathbb{R}^2} L\varphi_n e^{\alpha|x|}\bar{\varphi}_n\,\mathrm{d}x\right| &= \left|2\varepsilon_n^2\int_{\mathbb{R}^2} e^{\alpha|x|/2}\bar{\varphi}_n L(e^{\alpha|x|/2}\varphi_n)\,\mathrm{d}x\right|\\ &\leq 2\varepsilon_n^2\|x^\perp e^{\alpha|x|/2}\varphi_n\|_{L^2}\|\nabla(e^{\alpha|x|/2}\varphi_n)\|_{L^2}\\ &\leq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla(e^{\alpha|x|/2}\varphi_n)|^2\,\mathrm{d}x + 2\varepsilon_n^4\int_{\mathbb{R}^2} |x|^2 e^{\alpha|x|}|\varphi_n|^2\,\mathrm{d}x.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^2} |\nabla(e^{\alpha|x|/2}\varphi_n)|^2\,\mathrm{d}x \leq\ &\varepsilon_n^4\int_{\mathbb{R}^2} |x|^2 e^{\alpha|x|}|\varphi_n|^2\,\mathrm{d}x\\ &+ \int_{\mathbb{R}^2} e^{\alpha|x|}\Big(a_n|\varphi_n|^2 + |\varepsilon_n^2\mu_n| + \frac{\alpha^2}{4}\Big)|\varphi_n|^2\,\mathrm{d}x.\end{aligned}$$

By choosing $\alpha = \frac{1}{4}$, using (2.17), (2.18), and the fact that $\varepsilon_n^2 \mu_n \to -1$, we obtain

$$\int_{\mathbb{R}^2} |\nabla(e^{|x|/8}\varphi_n)|^2 \,\mathrm{d}x \leq C \tag{2.29}$$

for all n sufficiently large. Note that, by the triangle inequality,

$$\|\nabla(e^{|x|/8}\varphi_n)\|_{L^2} = \left\| e^{|x|/8}\nabla\varphi_n + \frac{x}{8|x|}e^{|x|/8}\varphi_n \right\|_{L^2} \geq \|e^{|x|/8}\nabla\varphi_n\|_{L^2} - \frac{1}{8}\|e^{|x|/8}\varphi_n\|_{L^2}.$$

Then claim (2.28) follows directly from (2.29) and (2.18).

We next show that $\{\varphi_n\}_n$ is bounded uniformly in $H^2(\mathbb{R}^2)$. To see this, we rewrite (2.10) as

$$-\Delta\varphi_n + \varphi_n = (1 + \varepsilon_n^2\mu_n)\varphi_n - \varepsilon_n^4|x|^2\varphi_n - 2\varepsilon_n^2 L\varphi_n + a_n|\varphi_n|^2\varphi_n.$$

Since $\{\varphi_n\}_n$ is bounded uniformly in $H^1(\mathbb{R}^2)$, the uniform exponential decay in (2.18) and (2.28) imply that the right-hand side is bounded uniformly in $L^2(\mathbb{R}^2)$. This shows that $\{\varphi_n\}_n$ is bounded uniformly in $H^2(\mathbb{R}^2)$. By the Sobolev embedding $H^{3/2}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$, the strong convergence $\varphi_n \to Q_0$ in $H^1(\mathbb{R}^2)$, and the uniformly boundedness of $(\varphi_n)_n$ in $H^2(\mathbb{R}^2)$, we have that φ_n converges strongly to Q_0 in $L^\infty(\mathbb{R}^2)$ and hence (1.8). ∎

2.2. Collapse with an almost critical speed

We now study the blow-up behavior of minimizers for $E_{\Omega,a}$ when both $\Omega \nearrow 1$ and $a \nearrow a_*$ at the same time. To this end, we recall the following energy asymptotic formula when $\Omega = 0$ (see [20]):

$$E_{0,a}^{\mathrm{NLS}} = \sqrt{a_* - a}\left(2\frac{\|xQ_0\|_{L^2}}{a_*^{1/2}} + o(1)\right) \quad \text{as } a \nearrow a_*. \tag{2.30}$$

Proof of Corollary 1.2. Let $\Omega_n \nearrow 1$, $a_n \nearrow a_*$ as $n \to \infty$, and ϕ_n be a minimizer for E_{Ω_n,a_n}. We rewrite the energy functional as

$$\begin{aligned} E_{\Omega_n,a_n}^{\mathrm{NLS}} = \mathcal{E}_{\Omega_n,a_n}^{\mathrm{NLS}}(\phi_n) &= \Omega_n \mathcal{E}_{1,a_n}^{\mathrm{NLS}}(\phi_n) + (1-\Omega_n)\mathcal{E}_{0,a_n}^{\mathrm{NLS}}(\phi_n) \\ &\geq \Omega_n E_{1,a_n}^{\mathrm{NLS}} + (1-\Omega_n)E_{0,a_n}^{\mathrm{NLS}}, \end{aligned} \tag{2.31}$$

where we have used that $\mathcal{E}_{1,a_n}^{\mathrm{NLS}}(\phi_n) \geq E_{1,a_n}^{\mathrm{NLS}}$ and $\mathcal{E}_{0,a_n}^{\mathrm{NLS}}(\phi_n) \geq E_{0,a_n}^{\mathrm{NLS}}$. Since both E_{1,a_n}^{NLS} and E_{0,a_n}^{NLS} have the same asymptotic formula (see (1.7) and (2.30)), we obtain

$$E_{\Omega_n,a_n}^{\mathrm{NLS}} = (a_* - a_n)^{1/2}\left(2\frac{\|xQ_0\|_{L^2}}{a_*^{1/2}} + o(1)\right),$$

where the upper bound follows from (2.2). Let ψ_n be a ground state for E_{1,a_n}^{NLS}. By Theorem 1.1, there exist sequences $\{x_n\}_n \subset \mathbb{R}^2$ and $(\vartheta_n)_n \subset [0, 2\pi)$ such that

$$\varphi_n(x) := \varepsilon_n\psi_n(\varepsilon_n x + x_n)e^{i\varepsilon_n x_n^\perp \cdot x + i\vartheta_n} \to Q_0(x)$$

strongly in $H^1 \cap L^\infty(\mathbb{R}^2)$ as $n \to \infty$. We choose $\tilde{\psi}_n(x) := \psi_n(x + x_n)e^{ix_n^\perp \cdot x + i\vartheta_n}$ as a trial state for $E^{\mathrm{NLS}}_{\Omega_n,a_n}$ and obtain

$$\begin{aligned} E^{\mathrm{NLS}}_{\Omega_n,a_n} \le \mathcal{E}^{\mathrm{NLS}}_{\Omega_n,a_n}(\tilde{\psi}_n) &= \Omega_n \mathcal{E}^{\mathrm{NLS}}_{1,a_n}(\tilde{\psi}_n) + (1-\Omega_n)\mathcal{E}^{\mathrm{NLS}}_{0,a_n}(\tilde{\psi}_n) \\ &= \Omega_n E^{\mathrm{NLS}}_{1,a_n} + (1-\Omega_n)\mathcal{E}^{\mathrm{NLS}}_{0,a_n}(\tilde{\psi}_n). \end{aligned} \tag{2.32}$$

Here we have used the magnetic translation invariance of the energy functional $\mathcal{E}^{\mathrm{NLS}}_{1,a_n}$. Putting together (2.31) and (2.32), we obtain

$$\mathcal{E}^{\mathrm{NLS}}_{0,a_n}(\phi_n) \le \mathcal{E}^{\mathrm{NLS}}_{0,a_n}(\tilde{\psi}_n).$$

By (2.2) and the arguments in the proof of Theorem 1.1 (especially of (2.27) and $\varepsilon_n \simeq (a_* - a_n)^{1/4}$), we have

$$\begin{aligned} \mathcal{E}^{\mathrm{NLS}}_{0,a_n}(\tilde{\psi}_n) = \mathcal{E}^{\mathrm{NLS}}_{1,a_n}(\tilde{\psi}_n) - 2\langle \tilde{\psi}_n, L\tilde{\psi}_n\rangle &= E^{\mathrm{NLS}}_{1,a_n} - 2\langle \varphi_n, L\varphi_n\rangle \\ &\le (a_* - a_n)^{1/2}\Big(2\frac{\|xQ_0\|_{L^2}}{a_*^{1/2}} + o(1)\Big). \end{aligned}$$

This together with (2.30) shows that ϕ_n is an approximate ground state for E^{NLS}_{0,a_n}. We then conclude (see e.g. [25, Step 5 in Section 3]) that there exists a sequence of phases $\{\theta_n\}_n \subset [0, 2\pi)$ such that

$$\lim_{n\to\infty} \frac{(a_* - a_n)^{1/4}}{a_*^{1/4}\|xQ_0\|^{1/2}_{L^2}} \phi_n\Big(\frac{(a_* - a_n)^{1/4}}{a_*^{1/4}\|xQ_0\|^{1/2}_{L^2}} x\Big)e^{i\theta_n} = Q_0(x) \tag{2.33}$$

strongly in $H^1(\mathbb{R}^2)$. In fact, we obtain the strong convergence in $L^\infty(\mathbb{R}^2)$, by the same arguments as in the proof of (1.8). ■

3. Collapse of many-body ground states

In this section, we prove the large-N behavior of ground states for (1.11) given in Theorem 1.4.

Proof of Theorem 1.4. Following arguments from [25], we have

$$CN^{-\beta}\|\nabla Q_N\|_{L^2}\|Q_N\|^3_{L^6} + E^{\mathrm{NLS}}_{\Omega,a_N} \ge E^{\mathrm{QM}}_{\Omega,a_N}(N) \ge E^{\mathrm{NLS}}_{\Omega,a_N} - CN^{2\beta-1},$$

where Q_N is given in Theorem 1.4. Note that the above energy estimates as well as the asymptotic formula of $E^{\mathrm{NLS}}_{\Omega,a_N}$ are independent of Ω. Therefore, we obtain (1.12) for every $0 \le \Omega \le 1$.

To prove convergence of ground states as $\Omega = \Omega_N \nearrow 1$ we consider the perturbed Hamiltonian

$$H_{\Omega_N,a_N,N,\eta_N} = H_{\Omega_N,a_N,N} + \eta_N \sum_{j=1}^N A_j$$

with ground-state energy per particle denoted $E^{\rm QM}_{\Omega_N,a_N,\eta_N}(N)$. Here $\eta_N > 0$ is a small parameter to be chosen later and A is a bounded self-adjoint operator on $L^2(\mathbb{R}^2)$. The associated NLS energy functional is

$$\mathcal{E}^{\rm NLS}_{\Omega_N,a_N,\eta_N}(u) = \mathcal{E}^{\rm NLS}_{\Omega_N,a_N}(u) + \eta_N \langle Au, u\rangle.$$

Denote by $E^{\rm NLS}_{\Omega_N,a_N,\eta_N}$ the corresponding ground-state energy and u_{η_N} its ground state. Let Φ_N be a ground state for $H_{\Omega_N,a_N,N} = H_{\Omega_N,a_N,N,0}$ and $\gamma^{(1)}_{\Phi_N}$ its one-body reduced density matrix. As in [25, Step 2 in Section 4] we obtain

$$\eta_N \operatorname{Tr}[A\gamma^{(1)}_{\Phi_N}] \geq \eta_N \langle u_{\eta_N}|A|u_{\eta_N}\rangle + O(N^{2\beta-1}) + O(N^{3\alpha/4-\beta}). \tag{3.1}$$

Again the above estimate is independent of Ω_N. Under the assumption that $a_* - a_N = N^{-\alpha}$ with

$$0 < \alpha < \min\Big\{\frac{4\beta}{5}, 2(1-2\beta)\Big\},$$

one can choose $\eta_N = N^{-\alpha/2-\sigma}$ with

$$0 < \sigma < \min\Big\{1 - 2\beta - \frac{\alpha}{2}, \beta - \frac{5\alpha}{4}\Big\}$$

in such a way that

$$\eta_N = o(E^{\rm NLS}_{0,a_N}) = o((a_* - a_N)^{1/2}) = o(N^{-\alpha/2})$$

and also

$$\eta_N^{-1} N^{2\beta-1} + \eta_N^{-1} N^{3\alpha/4-\beta} \xrightarrow[N\to\infty]{} 0.$$

Then dividing (3.1) by η_N and repeating the argument with A changed to $-A$ yields

$$\langle u_{\eta_N}|A|u_{\eta_N}\rangle + o(1) \leq \operatorname{Tr}[A\gamma^{(1)}_{\Phi_N}] \leq \langle u_{-\eta_N}|A|u_{-\eta_N}\rangle + o(1). \tag{3.2}$$

On the other hand, with the above choice of η_N, we have

$$\begin{aligned}\mathcal{E}^{\rm NLS}_{\Omega_N,a_N}(u_{\eta_N}) = \mathcal{E}^{\rm NLS}_{\Omega_N,a_N,\eta_N}(u_{\eta_N}) + O(\eta_N\|A\|) &\leq \mathcal{E}^{\rm NLS}_{\Omega_N,a_N}(u_0) + O(\eta_N\|A\|)\\ &= E^{\rm NLS}_{\Omega_N,a_N} + O(\eta_N\|A\|).\end{aligned}$$

By the argument in the proof of (1.9), the above implies that

$$\mathcal{E}^{\rm NLS}_{0,a_N}(u_{\eta_N}) \leq (a_* - a_n)^{1/2}\Big(2\frac{\|xQ_0\|_{L^2}}{a_*^{1/2}} + o_N(1)\Big) + O\Big(\frac{\eta_N}{1-\Omega_N}\|A\|\Big).$$

It then follows that (u_{η_N}) and $(u_{-\eta_N})$ are sequences of quasi-ground states for $E^{\rm NLS}_{0,a_N}$, under the assumption on Ω_N in Theorem 1.4. Thus both sequences satisfy (2.33). Combining with (3.2), we get, after a dilation of space variables, trace-class weak-$\star$ convergence of $\gamma^{(1)}_{\Phi_N}$ to $|Q_N\rangle\langle Q_N|$. Since no mass is lost in the limit, this convergence must hold in trace-class norm (see e.g. [11] or [38, Appendix H]). The limit being rank 1, this implies the convergence of higher-order density matrices to tensor powers of the limiting operator by well-known arguments (recalled e.g. in [36, Section 2.2]). ∎

Acknowledgments. V.D.D. would like to express his deep gratitude to his wife, Uyen Cong, for her encouragement and support. The authors would like to thank the reviewers for their helpful comments and suggestions.

Funding. This work was supported by the European Union's Horizon 2020 Research and Innovation Programme (Grant agreement CORFRONMAT No. 758620).

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Received 17 August 2022; revised 29 March 2023; accepted 21 April 2023.

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